Supplementary Material for:
Locally Robust Inference for
Non-Gaussian SVAR models

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Abstract

In this supplementary material we provide the following additional results.

S1: Choice for the parametrization

S2: Technical details for the main proofs

S3: Some technical tools

S4: Details log density score estimation

S5: Additional simulation results

S6: Additional empirical results

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S1 Parametrization of the semi-parametric SVAR model

Under the main assumptions of the paper (i.e. Assumptions 2.1 and 2.2) the parameters of the SVAR are generally not locally identified. Even under the *additional* assumption that the errors $\epsilon_{k,t}$ follow non-Gaussian distributions, we have that $A(\alpha, \sigma)$ can only be identified up to permutation and sign changes of its rows (e.g. Comon, 1994).

Therefore, to ensure that we study economically interesting permutations we typically need to impose additional identifying restrictions, such as zero or sign restrictions. The choice for such restrictions interacts with the chosen parametrization for $A(\alpha, \sigma)$ for which we give a few examples.

EXAMPLE S1.1 (Supply and demand): *Following Baumeister and Hamilton (2015), when the SVAR defines a demand and a supply equation we can set*

$$A^{-1}(\alpha, \sigma) = \begin{pmatrix} -\alpha^d & 1 \\ -\alpha^s & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} , \tag{S1}$$

where $\alpha = (\alpha^d, \alpha^s)'$ are the short run demand and supply elasticities, and $\sigma = (\sigma_1, \sigma_2)'$ scales the structural shocks. With independent non-Gaussian errors A is identified up to permutation and sign changes of its rows. To pin down an economically interesting rotation we can impose the sign restrictions $\alpha^d \leq 0$, $\alpha^s \geq 0$ and $\sigma_1, \sigma_2 > 0$.

EXAMPLE S1.2 (Rotation matrix): *A canonical choice sets*

$$A^{-1}(\alpha, \sigma) = \Sigma^{1/2}(\sigma) R(\alpha) , \tag{S2}$$

where $\Sigma^{1/2}(\sigma)$ is a lower triangular matrix (with positive diagonal elements) defined by the vector σ and $R(\alpha)$ is a rotation matrix that is parametrized by the vector α. Different parametrizations for the rotation matrix are possible, see Magnus et al. (2021) for a detailed discussion. Similar to in Example S1.1, even with independent non-Gaussian errors $R(\alpha)$ is not uniquely identified and additional zero-, sign-, or long-run-restrictions are needed to pin down the desired rotation.

As the above examples make clear, several commonly used parametrizations can be adopted. Three general comments apply.

First, pinning down a specific permutation, as in the first example, is necessary for the economic interpretation of the results, but it is not necessary for the score testing methodology of the paper which fixes α under the null.

Second, the robust non-Gaussian approach of this paper can be combined with any of the existing SVAR identification approaches to obtain an economically interesting specification. Besides zero and sign restrictions one can also think of combining with external instruments or more general prior information as in Baumeister and Hamilton (2015) or Braun (2021).

Third, often multiple parametrizations are possible. We recommend jointly testing the possibly weakly identified parameters when they are of direct economic interest (e.g. Example 1). In contrast, when the interest is in more general functions, such as impulse responses or forecast error variances, we suggest to parameterize A such that α is as low-dimensional as possible, e.g. via the rotation matrix specification as in Example 2. In this way the Bonferroni procedure of **Algorithm 2** can be executed over the smallest possible grid for α, which reduces the computational burden.

S2 Technical details for the main proofs

Here we establish some technical details utilised in the proofs in section A of the main text.

S2.1 Markov structure

Define $Z_t \coloneqq (Y_t', Y_{t-1}', \ldots, Y_{t-p+1}')'$, $\mathsf{C}_\theta \coloneqq (c_\theta', 0', \ldots, 0')'$,

$$
\mathsf{B}_\theta \coloneqq \begin{bmatrix} B_{\theta,1} & B_{\theta,2} & \cdots & B_{\theta,p-1} & B_{\theta,p} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \mathsf{D}_\theta \coloneqq \begin{bmatrix} A_\theta^{-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

and note that we can write

$$
Z_t = \mathsf{C}_\theta + \mathsf{B}_\theta Z_{t-1} + \mathsf{D}_\theta \epsilon_t. \tag{S3}
$$

This can be re-written in de-meaned form as

$$
\tilde{Z}_t = \mathsf{B}_\theta \tilde{Z}_{t-1} + \mathsf{D}_\theta \epsilon_t \tag{S4}
$$

with $\tilde{Z}_t \coloneqq Z_t - m_\theta$, for $m_\theta \coloneqq (\sum_{i=0}^{\infty} \mathsf{B}_\theta) \mathsf{C}_\theta = (I - \mathsf{B}_\theta)^{-1} \mathsf{C}_\theta$.

LEMMA S2.1: *Suppose that assumption 2.1 holds. Define $U_{\theta,t}$ as the (unique, strictly) stationary*

solution to (S3). *Then* $U_{\theta,t}$ *has the representation*

$$U_{\theta,t} = m_\theta + \sum_{j=0}^\infty \mathsf{B}_\theta^j \mathsf{D}_\theta \epsilon_{t-j}, \quad m_\theta \coloneqq (I - \mathsf{B}_\theta)^{-1} \mathsf{C}_\theta, \quad \sum_{j=0}^\infty \|\mathsf{B}_\theta^j\| < \infty.$$

If ρ_θ *is the largest absolute eigenvalue of the companion matrix* B_θ *and* $\upsilon > 0$ *is such that* $\rho_\theta + \upsilon < 1$, *then*

$$\mathbb{E}\,\|U_{\theta,t} - m_\theta\|^\rho \le \frac{\mathbb{E}\,\|\mathsf{D}_\theta\epsilon_t\|^\rho}{1 - (\rho_\theta + \upsilon)^\rho}, \quad \rho \in [1, 4+\delta].$$

Proof. Rewriting (S3) as (S4) and applying Theorem 11.3.1 in Brockwell and Davis (1991) yields the first part. For the second part,

$$\|U_{\theta,t} - m_\theta\| \le \sum_{j=0}^\infty \|\mathsf{B}_\theta^j\|\|\mathsf{D}_\theta\epsilon_{t-j}\| \le \sum_{j=0}^\infty \|\mathsf{B}_\theta\|^j\|\mathsf{D}_\theta\epsilon_{t-j}\| \le \sum_{j=0}^\infty (\rho_\theta + \nu)^j\|\mathsf{D}_\theta\epsilon_{t-j}\|.$$

Since $\mathbb{E}\,\|\mathsf{D}_\theta\epsilon_{t-j}\|^\rho = \mathbb{E}\,\|\mathsf{D}_\theta\epsilon_t\|^\rho < \infty$ for all $t \in \mathbb{N}$, all $j \ge 0$ and $\rho \in [1, 4+\delta]$, it follows that

$$\mathbb{E}\,\|U_{\theta,t} - m_\theta\|^\rho \le \sum_{j=0}^\infty (\rho_\theta + \nu)^{j\rho}\,\mathbb{E}\,\|\mathsf{D}_\theta\epsilon_{t-j}\|^\rho = \frac{\mathbb{E}\,\|\mathsf{D}_\theta\epsilon_t\|^\rho}{1 - (\rho_\theta + \nu)^\rho}. \qquad \square$$

LEMMA S2.2: *Let* $Q_{n,\theta}$ *be the probability measure corresponding to* $\bar{q}_{n,\theta} \coloneqq \frac{1}{n}\sum_{t=1}^n q_{\theta,t}$, *where* $q_{\theta,t}$ *is the density of* X_t *under* P_θ^n $(1 \le t \le n).$[S1] *Then* $Q_{n,\theta} \xrightarrow{TV} Q_\theta$, *where* Q_θ *is the distribution of the (unique, strictly) stationary solution to* (1).

Proof. By Lemma S2.1, (S4) has a (unique, strictly) stationary solution with finite second moments. Applying Theorem 2 in Saikkonen (2007) gives that the Markov chain $(\tilde{Z}_t)$ is V-geometrically ergodic with $\mathsf{V}(x) = 1 + \|x\|^2$. That is, for an invariant probability measure $\tilde{\pi}_\theta$, some $r \in (1, \infty)$ and some $R < \infty$

$$\sum_{n=1}^\infty r^n \|\tilde{P}_\theta^n(\cdot, \tilde{z}) - \tilde{\pi}_\theta\|_{TV} \le \sum_{n=1}^\infty r^n \|\tilde{P}_\theta^n(\cdot, \tilde{z}) - \tilde{\pi}_\theta\|_\mathsf{V} \le R\mathsf{V}(\tilde{z}) = R(\|\tilde{z}\|^2 + 1) < \infty, \qquad (\text{S5})$$

where $\tilde{P}_\theta^n(\cdot, \tilde{z})$ is the n-step transition probability and $\tilde{z}$ is the initial condition.[S2] $\tilde{\pi}_\theta$ is the distribution of $U_{\theta,t} - m_\theta$ as defined in Lemma S2.1 (Kallenberg, 2021, Theorem 11.11).

Let $f_\theta : \mathbb{R}^{Kp} \to \mathbb{R}^K$ be defined as

$$f_\theta(x) \coloneqq \begin{bmatrix} I_K & 0_{K \times K(p-1)} \end{bmatrix}(x + m_\theta),$$

[S1]Here, and throughout the appendix, any reference to the density of X_t is to be understood as to the density of the non-deterministic parts of X_t.

[S2]The norm $\|\nu\|_\mathsf{V}$ is defined by $\|\nu\|_\mathsf{V} \coloneqq \sup_{f \le \mathsf{V}} |\int f\, d\nu|$ where the supremum is taken over all measurable functions dominated by V for any probability measure ν.

i.e. the function which adds m_θ to its argument and then returns the first K elements. The distribution of X_t under P_θ^n (given the initial condition $\tilde{z}$) is then $Q_\theta^{t-1}(\cdot, \tilde{z}) = \tilde{P}_\theta^{t-1}(\cdot, \tilde{z}) \circ f_\theta^{-1}$, i.e. the pushforward of $\tilde{P}_\theta^{t-1}(\cdot, \tilde{z})$ under f_θ. Henceforth we shall omit the $\tilde{z}$ in the notation. Similarly let $Q_\theta = \tilde{\pi}_\theta \circ f_\theta^{-1}$, i.e.the pushforward of $\tilde{\pi}_\theta$ under f. That Q_θ is the distribution of the (unique, strictly) stationary solution to (1) can be seen by noting that the first K elements of $U_{\theta,t}$ form a (strictly) stationary time series and satisfy the defining equation (1); by Theorem 11.3.1 in Brockwell and Davis (1991) it is therefore the unique solution. Then by (S5),

$$
\begin{aligned}
\left\| \frac{1}{n} \sum_{t=1}^n Q_\theta^t - Q_\theta \right\|_{TV} &\leq \frac{1}{n} \sum_{t=1}^n \left\| Q_\theta^t - Q_\theta \right\|_{TV} \\
&\leq \frac{1}{n} \sum_{t=1}^n \left\| \tilde{P}_\theta^{t-1} - \tilde{\pi}_\theta \right\|_{TV} \\
&\leq \frac{1}{n} \sum_{t=1}^n \left\| \tilde{P}_\theta^t - \tilde{\pi}_\theta \right\|_{TV} + o(1) \\
&\to 0. \qquad \qquad \qquad \square
\end{aligned}
$$

S2.2 Moment bounds

LEMMA S2.3: *Suppose that assumption 2.1 holds. Then for any sequence $\theta_n = (\gamma + g_n/\sqrt{n}, \eta)$ with $g_n \to g \in \mathbb{R}^L$, for some $\rho > 0$, under $P_{\theta_n}^n$*

(i) $\sup_{n \in \mathbb{N}} \mathbb{E}\left[\|\dot{\ell}_{\theta_n}\|^{2+\rho} \right] < \infty$;

(ii) $\sup_{n \in \mathbb{N}} \mathbb{E}\left[\|\tilde{\ell}_{\theta_n}\|^{2+\rho} \right] < \infty$.

Proof. Since the deterministic terms in $\dot{\ell}_{\theta_n}$ and $\tilde{\ell}_{\theta_n}$ are either constants or continuous functions of γ (by Assumption 2.1(iii)), they are uniformly bounded, since $\{\gamma + g_n/\sqrt{n} : n \in \mathbb{N}\} \cup \{\gamma\}$ is compact. It is therefore sufficient to show that under $P_{\theta_n}^n$, each of

$$
\sup_{n \in \mathbb{N}, 1 \leq t \leq n} \mathbb{E}\left[|A(\theta_n)_{k\bullet} V_{\theta_n,t}|^{4+\delta} \right], \quad \sup_{n \in \mathbb{N}, 1 \leq t \leq n} \mathbb{E}\left[|\phi_k(A(\theta_n)_{k\bullet} V_{\theta_n,t})|^{4+\delta} \right], \quad \sup_{n \in \mathbb{N}, 1 \leq t \leq n} \mathbb{E}\left[\|X_t\|^{4+\delta} \right],
$$

is finite. Since under $P_{\theta_n}^n$, each $A(\theta_n)_{k\bullet} V_{\theta_n,t} \sim \eta_k$, finiteness of the first two follow directly from Assumption 2.1(ii). For the third, recurse equation (S3) backwards under $\theta = \theta_n$, to obtain

$$
Z_t = \sum_{j=0}^{t-1} \mathsf{B}_{\theta_n}^j \mathsf{C}_{\theta_n} + \sum_{j=0}^{t-1} \mathsf{B}_{\theta_n}^j \mathsf{D}_{\theta_n} \epsilon_{t-j} + \mathsf{B}_{\theta_n}^t Z_0.
$$

Each of B_θ, C_θ, D_θ (depend on θ only through γ and) are continuous functions of γ, hence

$$\varrho := \sup_{n\in\mathbb{N}} \|\mathsf{B}_{\theta_n}\|_2 < 1, \ \sup_{n\in\mathbb{N}} \|\mathsf{C}_{\theta_n}\|_2 < C_1, \ \sup_{n\in\mathbb{N}} \|\mathsf{D}_{\theta_n}\|_2 < C_2,$$

where the first is due to Assumption 2.1(i). Since we condition on Z_0, by Assumption 2.1(ii),

$$\mathbb{E}\,\|Z_t\|^{4+\delta} \lesssim \left(\frac{C_1}{1-\varrho}\right)^{4+\delta} + \left(\frac{C_2}{1-\varrho}\right)^{4+\delta} \mathbb{E}\,|\epsilon_1|^{4+\delta} + \|Z_0\|^{4+\delta} < \infty. \tag{S6}$$

As the bound on the right hand side is independent of t or n, the claim follows. $\qquad\square$

LEMMA S2.4: *Let $W_{n,t}$ be as in the Proof of Proposition A.1 and suppose the conditions of that Proposition hold. Then, $P_\theta^n[|\sqrt{n}W_{n,t}|^{2+\rho}]$ is uniformly bounded for some $\rho > 0$. In consequence, under P_θ^n, $W_{n,t}$ satisfies:*

$$\lim_{n\to\infty} \sum_{t=1}^n \mathbb{E}\left[W_{n,t}^2 \mathbf{1}\{|\sqrt{n}W_{n,t}| > \varepsilon\sqrt{n}\}\right] = 0, \quad \textit{for any } \varepsilon > 0. \tag{S7}$$

Proof. Uniform boundedness of $P_\theta^n[|\sqrt{n}W_{n,t}|^{2+\rho}]$ implies:

$$\lim_{n\to\infty} \sum_{t=1}^n W_{n,t}^{2+\rho} = 0,$$

which in turns implies (S7) (cf. Billingsley, 1995, page 362). For the uniform boundedness, as

$$2\sqrt{n}W_{n,t} = g'\dot\ell_\theta(Y_t, X_t) + \sum_{k=1}^K h_k(A_{k\bullet}(\alpha, \sigma)V_{\theta,t}),$$

and the h_k are bounded, it suffices to note that by Lemma S2.3 $\mathbb{E}[(g'\dot\ell_\theta(X_t, Y_t))^{2+\rho}] \le C$ under P_θ^n for some $\rho > 0$. $\qquad\square$

S2.3 Log-likelihood ratios

LEMMA S2.5 (DQM): *Suppose that assumption 2.1 holds. Then with $W_{n,t}$ and $U_{n,t}$ defined as in the proof of Proposition A.1,*

$$\lim_{n\to\infty} \mathbb{E}\sum_{t=1}^n (W_{n,t} - U_{n,t})^2 = 0,$$

where the expectation is taken under P_θ^n.

Proof. We argue similarly to Lemma 7.6 in van der Vaart (1998). Let $V_{\theta,t} := Y_t - BX_t$ and

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$\varphi(v) = (g, \eta_1 h_1, \ldots, \eta_K h_K)$ for $v = (g, h)$ with $g \in \mathbb{R}^L$, $h \in \dot{\mathscr{H}}$. Let

$$p_\theta(Y_t, X_t) := |A(\theta)| \prod_{k=1}^{K} \eta_k(A_{k\bullet}(\theta)V_{\theta,t})$$

$$s_{\theta,u}(Y_t, X_t) := g'\dot{\ell}_{\theta+u\varphi(v)}(Y_t, X_t) + \sum_{k=1}^{K} \frac{h_k(A_{k\bullet}(\theta + u\varphi(v))V_{\theta+u\varphi(v),t})}{1 + uh_k(A_{k\bullet}(\theta + u\varphi(v))V_{\theta+u\varphi(v),t})}$$

$$+ \sum_{k=1}^{K} \frac{uh_k'(A_{k\bullet}(\theta + u\varphi(v))V_{\theta+u\varphi(v),t}) \left[\mathsf{D}_{1,k,u}V_{\theta+u\varphi(v),t} + \mathsf{D}_{2,k,u}X_t\right]}{1 + uh_k(A_{k\bullet}(\theta + u\varphi(v))V_{\theta+u\varphi(v),t})},$$

with

$$\mathsf{D}_{1,k,u} := e_k' \sum_{l=1}^{L_\alpha} g_{\alpha,l} D_{\alpha,l}(\theta + u\varphi(v)) + e_k' \sum_{l=1}^{L_\sigma} g_{\sigma,l} D_{\sigma,l}(\theta + u\varphi(v))$$

$$\mathsf{D}_{2,k,u} := -A_{k\bullet}(\theta + u\varphi(v)) \sum_{l=1}^{L_b} D_{b,l}(\theta + u\varphi(v)).$$

By Assumption 2.1 and standard computations, the derivative of $u \mapsto \sqrt{p_{\theta+u\varphi(v)}}$ at $u = \mathsf{u}$ is $\frac{1}{2}s_{\theta,\mathsf{u}}\sqrt{p_{\theta+\mathsf{u}\varphi(v)}}$ (everywhere). Inspection reveals that this is continuous in u.

For $q_{\theta,t}$ the density of X_t under P_θ^n and $s_\theta := s_{\theta,0}$,

$$\mathbb{E}\sum_{t=1}^{n}(W_{n,t} - U_{n,t})^2 = \frac{1}{n}\sum_{t=1}^{n}\int\left(\sqrt{n}\left[\sqrt{\frac{p_{\theta_n}}{p_\theta}} - 1\right] - \frac{1}{2}s_\theta\right)^2 p_\theta q_{\theta,t}\,\mathrm{d}\lambda$$

$$= \int\left(\sqrt{n}\left[\sqrt{p_{\theta_n}} - \sqrt{p_\theta}\right] - \frac{1}{2}s_\theta\sqrt{p_\theta}\right)^2 \bar{q}_{n,\theta}\,\mathrm{d}\lambda,$$

with $\bar{q}_{n,\theta} := \frac{1}{n}\sum_{t=1}^{n} q_{\theta,t}$. The integrand converges to zero as $n \to \infty$ by the differentiability of $u \mapsto \sqrt{p_{\theta+u\varphi(v)}}$ at $u = 0$.[S3] Let

$$I_{\theta,u,n} := \int s_{\theta,u}^2 p_{\theta+u\varphi(v)}\bar{q}_{n,\theta}\,\mathrm{d}\lambda = \int s_{\theta,u}^2\,\mathrm{d}G_{\theta,u,n},$$

where $G_{\theta,u,n}$ is the distribution of (Y_t, X_t) corresponding to the density $p_{\theta+u\varphi(v)}\bar{q}_{n,\theta}$. By Lemma S3.2 $G_{\theta,u/\sqrt{n},n} \xrightarrow{TV} G_\theta$, defined by

$$G_\theta(A) := \int_A p_\theta\,\mathrm{d}(\lambda(y) \otimes Q_\theta(x)).$$

For any $(u_n) \subset [0, 1]$ we have that $s_{\theta,u_n/\sqrt{n}}^2 \to s_\theta^2$ (pointwise). By Lemma S2.6 and Corollary

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[S3]Note that $p_{\theta_n} = p_{\theta_n(g,h)} = p_{\theta+\varphi(v)/\sqrt{n}}$.

2.9 in Feinberg et al. (2016), $\lim_{n\to\infty} I_{\theta,u_n/\sqrt{n},n} = \int s_\theta^2 \, \mathrm{d}G_\theta < \infty$ and hence

$$\left| \int_0^1 I_{\theta,u/\sqrt{n},n} \, \mathrm{d}u - \int_0^1 \int s_\theta^2 \, \mathrm{d}G_\theta \, \mathrm{d}u \right| \leq \sup_{u\in[0,1]} \left| I_{\theta,u/\sqrt{n},n} - \int s_\theta^2 \, \mathrm{d}G_\theta \right| \to 0.$$

By absolute continuity, Jensen's inequality and the Fubini – Tonelli theorem,

$$\int \left(\sqrt{n} \left[\sqrt{p_{\theta_n}} - \sqrt{p_\theta} \right] \right)^2 \bar{q}_{n,\theta} \, \mathrm{d}\lambda \leq \frac{1}{4} \int \int_0^1 \left(s_{\theta,u/\sqrt{n}} \sqrt{p_{\theta+u\varphi(v)/\sqrt{n}}} \right)^2 \bar{q}_{n,\theta} \, \mathrm{d}u \, \mathrm{d}\lambda \leq \int_0^1 I_{\theta,u/\sqrt{n},n} \, \mathrm{d}u.$$

Combine these observations with Proposition 2.29 in van der Vaart (1998). $\qquad\square$

LEMMA S2.6: *Suppose that assumption 2.1 holds. Let $s_{\theta,u}$ and $G_{\theta,u,n}$ be as in the proof of Proposition S2.5. Then for any $(u_n)_{n\in\mathbb{N}} \subset [0,1]$, $s_{\theta,u_n/\sqrt{n}}^2$ is asymptotically uniformly $G_{\theta,u_n/\sqrt{n},n}$-integrable and $s_\theta \in L_2(G_\theta)$.*

Proof. That $s_\theta \in L_2(G_\theta)$ follows from the moment bounds in Assumption 2.1(ii), the boundedness of the h_k, the form of $\dot{\ell}_\theta$ given in equations (7) – (9) and Lemma S2.1 given that Q_θ is the law of the stationary solution to (1).

For the uniform integrability, let $\vartheta_n := \theta + u_n\varphi(v)/\sqrt{n} \to \theta$ and

$$s_{\vartheta_n,1}(Y_t, X_t) := g'\dot{\ell}_{\vartheta_n}(Y_t, X_t)$$

$$s_{\vartheta_n,2}(Y_t, X_t) := \sum_{k=1}^K \frac{h_k(A_{k\bullet}(\vartheta_n)V_{\vartheta_n,t})}{1 + u_n h_k(A_{k\bullet}(\vartheta_n)V_{\vartheta_n,t})/\sqrt{n}}$$

$$s_{\vartheta_n,3}(Y_t, X_t) := \sum_{k=1}^K \frac{u_n h_k'(A_{k\bullet}(\vartheta_n)V_{\vartheta_n,t}) \left[\mathsf{D}_{1,k,u_n/\sqrt{n}} V_{\vartheta_n,t} + \mathsf{D}_{2,k,u_n/\sqrt{n}} X_t \right]/\sqrt{n}}{1 + u_n h_k(A_{k\bullet}(\vartheta_n)V_{\vartheta_n,t})/\sqrt{n}}$$

It suffices to show that under $G_{\theta,u_n/\sqrt{n},n}$ each $s_{\vartheta_n,i}$ $(i = 1, 2, 3)$ has uniformly bounded $2 + \rho$ moments for some $\rho > 0$ for all sufficiently large n.

We start with $s_{\vartheta_n,2}$: since each h_k is bounded, for all large enough n, each numerator is uniformly bounded above and each denominator is uniformly bounded below, away from zero. Thus there is a M such that $|s_{\vartheta_n,2}(Y_t, X_t)| \leq M$ for all such n.

