# Robust Estimation and Inference for Time-varying Unconditional Volatility[1]

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**Abstract**

We derive a general and robust estimator of a large class of parametric specifications of time-varying unconditional volatility of financial returns, both univariate and multivariate, and establish the Consistency and Asymptotic Normality (CAN) of the estimator. A number of well-known and widely used parametric specifications, for many of which asymptotic results have not been specifically established, are contained in the class. The estimator is robust in the sense that the exact specification of the conditional volatility dynamics need not be known or estimated, and in the sense that the stochastic component need not be strictly stationary. The latter is especially important in light of recent findings, which document that financial returns are frequently characterised by a non-stationary zero-process. Our estimator is also robust to the well-known "curse of dimensionality" in multivariate models due to its equation-by-equation nature. While our estimator does not require the exact specification of the conditional volatility dynamics to be known or estimated, our results imply that the scaled GARCH(1,1) specification is well-defined under both correct and incorrect specification. So we provide methods for its estimation in a second step. Also, due to the assumptions we rely upon, our results extend directly to the Multiplicative Error Model (MEM) interpretation of volatility models. This means our results can also be applied to other non-negative processes like volume, duration, realised volatility, dividends, unemployment and so on. Three numerical applications illustrate the versatility of our results.

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# 1 Introduction

Financial returns are frequently characterised by a time-varying unconditional volatility, and it has long been known that this has important implications for statistical inference and economic decision making. Lamoureux and Lastrapes (1990), Mikosch and Starica (2004), and Hillebrand (2005), for example, document that ignoring changes in the unconditional volatility can lead to spurious persistence and long-memory effects. In turn, the distortions induced by faulty estimates and inference, affect quantities that are key in economic decision making. Examples include risk estimation (e.g. Andreou and Ghysels, 2008), asset allocation (e.g. Pettenuzzo and Timmermann, 2011), the equity premium (e.g. Pastor and Stambaugh, 2001) and the shape of the option volatility smile (e.g. Bates, 2000), to name but a few.

Let $\epsilon_t$ denote an observed financial return at $t$. If $0 < E(\epsilon_t^2) < \infty$ for all $t$, then $\epsilon_t^2$ can be decomposed multiplicatively as

$$\epsilon_t^2 = g_t \phi_t^2 \qquad \text{with} \qquad g_t := E(\epsilon_t^2) \quad \text{and} \quad \phi_t^2 := \epsilon_t^2 / E(\epsilon_t^2) \tag{1}$$

for all $t$. Henceforth, we refer to $g_t$ as the unconditional volatility at $t$. The decomposition in (1) implies $E(\phi_t^2) = 1$ for all $t$. For some of the implications of this, see our discussion in relation with Assumption 4 further below. While a model of the conditional volatility dynamics is not needed for our main results, it is often of interest in empirical applications, since volatiltiy prediction is commonly an important objective. A leading example is the scaled version of the GARCH(1,1) model of Bollerslev (1986):

$$\phi_t^2 = h_t \eta_t^2, \qquad E(\eta_t^2 | \mathcal{F}_{t-1}) = 1, \qquad h_t = \omega + \alpha \phi_t^2 + \beta h_{t-1}, \qquad t \in \mathbb{Z}, \tag{2}$$

where $\mathcal{F}_{t-1}$ is a suitable $\sigma$-algebra generated by past $\phi_t^2$'s. The conditional volatility or variance at $t$ is thus $\sigma_t^2 = g_t h_t$, and the unconditional volatility or variance at $t$ is $E(\sigma_t^2) = g_t$. In Section 4 we show that the scaled GARCH(1,1) is well-defined in our setup under both correct and incorrect specification. Other examples of $h_t$ include scaled versions of Stochastic Volatility (SV) models (in which $\sigma_t^2$ need not equal the conditional variance), and scaled versions of Dynamic Conditional Score (DCS) models. Henceforth, to simplify the exposition, we refer to both $\sigma_t$ and $\sigma_t^2$ as conditional volatility, since one is obtained from the other via a straightforward transformation.

Broadly, there are two approaches to the specification and estimation of time-varying unconditional volatility $g_t$. In the first approach, estimation of $g_t$ is nonparametric. Examples include Feng (2004), the "Lip" specification in Van Bellegem and Von Sachs (2004), Feng and McNeil (2008), Hafner and Linton (2010), Koo and Linton (2015), Kim and Kim (2016), and Jiang et al. (2021). In the second approach, which we follow here, $g_t$ is governed by a finite-dimensional parameter $\boldsymbol{\theta}$. An early example is the piecewise constant specification in Van Bellegem and Von Sachs (2004). For estimation, they proposed the sample variance in each constant period under the assumption that break-locations are known. However, asymptotic methods for the joint estimation and inference of multiple break-sizes were not considered. Engle and Rangel (2008), in their specification without regressors, and Brownlees and Gallo (2010), specify $g_t$ as a deterministic spline function. The former use Gaussian Maximum Likelihood (ML) for estimation, whereas the latter employ penalised ML. No asymptotic results are established in either work, but in later work Zhang et al. (2020) derive asymptotic results for a least squares estimator of B-splines. In a series of papers, see e.g. Amado and Teräsvirta (2013, 2014, 2017), and Silvennoinen and Teräsvirta (2024), $g_t$ is specified as a smooth transition function, and $\phi_t^2$ is governed by a first-order GARCH model. In these papers the Gaussian Quasi ML Estimator (QMLE) is used to estimate the parameter $\boldsymbol{\theta}$ in the first step of an iterative estimation algorithm. However, in the former consistency of the first step Gaussian QMLE is proved under the restrictive and unrealistic assumption that $\phi_t^2$ is *iid*, and in the latter the standardised error is assumed to be *iid*.[5] Next, Consistency and Asymptotic Normality (CAN) of the parameters of a GARCH model that governs $\phi_t^2$ is established in the infeasible case where $\boldsymbol{\theta}$ is known from the first step.[6] Theorem 7 in Silvennoinen and Teräsvirta (2024) uses Theorem 3 in Song et al. (2005) to establish joint CAN of all the parameters of the multivariate model, but the proof of Theorem 7 appears to be incomplete.[7] To accommodate the possibility of cyclical patterns

[5]See the assumption that $h_t = 1$ for all $t$ in Theorem 1 of (Amado and Teräsvirta, 2013, p. 145), and Theorem 3 in Silvennoinen and Teräsvirta (2024).

[6]See Amado and Teräsvirta (2013, Theorem 2), and Silvennoinen and Teräsvirta (2024, Theorem 6).

[7]Theorem 3 in Song et al. (2005) assumes a set of unstated regularity conditions hold, and that consistency of the first and second step estimators have been established, see the proof of Theorem 3 on p. 1156 in Song et al. (2005). What the unstated regularity conditions are is particularly important in the current context due to the triangular nature of the sequence of $g_{t,T}$'s and how this may affect invertibility (i.e. the asymptotic irrelevance of the initial values of the GARCH recursion at the true parameter value), and due to the estimation error in the second step estimation of the parameters of $\phi_t^2$ (cf. Francq and Zakoïan, 2019, p. 190, Francq et al., 2011, and Francq et al., 2016).

in volatility, which is a common feature of intraday financial returns, Andersen and Bollerslev (1997), and Mazur and Pipien (2012), specify (3) as a Fourier Flexible Form (FFF). In the former estimation is by a least squares procedure (see their Appendix B), and in the latter Bayesian methods are used. No asymptotic results are established in either work. Escribano and Sucarrat (2018) propose a log-linear version of $g_t$, and use least squares methods to estimate the parameter $\boldsymbol{\theta}$. However, they do not establish any asymptotic results. In He et al. (2019), $g_t$ is specified as a seasonal smooth transition function, and CAN is established for a likelihood-based estimator under the assumption that $\phi_t \overset{iid}{\sim} N(0,1)$ (see their assumption A5 in Section 5.2). This is generalised to the multivariate case in He et al. (in press), but $\phi_t$ is still required to be $\overset{iid}{\sim} N(0,1)$ for all $t$ conditional on the past (see assumption AV6 in the supplement to He et al., 2024).

In this paper, $g_t$ is parametrised by a finite dimensional parameter $\boldsymbol{\theta}$ and the sample size $T$. Specifically,

$$g_t = g_{t,T}(\boldsymbol{\theta}), \tag{3}$$

so $\{g_{t,T} : T \in \mathbb{N},\ 1 \leq t \leq T\}$ forms a triangular array of functions from $\boldsymbol{\Theta}$ to $(0,\infty)$. We prove that the equation-by-equation Gaussian QMLE provides Consistent and Asymptotically Normal (CAN) estimators of $\boldsymbol{\theta}$ for a large number and widely used specifications in (3), both univariate and multivariate versions. In particular, most of the parametric specifications in the literature reviewed above are covered by our theory, since we allow the $g_t$ functions to change with $T$. A sub-class of special interest contained in (3) is $g_{t,T}(\boldsymbol{\theta}) = g(\boldsymbol{\theta}, t/T)$, $g : \Theta \times [0,1] \to (0,\infty)$, where time enters in the re-scaled form $t/T$.

Our results are characterised by several attractive properties. First, there is no need to specify – or know – the exact specification of the stochastic component $\phi_t^2$ in the estimation of $\boldsymbol{\theta}$. Also, the $\phi_t^2$'s can be dependent (strongly mixing) over time. Our results thus hold for a large class of specifications of $\phi_t^2$, including the most common GARCH and Stochastic Volatility (SV) models, both univariate and multivariate. This contrasts with previous results, which rely on specific and often restrictive specification assumptions on $\phi_t^2$.[8] Second, contrary to the parametric works cited above, we do not rely on the assumption that $\{\phi_t^2\}$ or $\{\eta_t^2\}$ is strictly stationary.

[8] Our dependence assumption could be relaxed further at the cost of increased conceptual complexity. Overall, we believe that the conditions we impose strike a good balance in this regard: They are relatively simple whilst being sufficiently weak for the theory to apply to many cases of practical interest.

Relaxing the strict stationarity assumption on $\{\phi_t^2\}$ and $\{\eta_t^2\}$ is important, since recent studies reveal that the zero-process of financial returns – both daily and intradaily – is frequently non-stationary, see e.g. Kolokolov et al. (2020), Sucarrat and Grønneberg (2022), Francq and Sucarrat (2023), Stauskas and Sucarrat (2025), Patilea and Raïssi (2024), and Kolokolov and Reno (2024). In particular, the standard assumption that $\eta_t^2$ is *iid* is *not* compatible with a non-stationary zero-process. Sections 5.2 and 5.3 contain illustrations. A third attractive property of our estimator is its equation-by-equation nature (cf. Francq and Zakoïan, 2016). This reduces the numerical challenges ("the curse of dimensionality") typically associated with multivariate models. Fourth, while our results do not require the estimation and explicit specification of a model of $\phi_t^2$, a model can nevertheless be estimated in a second step. In particular, an especially interesting implication of our results is that the scaled GARCH(1,1) prediction is well-defined under both correct and incorrect specification within our framework, even under certain types of non-stationarities of the stochastic component $\phi_t^2$. This is very useful in practice, since it means the user is not required to know the exact DGP of the conditional volatility dynamics, or to rely on restrictive assumptions like strict stationarity of $\{\phi_t^2\}$ or that the scaled error $\phi_t^2/h_t$ is *iid*. A fifth attractive property pertains to the challenge of modelling non-stationary periodic volatility (e.g. as in intraday returns). Standard ways of describing periodicity do not readily lend themselves to tractable re-formulations in terms of re-scaled time. By instead approaching the problem in terms of the vector-of-seasons representation, this problem is side-stepped. Section 5.3 gives an empirical illustration. Sixth, for parameter identification, previous theoretical results either rely on the high-level assumption that the true parameter is the unique optimiser, see e.g. Amado and Teräsvirta (2013, Assumption AG2 on p. 145), or on restrictive density and *iid* assumptions on the scaled error $\phi_t^2/h_t$, see e.g. He et al. (2019), He et al. (2024), and Silvennoinen and Teräsvirta (2024). Here, we establish milder, more primitive and verifiable sufficient conditions for important sub-classes of $g_t$, see Section 3. This is possible due to the nature of our estimator. Finally, due to the assumptions we rely on, the Multiplicative Error Model (MEM) interpretation of volatility models holds straightforwardly. The reason is that our assumptions are on $\epsilon_t^2$ and $\phi_t^2$, not on $\epsilon_t$ and $\phi_t$. Accordingly, our results also apply to models of the time-varying unconditional mean of non-negative processes like volume, duration, realised volatility, dividends, unemployment and so on by simply interpreting $\epsilon_t^2$ as the non-negative variable in question.

The rest of the paper is organised as follows. The next section, Section 2, contains our main theoretical results and the assumptions they rely on. Section 3 gives examples of $g_{t,T}$ specifications contained in (3), and derive primitive sufficient conditions for a unique optimiser for three important sub-classes of $g_t$. Section 4 outlines how a GARCH(1,1) specification can be used to estimate the conditional volatility dynamics in a second step under both correct and incorrect specification, and how time-varying correlations can be estimated subsequently. Section 5 contains numerical illustrations of our results, whereas Section 6 concludes. The proofs of our results are contained in the Appendix.

# 2 Consistency and Asymptotic Normality

## 2.1 Consistency

Let $\boldsymbol{\epsilon}_{t,T} = (\epsilon_{1,t,T}, \ldots, \epsilon_{M,t,T})'$ denote an $M$-dimensional multivariate return with $M \in \mathbb{N}$, and let

$$\epsilon^2_{m,t,T} = g_{m,t,T}(\boldsymbol{\theta}^\star_m)\phi^2_{m,t,T}, \qquad m = 1, \ldots, M, \quad 1 \le t \le T, \quad T \in \mathbb{N}, \tag{4}$$

where $g_{m,t,T}$ is a deterministic function. Our estimator of $\boldsymbol{\theta}^\star = (\boldsymbol{\theta}^{\star\prime}_1, \ldots, \boldsymbol{\theta}^{\star\prime}_M)' \in \prod_{m=1}^M \boldsymbol{\Theta}_m = \boldsymbol{\Theta}$ is derived from the objective function

$$L_T(\boldsymbol{\theta}) = \sum_{m=1}^M L_{m,T}(\boldsymbol{\theta}_m) \qquad \text{with} \qquad L_{m,T}(\boldsymbol{\theta}_m) = \frac{1}{T}\sum_{t=1}^T l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}), \tag{5}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}'_1, \ldots, \boldsymbol{\theta}'_M)'$ and

$$l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) = \ln g_{m,t,T}(\boldsymbol{\theta}_m) + \frac{\epsilon^2_{m,t,T}}{g_{m,t,T}(\boldsymbol{\theta}_m)}, \qquad m = 1, \ldots, M.$$

Minimisation of (5) leads to the Equation-by-Equation (EBE) Quasi Maximum Likelihood Estimator (QMLE):

$$\widehat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\arg\min}\ L_T(\boldsymbol{\theta}) = (\widehat{\boldsymbol{\theta}}'_1, \ldots, \widehat{\boldsymbol{\theta}}'_M)', \quad \widehat{\boldsymbol{\theta}}_m = \underset{\boldsymbol{\theta}_m\in\boldsymbol{\Theta}_m}{\arg\min}\ L_{m,T}(\boldsymbol{\theta}_m), \quad m = 1, \ldots, M. \tag{6}$$

This is an EBE estimator, since the parameters of equation $m$, i.e. $\boldsymbol{\theta}_m$, can be estimated separately from the parameters of the other equations. An attractive property of the EBE estimator is that it provides a solution to the "curse of dimensionality". Note in that regard that an EBE estimator is not necessarily less efficient asymptotically than a system estimator (Francq and Zakoïan 2016). Note also that a single-equation cannot be used to model unconditional volatility that is periodic (as in intraday data); this can be achieved however with a multiple-equation approach via the vector-of-seasons representation (this is illustrated in Section 5.3).

In establishing consistency of the EBE-QMLE, we rely on the following assumptions.

**Assumption 1.** $\boldsymbol{\Theta} \subset \mathbb{R}^{d_{\boldsymbol{\theta}}}$ *is compact.*

**Assumption 2.** *For each* $m = 1, \ldots, M$, *let* $\boldsymbol{\Theta}_m^*$ *be an open, convex set containing* $\boldsymbol{\Theta}_m$. *For all* $1 \leq t \leq T$, $T \in \mathbb{N}$,

*(i)* $g_{m,t,T}(\boldsymbol{\theta}_m)$ *is bounded away from zero and infinity, i.e.*

$$0 < \inf_{\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m^*,\, 1 \leq t \leq T,\, T \in \mathbb{N}} g_{m,t,T}(\boldsymbol{\theta}_m) \leq \sup_{\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m^*,\, 1 \leq t \leq T,\, T \in \mathbb{N}} g_{m,t,T}(\boldsymbol{\theta}_m) < \infty.$$

*(ii)* $\boldsymbol{\theta}_m \mapsto g_{m,t,T}(\boldsymbol{\theta}_m)$ *is continuously differentiable on* $\boldsymbol{\Theta}_m^*$ *and the derivatives* $\dot{\boldsymbol{g}}_{m,t,T}$ *are uniformly bounded:*

$$\sup_{\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m^*,\, 1 \leq t \leq T,\, T \in \mathbb{N}} \|\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)\| < \infty.$$

**Assumption 3.** *For each* $m = 1, \ldots, M$, $\{\epsilon_{m,t,T}^2 : t \in \mathbb{Z},\, T \in \mathbb{N}\}$ *forms a triangular array of a.s. non-negative random variables. Let* $\alpha_{m,T}(k)$ *be the* $\alpha$*-mixing coefficients corresponding to* $\{\epsilon_{m,t,T}^2 : t \in \mathbb{Z}\}$ *and suppose that as* $k \to \infty$,

$$\sup_{T \in \mathbb{N}} \alpha_{m,T}(k) \to 0,$$

*where* $\alpha_{m,T}(k) := \sup_{t \in \mathbb{Z}} \sup\{|P(F \cap G) - P(G)P(H)| : F \in \mathcal{F}_{m,-\infty,T}^t, G \in \mathcal{F}_{m,t+k,T}^\infty\}$ *with* $\mathcal{F}_{m,-\infty,T}^t := \sigma(\epsilon_{m,s,T}^2 : s \leq t)$ *and* $\mathcal{F}_{m,t+k,T}^\infty := \sigma(\epsilon_{m,s,T}^2 : s \geq t + k)$.

**Assumption 4.** *For each* $m = 1, \ldots, M$, $\phi_{m,t,T}^2 := \epsilon_{m,t,T}^2 / g_{m,t,T}(\boldsymbol{\theta}_m^\star)$ *is a non-degenerate random variable such that:*

*(i)* $E(\phi_{m,t,T}^2) = 1$ *for all* $1 \leq t \leq T$, $T \in \mathbb{N}$;

*(ii)* $\sup_{1\leq t\leq T,\, T\in\mathbb{N}} E|\phi^2_{m,t,T}|^{1+\delta} < \infty$ *for some* $\delta > 0$.

**Assumption 5.** *For each* $m = 1, \ldots, M$, $L_m(\boldsymbol{\theta}_m) := \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left(l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right)$ *exists and attains a unique minimum at* $\boldsymbol{\theta}^\star_m \in \boldsymbol{\Theta}_m$.

Assumption 1 is standard. Assumption 2 defines the general class of $g_m$ functions that we consider. Section 3 gives specific examples. Assumption 3 is a mild dependence assumption. In particular, it is substantially milder than the assumptions used by Amado and Teräsvirta (2013) in their univariate derivations, since they rely – in our notation – on $\{\phi^2_{m,t,T}\}$ being *iid*, see their Theorem 1 on p. 145 just below equation (15). Here, by contrast, Assumption 3 is compatible with any volatility model of $\phi^2_{m,t,T}$, stationary or not, as long strong mixing holds. This means our results apply not only to standard models within the ARCH, GAS and SV classes, but also to semi-strong volatility models, see e.g. Escanciano (2009) and Francq and Thieu (2019), and to models that are only weakly identified as models of the variance (e.g. intraday high-frequency measures of volatility), see Sucarrat (2021). Specific examples of GARCH and SV models that are compatible with Assumption 3 are studied in Carrasco and Chen (2002), Lindner (2009), Davis and Mikosch (2009), and Francq and Zakoïan (2019, Ch. 3). Note that, in the definition of mixing size, the underlying mixing coefficients are defined across $\sigma$-fields generated by the $\epsilon^2_{m,t,T}$'s and *not* the $\boldsymbol{\epsilon}^2_{t,T}$'s. Since the $\sigma$-fields generated by the former are contained in those of generated by the latter, the dependence as measured by mixing is stronger for the latter than for the former (cf. the discussion of Assumption 9).

Assumption 4(i) is a very mild identification assumption. The reason is that most conditional volatility models are stable by scaling (cf. Francq et al. 2018). For conditional volatility models that are not stable by scaling, the condition $E(\phi^2_{m,t,T}) = 1$ may be restrictive. Note that Assumption 4(i) is compatible with $\{\phi^2_{m,t,T}\}$ being non-stationary. A case in point is the common situation where the zero-process of a financial return is non-stationary, see e.g. Sucarrat and Grønneberg (2022), and Francq and Sucarrat (2023). In particular, Proposition 2.1(ii) in Sucarrat and Grønneberg (2022) implies $E(\phi^2_{m,t,T})$ can be constant over time even though the zero-process of a financial return is non-stationary. Another implication of Assumption 4(i) is that $E(\epsilon^2_{m,t,T}) = g_{m,t,T}(\boldsymbol{\theta}^\star_m)$. This facilitates interpretation. Assumption 4(ii) is a fairly mild moment assumption. For example, it holds when $\{\phi^2_{m,t,T}\}$ is governed by a stationary GARCH(1,1), as in (2), with finite $E(\phi^4_{m,t,T})$. Finally, Assumption 5 ensures the existence of a minimising $\boldsymbol{\theta}_m^\star$ in the asymptotic analogue of the minimisation problem in equation (6). These assumptions are sufficient for consistency of the $\widehat{\boldsymbol{\theta}}_m$ estimators as defined in (6).

**Theorem 1** (Consistency)**.** *If Assumptions 1 – 5 hold, then $\widehat{\boldsymbol{\theta}}_m \xrightarrow{P} \boldsymbol{\theta}_m^\star$ for each $m = 1, \dots, M$.*

## 2.2 Asymptotic Normality Equation-by-Equation

We now establish asymptotic normality of $\widehat{\boldsymbol{\theta}}_m$, for each $m = 1, \dots, M$ separately. For this we have to strengthen the imposed conditions.

**Assumption 6.** $\boldsymbol{\theta}^\star \in \text{int}(\boldsymbol{\Theta})$.

**Assumption 7.** *For each $m = 1, \dots, M$, $\sup_{1 \le t \le T,\, T \in \mathbb{N}} E|\phi_{m,t,T}^2|^{2+\delta_m} < \infty$ for some $\delta_m > 0$.*

**Assumption 8.** *For each $m = 1, \dots, M$, $1 \le t \le T$, $T \in \mathbb{N}$,*

*(i)* $\boldsymbol{\theta}_m \mapsto g_{m,t,T}(\boldsymbol{\theta}_m)$ *is twice continuously differentiable on a neighbourhood $\mathcal{V}_m$ of $\boldsymbol{\theta}_m^\star$ in $\boldsymbol{\Theta}_m$.*

*(ii) On $\mathcal{V}_m$, define*

$$\boldsymbol{S}_{m,T}(\boldsymbol{\theta}_m) := \frac{1}{T}\sum_{t=1}^{T} \dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2),$$

*and*

$$\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) := \frac{1}{T}\sum_{t=1}^{T} \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2),$$

*where $\dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)$ and $\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)$ are respectively the first and second derivative of $l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)$ with respect to $\boldsymbol{\theta}_m$.*

*(iii) There are deterministic functions $\varphi_{m,t,T} : \mathcal{V}_m \to \mathbb{R}$ and random variables $\upsilon_{m,t,T}$ such that,*

$$\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)\| \le \varphi_{m,t,T}(\boldsymbol{\theta}_m)\upsilon_{m,t,T}, \qquad \boldsymbol{\theta}_m \in \mathcal{V}_m,$$

*where*

$$\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \sup_{1 \le t \le T, T \in \mathbb{N}} \varphi_{m,t,T}(\boldsymbol{\theta}_m) < \infty, \qquad \sup_{1 \le t \le T, t \in \mathbb{N}} E\upsilon_{m,t,T}^2 < \infty.$$

*(iv) There exist random variables $\psi_{m,t,T}$ such that for $\boldsymbol{\theta}_m, \boldsymbol{\theta}'_m \in \mathcal{V}_m$,*

$$\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2) - \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}'_m, \epsilon_{m,t,T}^2)\| \le \psi_{m,t,T}\|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|,$$

*where*

$$\sup_{1\le t\le T, T\in\mathbb{N}} E\left|\psi_{m,t,T}\right| < \infty.$$

**Assumption 9.** *For each $m = 1,\ldots,M$, the strong mixing coefficients $\alpha_{m,T}(k)$ satisfy*

$$\sup_{T\in\mathbb{N}} \alpha_{m,T}(k) = O(k^{-\rho_m-\varepsilon}),$$

*for some $\varepsilon > 0$, where $\rho_m := r_m/(r_m-2)$, $r_m = 2+\delta_m$ with $\delta_m > 0$ as in Assumption 7.*

**Assumption 10.** *For each $m = 1,\ldots,M$, as $T\to\infty$*

$$\boldsymbol{B}_{m,T} := \mathrm{Var}\left(T^{-1/2}\sum_{t=1}^{T} \dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m^\star, \epsilon_{m,t,T}^2)\right) \to \boldsymbol{B}_m^\star,$$

*with $\boldsymbol{B}_m^\star$ positive definite.*

**Assumption 11.** *For each $m = 1,\ldots,M$, as $T\to\infty$,*

$$\boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m) := \frac{1}{T}\sum_{t=1}^{T} E\left[\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)\right] \to \boldsymbol{A}_m(\boldsymbol{\theta}_m), \quad \boldsymbol{\theta}_m \in \mathcal{V}_m,$$

*where $\mathcal{V}_m$ is as in assumption 8. $\boldsymbol{A}_m^\star := \boldsymbol{A}_m(\boldsymbol{\theta}_m^\star)$ is positive definite.*

Assumption 6 is standard. Assumption 7 is a strengthened version of Assumption 4(ii). Assumption 8 imposes twice continuous differentiability of each $g_{m,t,T}$ in a neighbourhood of the true parameter and assumes that the second derivative satisfies (iii) a domination condition and (iv) a Lipschitz-type condition. If the second derivative matrix of $g_{m,t,T}$ is bounded on $\mathcal{V}_m$, uniformly in $t,T$, (iii) holds (see Lemma 1 in the Appendix). A sufficient condition for Assumption 8 (given Assumptions 2 and 4) is that $g_{m,t,T}$ is three-times differentiable on $\mathcal{V}_m$ with its second and third derivatives uniformly bounded (over $t,T$ and $\mathcal{V}_m$); see Lemma 5 in the Appendix. In Assumption 9, the mixing size $r_m = 2+\delta_m$ is connected to the moments requirements in Assumption 7. The more dependence (i.e. the higher $r_m$ is), the more moments are required. Assumptions 2, 4 and 9 are sufficient for $\boldsymbol{B}_{m,T} = O(1)$ (cf. Lemma 2); Assumption 10 further ensures that $\boldsymbol{B}_{m,T}$ converges to a positive definite limit. Similarly Assumptions 2, 4 and 8 suffice that each $A_{m,T}(\boldsymbol{\theta}_m) = O(1)$ (cf. Lemma 3); existence of the limit is assumed in Assumption 11. Note that the limits in Assumptions 10 and 11 will not exist if $g_{m,t,T}$ is non-stationary periodic, as is common in intraday data. However, as we illustrate in Section 5.3, in that case our multivariate results can be used in combination with the vector-of-seasons representation. Note also that even if the limits in Assumptions 10 and 11 exist, they may not be positive definite. So an important role played by the assumptions is to ensure positive definiteness holds.

These assumptions are sufficient for marginal asymptotic normality of each $\hat{\boldsymbol{\theta}}_m$ and that $\hat{\boldsymbol{A}}_{m,T}(\hat{\boldsymbol{\theta}}_m)$ is consistent for $\boldsymbol{A}^{\star}_m$.

