Locally regular and efficient tests in non-regular semiparametric models

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Abstract

This paper considers hypothesis testing in semiparametric models which may be non-regular. I show that C(α) style tests are locally regular under mild conditions, including in cases where locally regular estimators do not exist, such as models which are (semi-parametrically) weakly identified. I characterise the appropriate limit experiment in which to study local (asymptotic) optimality of tests in the non-regular case, permitting the generalisation of classical power bounds to this case. I give conditions under which these power bounds are attained by the proposed C(α) style tests. The application of the theory to a single index model and an instrumental variables model is worked out in detail.

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1 Introduction

It is often considered desirable that estimators are "locally regular" in that they exhibit the same limiting behaviour under the true parameter as they do under sequences of "local alternatives" which cannot be consistently distinguished from the true parameter, even asymptotically.[1] Unfortunately, there are many semiparametric models in which locally regular estimators do not exist.[2] One necessary condition is given by Chamberlain (1986), who shows that if the efficient information for a scalar parameter is 0, then no locally regular estimator of that parameter exists. This result can be extended to singularity of the efficient information matrix implying the non-existence of locally regular estimators of Euclidean parameters. Models in which this may occur are called "non-regular". There are many widely used models in econometrics which are non-regular (at certain parameter values). Prominent examples include instrumental variables models, discrete choice models, single index models, mixed proportional hazard models, sample selection models and errors-in-variables models.

In this paper, I demonstrate that locally regular *tests* exist in a broad class of non-regular models, despite the non-existence of locally regular estimators. This result is constructive: I show that a class of tests based on the C(α) idea of Neyman (1959, 1979) are locally regular, i.e. they exhibit the same limiting behaviour under both the true parameter and local alternatives.

These tests are based on a quadratic form of moment conditions evaluated under the null hypothesis. The key [C(α)] idea which ensures the local regularity is that the moment conditions must be (asymptotically) orthogonal to the collection of score functions for all nuisance parameters. Such moment conditions can always be constructed from any initial moment conditions by an orthogonal projection.

A key advantage of these C(α) tests is that they do not (asymptotically) over-reject under (semiparametric) weak identification asymptotics, i.e. under local alternatives to a point of under- /un-identification.[3] The local regularity of these tests ensures that if the test is asymptotically of level α under any fixed parameter consistent with the null, it is also asymptotically of level α under any sequence of local alternatives consistent with the null, i.e. under weak identification asymptotics. In addition to the well-studied case where weak identification is due to

[1] Precise definitions will be given below. See Bickel, Klaassen, Ritov, and Wellner (1998); van der Vaart (1998), for example, for textbook treatments.

[2] See e.g. Chamberlain (1986, 1992); Newey (1990); Ritov and Bickel (1990) for some examples.

[3] The semiparametric weak identification asymptotics used are those of Kaji (2021) (see also Andrews and Mikusheva (2022)), suitably generalised to permit non-i.i.d models.

potential identification failure due to a finite dimensional nuisance parameter, the results in this paper also cover the case where identification failure is due to an infinite dimensional nuisance parameter and thus provides a generally applicable approach to weak-identification robust inference in semiparametric models.[4]

Achieving this local regularity does *not* come at the expense of (local asymptotic) power. I characterise (local asymptotic) power bounds for tests in non-regular models and show that the C(α) tests proposed in this paper acheive these power bounds provided the moment conditions are chosen optimally. These power bounds contain those for regular models as a special case and, moreover, the conditions imposed are weaker than those in the literature.[5]

Following the theoretical development, I give details of its application to two examples: (i) a single index model which may be weakly identified when the link function is too flat and (ii) an instrumental variables model which may be weakly identified when the (nonparametric) first stage is too close to the zero function. Simulation experiments based on these examples demonstrate that the proposed tests display good finite sample performance.

This paper is connected to three main strands of the literature: the first is that concerned with general results on estimation and testing in semiparametric models. Much of this is now textbook material: see e.g. Newey (1990); Choi et al. (1996); Bickel et al. (1998); van der Vaart (1998). The second is the literature on C(α) tests. These were introduced by Neyman (1959, 1979) and have seen many useful applications, most recently as a way to handle machine learning or otherwise high dimensional first steps (see e.g. Chernozhukov, Hansen, and Spindler, 2015; Bravo, Escanciano, and Van Keilegom, 2020; Chernozhukov, Escanciano, Ichimura, Newey, and Robins, 2022). In this paper, the same structure which ensures the good performance of these tests in such settings is used for a different purpose – to create tests which remain robust in non-regular settings. Lastly, the literature on robust testing in non-regular or otherwise non-standard settings is closely related to this paper (e.g. Andrews and Guggenberger, 2009; Romano and Shaikh, 2012; Elliott, Müller, and Watson, 2015; McCloskey, 2017). In particular, the locally regular tests derived in this paper have desirable theoretical properties under semiparametric weak identification and therefore this paper is closely re-

[4]These C(α) tests also behave well in other non-standard settings, such as when nuisance functions are estimated under shape constraints; see Section S2.1 for a discussion.