For $s_{\vartheta_n,3}$, by assumption 2.1 part (iii), each $\mathsf{D}_{1,k,u_n/\sqrt{n}}$ and $\mathsf{D}_{2,k,u_n/\sqrt{n}}$ are uniformly bounded for all large enough n; the same is true of $\|A(\vartheta_n)^{-1}\|_2$. Using this, the fact that $V_{\vartheta_n,t} = A(\vartheta_n)^{-1} \epsilon_t$ and arguing similarly to as in the preceding paragraph we have that for some M and all large enough n, $|s_{\vartheta_n,3}(Y_t, X_t)| \leq M [\|\epsilon_t\| + \|X_t\|]$. Thus it is enough to verify that

$$\sup_{n\geq N, 1\leq t\leq n} G_{\theta,u_n/\sqrt{n},n}\|\epsilon_t\|^{4+\delta} < \infty, \qquad \sup_{n\geq N, 1\leq t\leq n} G_{\theta,u_n/\sqrt{n},n}\|X_t\|^{4+\delta} < \infty. \qquad (\mathrm{S8})$$

Under $G_{\theta, u_n/\sqrt{n}, n}$, the elements $\epsilon_{t,k}$ are (independently across k) distributed according to $\eta_k(1 + u_n h_k/\sqrt{n})$, so there are $c, C < \infty$ such that

$$G_{\theta, u_n/\sqrt{n}, n}\|\epsilon_t\|^{4+\delta} \leq G_{\theta, u_n/\sqrt{n}, n}\left[\sum_{k=1}^{K} \epsilon_{t,k}^2\right]^{\frac{4+\delta}{2}} \leq c \sum_{k=1}^{K}\left[\left(1 + \frac{\bar{h}_k}{\sqrt{n}}\right)\int |x_k|^{4+\delta}\eta_k(x_k)\,\mathrm{d}x_k\right] \leq C,$$

where $|h_k(x)| \leq \bar{h}_k$. By arguing analogously to as in in Lemma S2.3, one has (cf. (S6))

$$G_{\theta, u_n/\sqrt{n}, n}\|Z_t\|^{4+\delta} \lesssim \left(\frac{C_1}{1-\varrho}\right)^{4+\delta} + \left(\frac{C_2}{1-\varrho}\right)^{4+\delta} G_{\theta, u_n/\sqrt{n}, n}|\epsilon_1|^{4+\delta} + \|Z_0\|^{4+\delta},$$

which is uniformly bounded given the penultimate display.

Finally consider $s_{\vartheta_n, 1}$. It suffices to show that each component of $\dot{\ell}_{\vartheta_n}$ has $4 + \delta$ moment bounded uniformly for all $n \geq N$.[S4] By Assumption 2.1(iii), by increasing N if necessary, $\sup_{\vartheta \in \mathsf{T}} |\zeta_{l,k,j}^x(\vartheta)| \leq M$ for all l, k, j and $x \in \alpha, \sigma$ and likewise $\sup_{\vartheta \in \mathsf{T}} \|A_{k\bullet}(\vartheta)D_{b_l}(\vartheta)\| \leq M$. Recall that $V_{\vartheta_n, t} = A(\vartheta_n)^{-1}\epsilon_t$. Given (S8) and the observations in footnote S4 to complete the proof it suffices to note that (for $\phi_k = \frac{\mathrm{d}\log \eta_k(x)}{\mathrm{d}x}$) and some $C < \infty$,

$$G_{\theta, u_n/\sqrt{n}, n}|\phi_k|^{4+\delta} \leq \left(1 + \frac{\bar{h}_k}{\sqrt{n}}\right)\int |\phi(x)|^{4+\delta}\eta_k(x)\,\mathrm{d}x \leq C. \qquad \square$$

LEMMA S2.7: *Let $W_{n,t}$ be as in the Proof of Proposition A.1 and suppose the conditions of that Proposition hold. Let G_θ be defined as in the Proof of Lemma S2.5. Then, under P_θ^n,*

$$\lim_{n\to\infty} \mathbb{E}\left|\sum_{t=1}^{n} W_{n,t}^2 - \frac{\tau^2}{4}\right| = 0, \quad \text{with} \quad \tau^2 := G_\theta\left(g'\dot{\ell}_\theta(Y, X) + \sum_{k=1}^{K} h_k(A_{k\bullet}(\theta)V_\theta)\right)^2.$$

Proof. Define

$$r_\theta(X_t) := \mathbb{E}[s_\theta(Y_t, X_t)^2 | X_t], \quad s_\theta(Y, X) := g'\dot{\ell}_\theta(Y, X) + \sum_{k=1}^{K} h_k(A_{k\bullet}(\theta)V_\theta),$$

where the conditional expectation is taken under P_θ^n. Since conditional expectations are L_1 con-

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[S4]The form each such component is that given in equations equations (7) – (9). Note here that each ϕ_k is (implicitly) a function of η_k and thus when evaluating equations (7) – (9) at ϑ_n, the ϕ_k that appear are $\phi_{k, u_n, n}$, defined as

$$\phi_{k,u,n} := \frac{\mathrm{d}(\log \eta_k(x) + \log(1 + uh_k(x)/\sqrt{n}))}{\mathrm{d}x} = \phi_k + \frac{uh_k'/\sqrt{n}}{1 + uh_k/\sqrt{n}}.$$

Since each h_k, and h_k' are bounded, increasing N if necessary, one has for $n \geq N$,

$$|\phi_{k,u_n,n}| \leq |\phi_k| + M.$$

tractions, by Lemma S2.4, we have that $P_\theta^n[|r_\theta(X_t)|^{1+\rho/2}] \lesssim C < \infty$ and hence $(|r_\theta(X_t)|^{1+\rho/2})_{t\in\mathbb{N}}$ is uniformly P_θ^n–integrable. Moreover we have for $\mathscr{F}_t := \sigma(\epsilon_1, \ldots, \epsilon_t)$,

$$r_\theta(X_t) = \mathbb{E}[s_\theta(Y_t, X_t)^2 | X_t] = \mathbb{E}[s_\theta(Y_t, X_t)^2 | \mathscr{F}_{t-1}],$$

as is clear from the definition of s_θ.[S5] Hence $(s_\theta(Y_t, X_t)^2 - r_\theta(X_t), \mathscr{F}_t)$ is a martingale difference squence and by Theorem 19.7 in Davidson (1994)

$$\lim_{n\to\infty} \mathbb{E}\left| \frac{1}{n} \sum_{t=1}^n [s_\theta(Y_t, X_t)^2 - r_\theta(X_t)] \right|^{1+\rho/2} = 0.$$

Now define $u_\theta(X_t) := r_\theta(X_t) - \mathbb{E}[r_\theta(X_t)]$, which satisfies $P_\theta^n[|u_\theta(X_t)|^{1+\rho/2}] \lesssim C < \infty$ and is evidently mean zero. By Theorem 3 in Saikkonen (2007), Z_t and hence $u_\theta(X_t)$ (e.g. Davidson, 1994, Theorem 14.1) has geometrically decaying β-mixing coefficients. Therefore, by Theorem 14.2 in Davidson (1994), $(u_\theta(X_t)/n)_{n\in\mathbb{N},1\le t\le n}$ is an L_1–mixingale array with respect to the filtration formed by $\mathsf{F}_{n,t} := \sigma(X_1, \ldots, X_t)$ relative to the sequence of positive constants

$$n^{-1} \le c_{n,t} = \max\left\{ 1/n, \left(P_\theta^n\left[|u_\theta(X_t)/n|^{1+\rho/2} \right] \right)^{1/(1+\rho/2)} \right\} \le n^{-1} \max\{C, 1\}.$$

By Theorem 19.11 in Davidson (1994),

$$\lim_{n\to\infty} \mathbb{E}\left| \frac{1}{n} \sum_{t=1}^n u_\theta(Y_t, X_t) \right| = 0.$$

It remains to show that $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[r_\theta(X_t)] \to \tau^2$. Since $\mathbb{E}[r_\theta(X_t)] = \mathbb{E}[s_\theta(Y_t, X_t)]$,

$$\tau_n^2 := G_{\theta,0,n}\left[s_\theta(Y, X)^2 \right] = \frac{1}{n} \sum_{t=1}^n \mathbb{E}\, s_\theta(Y_t, X_t)^2 = \frac{1}{n} \sum_{t=1}^n \mathbb{E}[r_\theta(X_t)],$$

where $G_{\theta,0,n}$ is as defined in the Proof of Lemma S2.5. That $\mathbb{E}\frac{1}{n}\sum_{t=1}^n s_\theta(Y_t, X_t)^2 \lesssim C$ follows from Lemma S2.4. Therefore, by Lemma S2.6, $s_\theta(Y, X)^2$ is uniformly $G_{\theta,0,n}$–integrable and also $\tau^2 < \infty$. Then, by Corollary 2.9 in Feinberg et al. (2016) and Lemma S3.2, $\tau_n^2 \to \tau$. $\qquad\square$

LEMMA S2.8: *In the setting of Proposition A.2,*

$$\log \frac{p_{\theta_n(g_n,h)}^n}{p_{\theta_n(g,h)}^n} = o_{P_{\theta_n(g,h)}^n}(1).$$

[S5]See e.g. Theorem 7.3.1 in Chow and Teicher (1997) for the (almost sure) equality of the conditional expectations.

Proof. Since by Proposition A.1 and Example 6.5 in van der Vaart (1998) $P^n_{\theta_n(g,h)} \lhd\rhd P^n_\theta$ it suffices to show that the left hand side is $o_{P^n_\theta}(1)$. We first show that

$$\log \frac{p^n_{\theta_n(g_n,0)}}{p^n_\theta} = \frac{1}{\sqrt{n}}\sum_{t=1}^n g'\dot{\ell}_\theta(Y_t, X_t) - \mathbb{E}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^n g'\dot{\ell}_\theta(Y_t, X_t)\right)^2 + o_{P^n_\theta}(1)$$

$$\log \frac{p^n_{\theta_n(g,0)}}{p^n_\theta} = \frac{1}{\sqrt{n}}\sum_{t=1}^n g'\dot{\ell}_\theta(Y_t, X_t) - \mathbb{E}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^n g'\dot{\ell}_\theta(Y_t, X_t)\right)^2 + o_{P^n_\theta}(1)$$

For these log–likelihood expansions we may appeal to Lemma 1 in Swensen (1985). The required Conditions (1.3) - (1.7) and (iii) of his Theorem 1 are all established in the proof of Proposition A.1 (take each $h_k = 0$). It remains to show condition (1.2) for each of the cases in the above display. In particular, set

$$W_{n,t} := \frac{1}{2\sqrt{n}}g'\dot{\ell}_\theta(Y_t, X_t)$$

and (cf. equations (37), (38))

$$U_{n,t} := \left[\left(\frac{|A(\theta_n(g_n, h))|}{|A(\theta)|}\right) \times \prod_{k=1}^K \frac{\eta_k(A_{k\bullet}(\theta_n(g_n, h))V_{\theta_n(g_n,h),t})}{\eta_k(A_{k\bullet}(\theta)V_{\theta,t})}\right]^{1/2} - 1$$

where we note that $A(\theta) = A(\theta_n(0, h))$ and $V_\theta = V_{\theta_n(0,h)}$. We verify (1.2), i.e. that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=1}^n (W_{n,t} - U_{n,t})^2\right] = 0,$$

under P^n_θ.[S6] The argument now follows similarly to that in Lemma S2.5. To simplify the notation, let $p_\gamma := p_{(\gamma,\eta)}$ and $\dot{\ell}_\gamma := \dot{\ell}_{(\gamma,\eta)}$ where $\eta = (\eta_1, \ldots, \eta_K)$ will remain fixed. By Assumption 2.1 and standard computations, the derivative of $\gamma \mapsto \sqrt{p_\gamma}$ is $\frac{1}{2}\dot{\ell}_\gamma\sqrt{p_\gamma}$ (everywhere). Inspection reveals that this is continuous in γ.

Let $\gamma_n := \gamma + g_n/\sqrt{n}$. For $q_{\theta,t}$ the density of X_t under P^n_θ,

$$\mathbb{E}\sum_{t=1}^n (W_{n,t} - U_{n,t})^2 = \frac{1}{n}\sum_{t=1}^n \int \left(\sqrt{n}\left[\sqrt{\frac{p_{\gamma_n}}{p_\gamma}} - 1\right] - \frac{1}{2}g'\dot{\ell}_\gamma\right)^2 p_\gamma q_{\theta,t}\,d\lambda$$

$$= \int \left(\sqrt{n}\left[\sqrt{p_{\gamma_n}} - \sqrt{p_\gamma}\right] - \frac{1}{2}g'\dot{\ell}_\gamma\sqrt{p_\gamma}\right)^2 \bar{q}_{n,\theta}\,d\lambda,$$

with $\bar{q}_{n,\theta} := \frac{1}{n}\sum_{t=1}^n q_{\theta,t}$. The term inside the parentheses converges to zero as $n \to \infty$ by the

---

[S6]This suffices as the second expansion is just the special case $g_n = g$ for each $n \in \mathbb{N}$.

differentiability of $\gamma \mapsto \sqrt{p_\gamma}$ and that $(g_n - g)'\dot{\ell}_\gamma \sqrt{p_\gamma} \to 0$ pointwise. Let

$$I_{\theta,u,n} := \int (g'\dot{\ell}_{\gamma+ug_n})^2 \, p_{\gamma+ug_n} \, \bar{q}_{n,\theta} \, d\lambda = \int (g'\dot{\ell}_{\gamma+ug_n})^2 \, dG_{\theta,u,n},$$

where $G_{\theta,u,n}$ is the distribution of (Y_t, X_t) corresponding to the density $p_{\gamma+ug_n} \bar{q}_{n,\theta}$. By Lemma S3.2 $G_{\theta,u_n/\sqrt{n},n} \xrightarrow{TV} G_\theta$, defined as in the proof of Lemma S2.5. For any $(u_n) \subset [0,1]$ we have that $(g'\dot{\ell}_{\gamma+u_ng_n/\sqrt{n}})^2 \to (g'\dot{\ell}_\gamma)^2$ (pointwise). Each component of $\dot{\ell}_\gamma \in L_2(G_\theta)$ by Lemma S2.6 and moreover $\sup_{n\geq N} G_{\theta,u_n/\sqrt{n},n}\|\dot{\ell}_{\gamma+u_ng_n/\sqrt{n}}\|^{2+\rho} \leq C$ for some $\rho > 0$.[S7] Therefore, by Corollary 2.9 in Feinberg et al. (2016), $\lim_{n\to\infty} I_{\theta,u_n/\sqrt{n},n} = \int (g'\dot{\ell}_\gamma)^2 \, dG_\theta < \infty$ and hence

$$\left| \int_0^1 I_{\theta,u/\sqrt{n},n} \, du - \int_0^1 \int s_\theta^2 \, dG_\theta \, du \right| \leq \sup_{u\in[0,1]} \left| I_{\theta,u/\sqrt{n},n} - \int (g'\dot{\ell}_\gamma)^2 \, dG_\theta \right| \to 0.$$

By the continuous differentiability of $\sqrt{p_\gamma}$, Jensen's inequality and the Fubini – Tonelli theorem,

$$\int \left(\sqrt{n} \left[\sqrt{p_{\gamma_n}} - \sqrt{p_\gamma} \right] \right)^2 \bar{q}_{n,\theta} \, d\lambda \leq \frac{1}{4} \int \int_0^1 \left((g'\dot{\ell}_{\gamma+ug_n/\sqrt{n}}) \sqrt{p_{\gamma+ug_n/\sqrt{n}}} \right)^2 \bar{q}_{n,\theta} \, du \, d\lambda$$

$$\leq \int_0^1 I_{\theta,u/\sqrt{n},n} \, du.$$

Combining these observations with Proposition 2.29 in van der Vaart (1998) verifies (1.2) and hence the claimed log – likelihood expansions follow from Lemma 1 in Swensen (1985).

To complete the proof set

$$\tilde{u}_{k,n,t} := A_{k\bullet}(\theta_n(g_n,h))V_{\theta_n(g_n,h),t}, \qquad u_{k,n,t} := A_{k\bullet}(\theta_n(g,h))V_{\theta_n(g,h),t},$$

and observe that

$$\log \frac{p_{\theta_n(g_n,h)}^n}{p_{\theta_n(g,h)}^n} - \left[\log \frac{p_{\theta_n(g_n,0)}^n}{p_\theta^n} - \log \frac{p_{\theta_n(g,0)}^n}{p_\theta^n} \right]$$

$$= \sum_{k=1}^K \sum_{i=1}^n \log \left(1 + \frac{h_k(\tilde{u}_{k,n,t})}{\sqrt{n}} \right) - \log \left(1 + \frac{h_k(u_{k,n,t})}{\sqrt{n}} \right),$$

where the bracketed term is $o_{P_\theta^n}(1)$ by the preceding argument. Hence it suffices to show that an arbitrary k-th element of the outer sum on the right hand side is also $o_{P_\theta^n}(1)$. Let $\varepsilon \in (0,1)$

[S7]This follows from (a) the continuity requirements in Assumption 2.1(iii), (b) under $G_{\theta,u_n/\sqrt{n},n}$ we have that $e_k'A(\theta_n(u_ng_n,0))^{-1}V_{\theta_n(u_ng_n),0} = \epsilon_k \sim \eta_k$ and (c) $\sup_{n\geq N,1\leq t\leq n} G_{\theta,u_n/\sqrt{n},n}\|X_t\|^{4+\delta} < \infty$, which can be shown by an argument analogous to that which is established in the proof of Lemma S2.6.

be fixed and define

$$E_n := \left\{ \max_{1 \le i \le n} |h_k(\tilde{u}_{k,n,t})|/\sqrt{n} \le \varepsilon \right\}, \quad F_n := \left\{ \max_{1 \le i \le n} |h_k(u_{k,n,t})|/\sqrt{n} \le \varepsilon \right\}.$$

Since h_k is bounded $P_\theta^n(E_n \cap F_n) \to 1$. On this set we may perform a two-term Taylor expansion of $\log(1+x)$ to obtain

$$\log\left(1 + \frac{h_k(\tilde{u}_{k,n,t})}{\sqrt{n}}\right) - \log\left(1 + \frac{h_k(u_{k,n,t})}{\sqrt{n}}\right)$$
$$= \frac{h_k(\tilde{u}_{k,n,t}) - h_k(u_{k,n,t})}{\sqrt{n}} - \frac{1}{2}\frac{h_k(\tilde{u}_{k,n,t})^2 - h_k(u_{k,n,t})^2}{n} + R\left(\frac{h_k(\tilde{u}_{k,n,t})}{\sqrt{n}}\right) - R\left(\frac{h_k(u_{k,n,t})}{\sqrt{n}}\right),$$

where $|R(x)| \le |x|^3$. For the remainder terms one has for any u_i,

$$\sum_{i=1}^n \left| R\left(\frac{h_k(u_i)}{\sqrt{n}}\right) \right| \le \max_{1 \le i \le n} \frac{h_k(u_i)}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n h_k(u_i)^2 \lesssim \frac{1}{\sqrt{n}},$$

since h_k is bounded. For the first term in Taylor expansion, note that the derivative (in θ, σ) of $A(\theta, \sigma)$ is bounded on a neighbourhood of (θ, σ) (by Assumption 2.1). Combine this with the boundedness of h_k' and the mean value theorem to conclude that

$$|h_k(\tilde{u}_{k,n,t}) - h_k(u_{k,n,t})| \lesssim n^{-1/2} \|g_n - g\| \left[\|\epsilon_t\| + \|X_t\| \right].$$

Using this, since h_k is bounded,

$$|h_k(\tilde{u}_{k,n,t})^2 - h_k(u_{k,n,t})^2| \lesssim n^{-1/2} \|g_n - g\| \left[\|\epsilon_t\| + \|X_t\| \right].$$

Therefore, using (S6) and Assumption 2.1(ii)

$$\sum_{i=1}^n \left| \frac{h_k(\tilde{u}_{k,n,t}) - h_k(u_{k,n,t})}{\sqrt{n}} - \frac{1}{2}\frac{h_k(\tilde{u}_{k,n,t})^2 - h_k(u_{k,n,t})^2}{n} \right|$$
$$\lesssim \|g_n - g\| \left(1 + \frac{1}{\sqrt{n}}\right) \frac{1}{n} \sum_{i=1}^n \left[\|\epsilon_t\| + \|X_t\| \right] = o_{P_\gamma^n}(1). \qquad \square$$

LEMMA S2.9: *In the setting of Proposition A.2,*

$$\log \frac{p_{\theta_n(g_n,h_n)}^n}{p_{\theta_n(g_n,h)}^n} = o_{P_{\theta_n(g_n,h)}^n}(1).$$

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Proof. For notational ease, set

$$u_{k,n,t} := e_k' A(\theta_n(g_n, h)) V_{\theta_n(g_n,h),t} = e_k' A(\theta_n(g_n, h_n)) V_{\theta_n(g_n,h_n),t}.$$

One has that

$$\log \frac{p_{\theta_n(g_n,h_n)}^n}{p_{\theta_n(g_n,h)}^n} = \sum_{k=1}^{K} \sum_{t=1}^{n} \log(1 + h_{k,n}(u_{k,n,t})/\sqrt{n}) - \log(1 + h_k(u_{k,n,t})/\sqrt{n}),$$

hence it suffices to show that each

$$l_{n,k} := \sum_{t=1}^{n} \log(1 + h_{k,n}(u_{k,n,t})/\sqrt{n}) - \log(1 + h_k(u_{k,n,t})/\sqrt{n}) \xrightarrow{P_{\theta_n(g_n,h)}^n} 0.$$

Let $\varepsilon \in (0, 1)$ be fixed and define

$$E_n := \left\{ \max_{1 \le t \le n} |h_{k,n}(u_{k,n,t})|/\sqrt{n} \le \varepsilon \right\};$$

$$F_n := \left\{ \max_{1 \le t \le n} |h_k(u_{k,n,t})|/\sqrt{n} \le \varepsilon \right\}.$$

Since h_k is bounded, $P_{\theta_n(g_n,h)}^n F_n \to 1$; $P_{\theta_n(g_n,h)}^n E_n \to 1$ follows from Lemma S2.11. Hence $P_{\theta_n(g_n,h)}^n F_n \cap E_n \to 1$. On $E_n \cap F_n$ we can perform a two-term Taylor expansion of $\log(1 + x)$ to obtain

$$\begin{aligned}
\log(1 &+ h_{k,n}(u_{k,n,t})/\sqrt{n}) - \log(1 + h_k(u_{k,n,t})/\sqrt{n}) \\
&= \frac{h_{k,n}(u_{k,n,t})}{\sqrt{n}} - \frac{1}{2}\frac{h_{k,n}(u_{k,n,t})^2}{n} - \frac{h_k(u_{k,n,t})}{\sqrt{n}} + \frac{1}{2}\frac{h_k(u_{k,n,t})^2}{n} \\
&\quad + R\left(\frac{h_{k,n}(u_{k,n,t})}{\sqrt{n}}\right) - R\left(\frac{h_k(u_{k,n,t})}{\sqrt{n}}\right),
\end{aligned}$$

where $|R(x)| \le |x|^3$. It follows that

$$\begin{aligned}
l_{n,k} &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t}) - \frac{1}{2}\frac{1}{n}\sum_{t=1}^{n}[h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2] \\
&\quad + \sum_{t=1}^{n} R\left(\frac{h_{k,n}(u_{k,n,t})}{\sqrt{n}}\right) - R\left(\frac{h_k(u_{k,n,t})}{\sqrt{n}}\right).
\end{aligned}$$

We will show that the remainder terms vanish. In particular, one has

$$\sum_{t=1}^{n} \left| R\left(\frac{h_{k,n}(u_{k,n,t})}{\sqrt{n}}\right) \right| \le \sum_{t=1}^{n} \left| \frac{h_{k,n}(u_{k,n,t})}{\sqrt{n}} \right| \left| \frac{h_{k,n}(u_{k,n,t})^2}{n} \right| \le \max_{1 \le t \le n} \frac{|h_{k,n}(u_{k,t,n})|}{\sqrt{n}} \frac{1}{n}\sum_{t=1}^{n} h_{k,n}(u_{k,n,t})^2.$$