**Theorem 2.** *Suppose Assumptions 1 – 11 hold. Then* $\sqrt{T}(\widehat{\boldsymbol{\theta}}_m - \boldsymbol{\theta}^{\star}_m) \xrightarrow{D} \mathcal{N}(\mathbf{0}, [\boldsymbol{A}^{\star}_m]^{-1}\boldsymbol{B}^{\star}_m[\boldsymbol{A}^{\star}_m]^{-1})$ *for* $m = 1, \ldots, M$*.*

**Corollary 1.** *Suppose Assumptions 1 – 11 hold. Then* $\widehat{\boldsymbol{A}}_{m,T}(\widehat{\boldsymbol{\theta}}_m) \xrightarrow{P} \boldsymbol{A}^{\star}_m$ *for* $m = 1, \ldots, M$*.*

Proposition 6 in the Appendix demonstrates how to verify the Assumptions required by Theorem 2 in a specific class of models (a piecewise constant specification with $\{\phi^2_{m,t,T}\}$ covariance stationary). In particular, this Proposition applies when the DGP of $\{\phi^2_{m,t,T}\}$ is a strictly stationary scaled GARCH process.

It is worth emphasising that the autocovariance structure of the $\phi^2_{m,t,T}$'s only affects the asymptotic variance in Theorem 2 via $\boldsymbol{B}^{\star}_m$, not via $\boldsymbol{A}^{\star}_m$. To see this, note that the term $E\big(\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}, \epsilon^2_{m,t,T})\big)$ in Assumption 11 depends only on $g_{m,t,T}$ and its first derivatives when evaluated at $\boldsymbol{\theta}^{\star}_m$. Accordingly, the limit $\boldsymbol{A}^{\star}_m$ does not depend on the specific model that governs $\phi^2_{m,t,T}$, or on any other of the statistical properties of $\{\phi^2_{m,t,T}\}$. By contrast, it can be shown that each entry in $\boldsymbol{B}_{m,T}$ in Assumption 10 are sums made up of the autocovariances of $\phi^2_{m,t,T}$. If the autocovariances are all non-negative, as in the standard GARCH (cf. Section 2.5.1 in Francq and Zakoïan 2019), the diagonal entries of the limit $\boldsymbol{B}^{\star}_m$ will be higher the stronger and more persistent the autocovariances of $\phi^2_{m,t,T}$ are. Still, since the off-diagonals of $\boldsymbol{B}^{\star}_m$ and the entries in $\boldsymbol{A}^{\star -1}_m$ can be negative, it is not clear what the overall effect is on the asymptotic variance by stronger and more persistent autocovariances. Under more specific assumptions the asymptotic variance simplifies; see Proposition 6 in Appendix B for the case where $g_{m,t,T}$ is piecewise constant and $\{\phi^2_{m,t,T}\}$ is covariance stationary.

In order to operationalise inference based on the asymptotic approximation of Theorem 2, beyond Corollary 1 we require a consistent estimator of $\boldsymbol{B}^{\star}_m$. We can consistently estimate this matrix using kernel weighted sample autocovariances. The general form of our estimator is

$$
\begin{aligned}
\widehat{\boldsymbol{B}}_{m,T} &\coloneqq \sum_{j=-T}^{T} k_m(j/\kappa_{m,T})\widehat{\boldsymbol{\Gamma}}_{m,T}(j), \\
\widehat{\boldsymbol{\Gamma}}_{m,T}(j) &\coloneqq \frac{1}{T}\sum_{t=1}^{T-j} \dot{\boldsymbol{l}}_{m,t,T}(\widehat{\boldsymbol{\theta}}_m, \epsilon^2_{m,t,T})\dot{\boldsymbol{l}}_{m,t,T}(\widehat{\boldsymbol{\theta}}_m, \epsilon^2_{m,t+j,T})' \qquad (j \geq 0), \\
\widehat{\boldsymbol{\Gamma}}_{m,T}(j) &\coloneqq \widehat{\boldsymbol{\Gamma}}_{m,T}(-j)' \qquad (j < 0).
\end{aligned} \tag{7}
$$

where the $k_m(\cdot)$'s are kernel weights, and $\kappa_{m,T}$ is the bandwidth. The permitted kernel functions are those which belong to the class $\mathscr{K}$ of De Jong and Davidson (2000, p. 409), defined as:

$$
\mathscr{K} \coloneqq \Bigg\{ k : \mathbb{R} \to [-1,1] : k(0) = 1,\ k(x) = k(-x),\ \int |k(x)|\,\mathrm{d}x < \infty,\ \int |\phi(\xi)|\,\mathrm{d}\xi < \infty, \\
k \text{ is continuous at 0 and at all but a finite number of points} \Bigg\},
$$

where $\phi(\xi) \coloneqq \frac{1}{2\pi}\int k(x)e^{i\xi x}\,\mathrm{d}x$.

**Assumption 12** (Kernel)**.** *For each $m = 1, \ldots, M$, $k_m \in \mathscr{K}$.*

**Assumption 13** (Bandwidth)**.** *$\kappa_{m,T} \to \infty$ and $\kappa_{m,T} = o(T^{1/2})$ for each $m = 1, \ldots, M$.*

Most kernels considered in the literature satisfy Assumption 12. This includes, amongst other, the Bartlett, Parzen and Quadratic Spectral kernels. Assumption 13 governs the divergence rate of the bandwidth.

The following Proposition is proven by verifying the conditions of Theorem 2.2 of De Jong and Davidson (2000), demonstrating that – under our Assumptions – $\widehat{\boldsymbol{B}}_{m,T}$ is consistent for $\boldsymbol{B}^{\star}_m$.

**Proposition 1.** *Suppose Assumptions 1 – 13 hold. Then $\widehat{\boldsymbol{B}}_{m,T} \xrightarrow{P} \boldsymbol{B}^{\star}_m$ for $m = 1, \ldots, M$.*

### 2.3 Joint Asymptotic Normality

We next establish the joint asymptotic normality of $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}'_1, \ldots, \widehat{\boldsymbol{\theta}}'_M)'$. This is required for hypothesis tests on coefficients across equations. Let $\boldsymbol{\epsilon}^2_{t,T} \coloneqq (\epsilon^2_{1,t,T}, \ldots, \epsilon^2_{M,t,T})'$. We can re-write the objective function in (5) as

$$
L_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=1}^{T} l_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T}), \qquad l_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T}) \coloneqq \sum_{m=1}^{M} l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}). \tag{8}
$$

Note that (under Assumption 2) the first and second derivatives of $\boldsymbol{\theta} \mapsto l_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T})$ are

$$\dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T}) = \left(\dot{\boldsymbol{l}}_{1,t,T}(\boldsymbol{\theta}_1, \epsilon^2_{1,t,T})', \ldots, \dot{\boldsymbol{l}}_{M,t,T}(\boldsymbol{\theta}_M, \epsilon^2_{M,t,T})'\right)',$$
$$\ddot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T}) = \operatorname{diag}\left(\ddot{\boldsymbol{l}}_{1,t,T}(\boldsymbol{\theta}_1, \epsilon^2_{1,t,T}), \ldots, \ddot{\boldsymbol{l}}_{M,t,T}(\boldsymbol{\theta}_M, \epsilon^2_{M,t,T})\right).$$

Note that under Assumption 11,

$$\boldsymbol{A}_T(\boldsymbol{\theta}) \coloneqq \frac{1}{T}\sum_{t=1}^{T} E\left[\ddot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T})\right] \to \boldsymbol{A}(\boldsymbol{\theta}) \coloneqq \operatorname{diag}\left(\boldsymbol{A}_1(\boldsymbol{\theta}_1), \ldots, \boldsymbol{A}_M(\boldsymbol{\theta}_M)\right), \quad \boldsymbol{\theta} \in \mathcal{V}, \tag{9}$$

where $\mathcal{V} \coloneqq \prod_{m=1}^{M} \mathcal{V}_m$ and $\boldsymbol{A}^\star \coloneqq \boldsymbol{A}(\boldsymbol{\theta}^\star)$ is positive definite. Define

$$\widehat{\boldsymbol{A}}_T(\boldsymbol{\theta}) \coloneqq \operatorname{diag}\left(\widehat{\boldsymbol{A}}_{1,T}(\boldsymbol{\theta}_1), \ldots, \widehat{\boldsymbol{A}}_{M,T}(\boldsymbol{\theta}_M)\right). \tag{10}$$

To establish joint asymptotic normality we need to strengthen Assumptions 9 and 10 to (respectively) Assumptions 14 and 15 below.

**Assumption 14.** *If $\alpha_T(k)$ are the strong mixing coefficients of $\{\boldsymbol{\epsilon}^2_{t,T} : t \in \mathbb{Z},\ T \in \mathbb{N}\}$, i.e. $\alpha_T(k) \coloneqq \sup_{t\in\mathbb{Z}} \sup\{|P(F \cap G) - P(G)P(H)| : F \in \mathcal{F}^t_{-\infty,T}, G \in \mathcal{F}^\infty_{t+k,T}\}$ with $\mathcal{F}^t_{-\infty,T} \coloneqq \sigma(\boldsymbol{\epsilon}^2_{s,T} : s \leq t)$ and $\mathcal{F}^\infty_{t+k,T} \coloneqq \sigma(\boldsymbol{\epsilon}^2_{s,T} : s \geq t+k)$, then*

$$\sup_{T\in\mathbb{N}} \alpha_T(k) = O(k^{-\rho-\varepsilon}),$$

*for some $\varepsilon > 0$, where $\rho \coloneqq \frac{r}{r-2}$, $r \coloneqq 2 + \min\{\delta_1, \ldots, \delta_M\}$ with $\delta_m$ as in Assumption 7.*

**Assumption 15.** *As $T \to \infty$*

$$\boldsymbol{B}_T \coloneqq \operatorname{Var}\left(T^{-1/2}\sum_{t=1}^{T} \dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}^\star, \boldsymbol{\epsilon}^2_{t,T})\right) \to \boldsymbol{B}^\star,$$

*with $\boldsymbol{B}^\star$ positive definite.*

Assumptions 2, 4 and 14 are sufficient for $\boldsymbol{B}_T = O(1)$ (cf. Lemma 4 in the Appendix); Assumption 15 further ensures that $\boldsymbol{B}_T$ converges to a positive definite limit.

**Theorem 3.** *Suppose Assumptions 1 – 8, 11, 14 and 15 hold. Then $\sqrt{T}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^\star) \xrightarrow{D} \mathcal{N}(\boldsymbol{0}, [\boldsymbol{A}^\star]^{-1}\boldsymbol{B}^\star[\boldsymbol{A}^\star]^{-1})$.*

We can consistently estimate $\boldsymbol{B}^\star$ in the same manner as $\boldsymbol{B}_m^\star$. Let

$$\begin{aligned}\widehat{\boldsymbol{B}}_T &\coloneqq \sum_{j=-T}^{T} k(j/\kappa_T)\widehat{\boldsymbol{\Gamma}}_T(j),\\ \widehat{\boldsymbol{\Gamma}}_T(j) &\coloneqq \frac{1}{T}\sum_{t=1}^{T-j} \dot{\boldsymbol{l}}_{t,T}(\widehat{\boldsymbol{\theta}}, \boldsymbol{\epsilon}_{t,T}^2)\dot{\boldsymbol{l}}_{t,T}(\widehat{\boldsymbol{\theta}}, \boldsymbol{\epsilon}_{t+j,T}^2)' \quad (j \geq 0),\\ \widehat{\boldsymbol{\Gamma}}_T(j) &\coloneqq \widehat{\boldsymbol{\Gamma}}_T(-j)' \quad (j < 0).\end{aligned} \tag{11}$$

where the $k(\cdot)$'s are kernel weights, and $\kappa_T$ is the bandwidth. We replace Assumptions 12 and 13 by Assumptions 16 and 17 below.

**Assumption 16.** *$k \in \mathscr{K}$, with $\mathscr{K}$ defined as in Assumption 12.*

**Assumption 17.** *$\kappa_T \to \infty$ and $\kappa_T = o(T^{1/2})$.*

**Proposition 2.** *Suppose Assumptions 1 – 8, 11 and 14 – 17 hold. Then $\widehat{\boldsymbol{B}}_T \xrightarrow{P} \boldsymbol{B}^\star$.*

## 3 Examples of $g_{t,T}$

Here we provide examples of $g_{m,t,T}(\boldsymbol{\theta}_m)$ and derive verifiable conditions that ensure the high-level Assumptions 5 and 8 hold. Our choice of examples comprise the most common classes of parametric versions of $g_{m,t,T}$. In Section 3.1 we establish conditions for the piecewise constant class of models. Specifications within this class have proved particularly useful in testing whether a break occurred at a known time-point due to, say, a policy decision, and in quantifying that break. The piecewise constant model is especially popular among practitioners due to its ease of estimation and interpretability. In Section 3.2 we establish conditions for the most common variant of smooth transition models. In contrast to the piecewise constant model, smooth transition models allow breaks to be gradual or "smooth". Moreover, in addition to an estimate of the (total) break-size, the smooth transition model also provides estimates of the locations of the breakpoints, and estimates of the speed of transitions. Sums of smooth transition terms are thus capable of providing a detailed and interpretable characterisation of virtually any dynamics of $g_{m,t,T}$. The smooth transition model can also be viewed as a generalisation of the piecewise constant model, since the latter is obtained as a special case (in the limit) when the speed of transition parameters tend to infinity. Finally, in Section 3.3 we establish conditions for a class of splines that have proved particularly useful and flexible in the econometric literature.

### 3.1 Piecewise constant models

Van Bellegem and Von Sachs (2004) specify $g_{m,t,T}$ as piecewise constant. This amounts to

$$g_{m,t,T}(\boldsymbol{\theta}_m) = \delta_{m,0} + \sum_{l=1}^{s_m} \delta_{m,l} \cdot I(t/T \geq c_{m,l}), \qquad \boldsymbol{\theta}_m = (\delta_{m,0}, \delta_{m,1}, \dots, \delta_{m,s_m})', \tag{12}$$

where $I(A)$ is an indicator function equal to 1 if $A$ holds and 0 otherwise. The values of the possible break-locations $c_{m,1}, \dots, c_{m,s}$ are thus known and not estimated. To estimate $\boldsymbol{\theta}_m$, Van Bellegem and Von Sachs (2004) proposed the sample variance of each constant period. This does not allow for the joint estimation and inference of multiple break-sizes. Our results, by contrast, permit this.

The log-linear version of a piecewise constant specification (cf. Escribano and Sucarrat 2018) is given by

$$\ln g_{m,t,T}(\boldsymbol{\theta}_m) = \delta_{m,0} + \sum_{l=1}^{s_m} \delta_{m,l} \cdot I(t/T \geq c_{m,l}), \qquad \boldsymbol{\theta}_m = (\delta_{m,0}, \delta_{m,1}, \dots, \delta_{m,s_m})'. \tag{13}$$

Notice that (12) can always be re-written as (13). The advantage of this is that non-negativity constraints on $\boldsymbol{\theta}_m$ are not needed in (13). This simplifies estimation and inference under the null hypothesis that one or more of the coefficients are zero. The following result ensures that the high-level Assumptions 5 and 8 hold.

**Proposition 3.** *Suppose $g_{mt,T}(\boldsymbol{\theta}_m)$ is given by (13) with $c_{m,0} < c_{m,1} < \cdots < c_{m,s_m} < c_{m,s_m+1}$, where $c_{m,0} = 0$ and $c_{m,s_m+1} = 1$. Suppose further that Assumption 1 holds, that $\boldsymbol{\theta}_m^\star \in \boldsymbol{\Theta}_m$, that $\boldsymbol{\Theta}_m^*$ is an open, bounded and convex set that contains $\boldsymbol{\Theta}_m$, that Assumption 4 holds, and that $\mathcal{V}_m$ in Assumption 8 is contained in $\boldsymbol{\Theta}_m$. Then Assumptions 2 and 8 hold, and the limit $L_m(\boldsymbol{\theta}_m)$ in Assumption 5 exists and attains a unique minimum at $\boldsymbol{\theta}_m^\star \in \boldsymbol{\Theta}_m^*$.*

### 3.2 Smooth transition models

A variety of smooth transition models have been considered, see Amado and Teräsvirta (2013) for a survey. Amado and Teräsvirta (2013) consider the following in more detail:

$$g_{m,t,T}(\boldsymbol{\theta}_m) = \delta_{m,0} + \sum_{l=1}^{s_m} \frac{\delta_{m,l}}{1 + \exp\big(-\gamma_{m,l}(t/T - c_{m,l})\big)}, \tag{14}$$

where $\boldsymbol{\theta}_m = (\boldsymbol{\delta}_m', \boldsymbol{\gamma}_m', \boldsymbol{c}_m')'$ with $\boldsymbol{\delta}_m = (\delta_{m,0}, \delta_{m,1}, \ldots, \delta_{m,s_m})'$, $\boldsymbol{\gamma}_m = (\gamma_{m,1}, \ldots, \gamma_{m,s_m})'$ and $\boldsymbol{c}_m = (c_{m,1}, \ldots, c_{m,s_m})'$. For $l = 1, \ldots, s_m$, the $\delta_{m,l}$ is the total size of break $l$, $\gamma_{m,l}$ is the speed of transition of break $l$, $c_l$ is the centre of break location $l$ and $s_m$ is the number of breaks. There are no breaks if $\delta_{m,1} = \cdots = \delta_{m,s_m} = 0$. Note that, for Assumption 5 to hold, the $\delta_{m,l}$'s and $\gamma_{m,l}$'s must all differ from zero. The following result ensures that the high-level Assumptions 5 and 8 hold.

**Proposition 4.** *Suppose $g_{m,t,T}(\boldsymbol{\theta}_m)$ is given by (14) with $\delta_{m,1} \neq 0, \ldots, \delta_{m,s_m} \neq 0$, with $\gamma_{m,1} \neq 0, \ldots, \delta_{m,s_m} \neq 0$, and with $c_{m,0} < c_{m,1} < \cdots < c_{m,s_m} < c_{m,s_m+1}$ where $c_{m,0} = 0$ and $c_{m,s_m+1} = 1$. Suppose further that Assumption 1 holds, that $\boldsymbol{\theta}_m^\star \in \boldsymbol{\Theta}_m$, that $\boldsymbol{\Theta}_m^*$ is an open, bounded and convex set that contains $\boldsymbol{\Theta}_m$, that Assumption 4 holds, and that $\mathcal{V}_m$ in Assumption 8 is contained in $\boldsymbol{\Theta}_m$. Then Assumptions 2 and 8 hold and the limit $L_m(\boldsymbol{\theta}_m)$ in Assumption 5 exists. Moreover, if the Hessian $\ddot{\boldsymbol{L}}_m(\boldsymbol{\theta}_m)$ is positive definite on $\boldsymbol{\Theta}_m^*$, $L_m(\boldsymbol{\theta}_m)$ attains a unique minimum at $\boldsymbol{\theta}_m^\star \in \boldsymbol{\Theta}_m^*$.*

Establishing conditions under which $\ddot{\boldsymbol{L}}_m(\boldsymbol{\theta}_m)$ is positive definite is tedious, even when there is only one transition ($s_m = 1$). However, numerical verification is straightforward.

### 3.3 Splines

Engle and Rangel (2008), and Brownlees and Gallo (2010), specify $g_{m,t,T}$ as a deterministic spline. The former use Gaussian ML for estimation, whereas the latter employs penalised ML. However, no asymptotic results are established in either work. Zhang et al. (2020) derive asymptotic results for a least squares estimator of B-splines.

Splines that are suitably expressed in terms of re-scaled time can satisfy Assumptions 2 and 8. An example is the exponential quadratic spline function considered by Engle and Rangel (2008) (without regressors). If we remove the trend and replace nominal time with re-scaled time, then we obtain

$$\ln g_{m,t,T}(\boldsymbol{\theta}_m) = \delta_{m,0} + \sum_{l=1}^{s_m} \delta_{m,l}(t/T - c_{m,l})^2 I(t/T \geq c_{m,l}), \quad \boldsymbol{\theta}_m = (\delta_{m,0}, \delta_{m,1}, \ldots, \delta_{m,s_m})', \quad (15)$$

where $I(A)$ is an indicator function equal to 1 if $A$ holds and 0 otherwise, and the $c_{m,l}$'s are given knot-locations that are assumed known and therefore not estimated. The value $s_m$ is the number of knots, and $\delta_{m,1}, \ldots, \delta_{m,s_m}$ are the knot-coefficients. Large values of $s_m$ imply more frequent cycles, and the sharpness of each cycle is governed by the knot-coefficients. The following result ensures that the high-level Assumptions 5 and 8 hold.

**Proposition 5.** *Suppose $g_{t,T}(\boldsymbol{\theta})$ is given by (15) with $c_0 < c_1 < \cdots < c_s < c_{s+1}$, where $c_0 = 0$ and $c_{s+1} = 1$. Suppose further that Assumption 1 holds, that $\boldsymbol{\theta}_m^{\star} \in \boldsymbol{\Theta}_m$, that $\boldsymbol{\Theta}_m^{*}$ is an open, bounded and convex set that contains $\boldsymbol{\Theta}_m$, that Assumption 4 holds, and that $\mathcal{V}_m$ in Assumption 8 is contained in $\boldsymbol{\Theta}_m$. Then Assumptions 2 and 8 hold and the limit $L_m(\boldsymbol{\theta}_m)$ in Assumption 5 exists. Moreover, if the Hessian $\ddot{\boldsymbol{L}}_m(\boldsymbol{\theta}_m)$ is positive definite on $\boldsymbol{\Theta}_m^{*}$, $L_m(\boldsymbol{\theta}_m)$ attains a unique minimum at $\boldsymbol{\theta}_m^{\star} \in \boldsymbol{\Theta}_m^{*}$*

## 4 Estimation of conditional volatility

In empirical applications, it is often of interest to obtain estimates of the full conditional covariance matrix $E(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'|\mathcal{F}_{t-1})$, where $\mathcal{F}_{t-1} = \sigma\{\boldsymbol{\epsilon}_u, u < t\}$. The conditional volatilities, the $\sigma_{m,t}^2$'s with $\sigma_{m,t}^2 = g_{m,t}h_{m,t}$, are on the diagonal of this matrix. In portfolio analysis, under the unpredictability of returns assumption $E(\boldsymbol{\epsilon}_t|\mathcal{F}_{t-1}) = \mathbf{0}$, the matrix must be positive definite to ensure the conditional variance (i.e. a measure of risk) of a weighted portfolio of asset returns is non-negative. In this case, the conditional covariance matrix can be written as

$$E(\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'|\mathcal{F}_{t-1}) = \boldsymbol{G}_t^{1/2}\boldsymbol{H}_t^{1/2}\boldsymbol{R}_t\boldsymbol{H}_t^{1/2}\boldsymbol{G}_t^{1/2}, \tag{16}$$

where $\boldsymbol{G}_t^{1/2} := \text{diag}\big(g_{1,t}^{1/2}, \ldots, g_{M,t}^{1/2}\big)$, $\boldsymbol{H}_t^{1/2} := \text{diag}\big(h_{1,t}^{1/2}, \ldots, h_{M,t}^{1/2}\big)$ and $\boldsymbol{R}_t$ is a conditional correlation matrix that is either constant or time-varying. Both $\boldsymbol{G}_t^{1/2}$ and $\boldsymbol{H}_t^{1/2}$ are positive definite if their diagonal elements are strictly positive, and $E(\boldsymbol{\phi}_t\boldsymbol{\phi}_t'|\mathcal{F}_{t-1}) = \boldsymbol{H}_t^{1/2}\boldsymbol{R}_t\boldsymbol{H}_t^{1/2}$. As a consequence, if $\boldsymbol{R}_t$ is also positive definite, then $E(\boldsymbol{\phi}_t\boldsymbol{\phi}_t'|\mathcal{F}_{t-1})$ and (16) are also positive definite.[9]

Given first step estimates of the $g_{m,t}$'s, estimates of the $h_{m,t}$'s can be obtained in a second step. While, in principle, any model can be fitted in a second step under suitable assumptions, here we study the second step estimation of the scaled GARCH(1,1) specification under both correct and incorrect specification. In other words, $g_{m,t}h_{m,t}$ constitutes a prediction of $\sigma_{m,t}^2 = g_{m,t}E(\phi_{m,t}^2|\mathcal{F}_{t-1})$, where $h_{m,t}$ is either correctly or incorrectly specified for $E(\phi_{m,t}^2|\mathcal{F}_{t-1})$. We

[9] If two square matrices of the same size $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite, then also $\boldsymbol{ABA}$ is positive definite.

also outline how Dynamic Conditional Correlations (DCCs) can be estimated in a third step while ensuring positive definiteness of (16).

### 4.1 Estimation of a scaled GARCH(1,1) model

In the first part of this subsection, the properties of $\{\phi^2_{m,t,T}\}$ do not vary with $T$. To emphasise this, we omit the subscript $T$. Suppose $\phi^2_{m,t} := \epsilon^2_t / g_{m,t}(\boldsymbol{\theta}^\star_m)$ is governed by a strictly stationary scaled GARCH(1,1), where $\boldsymbol{\theta}^\star_m$ is the true parameter. This means

$$\phi^2_{m,t} = h_{m,t}\eta^2_{m,t}, \qquad E(\eta^2_{m,t}|\mathcal{F}_{m,t-1}) = 1, \tag{17}$$

$$h_{m,t} = \omega^\star_m + \alpha^\star_m \phi^2_{m,t-1} + \beta^\star_m h_{m,t-1}, \qquad \omega^\star_m, \alpha^\star_m, \beta^\star_m > 0, \quad \omega^\star_m = 1 - \alpha^\star_m - \beta^\star_m, \tag{18}$$

for all $T$, where $\mathcal{F}_{m,t-1} = \sigma\{\phi^2_{m,u}, u < t\}$. It is the condition $\omega^\star_m = (1 - \alpha^\star_m - \beta^\star_m)$ in (18) which makes the GARCH(1,1) a scaled version, i.e. $E(\phi^2_{m,t}) = 1$ for all $t$. An implication of the condition is that only two parameters need to be estimated in the second step, namely $\boldsymbol{\vartheta}^\star_m := (\alpha^\star_m, \beta^\star_m)'$. In the infeasible case, $\{\phi^2_{m,t}\}$ is observed and the consistency of the standard GARCH QMLE follows trivially under suitable assumptions, see Appendix D.1. In the feasible case, the second step QMLE estimator is similar – but *not* identical – to the target variance estimator of Francq et al. (2011). There, $g_{m,t}$ is constant (i.e. $g_{m,t} = g_m$ for all $t$), and the asymptotic variance can differ from that of the Ordinary QMLE. Here, the recursive parametrisation differs from that of the target variance estimator. The feasible second step QMLE of $\boldsymbol{\vartheta}^\star_m = (\alpha^\star_m, \beta^\star_m)'$ is

$$\widehat{\boldsymbol{\vartheta}}_{m,T} = \underset{\boldsymbol{\vartheta}_m \in \boldsymbol{\Xi}_m}{\arg\min} \, \frac{1}{T}\sum_{t=1}^{T} \ln \widehat{h}_{m,t} + \frac{\widehat{\phi}^2_{m,t}}{\widehat{h}_{m,t}},$$

where $\boldsymbol{\vartheta}_m = (\alpha_m, \beta_m)'$, $\widehat{\phi}^2_{m,t} = \frac{\epsilon^2_{m,t}}{\widehat{g}_{m,t}}$ and $\widehat{h}_{m,t} = (1 - \alpha_m - \beta_m) + \alpha_m \widehat{\phi}^2_{m,t-1} + \beta_m \widehat{h}_{m,t-1}$. Appendix D.2 contains the simulation results of the Two-step QMLE. Based on the results a reasonable conjecture is that the asymptotic properties of the two-step QMLE are the same as those of the ordinary QMLE, both when $g_{m,t}$ is constant and when it is time-varying, in the experiments investigated.