[5]In particular, in regular models the attainment result is well known if either (a) the observations are i.i.d. (cf. van der Vaart, 1998, Chapter 25) or (b) the information operator (as defined in Choi, Hall, and Schick, 1996, p. 846) is boundedly invertible (Choi et al., 1996). The result in this paper does not require either of these conditions.

lated to the literature on weak identification robust inference in econometrics (e.g. Staiger and Stock, 1997; Dufour, 1997; Stock and Wright, 2000; Kleibergen, 2005; Andrews and Cheng, 2012; Andrews and Mikusheva, 2015, 2016). More specifically, this paper is most closely related to the recent work on semiparametric weak identification (Kaji, 2021; Andrews and Mikusheva, 2022) and extends the notion of semiparametric weak identification considered therein to non-i.i.d. models.[6]

2 Locally regular testing

2.1 The local setup

The goal considered throughout this paper is to construct tests of the hypothesis $\mathrm{H}_0 : \theta = \theta_0$ against $\mathrm{H}_1 : \theta \neq \theta_0$ in the sequence of models $\mathcal{P}_n = \{P_{n,\gamma} : \gamma \in \Gamma\}$ where $\gamma = (\theta, \eta) \in \Gamma = \Theta \times \mathcal{H}$ for some open $\Theta \subset \mathbb{R}^{d_\theta}$ and $\mathcal{H}$ an arbitrary set. Each $\mathcal{P}_n$ consists of probability measures on a measurable space $(\mathcal{W}_n, \mathcal{B}(\mathcal{W}_n))$ and is dominated by a σ-finite measure ν_n.[7]

Let $H_\gamma = \mathbb{R}^{d_\theta} \times B_\gamma$ be a subset of a linear space containing 0, and suppose that $\{P_{n,\gamma,h} : h \in H_\gamma\} \subset \mathcal{P}_n$ are such that $P_{n,\gamma} = P_{n,\gamma,0}$. Elements of H_γ will be written as $h = (\tau, b) \in \mathbb{R}^{d_\theta} \times B_\gamma$.[8] The measures $P_{n,\gamma,h}$ should be interpreted as local perturbations (local alternatives) to the measure $P_{n,\gamma}$ in a direction $h \in H_\gamma$. These local perturbations can be split in two groups. The null hypothesis $\mathrm{H}_0 : \theta = \theta_0$ corresponds to the set of perturbations $H_{\gamma,0} \coloneqq \{(0, b) : b \in B_\gamma\}$ and the alternative $\mathrm{H}_1 : \theta \neq \theta_0$ to $H_{\gamma,1} \coloneqq \{h = (\tau, b) : 0 \neq \tau \in \mathbb{R}^{d_\theta}, b \in B_\gamma\}$. As such, $P_{n,\gamma,h}$ with $h \in H_{\gamma,0}$ will be referred to as local perturbations consistent with the null hypothesis, whilst $P_{n,\gamma,h}$ for $h \in H_{\gamma,1}$ are local alternatives.

The theoretical analysis below is local with the "global" parameter γ being considered fixed at a γ consistent with H_0. As such, to keep the notation as light as possible, dependence on γ will be mostly left implicit: I write $P_{n,h}$ for $P_{n,\gamma,h}$, H for H_γ, H_i for $H_{\gamma,i}$ $(i = 0, 1)$ and similarly for other objects. I also write P_n for $P_{n,0}$.

[6]Failure of local identification and singularity of the information matrix are closely linked in parametric models, see Rothenberg (1971). In the semiparametric case, parameters may be identified but nevertheless have a singular efficient information matrix. The relationship between the efficient information matrix and identification is considered by Escanciano (2022).

[7]Typically the index n is sample size and $\mathcal{W}_n$ is the space in which a sample of size n takes its values. This is the situation considered in Section 4 as well as in the examples in Section 5.

[8]In most examples, H_γ will be a linear space. The more general situation as considered here is nevertheless important to allow for, for example, Euclidean nuisance parameters subject to boundary constraints. In such a setting, if the constraint is binding at γ, then γ can only be perturbed in certain directions if $P_{n,\gamma,h}$ is to remain within the model.

I will use the single-index model as a running example throughout the paper.[9]

EXAMPLE 1 (Single-index model): *Suppose that the researcher observes n i.i.d. copies of $W = (Y, X_1, X_2) \in \mathbb{R}^{2+K}$ where*

$$Y = f(X_1 + X_2'\theta) + \epsilon, \qquad \mathbb{E}[\epsilon|X] = 0, \tag{1}$$

and where f belongs to some set of continuously differentiable functions $\mathscr{F}$. The description of the model is completed by $\zeta \in \mathscr{Z}$, the density function of (ϵ, X) with respect to some σ-finite measure. $P_n = P^n$ where P corresponds to the density

$$p(W) = p_\gamma(W) := \zeta(\epsilon_{f,\theta},\, X), \qquad \epsilon_{f,\theta} := Y - f(V_\theta), \quad V_\theta := X_1 + X_2'\theta, \tag{2}$$

for $\gamma = (\theta, f, \zeta) \in \Theta \times \mathscr{F} \times \mathscr{Z} = \Gamma$. One example of local perturbations to this model are the probability measures $P_{n,h} = P_h^n$ where P_h has density $p_{\gamma+\varphi_n(h)}$ with

$$\varphi_n(h) = (\tau,\, b_1,\, b_2\zeta)\,/\sqrt{n}, \qquad h = (\tau, (b_1, b_2)) \in H := \mathbb{R}^{d_\theta} \times (B_1 \times B_2), \tag{3}$$

where B_1 is a subset of the bounded, continuously differentiable functions with bounded derivative and B_2 a subset of the bounded functions $b_2 : \mathbb{R}^{1+K} \to \mathbb{R}$, continuously differentiable in the first argument with bounded derivative.

2.2 Local asymptotic normality

The key technical condition under which the theory in this paper is developed is local asymptotic normality (LAN; see e.g. van der Vaart, 1998, Chapter 7 or Le Cam and Yang, 2000, Chapter 6). Define the log-likelihood ratios

$$L_n(h) := \log \frac{p_{n,h}}{p_{n,0}}, \qquad \text{where } \; p_{n,h} := \frac{\mathrm{d}P_{n,h}}{\mathrm{d}\nu_n}, \text{ for } h \in H. \tag{4}$$

ASSUMPTION 1 (LAN): *For bounded linear maps $\Delta_n : \overline{\mathrm{lin}}\, H \to L_2^0(P_n)$,*

$$L_n(h) = \Delta_n h - \frac{1}{2}\|\Delta_n h\|^2 + R_n(h), \qquad h \in H \tag{5}$$

with $R_n(h) \xrightarrow{P_n} 0$ for all $h \in H$. Additionally, for each $h \in H$, the law of $\Delta_n h$ (under P_n) converges to $\mathcal{N}(0, \sigma(h))$ in the Mallows-2 metric, d_2.