By Markov's inequality with Lemmas S2.10 and S2.11, this converges to zero in $P^n_{\theta_n(g_n,h)}$ probability. The same evidently holds for the case where $h_{k,n} = h_k$ for each $n \in \mathbb{N}$. Thus,

$$l_{n,k} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t}) - \frac{1}{2}\frac{1}{n} \sum_{t=1}^{n} [h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2] + o_{P^n_{\theta_n(g_n,h)}}(1),$$

and it remains to show that $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t})$ and $\frac{1}{n} \sum_{t=1}^{n} [h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2]$ also converge to zero in probability. The second of these follows directly from Lemma S2.10, Markov's inequality and the reverse triangle inequality since

$$P^n_{\theta_n(g_n,h)}\left(\left| \frac{1}{n} \sum_{t=1}^{n} [h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2] \right| > \varepsilon \right) \le \varepsilon^{-1}\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2 \right]$$

$$= \varepsilon^{-1} \mathbb{E}\left[h_{k,n}(u_{k,n,t})^2 - h_k(u_{k,n,t})^2 \right]$$

$$\to 0.$$

For the remaining term, we start by noting that

$$\mathbb{E}[h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t})] = \frac{\mathbb{E}[(h_{k,n}(\epsilon_k) - h_k(\epsilon_k))h_k(\epsilon_k)]}{\sqrt{n}}$$

so

$$\left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \mathbb{E}[h_{k,n}(u_{k,n,t})] - \mathbb{E}[h_k(u_{k,n,t})] \right| \le \frac{1}{n} \sum_{t=1}^{n} \|h_{k,n} - h_k\|_{L_2(P^n_\theta)} \|h_k\|_{L_2(P^n_\theta)} \to 0.$$

Thus it suffices to show that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \tilde{h}_{k,n}(u_{k,n,t}) - \tilde{h}_k(u_{k,n,t}) \xrightarrow{P^n_{\theta_n(g_n,h)}} 0,$$

for $\tilde{h}_{k,n}(u_{k,n,t}) := \tilde{h}_{k,n}(u_{k,n,t}) - \mathbb{E}\left[\tilde{h}_{k,n}(u_{k,n,t})\right]$ and $\tilde{h}_k(u_{k,n,t}) := \tilde{h}_k(u_{k,n,t}) - \mathbb{E}\left[\tilde{h}_k(u_{k,n,t})\right]$. By the reverse triangle inequality and Lemma S2.10,

$$\mathbb{E}\left[\left(\tilde{h}_{k,n}(u_{k,n,t}) - \tilde{h}_k(u_{k,n,t}) \right)^2 \right] \to 0, \quad \text{uniformly in } t.$$

Using this, the independence of the $u_{k,t,n}$ and Markov's inequality:

$$P^n_{\theta_n(g_n,h)}\left(\left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \tilde{h}_{k,n}(u_{k,n,t}) - \tilde{h}_k(u_{k,n,t}) \right| > \varepsilon \right) \le \frac{1}{\varepsilon^2}\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[\left(\tilde{h}_{k,n}(u_{k,n,t}) - \tilde{h}_k(u_{k,n,t}) \right)^2 \right] \to 0.$$

This establishes that $\sum_{k=1}^{K} l_{n,k} \xrightarrow{P^n_{\theta_n(g_n,h)}} 0$, as required. □

LEMMA S2.10: *In the setting of Proposition A.2, let $u_{k,n,t} := e_k' A_{\theta_n(g_n,h)} V_{\theta_n(g_n,h),t}$. Under $P^n_{\theta_n(g_n,h)}$,*

$$\mathbb{E}\left[h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t})\right]^2 \leq \|h_{n,k} - h_k\|_{L_2(P^n_\theta)} \left(1 + \frac{\|h_k\|_{L_\infty(P^n_\theta)}}{\sqrt{n}}\right).$$

Proof. Under $P^n_{\theta_n(g_n,h)}$, $u_{k,n,t} \sim \eta_k(1 + h_k/\sqrt{n})$, so for $\epsilon_k \sim \eta_k$, since h_k is bounded,

$$
\begin{aligned}
&\mathbb{E}\left[h_{k,n}(u_{k,n,t}) - h_k(u_{k,n,t})\right]^2 \\
&= \int \left[h_{n,k}(x) - h_k(x)\right]^2 \eta_k(x)(1 + h_k(x)/\sqrt{n})\,\mathrm{d}x \\
&\leq \mathbb{E}[h_{k,n}(\epsilon_k) - h_k(\epsilon_k)]^2 + \frac{1}{\sqrt{n}}\mathbb{E}[h_{k,n}(\epsilon_k) - h_k(\epsilon_k)]^2 \|h_k\|_{L_\infty(P^n_\theta)} \\
&\leq \|h_{n,k} - h_k\|_{L_2(P^n_\theta)} + \|h_{n,k} - h_k\|_{L_2(P^n_\theta)}\|h_k\|_{L_\infty(P^n_\theta)}/\sqrt{n}. \qquad \square
\end{aligned}
$$

LEMMA S2.11: *In the setting of Proposition A.2, let $u_{k,n,t} := e_k' A_{\theta_n(g_n,h)} V_{\theta_n(g_n,h),t}$. Then*

$$\max_{1 \leq t \leq n} \frac{|h_{k,n}(u_{k,n,t})|}{\sqrt{n}} \xrightarrow{P^n_{\theta_n(g_n,h)}} 0.$$

Proof. Under $P^n_{\theta_n(g_n,h)}$, $u_{k,n,t} \sim \eta_k(1 + h_k/\sqrt{n})$. By Lemma S2.10, $h_{k,n}(u_{k,n,t})$ is uniformly square $P^n_{\theta_n(g_n,h)}$–integrable and hence the Lindeberg condition holds for $h_{k,n}(u_{k,n,t})/\sqrt{n}$:

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{t=1}^n \mathbb{E}\left[\frac{h_{k,n}(u_{k,n,t})^2}{n}\mathbf{1}\left\{|h_{n,k}(u_{k,n,t})| > \delta\sqrt{n}\right\}\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^n \mathbb{E}\left[h_{k,n}(u_{k,n,t})^2\mathbf{1}\left\{|h_{n,k}(u_{k,n,t})| > \delta\sqrt{n}\right\}\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[h_{k,n}(u_{k,n,t})^2\mathbf{1}\left\{|h_{n,k}(u_{k,n,t})| > \delta\sqrt{n}\right\}\right] \\
&= 0,
\end{aligned}
$$

for any $\delta > 0$. This implies the claimed uniform asymptotic negligability condition (e.g. Gut, 2005, Remark 7.2.4):

$$\max_{1 \leq t \leq n} \frac{|h_{k,n}(u_{k,n,t})|}{\sqrt{n}} \xrightarrow{P^n_{\theta_n(g_n,h)}} 0. \qquad \square$$

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S2.4 Scores

LEMMA S2.12: *Suppose Assumption 2.1 holds. Let p_θ and $\bar{q}_{n,\theta}$ be as in the Proof of Proposition S2.5 and suppose that $\theta_n = (\gamma_n, \eta) \to (\gamma, \eta) = \theta$. Then*

$$\lim_{n \to \infty} \int \left\| \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \bar{q}_{n,\theta}^{1/2} - \tilde{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \right\|^2 \, d\lambda = 0. \tag{S9}$$

Proof. The integral in (S9) can be re-written as

$$\sum_{l=1}^{L} \int \left(\tilde{\ell}_{\theta_n,l}(y,x) p_{\theta_n}(y,x)^{1/2} - \tilde{\ell}_{\theta,l}(y,x) p_\theta(y,x)^{1/2} \right)^2 \, d(\lambda(y) \otimes Q_{n,\theta}(x))$$

Inspection of the forms of $\tilde{\ell}_\vartheta$ and p_ϑ reveals that each integrand in the preceding display converges to zero as $n \to \infty$. If we show that

$$\limsup_{n \to \infty} \int \tilde{\ell}_{\theta_n,l}^2 p_{\theta_n} \, d(\lambda \otimes Q_{n,\theta}) \leq \int \tilde{\ell}_{\theta,l}^2 p_\theta \, d(\lambda \otimes Q_\theta) < \infty, \tag{S10}$$

the proof will be complete in view of Lemma S2.2, Proposition S3.1 and Remark S3.1.[S8] The preceding display is equivalent to

$$\limsup_{n \to \infty} \int \tilde{\ell}_{\theta_n,l}^2 \, dG_{\theta_n,\theta,n} \leq \int \tilde{\ell}_{\theta,l}^2 \, dG_\theta < \infty,$$

for $G_{\vartheta,\theta,n}$ the distribution of (Y,X) corresponding to the density $p_\vartheta \bar{q}_{n,\theta}$ and G_θ as defined in the proof of Lemma S2.5. That $\tilde{\ell}_{\theta_n,l}^2 \to \tilde{\ell}_{\theta,l}^2$ pointwise is clear from its form, as given in Lemma 3.1. The finiteness of each of the integrals in the above display along with the fact that for some $N \in \mathbb{N}$ and some $\rho > 0$,

$$\sup_{n \geq N} \int \tilde{\ell}_{\theta_n,l}^{2+\rho} \, dG_{\theta_n,\theta,n} < \infty$$

follows from the form of $\tilde{\ell}_{\vartheta,l}^2$ (as given in Lemma 3.1) along with Assumption 2.1.[S9] □

LEMMA S2.13 (Smoothness): *Suppose that Assumption 2.1 holds. Then for any sequence $\theta_n = (\gamma + g_n/\sqrt{n}, \eta)$ with $g_n \to g \in \mathbb{R}^L$,*

$$R_n := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[\tilde{\ell}_{\theta_n}(Y_t, X_t) - \tilde{\ell}_\theta(Y_t, X_t) \right] + \tilde{I}_{\theta,n} g_n \xrightarrow{P_\theta^n} 0.$$

[S8]Note that the product structure of $\lambda \otimes Q_{n,\theta}$ and Lemma S2.2 ensure that $\lambda \otimes Q_{n,\theta} \to \lambda \otimes Q_\theta$ setwise.

[S9]Cf. the proof of Lemma S2.3: arguing in essentially the same manner as there allows one to obtain uniform boundedness of the $4 + \delta$ moments of ϵ_k, $\phi_k(\epsilon_k)$, X_t (uniformly in t) and all the non-stochastic terms in $\tilde{\ell}_{\theta_n,l}^2$.

Proof. From (the proof of) Lemma S2.8 we have

$$\lim_{n \to \infty} \int \left[\sqrt{n} \left(p_{\theta_n}^{1/2} - p_\theta^{1/2} \right) \bar{q}_{n,\theta}^{1/2} - \frac{1}{2} g' \dot{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \right]^2 \mathrm{d}\lambda = 0, \tag{S11}$$

whilst by Lemma S2.12 we have

$$\lim_{n \to \infty} \int \left\| \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \bar{q}_{n,\theta}^{1/2} - \tilde{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \right\|^2 \mathrm{d}\lambda = 0. \tag{S12}$$

Define

$$c_n^{-1} := \int p_{\theta_n}^{1/2} p_\theta^{1/2} \bar{q}_{n,\theta} \, \mathrm{d}\lambda = 1 - \frac{1}{2} \int (p_\theta^{1/2} - p_{\theta_n}^{1/2})^2 \bar{q}_{n,\theta} \, \mathrm{d}\lambda.$$

We have

$$-n \left(p_\theta^{1/2} - p_{\theta_n}^{1/2} \right)^2 = - \left(\sqrt{n} \left[p_{\theta_n}^{1/2} - p_\theta^{1/2} \right] - \frac{1}{2} g' \dot{\ell}_\theta p_\theta^{1/2} \right)^2 + \left(\frac{1}{2} g' \dot{\ell}_\theta p_\theta^{1/2} \right)^2$$
$$- g' \dot{\ell}_\theta p_\theta^{1/2} \sqrt{n} \left(p_{\theta_n}^{1/2} - p_\theta^{1/2} \right),$$

and so by (S11) and the continuity of the inner product

$$\int (p_\theta^{1/2} - p_{\theta_n}^{1/2})^2 \bar{q}_{n,\theta} \, \mathrm{d}\lambda = \frac{1}{n} \int g' \dot{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \sqrt{n} \left(p_{\theta_n}^{1/2} - p_\theta^{1/2} \right) \bar{q}_{n,\theta}^{1/2} \, \mathrm{d}\lambda$$
$$- \frac{1}{n} \int \left(\frac{1}{2} g' \dot{\ell}_\theta p_\theta^{1/2} \right)^2 \bar{q}_{n,\theta} \, \mathrm{d}\lambda + o(n^{-1})$$
$$= \frac{1}{4} (n^{-1/2} g)' \dot{I}_{n,\theta} (n^{-1/2} g) + o(n^{-1}),$$

where $\dot{I}_{n,\theta} := \int \dot{\ell}_\theta \dot{\ell}_\theta' p_\theta \bar{q}_{n,\theta} \, \mathrm{d}\lambda = O(1).$[S10] It follows that $c_n^{-1} = 1 - a_n$ with $a_n \to 0$ and $na_n = \frac{1}{4} g' \dot{I}_\theta g + o(1)$.

R_n is equal to the sum of

$$R_{1,n}' := \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[\tilde{\ell}_{\theta_n}(Y_t, X_t) \left(1 - \frac{p_{\theta_n}(Y_t, X_t)^{1/2}}{p_\theta(Y_t, X_t)^{1/2}} \right) \right] + \frac{1}{2} \tilde{I}_{n,\theta} g_n \; ;$$

$$R_{2,n}' := \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[\tilde{\ell}_{\theta_n}(Y_t, X_t) \frac{p_{\theta_n}(Y_t, X_t)^{1/2}}{p_\theta(Y_t, X_t)^{1/2}} - \tilde{\ell}_\theta(Y_t, X_t) \right] + \frac{1}{2} \tilde{I}_{n,\theta} g_n \; .$$

Since $\tilde{I}_{n,\theta}$ is $O(1)$ by Lemma S2.3 it suffices to prove that these converge in probability to zero with g_n replaced by g; let the corresponding expressions be called $R_{i,n}$ for $i = 1, 2$.

[S10]This follows by noting that $\|\dot{\ell}_\theta\|^2$ is uniformly integrable under $p_\theta \bar{q}_{n,\theta}$ which is a consequence of Lemma S2.3.

For $R_{1,n}$ we note that (omitting the arguments of the functions)

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \left(1 - \frac{p_{\theta_n}^{1/2}}{p_\theta^{1/2}}\right) + \frac{1}{2}\frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \dot{\ell}'_\theta g = \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \sqrt{n} \left(1 - \frac{p_{\theta_n}^{1/2}}{p_\theta^{1/2}} + \frac{1}{2\sqrt{n}}\dot{\ell}'_\theta g\right)$$

$$\leq \frac{1}{n} \sum_{t=1}^{n} \|\tilde{\ell}_{\theta_n}\|^2 \times \frac{1}{n} \sum_{t=1}^{n} \left[\sqrt{n}\left(1 - \frac{p_{\theta_n}^{1/2}}{p_\theta^{1/2}} + \frac{1}{2\sqrt{n}}\dot{\ell}'_\theta g\right)\right]^2.$$

The first term on the second line is $O_{P^n_{\theta_n}}(1)$ hence $O_{P^n_\theta}(1)$ (by contiguity). The second has $L_1(P^n_\theta)$ norm

$$\mathbb{E}\left|\frac{1}{n} \sum_{t=1}^{n} \left[\sqrt{n}\left(1 - \frac{p_{\theta_n}^{1/2}}{p_\theta^{1/2}} + \frac{1}{2\sqrt{n}}\dot{\ell}'_\theta g\right)\right]^2\right| \leq \int \left[\sqrt{n}\left(p_\theta^{1/2} - p_{\theta_n}^{1/2} + \frac{1}{2\sqrt{n}}\dot{\ell}'_\theta g p_\theta^{1/2}\right)\right]^2 \bar{q}_{n,\theta}\, \mathrm{d}\lambda \to 0,$$

where the convergence is by (S11). Therefore, it suffices to show that

$$\frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \dot{\ell}'_\theta - \tilde{I}_{n,\theta} \xrightarrow{P^n_\theta} 0. \tag{S13}$$

We may replace $\tilde{I}_{n,\theta}$ in (S13) with $\tilde{I}_\theta := \int \tilde{\ell}_\theta \dot{\ell}'_\theta\, \mathrm{d}G_\theta$ with G_θ as defined in the proof of Lemma S2.5. In particular, let $G_{\theta,n} := G_{\theta,0,n}$ as defined in the proof of Lemma S2.5. Then, since $\|\tilde{\ell}_\theta(Y_t, X_t)\dot{\ell}_\theta(Y_t, X_t)'\|^{1+\rho/2}$ is uniformly $L_1(P^n_\theta)$ bounded (Lemma S2.3) one has

$$\sup_{n \in \mathbb{N}} \int \|\tilde{\ell}_\theta \dot{\ell}'_\theta\|^{1+\rho/2}\, \mathrm{d}G_{n,\theta} < \infty,$$

and so $\|\tilde{\ell}_\theta \dot{\ell}'_\theta\|$ is uniformly $G_{\theta,n}$–integrable. By Lemma S3.2 and Theorem 2.8 of Serfozo (1982),

$$\tilde{I}_{n,\theta} = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[\tilde{\ell}_\theta(Y_t, X_t)\dot{\ell}_\theta(Y_t, X_t)'\right] = \int \tilde{\ell}_\theta \dot{\ell}'_\theta\, \mathrm{d}G_{n,\theta} \to \int \tilde{\ell}_\theta \dot{\ell}'_\theta\, \mathrm{d}G_\theta = \tilde{I}_\theta. \tag{S14}$$

For any $M > 0$, one has the decompositions

$$E_{n,1}^M := \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \dot{\ell}'_\theta - \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \leq M\}\dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\}$$

$$= \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\}\dot{\ell}'_\theta + \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \leq M\}\dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| > M\}$$

and

$$E_2^M := \tilde{I}_\theta - \int \tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| \le M\} \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} \, dG_\theta$$
$$= \int \tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\} \, dG + \int \tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\} \mathbf{1}\{\|\dot{\ell}_\theta\| > M\} \, dG_\theta.$$

Additionally, for $\mathbb{E}$ taken under P_θ^n, define

$$E_{n,3}^M := \frac{1}{n} \sum_{t=1}^n \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \le M\} \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} - \mathbb{E}\left[\tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \le M\} \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\}\right];$$

$$E_{n,4}^M := \mathbb{E} \frac{1}{n} \sum_{t=1}^n \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \le M\} \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} - \int \tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| \le M\} \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} \, dG_\theta.$$

Since $\|\tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\}\| \le \|\tilde{\ell}_\theta \dot{\ell}_\theta'\|$, $\|\tilde{\ell}_\theta \dot{\ell}_\theta' \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\} \mathbf{1}\{\|\dot{\ell}_\theta\| > M\}\| \le \|\tilde{\ell}_\theta \dot{\ell}_\theta'\|$ and $\|\tilde{\ell}_\theta \dot{\ell}_\theta'\|$ is G_θ–integrable by Lemma S2.3, by the dominated convergence theorem, for any $\delta > 0$ there is an M such that $E_2^{M'} < \delta$ for $M' \ge M$. For any $M > 0$, by Theorem 3 in Saikkonen (2007), Theorem 14.1 in Davidson (1994) and Theorem 2 in Kanaya (2017) one has (cf. Lemma S2.14 below)

$$E_{n,3}^M = O_{P_\theta^n}(M^2/\sqrt{n}).$$

For $E_{n,4}^M$ we introduce a new measure: define μ_n as

$$\mu_n(A) := \int_A c_n p_{\theta_n}(x,y)^{1/2} p_\theta(x,y)^{1/2} \, d(\lambda(y) \otimes Q_n(x)).$$

By Lemma S3.2 one has that $\mu_n \to G$, as well as $G_{n,\theta} \to G$, in TV. Then, by Cauchy – Schwarz and Lemma S2.3

$$c_n^{-1} \int \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \le M\} \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} \, d\mu_n - \int \tilde{\ell}_\theta \mathbf{1}\{\|\tilde{\ell}_\theta\| \le M\} \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} \, dG_{n,\theta}$$
$$= \int \left(\tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \le M\} p_{\theta_n}^{1/2} - \tilde{\ell}_\theta \mathbf{1}\{\|\tilde{\ell}_\theta\| \le M\} p_\theta^{1/2}\right) \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} p_\theta^{1/2} \, d(\lambda \otimes Q_{\theta,n})$$
$$= \int \left(\tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\} p_{\theta_n}^{1/2} - \tilde{\ell}_\theta \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\} p_\theta^{1/2}\right) \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} p_\theta^{1/2} \, d(\lambda \otimes Q_{\theta,n})$$
$$+ \int \left(\tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} - \tilde{\ell}_\theta p_\theta^{1/2}\right) \dot{\ell}_\theta' \mathbf{1}\{\|\dot{\ell}_\theta\| \le M\} p_\theta^{1/2} \, d(\lambda \otimes Q_{\theta,n})$$
$$\lesssim o(1) + \sup_{n \in \mathbb{N}} \mathbb{E}_{\theta_n}\left[\|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\}\right] + \sup_{n \in \mathbb{N}} \mathbb{E}_\theta\left[\|\tilde{\ell}_\theta\|^2 \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\}\right].$$

The last two right hand side terms can be made arbitrarily small, uniformly in n, by taking M

large enough; the $o(1)$ term follows from (S12) and is uniform in M. Now, by $G_{n,\theta} \xrightarrow{TV} G_\theta$,

$$\left| \int \tilde{\ell}_\theta \mathbf{1}\{\|\tilde{\ell}_\theta\| \leq M\} \dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\} \, dG_{\theta,n} - \int \tilde{\ell}_\theta \mathbf{1}\{\|\tilde{\ell}_\theta\| \leq M\} \dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\} \, dG_\theta \right|$$

$$\leq M^2 \|G_{n,\theta} - G_\theta\|_{TV}.$$

Since $\mu_n \to G_\theta$ and $G_{n,\theta} \to G_\theta$ in total variation, one has that $\|\mu_n - G_{n,\theta}\|_{TV} \to 0$. Since $\tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \leq M\} \dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\}$ is uniformly bounded, one has that

$$\left| \int \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \leq M\} \dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\} \, d\mu_n - \int \tilde{\ell}_{\theta_n} \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| \leq M\} \dot{\ell}'_\theta \mathbf{1}\{\|\dot{\ell}_\theta\| \leq M\} \, dG_{n,\theta} \right|$$

$$\leq M^2 \|\mu_n - G_{n,\theta}\|_{TV}.$$

As $c_n^{-1} - 1 = -a_n \to 0$, it follows that

$$E_{n,4}^M \leq M^2 \left[\|\mu_n - G_{n,\theta}\|_{TV} + \|G_{n,\theta} - G_\theta\|_{TV} \right] + e_n + M^2 |a_n| + r(M),$$

where $0 \leq r(M) := \sup_{n \in \mathbb{N}} \mathbb{E}_{P^n_{\theta_n}} \left[\|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\} \right] + \sup_{n \in \mathbb{N}} \mathbb{E}_{P^n_\theta} \left[\|\tilde{\ell}_\theta\|^2 \mathbf{1}\{\|\tilde{\ell}_\theta\| > M\} \right] \to 0$ as $M \to \infty$ and r does not depend on n and $e_n = o(1)$. For $E_{n,1}^M$ note that since $\|\dot{\ell}_\theta\|^2$ is uniformly P^n_θ–integrable (Lemma S2.3), $\frac{1}{n} \sum_{t=1}^n \|\dot{\ell}_\theta\|^2 = O_{P^n_\theta}(1)$. By Markov's inequality, for any $\delta > 0$

$$P^n_{\theta_n} \left(\left| \frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\} \right| > \delta \right) \leq \delta^{-1} \mathbb{E} \left[\left| \frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\} \right| \right]$$

$$\leq \delta^{-1} \sup_{n \in \mathbb{N}} \mathbb{E} \, \|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\}$$

$$\leq \delta^{-1} r(M).$$

Thus by taking $M \to \infty$, the probability on the left hand side of the preceding display vanishes. Therefore, the same is true of

$$P^n_\theta \left(\left| \frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M\} \right| > \delta \right),$$

by contiguity. That is, we can take a large enough M such that the probability in the display above is arbitrarily small (for all large enough $n \in \mathbb{N}$).