We now turn to the case where $\{\phi^2_{m,t,T}\}$ is not governed by a GARCH, where it is not necessarily strictly stationary, and where the properties of $\{\phi^2_{m,t,T}\}$ can vary with $T$. In this case, a scaled GARCH(1,1) specification provides mis-specified predictions of volatility. However, even though the predictions are generated by a mis-specified model, they nevertheless possess several desirable properties that are typically associated with the predictions of a correctly specified conditional expectation. First, the prediction is unbiased for volatility in the unconditional sense, just *as if* it were the correct specification. To see this, suppose $\{\phi^2_{m,t,T}\}$ is not governed by a GARCH(1,1), and let

$$h_{m,t,T} = \omega_m + \alpha_m \phi^2_{m,t-1,T} + \beta_m h_{m,t-1,T}, \qquad \omega_m, \alpha_m, \beta_m > 0, \quad \omega_m = 1 - \alpha_m - \beta_m, \tag{19}$$

denote the scaled GARCH(1,1) prediction, where $\alpha_m$ and $\beta_m$ are real-valued parameters that satisfy the constraints in (19). It is straightforward to verify by backwards recursion that, for any pair $(\alpha_m, \beta_m)$ that satisfies the parameter constraints in (19),

$$E(h_{m,t,T}) \;=\; \frac{\omega_m}{1-\beta_m} + \alpha_m \sum_{i=1}^{\infty} \beta_m^{i-1} E(\phi^2_{m,t-i,T}) \;=\; 1 \qquad \text{for all } t \text{ and } T.$$

Accordingly, $g_{m,t,T} h_{m,t,T}$ is unbiased for $E(\epsilon^2_{m,t,T})$ in the unconditional sense, since $E(g_{m,t,T} h_{m,t,T}) = g_{m,t,T} = E(\epsilon^2_{m,t,T})$ for all $t$ and $T$. We emphasise that $E(h_{m,t,T}) = 1$ holds under certain types of non-stationarities of $\{\phi^2_{m,t,T}\}$, e.g. when the zero process is non-stationary (as in the illustrations in Sections 5.2 and 5.3). A second desirable property that characterises the scaled GARCH(1,1) prediction is that the $S$-steps-ahead prediction satisfies $\lim_{S\to\infty} E(h_{m,t+S,T}) = E(h_{m,t,T}) = 1$ for all $t$ and $T$, just as if $h_{m,t,T}$ were the true DGP. Finally, a third desirable property the scaled GARCH(1,1) predictions possess under suitable regularity conditions when $(\alpha_m, \beta_m)$ are estimated by QML, is weak identification in the sense of Sucarrat (2021), i.e. $E(\phi^2_{m,t,T}/h_{m,t,T}) = 1$, see exercise 7.6 in Francq and Zakoïan (2019). In other words, under mis-specification, QML estimation under suitable assumptions ensures a necessary condition for weak identification holds.

Finally, it is worth mentioning that the moment estimators of Kristensen and Linton (2006) can also be used to estimate the parameters of scaled GARCH(1,1) specification, both under correct and incorrect specifications. The details of this is contained in Section D.3 of the appendix.

### 4.2 Estimation of conditional correlations

Let $\eta_{m,t} = \epsilon_{m,t}/\sqrt{g_{m,t}h_{m,t}}$, $m = 1, \ldots, M$, and let $\boldsymbol{\eta}_t = (\eta_{1,t}, \ldots, \eta_{M,t})'$. Accordingly, $\boldsymbol{\epsilon}_t = \boldsymbol{G}_t^{1/2}\boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t$ and $E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t'|\mathcal{F}_{t-1}) = \boldsymbol{R}_t$. Note also that $Corr(\boldsymbol{\epsilon}_t|\mathcal{F}_{t-1}) = \boldsymbol{R}_t$ under the assumption that $E(\boldsymbol{\epsilon}_t|\mathcal{F}_{t-1}) = \mathbf{0}$ for all $t$. In applications, an estimator of $\boldsymbol{R}_t$ can be built with the standardised residuals $\widehat{\eta}_{m,t}$, where $\widehat{\eta}_{m,t} = \epsilon_{m,t}/\sqrt{\widehat{g}_{m,t}\widehat{h}_{m,t}}$. If $\boldsymbol{R}_t$ is constant over time, for example, the natural estimator is the sample estimator $\widehat{\boldsymbol{R}} = T^{-1}\sum_{t=1}^{T}\widehat{\boldsymbol{\eta}}_t\widehat{\boldsymbol{\eta}}_t'$, where $\widehat{\eta}_{m,t} = \epsilon_{m,t}/\sqrt{\widehat{g}_{m,t}\widehat{h}_{m,t}}$, $m = 1, \ldots, M$. If $\boldsymbol{R}_t$ is time-varying, then a natural candidate is the Dynamic Conditional Correlations (DCCs) specification of Engle (2002) or Aielli (2013). Under mis-specification, $\boldsymbol{R}_t$ must be interpreted as a prediction that is not necessarily equal to $E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t'|\mathcal{F}_{t-1})$. Finally, note that our framework is compatible with time-varying unconditional correlations $E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t')$.

## 5 Numerical illustrations

### 5.1 An efficiency comparison

A question of practical interest is how the efficiency of the Two-step QMLE discussed in Section 4.1 compares with that of the Iterative QMLE proposed by Amado and Teräsvirta (2013), since the latter is substantially more demanding computationally. The latter is also more prone to the "curse of dimensionality" in multiple equation specifications. In single equation specifications, Step 1 of the Two-step QMLE coincides with the first-step of the first iteration of the Iterative QMLE. Table 1 contains the simulation results from a comparison, where the parameter values of the DGP correspond to what is commonly found empirically. The upper part of Table 1 contains the results of the $g_t$ parameters, whereas the lower part contains the results of the $h_t$ parameters. For the $g_t$ parameters, the Iterative QMLE is not always more efficient for $T \leq 10000$. As the sample size grows very large, however, the results suggest the Iterative QMLE is slightly more efficient. For the $h_t$ parameters, the numerical efficiency of the two estimators is similar across all sample sizes. Interestingly, the standard errors are very close to the asymptotic standard errors of the infeasible QMLE (see Appendix D.2), which suggests the prior estimation of the $g_t$ parameters does not affect the efficiency of the $h_t$ parameters in a second step.

## 5.2 Daily return with a non-stationary zero-process

An attractive feature of our estimator is that the stochastic component $\phi_t^2$ need not be stationary. To illustrate this, we revisit one of the daily stock returns investigated by Sucarrat and Grønneberg (2022). Eros International plc. (EROS) was an Indian multinational mass media conglomerate (a "Bollywood" company) that merged with the US company STX Entertainment in April 2020. The left graph of Figure 1 depicts the daily returns at the New York Stock Exchange (NYSE) from 21 December 2009 to 4 February 2019 ($T = 2295$). The datasource is Bloomberg. In the beginning of the period the primary listing of the stock was in India. This explains all the zeros until November 2013. Thereafter, there are few zeros. The return series thus exhibits a clear break in the unconditional zero-probability, so the zero-process is non-stationary over the sample. The return process $\epsilon_t$ and the transformation $\phi_t^2 = \epsilon_t^2/E(\epsilon_t^2)$ are therefore also non-stationary. Again, to keep notation simple, we suppress the subscripts $m$ (since $m = 1$) and $T$.

Interestingly, the 500-day moving average of squared return in the right graph of Figure 1 does not suggest in a clear way that there is a break in the unconditional volatility $E(\epsilon_t^2)$ in November 2013. Instead, the graph suggests the break or breaks occur later, namely in October 2015 and in October 2017. To illustrate the estimation of a piecewise constant log-linear specification $g_t$, we use it to investigate whether there are breaks at the aforementioned points of time. More precisely, the data suggest the possible break-locations are 11 November 2013, 14 October 2015 and 6 October 2017, respectively. In terms of re-scaled time these correspond to $(c_1, c_2, c_3)' = (0.427, 0.638, 0.855)$. Our estimated model is

$$\widehat{\ln g_t} = \underset{(0.4174)}{1.795} + \underset{(0.4355)}{0.351}\, I(t/T \geq c_1) + \underset{(0.2342)}{1.215}\, I(t/T \geq c_2) - \underset{(0.2545)}{0.912}\, I(t/T \geq c_3).$$

The numbers in parentheses are the standard errors of the estimates. These are computed as the square root of the diagonal of $\widehat{\mathbf{\Sigma}}/T$, where $\widehat{\mathbf{\Sigma}} = \widehat{\boldsymbol{A}}^{-1}\widehat{\boldsymbol{B}}\widehat{\boldsymbol{A}}^{-1}$ is the estimate of the asymptotic coefficient covariance. A Bartlett kernel is used in the computation of $\widehat{\boldsymbol{B}}$, and the truncation lag is obtained as the integer part of $4(T/100)^{(2/9)}$. The $t$-ratios of the break-size estimates are 0.806, 5.181 and $-3.583$, respectively. So two-sided $t$-tests at common significance levels (i.e. 10%, 5% and 1%) suggest there are breaks at $c_2$ and $c_3$, but not at $c_1$. Finally, the second step QMLE (see Section 4.1) returns an estimated scaled GARCH(1,1) specification equal to $\widehat{h}_t = 0.873 + 0.127\widehat{\phi}_{t-1}^2 + 0.000\widehat{h}_{t-1}$ with $T^{-1}\sum_{t=1}^{T}\widehat{\phi}_t^2/\widehat{h}_t = 1.044$. In other words, the optimal scaled GARCH(1,1) prediction – optimal in the sense that it is both unbiased unconditionally and satisfies the necessary condition for weak identification – is characterised by an ARCH parameter aqual to 0.127, and a GARCH parameter close to zero. Second step estimation with the moment method (see the appendix) gives estimates that violate the parameter conditions, and the value $T^{-1}\sum_{t=1}^{T}\widehat{\phi}_t^2/\widehat{h}_t$ is far from 1 (i.e. the necessary condition for weak identification fails). So we do not report these estimates.

### 5.3 Time-varying intraday periodic volatility

To model intraday periodic volatility, Andersen and Bollerslev (1997), and Mazur and Pipien (2012) specify $g_t(\boldsymbol{\theta})$ as a Fourier Flexible Form (FFF) in terms of nominal time $t$. No asymptotic results are established in either work. More recently, He et al. (2019), and He et al. (in press), establish asymptotic results for two classes of periodic smooth transition models. But they do so under restrictive assumptions on $\phi_t$ (it is assumed *iid* normal in the former, conditionally *iid* normal in the latter). In our case, by contrast, it can be substantially dependent in unknown ways, both intradaily and interdaily. Also, it can be non-stationary. To illustrate our results, we use the vector of seasons representation to model the evolution of intraday unconditional volatility. In effect, our EBE estimator becomes a "period-by-period" estimator (see e.g. Escribano and Sucarrat 2018).

The common practice of estimating the intraday unconditional volatilities with cross-day averages of squared return is a special case of period-by-period estimation via the vector of seasons representation. Consider, for example, the intraday returns $\epsilon_{m,t}$, $m = 1, \ldots, M$, of day $t$. Often, the sample averages $T^{-1}\sum_{t=1}^{T}\epsilon_{m,t}^2$, $m = 1, \ldots, M$, are used to estimate the intraday unconditional volatilities $E(\epsilon_{1,t}^2), \ldots, E(\epsilon_{M,t}^2)$. The collection of sample averages is a special case of the period-by-period estimator. But it is only consistent in the special case where the unconditional intraday volatilities are constant across days, i.e. for each $m = 1, \ldots, M$ we have $E(\epsilon_{m,t_1}^2) = E(\epsilon_{m,t_2}^2)$ for all $t_1, t_2$. By contrast, period-by-period estimation as sketched here can also be used to estimate unconditional intraday volatilities that vary across days. Again, to simplify notation, we suppress the subscript $T$.

For illustration we use intraday hourly USD/EUR exchange rate returns. Let $S_{m,t}$ denote the exchange rate at the end of hour $m$ in day $t$, and let $\epsilon_{m,t} = 100^2 \cdot (\ln S_{m,t} - \ln S_{m-1,t})$ denote the hour $m$ log-return denominated in basis points. The left graph of Figure 2 plots the hourly returns at Forexite (`https://www.forexite.com`), a currency trading platform, from 2 January 2017 to 31 December 2018. This corresponds to 12 184 hourly returns. Only trading days are included in the sample (i.e. weekends are excluded), and a trading day contains $M = 24$ returns. The first return of a trading day covers the interval from 00:00 CET to 01:00 CET, whereas the last covers 23:00 CET to 00:00 CET. The right graph of Figure 2 contains the sample averages of squared returns across days, i.e. $T_m^{-1} \sum_{t=1}^{T_m} \epsilon_{m,t}^2$, where $T_m$ is the number of observations available for period $m$. As is clear from the graph, the intraday hourly unconditional volatility is time-varying. It is at its lowest at the end of the day at 24h CET, and it is at its highest at 15h CET.

To shed light on whether the intraday unconditional volatilites are constant across days, we estimate a quadratic spline function similar to that of Engle and Rangel (2008) with re-scaled time and four knots at equidistant locations, i.e.

$$\ln g_{m,t} = \delta_{m,0} + \sum_{l=1}^{4} \delta_{m,l}(t/T - c_l)^2 I(t/T \geq c_l), \qquad (c_1, c_2, c_3, c_4) = (0.2, 0.4, 0.6, 0.8),$$

for each period $m = 1, \ldots, M$. Table 2 contains the estimation results together with a Wald-test of $H_0 : \delta_{m,1} = \cdots = \delta_{m,4} = 0$. Under the null the unconditional volatility of period $m$ is thus constant and equal to $g_{m,t} = \exp(\delta_{m,0})$ for all $t$. The $p$-values of the test are contained in the square brackets of the last column. Out of the 24 tests, 8 reject the null at the 5% significance level, and 4 reject the null at 1%. Without time-varying period $m$ volatilities, we should on average expect 1.2 rejections at 5%, and 0.24 rejections at 1%. Accordingly, the results support the hypothesis that some of the unconditional intraday volatilites are time-varying across days.

Since $E(\phi_{m,t}^2) = 1$ for all $m$ and $t$, the intraday or "cross-sectional" scaled GARCH(1,1) prediction $h_{m,t} = \omega + \alpha\phi_{m-1,t} + \beta h_{m-1,t}$ is well-defined and characterised by the properties sketched in Section 4.1. In other words, it is straightforward to estimate a single, scaled GARCH(1,1) prediction of volatility for both within and across days, even when the $g_{m,t}$'s are time-varying and the $\phi_t^2$'s are non-stationary. The QML estimated specification is $\widehat{h}_{m,t} = 0.106 + 0.052\widehat{\phi}_{m-1,t}^2 + 0.8418\widehat{h}_{m-1,t}$ with $(T \cdot M)^{-1} \sum_{t=1}^{T} \sum_{m=1}^{M} \widehat{\phi}_{m,t}^2 / \widehat{h}_{m,t} = 1.0004$. Second step estimation with the moment method (see the appendix), by contrast, returns estimates that violate the parameter conditions, and the value $(T \cdot M)^{-1} \sum_{t=1}^{T} \sum_{m=1}^{M} \widehat{\phi}_{m,t}^{2}/\widehat{h}_{m,t}$ is not close to 1 (i.e. the necessary condition for weak identification fails). So we do not report these estimates.

# 6 Conclusions

We conclude by summarising our contributions. We derive a general and robust estimator of a large class of parametric models of time-varying unconditional volatility, both univariate and multivariate, and establish its consistency and asymptotic normality. Our estimator is based on the equation-by-equation version of the Gaussian QMLE, and it is characterised by several attractive properties. One is its ease of implementation, since the equation-by-equation nature of the estimator reduces the curse of dimensionality in multivariate models. Another attractive property is that the exact specification of the conditional volatility dynamics need not be known or estimated. However, in empirical applications, models of the conditional volatility dynamics can nevertheless be fitted in a second step, if desired. In particular, as we show, within our framework the scaled GARCH(1,1) is well-defined under both correct and incorrect specification, in both the univariate and multivariate cases. Our multivariate results can also be used to estimate non-stationary periodic volatility by framing the problem via the vector of seasons representation. This leads to a period-by-period estimator, whereby not only the variation in intraday unconditional volatility is modelled, but also the variation over days for each intraday period. Another novel property of our results is that they are valid when the zero-process of financial returns is non-stationary. This is important, since recent studies document that financial returns, both daily and intradaily, are widely characterised by a non-stationary zero-process. In the multivariate case, our results are also valid when the time-varying correlations are non-stationary, even when this is not due to a non-stationary zero-process. Next, due to the assumptions we rely upon, our results extend directly to the Multiplicative Error Model (MEM) interpretation of volatility models. Finally, we illustrated the usefulness of our results in three applications.

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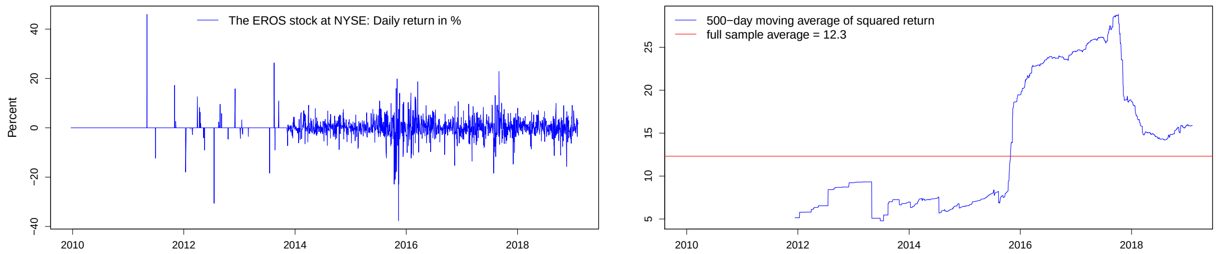


Figure 1: Daily log-returns in % of the EROS stock at NYSE (left) and 500-day moving average of squared returns (right), 21 December 2009 – 4 February 2021 (see Section 5.2). Datasource: Bloomberg

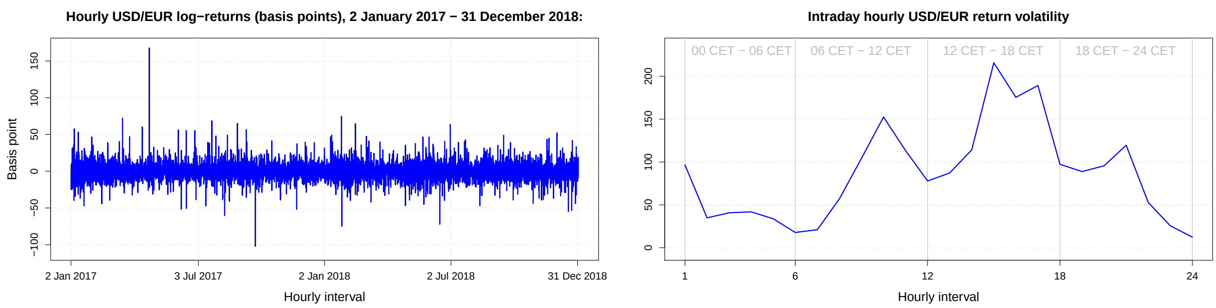


Figure 2: Hourly log-returns in basis points of the USD/EUR exchange rate (left) and estimates (assuming constancy over $t$) of its intraday hourly volatility (right), 2 January 2017 – 31 December 2018 (see Section 5.3). Datasource: Forexite

Table 1: Comparison of the Two-step QMLE and the Iterative QMLE of Amado and Teräsvirta (2013), see Section 5.1

| | $T$ | $m(\widehat{\delta}_0)$ | $se(\widehat{\delta}_0)$ | $m(\widehat{\delta}_1)$ | $se(\widehat{\delta}_1)$ | $m(\widehat{\gamma})$ | $se(\widehat{\gamma})$ | $m(\widehat{c})$ | $se(\widehat{c})$ |
|---|---|---|---|---|---|---|---|---|---|
| $g_t$: | Two-step QMLE (step 1): | | | | | | | | |
| | 2000 | −0.0306 | 0.1421 | 0.1928 | 0.7559 | 6.8617 | 19.326 | 0.0197 | 0.1180 |
| | 5000 | −0.0146 | 0.0933 | 0.0848 | 0.4203 | 1.6698 | 6.8159 | 0.0099 | 0.0686 |
| | 10000 | −0.0050 | 0.0569 | 0.0269 | 0.1961 | 0.6592 | 2.9829 | 0.0031 | 0.0339 |
| | 20000 | −0.0039 | 0.0386 | 0.0170 | 0.1260 | 0.2612 | 1.8592 | 0.0011 | 0.0237 |
| | 40000 | −0.0032 | 0.0262 | 0.0156 | 0.0898 | 0.0720 | 1.3271 | 0.0013 | 0.0167 |
| | 80000 | −0.0018 | 0.0171 | 0.0062 | 0.0572 | 0.0337 | 0.8697 | 0.0002 | 0.0111 |
| | Iterative QMLE: | | | | | | | | |
| | 2000 | −0.0380 | 0.1548 | 7.9542 | 145.1647 | 10.4640 | 40.8952 | 0.0240 | 0.1189 |
| | 5000 | −0.0206 | 0.1086 | 0.1939 | 2.4315 | 1.3270 | 10.5000 | 0.0089 | 0.0570 |
| | 10000 | −0.0088 | 0.0718 | 0.0248 | 0.2045 | 0.4501 | 2.6239 | 0.0030 | 0.0295 |
| | 20000 | −0.0055 | 0.0490 | 0.0127 | 0.1459 | 0.1787 | 1.6676 | 0.0015 | 0.0223 |
| | 40000 | −0.0037 | 0.0306 | 0.0106 | 0.0863 | 0.0476 | 1.1624 | 0.0010 | 0.0144 |
| | 80000 | −0.0018 | 0.0171 | 0.0044 | 0.0510 | 0.0177 | 0.7819 | 0.0003 | 0.0097 |

| | $T$ | $m(\widehat{\omega})$ | $se(\widehat{\omega})$ | $m(\widehat{\alpha})$ | $se(\widehat{\alpha})$ | $m(\widehat{\beta})$ | $se(\widehat{\beta})$ |
|---|---|---|---|---|---|---|---|
| $h_t$: | Two-step QMLE (step 2): | | | | | | |
| | 2000 | – | – | −0.0006 | 0.0197 | −0.0180 | 0.0525 |
| | 5000 | – | – | −0.0003 | 0.0121 | −0.0057 | 0.0271 |
| | 10000 | – | – | 0.0000 | 0.0087 | −0.0031 | 0.0192 |
| | 20000 | – | – | 0.0000 | 0.0059 | −0.0017 | 0.0135 |
| | 40000 | – | – | 0.0001 | 0.0042 | −0.0009 | 0.0092 |
| | 80000 | – | – | 0.0001 | 0.0032 | −0.0005 | 0.0068 |
| | Iterative QMLE: | | | | | | |
| | 2000 | 0.0189 | 0.0587 | 0.0000 | 0.0200 | −0.0195 | 0.0524 |
| | 5000 | 0.0069 | 0.0231 | 0.0000 | 0.0122 | −0.0064 | 0.0271 |
| | 10000 | 0.0037 | 0.0144 | 0.0001 | 0.0087 | −0.0034 | 0.0192 |
| | 20000 | 0.0022 | 0.0103 | 0.0001 | 0.0059 | −0.0019 | 0.0135 |
| | 40000 | 0.0011 | 0.0069 | 0.0001 | 0.0042 | −0.0010 | 0.0092 |
| | 80000 | 0.0006 | 0.0049 | 0.0001 | 0.0032 | −0.0006 | 0.0068 |

DGP: $\epsilon_t = \sqrt{g_t}\phi_t$, $\phi_t = \sqrt{h_t}\eta_t$, $\eta_t \overset{iid}{\sim} N(0,1)$, $h_t = 0.1 + 0.1\phi_{t-1}^2 + 0.8h_{t-1}$, $g_t = 0.5 + 1.5\big(1 + \exp(-10(t/T - 0.5))\big)^{-1}$. $T$, sample size. $m(\widehat{x})$, average bias of estimate $\widehat{x}$ across replications (no. of replications = 1000). $se(\widehat{x})$, sample standard deviation of estimate $\widehat{x}$ across replications. All computations in R (R Core Team, 2021). The Two-step QMLE is implemented with own code. The Iterative QMLE is implemented with the `tvgarch()` function of the CRAN package **tvgarch** (Campos-Martins and Sucarrat, 2024).

Table 2: Spline estimates of intraday hourly volatility (see Section 5.3)

| $m$ | $\widehat{\delta}_{m,0}$ (*s.e.*) | $\widehat{\delta}_{m,1}$ (*s.e.*) | $\widehat{\delta}_{m,2}$ (*s.e.*) | $\widehat{\delta}_{m,3}$ (*s.e.*) | $\widehat{\delta}_{m,4}$ (*s.e.*) | $T_m$ | $\chi^2(4)$ [$p$-value] |
|---|---|---|---|---|---|---|---|
| 1 | 5.299 (0.7500) | −15.810 (13.4496) | 21.374 (35.3855) | −6.012 (42.0653) | 53.597 (52.1279) | 515 | 21.853 [0.0004] |
| 2 | 3.775 (0.2522) | 1.133 (5.5681) | −11.494 (17.2062) | 9.289 (27.5929) | 53.769 (44.4827) | 516 | 13.659 [0.0085] |
| 3 | 3.643 (0.1799) | 2.215 (5.6773) | −8.954 (18.0857) | 18.192 (29.0796) | −22.852 (51.8098) | 516 | 0.518 [0.9717] |
| 4 | 3.881 (0.2042) | −1.513 (5.8472) | −0.526 (17.8325) | 10.394 (24.7791) | −31.197 (36.2248) | 516 | 3.739 [0.4425] |
| 5 | 3.374 (0.2480) | −3.385 (5.6155) | 17.916 (25.1050) | −22.709 (50.8935) | −58.255 (65.4081) | 516 | 11.252 [0.0239] |
| 6 | 2.735 (0.1639) | 4.456 (4.8265) | −13.627 (14.8996) | 17.628 (21.9421) | −6.619 (35.6283) | 516 | 2.678 [0.6131] |
| 7 | 2.828 (0.1880) | 2.119 (4.8305) | 2.422 (14.8866) | −18.812 (23.0809) | 24.475 (41.8878) | 516 | 4.935 [0.2940] |
| 8 | 3.857 (0.1761) | 0.895 (4.7521) | −5.108 (15.6562) | 27.629 (27.3992) | −85.692 (50.4148) | 515 | 4.604 [0.3303] |
| 9 | 4.521 (0.1046) | 1.519 (3.3635) | 1.150 (11.0441) | −10.082 (17.3547) | −6.229 (26.6335) | 516 | 9.899 [0.0422] |
| 10 | 5.017 (0.1083) | −4.345 (4.1078) | 20.107 (14.1018) | −44.509 (23.5979) | 86.918 (39.2136) | 517 | 5.068 [0.2804] |
| 11 | 4.639 (0.1883) | −6.359 (4.2830) | 31.118 (13.0861) | −61.819 (19.7582) | 82.749 (32.9971) | 517 | 12.851 [0.0120] |
| 12 | 4.270 (0.1340) | 3.207 (6.0620) | −9.070 (19.7664) | 7.739 (27.8529) | 8.046 (31.5354) | 517 | 1.285 [0.8638] |
| 13 | 4.557 (0.2465) | −1.464 (5.8333) | 4.095 (16.2241) | −6.101 (20.1854) | −3.786 (27.0236) | 517 | 6.345 [0.1748] |
| 14 | 4.543 (0.1308) | −2.035 (5.9661) | 19.024 (20.3574) | −47.267 (30.2696) | 52.408 (33.8309) | 517 | 6.326 [0.1761] |
| 15 | 5.459 (0.1745) | −2.838 (6.1909) | 10.719 (19.1024) | −19.411 (26.1782) | 11.935 (39.5786) | 517 | 4.168 [0.3838] |
| 16 | 5.119 (0.1370) | 2.594 (3.5312) | −9.415 (11.2713) | 11.557 (18.2850) | 10.993 (29.4824) | 517 | 4.806 [0.3078] |
| 17 | 5.409 (0.2083) | −4.002 (6.1470) | 10.676 (18.9849) | −19.079 (26.0639) | 51.860 (33.0372) | 517 | 6.545 [0.1620] |
| 18 | 4.525 (0.1408) | 2.129 (4.2691) | −5.967 (14.9249) | 8.191 (25.3200) | −22.307 (35.8577) | 517 | 1.757 [0.7804] |
| 19 | 4.538 (0.2072) | −8.461 (4.6763) | 28.788 (14.3697) | −45.468 (22.1037) | 89.723 (37.5709) | 517 | 11.391 [0.0225] |
| 20 | 4.812 (0.2824) | −12.499 (6.1907) | 42.377 (18.0686) | −66.247 (25.3002) | 78.278 (37.1904) | 515 | 7.317 [0.1201] |
| 21 | 4.762 (0.2543) | 12.607 (8.3223) | −51.998 (27.7935) | 68.146 (41.7346) | 13.428 (60.2769) | 516 | 13.748 [0.0081] |
| 22 | 4.247 (0.2998) | −9.138 (6.8886) | 26.477 (22.1289) | −37.022 (36.0346) | 50.927 (50.1777) | 516 | 3.522 [0.4746] |
| 23 | 3.192 (0.1530) | −0.677 (6.5806) | −1.918 (21.7796) | 17.689 (34.8812) | −34.275 (58.9946) | 412 | 3.196 [0.5257] |
| 24 | 2.203 (0.1849) | 20.118 (4.9578) | −68.402 (15.6431) | 80.915 (24.5484) | −18.442 (37.1556) | 413 | 34.318 [0.0000] |

The estimated model is $\ln g_{m,t} = \delta_{m,0} + \sum_{l=1}^{4} \delta_{m,l}(t/T - c_l)^2 I(t/T \geq c_l)$ with $(c_1, c_2, c_3, c_4)' = (0.2, 0.4, 0.6, 0.8)$. $m$, intraday period/hour. *s.e.*, standard error of estimate. $T$, number of observations. $\chi^2(4)$, the test statistic of a Wald-test with $H_0 : \delta_{m,1} = \cdots = \delta_{m,4} = 0$ ($p$-value in square brackets).