The requirement that (the law of) $\Delta_n h$ converges in d_2 is equivalent to requiring

[9] Technical details for this example are deferred to Sections 5.1 and S4.1.

that it converges weakly and $(\Delta_n h)_{n\in\mathbb{N}}$ is uniformly square P_n-integrable (e.g. Bickel et al., 1998, Appendix A.6). This implies that $\sigma(h) = \lim_{n\to\infty} \|\Delta_n h\|^2$. Δ_n is the *score operator* (cf. van der Vaart, 1998, p. 371): it produces score functions (or "scores") from "directions" $h \in H$.

Remark 1: *Assumption 1 ensures that the sequences $(P_n)_{n\in\mathbb{N}}$ and $(P_{n,h})_{n\in\mathbb{N}}$ are mutually contiguous for any $h \in H$ (see e.g. van der Vaart, 1998, Example 6.5).*

Remark 2: *If H is (pseudo-)metrised one may consider a uniform version of Assumption 1, i.e. uniform local asymptotic normality (ULAN). Such a version is given in Assumption S1 and is equivalent to Assumption 1 plus asymptotic equicontinuity on compact sets of $h \mapsto \Delta_n h$ (in $L_2(P_n)$) and $h \mapsto P_{n,h}$ (in total variation) (Proposition S1). The latter equicontinuity condition is of interest regarding local uniformity of size control, see Corollary 1 and Lemma 1 below.*

Example 1 (continued): *Under regularity conditions, the single-index model satisfies Assumption 1 with*

$$\Delta_n h := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tau' \dot{\ell}(W_i) + [Db](W_i), \tag{6}$$

where for $\phi(e, x) := \frac{\partial \log \zeta(e,x)}{\partial e}$,

$$\dot{\ell}(W) := -\phi(\epsilon_{f,\theta}, X) f'(V_\theta) X_2, \quad [D_\gamma b](W) := -\phi(\epsilon_{f,\theta}, X) b_1(V_\theta) + b_2(\epsilon_{f,\theta}, X). \tag{7}$$

2.3 Local regularity for tests

Definition 1: *A sequence of tests $\phi_n : \mathcal{W}_n \to [0,1]$ of the hypothesis $\mathrm{H}_0 : \theta = \theta_0$ against $\mathrm{H}_1 : \theta \neq \theta_0$ is asymptotically level α and locally regular if*

$$\pi_n(\tau, b) := P_{n,h}\phi_n \to \pi(\tau), \quad h = (\tau, b) \in H \quad \textit{and} \quad \pi(0) \leq \alpha. \tag{8}$$

That is, the finite sample (local) power function of the test, π_n converges under each $P_{n,h}$ to a function π which may depend on τ (and, implicitly, γ) but not on b, the parameter which describes local deviations from the nuisance parameter η.[10] If a sequence of tests does *not* satisfy (8) it is *(locally) non – regular*.

Local regularity of test sequences as in (8) is a pointwise concept. It is also of interest to consider a version which holds uniformly over certain subsets.

[10]Cf. the definition of a (locally) regular estimator in e.g. van der Vaart, 1998, p. 365.

Definition 2: *A sequence of tests* $\phi_n : \mathcal{W}_n \to [0,1]$ *of the hypothesis* $\mathrm{H}_0 : \theta = \theta_0$ *against* $\mathrm{H}_1 : \theta \neq \theta_0$ *is asymptotically level* α *and locally uniformly regular on* $K \subset H$ *if* (8) *holds with the convergence uniform on* K.

If H is a (pseudo-)metric space and K is a compact set, for the pointwise convergence in (8) to hold uniformly on K it is necessary and sufficient to show that the sequence of functions π_n is asymptotically equicontinuous on K.[11]

Directly working with the power functions $\pi_n(\tau, b)$ to show their asymptotic equicontinuity may be complicated in many cases. It is, however, often possible to show results which imply this property. For instance, the functions $h \mapsto P_{n,h}$ being asymptotically equicontinuous in d_{TV} implies the required asymptotic equicontinuity of the power functions. Despite being (much) stronger, this often holds.[12]

Weak identification asymptotics and local regularity In many models there are parameter values, γ, at which locally regular estimators do not exist. Points where the parameter of interest is un- or under-identified provide an important class of examples. Moreover, as is well known from the literature on weak identification, even if θ is identified at γ, finite sample inference may be poor if γ is too close to a point of identification failure relative to the amount of information contained in the sample. Such behaviour has been widely studied in models where the part of γ causing the identification failure is finite dimensional (e.g. Andrews and Cheng, 2012; Andrews and Mikusheva, 2015).

There are also many examples where weak identification may occur due to the value of *infinite-dimensional* nuisance parameters. Kaji (2021) and Andrews and Mikusheva (2022) use a differentiability in quadratic mean (DQM) condition to define "weak identification asymptotics" in i.i.d. models. In particular, they consider the behaviour under sequences of models $P_{n,h}^n$ which satisfy

$$\lim_{n\to\infty} \int \left[\sqrt{n}\left(\sqrt{p_{n,h}} - \sqrt{p_0}\right) - \frac{1}{2} f \sqrt{p_0}\right]^2 \mathrm{d}\nu_n = 0 \tag{9}$$

at a point P_0 where the parameter of interest is unidentified. In the i.i.d. case, (9) implies the LAN expansion in Assumption 1 with $\Delta_n h = \frac{1}{\sqrt{n}} \sum_{i=1}^n f(W_i)$ (e.g. van der Vaart, 1998, Lemma 25.14). Working with Assumption 1 in place of (9) broadens the applicability of this class of semiparametric weak identification asymptotics to non-i.i.d. models. It is clear from Definition 1 that a test sequence

[11] The same is true if K is totally bounded. See e.g. Davidson (2021), p. 123, for the definition of asymptotic equicontinuity.