Now, fix $\varepsilon > 0, \delta > 0$. By Lemma S2.3, $\frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_\theta\|^2 = O_{P^n_\theta}(1)$ and also $\frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 =$

$O_{P_{\theta_n}^n}(1)$. By this and contiguity, we can choose $R > 0$ be such that for all $n \geq N_1$,

$$P_\theta^n \left(\frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_\theta\|^2 > R \right) < \varepsilon/4, \qquad P_\theta^n \left(\frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 > R \right) < \varepsilon/4.$$

Take M large enough that $\|E_2^M\| < \delta$, $r(M) < \delta$ and for all $n \geq N_2$

$$P_\theta^n \left(\left| \frac{1}{n} \sum_{t=1}^n \|\tilde{\ell}_{\theta_n}\|^2 \mathbf{1}\{\|\tilde{\ell}_{\theta_n}\| > M_n\} \right| > \delta/R \right) < \varepsilon/4$$

$$P_\theta^n \left(\left| \frac{1}{n} \sum_{t=1}^n \|\dot{\ell}_\theta\|^2 \mathbf{1}\{\|\dot{\ell}_\theta\| > M_n\} \right| > \delta/R \right) < \varepsilon/4$$

where $M_n \geq M$ and $M_n \to \infty$ slowly. This ensures that $\|E_2^{M_n}\| < \delta$, $P_\theta^n(\|E_{n,1}^{M_n}\| > 2\delta) < \varepsilon$ for all $n \geq \max\{N_1, N_2\}$. Then, let N be large enough such that $N \geq \max\{N_1, N2\}$, and for all $n \geq N$, $P_\theta^n(\|E_{n,3}^{M_n}\| > \delta) < \varepsilon$ and $\|E_{n,4}^{M_n}\| \leq 3\delta$.[S11] Combining these ensures that for all such n,

$$P_\theta^n \left(\left\| \frac{1}{n} \sum_{t=1}^n \tilde{\ell}_{\theta_n} \dot{\ell}_\theta' - \tilde{I}_\theta \right\| > 7\delta \right) < 2\varepsilon.$$

In conjunction with (S14) this establishes (S13).

We next show that $R_{2,n}$ converges to zero in P_θ^n–probability. Define

$$Z_{n,t} := \tilde{\ell}_{\theta_n}(Y_t, X_t) \frac{p_{\theta_n}(Y_t, X_t)^{1/2}}{p_\theta(Y_t, X_t)^{1/2}}, \quad m_n(X_t) := \int \tilde{\ell}_{\theta_n}(y, X_t) p_{\theta_n}(y, X_t)^{1/2} p_\theta(y, X_t)^{1/2} dy,$$

and note that $m_n(X_t) = \mathbb{E}[Z_{n,t}|X_t]$ (P_θ^n–a.s.). Since $\mathbb{E}[\tilde{\ell}_{\theta_n}(Y_t, X_t)|X_t] = 0$ under $P_{\theta_n}^n$ (which is clear from its form),

$$
\begin{aligned}
m_n(X_t) &= \int \tilde{\ell}_{\theta_n}(y, X_t) p_{\theta_n}(y, X_t)^{1/2} p_\theta(y, X_t)^{1/2} \, dy \\
&= \int \tilde{\ell}_{\theta_n}(y, X_t) p_{\theta_n}(y, X_t)^{1/2} \left[p_\theta(y, X_t)^{1/2} - p_{\theta_n}(y, X_t)^{1/2} \right] \, dy.
\end{aligned}
$$
(S15)

Using (S11), (S12) and Cauchy-Schwarz yields

$$\lim_{n\to\infty} \left| \left\langle \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \bar{q}_{\theta,n}^{1/2}, \sqrt{n} \left(p_\theta^{1/2} - p_{\theta_n}^{1/2} \right) \bar{q}_{n,\theta}^{1/2} \right\rangle_\lambda - \left\langle \tilde{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2}, -\frac{1}{2} g' \dot{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \right\rangle_\lambda \right| = 0,$$

which implies that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n m_n(X_t) + \frac{1}{2} \tilde{I}_{n,\theta} g \xrightarrow{P_\theta^n} 0,$$

[S11] I.e. n such that $M_n^2|a_n| < \delta$, $|e_n| < \delta$, $M_n^2 [\|\mu_n - G_{n,\theta}\|_{TV} + \|G_{n,\theta} - G_\theta\|_{TV}] < \delta$. Here one needs to take $M_n \to \infty$ slowly enough that these sequences still converge to zero and $M_n^2/\sqrt{n} \to 0$.

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given the representation of m_n in (S15). In consequence it remains to show that

$$R_{2,n}^* := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} Z_{t,n} - m_n(X_t) - \tilde{\ell}_\theta(Y_t, X_t) \xrightarrow{P_\theta^n} 0.$$

Put $\mathcal{F}_{n,t} = \sigma(Y_t, X_t)$. Then, as is straightforward to verify, $(Z_{t,n} - m_n(X_t) - \tilde{\ell}_\theta(Y_t, X_t), \mathcal{F}_{n,t})_{n\in\mathbb{N}, 1\le t\le n}$ forms a martingale difference array. Hence it suffices to show that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left\| Z_{t,n} - m_n(X_t) - \tilde{\ell}_\theta(Y_t, X_t) \right\|^2 \xrightarrow{P_\theta^n} 0.$$

The left hand side of this display can be written as

$$\int \left\| \tilde{\ell}_{\theta_n} \frac{p_{\theta_n}^{1/2}}{p_\theta^{1/2}} - m_n - \tilde{\ell}_\theta \right\|^2 p_\theta \bar{q}_{n,\theta} \, d\lambda \le 2 \int \left\| \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \bar{q}_{n,\theta}^{1/2} - \tilde{\ell}_\theta p_\theta^{1/2} \bar{q}_{n,\theta}^{1/2} \right\|^2 d\lambda + 2 \int \|m_n\|^2 \, dQ_{n,\theta},$$

and so, given (S12) it suffices to show that the second term on the right hand side converges to zero. For this note that by Fubini's theorem and the Cauchy-Schwarz inequality

$$\int \|m_n\|^2 \, dQ_{n,\theta} \le \int \left\| \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \left[p_\theta^{1/2} - p_{\theta_n}^{1/2} \right] \right\|^2 \bar{q}_{n,\theta} \, d\lambda$$

$$\le \int \left\| \tilde{\ell}_{\theta_n} p_{\theta_n}^{1/2} \bar{q}_{n,\theta}^{1/2} \right\|^2 d\lambda \int \left[\left(p_{\theta_n}^{1/2} - p_\theta^{1/2} \right) \bar{q}_{n,\theta}^{1/2} \right]^2 d\lambda.$$

The first term on the right hand side is $O(1)$ by equation (S10), whilst the second converges to zero by (S11) and the uniform $G_{\theta,0,n}$ – integrability of $g'\dot{\ell}_\theta$ as established in Lemma S2.6. $\square$

S2.4.1 Estimation

LEMMA S2.14: *Suppose that Assumption 2.1 holds and g_n are ϱ – integrable functions for some $\varrho > 2$ such that $\max_{t=1,\ldots,n} \|g_n(Y_t, X_t)\|_{L_\varrho} \le M_n$ (all under P_θ^n). Then,*

$$\frac{1}{n} \sum_{t=1}^{n} g_n(Y_t, X_t) - \mathbb{E}\left[g_n(Y_t, X_t)\right] = O_{P_\theta}(M_n/\sqrt{n}).$$

Proof. Let $\alpha_n(m)$ be the α – mixing coefficients of the array $\{g_n(Y_t, X_t) - \mathbb{E}[g_n(Y_t, X_t)] : n \in \mathbb{N}, 1 \le t \le n\}$. By (the proof of) Theorem 14.1 in Davidson (1994), $\alpha_n(m) \le \tilde{\alpha}(m - p)$ (for $m \ge p$) where $\tilde{\alpha}(m)$ are the mixing coefficients of $\{Y_t : t \in \mathbb{N}\}$. By Theorem 3 in Saikkonen (2007) and Proposition 1.1.1 in Doukhan (1994) $\tilde{\alpha}(m) = O(a^m)$ for some $a \in (0,1)$. Condition A1 in Kanaya (2017) then holds (with $\Delta = 1$) with $\beta > \varrho/(\varrho - 2)$. To see this note that for all

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$m \geq M_1$ we have $\tilde{\alpha}(m - p) \leq Ca^m$ whilst $Ca^m \leq Am^{-\beta}$ whenever

$$\beta \leq \frac{\log(A) - \log(C) + m|\log(a)|}{\log(m)}.$$

As the right hand side diverges as $m \to \infty$, for all m larger than some $M \geq M_1$, the inequality will hold for some $\beta > \varrho/(\varrho - 2)$. Noting that the inequality above continues to hold if we increase A, we may then choose A such that each $\tilde{\alpha}(m) \leq Am^{-\beta}$ for all $1 \leq m \leq M$. The result then follows by Theorem 2 in Kanaya (2017). □

LEMMA S2.15: *Suppose that Assumptions 2.1 and 2.2 hold. Then*

(i) If $Z_{n,1} := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \tilde{\ell}_\theta(Y_t, X_t)$ and $Z_{n,2} := \Lambda^n_{\theta_n(g,h)}(Y^n)$, then under P^n_θ,

$$Z_n \rightsquigarrow Z \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^2_{g,h} \end{pmatrix}, \begin{pmatrix} \tilde{I}_\theta & \tilde{I}_\theta g \\ g'\tilde{I}_\theta & \sigma^2_{g,h} \end{pmatrix}\right).$$

Additionally, let $\theta_n := \theta_n(g_n, 0) = (\gamma + g_n/\sqrt{n}, \eta)$ for $g_n \to g \in \mathbb{R}^L$. Then

(ii) We have that

$$\frac{1}{n} \sum_{t=1}^{n} \left(\hat{\ell}_{\theta_n}(Y_t, X_t) - \tilde{\ell}_{\theta_n}(Y_t, X_t) \right) = o_{P^n_{\theta_n}}(n^{-1/2}).$$

(iii) $\|\hat{I}_{n,\theta_n} - \tilde{I}_\theta\| = o_{P^n_{\theta_n}}(\nu_n^{1/2})$ where ν_n is defined in Assumption 2.2, and $\tilde{I}_\theta := G_\theta \tilde{\ell}_\theta \tilde{\ell}'_\theta$ with G_θ as in the proof of Lemma S2.5.

Proof. For part (i), let z_t be

$$z_t := \left(\tilde{\ell}_\theta(Y_t, X_t)', g'\dot{\ell}_\theta(Y_t, X_t) + \sum_{k=1}^{K} h_k(A_{k\bullet} V_{\theta,t}) \right)',$$

and $\mathcal{F}_t := \sigma(\epsilon_1, \ldots, \epsilon_t)$. Under P^n_θ, $\{z_t, \mathcal{F}_t : t \in \mathbb{N}\}$ is a martingale difference sequence such that

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[z_t z_t'\right] = \begin{bmatrix} \tilde{I}_{n,\theta} & \tilde{I}_{\theta,\theta} g \\ g'\tilde{I}_{n,\theta} & \sigma^2_{g,h,n} \end{bmatrix} \to \begin{bmatrix} \tilde{I}_\theta & \tilde{I}_\theta g \\ g'\tilde{I}_\theta & \sigma^2_{g,h} \end{bmatrix},$$

noting Lemma 3.1 and Theorem 12.14 of Rudin (1991). That $\sigma^2_{g,h,n}$ converges to a $\sigma^2_{g,h}$ is part of the conclusion of Proposition A.1. That $\tilde{I}_{\theta,n} \to \tilde{I}_\theta$ follows by combining Lemma S2.3, the fact that $G_{\theta,0,n}$ as defined in the proof of Lemma S2.5 converges in total variation to G_θ (cf. Lemma S3.2), and Corollary 2.9 in Feinberg et al. (2016). Lindeberg's condition is satisfied since $\{\|z_t\|^2 : t \in \mathbb{N}\}$ is uniformly P^n_θ-integrable (by Lemma S2.3 and the fact that each h_k

is bounded) and the variance convergence in the preceding display. Part (i) then follows from Proposition A.1 and the central limit theorem for martingale differences.

Define $A_n := A(\theta_n)$, $B_n := B(\theta_n)$, and $\zeta^x_{n,l,k,j} := \zeta^x_{l,k,j}(\theta_n)$ for each triple (l, j, k) of indicies and $x \in \{\alpha, \sigma\}$. Note that each $A_{n,k}(Y_t - B_n X_t) \approx \epsilon_{k,t} \sim \eta_k$ under $P^n_{\theta_n}$. Hence

$$\tilde{\ell}_{\theta_n, \alpha_l}(Y_t, X_t) \approx \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta^\alpha_{l,k,j} \phi_k(\epsilon_{k,t}) \epsilon_{j,t} + \sum_{k=1}^{K} \zeta^\alpha_{n,l,k,k} \left[\tau_{k,1} \epsilon_{k,t} + \tau_{k,2} \kappa(\epsilon_{k,t}) \right] \tag{S16}$$

$$\tilde{\ell}_{\theta_n, \sigma_l}(Y_t, X_t) \approx \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta^\sigma_{n,l,k,j} \phi_k(\epsilon_{k,t}) \epsilon_{j,t} + \sum_{k=1}^{K} \zeta^\sigma_{l,k,k} \left[\tau_{k,1} \epsilon_{k,t} + \tau_{k,2} \kappa(\epsilon_{k,t}) \right] \tag{S17}$$

$$\tilde{\ell}_{\theta_n, b_l}(Y_t, X_t) \approx \sum_{k=1}^{K} -A_{n,k\bullet} D_{b,l} \left[\phi_k(\epsilon_{k,t})(X_t - \mathbb{E}\, X_t) - \mathbb{E}\, X_t \left(\varsigma_{k,1} \epsilon_{k,t} + \varsigma_{k,2} \kappa(\epsilon_{k,t}) \right) \right] \tag{S18}$$

By Assumption 2.1(iii), $\zeta^x_{n,l,k,j} \to \zeta^\alpha_{\infty,l,k,j} := [D_{x_l}(\alpha, \sigma)]_{k\bullet} A(\alpha, \sigma)^{-1}_{\bullet j}$ for $x \in \{\alpha, \sigma\}$. Note that the entries of $D_{b,l}$ are all zero except for entry l (corresponding to b_l) which is equal to one.

We verify (ii) for each component of the efficient score (S16) – (S18). For components (S16) and (S17), we define for x either of α, σ

$$\varphi_{1,n,t} := \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta^x_{l,k,j,n} \phi_k(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} \,,$$

and

$$\hat{\varphi}_{1,n,t} := \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta^x_{l,k,j,n} \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} \,,$$

with $V_{n,t} = Y_t - B_n X_t$, and let $\overline{\zeta}_n := \max_{l \in [L], j \in [K], k \in [K]} |\zeta^x_{l,j,k,n}|$ which converges to $\overline{\zeta} := \max_{l \in [L], j \in [K], k \in [K]} |\zeta^x_{l,j,k,\infty}| < \infty$. We have that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\hat{\varphi}_{1,n,t} - \varphi_{1,n,t}) \leq \sqrt{n} \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \overline{\zeta}_n \left| \frac{1}{n} \sum_{t=1}^{n} \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} - \phi_k(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} \right| ,$$

Each $\left| \frac{1}{n} \sum_{t=1}^{n} \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} - \phi_k(A_{n,k\bullet} V_{n,t}) A_{n,j\bullet} V_{n,t} \right| = o_{P_{\theta_n}}(n^{-1/2})$ by applying Lemma A.1 with $W_{n,t} = A_{n,j\bullet} V_{n,t}$ (noting that $A_{n,k\bullet} V_{n,s} \simeq \epsilon_{k,s}$ and $A_{n,j\bullet} V_{n,t} \simeq \epsilon_{j,t}$ with are independent for any s, t with $\mathbb{E}_{\theta_n}(A_{n,j\bullet} V_{n,t})^2 = 1$ by Assumption 2.1(ii)), and the outside summations are finite, it follows that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\hat{\varphi}_{1,n,t} - \varphi_{1,n,t}) = o_{P^n_{\tilde{\theta}_n}}(1) \,. \tag{S19}$$

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That $\hat{\tau}_{k,n} \xrightarrow{P_{\theta_n}^n} \tau_k$ follows from Lemma S2.16. Now, consider $\varphi_{2,\tau,n,t}$ defined by

$$\varphi_{2,\tau,n,t} := \sum_{k=1}^{K} \zeta_{n,l,k,k}^z \left[\tau_{k,1} A_{n,k\bullet} V_{n,t} + \tau_{k,2} \kappa(A_{n,k\bullet} V_{n,t}) \right],$$

for x equal to either α or σ. Since sum is finite and each $|\zeta_{n,l,k,k}^x| \to |\zeta_{\infty,l,k,k}^x| < \infty$ it is sufficient to consider the convergence of the summands. In particular we have that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} [\hat{\tau}_{k,n,1} - \tau_{k,1}] A_{n,k\bullet} V_{n,t} = [\hat{\tau}_{k,n,1} - \tau_{k,1}] \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_{n,k\bullet} V_{n,t} \to 0,$$

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} [\hat{\tau}_{k,n,2} - \tau_{k,2}] \kappa(A_{n,k\bullet} V_{n,t}) = [\hat{\tau}_{k,n,2} - \tau_{k,2}] \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \kappa(A_{n,k\bullet} V_{n,t}) \to 0,$$

in probability, since $A_{n,k\bullet} V_{n,t} \approx \epsilon_{k,t} \sim \eta_k$ and $(\epsilon_{k,t})_{t\geq 1}$ and $(\kappa(\epsilon_{k,t}))_{t\geq 1}$ are i.i.d. mean-zero sequences with finite second moments such that the central limit theorem holds.

Together these yield that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\varphi_{2,\hat{\tau}_n,n,t} - \varphi_{2,\tau,n,t}) \xrightarrow{P_{\theta_n}^n} 0. \tag{S20}$$

Combination of (S19) and (S20) yields (ii) for components of the type (S16), (S17).

For components (S18) let $a_{n,k,l} := -A_{n,k\bullet} D_{b_l}$, $\tilde{\varsigma}_{k,n} := \hat{\varsigma}_{k,n} - \varsigma_k$, $c_{n,t} := \mathbb{E}_{\theta_n} X_t$ and $\bar{c}_n := \frac{1}{n} \sum_{t=1}^{n} c_{n,t}$. Since $a_{n,k,l} \to a_{\infty,k,l} := A(\alpha,\sigma)_{k\bullet} D_{b_l}(\alpha,\sigma)$, it suffices to show that

(i) $\frac{1}{n} \sum_{t=1}^{n} \left[\phi_k(A_{n,k\bullet} V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) \right] (X_t - c_{n,t}) = o_{P_{\theta_n}^n}(n^{-1/2})$;

(ii) $\frac{1}{n} \sum_{t=1}^{n} \left[\phi_k(A_{n,k\bullet} V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) \right] (\bar{X}_n - \bar{c}_n) = o_{P_{\theta_n}^n}(n^{-1/2})$;

(iii) $\frac{1}{n} \sum_{t=1}^{n} \left[\phi_k(A_{n,k\bullet} V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet} V_{n,t}) \right] (\bar{c}_n - c_{n,t}) = o_{P_{\theta_n}^n}(n^{-1/2})$;

(iv) $\frac{1}{n} \sum_{t=1}^{n} \phi_k(A_{n,k\bullet} V_{n,t}) (\bar{X}_n - \bar{c}_n) = o_{P_{\theta_n}^n}(n^{-1/2})$;

(v) $\frac{1}{n} \sum_{t=1}^{n} \phi_k(A_{n,k\bullet} V_{n,t}) (\bar{c}_n - c_{n,t}) = o_{P_{\theta_n}^n}(n^{-1/2})$;

(vi) $\frac{1}{n} \sum_{t=1}^{n} \bar{X}_n [\tilde{\varsigma}_{k,n,1} A_{n,k\bullet} V_{n,t} + \tilde{\varsigma}_{k,n,2} \kappa(A_{n,k\bullet} V_{n,t})] = o_{P_{\theta_n}^n}(n^{-1/2})$;

(vii) $\frac{1}{n} \sum_{t=1}^{n} (\bar{X}_n - \bar{c}_n) [\varsigma_{k,1} A_{n,k\bullet} V_{n,t} + \varsigma_{k,2} \kappa(A_{n,k\bullet} V_{n,t})] = o_{P_{\theta_n}^n}(n^{-1/2})$;

(viii) $\frac{1}{n} \sum_{t=1}^{n} (\bar{c}_n - c_{n,t}) [\varsigma_{k,1} A_{n,k\bullet} V_{n,t} + \varsigma_{k,2} \kappa(A_{n,k\bullet} V_{n,t})] = o_{P_{\theta_n}^n}(n^{-1/2})$

(i) follows by (the first part of) Lemma A.1 applied with $W_{n,t} = X_t - c_{n,t}$. This is mean-zero, independent of all $A_{n,k\bullet} V_{n,s}$ with $s \geq t$ and has uniformly bounded second moments (cf. (S6)).

(ii) follows by Jensen's inequality, (the second part of) Lemma A.1 applied with $W_{n,t} = 1$, (S6), Lemma S2.14 and Corollary 3.1.

(iii) follows by Cauchy – Schwarz, (the second part of) Lemma A.1 applied with $W_{n,t} = 1$ and Lemma S2.17.

For (iv), $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \phi_k(A_{n,k\bullet} V_{n,t}) = O_{P_{\theta_n}^n}(1)$ by the central limit theorem and $\bar{X}_n - \bar{c}_n = \frac{1}{n} \sum_{t=1}^{n} [X_t - c_{n,t}] \xrightarrow{P_{\theta_n}} 0$, which follows by (S6), Lemma S2.14 and Corollary 3.1.

(v) follows by Cauchy – Schwarz, the fact that $\mathbb{E}\, \phi_k(A_{n,k\bullet} V_{n,t})^2 = \mathbb{E}\, \phi_k(\epsilon_{k,t})^2$ is uniformly bounded hence $\frac{1}{n} \sum_{t=1}^{n} \phi_k(A_{n,k\bullet} V_{n,t})^2 = O_{P_{\theta_n}^n}(1)$ by Markov's inequality and Lemma S2.17.

For (vi), $\bar{X}_n = O_{P_{\theta_n}}(1)$ by e.g. Markov's inequality and (S6). By the central limit theorem also $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} U_t = O_{P_{\theta_n}^n}(1)$ for U_t equal to either $A_{n,k\bullet} V_{n,t}$ or $\kappa(A_{n,k\bullet} V_{n,t})$. The result therefore follows from Lemma S2.16.

For (vii), as for (vi), $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} U_t = O_{P_{\theta_n}^n}(1)$ for U_t equal to either $A_{n,k\bullet} V_{n,t}$ or $\kappa(A_{n,k\bullet} V_{n,t})$. Therefore it suffices to note that $\bar{X}_n - \bar{c}_n \xrightarrow{P_{\theta_n}} 0$, as noted for (iv).

For (viii), for U_t equal to either $\varsigma_{k,1} A_{n,k\bullet} V_{n,t}$ or $\varsigma_{k,2} \kappa(A_{n,k\bullet} V_{n,t})$, by Markov's inequality

$$P_{\theta_n}^n \left(\left\| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\bar{c}_n - c_{n,t}) U_t \right\| > \varepsilon \right) \leq \varepsilon^{-2}\, \mathbb{E}\, U_t^2 \frac{1}{n} \sum_{t=1}^{n} \|\bar{c}_n - c_{n,t}\|^2 \lesssim \frac{1}{n} \sum_{t=1}^{n} \|\bar{c}_n - c_{n,t}\|^2 \to 0,$$

by Lemma S2.17.

To verify (iii) we note that

$$\left\| \hat{I}_{n,\theta_n} - \tilde{I}_\theta \right\|_2 \leq \left\| \hat{I}_{n,\theta_n} - \breve{I}_{n,\theta_n} \right\|_2 + \left\| \breve{I}_{n,\theta_n} - \tilde{I}_{n,\theta_n} \right\|_2 + \left\| \tilde{I}_{n,\theta_n} - \tilde{I}_\theta \right\|_2 \tag{S21}$$

where $\tilde{I}_\theta := \mathbb{E}[\tilde{\ell}_\theta(Y_t, X_t)\tilde{\ell}_\theta(Y_t, X_t)'] = \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[\tilde{\ell}_\theta(Y_t, X_t)\tilde{\ell}_\theta(Y_t, X_t)']$ with the expectation taken under G_θ, $\hat{I}_{n,\theta} := \frac{1}{n} \sum_{t=1}^{n} \hat{\ell}_\theta(Y_t, X_t)\hat{\ell}_\theta(Y_t, X_t)'$ and $\breve{I}_{n,\theta} := \frac{1}{n} \sum_{t=1}^{n} \tilde{\ell}_\theta(Y_t, X_t)\tilde{\ell}_\theta(Y_t, X_t)'$. We will show each right hand side term is $o_{P_{\theta_n}^n}(\nu_n^{1/2})$.

For the first right hand side term in (S21) let $r \in \{\alpha, \sigma, b\}$ and let l denote an index, we write $\hat{U}_{n,t,r_l} := \hat{\ell}_{\theta_n,r_l}(Y_t, X_t)$, $\tilde{U}_{t,r_l} := \tilde{\ell}_{\theta_n,r_l}(Y_t, X_t)$ and $D_{n,t,r_l} := \hat{\ell}_{\theta_n,r_l}(Y_t, X_t) - \tilde{\ell}_{\theta_n,r_l}(Y_t, X_t)$.