Supplemental appendix to:

# Robust Estimation and Inference for Time-varying Unconditional Volatility

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# A Proofs of main results

## A.1 Proof of Theorem 1

By Theorem 5.7 in van der Vaart (1998) it suffices to show (i) uniform convergence in probability of $L_{m,T}$ to $L_m$ over $\boldsymbol{\Theta}_m$ and (ii) that the minimiser of $L_m$ (over $\boldsymbol{\Theta}_m$) is well-separated, i.e. for each $\varepsilon > 0$, $\inf_{\boldsymbol{\theta}_m:\|\boldsymbol{\theta}_m-\boldsymbol{\theta}_m^\star\|\geq\varepsilon} L_m(\boldsymbol{\theta}_m) > L_m(\boldsymbol{\theta}_m^\star)$.[4]

*(i) Uniform convergence in probability*

Let $\bar{L}_{m,T}(\boldsymbol{\theta}_m) := \frac{1}{T}\sum_{t=1}^T E\left[l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right]$. We first show that $L_{m,T}(\boldsymbol{\theta}_m) - \bar{L}_{m,T}(\boldsymbol{\theta}_m)$ converges to zero in probability, pointwise in $\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m$. Let $X_{t,T} := l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - E\left[l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right]$, $Y_{t,T} = X_{t,T}/T$ for $1 \leq t \leq T$ and zero otherwise, and $\mathcal{F}_{t,T} := \sigma(\{X_{i,T} : i \leq t\})$. We will apply Theorem 19.11 in Davidson (1994) to conclude that

$$\sum_{t=1}^T Y_{t,T} = \frac{1}{T}\sum_{t=1}^T X_{t,T} \xrightarrow{L_1} 0.$$

Application of this Theorem requires us to establish the following conditions:[5]

(i) $(Y_{t,T}/c_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ is uniformly integrable

(ii) $\text{limsup}_{T\to\infty}\sum_{t=1}^T < \infty$

(iii) $\lim_{T\to\infty}\sum_{t=1}^T c^2_{t,T} = 0$

(iv) $(Y_{t,T}, \mathcal{F}_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ is an $L_1$ mixingale with respect to $(c_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$.

(i) since $Y_{t,T}/c_{t,T}$ is either equal to $X_{t,T}$ $(1 \leq t \leq T)$ or zero (otherwise) it suffices to establish that $\kappa = \sup\{E|X_{t,T}|^p : 1 \leq t \leq T, T \in \mathbb{N}\} < \infty$ for a $p > 1$. Given the definition of $X_{t,T}$, it suffices to show that $E|l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})|^p$ is uniformly bounded over $1 \leq t \leq T$ and $T \in \mathbb{N}$. By the display immediately following equation (5)

$$l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) = \ln g_{m,t,T}(\boldsymbol{\theta}_m) + \frac{\epsilon^2_{m,t,T}}{g_{m,t,T}(\boldsymbol{\theta}_m)}$$

By Assumption 2 $g_{m,t,T}(\boldsymbol{\theta}_m)$ is uniformly bounded above and below for all $\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m^\star$ and all $t, T$. As such it is sufficient to establish that $\sup_{1\leq t\leq T, T\in\mathbb{N}} E|\epsilon^2_{m,t,T}|^p < \infty$. Since $\epsilon^2_{m,t,T} = g_{m,t,T}(\boldsymbol{\theta}_m^\star)\phi^2_{m,t,T}$, this follows from the uniform boundedness of $g_{m,t,T}(\boldsymbol{\theta}_m)$ along with part (ii) of Assumption 4.

(ii) follows from $\text{limsup}_{T\to\infty}\sum_{t=1}^T 1/T = 1$ and (iii) from $\sum_{t=1}^T (1/T)^2 = 1/T \to 0$ as $T \to \infty$.

(iv) The definition of an $L_1$ mixingale requires that (cf. Definition 16.5, p. 249 in Davidson (1994)) (a) for each $T$, the $\mathcal{F}_{t,T}$ are increasing in $t$, which is clear by their definition; and (b) that there exists a non-negative sequence $(\xi_l)_{l=0}^\infty$ satisfying $\xi_l \to 0$ as $l \to \infty$ such that

$$\begin{aligned} E|E(Y_{t,T}|\mathcal{F}_{t-l,T})| &\leq c_{t,T}\xi_l \\ E|Y_{t,T} - E(Y_{t,T}|\mathcal{F}_{t+l,T})| &\leq c_{t,T}\xi_{l+1} \end{aligned} \tag{20}$$

for all $1 \leq t \leq T$, $T \in \mathbb{N}$ and $l \geq 0$.[6] Since $Y_{t,T}$ is $\mathcal{F}_{t,T} \subset \mathcal{F}_{t+l,T}$ measurable, $E|Y_{t,T} - E(Y_{t,T}|\mathcal{F}_{t+l,T})| = 0$ and thus the second requirement in (20) places no restrictions on the sequence $\xi_l$.

[4]Theorem 5.7 in van der Vaart (1998) applies to maximisers. Multiplying by $-1$ in the appropriate places permits the application of the result under the conditions given.

[5]We refer the reader to Chapter 16 of Davidson (1994) for further details on mixingales.

[6]For $t \notin \{1, \ldots, T\}$, $Y_{t,T} = 0$ and so these requirements are automatically satisfied.

For the first part of (20), we use Theorems 14.1 & 14.2 in Davidson (1994). The argument used to prove Theorem 14.1 and Assumption 3 ensures that for each $T$, $(X_{t,T} : t \in \mathbb{Z})$ is $\alpha$-mixing with the same mixing coefficients $\alpha_T(k)$ as in Assumption 3. Since $E[X_{t,T}] = 0$ we may apply Theorem 14.2 to conclude that, for each $T$, each $1 \le t \le T$ and $m \ge 0$,

$$E|E(Y_{t,T}|\mathcal{F}_{t-l,T})| \le \frac{1}{T}E|E(X_{t,T}|\mathcal{F}_{t-l,T})| \le \frac{1}{T}2(2+1)\alpha_T(l)^{1-1/p}\kappa.$$

Taking $\xi_l = 6\kappa \sup_{T\in\mathbb{N}} \alpha_{m,T}(l)^{1-1/p}$ ensures that the first condition in (20) holds. By Assumption 3 $\sup_{T\in\mathbb{N}} \alpha_{m,T}(l) \to 0$ as $l \to \infty$ and hence (as $p > 1$) $\xi_l \to 0$ as $l \to \infty$. Thus $L_{m,T}(\boldsymbol{\theta}_m) - \bar{L}_{m,T}(\boldsymbol{\theta}_m)$ converges in probability to zero, pointwise in $\boldsymbol{\theta}_m$.

Next observe that for $\boldsymbol{\theta}_m, \boldsymbol{\theta}'_m \in \boldsymbol{\Theta}^*_m$, by the mean value theorem there is a $c \in [0,1]$ such that for $\boldsymbol{\theta}^\dagger_m := \boldsymbol{\theta}_m(1-c) + c\,\boldsymbol{\theta}'_m \in \Theta^*_m$,

$$\begin{aligned} &\left|l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - l_{m,t,T}(\boldsymbol{\theta}'_m, \epsilon^2_{m,t,T})\right| \\ &\quad\le \|\theta_m - \theta'_m\| \left[\frac{\left\|\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}^\dagger_m)\right\|}{|g_{m,t,T}(\boldsymbol{\theta}^\dagger_m)|} + \frac{\epsilon^2_{m,t,T}\left\|\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}^\dagger_m)\right\|}{g^2_{m,t,T}(\boldsymbol{\theta}^\dagger_m)}\right] \\ &\quad\le C\|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\| \times (1 + \epsilon^2_{m,t,T}), \end{aligned} \tag{21}$$

where $C$ is an absolute constant and the last line follows from Assumption 2. By (21) and Jensen's inequality

$$\begin{aligned} |\bar{L}_{m,T}(\boldsymbol{\theta}_m) - \bar{L}_{m,T}(\boldsymbol{\theta}'_m)| &\le \frac{1}{T}\sum_{t=1}^{T}\left|E\left[l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - l_{m,t,T}(\boldsymbol{\theta}'_m, \epsilon^2_{m,t,T})\right]\right| \\ &\le C\|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|\frac{1}{T}\sum_{t=1}^{T}E\left(1+\epsilon^2_{m,t,T}\right) \\ &\le \mathsf{L}\|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|, \end{aligned}$$

where the last line follows from $\sup_{1\le t\le T,\,T\in\mathbb{N}} E\epsilon^2_{m,t,T} < \infty$ which is ensured by Assumptions 2 and 4. Hence $\{\bar{L}_{m,T} : T \in \mathbb{N}\}$ is uniformly equicontinuous (on $\Theta^*_m$).

We next establish a Lipschitz–type bound on $L_{m,T}$. By (21)

$$\begin{aligned} |L_{m,T}(\boldsymbol{\theta}_m) - L_{m,T}(\boldsymbol{\theta}'_m)| &\le \frac{1}{T}\sum_{t=1}^{T}\left|l_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - l_{m,t,}(\boldsymbol{\theta}'_m, \epsilon^2_{m,t,T})\right| \\ &\le C\|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\| \times \frac{1}{T}\sum_{t=1}^{T}1 + \epsilon^2_{m,t,T}. \end{aligned}$$

Assumptions 2 and 4 together ensure that $\sup_{1\le t\le T,\,T\in\mathbb{N}} E\epsilon^2_{m,t,T} < \infty$ and so $\frac{1}{T}\sum_{t=1}^{T} 1+\epsilon^2_{m,t,T} = O_P(1)$ by Markov's inequality. This, along with the preceding paragraph, verifies Assumption SE-1 in Andrews (1992) and hence $\{L_{m,T} - \bar{L}_{m,T} : T \in \mathbb{N}\}$ is stochastically equicontinuous by Lemma 1 of Andrews (1992). In view of Assumption 1 and the pointwise convergence in probability previously established, Theorem 1 in Andrews (1992) therefore yields that

$$\sup_{\boldsymbol{\theta}_m\in\boldsymbol{\Theta}_m} |L_{m,T}(\boldsymbol{\theta}_m) - \bar{L}_{m,T}(\boldsymbol{\theta}_m)| \xrightarrow{P} 0.$$

By Assumption 5, $\bar{L}_{m,T} \to L_m$ pointwise on $\boldsymbol{\Theta}_m$. Since $\{\bar{L}_{m,T} : T \in \mathbb{N}\}$ is equicontinuous on $\boldsymbol{\Theta}_m$, this convergence is in fact uniform on $\boldsymbol{\Theta}_m$:

$$\sup_{\boldsymbol{\theta}_m\in\boldsymbol{\Theta}_m} |\bar{L}_{m,T}(\boldsymbol{\theta}_m) - L_m(\boldsymbol{\theta}_m)| \to 0.$$

Combination of the last two displays yields (i).

*(ii) Well-separated point of minimum*

By Assumption 5, $L_m$ has a unique minimiser. Since $\boldsymbol{\Theta}_m$ is compact by Assumption 1, the same is true of $\{\boldsymbol{\theta} \in \boldsymbol{\Theta}_m : \|\boldsymbol{\theta} - \boldsymbol{\theta}_m^\star\| \geq \varepsilon\}$ for any $\varepsilon > 0$. It is therefore sufficient to note that $L_m$ is continuous on $\boldsymbol{\Theta}_m$, which follows by the uniform limit theorem (e.g. Theorem 7.12 in Rudin (1976)) since each $\bar{L}_{m,T}$ is continuous and $\bar{L}_{m,T} \to L_m$ uniformly on $\boldsymbol{\Theta}_m$.

## A.2 Proof of Theorem 2

We verify the conditions in Theorem 3.1 of Newey and McFadden (1994).[7] $\hat{\boldsymbol{\theta}}_m$ minimises $L_{m,T}(\boldsymbol{\theta}_m)$ over $\boldsymbol{\Theta}_m$ by definition and is consistent for $\boldsymbol{\theta}_m^\star$ by Theorem 1. Condition (i) holds by Assumption 6. Condition (ii) holds by Assumption 8, the chain rule and the definition of $l_{m,t,T}$. For condition (iii) we show that

$$\boldsymbol{S}_{m,T}(\boldsymbol{\theta}_m^\star) = T^{-1/2} \sum_{t=1}^{T} \dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m^\star, \epsilon_{m,t,T}^2) \rightsquigarrow \mathcal{N}(0, \boldsymbol{B}_m^\star). \tag{22}$$

For this, firstly note that

$$\dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m^\star, \epsilon_{m,t,T}^2) = \frac{\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)}{g_{m,t,T}(\boldsymbol{\theta}_m^\star)} - \frac{\epsilon_{m,t,T}^2 \dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)}{g_{m,t,T}^2(\boldsymbol{\theta}_m^\star)} = \frac{\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)}{g_{m,t,T}(\boldsymbol{\theta}_m^\star)} \left(1 - \phi_{m,t,T}^2\right), \tag{23}$$

and hence by Assumption 4,

$$E\left[\dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m^\star, \epsilon_{m,t,T}^2)\right] = \frac{\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)}{g_{m,t,T}(\boldsymbol{\theta}_m^\star)} E\left[1 - \phi_{m,t,T}^2\right] = 0.$$

By Assumption 2

$$\|\dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m^\star, \epsilon_{m,t,T}^2)\| = \left\| \frac{\|\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)\|}{|g_{m,t,T}(\boldsymbol{\theta}_m^\star)|} + \frac{\epsilon_{m,t,T}^2 \, \|\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m^\star)\|}{g_{m,t,T}^2(\boldsymbol{\theta}_m^\star)} \right\| \leq C(1 + \epsilon_{m,t,T}^2). \tag{24}$$

For notational convenience we drop the arguments of $\dot{\boldsymbol{l}}_{m,t,T}$, $g_{m,t,T}$ and $\dot{\boldsymbol{g}}_{m,t,T}$ for the remainder of this part of the proof. Let $Z_{t,T} := T^{-1/2}\lambda' \dot{\boldsymbol{l}}_{m,t,T}$ for $\|\lambda\|_2 = 1$. $\sigma_T := \| \sum_{t=1}^T Z_{t,T} \|_{L_2}$ and $X_{t,T} := Z_{t,T}/\sigma_T$ for $1 \leq t \leq T$ and zero otherwise.[8] Let $\mathcal{F}_{m,t,T} := \sigma(\epsilon_{m,s,T}^2, s \leq t)$. We first establish that the conditions of Corollary 1 in de Jong (1997) are satisfied (with $\beta = \gamma = 0$), which will permit us to conclude that $\sum_{t=1}^T X_{t,T} \rightsquigarrow \mathcal{N}(0,1)$. In particular, we will demonstrate that

(a) $X_{t,T}$ is mean-zero with $\| \sum_{t=1}^T X_{t,T} \| = 1$;

(b) there exists an array of positive constants $(c_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ such that $(X_{t,T}/c_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ is $L_r$ bounded for a $r > 2$ uniformly in $t, T$;

(c) $(X_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ is $L_2$-NED of size $-1/2$ on an $\alpha$-mixing array of size $-r/(r-2)$ with constants $(d_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$ such that $d_{t,T}/c_{t,T}$ is uniformly bounded in $t$ and $T$;

(d) The constants $c_{t,T}$ satisfy $c_{t,T}^2 = O(T^{-1})$.

[7] Theorem 3.1 in Newey and McFadden (1994) applies to maximisers; our estimator is a minimiser. Multiplying by $-1$ in the appropriate places allows the application of this result under the conditions shown.

[8] That $\sigma_T$ is finite follows by Assumption 7; that it is (at least eventually) non-zero follows from (27) below.

(a) follows as $X_{t,T}$ is a mean-zero random variable with $\|\sum_{t=1}^T X_{t,T}\|_{L_2} = 1$ by construction. For (b) set $c_{t,T} := \max\{\|Z_{t,T}\|_{L_2}, 1\}/\sigma_T$ for $1 \leq t \leq T$ and $1/\sigma_T$ otherwise. By the moment bounds in Assumption 7 and (24), one concludes that

$$\sup_{1\leq t\leq T, T\in\mathbb{N}} \|X_{t,T}/c_{t,T}\|_{L_{r_m}} \leq \sup_{1\leq t\leq T, T\in\mathbb{N}} \sigma_T \|X_{t,T}\|_{L_{r_m}} = \sup_{1\leq t\leq T, T\in\mathbb{N}} \|Z_{t,T}\|_{L_{r_m}} < \infty, \tag{25}$$

where we note that by definition $r_m = 2 + \delta_m > 2$, as $\delta_m > 0$ (Assumption 7).[9] For (c), we will establish that $X_{t,T}$ is $L_2$-NED of size -1/2 on $(\epsilon^2_{m,t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ (which is $\alpha$-mixing of size $-\rho_m$, with $\rho_m := r_m/(r_m - 2)$ by Assumption 9): i.e. for a sequence $(v_l)_{l=0}^{\infty}$ with $v_l = O(l^{-1/2-\varepsilon})$ for some $\varepsilon > 0$ as $l \to \infty$ and an array of constants $(d_{t,T} : t \in \mathbb{Z}, T \in \mathbb{N})$, for all $m \geq 0$,[10]

$$\|X_{t,T} - E[X_{t,T}|\epsilon_{t-m,T}, \ldots, \epsilon_{t+m,T}]\|_{L_2} \leq d_{t,T}\nu_l. \tag{26}$$

Since $X_{t,T}$ is $\sigma(\epsilon^2_{m,t,T})$-measurable, for any $m \geq 0$, $X_{t,T} = E[X_{t,T}|\epsilon^2_{t-m,T}, \ldots, \epsilon^2_{t+m,T}]$ a.s. and therefore, setting $d_{t,T} = c_{t,T}$ we may take any sequence $v_l = o(l^{-1/2-\varepsilon})$ for a $\varepsilon > 0$. Since such a sequence is of size $-1/2$, (26) holds and $d_{t,T}/c_{t,T} = 1$ this establishes (c). Finally, for (d), we note that by the moment bounds in Assumption 4 and (24) $Tc^2_{t,T} \leq \frac{1}{\sigma_T^2}\max\{\|\dot{\boldsymbol{l}}_{m,t,T}\|^2_{L_2}, 1\} \lesssim \frac{1}{\sigma_T^2}$. By Assumption 10,

$$\sigma_T^2 = \left\|\sum_{t=1}^T Z_{t,T}\right\|^2_{L_2} = \lambda'\boldsymbol{B}_{m,T}\lambda \to \lambda'\boldsymbol{B}^{\star}_m\lambda > 0. \tag{27}$$

Thus $c^2_{t,T} = O(T^{-1})$ as required. Application of Corollary 1 of de Jong (1997) with $\beta = \gamma = 0$ yields $\sum_{t=1}^T X_{t,T} \rightsquigarrow \mathcal{N}(0, 1)$. In conjunction with (27) and Slutsky's Theorem this implies $\sum_{t=1}^T Z_{t,T} \rightsquigarrow \mathcal{N}(0, \lambda'\boldsymbol{B}^{\star}_m\lambda)$. Hence (22) holds by the Cramér – Wold Theorem.

For condition (iv) we will first show the pointwise convergence of $\hat{\boldsymbol{A}}_{m,T} - \boldsymbol{A}_{m,T}$ to zero on $\mathcal{V}_m$. By Jensen's inequality and part iii of Assumption 8,

$$E\left\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right\| \leq \varphi_{m,t,T}(\boldsymbol{\theta}_m)\mathbb{E}|\upsilon_{m,t,T}| \leq C_1\mathbb{E}|\upsilon_{m,t,T}|$$

for some $C_1 \in (0, \infty)$ and where $\sup_{1\leq t\leq T,T\in\mathbb{N}} E\upsilon^2_{m,t,T} < \infty$. Hence for $\delta \in (0, 1]$,

$$E\left\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right\|^{1+\delta} \leq C, \tag{28}$$

for some $C \in (0, \infty)$, implying that $(\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}))_{1\leq t\leq T,T\in\mathbb{N}}$ is uniformly integrable. The same is therefore true of $(X_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ for $X_{t,T} := \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - E\left[\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right]$ when $1 \leq t \leq T$ and zero otherwise. Equation (28) along with Assumption 3 and Theorems 14.1 and 14.2 in Davidson (1994) imply that $(X_{t,T}/T)_{t\in\mathbb{Z},T\in\mathbb{N}}$ forms a $L_1$ – mixingale array with respect to the constants $c_{t,T} = 1/T$.[11] Therefore

$$\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m) = \frac{1}{T}\sum_{t=1}^T \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - E\left[\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\right] \xrightarrow{P} 0 \tag{29}$$

by Theorem 19.11 in Davidson (1994). By Assumption 8 iv, for each pair of indices $(k, j)$ and

[9] The bound for the case with $t \notin \{1, \ldots, T\}$ is trivial as $X_{t,T}/ct, T = 0$.

[10] See e.g. Chapter 17 of Davidson (1994) for an introduction the concept of near-epoch dependence ("NED"). The definition of "size" as used here is given on e.g. p. 355 of de Jong (1997).

[11] The details of the application of Theorem 19.11 of Davidson (1994) are analogous to those given in the proof of Theorem 1 and so are omitted.

$\boldsymbol{\theta}_m, \boldsymbol{\theta}'_m \in \mathcal{V}_m$,

$$
\begin{aligned}
|\boldsymbol{A}_{m,T,k,j}(\boldsymbol{\theta}_m) - \boldsymbol{A}_{m,T,k,j}(\boldsymbol{\theta}'_m)| &= \left| \frac{1}{T} \sum_{t=1}^{T} e'_k E \left[ \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}'_m, \epsilon^2_{m,t,T}) \right] e_j \right| \\
&\le \frac{1}{T} \sum_{t=1}^{T} E \left\| \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) - \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}'_m, \epsilon^2_{m,t,T}) \right\| \\
&\le \frac{1}{T} \sum_{t=1}^{T} E|\psi_{m,t,T}| \|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\| \\
&\le C \|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|,
\end{aligned}
$$

for some constant $C \in (0, \infty)$. This implies that for each $(k, j)$ pair, $\{\boldsymbol{A}_{m,T,k,j} : T \in \mathbb{N}\}$ is uniformly equicontinuous on $\mathcal{V}_m$. Similarly,

$$
|\hat{\boldsymbol{A}}_{m,T,k,j}(\boldsymbol{\theta}_m) - \hat{\boldsymbol{A}}_{m,T,k,j}(\boldsymbol{\theta}'_m)| \le \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_{m,t,T} \right] \|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|.
$$

with $\frac{1}{T} \sum_{t=1}^{T} \psi_{m,t,T} = O_P(1)$. In combination with the uniform equicontinuity of $\{\boldsymbol{A}_{m,T,k,j} : T \in \mathbb{N}\}$, this verifies Assumption SE-1 in Andrews (1992). Hence, for each pair of indices $(k, j)$, $\{\hat{\boldsymbol{A}}_{m,T,k,j} - \boldsymbol{A}_{m,T,k,j} : T \in \mathbb{N}\}$ is stochastically equicontinuous by Lemma 1 of Andrews (1992). In view of (29) and since $\mathcal{V}_m$ is totally bounded as a subset of a compact metric space, Theorem 1 in Andrews (1992) applied to each pair $(k, j)$ therefore yields that

$$
\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m)\| \xrightarrow{P} 0. \tag{30}
$$

By Assumption 11, $\|\boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}_m)\| \to 0$. Since $\{\boldsymbol{A}_{m,T} : T \in \mathbb{N}\}$ is uniformly equicontinuous on $\mathcal{V}_m$ (as noted above), the convergence is uniform:

$$
\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}_m)\| \to 0 \tag{31}
$$

Combining this with equation (30) demonstrates that the convergence in condition (iv) holds. That $\boldsymbol{A}_m$ is continuous at $\boldsymbol{\theta}^\star_m$ follows from the uniform limit theorem.

Condition (v) holds by Assumption 11. The claimed result follows by Theorem 3.1 in Newey and McFadden (1994).

### A.3 Proof of Corollary 1

By Theorem 1, with probability approaching one, $\hat{\boldsymbol{\theta}}_m \in \mathcal{V}_m$. Therefore,

$$
\|\hat{\boldsymbol{A}}_{m,T}(\hat{\boldsymbol{\theta}}_m) - \boldsymbol{A}^\star_m\| \le \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}_m)\| + \|\boldsymbol{A}_m(\hat{\boldsymbol{\theta}}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}^\star_m)\|,
$$

with probability approaching one. By (30) and (31) in the Proof of Theorem 2,

$$
\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}_m)\| \xrightarrow{P} 0.
$$

The Proof of Theorem 2 also demonstrated that $\boldsymbol{A}_m$ is continuous at $\boldsymbol{\theta}^\star_m$. Combination of this, $\hat{\boldsymbol{\theta}}_m \xrightarrow{P} \boldsymbol{\theta}^\star_m$ and the preceding two displays proves the claim.

## A.4 Proof of Proposition 1

We verify Assumptions 1 – 4 of De Jong and Davidson (2000), in order to apply their Theorem 2.2 with

$$X_{t,T}(\boldsymbol{\theta}_m) \coloneqq T^{-1/2} \dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}), \quad X_{t,T} \coloneqq X_{t,T}(\boldsymbol{\theta}^\star_m),$$

for $1 \le t \le T$ and $X_{t,T} = 0$ otherwise. Their Assumption 1 holds by our Assumption 12. For their Assumption 2, let $d_{t,T} \coloneqq c_{t,T} = T^{-1/2}$. Since each $X_{t,T}$ is $\sigma(\epsilon^2_{m,t,T})$–measurable (and in $L_2$), it is $L_2$ – NED of size -1/2 on $(\epsilon^2_{m,t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ and by Assumption 9 this latter array is $\alpha$-mixing of size $-\rho_m$, with $\rho_m \coloneqq r_m/(r_m - 2)$.[12] Their condition (2.7) is satisfied since $\sum_{t=1}^T c^2_{t,T} = 1$ for all $T$, whilst (2.6) holds by Assumptions 2 and 7 since for $1 \le t \le T$ (cf. (24))

$$\left(\|X_{t,T}\|_{L_{r_m}} + d_{t,T}\right) c^{-1}_{t,T} = \|\dot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T})\|_{L_{r_m}} + 1.$$

Their Assumption 3 holds by Assumption 13. Part (a) of their Assumption 4 is implied by Theorem 2; part (b) of their Assumption 4 is implied by the (uniform) equicontinuity on $\mathcal{V}_m$ of $\{\boldsymbol{A}_{m,T} : T \in \mathbb{N}\}$, which was noted to hold in the Proof of Theorem 2. For part (c) of their Assumption 4, we note that the rate condition in (ii) is satisfied under Assumption 13, whilst (2.10) holds by Assumption 7 and Assumption 8 iii since for some $C \in (0, \infty)$,

$$\begin{aligned}
\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \left\| \sum_{t=1}^T \left[\nabla_{\boldsymbol{\theta}_m} X_{t,T}(\boldsymbol{\theta}_m)'\right] \left[\nabla_{\boldsymbol{\theta}_m} X_{t,T}(\boldsymbol{\theta}_m)\right] \right\| &\le \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \frac{1}{T} \sum_{t=1}^T \left\| \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right\|^2 \\
&\le \frac{1}{T} \sum_{t=1}^T C^2 \upsilon^2_{m,t,T} = O_P(1).
\end{aligned}$$

Finally, their (2.8) holds as by Assumption 7 and Assumption 8 iii, uniformly in $1 \le t \le T$,

$$E \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\nabla_{\boldsymbol{\theta}_m} X_{t,T}(\boldsymbol{\theta}_m)\|^2 = E \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \frac{1}{T} \left\| \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right\|^2 \le \frac{C^2}{T} E\upsilon^2_{m,t,T} = O(T^{-1}).$$

Hence the claim follows by Theorem 2.2 in De Jong and Davidson (2000).

## A.5 Proof of Theorem 3

The proof is essentially analogous to that of Theorem 2; we give an abridged version for completeness.