[12] See the discussion following Remark 4 below.

will have asymptotic null rejection probability (NRP) which does not exceed the nominal level under weak identification asymptotics $P_{n,h}$ if it is locally regular.[13]

I now give two examples of semiparametric models where the parameter of interest θ may be un- or under- identified depending on the value of an infinite dimensional nuisance parameter. The first is the running example.

Example 1 (continued): *As is clear from the model equation* $Y = f(X_1+X_2'\theta)+\epsilon$, *if* f *is flat, i.e.* $f' = 0$, *then the parameter* θ *is unidentified. (The sequences given in* (3) *are weak identification asymptotic sequences if* $f' = 0$.*)*

Example 2 (Instrumental variables): *Suppose the researcher observes* n *i.i.d. copies of* $W = (Y, X, Z)$ *where*

$$Y = X'\theta + Z_1'\beta + \epsilon, \qquad \mathbb{E}[\epsilon|Z] = 0, \qquad Z = (Z_1', Z_2')'.$$

If $\pi(Z) \coloneqq \mathbb{E}[X|Z]$ *is zero,* θ *is unidentified; if some components of* $\pi(Z)$ *are zero,* θ *is underidentified.*

In Examples 1 and 2, at the point of identification failure, no locally regular estimator exists, however locally regular $C(\alpha)$ tests which satisfy $\pi(0) \leq \alpha$ are developed in Section 5. These examples consider i.i.d. setups for simplicity; this is not a requirement. See, for example, Hoesch, Lee, and Mesters (2024) who develop locally regular $C(\alpha)$ tests of the form proposed in this paper for the potentially un- / under- identified parameter in a structural vector autoregressive model.

2.4 A class of locally regular tests

To construct locally regular tests of $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, I use a generalisation of the class of $C(\alpha)$ tests introduced by Neyman (1959, 1979) to characterise optimal tests in regular parametric models. These tests are a based on a quadratic form of (estimators of) a vector of d_θ moment conditions $g_n \in L_2(P_n)$ which satisfy the following requirements.

Assumption 2 (Joint convergence): *For* $g_n \in L_2(P_n)^{d_\theta}$ *and each* $h = (\tau, b) \in H$,

$$(\Delta_n h,\ g_n')' \overset{P_n}{\rightsquigarrow} \mathcal{N}(0, \Sigma(h)),$$

[13] Of course, a (non-regular) test sequence may have asymptotic NRP which depends on b and yet is bounded by the nominal level under $P_{n,h}$ for all $h = (0, b) \in H_0$ and / or have an asymptotic power function which depends on b for $h = (\tau, b) \in H_1$. Restricting attention to locally regular test sequences may be justified by the power optimality results of Section 3.

$$\Sigma(h) \coloneqq \begin{bmatrix} \sigma(h) & \tau'\Sigma_{21}' \\ \Sigma_{21}\tau & V \end{bmatrix} = \lim_{n\to\infty} \begin{bmatrix} \|\Delta_n h\|^2 & \langle \Delta_n(\tau,0)\,,\, g_n' \rangle \\ \langle g_n\,,\, \Delta_n(\tau,0) \rangle & \langle g_n\,,\, g_n' \rangle \end{bmatrix}.$$

Built-in to Assumption 2 is a requirement of asymptotic orthogonality of g_n and the scores for the nuisance parameters η. This generalises the analogous condition in Neyman (1959, 1979) and is key to the local regularity of C(α) tests.

Remark 3: *For Assumption 2 to hold it is necessary that the g_n are approximately zero mean: since $(g_n)_{n\in\mathbb{N}}$ is uniformly integrable, $P_n g_n = o(1)$. It is also necessary that the g_n satisfy an approximate orthogonality property with the scores for nuisance parameters: as $([\Delta_n h]g_n)_{n\in\mathbb{N}}$ is uniformly integrable for each $h = (\tau, b) \in H$, $\lim_{n\to\infty} \langle \Delta_n h\,,\, g_n' \rangle = \tau'\Sigma_{\gamma,21}' = \lim_{n\to\infty} \langle \Delta_n(\tau,0)\,,\, g_n' \rangle$, and so*

$$\langle \Delta_n(0,b)\,,\, g_n' \rangle = \langle \Delta_n h\,,\, g_n' \rangle - \langle \Delta_n(\tau,0)\,,\, g_n' \rangle = o(1). \tag{10}$$

Given any d_θ moment conditions $f_n \in L_2^0(P_n)$ moment conditions which satisfy an exact version of the orthogonality condition (10) may be obtained as

$$g_n \coloneqq \Pi\left[f_n \middle| \{\Delta_n(0,b) : b \in B\}^\perp \right]. \tag{11}$$

An important special case of this construction is with $f_n = \dot{\ell}_n$, the score function for θ, i.e. $\dot{\ell}_n$ is such that $\tau'\dot{\ell}_n = \Delta_n(\tau, 0)$ for each $\tau \in \mathbb{R}^{d_\theta}$. The function

$$g_n = \tilde{\ell}_n \coloneqq \Pi\left[\dot{\ell}_n \middle| \{\Delta_n(0,b) : b \in B\}^\perp \right], \tag{12}$$

is called the *efficient score function*. This yields a power optimal choice of moment conditions satisfying (10) as shown in Section 3 below.