Since it is the absolute value of the $(r,l) - (s,m)$ component of $\hat{I}_{n,\theta_n} - \breve{I}_{n,\theta_n}$, it is sufficient to show that $\left| \frac{1}{n} \sum_{t=1}^{n} \hat{U}_{n,t,r_l} D_{n,t,s_m} + \frac{1}{n} \sum_{t=1}^{n} D_{n,t,r_l} \tilde{U}_{t,s_m} \right| = o_{P_{\theta_n}^n}(\nu_n^{1/2})$ as $n \to \infty$ for any $r, s \in \{\alpha, \sigma, b\}$ and l, m. By Cauchy-Schwarz and Lemma S2.19

$$\left| \frac{1}{n} \sum_{t=1}^{n} D_{n,t,r_l} \tilde{U}_{t,s_m} \right| \leq \left(\frac{1}{n} \sum_{t=1}^{n} \tilde{U}_{t,s_m}^2 \right)^{1/2} \left(\frac{1}{n} \sum_{t=1}^{n} D_{n,t,r_l}^2 \right)^{1/2} = O_{P_{\theta_n}^n}(1) \times o_{P_{\theta_n}^n}(\nu_n^{1/2}) = o_{P_{\theta_n}^n}(\nu_n^{1/2}),$$

$$\left| \frac{1}{n} \sum_{t=1}^{n} \hat{U}_{n,t,r_l} D_{n,t,s_m} \right| \leq \left(\frac{1}{n} \sum_{t=1}^{n} \hat{U}_{n,t,r_l}^2 \right)^{1/2} \left(\frac{1}{n} \sum_{t=1}^{n} D_{n,t,s_m}^2 \right)^{1/2} = O_{P_{\theta_n}^n}(1) \times o_{P_{\theta_n}^n}(\nu_n^{1/2}) = o_{P_{\theta_n}^n}(\nu_n^{1/2}),$$

for any $(r, l) - (s, m)$. It follows that

$$\left[\frac{1}{n}\sum_{t=1}^{n}\hat{U}_{n,t,r_l}D_{n,t,s_m} + D_{n,t,r_l}\tilde{U}_{t,s_m}\right]^2 \leq 2\left[\frac{1}{n}\sum_{t=1}^{n}\hat{U}_{n,t,r_l}D_{n,t,s_m}\right]^2 + 2\left[\frac{1}{n}\sum_{t=1}^{n}D_{n,t,r_l}\tilde{U}_{t,s_m}\right]^2 = o_{P_{\theta_n}^n}(\nu_n)$$

and hence $\|\hat{I}_{n,\theta_n} - \breve{I}_{n,\theta_n}\|_2 \leq \|\hat{I}_{n,\theta_n} - \breve{I}_{n,\theta_n}\|_F = o_{P_{\theta_n}^n}(\nu_n^{1/2})$

For the second right hand side term in (S21), Let $Q_{l,m,t,n}^{r,ls} = \tilde{\ell}_{\theta_n,r_l}(Y_t, X_t)\tilde{\ell}_{\theta_n,s_m}(Y_t, X_t)$, where $r, s \in \{\alpha, \sigma, b\}$ and l, m denote the indices of the components of the efficient scores. Fix any r, s and l, m and note that by the fact that $\tilde{\ell}_{\theta_n}$ has uniformly bounded $2 + \delta/2$ moments under $P_{\theta_n}^n$, Theorem 3 of Saikkonen (2007) and Theorem 1 of Kanaya (2017) together imply that (cf. Lemma S2.14)

$$\frac{1}{n}\sum_{t=1}^{n}Q_{l,m,t,n}^{r,s} - \mathbb{E}_{\theta_n}Q_{l,m,t,n}^{r,s} = O_{P_{\theta_n}^n}\left(n^{(1/p-1)/2}\right) = o_{P_{\theta_n}^n}(\nu_n^{1/2}), \quad p \in (1, 1 + \delta/4],$$

hence $\|\breve{I}_{n,\theta_n} - \tilde{I}_{n,\theta_n}\|_2 = o_{P_{\theta_n}^n}(\nu_n^{1/2})$.

That the last right hand side term in (S21) is $o(\nu_n^{1/2})$ follows from the assumed local Lipschitz continuity of the map defining the ζ's, that of each $\beta \mapsto A(\alpha, \sigma)_{k\bullet}$, Theorem 11.11 of Kallenberg (2021) and Lemma S2.18. $\qquad\square$

LEMMA S2.16: *If assumption 2.1 holds, then* $\|\hat{\varrho}_{k,n} - \varrho_{k,n}\|_2 = o_{P_{\tilde{\theta}_n}^n}(\nu_{n,p}) = o_{P_{\theta_n}^n}(\nu_n^{1/2})$, *where* $\tilde{\theta}_n$ *is as in Lemma S2.15 and* $\varrho \in \{\tau, \varsigma\}$.

Proof. Under $P_{\theta_n}^n$, $A_{n,k\bullet}V_{n,t} \approx \epsilon_{k,t} \sim \eta_k$, for $V_{n,t} := Y_t - B_n X_t$ and $A_n := A(\theta_n)$. Let $w \in \{(0, -2)', (1, 0)'\}$ Since the map $M \mapsto M^{-1}$ is Lipschitz at a positive definite matrix M_0, then for large enough n, with probability approaching one

$$\|\hat{\varrho}_{k,n} - \varrho_{k,n}\|_2 = \|(\hat{M}_{k,n}^{-1} - M_k^{-1})w\|_2 \leq 2\|\hat{M}_{k,n}^{-1} - M_k^{-1}\|_2 \leq 2C\|\hat{M}_{k,n} - M_k\|_2, \qquad \text{(S22)}$$

for some positive constant C. By Theorem 2.5.11 in Durrett (2019)

$$\frac{1}{n}\sum_{t=1}^{n}[(A_{n,k\bullet}V_{n,t})^3 - \mathbb{E}(A_{n,k\bullet}V_{n,t})^3] = o_{P_{\theta_n}^n}\left(n^{\frac{1-p}{p}}\right)$$

$$\frac{1}{n}\sum_{t=1}^{n}[(A_{n,k\bullet}V_{n,t})^4 - \mathbb{E}(A_{n,k\bullet}V_{n,t})^4] = o_{P_{\theta_n}^n}\left(n^{\frac{1-p}{p}}\right).$$

These together imply that

$$\|\hat{M}_{k,n} - M_k\|_2 \leq \|\hat{M}_{k,n} - M_k\|_F = o_{P_{\theta_n}^n}\left(n^{\frac{1-p}{p}}\right) = o_{P_{\theta_n}^n}(\nu_{n,p}).$$

Combining these convergence rates with equation (S22) yields the result. $\qquad\square$

LEMMA S2.17: *In the setting of Lemma S2.15, let $c_{n,t} := \mathbb{E}_{\theta_n} X_t$ and $\bar{c}_n := \frac{1}{n} \sum_{t=1}^{n} c_{n,t}$. Then*

$$\frac{1}{n} \sum_{t=1}^{n} \|\bar{c}_n - c_{n,t}\|^2 = O(n^{-1}).$$

Proof. Since $X_t = (1, Z'_{t-1})'$, it suffices to show that $\frac{1}{n} \sum_{t=1}^{n} \left\| \tilde{c}_{n,t} - \frac{1}{n} \sum_{t=1}^{n} \tilde{c}_{n,t} \right\|^2 = O(n^{-1})$ for $\tilde{c}_{n,t} := \mathbb{E}_{\theta_n} Z_{t-1}$. Let $\tilde{c}_{n,\infty} := \sum_{j=0}^{\infty} \mathsf{B}_{\theta_n}^j \mathsf{C}_{\theta_n}$. This converges uniformly in n since under Assumption 2.1 parts (i) & (iii), the sets $\{\|\mathsf{B}_{\theta_n}\|_2 : n \in \mathbb{N}\} \cup \{\|\mathsf{B}_\theta\|_2\}$ and $\{\|\mathsf{C}_{\theta_n}\|_2 : n \in \mathbb{N}\} \cup \{\|\mathsf{C}_\theta\|_2\}$ are bounded above by $\rho_\star < 1$ and $C_\star < \infty$ respectively. By Jensen's inequality

$$\frac{1}{n} \sum_{t=1}^{n} \left\| \tilde{c}_{n,t} - \frac{1}{n} \sum_{t=1}^{n} \tilde{c}_{n,t} \right\|^2 \lesssim \frac{1}{n} \sum_{t=1}^{n} \|\tilde{c}_{n,t} - \tilde{c}_{n,\infty}\|^2 + \frac{1}{n} \sum_{t=1}^{n} \left\| \frac{1}{n} \sum_{t=1}^{n} [\tilde{c}_{n,\infty} - \tilde{c}_{n,t}] \right\|^2$$

$$\leq \frac{2}{n} \sum_{t=1}^{n} \|\tilde{c}_{n,t} - \tilde{c}_{n,\infty}\|^2$$

so it suffices to show that $n/2$ times the last term is uniformly bounded above. One has:

$$\sum_{t=1}^{n} \|\tilde{c}_{n,t} - \tilde{c}_{n,\infty}\|^2 = \sum_{t=1}^{n} \left\| \sum_{j=t-1}^{\infty} \mathsf{B}_{\theta_n}^j \mathsf{C}_{\theta_n} - \mathsf{B}_{\theta_n}^{t-1} Z_0 \right\|^2$$

$$\lesssim \sum_{t=1}^{n} \left\| \sum_{j=t-1}^{\infty} \mathsf{B}_{\theta_n}^j \mathsf{C}_{\theta_n} \right\|^2 + \sum_{t=1}^{n} \|\mathsf{B}_{\theta_n}^{t-1} Z_0\|^2$$

$$\leq \sum_{t=1}^{n} \left[\sum_{j=t-1}^{\infty} \|\mathsf{B}_{\theta_n}\|_2^j \|\mathsf{C}_{\theta_n}\|_2 \right]^2 + \sum_{t=1}^{n} \|\mathsf{B}_{\theta_n}\|_2^{2(t-1)} \|Z_0\|^2$$

$$\leq C_\star^2 \sum_{t=1}^{n} \left[\frac{\rho_\star^{t-1}}{1 - \rho_\star} \right]^2 + \|Z_0\|^2 \sum_{t=1}^{n} \rho_\star^{2(t-1)}$$

$$\leq \left[\frac{C_\star^2}{(1 - \rho_\star)^2} + \|Z_0\|^2 \right] \frac{1}{1 - \rho_\star^2}. \qquad\square$$

LEMMA S2.18: *In the setting of Lemma S2.15, let $\tilde{X}_t = (1, \tilde{Y}'_{t-1}, \ldots, \tilde{Y}'_{t-p})'$ where $\tilde{Y}_t$ is a stationary solution to (1). Then,*

(i) $\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}_{\theta_n} X_t - \mathbb{E}_\theta \tilde{X}_t = o(\nu_n^{1/2})$,

(ii) $\frac{1}{n} \sum_{t=1}^{n} [\mathbb{E}_{\theta_n} X_t][\mathbb{E}_{\theta_n} X_t]' - [\mathbb{E}_\theta \tilde{X}_t][\mathbb{E}_\theta \tilde{X}_t]' = o(\nu_n^{1/2})$.

(iii) $\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}_{\theta_n} [X_t - \mathbb{E}_{\theta_n} X_t][X_t - \mathbb{E}_{\theta_n} X_t]' - \mathbb{E}_\theta [X_t - \mathbb{E}_\theta X_t][X_t - \mathbb{E}_\theta X_t]' = o(\nu_n^{1/2})$.

Proof. Note that $\| \mathbb{E}_{\theta_n} X_t - \mathbb{E}_{\theta_n} \tilde{X}_t \|^2 \leq \|\tilde{c}_{n,t} - \tilde{c}_{n,\infty}\|^2$ in the notation of (the proof of) Lemma

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S2.17, which shows that $\frac{1}{n}\sum_{t=1}^{n}\|\tilde{c}_{n,t} - \tilde{c}_{n,\infty}\|^2 = O(n^{-1})$. Hence by Jensen's inequality,

$$\frac{1}{n}\sum_{t=1}^{n}\|\mathbb{E}_{\theta_n} X_t - \mathbb{E}_{\theta_n} \tilde{X}_t\| = O(n^{-1/2}) = o(\nu_n^{1/2}),$$

Since $\beta \mapsto \mathbb{E}_\theta \tilde{X}_t = \text{vec}(\iota_K, (\iota_p \otimes (I_K - B_1 - \ldots - B_p)^{-1}c))$ is locally Lipschitz,

$$\frac{1}{n}\sum_{t=1}^{n}\|\mathbb{E}_{\theta_n} \tilde{X}_t - \mathbb{E}_\theta \tilde{X}_t\| = \|\mathbb{E}_{\theta_n} \tilde{X}_t - \mathbb{E}_\theta \tilde{X}_t\| = O(n^{-1/2}) = o(\nu_n^{1/2}).$$

Combination of the above two displays yields that $\frac{1}{n}\sum_{t=1}^{n}\|\mathbb{E}_{\theta_n} X_t - \mathbb{E}_\theta \tilde{X}_t\| = O(n^{-1/2}) = o(\nu_n^{1/2})$ which implies (i). Moreover, combined with the uniform moment bounds given in (S6) and Lemma S2.1 this yields

$$\frac{1}{n}\sum_{t=1}^{n}\|[\mathbb{E}_{\theta_n} X_t][\mathbb{E}_{\theta_n} X_t]' - [\mathbb{E}_\theta \tilde{X}_t][\mathbb{E}_\theta \tilde{X}_t]'\| \lesssim \frac{1}{n}\sum_{t=1}^{n}\|\mathbb{E}_{\theta_n} X_t - \mathbb{E}_\theta \tilde{X}_t\| = O(n^{-1/2}) = o(\nu_n^{1/2}),$$

which implies (ii).

For (iii) let $U_{\vartheta,t} := X_t - \mathbb{E}_\vartheta X_t$ and $\tilde{U}_{\vartheta,t} := \tilde{X}_t - \mathbb{E}_\vartheta \tilde{X}_t$. Note that as $U_{\vartheta,t} = \sum_{j=0}^{t-2} \mathsf{B}_\vartheta^j \mathsf{D}_\vartheta \epsilon_{t-j}$ and $\tilde{U}_{\vartheta,t} = \sum_{j=0}^{\infty} \mathsf{B}_\vartheta^j \mathsf{D}_\vartheta \epsilon_{t-j}$, $U_{\theta_n,t} - \tilde{U}_{\theta_n,t}$ and $U_{\theta_n,t}$ are independent. Additionally by Assumption 2.1 parts (i) and (iii) the sets the sets $\{\|\mathsf{B}_{\theta_n}\|_2 : n \in \mathbb{N}\}$ and $\{\|\mathsf{D}_{\theta_n}\|_2 : n \in \mathbb{N}\}$ are bounded above by $\rho_\star < 1$ and $D_\star < \infty$ respectively. Hence

$$\frac{1}{n}\sum_{t=1}^{n}\left\|\mathbb{E}_{\theta_n}\left[U_{\theta_n,t}U'_{\theta_n,t} - \tilde{U}_{\theta_n,t}\tilde{U}'_{\theta_n,t}\right]\right\|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n}\left\|\mathbb{E}_{\theta_n}\left[\left(U_{\theta_n,t} - \tilde{U}_{\theta_n,t}\right)U'_{\theta_n,t}\right]\right\| + \frac{1}{n}\sum_{t=1}^{n}\left\|\mathbb{E}_{\theta_n}\left[\left(U_{\theta_n,t} - \tilde{U}_{\theta_n,t}\right)\tilde{U}'_{\theta_n,t}\right]\right\|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n}\left\|\mathbb{E}_{\theta_n}\sum_{k=0}^{\infty}\sum_{j=t-1}^{\infty}\mathsf{B}_{\theta_n}^j\mathsf{D}_{\theta_n}\epsilon_{t-j}\epsilon'_{t-k}\mathsf{D}'_{\theta_n}(\mathsf{B}_{\theta_n}^j)'\right\|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n}\sum_{j=t-1}^{\infty}\|\mathsf{B}_{\theta_n}\|_2^{2j}\|\mathsf{D}_{\theta_n}\|_2^2$$

$$\leq D_\star^2 \times \frac{1}{n}\sum_{t=1}^{n}\sum_{j=t-1}^{\infty}\rho_\star^{2j}$$

$$\leq \frac{D_\star^2}{1-\rho_\star^2} \times \frac{1-\rho_\star^{2n}}{1-\rho_\star^2} \times \frac{1}{n}$$

$$= O(n^{-1}).$$

Additionally, we can write $\text{vec}(\mathbb{E}_\vartheta \tilde{U}_{\vartheta,t}\tilde{U}'_{\vartheta,t}) = (I - \mathsf{B}_\vartheta \otimes \mathsf{B}_\vartheta)^{-1}\text{vec}(\mathsf{D}_\vartheta\mathsf{D}'_\vartheta)$, which is locally

Lipschitz in β at θ. This implies that

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}_{\theta_n} \tilde{U}_{\theta_n,t}\tilde{U}'_{\theta_n,t} - \mathbb{E}_\theta \tilde{U}_{\theta,t}\tilde{U}'_{\theta,t} = O(n^{-1/2}) = o(\nu_n^{1/2}).$$

The previous two displays suffice for (iii). $\qquad\square$

LEMMA S2.19: *In the setting of Lemma S2.15, for each $r \in \{\alpha, \sigma, b\}$ and l*

$$\frac{1}{n}\sum_{t=1}^{n}\left(\hat{\ell}_{\tilde{\theta}_n,r_l}(Y_t, X_t) - \tilde{\ell}_{\tilde{\theta}_n,r_l}(Y_t, X_t)\right)^2 = o_{P^n_{\tilde{\theta}_n}}(\nu_n).$$

Proof. We start by considering elements in $\frac{1}{n}\sum_{t=1}^{n}\left(\hat{\ell}_{\tilde{\theta}_n,\alpha_l}(Y_t, X_t) - \tilde{\ell}_{\tilde{\theta}_n,\alpha_l}(Y_t, X_t)\right)^2$. Define $\tilde{\tau}_{k,n,q} := \hat{\tau}_{k,n,q} - \tau_{k,q}$ and $V_{n,t} = Y_t - B_n X_t$. Since each $|\zeta^\alpha_{n,l,k,j}| < \infty$ and the sums over k, j are finite, it is sufficient to demonstrate that for every $k, j, m, s \in [K]$, with $k \neq j$ and $s \neq m$,

$$\frac{1}{n}\sum_{t=1}^{n}\left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t})\right]\left[\hat{\phi}_{s,n}(A_{n,s\bullet}V_{n,t}) - \phi_s(A_{n,s\bullet}V_{n,t})\right]A_{n,j\bullet}V_{t,n}A_{n,m\bullet}V_{n,t} \tag{S23}$$

$$\frac{1}{n}\sum_{t=1}^{n}\left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t})\right]A_{n,j\bullet}V_{n,t}\left[\tilde{\tau}_{s,n,1}A_{n,s\bullet}V_{n,t} + \tilde{\tau}_{s,n,2}\kappa(A_{n,s\bullet}V_{n,t})\right] \tag{S24}$$

$$\frac{1}{n}\sum_{t=1}^{n}\left[\tilde{\tau}_{s,n,1}A_{n,s\bullet}V_{n,t} + \tilde{\tau}_{s,n,2}\kappa(A_{n,s\bullet}V_{n,t})\right]\left[\tilde{\tau}_{k,n,1}A_{n,k\bullet}V_{n,t} + \tilde{\tau}_{k,n,2}\kappa(A_{n,k\bullet}V_{n,t})\right] \tag{S25}$$

are each $o_{P^n_{\tilde{\theta}_n}}(\nu_n)$.

For (S25), let $\xi_1(x) = x$ and $\xi_2(x) = \kappa(x)$. Then, we can split the sum into 4 parts, each of which has the following form for some $q, w \in \{1, 2\}$

$$\frac{1}{n}\sum_{t=1}^{n}\tilde{\tau}_{s,n,q}\tilde{\tau}_{k,n,w}\xi_q(A_{n,s\bullet}V_{n,t})\xi_w(A_{n,k\bullet}V_{n,t}) = \tilde{\tau}_{s,n,q}\tilde{\tau}_{k,n,w}\frac{1}{n}\sum_{t=1}^{n}\xi_q(A_{n,s\bullet}V_{n,t})\xi_w(A_{n,k\bullet}V_{n,t}) = o_{P^n_{\tilde{\theta}_n}}(\nu_n),$$

since we have that each $\tilde{\tau}_{s,n,q}\tilde{\tau}_{k,n,w} = o_{P^n_{\tilde{\theta}_n}}(\nu_n)$ by lemma S2.16.[S12] For (S24) we can argue similarly. Again let $\xi_1(x) = x$ and $\xi_2(x) = \kappa(x)$. Then, we can split the sum into 2 parts, each

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[S12]The fact that $\frac{1}{n}\sum_{t=1}^{n}\xi_q(A_{n,s\bullet}V_{n,t})\xi_w(A_{n,k\bullet}V_{n,t}) = O_{P^n_{\tilde{\theta}_n}}(1)$ can be seem to hold using the moment and i.i.d. assumptions from assumption 2.1 and Markov's inequality, noting once more that $A_{n,k\bullet}V_{n,t} \simeq \epsilon_{k,t}$ under $P^n_{\tilde{\theta}_n}$.

of which has the following form for some $q \in \{1, 2\}$

$$\frac{1}{n} \sum_{t=1}^n \left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t}) \right] A_{n,j\bullet}V_{n,t}\tilde{\tau}_{s,n,q}\xi_q(A_{n,s\bullet}V_{n,t})$$

$$\leq \tilde{\tau}_{s,n,q} \left(\frac{1}{n} \sum_{t=1}^n \left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t}) \right]^2 (A_{n,j\bullet}V_{n,t})^2 \right)^{1/2} \left(\frac{1}{n} \sum_{t=1}^n \xi_q(A_{n,s\bullet}V_{n,t})^2 \right)^{1/2}$$

$$= o_{P^n_{\hat{\theta}_n}}(\nu_n).$$

by Lemma A.1 applied with $W_{n,t} = A_{n,j\bullet}V_{n,t}$ and $\tilde{\tau}_{s,n,q} = o_{P^n_{\hat{\theta}_n}}(\nu_n^{1/2})$.[S13] For (S23) use Cauchy-Schwarz with Lemma A.1

$$\frac{1}{n} \sum_{t=1}^n \left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t}) \right] \left[\hat{\phi}_{s,n}(A_{n,s\bullet}V_{n,t}) - \phi_s(A_{n,s\bullet}V_{n,t}) \right] A_{n,j\bullet}V_{n,t} A_{n,m\bullet}V_{n,t}$$

$$\leq \left(\frac{1}{n} \sum_{t=1}^n \left[\hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) - \phi_k(A_{n,k\bullet}V_{n,t}) \right]^2 (A_{n,j\bullet}V_{n,t})^2 \right)^{1/2}$$

$$\times \left(\frac{1}{n} \sum_{t=1}^n \left[\hat{\phi}_{s,n}(A_{n,s\bullet}V_{n,t}) - \phi_s(A_{n,s\bullet}V_{n,t}) \right]^2 (A_{n,m\bullet}V_{n,t})^2 \right)^{1/2}$$

$$= o_{P^n_{\hat{\theta}_n}}(\nu_n).$$

This completes the proof for the components corresponding to α_l. We note that the components corresponding to σ_l follow analogously.