We verify the conditions in Theorem 3.1 of Newey and McFadden (1994).[13] $\hat{\boldsymbol{\theta}}$ minimises $L_T(\boldsymbol{\theta})$ over $\Theta$ by definition and is consistent for $\boldsymbol{\theta}^\star$ by Theorem 1. Condition (i) holds by Assumption 6. Condition (ii) holds by Assumption 8, the chain rule and the definition of $l_{m,t,T}$. For condition (iii) we show that

$$S_T(\boldsymbol{\theta}^\star) = T^{-1/2} \sum_{t=1}^T \dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}^\star, \boldsymbol{\epsilon}^2_{t,T}) \rightsquigarrow \mathcal{N}(0, \boldsymbol{B}^\star). \tag{32}$$

For this, firstly note that by equation (23) and Assumption 4, $E\left[\dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}^\star, \boldsymbol{\epsilon}^2_{t,T})\right] = 0$. For notational convenience we drop the arguments of $\dot{\boldsymbol{l}}_{t,T}$ for the remainder of this part of the proof. Let $Z_{t,T} \coloneqq T^{-1/2}\lambda'\dot{\boldsymbol{l}}_{t,T}$ for $\|\lambda\|_2 = 1$. $\sigma_T \coloneqq \|\sum_{t=1}^T Z_{t,T}\|_{L_2}$ and $X_{t,T} = Z_{t,T}/\sigma_T$ for A1 $\le t \le T$ and $X_{t,T} = 0$ otherwise.[14] We verify the conditions of Corollary 1 in de Jong

[12] This argument is analogous to that given in the proof of Theorem 2 and so the details are omitted.

[13] Cf. footnote 7.

[14] That $\sigma_T$ is finite follows by Assumption 7; that it is (at least eventually) non-zero follows from (34) below.

(1997) as in the proof of Theorem 2.[15] (a) follows as $X_{t,T}$ is a mean-zero random variable with $\|\sum_{t=1}^{T} X_{t,T}\|_{L_2} = 1$ by construction. For (b) set $c_{t,T} := \max\{\|Z_{t,T}\|_{L_2}, 1\}/\sigma_T$ for $1 \leq t \leq T$ and $c_{t,T} = 1/\sigma_T$ otherwise. By the moment bounds in Assumption 7 and (24), one concludes that

$$\sup_{1\leq t\leq T, T\in\mathbb{N}} \|X_{t,T}/c_{t,T}\|_{L_r} \leq \sup_{1\leq t\leq T, T\in\mathbb{N}} \sigma_T\|X_{t,T}\|_{L_r} = \sup_{1\leq t\leq T, T\in\mathbb{N}} \|Z_{t,T}\|_{L_r} < \infty. \tag{33}$$

For (c), since each $X_{t,T}$ is $\sigma(\boldsymbol{\epsilon}^2_{t,T})$–measurable (and in $L_2$), taking $d_{t,T} = c_{t,T}$ and any sequence $v_l = o(l^{-1/2-\varepsilon})$ for a $\varepsilon > 0$ we conclude that $X_{t,T}$ is $L_2$ – NED of size $-1/2$ on $(\boldsymbol{\epsilon}^2_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ and $d_{t,T}/c_{t,T} = 1$ is uniformly bounded. By Assumption 14 $(\boldsymbol{\epsilon}^2_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ is $\alpha$-mixing of size $-\rho$, with $\rho := r/(r-2)$. Finally, we note that by the moment bounds in Assumption 4 and (24), $Tc^2_{t,T} \leq \frac{1}{\sigma_T^2}\max\{\|\dot{\boldsymbol{l}}_{t,T}\|^2_{L_2}, 1\} \lesssim \frac{1}{\sigma_T^2}$ and by Assumption 15,

$$\sigma_T^2 = \left\|\sum_{t=1}^{T} Z_{t,T}\right\|^2_{L_2} = \lambda'\boldsymbol{B}_T\lambda \to \lambda'\boldsymbol{B}^\star\lambda > 0. \tag{34}$$

Thus $c^2_{t,T} = O(T^{-1})$, establishing the final condition of Corollary 1 of de Jong (1997) with $\beta = \gamma = 0$. Therefore, $\sum_{t=1}^{T} X_{t,T} \rightsquigarrow \mathcal{N}(0,1)$. In conjunction with (34) and Slutsky's Theorem this implies $\sum_{t=1}^{T} Z_{t,T} \rightsquigarrow \mathcal{N}(0, \lambda'\boldsymbol{B}^\star\lambda)$. Hence (32) holds by the Cramér – Wold Theorem.

For condition (iv) we note that by (30) and (31) (which can be established in the present setting in exactly the same way as in the proof of Theorem 2) we have

$$\sup_{\boldsymbol{\theta}_m\in\mathcal{V}_m} \|\hat{\boldsymbol{A}}_{m,T}(\boldsymbol{\theta}_m) - \boldsymbol{A}_m(\boldsymbol{\theta}_m)\| \xrightarrow{P} 0,$$

hence (cf. (10))

$$\sup_{\theta\in\mathcal{V}} \|\hat{\boldsymbol{A}}_T(\boldsymbol{\theta}) - \boldsymbol{A}(\boldsymbol{\theta})\| \xrightarrow{P} 0.$$

Condition (v) holds by Assumption 11. The claimed result follows by Theorem 3.1 in Newey and McFadden (1994).

### A.6 Proof of Proposition 2

We verify Assumptions 1 – 4 of De Jong and Davidson (2000), in order to apply their Theorem 2.2 with

$$X_{t,T}(\boldsymbol{\theta}) := T^{-1/2}\dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T}), \quad X_{t,T} := X_{t,T}(\boldsymbol{\theta}^\star),$$

for $1 \leq t \leq T$ and $X_{t,T} = 0$ otherwise. Their Assumption 1 holds by our Assumption 16. For their Assumption 2, let $d_{t,T} := c_{t,T} = T^{-1/2}$. Since each $X_{t,T}$ is $\sigma(\boldsymbol{\epsilon}^2_{t,T})$–measurable (and in $L_2$), it is $L_2$ – NED of size -1/2 on $(\boldsymbol{\epsilon}^2_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ and by Assumption 14 this latter array is $\alpha$-mixing of size $-\rho$, with $\rho := r/(r-2)$.[16]

Their condition (2.7) is evidently satisfied by this choice of $c_{t,T}$ since $\sum_{t=1}^{T} c^2_{t,T} = 1$ for all $T$, whilst (2.6) holds by Assumptions 2 and 7 since for $1 \leq t \leq T$,

$$(\|X_{t,T}\|_{L_r} + d_{t,T})\, c^{-1}_{t,T} = \|\dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}^2_{t,T})\|_{L_{rm}} + 1.$$

Their Assumption 3 holds by Assumption 17. Part (a) of their Assumption 4 is implied by Theorem 3; part (b) of their Assumption 4 is implied by the (uniform) equicontinuity on $\mathcal{V}$ of $\{\boldsymbol{A}_T : T \in \mathbb{N}\}$, which follows from the uniform equicontinuity on $\mathcal{V}_m$ of $\{\boldsymbol{A}_{m,T} : T \in \mathbb{N}\}$ for

[15] Here (a), (b) and (c) are as listed in the proof of Theorem 2; they (respectively) correspond to parts (a), (b) and (c) of Assumption 2 in de Jong (1997). The argument given here is entirely analogous to that in the proof of Theorem 2 and so some details are omitted.

[16] This argument is analogous to that given in the proof of Theorem 2 and so some details are omitted.

$m = 1, \dots, M$ (as was noted to hold in the Proof of Theorem 2 and can be established in exactly the same way in the present setting). For part (c) of their Assumption 4, we note that the rate condition in (ii) is satisfied under Assumption 17, whilst (2.10) holds by Assumption 7 and Assumption 8 iii since for some $C \in (0, \infty)$,

$$\begin{aligned}\sup_{\boldsymbol{\theta}\in\mathcal{V}} \left\| \sum_{t=1}^{T} \left[\nabla_{\boldsymbol{\theta}} X_{t,T}(\boldsymbol{\theta})'\right] \left[\nabla_{\boldsymbol{\theta}} X_{t,T}(\boldsymbol{\theta})\right] \right\| &\leq \sup_{\boldsymbol{\theta}\in\mathcal{V}} \frac{1}{T} \sum_{t=1}^{T} \left\| \ddot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}_{t,T}^2) \right\|^2 \\ &\leq \frac{1}{T} \sum_{t=1}^{T} C^2 \sum_{m=1}^{M} \upsilon_{m,t,T}^2 = O_P(1).\end{aligned}$$

Finally, their (2.8) holds as by Assumption 7 and Assumption 8 iii, uniformly in $1 \leq t \leq T$,

$$E \sup_{\boldsymbol{\theta}\in\mathcal{V}} \|\nabla_{\boldsymbol{\theta}} X_{t,T}(\boldsymbol{\theta})\|^2 = E \sup_{\boldsymbol{\theta}\in\mathcal{V}} \frac{1}{T} \left\| \ddot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}, \boldsymbol{\epsilon}_{t,T}^2) \right\|^2 \leq \frac{C^2}{T} \sum_{m=1}^{M} E\upsilon_{m,t,T}^2 = O(T^{-1}).$$

Hence the claim follows by Theorem 2.2 in De Jong and Davidson (2000).

# B Auxiliary results

**Lemma 1.** *Suppose that Assumption 2 and part i of Assumption 8 hold and that*

$$\sup_{\boldsymbol{\theta}_m\in\mathcal{V}_m} \sup_{1\leq t\leq T, T\in\mathbb{N}} \|\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)\| < \infty. \tag{35}$$

*Then, Assumption 8 part iii holds with $\upsilon_{m,t,T} := 1 + \epsilon_{m,t,T}^2$.*

*Proof.* By direct calculation the $(i,j)$-th element of $\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)$ is

$$\begin{aligned}&[\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)]_{i,j} \\ &\quad= \epsilon_{m,t,T}^2 \left( \frac{2[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i [\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j}{g_{m,t,T}(\boldsymbol{\theta}_m)^3} - \frac{[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,j}}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} \right) \\ &\qquad+ \left( \frac{[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,j}}{g_{m,t,T}(\boldsymbol{\theta}_m)} - \frac{[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i [\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} \right).\end{aligned}$$

By Assumption 2, Assumption 8 part i, and equation (35)

$$0 < c_\rho \leq |g_{m,t,T}|^\rho \leq C_{0,\rho} < \infty, \qquad |[\dot{g}_{m,t,T}]_k| \leq C_1 < \infty, \qquad \text{and} \qquad |[\ddot{g}_{m,t,T}]_{k,l}| \leq C_2 < \infty$$

uniformly in $t, T$ and $\boldsymbol{\theta}_m \in \mathcal{V}_m$, $\rho = 1, 2, 3$ and $k, l = 1, \dots, K$, where $K$ is the dimension of $\boldsymbol{\theta}_m$. Therefore, for $C_0 := \max\{2C_1^2 c_3^{-1} + C_2 c_2^{-1}, C_2 c_1^{-1} + C_1^2 c_2^{-1}\} < \infty$,

$$\begin{aligned}\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)\|_\infty &= \max_{1\leq i,j\leq K} |[\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)]_{i,j}| \\ &\leq \epsilon_{m,t,T}^2 (2C_1^2 c_3^{-1} + C_2 c_2^{-1}) + C_2 c_1^{-1} + C_1^2 c_2^{-1} \\ &\leq C_0(1 + \epsilon_{m,t,T}^2).\end{aligned}$$

Since $\|\cdot\|_\infty$ is a norm on the space of $K \times K$ matrices (e.g. Horn and Johnson, 2013, p. 342) and all norms on the same finite dimensional vector space are equivalent, this implies that

$$\|\ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon_{m,t,T}^2)\| \leq C(1 + \epsilon_{m,t,T}^2),$$

for some constant $C > 0$ and hence the claim follows with $\varphi_{m,t,T}(\boldsymbol{\theta}_m) =: C$. ∎

**Lemma 2.** *Under Assumptions 2, 4 and 9, $\boldsymbol{B}_{m,T} = O(1)$ for $\boldsymbol{B}_{m,T}$ as in Assumption 10.*

*Proof.* By Davydov's inequality (e.g. Davidson, 1994, Corollary 14.3), (24) and Assumption 4 for $1 \le t, s \le T$, any indices $l, k$, and $r_m = 2 + \delta_m/2$,

$$|\mathrm{Cov}(\dot{\boldsymbol{l}}_{m,t,T,l}, \dot{\boldsymbol{l}}_{m,s,T,k})| \le 6\|\dot{\boldsymbol{l}}_{m,t,T,l}\|_{L_{r_m}} \|\dot{\boldsymbol{l}}_{m,s,T,k}\|_{L_{r_m}} \alpha_{m,T}(|t-s|)^{1-2/r_m} \lesssim \sup_{T\in\mathbb{N}} \alpha_{m,T}(|t-s|)^{1-2/r_m}.$$

Hence

$$\begin{aligned} e_T := \left| E\left[ T^{-1} \sum_{t=1}^T \sum_{s=1}^T \dot{\boldsymbol{l}}_{m,t,T,l} \dot{\boldsymbol{l}}_{m,s,T,k} \right] \right| &\lesssim T^{-1} \sum_{t=1}^T \sum_{s=1}^T \sup_{T\in\mathbb{N}} \alpha_{m,T}(|t-s|)^{1-2/r_m} \\ &= \sum_{k=-T}^{T} \left(1 - \frac{|k|}{T}\right) \sup_{T\in\mathbb{N}} \alpha_{m,T}(|k|)^{1-2/r_m} \\ &\le 2\sum_{k=0}^{T} \left(1 - \frac{k}{T}\right) \sup_{T\in\mathbb{N}} \alpha_{m,T}(k)^{1-2/r_m} \end{aligned}$$

By Assumption 9, for all sufficiently large $k \ge K \in \mathbb{N}$, $\sup_{T\in\mathbb{N}} \alpha_{m,T}(k)^{1-2/r_m} \lesssim k^{-r_m/(r_m-2)-\varepsilon}$. Hence for some constant $C > 0$

$$e_T \lesssim C + \sum_{k=K}^{T} k^{-r_m/(r_m-2)-\varepsilon} \le C + \sum_{k=0}^{\infty} k^{-r_m/(r_m-2)-\varepsilon} < \infty$$

since $r_m/(r_m - 2) + \varepsilon > 1$. Hence $\boldsymbol{B}_{m,T} = O(1)$. □

**Lemma 3.** *If Assumptions 2, 4 and parts i, iii of Assumption 8 hold, then $\boldsymbol{A}_{m,T}(\boldsymbol{\theta}_m) = O(1)$ for each $\boldsymbol{\theta}_m \in \mathcal{V}_m$.*

*Proof.* By Jensen's inequality and part iii of Assumption 8,

$$\left\| E\left[ \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right] \right\| \le E\left\| \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right\| \le \varphi_{m,t,T}(\boldsymbol{\theta}_m) E|\upsilon_{m,t,T}|,$$

with $E|\upsilon_{m,t,T}|^2 \le C_1 < \infty$ and $\varphi_{m,t,T}(\boldsymbol{\theta}_m) \le C_2 < \infty$. Putting $C = \sqrt{C_1}C_2$, one then has that

$$\left\| E\left[ \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right] \right\| \le C.$$

Hence $T^{-1}\sum_{t=1}^T \left\| E\left[ \ddot{\boldsymbol{l}}_{m,t,T}(\boldsymbol{\theta}_m, \epsilon^2_{m,t,T}) \right] \right\|$ is bounded above by $C$, which implies the result. □

**Lemma 4.** *Under Assumptions 2, 4 and 14, $\boldsymbol{B}_T = O(1)$ for $\boldsymbol{B}_T$ as in Assumption 15.*

*Proof.* The proof follows analogously to that of Lemma 2 on replacing $r_m$ with $r$, $\delta_m$ with $\min\{\delta_1, \dots, \delta_M\}$ and $\alpha_{m,T}$ with $\alpha_T$. □

**Proposition 6.** *Suppose $g_{m,t,T}(\boldsymbol{\theta}_m)$ is given by (13) with $0 = c_{m,0} < c_{m,1} < \cdots < c_{m,s_m} < c_{m,s_m+1} = 1$, that Assumptions 1 & 6 hold and that $\Theta_m$ is convex. Additionally suppose that $\{\phi^2_{m,t,T} : t \in \mathbb{Z}\}$ is covariance stationary for each $T \in \mathbb{N}$ with absolutely summable autocovariances $\gamma_{m,T}(h) := \gamma_m(h)$ (for all $T \in \mathbb{N}$), $\sum_{h=-\infty}^{\infty} \gamma_m(h) \neq 0$ and Assumptions 4, 7 and 9 hold.*

*Then Assumptions 2, 5, 8, 10 and 11 hold, with*

$$\boldsymbol{A}^\star_m := \begin{bmatrix} 1 & (1-c_{m,1}) & (1-c_{m,2}) & (1-c_{m,3}) & \cdots & (1-c_{m,s}) \\ (1-c_{m,1}) & (1-c_{m,1}) & (1-c_{m,2}) & (1-c_{m,3}) & \cdots & (1-c_{m,s}) \\ (1-c_{m,2}) & (1-c_{m,2}) & (1-c_{m,2}) & (1-c_{m,3}) & \cdots & (1-c_{m,s}) \\ (1-c_{m,3}) & (1-c_{m,3}) & (1-c_{m,3}) & (1-c_{m,3}) & \cdots & (1-c_{m,s}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-c_{m,s}) & (1-c_{m,s}) & (1-c_{m,s}) & (1-c_{m,s}) & \cdots & (1-c_{m,s}) \end{bmatrix}$$

*and* $\boldsymbol{B}^\star_m = \boldsymbol{A}^\star_m \sum_{h=-\infty}^{\infty} \gamma_m(h)$. *In consequence, Theorem 2 holds, i.e.*

$$\sqrt{T}(\widehat{\boldsymbol{\theta}}_m - \boldsymbol{\theta}^\star_m) \xrightarrow{D} \mathcal{N}\left(\mathbf{0},\, [\boldsymbol{A}^\star_m]^{-1} \sum_{h=-\infty}^{\infty} \gamma_m(h)\right).$$

*Proof.* To ease notation, we omit the subscript $m$ throughout the proof. Let $g_{t,T}(\boldsymbol{\theta}) = g(\boldsymbol{\theta}, t/T)$ and $\boldsymbol{\theta}$ be as in (13). We shall first verify Assumptions 10 and 11. At $\boldsymbol{\theta}^\star$, the expression $E\big(\ddot{\boldsymbol{l}}_t(\boldsymbol{\theta}, \epsilon_t^2)\big)$ in Assumption 11 is

$$E\big(\ddot{\boldsymbol{l}}_t(\boldsymbol{\theta}^\star, \epsilon_t^2)\big) \;=\; \frac{1}{\big(g(\boldsymbol{\theta}^\star, t/T)\big)^2} \cdot \dot{\boldsymbol{g}}_t(\boldsymbol{\theta}^\star)\dot{\boldsymbol{g}}_t(\boldsymbol{\theta}^\star)'$$

for all $t$ and $T$, where $\dot{\boldsymbol{g}}_t(\boldsymbol{\theta}^\star) = \big(\dot{g}_{1,t}(\boldsymbol{\theta}^\star), \ldots, \dot{g}_{s+1,t}(\boldsymbol{\theta}^\star)\big)'$ with $\dot{g}_{i,t}(\boldsymbol{\theta}) = \partial g(\boldsymbol{\theta}, t/T)/\partial\theta_i$. Due to the log-linear piecewise constant specification, we have (see the proof of Proposition 3) $\dot{\boldsymbol{g}}_t(\boldsymbol{\theta}^\star)' = g(\boldsymbol{\theta}^\star, t/T)\cdot\big(1, I(t/T \geq c_1), \ldots, I(t/T \geq c_s)\big)$, so

$$E\big(\ddot{\boldsymbol{l}}_t(\boldsymbol{\theta}^\star, \epsilon_t^2)\big) = [I(t/T \geq c_{i-1})I(t/T \geq c_{j-1})]_{i,j=1}^{s+1} = \big[I(t/T \geq c_{\max\{i-1,j-1\}})\big]_{i,j=1}^{s+1}.$$

since $I(t/T \geq c_i) \cdot I(t/T \geq c_j) = I(t/T \geq c_{\max(i,j)})$. At $\boldsymbol{\theta}^\star$, the limit in Assumption 11 is thus

$$\boldsymbol{A}^\star = \big[1 - c_{\min\{j-1,i-1\}}\big]_{i,j=1}^{s+1}$$

We now turn to $\boldsymbol{B}^\star$. A generic row $i$ and column $j$ entry of $\boldsymbol{B}_T$ in Assumption 10 is equal to $E\big[\big(b_i - E(b_i)\big)\big(b_j - E(b_j)\big)\big]$, where

$$b_i \;=\; T^{-1/2}\sum_{t=1}^{T} \dot{l}_{i,t}(\boldsymbol{\theta}^\star, \epsilon_t^2) \qquad \text{with} \qquad \dot{l}_{i,t} = \partial l_t/\partial\theta_i,$$

We have

$$\dot{l}_{i,t}(\boldsymbol{\theta}^\star, \epsilon_t^2) \;=\; \frac{\dot{g}_i(\boldsymbol{\theta}^\star, t/T)}{g(\boldsymbol{\theta}^\star, t/T)}(1 - \phi_t^2),$$

since $\phi_t^2 := \epsilon_t^2/g(\boldsymbol{\theta}^\star, t/T)$ by Assumption 4. Accordingly, $E(b_i) = E(b_j) = 0$ since $E(\phi_t^2) = 1$ for all pairs $(t, T)$, and so

$$\begin{aligned} E\big[\big(b_i - E(b_i)\big)\big(b_j - E(b_j)\big)\big] \;&=\; E(b_i b_j) \\ &=\; E\left[\frac{1}{T}\left(\sum_{t=1}^{T} \dot{l}_{i,t}(\boldsymbol{\theta}^\star, \epsilon_t^2)\right)\left(\sum_{t=1}^{T} \dot{l}_{j,t}(\boldsymbol{\theta}^\star, \epsilon_t^2)\right)\right] \\ &=\; \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} \widetilde{g}_{i,j,t}\widetilde{g}_{i,j,s}\, E\big[(1-\phi_t^2)(1-\phi_s^2)\big], \end{aligned}$$

where $\widetilde{g}_{i,j,x} = \frac{\dot{g}_i(\boldsymbol{\theta}^\star, x/T)\dot{g}_j(\boldsymbol{\theta}^\star, x/T)}{g(\boldsymbol{\theta}^\star, x/T)^2} = I(x/T \geq c_{i-1})I(x/T \geq c_{j-1})$. Recall that $c_0 = 0$, so $I(x/T \geq c_0) = 1$. Recall also that $c_i < c_j$ if $i < j$. This means

$$\widetilde{g}_{i,j,t}\widetilde{g}_{i,j,s} = \begin{cases} I(t/T \geq c_{j-1})I(s/T \geq c_{j-1}) & \text{for} \quad i \leq j \\ I(t/T \geq c_{i-1})I(s/T \geq c_{i-1}) & \text{for} \quad i > j \end{cases}$$

To unify the treatment of all elements of $\boldsymbol{B}_T$, denote

$$a_t = \begin{cases} I(t/T \geq c_{j-1}) & \text{for} \quad i \leq j \\ I(t/T \geq c_{i-1}) & \text{for} \quad i > j \end{cases}, \tag{36}$$

We now work out the limit for an arbitrary $(i,j)$-th element: for any pair $(i,j)$, we have $\widetilde{g}_{i,j,t}\widetilde{g}_{i,j,s} = a_t a_s$ and $a_t a_s = a_{\max(s,t)}$. So $a_t a_s = a_t$ if $t \geq s$ and $a_t a_s = a_s$ if $t < s$. Therefore

$$\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma(|t-s|)a_t a_s = \gamma(0)\sum_{t=1}^{T}a_t + 2\sum_{h=1}^{T-1}\gamma(h)\left(\sum_{t=1+h}^{T}a_t\right). \tag{37}$$

For the second term on the right hand side in (37) we have

$$\gamma(h)\left(\frac{1}{T}\sum_{t=1+h}^{T}a_t\right) = \gamma(h)\left(\frac{1}{T}\sum_{t=1}^{T}a_t\right) - \gamma(h)\left(\frac{1}{T}\sum_{t=1}^{h}a_t\right).$$

Since $0 \leq a_t \leq 1$, the absolute value of the last term in parenthesis on the right hand side is bounded above by $h/T \to 0$ (as $T \to \infty$). Thus, pointwise in $h$,

$$\lim_{T\to\infty}\gamma(h)\left(\frac{1}{T}\sum_{t=1+h}^{T}a_t\right) = \lim_{T\to\infty}\gamma(h)\left(\frac{1}{T}\sum_{t=1}^{T}a_t\right) = \gamma(h)\lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=1}^{T}a_t\right),$$

where (36) implies

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}a_t = \begin{cases}(1-c_{j-1}) & \text{if} \quad i \leq j\\ (1-c_{i-1}) & \text{if} \quad i > j\end{cases} \tag{38}$$

Since $0 \leq a_t \leq 1$ we have $\left|\frac{1}{T}\sum_{t=1}^{T-h}a_t\right| \leq 1$. Thus, as $\sum_{h=-\infty}^{\infty}|\gamma(h)| < \infty$ we may dominate the summands by $|\gamma(h)|$. Thus by the dominated convergence theorem,

$$\lim_{T\to\infty}\sum_{h=1}^{T-1}\gamma(h)\left(\frac{1}{T}\sum_{t=1+h}^{T}a_t\right) = \lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=1}^{T}a_t\right)\cdot\left(\sum_{h=1}^{\infty}\gamma(h)\right).$$

Combining this with the earlier results and using $\gamma(h) = \gamma(-h)$, we conclude that each entry $(i,j)$ in $\boldsymbol{B}^\star$ satisfies

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma(|t-s|)a_t a_s = \lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=1}^{T}a_t\right)\cdot\left(\sum_{h=-\infty}^{\infty}\gamma(h)\right).$$

As a consequence, recalling (38), the matrix for which entry $(i,j)$ equals $\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}a_t$ is $\boldsymbol{A}^\star$. We have thus established that

$$\lim_{T\to\infty}\boldsymbol{B}_T^\star = \boldsymbol{A}^\star\sum_{h=-\infty}^{\infty}\gamma(h) \qquad \text{and} \qquad \boldsymbol{A}^{\star-1}\boldsymbol{B}^\star\boldsymbol{A}^{\star-1} = \boldsymbol{A}^{\star-1}\sum_{h=-\infty}^{\infty}\gamma(h).$$

$\boldsymbol{A}^\star$ and $\boldsymbol{B}^\star$ are positive definite by Lemma 6 and $\sum_{h=-\infty}^{\infty}\gamma(h) \neq 0$. This establishes Assumptions 11 and 10. It remains to establish Assumptions 2, 5 and 8; these follow by applying Proposition 3 with $\mathcal{V}$ any neighbourhood of $\boldsymbol{\theta}^\star$ contained in $\Theta$ and $\Theta^*$ any open $\epsilon$-enlargement of $\Theta$. Since Assumption 9 implies Assumption 3, all of the conditions of Theorem 2 hold. $\square$

# C Proofs of the results in Section 3

## C.1 A useful lemma for verification of Assumption 8

Verifying Assumption 8 requires the choice of the functions $\varphi_{m,t,T}$ and $\psi_{m,t,T}$. For the class of functions that is the main target of our theory, third order differentiability with respect to $\boldsymbol{\theta}_m$ along with certain domination conditions simplifies the choices of $\varphi_{m,t,T}$ and $\psi_{m,t,T}$. This is recorded formally in Lemma 5 below. Next, the Lemma is used to show that Assumption 8 holds for the class of functions that is the main target of our theory in Sections C.3 – C.4.

**Lemma 5.** *Suppose that Assumptions 2 and 4 hold, and that each $g_{m,t,T}$ is three-times differentiable on a neighbourhood $\mathcal{V}_m$ of $\boldsymbol{\theta}_m^\star$. Let $\dddot{g}_{m,t,T,(j,l,k)} := \frac{\partial[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{j,l}}{\partial[\boldsymbol{\theta}_m]_k}$ denote the $(j,l,k)$ entry of the array of third order derivatives and suppose also that for some functions $\overline{g_m}(\boldsymbol{\theta}_m)$, $\overline{\overline{g_m}}(\boldsymbol{\theta}_m)$,*

$$\|\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)\| \leq \overline{g_m}(\boldsymbol{\theta}_m) \leq \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \overline{g_m}(\boldsymbol{\theta}_m) < \infty.$$

*and*

$$|\dddot{g}_{m,t,T,(j,l,k)}(\boldsymbol{\theta}_m)| \leq \overline{\overline{g_m}}(\boldsymbol{\theta}_m) \leq \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \overline{\overline{g_m}}(\boldsymbol{\theta}_m) < \infty.$$

*Then Assumption 8 holds.*

Note that the boundedness of the dominating functions $\overline{g_m}$ and $\overline{\overline{g_m}}$ is automatic if $\mathcal{V}_m$ is taken to be compact and $\overline{g_m}$ and $\overline{\overline{g_m}}$ are continuous.