Example 1 (continued): *Let $\omega : \mathbb{R}^K \to [\underline{\omega}, \overline{\omega}] \subset (0,\infty)$. Then $g_n \coloneqq \mathbb{G}_n g$,*

$$g(W) \coloneqq \omega(X)(Y - f(V_\theta))f'(V_\theta)\left(X_2 - \frac{\mathbb{E}[\omega(X)X_2|V_\theta]}{\mathbb{E}[\omega(X)|V_\theta]} \right), \tag{13}$$

has components which belong to $\{\Delta_n(0,b) : b \in B\}^\perp$ (where Δ_n is as in (6)).[14] *Under regularity conditions, g_n satisfies Assumption 2 (see Section 5.1 below).*

[14] g coincides with the efficient score function $\tilde{\ell}$ derived by Newey and Stoker, 1993,

$$\tilde{\ell}(W) = \tilde{\omega}(X)(Y - f(V_\theta))f'(V_\theta)\left(X_2 - \frac{\mathbb{E}[\tilde{\omega}(X)X_2|V_\theta]}{\mathbb{E}[\tilde{\omega}(X)|V_\theta]} \right), \quad \tilde{\omega}(X) \coloneqq \mathbb{E}[\epsilon^2|X]^{-1},$$

in the (typically infeasible) case with $\omega = \tilde{\omega}$.

To construct the test statistic, I assume that consistent estimators of g_n, $V^\dagger$ (the Moore-Penrose pseudo-inverse of V) and $r := \operatorname{rank}(V)$ are available, given θ.

Assumption 3 (Consistent estimation): *$\hat{g}_{n,\theta}$, $\hat{\Lambda}_{n,\theta}$, $\hat{r}_{n,\theta} \in \{0, 1, \ldots, d_\theta\}$ satisfy*

(i) $\hat{g}_{n,\theta} - g_n \xrightarrow{P_n} 0$;

(ii) $\hat{\Lambda}_{n,\theta} \xrightarrow{P_n} V^\dagger$;

(iii) If $r \geq 1$, *then* $\hat{r}_{n,\theta} \xrightarrow{P_n} r$; *if* $r = 0$, *then* $\operatorname{rank}(\hat{\Lambda}_{n,\theta}) \xrightarrow{P_n} 0$.

Assumption 3(i) typically must be verified by model specific arguments. One generally applicable approach to obtain an estimator which satisfies Assumption 3(ii) is to take an initial estimator which is consistent for V, threshold its eigenvalues at an appropriate rate and then take the pseudo-inverse.[15] If one uses the estimator $\hat{\Lambda}_{n,\theta} := \hat{V}_{n,\theta}^\dagger$ where $\hat{V}_{n,\theta} \xrightarrow{P_n} V$ and $\hat{r}_{n,\theta} := \operatorname{rank}(\hat{V}_{n,\theta})$ then condition (ii) holds if and only if condition (iii) holds (Andrews, 1987, Theorem 2). Nevertheless, as emphasised by the notation, it is not necessary that the estimate $\hat{\Lambda}_{n,\theta}$ be the pseudo-inverse of an initial estimate.

Example 1 (continued): *Given estimators $\hat{f}_{n,i}$, $\widehat{f'}_{n,i}$ of f, f' and $\hat{Z}_{1,n,i}, \hat{Z}_{2,n,i}$ of $Z_1 := \mathbb{E}[\omega(X)X_2|V_\theta]$, $Z_2 := \mathbb{E}[\omega(X)|V_\theta]$, define $g_{n,\theta} := \frac{1}{\sqrt{n}}\sum_{i=1}^n g_{n,\theta,i}$,*

$$\hat{g}_{n,\theta,i} := \omega(X_i)(Y_i - \hat{f}_{n,i}(V_{\theta,i}))\widehat{f'}_{n,i}(V_{\theta,i})\left(X_{2,i} - \hat{Z}_{1,n,i}(V_{\theta,i})/\hat{Z}_{2,n,i}(V_{\theta,i})\right). \tag{14}$$

Under regularity conditions (see Section 5.1), $\hat{g}_{n,\theta}$ satisfies part (i) of Assumption 3, thresholding the eigenvalues of $\frac{1}{n}\sum_{i=1}^n \hat{g}_{n,\theta,i}\hat{g}_{n,\theta,i}'$ at an appropriate rate yields an estimator $\hat{\Lambda}_{n,\theta}$ satisfies part (ii) and $\hat{r}_{n,\theta} := \operatorname{rank}(\hat{\Lambda}_{n,\theta})$ satisfies part (iii).

Given the estimators of Assumption 3, the C(α) - style test statistic is

$$\hat{S}_{n,\theta} := \hat{g}_{n,\theta}'\hat{\Lambda}_{n,\theta}\hat{g}_{n,\theta}, \tag{15}$$

and the C(α) – style test ψ_{n,θ_0} of H_0 against H_1 at level α is:

$$\psi_{n,\theta} := \mathbf{1}\left\{\hat{S}_{n,\theta} > c_n\right\}, \tag{16}$$

where c_n is the $1-\alpha$ quantile of a $\chi^2_{\hat{r}_n}$ random variable.

Local regularity Assumptions 1 – 3 suffice for local regularity of $\psi_{n,\theta}$.

[15] See Section S5 of Lee and Mesters (2024b) for full details of this approach. Other regularisation schemes are also possible (see e.g. Dufour and Valéry, 2016).