Finally, we consider the elements in $\frac{1}{n} \sum_{t=1}^n \left(\hat{\ell}_{\theta_n,b_l}(Y_t, X_t) - \tilde{\ell}_{\theta_n,b_l}(Y_t, X_t) \right)^2$. Let $a_{n,k,l} := -A_{n,k\bullet}D_{b_l}$, $\tilde{\varsigma}_{k,n} := \hat{\varsigma}_{k,n} - \varsigma_k$, $c_{n,t} := \mathbb{E}_{\theta_n} X_t$ and $\bar{c}_n := \frac{1}{n} \sum_{t=1}^n c_{n,t}$. Since $a_{n,k,l} \to a_{\infty,k,l} := A(\alpha,\sigma)_{k\bullet}D_{b_l}(\alpha,\sigma)$, it suffices to show that

(i) $\frac{1}{n} \sum_{t=1}^n \left[\phi_k(A_{n,k\bullet}V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) \right]^2 \|X_t - c_{n,t}\|^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(ii) $\frac{1}{n} \sum_{t=1}^n \left[\phi_k(A_{n,k\bullet}V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) \right]^2 \|\bar{X}_n - \bar{c}_n\|^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(iii) $\frac{1}{n} \sum_{t=1}^n \left[\phi_k(A_{n,k\bullet}V_{n,t}) - \hat{\phi}_{k,n}(A_{n,k\bullet}V_{n,t}) \right]^2 \|\bar{c}_n - c_{n,t}\|^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(iv) $\frac{1}{n} \sum_{t=1}^n \phi_k(A_{n,k\bullet}V_{n,t})^2 \|\bar{X}_n - \bar{c}_n\|^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(v) $\frac{1}{n} \sum_{t=1}^n \phi_k(A_{n,k\bullet}V_{n,t})^2 \|\bar{c}_n - c_{n,t}\|^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(vi) $\frac{1}{n} \sum_{t=1}^n \|\bar{X}_n\|^2 \left[\tilde{\varsigma}_{k,n,1} A_{n,k\bullet}V_{n,t} + \tilde{\varsigma}_{k,n,2}\kappa(A_{n,k\bullet}V_{n,t}) \right]^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(vii) $\frac{1}{n} \sum_{t=1}^n \|\bar{X}_n - \bar{c}_n\|^2 \left[\varsigma_{k,1} A_{n,k\bullet}V_{n,t} + \varsigma_{k,2}\kappa(A_{n,k\bullet}V_{n,t}) \right]^2 = o_{P^n_{\theta_n}}(\nu_n)$;

(viii) $\frac{1}{n} \sum_{t=1}^n \|\bar{c}_n - c_{n,t}\|^2 \left[\varsigma_{k,1} A_{n,k\bullet}V_{n,t} + \varsigma_{k,2}\kappa(A_{n,k\bullet}V_{n,t}) \right]^2 = o_{P^n_{\theta_n}}(\nu_n)$.

[S13]See footnote S12.

(i) follows from repeated application of Lemma A.1 with $W_{n,t} = e_j'(X_t - c_{n,t})$.

(ii) follows from application of Lemma A.1 with $W_{n,t} = 1$ and $\bar{X}_n - \bar{c}_n = \frac{1}{n}\sum_{t=1}^{n}[X_t - c_{n,t}] \xrightarrow{P_{\theta_n}} 0$, which follows by (S6), Lemma S2.14 and Corollary 3.1.

(iii) follows by Lemma A.1 applied repeatedly with $W_{n,t} = e_j'(\bar{c}_n - c_{n,t})$.[S14]

For (iv), $\frac{1}{n}\sum_{t=1}^{n}\phi_k(A_{n,k\bullet}V_{n,t})^2 = O_{P_{\theta_n}^n}(1)$ since $\phi_k(A_{n,k\bullet}V_{n,t})^2$ has uniformly bounded second moments and $\bar{X}_n - \bar{c}_n = O_{P_{\theta_n}^n}(n^{-1/2})$, by (S6), Lemma S2.14 and Corollary 3.1.

For (v) use Markov's inequality and Lemma S2.17 to conclude

$$P_{\theta_n}^n\left(\frac{1}{n}\sum_{t=1}^{n}\phi_k(A_{n,k\bullet}V_{n,t})^2\|\bar{c}_n - c_{n,t}\|^2 > \nu_n\varepsilon\right) \le \nu_n^{-1}\varepsilon^{-1}\,\mathbb{E}\left[\phi_k(\epsilon_k)^2\right]\frac{1}{n}\sum_{t=1}^{n}\|\bar{c}_n - c_{n,t}\|^2 \to 0.$$

For (vi), $\bar{X}_n = O_{P_{\theta_n}}(1)$ by e.g. Markov's inequality and (S6). Similarly, $\frac{1}{n}\sum_{t=1}^{n}U_{t,i}U_{t,j} = O_{P_{\theta_n}^n}(1)$ for $i,j \in \{1,2\}$ with $U_{t,1} = A_{n,k\bullet}V_{n,t}$ and $U_{t,2} = \kappa(A_{n,k\bullet}V_{n,t})$. The result then follows from Lemma S2.16.

For (vii), $\frac{1}{n}\sum_{t=1}^{n}U_{t,i}U_{t,j} = O_{P_{\theta_n}^n}(1)$ for $i,j \in \{1,2\}$ with $U_{t,1}$ and $U_{t,2}$ as in the preceding paragraph. Therefore it suffices to note that $\bar{X}_n - \bar{c}_n = O_{P_{\theta_n}}(n^{-1/2})$, as noted for (iv).

For (viii), for $U_{t,1}$ and $U_{t,2}$ as in the preceding paragraph and $i,j \in \{1,2\}$,

$$P_{\theta_n}^n\left(\left|\frac{1}{n}\sum_{t=1}^{n}\|\bar{c}_n - c_{n,t}\|^2\varsigma_{k,i}U_{t,i}\varsigma_{k,j}U_{t,j}\right| > \nu_n\varepsilon\right) \le \nu_n^{-1}\varepsilon^{-1}|\varsigma_{k,i}\varsigma_{k,j}|[\mathbb{E}\,U_{t,i}^2]^{1/2}[\mathbb{E}\,U_{t,j}^2]^{1/2}\frac{1}{n}\sum_{t=1}^{n}\|\bar{c}_n - c_{n,t}\|^2$$

$$\lesssim \nu_n^{-1}\frac{1}{n}\sum_{t=1}^{n}\|\bar{c}_n - c_{n,t}\|^2 \to 0,$$

by Markov's inequality and Lemma S2.17. $\qquad\square$

S2.5   Assumption 2.1-(ii)-(b)

We provide a sufficient condition under which Assumption 2.1 part (ii)-(b) holds, given part (ii)-(a). For convenience recall that part (ii) reads as

(ii) Conditional on the initial values $(Y'_{-p+1},\ldots,Y_0')'$, $\epsilon_t = (\epsilon_{1,t},\ldots,\epsilon_{K,t})'$ is independently and identically distributed across t, with independent components $\epsilon_{k,t}$. Each $\eta = (\eta_1,\ldots,\eta_K) \in \mathcal{H}$ is such that each η_k is nowhere vanishing, dominated by Lebesgue measure on $\mathbb{R}$, continuously differentiable with log density scores denoted by $\phi_k(z) := \partial\log\eta_k(z)/\partial z$, and for all $k = 1,\ldots,K$

    (a) $\mathbb{E}\epsilon_{k,t} = 0$, $\mathbb{E}\epsilon_{k,t}^2 = 1$, $\mathbb{E}\epsilon_{k,t}^{4+\delta} < \infty$, $\mathbb{E}(\epsilon_{k,t}^4) - 1 > \mathbb{E}(\epsilon_{k,t}^3)^2$, and $\mathbb{E}\phi_k^{4+\delta}(\epsilon_{k,t}) < \infty$ (for some $\delta > 0$);

[S14]That this is uniformly bounded follows from (S6).

(b) $\mathbb{E}\,\phi_k(\epsilon_{k,t}) = 0$, $\mathbb{E}\,\phi_k^2(\epsilon_{k,t}) > 0$, $\mathbb{E}\,\phi_k(\epsilon_{k,t})\epsilon_{k,t} = -1$, $\mathbb{E}\,\phi_k(\epsilon_{k,t})\epsilon_{k,t}^2 = 0$ and $\mathbb{E}\,\phi_k(\epsilon_{k,t})\epsilon_{k,t}^3 = -3$;

In this assumption part (a) is standard — only imposes that the shocks are mean zero with unit variance, and that certain $4 + \delta$ moments are finite —. In contrast, part (b) may seem strong at first sight.

An important observation is that (b) should not be understood independently from (a). Indeed, the following lemma shows that given (a), condition (b) follows if the structural shocks have densities that decays to zero at a polynomial rate.

LEMMA S2.20: *Let $a_k = \inf\{x \in \mathbb{R} \cup \{-\infty\} : \eta_k(x) > 0\}$ and $b_k = \sup\{x \in \mathbb{R} \cup \{\infty\} : \eta_k(x) > 0\}$. Suppose that, for $r = 0, 1, 2, 3$: (i) if $a_k = -\infty$ then $\eta_k(x) = o(x^{-3})$ as $x \to -\infty$, else $a_k^r \lim_{x\downarrow a_k} \eta_k(x) = 0$, and (ii) if $b_k = \infty$ then $\eta_k(x) = o(x^{-3})$ as $x \to \infty$, else $b_k^r \lim_{x\uparrow b_k} \eta_k(x) = 0$. Then, if part (a) of assumption 2.1-(ii) holds, part (b) is also satisfied.*

Proof. Let $r \in \{0, 1, 2, 3\}$, $b_k = \sup\{x \in \mathbb{R} : \eta_k(x) > 0\}$ and $a_k = \inf\{x \in \mathbb{R} : \eta_k(x) > 0\}$. We have, by integration by parts, with G_k denoting the measure on $\mathbb{R}$ corresponding to η_k,

$$\int \phi_k(z)z^r \, \mathrm{d}G_k = \int \frac{\eta_k'(z)}{\eta_k(z)}\eta_k(z)z^r \, \mathrm{d}z = \int \eta_k'(z)z^r \, \mathrm{d}z = \eta_k(z)z^r\Big|_{a_k}^{b_k} - \int \eta_k(z)\frac{\mathrm{d}z^r}{\mathrm{d}z} \, \mathrm{d}z.$$

Our hypothesis ensures that $z^r\eta_k(z)\big|_{a_k}^{b_k} = 0$. Therefore we have $G_k\phi_k(z)z^r = -G_k\frac{\mathrm{d}}{\mathrm{d}z}z^r$. For $r = 0$ this equals zero as $\frac{\mathrm{d}}{\mathrm{d}z}z^0 = \frac{\mathrm{d}}{\mathrm{d}z}1 = 0$. For $r \in \{1, 2, 3\}$ we have $\frac{\mathrm{d}z^r}{\mathrm{d}z} = rz^{r-1}$ and hence $G_k\phi_k(z)z^r = -rG_kz^{r-1}$. Since $G_k1 = 1$, $G_kz = 0$, and $G_kz^2 = 1$, the result follows. $\qquad\square$

We now provide two examples. The first is a mixture of normals. We directly verify the moment conditions in (a) and (b) are satisfied.

The second example is a normalised χ_2^2 distribution. We show that this does satisfy the moment conditions in (a) but not those in (b) (nor the conditions of Lemma S2.20).[S15]

EXAMPLE S2.1 (Normal mixtures): *Suppose that ϵ_k has the density function*

$$\eta_k(z) = \sum_{m=1}^{M} p_m f_m(z, \mu_m, \sigma_m^2), \quad p_m \geq 0, \ \sum_{m=1}^{M} p_m = 1, \ \sum_{m=1}^{M} p_m \mu_m = 0, \ \sum_{m=1}^{M} p_m(\sigma_m^2 + \mu_m^2) = 1,$$

where $f_m(z, \mu_m, \sigma_m^2)$ is the density function of a $e_m \sim \mathcal{N}(\mu_m, \sigma_m^2)$.

ϵ_k *has mean zero and unit variance. We first establish that each of the conditions in (a) are*

[S15]Additionally, the (normalised) χ_2^2 distribution does not have a nowhere vanishing Lebesgue density.

satisfied. In particular we first note that $\mathbb{E}\left[|\epsilon_k|^r\right]$ is finite for any positive integer r as

$$\mathbb{E}\left[|\epsilon_k|^r\right] = \sum_{m=1}^{M} p_m \, \mathbb{E}\left[|e_m|^r\right] < \infty, \tag{S26}$$

since the Normal distribution has finite moments of all orders. To establish that $\mathbb{E}[\epsilon_k^3]^2 < \mathbb{E}[\epsilon_k^4] - 1$ note that this is equivalent to the linear independence in L_2 of $1, \epsilon_k, \epsilon_k^2$ (e.g. Horn and Johnson, 2013, Theorem 7.2.10). This is equivalent to the condition that

$$a_1^2 + 2a_1 a_3 + a_2^2 + a_3^2 \, \mathbb{E}[\epsilon_k^4] = 0 \implies a_1 = a_2 = a_3 = 0.$$

This holds since $\mathbb{E}[\epsilon_k^4] \geq 1 = \mathbb{E}[\epsilon_k^2]$ by the fact that L_p norms are increasing and so

$$a_1^2 + 2a_1 a_3 + a_2^2 + a_3^2 \, \mathbb{E}[\epsilon_k^4] \geq a_1^2 + 2a_1 a_3 + a_3^2 = (a_1 + a_3)^2 \geq 0,$$

where equality is possible only if $a_1 = a_2 = a_3 = 0$. Next, note that

$$\phi_k(z) = -\frac{\sum_{m=1}^{M} p_m \sigma_m^{-2}(z - \mu_m) f_m(z, \mu_m, \sigma_m^2)}{\eta_k(z)}, \tag{S27}$$

and for any integer r and some $\mu \in \mathbb{R}$

$$|\phi_k(z)|^r \lesssim |\phi_k(z)|^{r-1} \left| \eta_k(z)^{-1}(|z| + |\mu|) \sum_{m=1}^{M} p_m f_m(z, \mu_m, \sigma_m^2) \right| = |\phi_k(z)|^{r-1}(|z| + |\mu|).$$

Recursively using this inequality from $r = 0$, yields (for some constant $C_r \in (0, \infty)$)

$$|\phi_k(z)|^r \leq C_r(|z|^r + |\mu|^r).$$

That $\mathbb{E}\,|\phi(\epsilon_k)|^r < \infty$ for any integer r then follows from (S26).

For the conditions in (b), note that by (S27),

$$\begin{aligned}
\mathbb{E}\left[\phi_k(\epsilon_k)\epsilon_k^r\right] &= -\sum_{m=1}^{M} p_m \int z^r \frac{\sigma_m^{-2}(z - \mu_m) f_m(\epsilon_k, \mu_m, \sigma_m^2)}{\eta_k(z)} \eta_k(z)\,\mathrm{d}z \\
&= -\sum_{m=1}^{M} p_m \sigma_m^{-2} \int z^r (z - \mu_m) f_m(\epsilon_k, \mu_m, \sigma_m^2)\,\mathrm{d}z \\
&= -\sum_{m=1}^{M} p_m \sigma_m^{-2} \left(\mathbb{E}\left[e_m^{r+1}\right] - \mathbb{E}\left[e_m^r\right] \mu_m \right).
\end{aligned}$$

Taking $r = 0, 1, 2, 3$ in the right hand expression respectively gives:

$$\mathbb{E}[\phi_k(\epsilon_k)] = -\sum_{m=1}^{M} p_m \sigma_m^{-2} (\mu_m - \mu_m) = 0 \ ,$$

$$\mathbb{E}[\phi_k(\epsilon_k)\epsilon_k] = -\sum_{m=1}^{M} p_m \sigma_m^{-2} (\sigma_m^2 + \mu_m^2 - \mu_m^2) = -1 \ ,$$

$$\mathbb{E}[\phi_k(\epsilon_k)\epsilon_k^2] = -\sum_{m=1}^{M} p_m \sigma_m^{-2} (\mu_m^3 + 3\mu_m \sigma_m^2 - (\sigma_m^2 + \mu_m^2)\mu_m) = 0 \ ,$$

$$\mathbb{E}[\phi_k(\epsilon_k)\epsilon_k^3] = -\sum_{m=1}^{M} p_m \sigma_m^{-2} (\mu_m^4 + 6\mu_m^2 \sigma_m^2 + 3\sigma_m^4 - \mu_m^4 - 3\mu_m^2 \sigma_m^2) = -3 \ .$$

EXAMPLE S2.2 (The normalised χ_2^2 distribution): *Suppose that $\tilde{\epsilon}_k \sim \chi_2^2$ and let $\epsilon_k = (\tilde{\epsilon}_k - 2)/2$.* *Then ϵ_k has mean zero, variance one and density function $\eta_k(z) = \exp(-z - 1)$ on its support* $[-1, \infty)$ *on which we also have that $\phi_k(z) = -1$. The χ_2^2 distribution has finite moments of all* *orders and has moment generating function (e.g. [Johnson et al., 1995](), p. 420)*

$$M_{\tilde{\epsilon}}(t) = (1 - 2t)^{-1}, \quad t < 1/2.$$

Hence ϵ_k has finite moments of all orders. The same is evidently true of $\phi_k(\epsilon_k) = -1$. Using *the above display, we have*

$$M_{\epsilon}(t) = e^{-t}(1 - t)^{-1}, \quad t < 1,$$

and therefore may directly calculate $\mathbb{E}[\epsilon_k^3] = 2$ and $\mathbb{E}[\epsilon_k^4] = 9$, hence $\mathbb{E}[\epsilon_k^3]^2 < \mathbb{E}[\epsilon_k^4] - 1$ holds. *The moment conditions in part (a) are therefore all satisfied.*

However, $\mathbb{E}\,\phi_k(z) = -1 \neq 0$, hence part (b) does not hold. Note also that this example does *not satisfy the requirements of Lemma S2.20: we have $a_k = -1, b_k = \infty$ and*

$$\lim_{z \downarrow a_k} \eta_k(x) = \lim_{z \downarrow -1} \exp(-z - 1) = 1 \neq 0,$$

and hence the required condition is violated for $r = 0$.

S3   Technical tools

This section records some technical tools used in the proofs for ease of reference.

LEMMA S3.1 (Discretisation): *Suppose that P_n is a sequence of probability measures and f_n :*

$\Gamma \to \mathbb{R}$, $\Gamma \subset \mathbb{R}^L$, *is a sequence of functions which satisfy*

$$f_n(\gamma_n) \xrightarrow{P_n} 0 \tag{S28}$$

for any $\gamma_n := \gamma + g_n/\sqrt{n}$, $g_n \to g \in \mathbb{R}^L$. *Suppose that the estimator sequence* $\bar{\gamma}_n$ *satisfies* $\sqrt{n}\|\bar{\gamma}_n - \gamma\| = O_{P_n}(1)$ *and* $\bar{\gamma}_n$ *takes values in* $\mathscr{S}_n := \{CZ/\sqrt{n} : Z \in \mathbb{R}^L\}$ *for some* $L \times L$ *matrix* C. *Then*

$$f_n(\bar{\gamma}_n) \xrightarrow{P_n} 0.$$

Proof. Since $\bar{\gamma}_n$ is $\sqrt{n}$-consistent there is an $M > 0$ such that $P_n\left(\sqrt{n}\|\bar{\gamma}_n - \gamma\| > M\right) < \varepsilon$. If $\sqrt{n}\|\bar{\gamma}_n - \gamma\| \leq M$ then $\bar{\gamma}$ is equal to one of the values in the finite set $\mathscr{S}_n^c = \{\gamma^* \in \mathscr{S}_n : \|\gamma^* - \gamma\| \leq n^{-1/2}M\}$. For each M this set has finite number of elements bounded independently of n, call this upper bound $\overline{B}$. For any $\upsilon > 0$

$$P_n\left(|f_n(\bar{\gamma}_n)| > \upsilon\right) \leq \varepsilon + \sum_{\gamma_n \in \mathscr{S}_n^c} P_n\left(\{|f_n(\gamma_n)| > \upsilon\} \cap \{\bar{\gamma}_n = \gamma_n\}\right)$$

$$\leq \varepsilon + \sum_{\gamma_n \in \mathscr{S}_n^c} P_n\left(|f_n(\gamma_n)| > \upsilon\right)$$

$$\leq \varepsilon + \overline{B} P_n(|f_n(\gamma_n^\star)| > \upsilon),$$

where $\gamma_n^\star \in \mathscr{S}_n^c$ maximises $\gamma \mapsto P_n\left(|f_n(\gamma)| > \upsilon\right)$. As $\gamma_n^\star \in \mathscr{S}_n^c$, $\|\gamma^\star - \gamma\| \leq n^{-1/2}M$. Hence letting $g_n := \sqrt{n}(\gamma_n^\star - \gamma)$, $\|g_n\| \leq M$. Arguing along subsequences if necessary, we may therefore assume that $g_n \to g \in \mathbb{R}^L$ and hence $f_n(\gamma_n^\star) \xrightarrow{P_n} 0$ by (S28). The proof is complete on combining this with the previously established bound on $P_n\left(|f_n(\bar{\gamma}_n)| > \upsilon\right)$. $\square$

LEMMA S3.2: *Let* $(X, \mathcal{B}(X))$ *be a measurable space, and* Q_n *a sequence of probability measures on* $(X, \mathcal{B}(X))$ *which converges to a probability measure* Q *in total variation. Let* $(Y, \mathcal{B}(Y), \lambda)$ *be a measure space and suppose that* $p_n : X \times Y \to [0, \infty)$ *is a sequence of functions and* $p : X \times Y \to [0, \infty)$ *a function such that (i)* $\int p_n(x, y) \, d\lambda(y) = 1 = \int p(x, y) \, d\lambda(y)$ *for each* $n \in \mathbb{N}$ *and each* $x \in X$ *and (ii)* $p_n \to p$ *pointwise. Then, if* G_n *and* G_n *are defined according to*

$$G_n(A) := \int_A p_n(x, y) \, d(\lambda(y) \otimes Q_n(x));$$

$$G(A) := \int_A p(x, y) \, d(\lambda(y) \otimes Q(x)),$$

it follows that $G_n \xrightarrow{TV} G$.

Proof. For any x, $p_n(x, \cdot) \to p(x, \cdot)$ pointwise and since each $p_n(\cdot, x)$, $p(\cdot, x)$ has integral one

under λ, by Proposition 2.29 in van der Vaart (1998),

$$\mathscr{Q}_n(x) \coloneqq \int |p_n(x,y) - p(x,y)| \, \mathrm{d}\lambda(y) \to 0,$$

pointwise. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $X \times Y$ with $\psi_n \in [0,1]$. Then

$$\left| \int \int \psi_n(x,y)(p_n(x,y) - p(x,y)) \, \mathrm{d}\lambda(y) \, \mathrm{d}Q_n(x) \right| \leq \int \mathscr{Q}_n(x) \, \mathrm{d}Q_n(x).$$

Since $\mathscr{Q}_n(x) \leq \int p_n(x,y) \, \mathrm{d}\lambda(y) + \int p(x,y) \, \mathrm{d}\lambda(y) = 2$, the $\mathscr{Q}_n(x)$ are uniformly Q_n – integrable and uniformly Q – integrable. By Theorem 2.8 of Serfozo (1982), $\int \mathscr{Q}_n(x) \, \mathrm{d}Q_n(x) \to 0$. $\qquad\square$

LEMMA S3.3: *Suppose that P_n and Q_n are probability measures (each pair (P_n, Q_n) is defined on a common measurable space) with corresponding densities p_n and q_n (with respect to some σ-finite measure ν_n). Let $l_n = \log q_n/p_n$ be the log-likelihood ratio.[S16] If*

$$l_n = o_{P_n}(1),$$

then $d_{TV}(P_n, Q_n) \to 0$.

Proof. By the continuous mapping theorem

$$\frac{q_n}{p_n} = \exp(l_n) \xrightarrow{P_n} 1.$$

Le Cam's first lemma (e.g. van der Vaart, 1998, Lemma 6.4) then implies that $Q_n \triangleleft P_n$. Let ϕ_n be arbitrary measurable functions valued in $[0,1]$. Since the ϕ_n are uniformly tight, Prohorov's theorem ensures that for any arbitrary subsequence $(n_j)_{j\in\mathbb{N}}$ there exists a further subsequence $(n_m)_{m\in\mathbb{N}}$ such that $\phi_{n_m} \rightsquigarrow \phi \in [0,1]$ under P_{n_m}. Therefore,

$$(\phi_{n_m}, \exp(l_{n_m})) \rightsquigarrow (\phi, 1) \quad \text{under } P_{n_m}.$$

By Le Cam's third Lemma (e.g. van der Vaart, 1998, Theorem 6.6), under Q_{m_n} the law of ϕ_{n_m} converges weakly to the law of ϕ. Since each $\phi_n \in [0,1]$

$$\lim_{m\to\infty} [Q_{n_m}\phi_{n_m} - P_{n_m}\phi_{n_m}] = 0.$$

As $(n_j)_{j\in\mathbb{N}}$ was arbitrary, the preceding display holds also along the original sequence. $\qquad\square$

[S16] l_n may be defined arbitrarily when $p_n = 0$.