*Proof.* That Assumption 8 part i holds follows immediately from the assumption of three – times differentiability. Given this, part ii is simply a definition and requires no proof.

For part iii it suffices to note that equation (35) holds as

$$\sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \|\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)\| \leq \sup_{\boldsymbol{\theta}_m \in \mathcal{V}_m} \overline{g_m}(\boldsymbol{\theta}_m) < \infty,$$

and hence we may apply Lemma 1.

Finally, for part iv, note that the derivative of the $(i,j)$ element of $\ddot{\boldsymbol{l}}_{m,t,T}$ with respect to $[\boldsymbol{\theta}_m]_k$ has the form

$$\begin{aligned}
&\epsilon_{m,t,T}^2 \left( \frac{2[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,k}[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j}{g_{m,t,T}(\boldsymbol{\theta}_m)^3} + \frac{2[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{j,k}}{g_{m,t,T}(\boldsymbol{\theta}_m)^3} + \frac{2[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,j}[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_k}{g_{m,t,T}(\boldsymbol{\theta}_m)^3} \right) \\
&\quad - \epsilon_{m,t,T}^2 \left( \frac{6[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_k}{g_{m,t,T}(\boldsymbol{\theta}_m)^4} + \frac{\dddot{g}_{m,t,T,(i,j,k)}(\boldsymbol{\theta}_m)}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} \right) \\
&\quad + \left( \frac{-[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,j}[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_k}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} + \frac{\dddot{g}_{m,t,T,(i,j,k)}(\boldsymbol{\theta}_m)}{g_{m,t,T}(\boldsymbol{\theta}_m)} - \frac{[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{i,k}[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} \right) \\
&\quad + \left( \frac{-[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i[\ddot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_{j,k}}{g_{m,t,T}(\boldsymbol{\theta}_m)^2} + \frac{2[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_i[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_j[\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)]_k}{g_{m,t,T}(\boldsymbol{\theta}_m)^3} \right)
\end{aligned}$$

which, given the boundedness of each 0th, 1st, 2nd and 3rd derivative uniformly over $\boldsymbol{\theta}_m$ and $t, T$ already established, this is bounded by $C(\epsilon_{m,t,T}^2 + 1)$ for some constant $C \in (0, \infty)$.[17] Moreover, this bound is uniform over all $i, j, k$ indices. Hence by the mean value theorem, the required inequality holds with $\psi_{m,t,T} := C(\epsilon_{m,t,T}^2 + 1)$. That $\sup_{1 \leq t \leq T, T \in \mathbb{N}} E|\psi_{m,t,T}| < \infty$ then follows directly from Assumptions 2 and 4. □

[17]Specifically $g_{m,t,T}(\boldsymbol{\theta}_m)$ is bounded above and below by Assumption 2; the first derivatives $\dot{\boldsymbol{g}}_{m,t,T}$ have their norm bounded above by Assumption 2. The boundedness of $\|\ddot{\boldsymbol{g}}_{m,t,T}\|$ holds by the $\overline{g_m}$ – domination hypothesis of this Lemma, whilst the boundedness of each $|\dddot{\boldsymbol{g}}_{m,t,T,(i,j,k)}(\boldsymbol{\theta}_m,)\|$ holds by the $\overline{\overline{g_m}}$ – domination hypothesis of this Lemma.

## C.2 Proof of Proposition 3

To simplify notation we omit the subscript $m$. The first, second and third partial derivatives of $g_{t,T}(\boldsymbol{\theta})$ are

$$\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta}) = g_{t,T}(\boldsymbol{\theta})\cdot\big(1, I(t/T \geq c_1), \ldots, I(t/T \geq c_s)\big)',$$

$$\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta}) = g_{t,T}(\boldsymbol{\theta})\cdot \begin{pmatrix} 1 & I(t/T \geq c_1) & \cdots & I(t/T \geq c_s) \\ I(t/T \geq c_1) & I(t/T \geq c_1)\cdot I(t/T \geq c_1) & \cdots & I(t/T \geq c_1)\cdot I(t/T \geq c_s) \\ \vdots & \vdots & \ddots & \vdots \\ I(t/T \geq c_s) & I(t/T \geq c_s)\cdot I(t/T \geq c_1) & \cdots & I(t/T \geq c_s)\cdot I(t/T \geq c_s) \end{pmatrix},$$

$$\frac{\partial^3 g_{t,T}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'\partial\theta_k} = \ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})\cdot\frac{\partial g_{t,T}(\boldsymbol{\theta})}{\partial\theta_k} \qquad \text{for} \qquad k = 1, \ldots, s+1.$$

It follows straightforwardly that $\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})$ satisfies Assumption 2 and that on a suitable compact subset of $\boldsymbol{\Theta}^*$, both $\|\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})\|$ and the $|\dddot{g}_{t,T,(j,l,k)}(\boldsymbol{\theta})|$ in Lemma 5 are uniformly bounded by sufficiently large constants that serve as dominating functions. In consequence, (13) satisfies Assumption 8.

The limit in Assumption 5 can be written as $L(\boldsymbol{\theta}) = L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta})$, where

$$L_1(\boldsymbol{\theta}) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln g_{t,T}(\boldsymbol{\theta}) \qquad \text{and} \qquad L_2(\boldsymbol{\theta}) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E\big(\epsilon_t^2/g_{t,T}(\boldsymbol{\theta})\big).$$

The first term is

$$L_1(\boldsymbol{\theta}) = \delta_0 + \int_{c_1}^{1}\delta_1 dx + \cdots + \int_{c_s}^{1}\delta_s dx \;=\; \delta_0 + \sum_{l=1}^{s}\delta_l(1-c_l) \;=\; \sum_{l=0}^{s}\delta_l(1-c_l),$$

where $c_0 = 0$. The second term is, using that $E(\epsilon_{t,T}^2) = g_{t,T}(\boldsymbol{\theta}^\star)$ for all $t, T$,

$$L_2(\boldsymbol{\theta}) \;=\; \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{E(\epsilon_t^2)}{g_{t,T}(\boldsymbol{\theta})} = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{g_{t,T}(\boldsymbol{\theta}^\star)}{g_{t,T}(\boldsymbol{\theta})} = \int_0^1\frac{g(\boldsymbol{\theta}^\star, x)}{g(\boldsymbol{\theta}, x)}dx.$$

Let $g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) = g(\boldsymbol{\theta}^\star, x)/g(\boldsymbol{\theta}, x)$, so that

$$g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) \;=\; \exp\big(\delta_0^\star - \delta_0\big)\cdot\exp\big((\delta_1^\star - \delta_1)I(x \geq c_1)\big) \;\cdots\; \exp\big((\delta_s^\star - \delta_s)I(x \geq c_s)\big)$$

and

$$\int_0^1 g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx \;=\; \int_0^{c_1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx + \int_{c_1}^{c_2} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx + \cdots + \int_{c_s}^{1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx,$$

where

$$\begin{aligned}
\int_0^{c_1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx &= \exp\big(\delta_0^\star - \delta_0\big)\cdot(c_1 - c_0) \\
\int_{c_1}^{c_2} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx &= \exp\big(\delta_0^\star - \delta_0\big)\cdot\exp\big(\delta_1^\star - \delta_1\big)\cdot(c_2 - c_1) \\
&\;\;\vdots \\
\int_{c_s}^{1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx &= \exp\big(\delta_0^\star - \delta_0\big)\cdot\exp\big(\delta_1^\star - \delta_1\big)\cdots\exp\big(\delta_s^\star - \delta_s\big)\cdot(1 - c_s).
\end{aligned}$$

The sum of these terms is

$$\int_0^1 g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)dx = \sum_{l=0}^{s}(c_{l+1} - c_l)\prod_{k=0}^{l}\exp(\delta_k^\star - \delta_k) \qquad \text{with} \qquad c_0 = 0 \text{ and } c_{s+1} = 1,$$

In conclusion,

$$L(\boldsymbol{\theta}) = \sum_{l=0}^{s}\delta_l(1 - c_l) + \sum_{l=0}^{s}(c_{l+1} - c_l)\prod_{k=0}^{l}\exp(\delta_k^\star - \delta_k),$$

and from Assumption 2(i) it follows that $\big|L(\boldsymbol{\theta})\big| < \infty$ on $\boldsymbol{\Theta}^*$.

For $i = 0, 1, 2, \ldots, s$

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \delta_i} = (1 - c_i) - \sum_{l=i}^{s}(c_{l+1} - c_l)\prod_{k=0}^{l}\exp(\delta_k^\star - \delta_k) \qquad \text{with} \qquad c_0 = 0 \text{ and } c_{s+1} = 1.$$

Hence, at $\boldsymbol{\theta}_m = \boldsymbol{\theta}_m^\star$ the derivatives are

$$\begin{aligned}
\frac{\partial L(\boldsymbol{\theta}^\star)}{\partial \delta_0} &= (1 - c_0) - \big((c_1 - c_0) + (c_2 - c_1) + \cdots + (1 - c_s)\big) \;=\; (1 - c_0) - 1 \;=\; 0 \\
\frac{\partial L(\boldsymbol{\theta}^\star)}{\partial \delta_1} &= (1 - c_1) - \big((c_2 - c_1) + \cdots + (1 - c_s)\big) \;=\; (1 - c_1) - (1 - c_1) \;=\; 0 \\
&\vdots \\
\frac{\partial L(\boldsymbol{\theta}^\star)}{\partial \delta_{s-1}} &= (1 - c_{s-1}) - \big((c_s - c_{s-1}) + (1 - c_s)\big) \;=\; (1 - c_{s-1}) - (1 - c_{s-1}) \;=\; 0 \\
\frac{\partial L(\boldsymbol{\theta}^\star)}{\partial \delta_s} &= (1 - c_s) - (1 - c_s) \;=\; 0.
\end{aligned}$$

Accordingly, $\boldsymbol{\theta}^\star$ is a stationary point of $L(\boldsymbol{\theta})$.

By differentiating $\partial L(\boldsymbol{\theta})/\partial \delta_i$ with respect to $\delta_j$, it follows that the generic $(i+1, j+1)$ entry of $\ddot{\boldsymbol{L}}(\boldsymbol{\theta})$, with $i, j \in \{0, 1, 2, \ldots, s\}$, is

$$\ddot{L}_{ij}(\boldsymbol{\theta}) = \sum_{l=i}^{s} I(l \geq j) \cdot a_l(\boldsymbol{\theta}) \;\; = \sum_{l=\max\{i,j\}}^{s} a_l(\boldsymbol{\theta}),$$

where $a_l(\boldsymbol{\theta}) \coloneqq (c_{l+1} - c_l)\prod_{k=0}^{l}\exp(\delta_k^\star - \delta_k) > 0$. Fix any $\boldsymbol{\theta}$ and set $a_l \coloneqq a_l(\boldsymbol{\theta})$. Then

$$\ddot{\boldsymbol{L}}(\boldsymbol{\theta}) = \begin{bmatrix}
\sum_{l=0}^{s} a_l & \sum_{l=1}^{s} a_l & \cdots & \sum_{l=s-1}^{s} a_l & \sum_{l=s}^{s} a_l \\
\sum_{l=1}^{s} a_l & \sum_{l=1}^{s} a_l & \cdots & \sum_{l=s-1}^{s} a_l & \sum_{l=s}^{s} a_l \\
\sum_{l=2}^{s} a_l & \sum_{l=2}^{s} a_l & \cdots & \sum_{l=s-1}^{s} a_l & \sum_{l=s}^{s} a_l \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sum_{l=s-1}^{s} a_l & \sum_{l=s-1}^{s} a_l & \cdots & \sum_{l=s-1}^{s} a_l & \sum_{l=s}^{s} a_l \\
\sum_{l=s}^{s} a_l & \sum_{l=s}^{s} a_l & \cdots & \sum_{l=s}^{s} a_l & \sum_{l=s}^{s} a_l
\end{bmatrix}.$$

As $a_l > 0$, we have $\sum_{l=0}^{s} a_l > \sum_{l=1}^{s} a_l > \cdots > \sum_{l=s}^{s} a_l = a_s > 0$. Let $x_i \coloneqq \sum_{l=i}^{s} a_l$. Then

$$\ddot{\boldsymbol{L}}(\boldsymbol{\theta}) = \begin{bmatrix}
x_0 & x_1 & \cdots & x_{s-1} & x_s \\
x_1 & x_1 & \cdots & x_{s-1} & x_s \\
x_2 & x_2 & \cdots & x_{s-1} & x_s \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{s-1} & x_{s-1} & \cdots & x_{s-1} & x_s \\
x_s & x_s & \cdots & x_s & x_s
\end{bmatrix}.$$

This is positive definite for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}^*$ by Lemma 6 (see below) and hence $L(\boldsymbol{\theta})$ is strictly convex since the set $\boldsymbol{\Theta}^*$ is open and convex by Assumption 2. Next, part 1 of Theorem 7.13 in Sundaram (1996) implies $\boldsymbol{\theta}^\star$ is a global minimum, and Theorem 7.14 (Sundaram, 1996) implies the set of minimisers of $L$ over $\boldsymbol{\Theta}^*$ is either empty or a singleton. In conclusion, $\boldsymbol{\theta}^\star$ is the unique minimiser of $L(\boldsymbol{\theta})$ on $\boldsymbol{\Theta}^*$.

**Lemma 6.** *Let $\mathcal{J} := \{x \in \mathbb{R}^s : x_1 > x_2 > \cdots > x_s > 0\}$ and $X : \mathbb{R}^s \to \mathbb{R}^{s\times s}$ be the function:*

$$X(x) := \begin{bmatrix} x_1 & x_2 & \dots & x_{s-1} & x_s \\ x_2 & x_2 & \dots & x_{s-1} & x_s \\ x_3 & x_3 & \dots & x_{s-1} & x_s \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{s-1} & x_{s-1} & \dots & x_{s-1} & x_s \\ x_s & x_s & \dots & x_s & x_s \end{bmatrix}.$$

*Then the image of $\mathcal{J}$ under $X$ is a subset of the convex cone of positive definite $s \times s$ matrices.*

*Proof.* It suffices to show that each $X(x)$ is positive definite. Any such $X(x)$ is clearly symmetric. If $s = 1$ then $x \in \mathcal{J}$ iff $x_1 > 0$ and $X(x) = [x_1] \succ 0$. Now suppose that the conclusion holds for matrices of size $k-1 \times k-1$. We will show that the image of $\mathcal{J}_k := \{x \in \mathbb{R}^k : x_1 > \cdots > x_k > 0\}$ under $X_k$ is a subset of convex cone of positive definite $k \times k$ matrices where

$$X_k(x) := \begin{bmatrix} x_1 & x_2 & \dots & x_{k-1} & x_k \\ x_2 & x_2 & \dots & x_{k-1} & x_k \\ x_3 & x_3 & \dots & x_{k-1} & x_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k-1} & x_{k-1} & \dots & x_{k-1} & x_k \\ x_k & x_k & \dots & x_k & x_k \end{bmatrix}.$$

In particular, by the induction hypothesis

$$A = \begin{bmatrix} x_2 & \dots & x_{k-1} & x_k \\ x_3 & \dots & x_{k-1} & x_k \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-1} & \dots & x_{k-1} & x_k \\ x_k & \dots & x_k & x_k \end{bmatrix},$$

(the matrix obtained by removing the first column and row of $X_k(x)$) is positive definite since $x_2 > x_3 > \cdots > x_k > 0$. Define $a := [x_2, \dots, x_k]'$. By Proposition 8.2.4 in Bernstein (2009) it suffices to show that $x_1 - a'A^{-1}a > 0$. As $x_1 > x_2$ it is enough to note that $a'A^{-1}a = x_2$ since

$$A^{-1}a = d \qquad \iff \qquad a = Ad,$$

and choosing $d = e_1$ satisfies the right hand side. $\square$

### C.3 Proof of Proposition 4

To simplify notation we omit the subscript $m$. Let $G(r, \gamma, c) := 1 + \exp(-\gamma(r - c))$. The gradient and Hessian of this function are

$$\dot{\boldsymbol{G}}(r, \gamma, c) = ((c - r), \gamma)' \exp(-\gamma(r - c)),$$

$$\ddot{\boldsymbol{G}}r, \gamma, c) = \begin{bmatrix} (c-r)^2 & \gamma(c-r)+1 \\ \gamma(c-r)+1 & \gamma^2 \end{bmatrix} \exp(-\gamma(r - c)).$$

Moreover, define $\dddot{\boldsymbol{G}}^{(1)}(r,\gamma,c)$ as

$$\frac{\partial \ddot{\boldsymbol{G}}(r,\gamma,c)}{\partial \gamma} = \begin{bmatrix} (c-r)^3 & 2(c-r)+\gamma(c-r)^2 \\ 2(c-r)+\gamma(c-r)^2+1 & 2\gamma+(c-r)\gamma^2 \end{bmatrix} \exp(-\gamma(r-c)),$$

and $\dddot{\boldsymbol{G}}^{(2)}(r,\gamma,c)$ as

$$\frac{\partial \ddot{\boldsymbol{G}}(r,\gamma,c)}{\partial c} = \begin{bmatrix} 2(c-r)+\gamma(c-r)^2 & 2\gamma+(c-r)\gamma^2 \\ 2\gamma+(c-r)\gamma^2 & \gamma^3 \end{bmatrix} \exp(-\gamma(r-c)).$$

Then with $r=t/T$, $\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta}) \coloneqq \dot{\boldsymbol{g}}(\boldsymbol{\theta},r) \coloneqq (\dot{\boldsymbol{g}}^{(\delta)}_{t,T}(\boldsymbol{\theta})', \dot{\boldsymbol{g}}^{(\gamma)}_{t,T}(\boldsymbol{\theta})', \dot{\boldsymbol{g}}^{(c)}_{t,T}(\boldsymbol{\theta})')'$ where,

$$\begin{aligned}
\dot{\boldsymbol{g}}^{(\delta)}_{t,T}(\boldsymbol{\theta}) &= \left(1, G(r,\gamma_1,c_1)^{-1}, \dots, G(r,\gamma_s,c_s)^{-1}\right) \\
\dot{\boldsymbol{g}}^{(\gamma)}_{t,T}(\boldsymbol{\theta}) &= -\left(\delta_1 G(r,\gamma_1,c_1)^{-2}\dot{\boldsymbol{G}}_1(r,\gamma_1,c_1), \dots, \delta_s G(r,\gamma_s,c_s)^{-2}\dot{\boldsymbol{G}}_1(r,\gamma_s,c_s)\right) \\
\dot{\boldsymbol{g}}^{(c)}_{t,T}(\boldsymbol{\theta}) &= -\left(\delta_1 G(r,\gamma_1,c_1)^{-2}\dot{\boldsymbol{G}}_2(r,\gamma_1,c_1), \dots, \delta_s G(r,\gamma_s,c_s)^{-2}\dot{\boldsymbol{G}}_2(r,\gamma_s,c_s)\right).
\end{aligned}$$

The Hessian $\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})$ is

$$\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{0}'_{1+s,1+s} & \mathbf{0}'_{1+s,s} & \mathbf{0}'_{1+s,s} \\ \mathbf{0}_{s,1+s} & \ddot{\boldsymbol{g}}^{(\gamma\gamma)}_{t,T}(\boldsymbol{\theta}) & \ddot{\boldsymbol{g}}^{(\gamma c)}_{t,T}(\boldsymbol{\theta}) \\ \mathbf{0}_{s,1+s} & \ddot{\boldsymbol{g}}^{(\gamma c)}_{t,T}(\boldsymbol{\theta}) & \ddot{\boldsymbol{g}}^{(cc)}_{t,T}(\boldsymbol{\theta}) \end{bmatrix},$$

where $\ddot{\boldsymbol{g}}^{(\gamma\gamma)}_{t,T}(\boldsymbol{\theta})$, $\ddot{\boldsymbol{g}}^{(\gamma c)}_{t,T}(\boldsymbol{\theta})$ and $\ddot{\boldsymbol{g}}^{(cc)}_{t,T}(\boldsymbol{\theta})$ are diagonal with (respectively) entries

$$\begin{aligned}
[\ddot{\boldsymbol{g}}^{(\gamma\gamma)}_{t,T}(\boldsymbol{\theta})]_{ll} &= \delta_l\left[2G(r,\gamma_l,c_l)^{-3}\dot{\boldsymbol{G}}_1(r,\gamma_l,c_l)^2 - G(r,\gamma_l,c_l)^{-2}\ddot{\boldsymbol{G}}_{11}(r,\gamma_l,c_l)\right] \\
[\ddot{\boldsymbol{g}}^{(\gamma c)}_{t,T}(\boldsymbol{\theta})]_{ll} &= \delta_l\left[2G(r,\gamma_l,c_l)^{-3}\dot{\boldsymbol{G}}_1(r,\gamma_l,c_l)\dot{\boldsymbol{G}}_2(r,\gamma_l,c_l) - G(r,\gamma_l,c_l)^{-2}\ddot{\boldsymbol{G}}_{12}(r,\gamma_l,c_l)\right] \\
[\ddot{\boldsymbol{g}}^{(cc)}_{t,T}(\boldsymbol{\theta})]_{ll} &= \delta_l\left[2G(r,\gamma_l,c_l)^{-3}\dot{\boldsymbol{G}}_2(r,\gamma_l,c_l)^2 - G(r,\gamma_l,c_l)^{-2}\ddot{\boldsymbol{G}}_{22}(r,\gamma_l,c_l)\right].
\end{aligned}$$

Each third order derivative is thus either zero or has the form

$$\begin{aligned}
\delta_l\Big[&-6G(r,\gamma_l,c_l)^{-4}\dot{\boldsymbol{G}}_i(r,\gamma_l,c_l)\dot{\boldsymbol{G}}_j(r,\gamma_l,c_l)\dot{\boldsymbol{G}}_k(r,\gamma_l,c_l) \\
&+2G(r,\gamma_l,c_l)^{-3}\dot{\boldsymbol{G}}_i(r,\gamma_l,c_l)\ddot{\boldsymbol{G}}_{jk}(r,\gamma_l,c_l) + 2G(r,\gamma_l,c_l)^{-3}\ddot{\boldsymbol{G}}_{ik}(r,\gamma_l,c_l)\dot{\boldsymbol{G}}_j(r,\gamma_l,c_l) \\
&+2G(r,\gamma_l,c_l)^{-3}\ddot{\boldsymbol{G}}_{ij}(r,\gamma_l,c_l)\dot{\boldsymbol{G}}_k(r,\gamma_l,c_l) - G(r,\gamma_l,c_l)^{-2}\dddot{\boldsymbol{G}}^{(k)}_{ij}(r,\gamma_l,c_l)\Big],
\end{aligned}$$

for $(i,j,k) \in \{1,2\}^3$. It follows straightforwardly that Assumption 2 holds and that on a suitable compact subset of $\boldsymbol{\Theta}^*$, both $\|\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})\|$ and the $|\dddot{g}_{t,T,(j,l,k)}(\boldsymbol{\theta})|$ in Lemma 5 are uniformly bounded by sufficiently large constants that serve as dominating functions. In consequence, (13) satisfies Assumption 8.

To establish the existence of the limit $L(\boldsymbol{\theta})$, decompose $E(l_{t,T}(\boldsymbol{\theta},\epsilon^2_{t,T})) = \ln g_{t,T}(\boldsymbol{\theta}) + g_{t,T}(\boldsymbol{\theta}^\star)/g_{t,T}(\boldsymbol{\theta})$. Note by the form in (14),

$$g_{t,T}(\boldsymbol{\theta}) \coloneqq g(\boldsymbol{\theta},r) \coloneqq \delta_0 + \sum_{l=1}^{s}\delta_l G(r,\gamma_l,c_l)^{-1}, \qquad r=t/T.$$

As $|g(\boldsymbol{\theta},r)|$ is bounded away from zero and infinity the same is true of $|\ln g(\boldsymbol{\theta},r) + g(\boldsymbol{\theta}^\star,r)/g(\boldsymbol{\theta},r)|$.

As $\ln g(\boldsymbol{\theta}, r) + g(\boldsymbol{\theta}^\star, r)/g_{t,T}(\boldsymbol{\theta}, r)$ is also continuous in $r$ it is Riemann integrable and hence

$$
\begin{aligned}
L(\boldsymbol{\theta}) &= \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} \left(\ln g_{t,T}(\boldsymbol{\theta}) + g_{t,T}(\boldsymbol{\theta}^\star)/g_{t,T}(\boldsymbol{\theta})\right) \\
&= \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} \left(\ln g(\boldsymbol{\theta}, t/T) + g(\boldsymbol{\theta}^\star, t/T)/g(\boldsymbol{\theta}, t/T)\right) \\
&= \int_0^1 \left(\ln g(\boldsymbol{\theta}, r) + g(\boldsymbol{\theta}^\star, r)/g(\boldsymbol{\theta}, r)\right) \mathrm{d}r.
\end{aligned}
$$

Given the continuity and boundedness of the partial derivatives noted above, we may calculate $\dot{\boldsymbol{L}}(\boldsymbol{\theta})$ by differentiation inside the integral (e.g. Folland, 1999, Theorem 2.27)

$$
\dot{\boldsymbol{L}}(\boldsymbol{\theta}) = \int_0^1 \dot{\boldsymbol{g}}(\boldsymbol{\theta}, r)\frac{1}{g(\boldsymbol{\theta}, r)} - \dot{\boldsymbol{g}}(\boldsymbol{\theta}, r)\frac{g(\boldsymbol{\theta}^\star, r)}{g(\boldsymbol{\theta}, r)^2}\,\mathrm{d}r.
$$

It is clear that $\dot{\boldsymbol{L}}(\boldsymbol{\theta}^\star) = \mathbf{0}$. Given continuity and boundedness of the second partial derivatives noted above, we may calculate $\ddot{\boldsymbol{L}}(\boldsymbol{\theta})$ by differentiating under the integral, hence $\ddot{\boldsymbol{L}}(\boldsymbol{\theta})$ exists.

## C.4 Proof of Proposition 5

To simplify notation we omit the subscript $m$. Let $G(r, \delta, c) := \exp\left(\delta_0 + \sum_{l=1}^{s} \delta_l (r - c_l)^2 I(r \geq c_l)\right)$, where $r = t/T$. The gradient is

$$
\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta}) = G(r, \delta, c)\left(1, (r - c_1)^2 I(r \geq c_1), \ldots, (r - c_s)^2 I(r \geq c_s), \right)',
$$

and the Hessian has $(i+1, j+1)$-th entry

$$
[\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})]_{i+1,j+1} = \begin{cases} G(r, \delta, c)^2 & \text{if } i = j = 0 \\ G(r, \delta, c)^2 (r - c_j)^2 I(r \geq c_j) & \text{if } i = 0, j > 0 \\ G(r, \delta, c)^2 (r - c_i)^2 I(r \geq c_i) & \text{if } i > 0, j = 0 \\ G(r, \delta, c)^2 (r - c_i)^2 I(r \geq c_i)(r - c_j)^2 I(r \geq c_j) & \text{otherwise} \end{cases}.
$$

The third partial derivatives are the product of $G(r, \delta, c)^3$ and three terms from $\{1\} \cup \{(r - c_l)^2 I(r \geq c_l) : l = 1, \ldots, s\}$. It follows straightforwardly that $\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})$ satisfies Assumption 2 and that on a suitable compact subset of $\boldsymbol{\Theta}^*$, both $\|\ddot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})\|$ and the $|\dddot{g}_{t,T,(j,l,k)}(\boldsymbol{\theta})|$ in Lemma 5 are uniformly bounded by sufficiently large constants that serve as dominating functions. In consequence, (13) satisfies Assumption 8.