Proposition 1: *Under Assumptions 1 and 2, for* $h = (\tau, b) \in H$

$$g_n \overset{P_{n,h}}{\rightsquigarrow} \mathcal{N}\left(\Sigma_{21}\tau, V\right).$$

If Assumption 3 also holds, then additionally

$$\hat{g}_{n,\theta} \overset{P_{n,h}}{\rightsquigarrow} \mathcal{N}\left(\Sigma_{21}\tau, V\right) \qquad \text{and} \qquad \hat{S}_{n,\theta} \overset{P_{n,h}}{\rightsquigarrow} \chi^2_r\left(\tau'\Sigma_{21}'V\Sigma_{21}\tau\right).$$

Theorem 1: *Suppose that Assumptions 1, 2 and 3 hold and* $h = (\tau, b) \in H$*. Then,*

$$\lim_{n\to\infty} P_{n,h}\psi_{n,\theta} = \pi(\tau) := \begin{cases} 1 - \mathrm{P}\left(\chi^2_r\left(\tau'\Sigma_{21}'V_\gamma^\dagger\Sigma_{21}\tau\right) \leq c_r\right) & \text{if } r \geq 1 \\ 0 & \text{if } r = 0 \end{cases},$$

where c_r *is the* $1-\alpha$ *quantile of the* χ^2_r *distribution.*

Theorem 1 immediately shows that $\psi_{n,\theta}$ is locally regular (cf. (8)). The asymptotic orthogonality in (10) is key to this result. If, instead, $\lim_{n\to\infty}\langle\Delta_n h\,,\, g_n'\rangle = \tau'\Sigma_{21}' + c(b)$, then (by Le Cam's third Lemma) the limiting distribution of g_n under $P_{n,h}$ would be $\mathcal{N}(\Sigma_{21}\tau + c(b), V)$ and hence the limiting power function of the test sequence would not be free of b.

Uniform local regularity The local regularity given by Theorem 1 may be "upgraded" to local uniform regularity (Definition 2) under various conditions. In the case where H posseses a (pseudo-)metric structure (e.g. if H is a subset of a (semi-)normed linear space), for $\psi_{n,\theta}$ to be locally uniformly regular on a compact (or totally bounded) $K \subset H$ it is necessary and sufficient that the functions $h = (\tau, b) \mapsto \pi_n(\tau, b)$ are asymptotically equicontinuous.[16]

Corollary 1: *Suppose that the conditions of Theorem 1 hold and that* (H, d) *is a pseudometric space. If the functions* $h = (\tau, b) \mapsto \pi_n(\tau, b) := P_{n,h}\psi_{n,\theta}$ *are asymptotically equicontinuous on a compact (or totally bounded) subset* $K \subset H$*,*

$$\lim_{n\to\infty} \sup_{(\tau,b)\in K} |\pi_n(\tau, b) - \pi(\tau)| = 0.$$

I now give a sufficient condition for the asymptotic equicontinuity condition in Corollary 1.

[16] If H posseses a (finite) measure structure and the functions $h = (\tau, b) \mapsto \pi_n(\tau, b)$ are measurable then ψ_{n,θ_0} is locally uniformly regular except on a "small" subset of H by Egorov's Theorem. See Section S2.3.2 for details.

Lemma 1: *If (H, d) is a pseudometric space and $(h \mapsto P_{n,h})_{n\in\mathbb{N}}$ is asymptotically equicontinuous in d_{TV} on $K \subset H$, then $(h \mapsto P_{n,h}\psi_{n,\theta})_{n\in\mathbb{N}}$ is asymptotically equicontinuous on K.*

Remark 4: *Lemma 1 requires asymptotic equicontinuity in total variation of the functions $(h \mapsto P_{n,h})_{n\in\mathbb{N}}$ on subsets $K \subset H$. This holds for any compact K under ULAN (Assumption S1), as shown in Proposition S1.*

In the parametric i.i.d. case LAN is often verified by establishing a differentiability in quadratic mean condition, e.g. equation (7.1) in van der Vaart (1998). This is sufficient for the uniform LAN (ULAN) expansion in Assumption S1 to hold (e.g. van der Vaart, 1998, Theorem 7.2). Semiparametric generalisations of this result are also available (e.g. combine Proposition S1 and Lemma S1).[17] Therefore the, condition in Lemma 1 is natural given its link with the ULAN condition. Nevertheless, it is (much) stronger than necessary for the condition required by Corollary 1; a weaker sufficient condition is given in Lemma S2.

3 Power optimality

The preceding section established the local regularity of the tests $\psi_{n,\theta}$ based on (estimates of) moment functions g_n satisfying asymptotic orthogonality conditions. Thus far, nothing has been said about the choice of g_n beyond these orthogonality requirements. The choice of the functions g_n determines the attainable power of the corresponding test. As such, they ought to be chosen such that the resulting test has good power against alternatives of interest.

One natural choice is the efficient score function (12). It is well known that tests based on the efficient score function have certain optimality properties in regular models when the observations are (a) i.i.d. (cf. Section 25.6 van der Vaart, 1998) or (b) when the information operator is boundedly invertible (Choi et al., 1996). I show below that this result holds in both regular and non-regular models without requiring (a) or (b).

The results in this section are derived using the limits of experiments framework of Le Cam (e.g. Le Cam, 1986; van der Vaart, 1998). In particular, I show that the local experiments consisting of the measures $P_{n,h}$ for $h \in H$ converge weakly to a limit experiment which has a close relationship to a Gaussian shift experiment on the Hilbert space formed by taking the quotient of H under the seminorm

[17] Lee and Mesters (2024a) and Hoesch et al. (2024) verify this asymptotic equicontinuity property in i.i.d. and time series semiparametric examples respectively.

induced by the variance function $\sigma(h)$. The relation between these experiments is sufficiently tight that power bounds derived in the latter transfer to the former.[18]

The limit experiment For this development H is required to be linear; I will therefore assume that B (hence H) is a linear space. Under LAN, there exists a positive semi-definite symmetric bilinear form $\langle \cdot \,, \cdot \rangle_K$ on $H = \mathbb{R}^{d_\theta} \times B$ such that $\sigma(h) = \langle h \,, h \rangle_K$. This can be seen as a by-product of the following Lemma.