PROPOSITION S3.1 (Cf. Proposition 2.29 in van der Vaart, 1998): *Suppose that on a measureable space $(S, \mathcal{S})$, $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of measures and μ a measure such that $\mu(A) \leq \liminf_{n \to \infty} \mu_n(A)$ for each $A \in \mathcal{S}$. If $(f_n)_{n \in \mathbb{N}}$ and f are (real-valued) measurable functions such that $f_n \to f$ in μ-measure and $\limsup_{n \to \infty} \int |f_n|^p \, d\mu_n \leq \int |f|^p \, d\mu < \infty$ for some $p \geq 1$, then $\int |f_n - f|^p \, d\mu_n \to 0$.*

Proof. $(a + b)^p \leq 2^p (a^p + b^p)$ for any $a, b \geq 0$ and hence, under our hypotheses,

$$0 \leq 2^p |f_n|^p + 2^p |f|^p - |f_n - f|^p \to 2^{p+1} |f|^p \quad \text{in } \mu \text{ - measure.}$$

By Lemma 2.2 of Serfozo (1982) and $\limsup_{n \to \infty} \int |f_n|^p \, d\mu_n \leq \int |f|^p \, d\mu < \infty$,

$$\int 2^{p+1} |f|^p \, d\mu \leq \liminf_{n \to \infty} \int 2^p |f_n|^p + 2^p |f|^p - |f_n - f|^p \, d\mu_n$$
$$\leq 2^{p+1} \int |f|^p \, d\mu - \limsup_{n \to \infty} \int |f_n - f|^p \, d\mu_n. \qquad \square$$

REMARK S3.1: *The condition that $\mu(A) \leq \liminf_{n \to \infty} \mu_n(A)$ for each $A \in \mathcal{S}$ in Propositions S3.1 is clearly satisfied if $\mu_n \to \mu$ setwise or in total variation.*

S4   Log density score estimation and optimal knot selection

In this section we provide more details for the estimation of the log density scores. Further, we discuss a data-driven way for selecting the number of knots following the approach of Chen and Bickel (2006). We evaluate the size and power of the test under optimal knot selection in some additional simulations that are presented below.

S4.1   B-spline based log density score estimation

For $\xi_1 < \cdots < \xi_N$ a knot sequence, the first order B-splines are defined according to $b_i^{(1)}(x) := \mathbf{1}_{[\xi_i, \xi_{i+1})}(x)$. Subsequent order B-splines can be computed according to the recurrence relation

$$b_i^{(l)}(x) = \frac{x - \xi_i}{\xi_{i+l-1} - \xi_i} b_i^{(l-1)}(x) + \frac{\xi_{i+l} - x}{\xi_{i+l} - \xi_{i+1}} b_{i+1}^{(l-1)}(x), \qquad \text{(S29)}$$

for $l > 1$ and $i = 1, \ldots, N - l$. A l-th order B-spline is $l - 2$ times differentiable in x with first derivative

$$c_i^{(l)}(x) = \frac{l - 1}{\xi_{i+l-1} - \xi_i} b_i^{(l-1)}(x) - \frac{l - 1}{\xi_{i+l} - \xi_{i+1}} b_{i+1}^{(l-1)}(x). \qquad \text{(S30)}$$

See de Boor (2001) for more details on B-splines.

Let $b_{k,n} = (b_{k,n,1}, \ldots, b_{k,n,\mathsf{B}_{k,n}})'$ be a collection of $B_{k,n}$ cubic (i.e. 4-th order) B-splines and let $c_{k,n} = (c_{k,n,1}, \ldots, c_{k,n,B_{k,n}})'$ be their derivatives: $c_{k,n,i}(x) := \frac{\mathrm{d}b_{k,n,i}(x)}{\mathrm{d}x}$ for each $i \in \{1, \ldots, B_{k,n}\}$. The knots of the splines, $\xi_{k,n} = (\xi_{k,n,i})_{i=1}^{K_{k,n}}$ are equally spaced in $[\Xi_{k,n}^L, \Xi_{k,n}^U]$ with $\delta_{k,n} := \xi_{k,n,i+1} - \xi_{k,n,i} > 0$.[S17] For each (k,n) pair the relationships between the number of knots $(K_{k,n})$, the number of spline functions $(B_{k,n})$ and $\delta_{k,n}$ are given by $B_{k,n} = K_{k,n} - 4$ and $K_{k,n} = 1 + (\Xi_{k,n}^U - \Xi_{k,n}^L)/\delta_{k,n}$.

Since the B-splines vanish at infinity for any $n \in \mathbb{N}$, integration by parts gives that

$$
\begin{aligned}
&\int (\phi_k(z) - \psi'_{k,n} b_{k,n}(z))^2 \eta_k(z)\, \mathrm{d}z \\
&= \int \phi_k(z)^2 \eta_k(z)\, \mathrm{d}z + \int (\psi'_{k,n} b_{k,n})^2 \eta_k(z)\, \mathrm{d}z + 2 \int \psi'_{k,n} c_{k,n}(z) \eta_k(z)\, \mathrm{d}z \qquad \text{(S31)} \\
&= \mathbb{E}\, \phi_k(\epsilon_k)^2 + \psi'_{k,n} \mathbb{E}[b_{k,n}(\epsilon_k) b_{k,n}(\epsilon_k)'] \psi_{k,n} + 2\psi'_{k,n} \mathbb{E}\, c_{k,n}(\epsilon_k),
\end{aligned}
$$

where we integrate over the support of $\phi_{k,n}$ (which is also the support of $b_{k,n}$ and $c_{k,n}$). This mean-squared error is minimized by:[S18]

$$
\psi_{k,n} := - \mathbb{E}[b_{k,n}(\epsilon_k) b_{k,n}(\epsilon_k)']^{-1} \mathbb{E}[c_{k,n}(\epsilon_k)]. \qquad \text{(S32)}
$$

Replace the population expectations with sample counterparts to define the estimator of $\psi_{k,n}$

$$
\hat{\psi}_{k,n} := - \left[\frac{1}{n} \sum_{t=1}^{n} b_{k,n}(A_{k\bullet} V_{\gamma,t}) b_{k,n}(A_{k\bullet} V_{\gamma,t})' \right]^{-1} \frac{1}{n} \sum_{t=1}^{n} c_{k,n}(A_{k\bullet} V_{\gamma,t}) \ .
$$

Our estimate for the log density score ϕ_k is given by

$$
\hat{\phi}_{k,n}(z) := \hat{\psi}'_{k,n} b_{k,n}(z) \ . \qquad \text{(S33)}
$$

As discussed in the main text, the knots of the splines, $\xi_{k,n} = (\xi_{k,n,i})_{i=1}^{K_{k,n}}$ are taken as equally spaced in $[\Xi_{k,n}^L, \Xi_{k,n}^U]$. In practice we take these points as the 95th and 5th percentile of the samples $\{A_{k\bullet} V_t\}_{i=1}^n$ adjusted by $\log(\log(n))$, where $A = A(\alpha, \sigma)$ and $V_t = Y_t - BX_t$ for a given parameter choice $\gamma = (\alpha, \beta)$. In our main simulations we used $B_{k,n} = 12$ splines.

S4.2   Data driven B-spline selection

The number of B-spline basis functions $B_{k,n}$ is a tuning parameter. In practice we can use cross-validation to choose $B_{k,n}$ for each k. A possible approach is as follows

---

[S17]For each $k = 1, \ldots, K$ the sequences $(\Xi_{k,n}^L)_{n \in \mathbb{N}}$, $(\Xi_{k,n}^U)_{n \in \mathbb{N}}$, $(B_{k,n})_{n \in \mathbb{N}}$ and $(\delta_{k,n})_{n \in \mathbb{N}}$ are deterministic.

[S18]This differs from the expression in Chen and Bickel (2006) by a factor of -1 as they estimate $-\phi_k$.

(i) Split the sample $A_{k\bullet}V_{\gamma,t}$ randomly into two halves, say n_1 and n_2.

(ii) For $B_{k,n} = 1, 2, \ldots$, use n_1 to estimate γ based on (S33), say $\hat{\phi}_{k,n_1}(z)$, and use n_2 to evaluate (S31) empirically, but omitting the first term $\mathbb{E}\,\phi_k(\epsilon_k)^2$, say $c_{n_2|n_1}(B_{k,n})$. Similarly calculate $c_{n_1|n_2}(B_{k,n})$.

(iii) Select the optimal $B_{k,n}$ as the largest value such that $\frac{1}{2}(c_{n_2|n_1}(B_{k,n}) + c_{n_1|n_2}(B_{k,n}))$ strictly decreases until $B_{k,n}$.

This method is taken from Jin (1992) and Chen and Bickel (2006). Jin (1992) proved its validity under an iid assumption. In the additional simulations of Section S5 we experiment with this cross-validation algorithm.

S5 Additional simulation results

S5.1 Alternative parametrizations

We show that the parametrization of $A(\alpha, \sigma)$ does not affect the size of the score test nor the alternative tests considered. Specifically, we repeat Tables 2 and 3 from the main text, respectively, for an upper triangular parameterization of A. Tables S1 and S2 below show that rejection rates are not affected by the change in parameterization.

S5.2 Data driven B-spline selection

In this section we evaluate the performance of the score test when the number of B-splines is selected using cross-validation following the approach of Jin (1992), see the discussion in Section S4. All specifications are the same as in the main text and we use the one-step efficient estimates to estimate the nuisance parameters β. The results are shown in Table S3.

We find that with cross validation the test becomes slight conservative. The empirical rejection frequencies for $n = 200, 500$ are nearly always below the nominal level. Only when $n = 1000$ the correct size is reached. A possible reason for this result is that the selection criteria from Jin (1992); Chen and Bickel (2006) is based on minimizing the mean squared error of the log density score estimate which may not be optimal for the size of the test. In future work it may be attractive to modify the selection criteria to directly target the size of the test.

S5.3 Size for larger SVARs

In the main text we presented simulation results for SVAR models of dimensions $K = 2$ and $K = 3$. Here we explore higher dimensional SVAR models. In such settings two bottlenecks

TABLE S1: EMPIRICAL REJECTION FREQUENCIES: TRIANGULAR A

K	p	n	N(0,1)	t(15)	t(10)	t(5)	SKU	KU	BM	SPB	SKB	TRI
One-Step Efficient Estimates												
2	1	200	5.0	5.8	5.8	6.0	5.2	7.2	4.2	4.6	4.6	4.6
2	1	500	7.2	5.8	6.0	5.9	5.1	6.1	4.8	5.1	6.2	5.0
2	1	1000	6.9	6.2	6.1	6.2	4.8	5.3	4.8	4.8	5.2	5.3
2	4	200	4.7	4.8	5.4	5.8	5.7	5.8	4.9	5.6	4.8	3.8
2	4	500	7.0	5.6	6.8	6.1	4.6	5.2	4.0	4.7	4.6	4.6
2	4	1000	6.1	6.5	6.4	5.8	5.0	5.1	4.3	4.6	4.5	5.3
2	12	200	6.1	5.6	5.2	6.4	5.8	4.5	4.4	4.7	4.7	4.2
2	12	500	6.6	7.0	6.2	7.1	6.2	5.3	4.7	5.1	5.6	4.6
2	12	1000	7.0	6.0	5.8	6.4	5.4	5.5	4.6	5.8	6.2	4.9
3	1	200	5.6	6.8	7.0	8.0	7.2	9.9	5.1	5.5	6.1	4.6
3	1	500	7.6	6.8	7.0	7.1	5.9	6.6	4.2	5.1	6.0	4.9
3	1	1000	7.5	7.2	6.1	6.2	5.0	6.2	4.8	5.2	4.9	5.2
3	4	200	5.4	7.4	8.2	8.9	7.1	6.8	4.1	4.6	5.7	3.8
3	4	500	8.0	6.4	7.2	8.8	6.8	7.7	6.4	6.1	5.8	4.9
3	4	1000	7.9	6.6	8.0	6.7	5.8	6.2	6.0	5.8	5.3	6.3
3	12	200	3.1	3.9	3.0	4.2	2.5	3.6	3.0	2.0	2.8	2.6
3	12	500	8.5	9.4	8.8	10.2	9.6	6.2	3.8	4.1	6.0	2.3
3	12	1000	8.8	7.8	8.2	8.7	7.4	6.6	5.4	5.5	6.2	4.7
OLS Estimates												
2	1	200	3.6	4.2	4.2	6.5	4.8	7.2	3.2	2.8	5.6	3.2
2	1	500	4.4	4.1	4.5	6.3	4.7	7.5	3.7	4.1	5.6	4.3
2	1	1000	4.5	5.0	4.8	6.0	4.8	6.2	4.2	4.1	5.0	5.1
2	4	200	3.6	5.4	6.1	6.4	5.2	4.6	3.6	2.9	5.0	3.1
2	4	500	4.8	4.5	5.5	6.6	4.8	4.6	3.4	3.2	4.3	3.5
2	4	1000	4.1	5.2	5.2	5.2	5.3	4.8	3.1	3.2	4.8	4.7
2	12	200	7.2	7.4	8.4	9.4	6.4	4.8	4.5	3.0	6.6	4.4
2	12	500	5.6	7.3	6.1	8.3	6.9	3.7	4.3	3.3	4.9	3.8
2	12	1000	5.2	5.1	5.2	6.4	8.2	4.2	3.4	2.6	5.5	3.4
3	1	200	3.6	5.2	6.0	9.5	6.0	7.4	2.8	2.4	6.0	2.5
3	1	500	3.8	5.0	5.3	9.4	5.2	6.8	3.2	3.0	5.5	3.2
3	1	1000	3.6	4.9	4.5	7.6	5.9	7.1	3.6	3.6	4.4	3.9
3	4	200	7.1	8.8	8.4	12.2	7.0	5.0	2.8	0.8	4.9	2.2
3	4	500	5.0	5.4	6.7	10.9	7.0	4.9	2.2	1.4	5.6	1.2
3	4	1000	4.7	4.8	6.0	7.8	7.6	4.7	3.6	2.8	4.2	2.8
3	12	200	15.6	14.6	17.1	21.6	11.1	10.7	8.2	6.3	14.4	6.7
3	12	500	9.7	10.4	11.5	16.0	11.0	3.5	2.8	1.5	7.5	2.8
3	12	1000	6.4	6.8	8.2	10.5	10.4	3.3	3.2	1.6	5.4	3.3

Note: The table reports empirical rejection frequencies for the semi-parametric score test of the hypothesis $H_0 : \alpha = \alpha_0$ vs. $H_1 : \alpha \neq \alpha_0$ in the K-variable SVAR(p) model with nominal size 5%. The nuisance parameter estimates $\hat{\beta}$ are either one-step efficient or OLS estimates. The columns correspond to the dimension K, the number of lags p, the sample size n and the different choices for the distributions of the structural shocks, $\epsilon_{k,t}$ for $k = 1, \ldots, K$. The distributions are reported in Table 1. Rejection rates are computed based on $M = 2,500$ Monte Carlo replications.

TABLE S2: EMPIRICAL REJECTION FREQUENCIES FOR ALTERNATIVE TESTS: TRIANGULAR A

Test	N(0,1)	t(15)	t(10)	t(5)	SKU	KU	BM	SPB	SKB	TRI
$\epsilon_{1,t} \sim \epsilon_{2,t}$										
$\hat{S}_{ols}$	4.8	4.9	5.4	7.1	4.2	7.1	3.7	3.9	6.3	4.7
$\hat{S}_{onestep}$	8.0	6.7	6.9	6.9	4.7	6.3	4.9	5.5	7.1	5.3
S^{DW}	4.2	3.8	4.0	6.6	4.7	3.7	3.2	3.8	4.1	4.0
BKR^{DW}	4.4	4.1	4.4	4.2	5.8	28.4	5.1	5.7	6.5	5.2
$LM^{PML,t}$	5.1	4.9	4.9	6.5	9.1	8.3	66.7	81.4	5.9	85.0
$LM^{GMM,LL}$	1.7	1.6	3.6	11.1	6.9	7.2	6.4	6.1	1.9	5.1
$LM^{GMM,Kew}$	1.4	2.1	3.8	17.2	9.9	8.0	6.1	5.9	1.2	5.2
$LR^{PML,t}$	25.9	11.3	6.5	4.8	4.9	1.6	100.0	100.0	11.5	100.0
$LR^{GMM,LL}$	3.5	7.4	8.9	15.9	12.4	9.8	5.9	5.8	7.5	4.6
$LR^{GMM,Kew}$	6.3	7.7	12.3	22.2	16.3	12.9	6.4	6.2	6.8	4.9
$W^{PML,t}$	4.5	7.4	9.2	10.4	11.2	8.0	66.2	69.9	8.3	69.6
$W^{GMM,LL}$	12.0	17.9	18.8	22.5	17.5	14.6	6.6	6.8	15.3	5.6
$W^{GMM,Kew}$	19.3	21.8	25.1	23.0	16.5	15.8	7.1	7.0	19.1	5.8

Note: The table reports empirical rejection frequencies for tests of the hypothesis $H_0 : \alpha = \alpha_0$ vs. $H_1 : \alpha \neq \alpha_0$ with 5% nominal size for the SVAR(1) model with $K = 2$ and $T = 500$, and $\alpha_0 = 0.5594$. $\hat{S}_{ols}$ denotes the semi-parametric score test using OLS estimates for β, $\hat{S}_{onestep}$ uses one-step efficient estimates. $LM^{PML,t}$, $W^{PML,t}$ and $LR^{PML,t}$ denote the pseudo-maximum likelihood tests based on Gouriéroux et al. (2017), assuming t-distributed shocks. $LM^{GMM,LL}$, $W^{GMM,LL}$ and $LR^{GMM,LL}$ denote the GMM-based tests based on Lanne and Luoto (2021) with one co-kurtosis condition based on $\epsilon_{1t}^3 \epsilon_{2t}$. $LM^{GMM,Kew}$, $W^{GMM,Kew}$ and $LR^{GMM,Kew}$ denote the corresponding GMM-based tests of Keweloh (2021) using both co-kurtosis conditions. Finally, S^{DW} and BKR^{DW} denote the bootstrapped GMM-based and non-parametric test of Drautzburg and Wright (2023), respectively. The columns correspond to different choices for the distributions of the structural shocks, $\epsilon_{k,t}$ for $k = 1, \ldots, K$. The distributions are reported in Table 1. The tests of Drautzburg and Wright (2023) use 500 bootstrap replications to simulate the null distribution of the test statistics. Rejection rates are computed based on $M = 1,000$ Monte Carlo replications.

TABLE S3: EMPIRICAL REJECTION FREQUENCIES: OPTIMAL KNOT SELECTION

K	p	n	N(0,1)	t(15)	t(10)	t(5)	SKU	KU	BM	SPB	SKB	TRI
2	1	200	3.2	3.4	4.2	5.2	3.6	5.2	4.1	5.5	3.3	4.4
2	1	500	3.8	4.7	4.2	5.7	5.1	6.4	4.4	6.1	4.8	4.7
2	1	1000	4.7	5.0	5.9	7.7	5.3	5.3	3.9	6.9	3.9	4.5
2	4	200	0.8	2.0	1.5	2.7	2.2	3.2	2.5	3.9	3.2	4.2
2	4	500	3.6	4.5	2.9	3.8	3.7	7.5	3.6	7.4	3.6	3.3
2	4	1000	4.4	5.1	4.9	5.6	4.0	8.0	3.2	5.6	4.6	4.3
2	12	200	1.2	2.6	2.0	1.3	1.9	1.3	2.1	2.3	2.5	2.3
2	12	500	2.4	2.8	2.9	2.5	3.5	6.7	2.6	3.7	2.0	3.6
2	12	1000	4.5	4.3	5.1	4.3	5.3	9.7	3.4	4.7	4.2	3.4
3	1	200	2.4	2.0	2.4	3.3	2.5	4.7	3.9	9.4	2.6	4.0
3	1	500	4.0	4.8	5.3	4.9	5.5	7.3	4.2	8.7	4.7	3.1
3	1	1000	5.3	5.5	5.4	6.2	5.7	7.6	4.2	7.7	4.5	4.8
3	4	200	1.1	0.4	0.8	0.7	1.1	2.7	2.0	1.2	1.8	3.1
3	4	500	3.8	4.9	3.1	4.0	3.9	8.7	3.4	4.8	3.2	3.1
3	4	1000	4.8	4.7	5.9	5.5	4.6	8.5	4.3	5.6	4.3	4.7
3	12	200	1.5	2.0	2.6	2.2	1.6	2.0	1.3	1.7	2.6	2.4
3	12	500	3.0	2.6	1.7	2.8	3.5	4.5	2.4	0.9	2.7	2.9
3	12	1000	4.9	4.5	5.7	5.8	4.2	7.5	5.3	5.6	5.3	5.7

Note: The table reports empirical rejection frequencies for the semi-parametric score test of the hypothesis $H_0 : \alpha = \alpha_0$ vs. $H_1 : \alpha \neq \alpha_0$ in the K-variable SVAR(p) model with nominal size 5%. The nuisance parameter estimates $\hat{\beta}$ are either one-step efficient or OLS estimates. For each density score the number of B-splines is determined by cross-validation. The columns correspond to the dimension K, the number of lags p, the sample size n and the different choices for the distributions of the structural shocks, $\epsilon_{k,t}$ for $k = 1, \ldots, K$. The distributions are reported in Table 1. Rejection rates are computed based on $M = 2,500$ Monte Carlo replications.

can arise. First, the computational costs for constructing the confidence sets in **Algorithm 1** increase substantially as one must evaluate the test at each point of the grid. Even in the most parsimonious specification for $K = 5$ such a grid is 10 dimensional. We note that this bottleneck is not specific to our approach but arises for most weak identification robust tests when constructing confidence sets.

Second the number of finite dimensional nuisance parameters β increases rapidly when the dimension of the SVAR model increases. For instance for $K = 5$ and $p = 12$ the number of nuisance parameters L_β is around 300. This has several consequences. First, when n is smaller than the number of nuisance parameters the test does not exist anymore as the inverse of $\hat{I}_{n,\gamma,\beta\beta}$ is not defined. Second, even when the number of nuisance parameters is proportional (but smaller) than the sample size the asymptotic theory of our paper may not provide a good approximation to the finite sample performance. The reason is that our theory is developed for L_β fixed (hence $L_\beta/n \to 0$). Extending the theory to the case where L_β may increase with n is an interesting topic for future work.

That said, it is of interest to explore the finite sample performance of the test in these settings. Table S4 reports the empirical rejection frequencies for the score test for larger SVARs with $K = 5$. All other settings for the simulation design are similar as above. We exclude $n = 200$ as the test is not defined for all specifications for this sample size. We find that the test based on one-step efficient estimates behaves well when $p = 1$, reasonably well when $p = 4$ and quite erratic when $p = 12$. We therefore recommend keeping the lag length modest when considering larger SVAR models.

TABLE S4: EMPIRICAL REJECTION FREQUENCIES FOR LARGER SVARS

K	p	n	N(0,1)	t(15)	t(10)	t(5)	SKU	KU	BM	SPB	SKB	TRI
One-Step Efficient Estimates												
5	1	500	3.4	5.5	5.4	5.1	5.1	12.0	2.9	11.0	4.7	4.1
5	1	1000	7.6	6.5	6.6	7.0	6.6	9.1	4.1	7.7	5.6	4.3
5	4	500	2.8	3.1	3.0	3.0	3.5	13.1	2.4	1.2	2.6	2.1
5	4	1000	5.0	6.2	6.7	6.9	7.1	15.4	4.5	3.4	4.0	4.6
5	12	500	0.4	0.3	1.1	0.6	0.4	0.8	0.9	0.1	1.2	0.9
5	12	1000	5.8	3.4	4.1	3.9	4.7	13.1	2.3	0.1	2.8	1.6
OLS Estimates												
5	1	500	8.2	11.5	11.3	9.8	9.1	9.5	4.9	11.0	7.0	6.6
5	1	1000	9.8	7.6	7.1	8.0	7.0	8.0	3.9	12.3	5.5	4.7
5	4	500	21.0	20.3	23.0	19.0	15.2	8.6	10.4	0.6	12.4	11.2
5	4	1000	12.0	11.0	11.1	12.9	10.7	6.6	5.6	1.6	7.2	6.3
5	12	500	68.8	64.0	65.4	64.6	52.1	23.8	43.2	0.0	53.1	29.7
5	12	1000	35.2	28.6	30.0	30.2	23.5	9.7	16.0	0.0	19.1	15.9

Note: The table reports empirical rejection frequencies for the semi-parametric score test of the hypothesis $H_0 : \alpha = \alpha_0$ vs. $H_1 : \alpha \neq \alpha_0$ in the K-variable SVAR(p) model with nominal size 5%. The nuisance parameters β are estimated using either one-step efficient estimates or OLS. For each density score the number of B-splines is determined by cross-validation. The columns correspond to different choices for the distributions of the structural shocks, $\epsilon_{k,t}$ for $k = 1, \ldots, K$. The distributions are reported in Table 1. Rejection rates are computed based on $M = 5,000$ Monte Carlo replications.

S5.4 Coverage and length of confidence sets

In this section, we consider evaluating our methodology for constructing confidence sets for smooth functions of the SVAR parameters as discussed in Section 5. We focus on evaluating the coverage and length of the confidence sets for structural impulse response functions, see Example 5.1 for the details.