The limit $L(\boldsymbol{\theta})$ in Assumption 5 is made up of two terms:

$$
L(\boldsymbol{\theta}) = L_1(\boldsymbol{\theta}) + L_2(\boldsymbol{\theta}) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \ln g_{t,T}(\boldsymbol{\theta}) + \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E\left(\epsilon_t^2 / g_{t,T}(\boldsymbol{\theta})\right).
$$

The first term is

$$
\begin{aligned}
L_1(\boldsymbol{\theta}) &= \delta_0 + \int_{c_1}^{1} \delta_1 (x - c_1)^2 dx + \cdots + \int_{c_s}^{1} \delta_s (x - c_s)^2 dx \\
&= \delta_0 + \sum_{l=1}^{s} \delta_l \int_{c_l}^{1} (x - c_l)^2 dx \\
&= \delta_0 - \frac{1}{3} \sum_{l=1}^{s} \delta_l (c_l - 1)^3.
\end{aligned}
$$

The second term in the limit $L(\boldsymbol{\theta})$ is, using that $E(\epsilon_{t,T}^2) = g_{t,T}(\boldsymbol{\theta}^\star)$ for all $t, T$,

$$L_2(\boldsymbol{\theta}) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{g_{t,T}(\boldsymbol{\theta}^\star)}{g_{t,T}(\boldsymbol{\theta})}.$$

Noting that we can write

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^{T} \frac{g_{t,T}(\boldsymbol{\theta}^\star)}{g_{t,T}(\boldsymbol{\theta})} &= \frac{1}{T} \sum_{t=1}^{T} \exp\bigl(\delta_0^\star - \delta_0\bigr) \cdot \prod_{l=1}^{s} \exp\bigl((\delta_l^\star - \delta_l)(t/T - c_l)^2 I(t/T \geq c_l)\bigr) \\
&= \frac{1}{T} \sum_{t=1}^{T} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, t/T) \\
&= \frac{1}{T} \sum_{c_0 \leq x < c_1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) + \cdots + \frac{1}{T} \sum_{c_s \leq x < c_{s+1}} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x)
\end{aligned}$$

where $x = t/T$, letting $T \to \infty$ gives

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, t/T) = \int_0^{c_1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx + \int_{c_1}^{c_2} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx + \cdots + \int_{c_s}^{1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx,$$

where

$$\begin{aligned}
\int_0^{c_1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx &= \exp\bigl(\delta_0^\star - \delta_0\bigr) \cdot c_1 \\
\int_{c_1}^{c_2} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx &= \exp\bigl(\delta_0^\star - \delta_0\bigr) \cdot \int_{c_1}^{c_2} \exp\bigl((\delta_1^\star - \delta_1)(x - c_1)^2\bigr) dx \\
&\vdots \\
\int_{c_s}^{1} g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx &= \exp\bigl(\delta_0^\star - \delta_0\bigr) \cdot \int_{c_s}^{1} \prod_{l=1}^{s} \exp\bigl((\delta_l^\star - \delta_l)(x - c_l)^2\bigr) dx.
\end{aligned}$$

Their sum is

$$\int_0^1 g(\boldsymbol{\theta}^\star, \boldsymbol{\theta}, x) dx = \exp\bigl(\delta_0^\star - \delta_0\bigr) \cdot \left( c_1 + \sum_{l=1}^{s} \int_{c_l}^{c_{l+1}} \prod_{k=1}^{l} \exp\bigl((\delta_k^\star - \delta_k)(x - c_k)^2\bigr) dx \right),$$

where $c_{s+1} = 1$. Hence

$$L(\boldsymbol{\theta}) = \delta_0 - \sum_{l=1}^{s} \frac{\delta_l}{3}(c_l - 1)^3 + \exp\bigl(\delta_0^\star - \delta_0\bigr) \left( c_1 + \sum_{l=1}^{s} \int_{c_l}^{c_{l+1}} \prod_{k=1}^{l} \exp\bigl((\delta_k^\star - \delta_k)(x - c_k)^2\bigr) dx \right).$$

We now show that $\dot{\boldsymbol{L}}(\boldsymbol{\theta}^\star) = \mathbf{0}$. For $i = 0$, $\frac{\partial L(\boldsymbol{\theta})}{\partial \delta_0} = 1 - L_2(\boldsymbol{\theta})$ and it is straightforward to verify that $L_2(\boldsymbol{\theta}^\star) = 1$ and therefore that $\frac{\partial L(\boldsymbol{\theta})}{\partial \delta_0} = 0$. Next, for $i = 1, \ldots, s$,

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \delta_i} = -\frac{1}{3}(c_i - 1)^3 - \exp(\delta_0^\star - \delta_0) \left( \sum_{l=i}^{s} \int_{c_l}^{c_{l+1}} (x - c_i)^2 \prod_{k=1}^{l} \exp\left((\delta_k^\star - \delta_k)(x - c_k)^2\right) \mathrm{d}x \right).$$

so, at $\boldsymbol{\theta} = \boldsymbol{\theta}^\star$,

$$\frac{\partial L(\boldsymbol{\theta})}{\partial \delta_i} = -\frac{1}{3}(c_i - 1)^3 - \sum_{l=i}^{s} \int_{c_l}^{c_{l+1}} (x - c_i)^2 dx = \int_{c_i}^{1} (x - c_i)^2 dx - \int_{c_i}^{1} (x - c_i)^2 dx = 0.$$

The second derivatives are ($j \geq 1$)

$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \delta_0^2} = L_2(\boldsymbol{\theta})$$
$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \delta_0 \partial \delta_j} = \exp(\delta_0^\star - \delta_0) \left( \sum_{l=j}^{s} \int_{c_l}^{c_{l+1}} (x - c_j)^2 \prod_{k=1}^{l} \exp\left((\delta_k^\star - \delta_k)(x - c_k)^2\right) \,\mathrm{d}x \right)$$
$$\frac{\partial^2 L(\boldsymbol{\theta})}{\partial \delta_i \partial \delta_j} = \exp(\delta_0^\star - \delta_0) \left( \sum_{l=i}^{s} I(j \leq l) \int_{c_l}^{c_{l+1}} (x - c_i)^2 (x - c_j)^2 \prod_{k=1}^{l} \exp\left((\delta_k^\star - \delta_k)(x - c_k)^2\right) \,\mathrm{d}x \right).$$

If $\ddot{\boldsymbol{L}}(\boldsymbol{\theta})$ is positive definite on $\boldsymbol{\Theta}^*$ then $L(\boldsymbol{\theta})$ has a unique minimum over this set by the same argument which concludes the proof of Proposition 3.

# D Auxiliary results and simulations referred to in Section 4

## D.1 The infeasible QMLE

In the infeasible case, $\{\phi_{m,t}^2\}$ is observed and the QMLE is

$$\widehat{\boldsymbol{\vartheta}}_m^\star = \arg\min_{\boldsymbol{\vartheta}_m \in \boldsymbol{\Xi}} \frac{1}{T} \sum_{t=1}^{T} l_t(\boldsymbol{\vartheta}_m, \phi_{m,t}^2), \qquad l_t(\boldsymbol{\vartheta}_m, \phi_{m,t}^2) = \ln h_t(\boldsymbol{\vartheta}_m) + \frac{\phi_{m,t}^2}{h_t(\boldsymbol{\vartheta}_m)}. \tag{39}$$

To emphasise that this estimator is infeasible, we add $\star$ as superscript. For this estimator to be consistent in the standard case (see Theorem 4 below), a compactness assumption is needed:

**Assumption 18** For each $m = 1, \ldots, M$, $\boldsymbol{\vartheta}_m^\star \in \boldsymbol{\Xi}_m$ and $\boldsymbol{\Xi}_m$ is compact.

Next, the following additional assumption ensures the process $\{\phi_{m,t}^2\}$ is $\beta$-mixing with exponential decay, which implies that also the $\alpha$-mixing assumption in Assumption 3 holds.

**Assumption 19** (condition $\eta$ in Carrasco and Chen 2002) For each $m = 1, \ldots, M$, the probability distribution of $\eta_{m,t}$ has a continuous density (with respect to the Lebesgue measure on the real line), and its density is positive on $(-\infty, \infty)$.

**Theorem 4** (consistency, infeasible case). *Suppose $\{\phi_{m,t}^2\}$ is governed by (17)–(18) with $\eta_t \sim iid(0,1)$, and that Assumption 18 holds. Then $\widehat{\boldsymbol{\vartheta}}_m^\star \xrightarrow{P} \boldsymbol{\vartheta}_m^\star$, $m = 1, \ldots, M$. If, in addition, Assumption 19 holds, then $\{\phi_{m,t}^2\}$ is $\beta$-mixing with exponential decay, $m = 1, \ldots, M$.*

*Proof of Theorem 4.* To simplify notation, we omit the subscript $m$. Consistency of $\widehat{\boldsymbol{\vartheta}}^\star$ follows if the four assumptions of Theorem 7.1 in Francq and Zakoïan (2019) hold. In the GARCH(1,1) case, the four assumptions are:

- A1 $\boldsymbol{\vartheta}^\star \in \boldsymbol{\Xi}$ and $\boldsymbol{\Xi}$ is compact
- A2 The top Lyapunov exponent is strictly negative
- A3 $\eta_t^2$ has non-degenerate distribution and $E(\eta_t^2) = 1$
- A4 If the GARCH order $p > 0$, then $\mathcal{A}_{\boldsymbol{\vartheta}^\star}(z) = \alpha^\star z$ and $\mathcal{B}_{\boldsymbol{\vartheta}^\star}(z) = 1 - \beta^\star z$ have no common roots, $\mathcal{A}_{\boldsymbol{\vartheta}^\star}(1) \neq 0$ and $(\alpha^\star + \beta^\star) \neq 0$

A1 holds by Assumption 18. A2 holds since $-\infty \leq E\big(\ln(\alpha^\star\eta_t^2 + \beta^\star)\big) < 0$ (see Francq and Zakoïan, 2019, Theorem 2.1, p. 22). The parameter restrictions in (18) imply $(\alpha^\star + \beta^\star) < 1$, and this in turn implies $-\infty \leq E\big(\ln(\alpha^\star\eta_t^2+\beta^\star)\big) < 0$, since $E\big(\ln(\alpha^\star\eta_t^2+\beta^\star)\big) \leq \ln E(\alpha^\star\eta_t^2+\beta^\star) = \ln(\alpha^\star+\beta^\star) < 0$ by Jensen's inequality. A3 holds due to the $\eta_t \sim iid(0,1)$ assumption. For $p > 0$ to occur in A4, we must have $\beta^\star > 0$. In this case, $\mathcal{A}_{\boldsymbol{\vartheta}^\star}(z) = \alpha^\star z$ and $\mathcal{B}_{\boldsymbol{\vartheta}^\star}(z) = 1 - \beta^\star z$ have no common roots, since the only root of the former is $z = 0$ and the latter has no roots. Also, $\mathcal{A}_{\boldsymbol{\vartheta}^\star}(1) \neq 0$ and $(\alpha^\star + \beta^\star) \neq 0$, since $\alpha^\star > 0$ in (18). So $\widehat{\boldsymbol{\vartheta}}^\star \xrightarrow{P} \boldsymbol{\vartheta}^\star$. If Assumption 19 also holds then $\{\phi_t^2\}$ is $\beta$-mixing with exponential decay by Corollary 6 in Carrasco and Chen (2002). □

## D.2 The feasible QMLE: Simulations

This subsection contains the details of the simulation results referred to in the first part of Section 4.1. The results are contained in Table 1. Our main objective is to investigate how the estimates of $\alpha$ and $\beta$ in a GARCH(1,1) are affected by the prior estimation of $g_t$.

Under $g$-DGP 1, $g_t$ is constant, so the DGP of $\epsilon_t^2$ corresponds to the standard case where $\{\epsilon_t^2\}$ is strictly stationary. The Ordinary QMLE thus proceeds in the usual way where the parameters $\omega$, $\alpha$ and $\beta$ are estimated in one step. The Two-step QMLE is a variant of the target-variance estimator of Francq et al. (2011): In step 1, $E(\epsilon_t^2)$ is estimated by the sample average $\widehat{g} := T^{-1}\sum_{t=1}^{T}\epsilon_t^2$, whereas in step 2 the estimate

$$\widehat{\boldsymbol{\vartheta}} = \arg\min_{\boldsymbol{\vartheta}\in\boldsymbol{\Xi}} \frac{1}{T}\sum_{t=1}^{T} \ln\widehat{h}_t + \frac{\widehat{\phi}_t^2}{\widehat{h}_t}, \qquad \text{with} \qquad \widehat{h}_t = (1-\widehat{\alpha}-\widehat{\beta}) + \widehat{\alpha}\widehat{\phi}_{t-1}^2 + \widehat{\beta}_{t-1}\widehat{h}_{t-1}$$

is obtained, where $\widehat{\phi}_t^2 = \epsilon_t^2/\widehat{g}$. The estimate of $\omega$ is obtained as $\widehat{\omega} = (1-\widehat{\alpha}-\widehat{\beta})\widehat{g}$. The simulation results of the two estimators are contained in the upper part of Table 1. The results suggest, as expected, that the asymptotic properties of the two estimators are very similar.

Under $g$-DGP 2, $g_t$ is governed by a smooth transition centred around a break location of 50%. In this case, the Two-step QMLE first estimates the parameters of the $g_t$ specification. Next, in step two, estimates of $\alpha$ and $\beta$ are obtained in the same way as for $g$-DGP 1, except that $\widehat{\phi}_t^2$ is now defined as $\widehat{\phi}_t^2 = \epsilon_t^2/\widehat{g}_t$. The simulation results are contained in the bottom part of Table 1. The results suggest the asymptotic properties of the second-step estimator are very similar to those in $g$-DGP 1.

Table 1: Comparison of the Ordinary QMLE and the Two-step QMLE (see Section 4.1)

| $g$-DGP | $T$ | $m(\widehat{\omega})$ | $se(\widehat{\omega})$ | $ase(\widehat{\omega})$ | $m(\widehat{\alpha})$ | $se(\widehat{\alpha})$ | $ase(\widehat{\alpha})$ | $m(\widehat{\beta})$ | $se(\widehat{\beta})$ | $ase(\widehat{\beta})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1: | Ordinary QMLE: | | | | | | | | | |
| | 10000 | 0.0017 | 0.0135 | 0.0132 | 0.0002 | 0.0087 | 0.0088 | −0.0019 | 0.0190 | 0.0190 |
| | 20000 | 0.0009 | 0.0094 | 0.0094 | 0.0002 | 0.0061 | 0.0062 | −0.0011 | 0.0133 | 0.0134 |
| | 40000 | 0.0003 | 0.0068 | 0.0066 | 0.0000 | 0.0045 | 0.0044 | −0.0002 | 0.0097 | 0.0095 |
| | 80000 | 0.0002 | 0.0047 | 0.0047 | −0.0001 | 0.0031 | 0.0031 | −0.0001 | 0.0068 | 0.0067 |
| | Two-step QMLE: | | | | | | | | | |
| | 10000 | 0.0014 | 0.0136 | – | −0.0001 | 0.0087 | – | −0.0016 | 0.0190 | – |
| | 20000 | 0.0087 | 0.0095 | – | −0.0003 | 0.0063 | – | −0.0005 | 0.0136 | – |
| | 40000 | 0.0004 | 0.0068 | – | 0.0000 | 0.0046 | – | −0.0005 | 0.0098 | – |
| | 80000 | 0.0002 | 0.0048 | – | −0.0001 | 0.0032 | – | −0.0001 | 0.0069 | – |

Relative efficiency (two-step *vs.* ordinary):

| $T$ | $\frac{se_2(\widehat{\omega})}{se_1(\widehat{\omega})}$ | $\frac{se_2(\widehat{\alpha})}{se_1(\widehat{\alpha})}$ | $\frac{se_2(\widehat{\beta})}{se_1(\widehat{\beta})}$ |
|---|---|---|---|
| 10000 | 1.0044 | 1.0021 | 0.9997 |
| 20000 | 1.0103 | 1.0377 | 1.0196 |
| 40000 | 1.0007 | 1.0127 | 1.0129 |
| 80000 | 1.0107 | 1.0377 | 1.0206 |

2: Two-step QMLE:

| $T$ | $m(\widehat{\delta}_0)$ $(se(\widehat{\delta}_0))$ | $m(\widehat{\delta}_1)$ $(se(\widehat{\delta}_1))$ | $m(\widehat{\gamma})$ $(se(\widehat{\gamma}))$ | $m(\widehat{c})$ $(se(\widehat{c}))$ | $m(\widehat{\alpha})$ | $se(\widehat{\alpha})$ | $m(\widehat{\beta})$ | $se(\widehat{\beta})$ |
|---|---|---|---|---|---|---|---|---|
| 10000 | −0.0050 (0.0569) | 0.0269 (0.1961) | 0.6592 (2.9829) | 0.0031 (0.0339) | 0.0000 | 0.0087 | −0.0031 | 0.0192 |
| 20000 | −0.0039 (0.0386) | 0.0170 (0.1260) | 0.2612 (1.8592) | 0.0011 (0.0237) | 0.0000 | 0.0059 | −0.0017 | 0.0135 |
| 40000 | −0.0032 (0.0262) | 0.0156 (0.0898) | 0.0720 (1.3271) | 0.0013 (0.0167) | 0.0001 | 0.0042 | −0.0009 | 0.0092 |
| 80000 | 0.0018 (0.0171) | 0.0062 (0.0572) | 0.0337 (0.8697) | 0.0002 (0.0111) | 0.0001 | 0.0032 | −0.0005 | 0.0068 |

DGP: $\epsilon_t = g_t\phi_t$, $\phi_t = \sqrt{h_t}\eta_t$, $\eta_t \overset{iid}{\sim} N(0,1)$, $h_t = 0.1 + 0.1\phi_{t-1}^2 + 0.8h_{t-1}$. $g$-DGP 1: $g_t = 1$ for all $t$. $g$-DGP 2: $g_t = 0.5 + 1.5\big(1 + \exp(-10(t/T - 0.5))\big)^{-1}$. $T$, sample size. $m(\widehat{x})$, average bias of estimate $\widehat{x}$ across replications (no. of replications = 1000). $se(\widehat{x})$, sample standard deviation of estimate $\widehat{x}$ across replications. $ase(\widehat{x})$, asymptotic standard deviation of Ordinary estimate $\widehat{x}$. $se_1(\widehat{x})$ and $se_2(\widehat{x})$, sample standard deviations of the Ordinary and Two-step QMLEs, respectively. All computations in R (R Core Team, 2021). The Ordinary estimator is estimated with the `garchx()` function of the CRAN package **garchx** (Sucarrat, 2021a). The asymptotic standard errors are obtained with the `garchxAvar()` function of the same package. The Two-step QMLE is implemented with own code.

## D.3 Moment estimation of the scaled GARCH(1,1) model

It is well-known that the standard GARCH(1,1) admits a heteroscedastic ARMA(1,1) representation. Kristensen and Linton (2006) used this representation to derive a closed form estimator of the GARCH(1,1) parameters based on the autocovariance functions of the ARMA(1,1) model. Interestingly, the limit of the closed form estimator can be interpreted as a GARCH(1,1) prediction under mis-specification along the lines of Section 4.1.

Suppose, initially, that $\phi^2_{m,t,T}$ is governed by a scaled GARCH(1,1) as in (17)–(18) in Section 4.1. The ARMA(1,1) representation is then

$$\phi^2_{m,t,T} = \omega_m + (\alpha^\star_m + \beta^\star_m)\phi^2_{m,t-1,T} - \beta^\star_m u_{m,t-1,T} + u_{m,t,T}, \qquad \omega^\star_m = 1 - \alpha^\star_m - \beta^\star_m, \tag{40}$$

where $(\alpha^\star_m + \beta^\star_m)$ is the AR-parameter, $(-\beta^\star_m)$ is the MA-parameter and $u_{m,t,T} = \phi^2_{m,t,T} - h_{m,t,T}$ is a heteroscedastic error. Let $\gamma^\star_{m,t,T,j} = E\big((\phi^2_{m,t,T} - 1)(\phi^2_{m,t-j,T} - 1)\big) = E(\phi^2_{m,t,T}\phi^2_{m,t-j,T}) - 1$ denote the $j$th. autocovariance, $j = 0, 1, 2$, and let $\rho^\star_m(j) = \gamma^\star_{m,j}/\gamma^\star_{m,0}$ denote the $j$th. autocorrelation of $\phi^2_{m,t,T}$ under the assumption that the $\gamma^\star_{m,t,Tj}$'s are constant over $t$ and $T$. For the scaled version of the standard GARCH(1,1), the expressions of $\alpha^\star_m$ and $\beta^\star_m$, respectively, are

$$\alpha^\star_m = \rho^\star_m(2)/\rho^\star_m(1) - \beta^\star_m, \tag{41}$$

$$\beta^\star_m = \frac{b^\star_m - \sqrt{(b^\star_m)^2 - 4}}{2}, \qquad b^\star_m = \frac{\rho^\star_m(2)^2/\rho^\star_m(1)^2 + 1 - 2\rho^\star_m(2)}{\rho^\star_m(2)/\rho^\star_m(1) - \rho^\star_m(1)}, \tag{42}$$

see Kristensen and Linton 2006, pp. 325-326. The sample counterparts of $\rho^\star_m(1)$ and $\rho^\star_m(2)$ can thus be used to obtain consistent estimators of $\alpha^\star_m$ and $\beta^\star_m$ under suitable assumptions. Note that, if $\phi^2_{m,t,T}$ is governed by (17)–(18), then the conditions in (18) imply that $\rho^\star_m(1) > 0$, $b^\star_m > 2$ and $b^\star_m < \infty$.

If $\phi^2_{m,t,T}$ is not governed by (17) – (18), the expressions for $\alpha^\star_m$ and $\beta^\star_m$ in (41) and (42) can be used to *define* a specific-valued scaled GARCH(1,1) prediction under mis-specification along the lines of Section 4.1 above. This is the practical consequence of Proposition 7.

**Proposition 7** (existence of the moment-based scaled GARCH(1,1) prediction). *Suppose Assumptions 1–4 hold and that $\gamma^\star_{m,t,T,j} = \gamma^\star_{m,j}$ for all $t, T$, $|\gamma^\star_{m,j}| < \infty$, $j = 0, 1, 2$, $m = 1, \ldots, M$. If, in addition,*

$$\gamma^\star_{m,1}, \gamma^\star_{m,2} > 0, \qquad \frac{\gamma^\star_{m,2}}{\gamma^\star_{m,1}} \neq \frac{\gamma^\star_{m,1}}{\gamma^\star_{m,0}}, \qquad \gamma^\star_{m,1} > \gamma^\star_{m,2}, \qquad \frac{\gamma^\star_{m,2}}{\gamma^\star_{m,1}} > 0, \qquad b_m > 2, \tag{43}$$

*then $\alpha^\star_m$ and $\beta^\star_m$ are given by (41) and (42), respectively, with*

$$\alpha^\star_m, \beta^\star_m > 0, \qquad \alpha^\star_m + \beta^\star_m < 1, \qquad m = 1, \ldots, M. \tag{44}$$

*Proof of Proposition 7.* For notational convenience we omit the subscript $m$. $\alpha^\star$ and $\beta^\star$ are well defined and finite provided $\gamma^\star_1 \neq 0$ and $(\gamma^\star_2/\gamma^\star_1) \neq (\gamma^\star_1/\gamma^\star_0)$ which ensure that $\rho^\star(1) \neq 0$ and $\rho^\star(2)/\rho^\star(1) - \rho^\star(1) \neq 0$. From (41) it follows that $\alpha^\star + \beta^\star = \gamma^\star_2/\gamma^\star_1$. Therefore $\gamma^\star_2 < \gamma^\star_1$ and $\gamma^\star_2/\gamma^\star_1 > 0$ suffice for $0 < \alpha^\star + \beta^\star < 1$. Next, setting the expression for $\beta^\star$ in (41) to 0 yields a contradiction, so $\beta^\star$ cannot be 0. As $b^\star > 2$, $(b^\star)^2 - 4 > 0$ and hence $\beta^\star$ is real, strictly positive and less than 1. □

The implication of Proposition 7 is that it is practically meaningful to *define* the (possibly mis-specified) scaled GARCH(1,1) specification $h_{m,t} = (1 - \alpha^\star_m - \beta^\star_m) + \alpha^\star_m\phi^2_{m,t-1} + \beta^\star_m h_{m,t-1}$ as a prediction, since feasible and consistent estimators of $\alpha^\star_m$ and $\beta^\star_m$ are available. To see this, note that feasible estimators of $\gamma^\star_{m,j}$, $j = 0, 1, 2$, are given by

$$\widehat{\gamma}_{m,j}(\widehat{\boldsymbol{\theta}}_{m,T}) = \frac{1}{T}\sum_{t=1}^{T}\Big(\widehat{\phi}^2_{m,t,T}(\widehat{\boldsymbol{\theta}}_{m,T}) - 1\Big)\Big(\widehat{\phi}^2_{m,t-j,T}(\widehat{\boldsymbol{\theta}}_{m,T}) - 1\Big), \tag{45}$$

where $\widehat{\phi}^2_{m,t,T}(\widehat{\boldsymbol{\theta}}_{m,T}) = \epsilon^2_{m,t,T}/g_{m,t,T}(\widehat{\boldsymbol{\theta}}_{m,T})$. Next, these can be used to obtain feasible estimators of $\alpha^\star_m$ and $\beta^\star_m$:

$$\widehat{\alpha}_m = \widehat{\rho}_m(2)/\widehat{\rho}_m(1) - \widehat{\beta}_m, \tag{46}$$

$$\widehat{\beta}_m = \frac{\widehat{b}_m - \sqrt{\widehat{b}^2_m - 4}}{2}, \qquad \widehat{b}_m = \frac{\widehat{\rho}_m(2)^2/\widehat{\rho}_m(1)^2 + 1 - 2\widehat{\rho}_m(2)}{\widehat{\rho}_m(2)/\widehat{\rho}_m(1) - \widehat{\rho}_m(1)}, \tag{47}$$

where $\widehat{\rho}_m(1) = \widehat{\gamma}_{m,1}/\widehat{\gamma}_{m,0}$ and $\widehat{\rho}_m(2) = \widehat{\gamma}_{m,2}/\widehat{\gamma}_{m,0}$. Consistency of the estimators for $(\alpha^\star_m, \beta^\star_m)'$ as defined in (41) and (42) is established in Theorem 5, whereas asymptotic normality is established in Theorem 6. It is worth underlining that the assumptions required for consistency are very mild, since consistency even holds under certain types of non-stationarities of $\{\phi^2_t\}$ and mis-specification. Drawbacks of the estimators, however, are that weak identification in the sense of Sucarrat (2021b) may not hold, that the random denominator of $\widehat{b}_m$ can be close to zero, and that simulations suggest the estimators are inefficient in small samples, see e.g. Kristensen and Linton (2006), and the Supplemental appendix of Francq and Sucarrat (2023).

**Theorem 5** (consistency of $\widehat{\alpha}_m$ and $\widehat{\beta}_m$). *Suppose the assumptions of Theorem 1, the assumptions of Proposition 7 and Assumption 7 hold. Then* $(\widehat{\alpha}_m, \widehat{\beta}_m)' \xrightarrow{P} (\alpha^\star_m, \beta^\star_m)'$ *for* $m = 1, \ldots, M$.

In Section D.5 we establish the asymptotic normality of these estimators under additional assumptions.