LEMMA 2: *Suppose Assumption 1 holds and B is a linear space. Let Δ be the square integrable stochastic process defined on H such that $\Delta_n h \overset{P_n}{\rightsquigarrow} \Delta h$. Then Δ is a mean-zero Gaussian linear process with covariance kernel K, where*

$$K(h, g) \coloneqq \lim_{n\to\infty} P_n \left[\Delta_n h \Delta_n g\right].$$

For $h, g \in H$, setting $\langle h \,, g \rangle_K \coloneqq K(h, g)$ gives a positive semi-definite symmetric bilinear form. Let $\|\cdot\|_K$ denote the seminorm induced by $\langle \cdot \,, \cdot \rangle_K$ on H.

REMARK 5: *Suppose that $\langle \cdot \,, \cdot \rangle_H$ is an inner product on $H = \mathbb{R}^{d_\theta} \times B$. The existence of the positive semi-definite symmetric bilinear form $\langle \cdot \,, \cdot \rangle_K$ is equivalent to the existence of a bounded, self-adjoint, positive semi-definite linear operator B such that $\langle h \,, h \rangle_K = \langle h \,, \mathsf{B} h \rangle_H$ for $h \in H$ (cf. Choi et al., 1996, p. 845).*

With this established, define $\mathbb{H}$ as the quotient of H by the subspace on which the semi-norm $\|\cdot\|_K$ vanishes:

$$\mathbb{H} \coloneqq H \,/\, \{h \in H : \|h\|_K = 0\}. \tag{17}$$

which is an inner product space when equipped with the natural inner product induced by $\langle \cdot \,, \cdot \rangle_K$, which I also denote by $\langle \cdot \,, \cdot \rangle_K$. An element of $\mathbb{H}$ corresponding to representative element $h \in H$ will be denoted by $[h]$.[19]

The (weak) limit of the sequence of experiments consisting of the measures $P_{n,h}$ can be obtained by standard results on weak convergence of experiments.

PROPOSITION 2: *Suppose that Assumption 1 holds, that B is a linear space and*

[18] That the local experiments do not converge to the mentioned Gaussian shift experiment is essentially a purely technical point: the Gaussian shift experiment is defined on a different parameter space to the local experiments, whilst (weak) convergence of experiments in the sense of Le Cam (1986) is defined for experiments with the same parameter space.

[19] Analogous comments apply to the related space $\mathbb{H}_1$, defined below. In both cases, to avoid an excess of parentheses / brackets, if $h = (\tau, b)$ I will write either $[h]$ or $[\tau, b]$, rather than $[(\tau, b)]$.

define the sequence of experiments $\mathscr{E}_n := (\mathcal{W}_n, \mathcal{B}(\mathcal{W}_n), (P_{n,h} : h \in H))$. *Let* Δ *be the Gaussian process of Lemma 2 and let* $(\Omega, \mathcal{F}, \mathrm{P})$ *be the probability space on which it is defined. Define the experiment* $\mathscr{E} := (\Omega, \mathcal{F}, (P_h : h \in H))$ *according to*

$$P_0 := \mathrm{P}, \qquad \frac{\mathrm{d}P_h}{\mathrm{d}P_0} = \exp\left(\Delta h - \frac{1}{2}\|h\|^2\right).$$

Then $\mathscr{E}_n$ *converges weakly to* $\mathscr{E}$.

Under the assumption that $\mathbb{H}$ is separable, the experiment $\mathcal{E}$ is equivalent to a Gaussian shift on $(\mathbb{H}, \langle\cdot\,,\,\cdot\rangle_K)$, in the sense given by Proposition 3 below.

Assumption 4: *B is a linear space and $\mathbb{H}$ as defined in (17) is separable.*

Proposition 3: *Suppose Assumptions 1 and 4 hold. If* $\mathscr{E}$ *is as in Proposition 2, there is a Gaussian shift experiment* $\mathscr{G} := (\Omega, \mathcal{F}, (G_{[h]} : [h] \in \mathbb{H}))$ *such that* $d_{TV}(P_h, G_{[h]}) = 0$ *for each* $h \in H$.

The efficient information matrix Power bounds for tests of $\mathrm{K}_0 : h \in H_0$ against $\mathrm{K}_1 : h \notin H_1$ can be expressed in terms of the *efficient information matrix*, $\tilde{\mathcal{I}}$, so named because in the i.i.d. setting it is the covariance matrix of the efficient score function for a single observation. Here I provide an alternative definition of this matrix which applies more generally, and reduces to the classical definition in the i.i.d. case (as shown in Lemma S4).

Let $\|\tau\| := \inf_{b \in B} \|(\tau, b)\|_K$; this is a seminorm on $\mathbb{R}^{d_\theta}$. Equipping the quotient $\mathbb{H}_1 := \mathbb{R}^{d_\theta} / \{\tau \in \mathbb{R}^{d_\theta} : \|\tau\| = 0\}$ with the natural norm induced by $\|\cdot\|$ (which I also denote by $\|\cdot\|$) turns it into a normed space. Define the linear map $\pi_1 : \mathbb{H} \to \mathbb{H}_1$ as $\pi_1([\tau, b]) := [\tau]$. As π_1 is continuous it may be uniquely extended to a continuous function defined on $\overline{\mathbb{H}}$, the completion of $\mathbb{H}$; this extension will also be called π_1. Since π_1 is continuous, $\ker \pi_1 \subset \overline{\mathbb{H}}$ is closed. Let Π be the orthogonal projection onto $\ker \pi_1$ and define $\Pi^\perp := I - \Pi$, the orthogonal projection onto $[\ker \pi_1]^\perp$.