We consider a similar simulation set up as above and discuss the results for the SVAR(1) model with $K = 2$, $T = 500$, and two independent shocks drawn from the same distribution, as listed in Table 1. In each case, the confidence set is calculated using **Algorithm 2** for the structural impulse response of the first variable to the second shock and we report the coverage rate and length for horizons 0-12. Further, we compare our approach to the identification robust methods of Drautzburg and Wright (2023), for which we change step (i) in **Algorithm 2** and replace the efficient score test by the tests of Drautzburg and Wright (2023).

Figure S1 shows the empirical coverage rates. Not surprising we generally find that the two-step Bonferroni approach is conservative; all empirical coverage rates are above the nominal 90% level. This holds for all horizons, densities and methods considered.

That said, we find that if the efficient score test, based on one-step efficient estimates, is used as the first step in the Bonferroni method the coverage becomes much closer to the nominal size. This holds for nearly all densities, the exception being the t densities that are very close to Gaussian, where there is generally very low power.

Figure S2 shows the length of the confidence intervals. We find that efficient score approach gives the smallest length among all procedures considered and for all densities. The differences between the methods varies; for some densities all methods give comparable intervals, but for others the efficient score approach can give intervals that are up to 30% shorter in length. This holds especially at longer horizons.

We conclude that the two-step Bonferroni method, where the first step is based on the efficient score test, gives substantial efficiency improvements when compared to existing methods.

FIGURE S1: COVERAGE RATES OF $\hat{C}_{n,g,\alpha,0.9}$
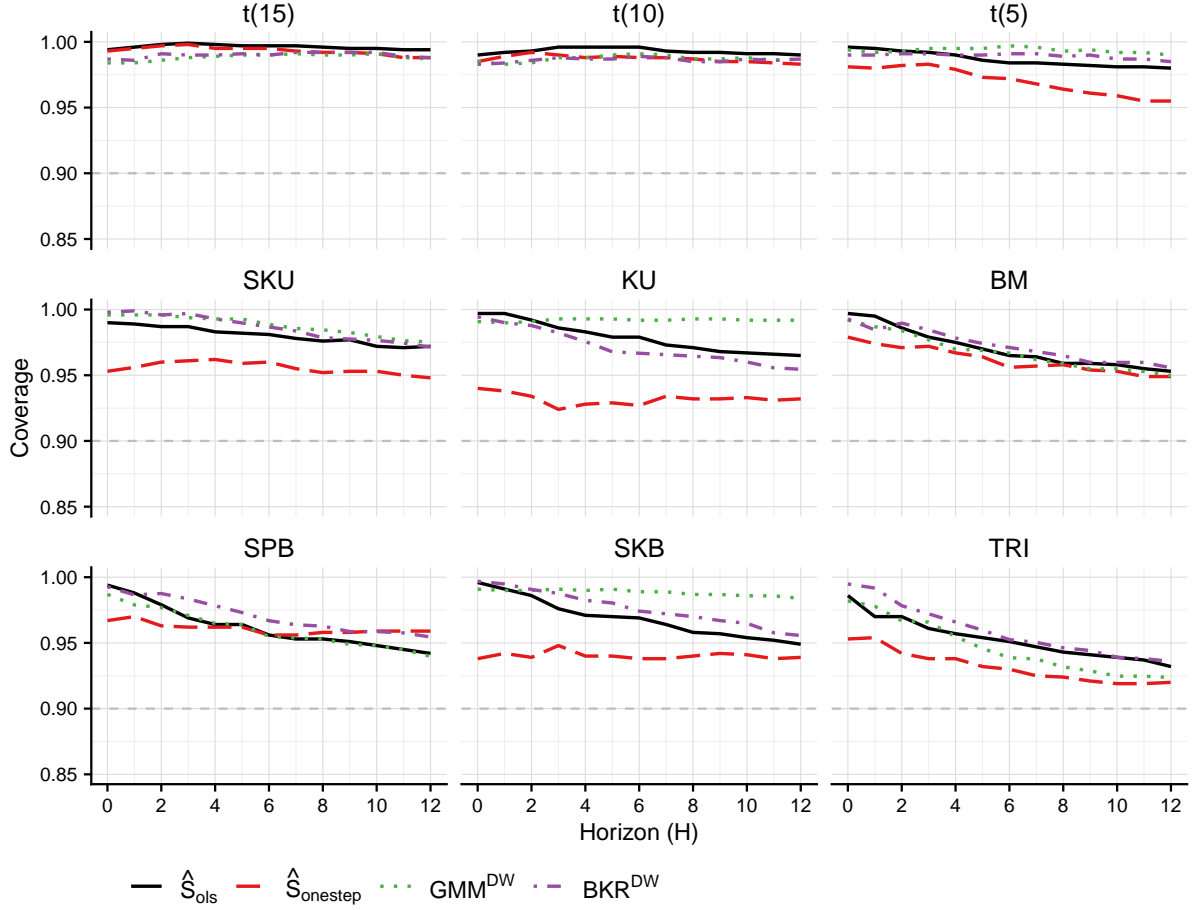
Note: The figure reports empirical coverage rates of confidence intervals at individual horizons for the impulse response of the first variable to the second shock with 90% nominal coverage for the SVAR(1) model with $K = 2$ and $T = 500$. $\hat{S}_{ols}$ denotes the semi-parametric score test using OLS estimates for β, $\hat{S}_{onestep}$ uses one-step efficient estimates. GMM^{DW} denotes the GMM-based test of Drautzburg and Wright (2023) and BKR^{DW} denotes the non-parametric test of Drautzburg and Wright (2023). The tests of Drautzburg and Wright (2023) use 500 bootstrap replications to obtain critical values. Coverage is computed using $M = 1,000$ Monte Carlo replications.

FIGURE S2: AVERAGE LENGTH OF $\hat{C}_{n,g,\alpha,0.9}$
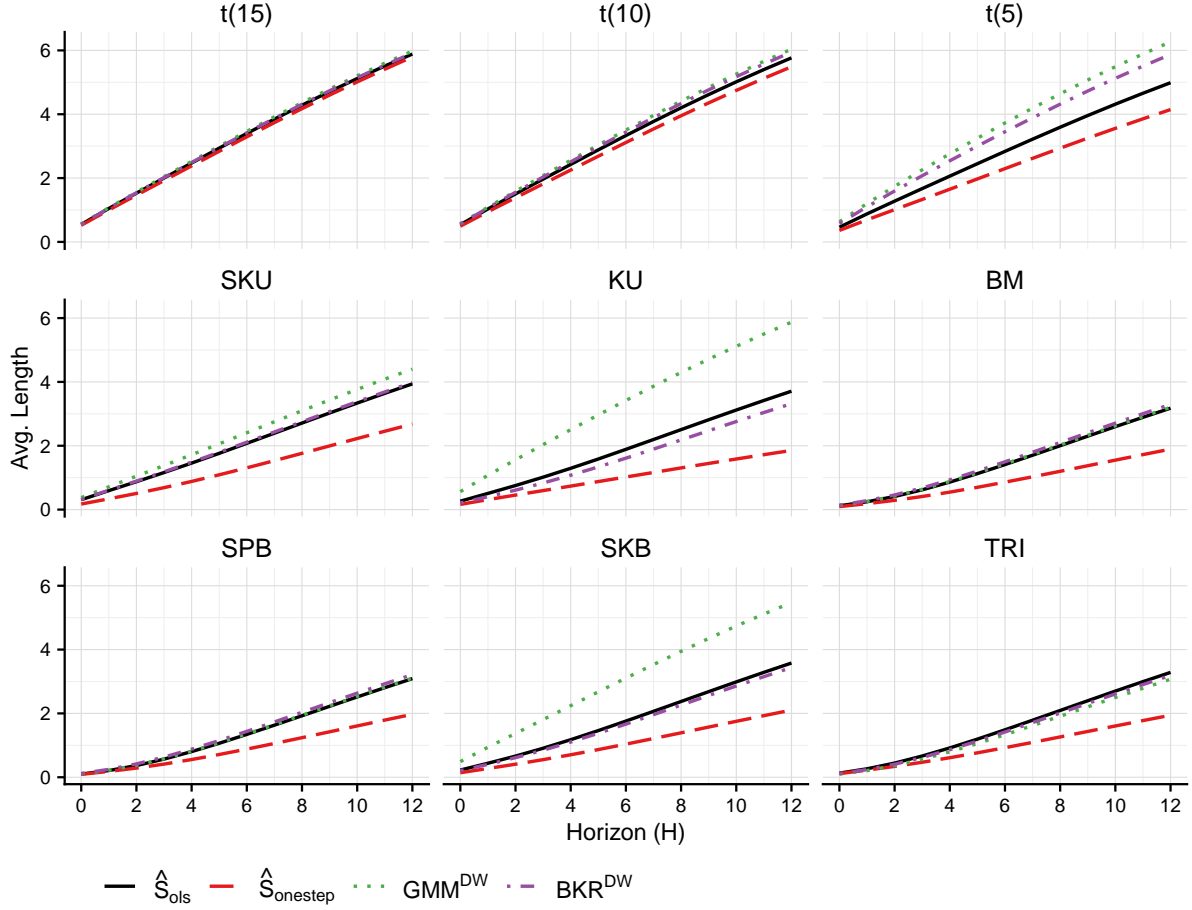
Note: The figure reports average length of confidence intervals at individual horizons for the impulse response of the first variable to the second shock with 90% nominal coverage for the SVAR(1) model with $K = 2$ and $T = 500$. $\hat{S}_{ols}$ denotes the semi-parametric score test using OLS estimates for β, $\hat{S}_{onestep}$ uses one-step efficient estimates. GMM^{DW} denotes the GMM-based test of Drautzburg and Wright (2023) and BKR^{DW} denotes the non-parametric test of Drautzburg and Wright (2023). The tests of Drautzburg and Wright (2023) use 500 bootstrap replications to obtain critical values. Average length is computed using $M = 1,000$ Monte Carlo replications.

S5.5 Point estimation results

Table S5 shows the Root Mean Squared Errors (RMSEs) for parameter estimates $\hat{\alpha}$ in the K-variable SVAR(1) model with $K = 2, T = 500$. We compare the performance of different estimators and their one-step efficient counterparts as discussed in Section 6. Specifically, we consider the psuedo maximum likelihood estimator of Gouriéroux et al. (2017) and the moment estimators of Lanne and Luoto (2019) and Keweloh (2021) as initial estimators. For each of these we compute the corresponding one-step efficient estimate from (29).

The results show that if the true density is Gaussian or close to Gaussian there is no advantage in doing a one-step efficient update. Intuitively, in these settings the efficient scores are noisy and add little additional information to the initial estimate, implying that the mean squared errors do not improve. In contrast, when the underlying density is away from the Gaussian (as imposed asymptotically by Assumption 6.1) the one-step efficient estimates always have lower RMSEs. The gains can be large, and appear to outweigh the small relative losses that are sometimes incurred for densities close to Gaussian.

TABLE S5: EFFICIENCY OF ONE-STEP UPDATED ESTIMATES $\hat{\alpha}$

η	PML^t		GMM^{LL}		GMM^{Kew}	
	$\hat{\alpha}_n$	$\hat{\alpha}_n^{onestep}$	$\hat{\alpha}_n$	$\hat{\alpha}_n^{onestep}$	$\hat{\alpha}_n$	$\hat{\alpha}_n^{onestep}$
N(0,1)	0.207	0.235	0.188	0.194	0.194	0.194
t(15)	0.137	0.146	0.156	0.147	0.154	0.148
t(10)	0.103	0.108	0.129	0.113	0.120	0.114
t(5)	0.051	0.056	0.082	0.061	0.070	0.061
SKU	0.042	0.032	0.071	0.037	0.058	0.035
KU	0.041	0.026	0.082	0.041	0.068	0.038
BM	0.250	0.070	0.030	0.016	0.016	0.015
SPB	0.250	0.090	0.027	0.012	0.013	0.012
SKB	0.138	0.067	0.163	0.074	0.160	0.074
TRI	0.250	0.113	0.025	0.012	0.012	0.012

Note: The table reports Root Mean Squared Errors (RMSEs) for parameter estimates $\hat{\alpha}$ in the K-variable SVAR(1) model with $K = 2, T = 500$. The rows correspond to different choices for the distributions of the structural shocks, $\epsilon_{k,t}$ for $k = 1, \ldots, K$. The distributions are reported in Table 1. RMSEs are computed based on $M = 2,500$ Monte Carlo replications.

S6 Additional empirical results

S6.1 Alternative parametrization of Baumeister and Hamilton (2015) model

This section presents results from an alternative parametrization of the Baumeister and Hamilton (2015) model where A^{-1} is expressed as follows:

$$A^{-1}(\xi,\sigma) = \begin{pmatrix} A_{11}^{-1} & A_{12}^{-1} \\ A_{21}^{-1} & A_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} \cos(\xi) & -\sin(\xi) \\ \sin(\xi) & \cos(\xi) \end{pmatrix}, \qquad \xi \in [0, 2\pi) \qquad \text{(S34)}$$

The sign restrictions of the original model, which are formulated on demand and supply elasticities as well as scaling parameters (see Example S1.1), imply that $A_{11}^{-1} \geq 0, A_{12}^{-1} \leq 0, A_{21}^{-1} \geq 0$ and $A_{22}^{-1} \geq 0$. The parametrization in (S34) translates the sign restrictions into constraints imposed on (ξ, σ). For the alternative parameterization, we assume that $\sigma_1, \sigma_2, \sigma_3 > 0$, which corresponds to imposing the identification restriction $\xi \in (0, \pi/2)$.[S19] Hence, for **Algorithm 1**, we set up a grid of 500 grid-points in $\xi \in (0, \pi/2)$. Note that there is a direct mapping between the two parametrizations that given (ξ, σ) lets us recover the elasticities (α^d, α^s) from the main parametrization discussed in the paper. Specifically, we can define $g(\xi, \sigma)$ as the following (smooth) vector-valued function which recovers the structural elasticities (α^d, α^s) from the rotation angle α.
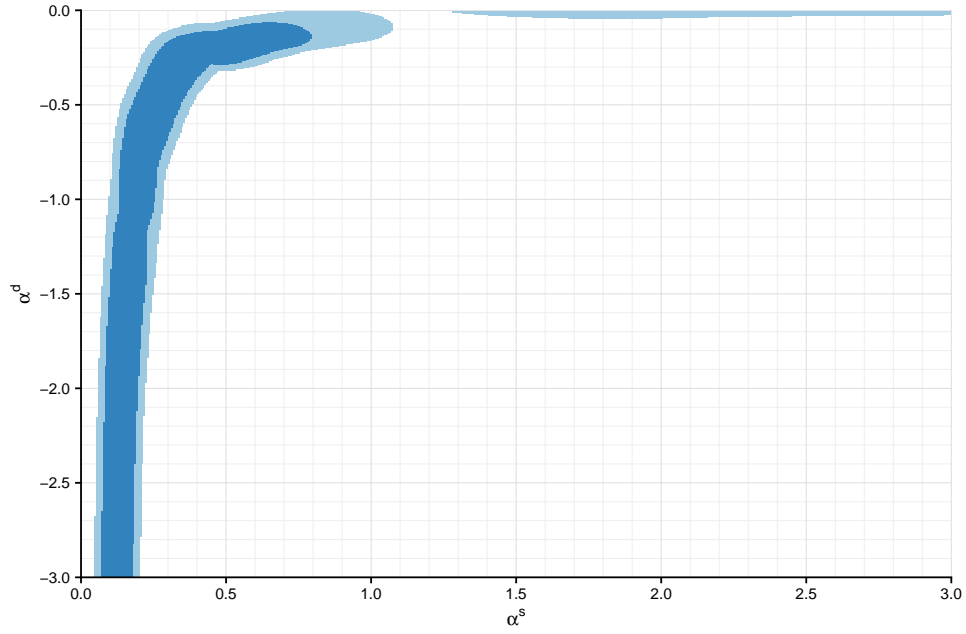
$$g(\xi, \sigma) = \begin{pmatrix} \alpha^d, & \alpha^s \end{pmatrix}' := \begin{pmatrix} \frac{\sigma_2 \cdot \sin(\xi) - \sigma_3 \cdot \cos(\xi)}{\sigma_1 \cdot \sin(\xi)}, & \frac{\sigma_2 \cdot \cos(\xi) + \sigma_3 \cdot \sin(\xi)}{\sigma_1 \cdot \cos(\xi)} \end{pmatrix}' \qquad \text{(S35)}$$

Since $g(\xi, \sigma)$ is a smooth function, we can use **Algorithm 2** to recover confidence sets for the structural elasticities (α^d, α^s). To this purpose, we define a grid with 250,000 equally-spaced grid points for $(\alpha^d, \alpha^s) \in [-3, 0) \times (0, 3]$, similar to the grid used in the main parametrisation. Similarly, we can use **Algorithm 2** to directly recover confidence bands for the impulse response functions.

Figures S3 and S4 report the confidence sets for labor demand and labor supply elasticities, as well as confidence bands for the impulse response functions, respectively, obtained using the alternative parametrization. Overall, the results are very close to the ones reported for the main parametrization. Due to the Bonferroni procedure of **Algorithm 2**, the confidence set for the elasticites is slightly wider than the one reported in the main text of the paper based on the alternative parametrization. For the IRF bands, there are also slight differences in the widths of the impulse response bands.


[S19] $\sigma_1, \sigma_3 > 0$ is trivial, since these capture standard deviations of the reduced form SVAR residuals. $\sigma_2 > 0$ can be verified from a Cholesky decomposition of the estimated reduced-form errors of the SVAR.

Note: 95% (light blue) and 67% (dark blue) confidence regions for labor demand and supply elasticities obtained using **Algorithm 2** with 250,000 equally-spaced grid points for $(\alpha^d, \alpha^s) \in [-3,0) \times (0,3]$.

FIGURE S4: IRF CONFIDENCE BANDS FOR LABOR DEMAND AND SUPPLY SHOCKS
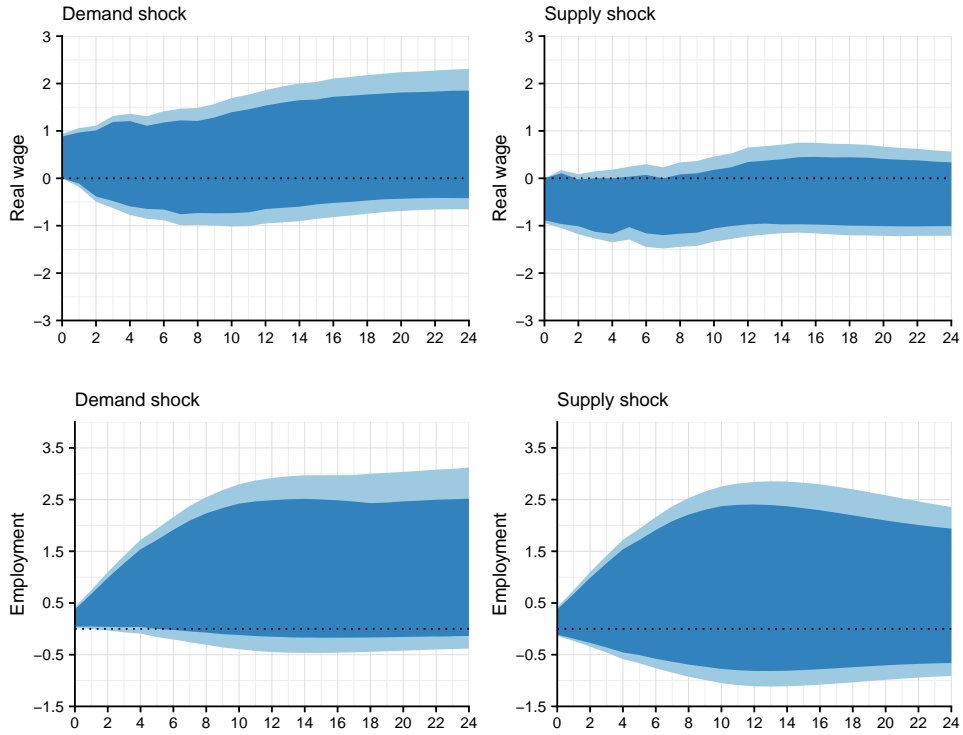
Note: 95% (light blue) and 67% (dark blue) confidence bands for impulse responses to labor supply and labor demand shocks, obtained using using **Algorithm 2** with 500 equally-spaced grid points for $\xi \in [0, \pi/2]$.

## S6.2 Distributions of recovered structural shocks

In this section, we present kernel density estimates for the structural errors recovered from the empirical studies in the paper. To obtain estimates of the structural shocks, we require an estimate of α, which we obtained using a GMM estimator employing the moment conditions of Keweloh (2021). Using the estimate, we can recover the structural shocks $\hat{\epsilon}_{k,t}(\hat{\alpha}, \hat{\beta})$ for $k = 1, \ldots, K$. We plot histograms of the structural errors in Figure S6.2 for the model of Baumeister and Hamilton (2015) and on Figure S6.2 for the model of Kilian and Murphy (2012), together with their kernel density estimates and an overlaid standard gaussian density.

FIGURE S5: DISTRIBUTIONS OF SHOCKS – BAUMEISTER AND HAMILTON (2015) MODEL
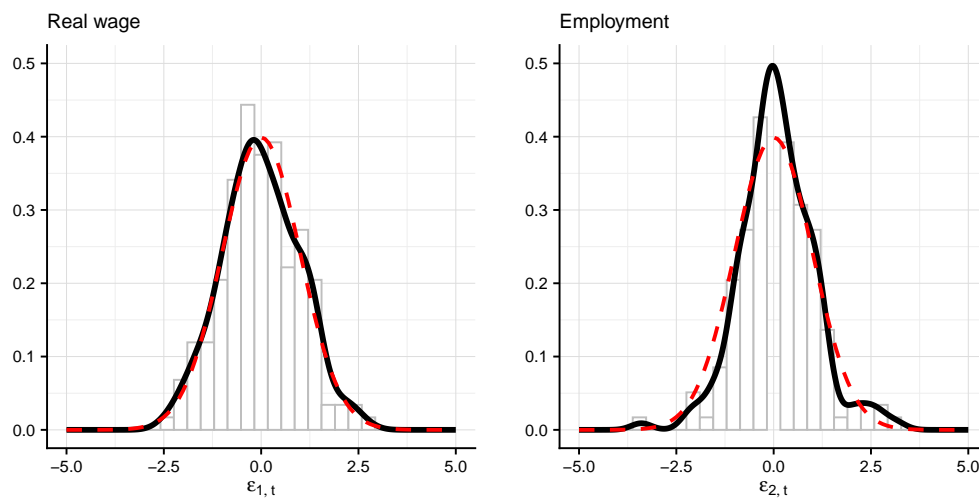
Note: Histogram (light gray) and kernel density estimates (black solid) of the recovered structural shocks $\hat{\epsilon}_{k,t}$ for $k = 1, 2$ from the Baumeister and Hamilton (2015) model, overlaid with a standard normal density (red dashed).

FIGURE S6: DISTRIBUTIONS OF SHOCKS – KILIAN AND MURPHY (2012) MODEL
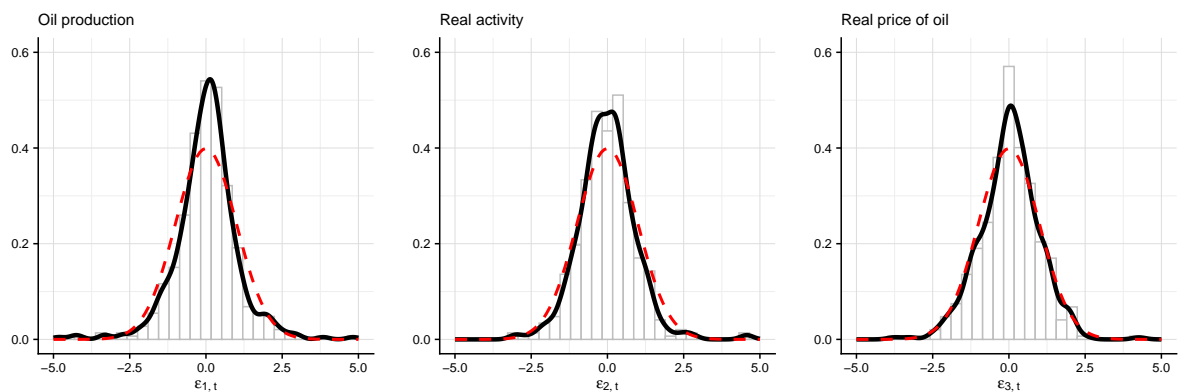
Note: Histogram (light gray) and kernel density estimates (black solid) of the recovered structural shocks $\hat{\epsilon}_{k,t}$ for $k = 1, 2, 3$ from the Kilian and Murphy (2012) model, overlaid with a standard normal density (red dashed).

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