### D.4 Proof of Theorem 5

For notational convenience we omit the subscript $m$. By a mean-value expansion,

$$\widehat{\gamma}_j(\widehat{\boldsymbol{\theta}}_T) = \widehat{\gamma}_j(\boldsymbol{\theta}^\star) + \frac{\partial \widehat{\gamma}_j(\overline{\boldsymbol{\theta}}_T)}{\partial \boldsymbol{\theta}'}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^\star),$$

where $\overline{\boldsymbol{\theta}}_T$ is a mean-value between $\widehat{\boldsymbol{\theta}}_T$ and $\boldsymbol{\theta}^\star$. Given Theorem 1 it follows that $\widehat{\gamma}_j(\widehat{\boldsymbol{\theta}}_T) \xrightarrow{P} \gamma^\star_j$ if $\widehat{\gamma}_j(\boldsymbol{\theta}^\star) \xrightarrow{P} \gamma^\star_j$ and $\frac{\partial \widehat{\gamma}_j(\overline{\boldsymbol{\theta}}_T)}{\partial \boldsymbol{\theta}'} = O_P(1)$. We first show that $\widehat{\gamma}_j(\boldsymbol{\theta}^\star)$ is consistent. At $\boldsymbol{\theta}^\star$,

$$\widehat{\gamma}_j(\boldsymbol{\theta}^\star) = \frac{1}{T}\sum_{t=1}^{T}\left(\phi^2_{t,T} - 1\right)\left(\phi^2_{t-j,T} - 1\right) = \frac{1}{T}\sum_{t=1}^{T} a(\phi^2_{t,T}, \phi^2_{t-j,T}),$$

where $a(\phi^2_{t,T}, \phi^2_{t-j,T})$ is defined to be the summand to make the notation more compact. The assumption in Proposition 7 implies that $\gamma^\star_{t,j,T}$ is constant across $t, T$ for $j = 0, 1, 2$ and so $Ea\big(\phi^2_{t,T}\phi^2_{t-j,T}\big) = \gamma^\star_j$ is constant over $t, T$ for $j = 0, 1, 2$. Let $X_{t,T} := a(\phi^2_{t,T}, \phi^2_{t-j,T}) - E\big(a(\phi^2_{t,T}, \phi^2_{t-j,T})\big)$ for $1 \leq t \leq T$ with $X_{t,T} = 0$ otherwise and let $\mathcal{F}_{t,T} := \sigma(\{X_{i,T} : i \leq t\})$. Assumption 7 yields that $\sup_{1\leq t\leq T, T\in\mathbb{N}} E|X_{t,T}|^p < \infty$ for some $p > 1$. Hence $\{X_{t,T} : t \in \mathbb{Z},\ T \in \mathbb{N}\}$ is uniformly integrable and, given Assumption 3, $\{X_{t,T}/T, \mathcal{F}_{t,T} : t \in \mathbb{Z},\ T \in \mathbb{N}\}$ forms a $L_1$-mixingale array with respect to the constants $c_{t,T} = 1/T$ by Theorems 14.1 and 14.2 in Davidson (1994).[18] Therefore, by Theorem 19.11 in Davidson (1994),

$$E\left|\frac{1}{T}\sum_{t=1}^{T} a(\phi^2_{t,T}, \phi^2_{t-j,T}) - \frac{1}{T}\sum_{t=1}^{T} E\big(a(\phi^2_{t,T}, \phi^2_{t-j,T})\big)\right| \to 0,$$

which implies $\widehat{\gamma}_j(\boldsymbol{\theta}^\star) \xrightarrow{P} \gamma^\star_j$, $j = 0, 1, 2$. We now show that $\frac{\partial \widehat{\gamma}_j(\overline{\boldsymbol{\theta}}_T)}{\partial \boldsymbol{\theta}'} = O_P(1)$. We have

$$\frac{\partial \widehat{\gamma}_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{1}{T}\sum_{t=1}^{T} \frac{\partial \widehat{\phi}^2_{t,T}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\widehat{\phi}^2_{t-j,T}(\boldsymbol{\theta}) + \widehat{\phi}^2_{t,T}(\boldsymbol{\theta})\frac{\partial \widehat{\phi}^2_{t-j,T}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - \frac{\partial \widehat{\phi}^2_{t,T}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - \frac{\partial \widehat{\phi}^2_{t-j,T}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

[18] The details of the application of Theorem 19.11 of Davidson (1994) are analogous to those given in the proof of Theorem 1 and so are omitted.

with

$$\widehat{\phi}^2_{t-j,T}(\boldsymbol{\theta}) = \frac{g_{t-j,T}(\boldsymbol{\theta}^\star)\phi^2_{t-j,T}}{g_{t-j,T}(\boldsymbol{\theta})}, \qquad \frac{\partial\widehat{\phi}^2_{t-j,T}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} = -\frac{g_{t-j,T}(\boldsymbol{\theta}^\star)\phi^2_{t-j,T}}{\big(g_{t-j,T}(\boldsymbol{\theta})\big)^2}\,\dot{\boldsymbol{g}}_{t-j,T}(\boldsymbol{\theta}), \qquad j = 0,1,2.$$

For more compact notation, let $\frac{\partial\widehat{\gamma}_j(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'} = F_{j,T}(\boldsymbol{\theta}) = T^{-1}\sum_{t=1}^T b_{t,T}(\phi^2_{t,T},\phi^2_{t-j,T},\boldsymbol{\theta})$, i.e.

$$\begin{aligned} b_{t,T}(\phi^2_{t,T},\phi^2_{t-j,T},\boldsymbol{\theta}) &\coloneqq \phi^2_{t,T}\left(\frac{g_{t,T}(\boldsymbol{\theta}^\star)\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})}{g_{t,T}(\boldsymbol{\theta})^2}\right) + \phi^2_{t-j,T}\left(\frac{g_{t-j,T}(\boldsymbol{\theta}^\star)\dot{\boldsymbol{g}}_{t-j,T}(\boldsymbol{\theta})}{g_{t-j,T}(\boldsymbol{\theta})^2}\right) \\ &\quad - \phi^2_{t,T}\phi^2_{t-j,T}\left(\frac{g_{t,T}(\boldsymbol{\theta}^\star)\dot{\boldsymbol{g}}_{t,T}(\boldsymbol{\theta})}{g_{t,T}(\boldsymbol{\theta})^2}\frac{g_{t-j,T}(\boldsymbol{\theta}^\star)}{g_{t-j,T}(\boldsymbol{\theta})} + \frac{g_{t-j,T}(\boldsymbol{\theta}^\star)\dot{\boldsymbol{g}}_{t-j,T}(\boldsymbol{\theta})}{g_{t-j,T}(\boldsymbol{\theta})^2}\frac{g_{t,T}(\boldsymbol{\theta}^\star)}{g_{t,T}(\boldsymbol{\theta})}\right). \end{aligned}$$

By Assumption 2 and the fact that $\overline{\boldsymbol{\theta}} \xrightarrow{P} \boldsymbol{\theta}^\star$ given Theorem 1 as it is a mean-value between $\hat{\boldsymbol{\theta}}_T$ and $\boldsymbol{\theta}^\star$, the absolute value of each of the terms in parenthesis is bounded above by some constant $C$ on sets $E_n$ with $P(E_n) \to 1$. On these sets

$$F_{j,T}(\overline{\boldsymbol{\theta}}) \le C\frac{1}{T}\sum_{t=1}^T \phi^2_{t,T} + \phi^2_{t-j,T} + \phi^2_{t,T}\phi^2_{t-j,T}.$$

By Markov's inequality and Assumption 7

$$P\left(\frac{1}{T}\sum_{t=1}^T \phi^2_{t,T} + \phi^2_{t-j,T} + \phi^2_{t,T}\phi^2_{t-j,T} > M\right) \le M^{-1}\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\phi^2_{t,T} + \phi^2_{t-j,T} + \phi^2_{t,T}\phi^2_{t-j,T}\right] \lesssim M^{-1}.$$

Combine these observations to conclude that $F_{j,T}(\overline{\boldsymbol{\theta}}) = O_P(1)$. Thus, $\widehat{\gamma}_{m,j}(\widehat{\boldsymbol{\theta}}_T) \xrightarrow{P} \gamma^\star_{m,j}$ and by the continuous mapping theorem $(\widehat{\alpha}_m,\widehat{\beta}_m)' \xrightarrow{P} (\alpha_m,\beta_m)'$.

## D.5 Asymptotic normality of the moment estimators

To establish asymptotic normality of the estimators in Theorem 5, we require the following additional conditions.

**Assumption 20** *For each $m = 1,\dots,M$, $\sup_{1\le t\le T, T\in\mathbb{N}} E|\phi^4_{m,t,T}|^{2+\tilde{\delta}_m}$ for some $\tilde{\delta}_m > 0$. Additionally, the strong mixing coefficients $\alpha_{m,T}(k)$ satisfy*

$$\sup_{T\in\mathbb{N}} \alpha_{m,T}(k) = O(k^{-\tilde{\rho}_m-\varepsilon}),$$

*for some $\varepsilon > 0$ where $\tilde{\rho}_m \coloneqq \tilde{r}_m/(\tilde{r}_m - 2)$ with $\tilde{r}_m \coloneqq 2 + \tilde{\delta}_m$.*

For the following condition, let $\overline{\boldsymbol{F}}_{m,T}$ be defined (on $\mathcal{V}_m$) as

$$\overline{\boldsymbol{F}}_{m,T}(\boldsymbol{\theta}_m) \coloneqq \frac{1}{T}\sum_{t=1}^T E[\boldsymbol{b}_{m,t,T}(\boldsymbol{\theta}_m)], \qquad \boldsymbol{b}_{m,t,T} \coloneqq \begin{pmatrix}\boldsymbol{b}_{m,t,T,0} & \boldsymbol{b}_{m,t,T,1} & \boldsymbol{b}_{m,t,T,2}\end{pmatrix}',$$

where for $j = 0,1,2$,

$$\begin{aligned} \boldsymbol{b}_{m,t,T,j}(\boldsymbol{\theta}_m) &\coloneqq \phi^2_{m,t,T}\left(\frac{g_{m,t,T}(\boldsymbol{\theta}^\star_m)\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)}{g_{m,t,T}(\boldsymbol{\theta}_m)^2}\right) + \phi^2_{m,t-j,T}\left(\frac{g_{m,t-j,T}(\boldsymbol{\theta}^\star_m)\dot{\boldsymbol{g}}_{m,t-j,T}(\boldsymbol{\theta}_m)}{g_{m,t-j,T}(\boldsymbol{\theta}_m)^2}\right) \\ &\quad - \phi^2_{m,t,T}\phi^2_{m,t-j,T}\left(\frac{g_{m,t,T}(\boldsymbol{\theta}^\star_m)\dot{\boldsymbol{g}}_{m,t,T}(\boldsymbol{\theta}_m)}{g_{m,t,T}(\boldsymbol{\theta}_m)^2}\frac{g_{m,t-j,T}(\boldsymbol{\theta}^\star_m)}{g_{m,t-j,T}(\boldsymbol{\theta}_m)}\right) \\ &\quad - \phi^2_{m,t,T}\phi^2_{m,t-j,T}\left(\frac{g_{m,t-j,T}(\boldsymbol{\theta}^\star_m)\dot{\boldsymbol{g}}_{m,t-j,T}(\boldsymbol{\theta}_m)}{g_{m,t-j,T}(\boldsymbol{\theta}_m)^2}\frac{g_{m,t,T}(\boldsymbol{\theta}^\star_m)}{g_{m,t,T}(\boldsymbol{\theta}_m)}\right). \end{aligned}$$

**Assumption 21** *For each $m = 1, \dots, M$, the limit*

$$\overline{\boldsymbol{F}}^{\star}_{m,j}(\boldsymbol{\theta}_m) \coloneqq \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E[\boldsymbol{b}_{m,t,T,j}(\boldsymbol{\theta}_m)], \qquad \boldsymbol{\theta}_m \in \mathcal{V}_m, \tag{48}$$

*exists where $\mathcal{V}_m$ is as in Assumption 8. Additionally, there are random variables $\tilde{\psi}_{m,t,T}$ such that for $\boldsymbol{\theta}_m, \boldsymbol{\theta}'_m \in \mathcal{V}_m$*

$$\|\boldsymbol{b}_{m,t,T}(\boldsymbol{\theta}_m) - \boldsymbol{b}_{m,t,T}(\boldsymbol{\theta}'_m)\| \le \tilde{\psi}_{m,t,T} \|\boldsymbol{\theta}_m - \boldsymbol{\theta}'_m\|,$$

*with $\sup_{1\le t\le T, T\in\mathbb{N}} E|\tilde{\psi}_{m,t,T}| < \infty$.*

The following Assumption strengthens Assumption 10.

**Assumption 22** *For each $m = 1, \dots, M$, as $T \to \infty$,*

$$\boldsymbol{D}_{m,T} \coloneqq \operatorname{Var}\left(T^{-1/2} \sum_{t=1}^{T} \boldsymbol{y}_{m,t,T}\right) \to \boldsymbol{D}^{\star}_m, \qquad \boldsymbol{D}^{\star}_m \textit{ positive definite},$$

*for*

$$\boldsymbol{y}_{m,t,T} \coloneqq \begin{bmatrix} \boldsymbol{i}_{m,t,T}(\boldsymbol{\theta}^{\star}_m, \epsilon^2_{m,t,T}) \\ \phi^4_{m,t,T} - E(\phi^4_{m,t,T}) \\ \phi^2_{m,t,T}\phi^2_{m,t-1,T} - E(\phi^2_{m,t,T}\phi^2_{m,t-1,T}) \\ \phi^2_{m,t,T}\phi^2_{m,t-2,T} - E(\phi^2_{m,t,T}\phi^2_{m,t-2,T}) \end{bmatrix}.$$

**Theorem 6** *Suppose the assumptions of Theorems 1, 2, Proposition 7 and Assumptions 20 – 22 hold. Then $\sqrt{T}(\widehat{\gamma}_m - \boldsymbol{\gamma}^{\star}_m) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{\Sigma}^{\star}_{m,\gamma})$ with $\boldsymbol{\Sigma}^{\star}_{m,\gamma} = \boldsymbol{C}^{\star}_m \boldsymbol{D}^{\star}_m [\boldsymbol{C}^{\star}_m]'$, $m = 1, \dots, M$, and*

$$\sqrt{T}\begin{pmatrix} \widehat{\alpha}_m - \alpha^{\star}_m \\ \widehat{\beta}_m - \beta^{\star}_m \end{pmatrix} \xrightarrow{D} N\big(\mathbf{0}, \boldsymbol{\Upsilon}_m(\boldsymbol{\gamma}^{\star}_m)\boldsymbol{\Sigma}_{m,\gamma}\boldsymbol{\Upsilon}_m(\boldsymbol{\gamma}^{\star}_m)'\big), \qquad m = 1, \dots, M, \tag{49}$$

*where $\boldsymbol{\Upsilon}_m(\boldsymbol{\gamma}^{\star}_m) = \left(\frac{\partial\alpha_m(\boldsymbol{\gamma}^{\star}_m)}{\partial\boldsymbol{\gamma}'_m}, \frac{\partial\beta_m(\boldsymbol{\gamma}^{\star}_m)}{\partial\boldsymbol{\gamma}'_m}\right)'$.*

The exact expressions for $\boldsymbol{C}^{\star}_m$ and $\boldsymbol{D}^{\star}_m$ are contained in the proof. A consistent estimator of $\boldsymbol{C}^{\star}_m$ can be derived by using the estimates of $g_{t,T}(\boldsymbol{\theta}^{\star}_m)$, $\gamma^{\star}_{m,0}$, $\gamma^{\star}_{m,1}$ and $\gamma^{\star}_{m,2}$ as ingredients; a consistent estimator of $\boldsymbol{D}^{\star}_m$ of the HAC type can be derived along the same lines as in Proposition 1. The proof requires the following lemma.

**Lemma 7.** *In the setting of Theorem 6, for any $\overline{\boldsymbol{\theta}}_T \xrightarrow{P} \boldsymbol{\theta}^{\star}$,*

$$\frac{\partial\boldsymbol{\gamma}_m(\overline{\boldsymbol{\theta}}_T)}{\partial\boldsymbol{\theta}'} \xrightarrow{P} \overline{\boldsymbol{F}}^{\star}_m = \big(\overline{\boldsymbol{F}}_{m,0}(\boldsymbol{\theta}^{\star}), \overline{\boldsymbol{F}}_{m,1}(\boldsymbol{\theta}^{\star}), \overline{\boldsymbol{F}}_{m,2}(\boldsymbol{\theta}^{\star})\big)'.$$

*Proof.* Dropping the $m$ in the notation, let $\boldsymbol{F}_T(\boldsymbol{\theta}) \coloneqq \frac{1}{T}\sum_{t=1}^{T} b_{t,T}(\boldsymbol{\theta})$ on $\mathcal{V}$. It is clearly sufficient to show that $\boldsymbol{F}_T$ converges to $\boldsymbol{F}^{\star}$ uniformly in probability. Initially, we show pointwise convergence in probability of $\boldsymbol{F}_T - \overline{\boldsymbol{F}}_T$ to zero on $\mathcal{V}$. Let $X_{t,T} \coloneqq \boldsymbol{b}_{t,T}(\boldsymbol{\theta}) - E[\boldsymbol{b}_{t,T}(\boldsymbol{\theta})]$ for $1 \le t \le T$ with $X_{t,T} = 0$ otherwise and $\mathcal{F}_{t,T} \coloneqq \sigma(X_{t,T}, X_{t-1,T}, \dots)$. Assumptions 2,20 allow the application of Theorems 14.1 & 14.2 in Davidson (1994) to permit the conclusion that $(X_{t,T}/T, \mathcal{F}_{t,T})_{t\in\mathbb{Z}, T\in\mathbb{N}}$ is an $L_1$ – mixingale arrray with respect to the constants $c_{t,T} = 1/T$.[19] Therefore Theorem 19.11 in Davidson (1994) yields that

$$\boldsymbol{F}_T(\boldsymbol{\theta}) - \overline{\boldsymbol{F}}_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{b}_{t,T}(\boldsymbol{\theta}) - E[\boldsymbol{b}_{t,T}(\boldsymbol{\theta})] \xrightarrow{P} 0. \tag{50}$$

[19] The details of the application of Theorem 19.11 of Davidson (1994) are analogous to those given in the proof of Theorem 1 and so are omitted.

By Assumption 21, one has

$$\|\overline{\boldsymbol{F}}_T(\boldsymbol{\theta}) - \overline{\boldsymbol{F}}_T(\boldsymbol{\theta}')\| \leq \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}\left\|b_{t,T}(\boldsymbol{\theta}) - b_{t,T}(\boldsymbol{\theta}')\right\| \leq \frac{1}{T}\sum_{t=1}^{T} E\tilde{\psi}_{t,T}\|\boldsymbol{\theta} - \boldsymbol{\theta}'\| \leq C\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|,$$

for some constant $C \in (0, \infty)$. Hence each $\overline{\boldsymbol{F}}_T$ is Lipschitz on $\mathcal{V}$ with a common Lipschitz constant and so uniformly equicontinuous on $\mathcal{V}$. Similarly,

$$\|\boldsymbol{F}_T(\boldsymbol{\theta}) - \boldsymbol{F}_T(\boldsymbol{\theta}')\| \leq \left[\frac{1}{T}\sum_{t=1}^{T} \tilde{\psi}_{t,T}\right] \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|,$$

with $\frac{1}{T}\sum_{t=1}^{T} \tilde{\psi}_{t,T} = O_P(1)$. In combination with the uniform equiconitinuity of $\overline{\boldsymbol{F}}_T$, this verifies Assumption SE-1 in Andrews (1992) (elementwise). Hence Lemma 1 of Andrews (1992) yields that for each co-ordinate $k$, $\{\boldsymbol{F}_{T,k} - \overline{\boldsymbol{F}}_{T,k} : T \in \mathbb{N}\}$ is stochastically equicontinuous on $\mathcal{V}$. Given (50) and that $\mathcal{V}$ is totally bounded as a subset of a compact metric space, Theorem 1 in Andrews (1992) applied to each co-ordinate $k$ yields that

$$\sup_{\boldsymbol{\theta}\in\mathcal{V}} \|\boldsymbol{F}_T(\boldsymbol{\theta}) - \overline{\boldsymbol{F}}_T(\boldsymbol{\theta})\| \xrightarrow{P} 0.$$

By Assumption 21, $\overline{\boldsymbol{F}}_T(\boldsymbol{\theta}) - \overline{\boldsymbol{F}}^{\star} \to 0$ as $T \to \infty$. Since, as noted above, $\{\overline{\boldsymbol{F}}_T : T \in \mathbb{N}\}$ is uniformly equicontinuous on the totally bounded set $\mathcal{V}$, this convergence is uniform. Combining this with the preceding display yields

$$\sup_{\boldsymbol{\theta}\in\mathcal{V}} \|\boldsymbol{F}_T(\boldsymbol{\theta}) - \overline{\boldsymbol{F}}^{\star}(\boldsymbol{\theta})\| \xrightarrow{P} 0.$$

Since $\bar{\theta}_T \xrightarrow{P} \boldsymbol{\theta}$, and $\mathcal{V}$ is a neighbourhood of $\boldsymbol{\theta}$, the claim follows. $\square$

*Proof of Theorem 6.* For notational convenience we omit the subscript $m$. Let $\widehat{\boldsymbol{\gamma}} = \big(\widehat{\gamma}_0(\widehat{\boldsymbol{\theta}}), \widehat{\gamma}_1(\widehat{\boldsymbol{\theta}}), \widehat{\gamma}_2(\widehat{\boldsymbol{\theta}})\big)'$, where the expression for $\widehat{\gamma}_j(\widehat{\boldsymbol{\theta}})$ is contained in (45). By a mean-value expansion,

$$\widehat{\boldsymbol{\gamma}}(\widehat{\boldsymbol{\theta}}_T) = \widehat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^{\star}) + \frac{\partial\boldsymbol{\gamma}(\overline{\boldsymbol{\theta}}_T)}{\partial\boldsymbol{\theta}'}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\star}),$$

where $\overline{\boldsymbol{\theta}}_T$ is a mean value between $\widehat{\boldsymbol{\theta}}_T$ and $\boldsymbol{\theta}^{\star}$. Let $\boldsymbol{\gamma}^{\star} = (\gamma_0^{\star}, \gamma_1^{\star}, \gamma_2^{\star})'$, so that we can write

$$\begin{aligned}
\sqrt{T}\big(\widehat{\boldsymbol{\gamma}}(\widehat{\boldsymbol{\theta}}_T) - \boldsymbol{\gamma}^{\star}\big) &= \sqrt{T}\big(\widehat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^{\star}) - \boldsymbol{\gamma}^{\star}\big) + \frac{\partial\boldsymbol{\gamma}(\overline{\boldsymbol{\theta}}_T)}{\partial\boldsymbol{\theta}'}\sqrt{T}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\star}) \\
&= \sqrt{T}\big(\widehat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^{\star}) - \boldsymbol{\gamma}^{\star}\big) + \overline{\boldsymbol{F}}^{\star}\sqrt{T}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\star}) + o_P(1),
\end{aligned}$$

since $\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\star}) = O_P(1)$ by Theorem 2 and where $\overline{\boldsymbol{F}}^{\star} = \big(\overline{F}_0(\boldsymbol{\theta}^{\star}), \overline{F}_1(\boldsymbol{\theta}^{\star}), \overline{F}_2(\boldsymbol{\theta}^{\star})\big)'$ is the probability limit of $\partial\boldsymbol{\gamma}(\overline{\boldsymbol{\theta}}_T)/\partial\boldsymbol{\theta}'$ as $T \to \infty$ which is established in Lemma 7.

Let $\boldsymbol{x}_{t,T} = \big(\phi_{t,T}^4 - E(\phi_{t,T}^4), \phi_{t,T}^2\phi_{t-1,T}^2 - E(\phi_{t,T}^2\phi_{t-1,T}^2), \phi_{t,T}^2\phi_{t-2,T}^2 - E(\phi_{t,T}^2\phi_{t-2,T}^2)\big)'$. Then

$$\sqrt{T}\big(\widehat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^{\star}) - \boldsymbol{\gamma}^{\star}\big) = \sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_{t,T}\right).$$

(The proof of) Theorem 2 (implicitly) uses the result that (cf. Theorem 3.1 in Newey and McFadden (1994))

$$\sqrt{T}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\star}) = [\boldsymbol{A}^{\star}]^{-1}\sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{i}_{t,T}(\boldsymbol{\theta}^{\star}, \epsilon_{t,T}^2)\right) + o_P(1).$$

Combine these to obtain

$$\begin{aligned}\sqrt{T}\big(\widehat{\gamma}(\widehat{\boldsymbol{\theta}}_T)-\boldsymbol{\gamma}^\star\big) &= \overline{\boldsymbol{F}}^\star\sqrt{T}(\widehat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^\star)+\sqrt{T}\big(\widehat{\gamma}(\boldsymbol{\theta}^\star)-\boldsymbol{\gamma}^\star\big)+o_P(1),\\ &= \boldsymbol{C}^\star\, T^{-1/2}\sum_{t=1}^{T}\boldsymbol{y}_{t,T}+o_P(1),\end{aligned}$$

where

$$\boldsymbol{C}^\star=\begin{pmatrix}\overline{\boldsymbol{F}}^\star[\boldsymbol{A}^\star]^{-1} & \underset{(3\times3)}{\boldsymbol{I}}\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{y}_{t,T}=\Big(\dot{\boldsymbol{l}}_{t,T}(\boldsymbol{\theta}^\star,\epsilon^2_{t,T})',\boldsymbol{x}'_{t,T}\Big)'.$$

with $E(\boldsymbol{y}_{t,T})=\mathbf{0}$ for all $t,T$ by Assumption 4. We now show $T^{-1/2}\sum_{t=1}^T\boldsymbol{y}_{t,T}\xrightarrow{D}N(\mathbf{0},\boldsymbol{D}^\star)$ with $\boldsymbol{D}^\star:=\lim_{T\to\infty}\mathrm{Var}\Big(T^{-1/2}\sum_{t=1}^T\boldsymbol{y}_{t,T}\Big)$. Notice that the upper-left $(K\times K)$ block of $\boldsymbol{D}^\star$, where $K$ is the dimension of $\boldsymbol{\theta}^\star$, is equal to $\boldsymbol{B}^\star$ in Assumption 10. Let $Z_{t,T}:=T^{-1/2}\lambda'\boldsymbol{y}_{t,T}$ for $\|\lambda\|_2=1$, and let $\sigma_T:=\|\sum_{t=1}^T Z_{t,T}\|_{L_2}$ and $X_{t,T}=Z_{t,T}/\sigma_T$ for $1\le t\le T$ and 0 otherwise. That $\sigma_T$ is finite follows from Assumption 20; that it is (at least eventually) non-zero follows from Assumption 22. Let $\mathcal{F}_{t,T}:=\sigma(\epsilon^2_{s,T}:s\le t)$. We verify the conditions of Corollary 1 in de Jong (1997) as in the proof of Theorem 2.[20] (a) follows as $X_{t,T}$ is a mean-zero random variable with $\|\sum_{t=1}^T X_{t,T}\|_{L_2}=1$. For (b) set $c_{t,T}:=\max\{\|Z_{t,T}\|_{L_2},1\}/\sigma_T$ for $1\le t\le T$ and $1/\sigma_T$ otherwise. By the moment bounds in Assumption 20

$$\sup_{1\le t\le T,T\in\mathbb{N}}\|X_{t,T}/c_{t,T}\|_{L_{r_m}}\le\sup_{1\le t\le T,T\in\mathbb{N}}\sigma_T\|X_{t,T}\|_{L_{r_m}}=\sup_{1\le t\le T,T\in\mathbb{N}}\|Z_{t,T}\|_{L_{r_m}}<\infty, \tag{51}$$

For (c), since each $X_{t,T}$ is $\mathcal{F}_{t,T}$–measurable (and in $L_2$), taking $d_{t,T}=c_{t,T}$ and any sequence $v_l=o(l^{-1/2-\varepsilon})$ for a $\varepsilon>0$ we conclude that $X_{t,T}$ is $L_2$ – NED of size $-1/2$ on $(\epsilon^2_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ and $d_{t,T}/c_{t,T}=1$ is uniformly bounded. By Assumption 20 $(\epsilon^2_{t,T})_{t\in\mathbb{Z},T\in\mathbb{N}}$ is $\alpha$-mixing of size $-\tilde{\rho}$, with $\tilde{\rho}:=\tilde{r}/(\tilde{r}-2)$.[21] Finally, we note that by the moment bounds in Assumption 20 $Tc^2_{t,T}\le\frac{1}{\sigma_T^2}\max\{\|\boldsymbol{y}_{t,T}\|^2_{L_2},1\}\lesssim\frac{1}{\sigma_T^2}$. By Assumption 22

$$\sigma_T^2=\left\|\sum_{t=1}^T Z_{t,T}\right\|^2_{L_2}=\lambda'\boldsymbol{D}_T\lambda\to\lambda'\boldsymbol{D}^\star\lambda>0. \tag{52}$$

Hence $c^2_{t,T}=O(T^{-1})$, establishing the final condition of Corollary 1 of de Jong (1997) with $\beta=\gamma=0$. Therefore, $\sum_{t=1}^T X_{t,T}\xrightarrow{D}N(0,1)$. In conjunction with (52) and Slutsky's Theorem this implies $\sum_{t=1}^T Z_{t,T}\xrightarrow{D}N(0,\lambda'\boldsymbol{D}^\star\lambda)$. Hence $T^{-1/2}\sum_{t=1}^T\boldsymbol{y}_{t,T}\xrightarrow{D}N(\mathbf{0},\boldsymbol{D}^\star)$ holds by the Cramér – Wold Theorem. Next, applying Slutsky's theorem again gives $\sqrt{T}(\widehat{\gamma}-\gamma^\star)\xrightarrow{D}N\big(\mathbf{0},\boldsymbol{C}^\star\boldsymbol{D}^\star[\boldsymbol{C}^\star]'\big)$. Apply the delta method to obtain (49). □

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[20]Here (a), (b) and (c) are as listed in the proof of Theorem 2; they (respectively) correspond to parts (a), (b) and (c) of Assumption 2 in de Jong (1997). The argument given here is entirely analogous to that in the proof of Theorem 2 and so some details are omitted.

[21]Here $\epsilon^2_{t,T}$, $\tilde{\rho}$ and $\tilde{r}$ are $\epsilon^2_{m,t,T}$, $\tilde{\rho}_m$ and $\tilde{r}_m$ with the subscript $m$ omitted.

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