Let e_i be the i-th canonical basis vector in $\mathbb{R}^{d_\theta}$ and define the *efficient information matrix* $\tilde{\mathcal{I}}$ as the $d_\theta \times d_\theta$ matrix with i, j-th entry $\tilde{\mathcal{I}}_{ij}$ given by[20]

$$\tilde{\mathcal{I}}_{ij} = \left\langle \Pi^\perp[e_i, 0]\,,\, \Pi^\perp[e_j, 0]\right\rangle_K. \tag{18}$$

Lemma 3: *Under Assumption 4,* $\|\tau\|^2 = \tau'\tilde{\mathcal{I}}\tau$ *and* $\ker \tilde{\mathcal{I}} = \{\tau \in \mathbb{R}^{d_\theta} : \|\tau\| = 0\}$.

[20] An alternative expression for $\tilde{\mathcal{I}}$ based on the Gaussian process Δ of Lemma 2 is given in Lemma S3.

3.1 Tests of a scalar parameter

The following Theorem records the power bound for (locally asymptotically) unbiased two-sided tests of a scalar θ. As previously mentioned, in the case where $d_\theta = 1$, the matrix $\tilde{\mathcal{I}}$ has rank either 0 or 1 and there is no intermediate case. Theorem 2 handles both cases simultaneously.

THEOREM 2: *Suppose that Assumptions 1 and 4 hold and $d_\theta = 1$. Let $\phi_n : \mathcal{W}_n \to [0,1]$ be a sequence of locally asymptotically unbiased level α tests of* $\mathrm{K}_0 : \tau = 0$ *against* $\mathrm{K}_1 : \tau \neq 0$. *That is,*

$$\limsup_{n\to\infty} P_{n,h}\phi_n \leq \alpha, \quad h \in H_0, \qquad \textit{and} \qquad \liminf_{n\to\infty} P_{n,h}\phi_n \geq \alpha, \quad h \in H_1.$$

Then, for any $h \in H$,

$$\limsup_{n\to\infty} P_{n,h}\phi_n \leq 1 - \Phi\left(z_{\alpha/2} - \tilde{\mathcal{I}}^{1/2}\tau\right) + 1 - \Phi\left(z_{\alpha/2} + \tilde{\mathcal{I}}^{1/2}\tau\right), \tag{19}$$

where z_α is the $1-\alpha$ quantile and Φ the CDF of the standard normal distribution.

That the power bound of Theorem 2 is achieved by the test $\psi_{n,\theta}$ provided $\Sigma_{21}V^\dagger\Sigma_{21} = \tilde{\mathcal{I}}$ and $r = 1$ follows from Theorem 1.

COROLLARY 2: *Suppose that Assumptions 1, 2 and 3 hold with $\Sigma_{21}V^\dagger\Sigma_{21}' = \tilde{\mathcal{I}}$ and $r = 1$. Then, for $h \in H$,*

$$\lim_{n\to\infty} P_{n,h}\psi_{n,\theta} = 1 - \Phi\left(z_{\alpha/2} - \tilde{\mathcal{I}}^{1/2}\tau\right) + 1 - \Phi\left(z_{\alpha/2} + \tilde{\mathcal{I}}^{1/2}\tau\right). \tag{20}$$

3.2 Tests of a multivariate parameter

When $d_\theta > 1$ there is a truly intermediate case where $0 < \operatorname{rank}(\tilde{\mathcal{I}}) < d_\theta$. Here I permit $0 < \operatorname{rank}(\tilde{\mathcal{I}}) \leq d_\theta$ and establish a maximin power bound for (potentially) non – regular models, which contains the regular full rank case as a special case.[21]

THEOREM 3: *Suppose that Assumptions 1 and 4 hold and $r := \operatorname{rank}(\tilde{\mathcal{I}}) \geq 1$. Let $\phi_n : \mathcal{W}_n \to [0,1]$ be a sequence of tests such that for each $h = (0,b) \in H_0$*

$$\limsup_{n\to\infty} P_{n,h}\phi_n \leq \alpha$$

[21] Section S2.5 shows that the most stringent test (in the sense of Wald, 1943) in the limit experiment has the same power function as the maximin test, and no sequence of asymptotically level α tests can correspond to a test in the limit experiment with smaller regret.

Let c_r the $1-\alpha$ quantile of a χ^2_r random variable. Then, if $a \geq 0$,

$$\limsup_{n\to\infty} \inf \left\{ P_{n,h}\phi_n : h = (\tau, b) \in H,\ \tau'\tilde{\mathcal{I}}\tau \geq a \right\} \leq 1 - \mathrm{P}(\chi^2_r(a) \leq c_r). \tag{21}$$

As in the two-sided case, by Theorem 1, the power bound on the right hand side of (21) is achieved by the test $\psi_{n,\theta}$ provided $\Sigma_{21}V^\dagger\Sigma_{21} = \tilde{\mathcal{I}}$ and $\operatorname{rank}(V) = \operatorname{rank}(\tilde{\mathcal{I}}) = r \geq 1$. In order that the test be asymptotically maximin over a compact subset K_a of $\{h = (\tau, b) \in H : \tau'\tilde{\mathcal{I}}\tau \geq a\}$, with $a = \inf\{\tau'\tilde{\mathcal{I}}\tau = a : h \in K_a\}$, some uniformity is required.[22]

Corollary 3: *Suppose that Assumptions 1, 2 and 3 hold with $\Sigma_{21}V^\dagger\Sigma_{21}' = \tilde{\mathcal{I}}$ and $r = \operatorname{rank}(\tilde{\mathcal{I}}) = \operatorname{rank}(V) \geq 1$. Then for $h = (\tau, b) \in H$*

$$\lim_{n\to\infty} P_{n,h}\psi_{n,\theta} = 1 - \mathrm{P}\left(\chi^2_r(a) \leq c_r\right), \quad a = \tau'\tilde{\mathcal{I}}\tau. \tag{22}$$

Additionally, suppose that (H, d) is a (pseudo-)metric space and let K_a be a compact subset of $\{h = (\tau, b) \in H : \tau'\tilde{\mathcal{I}}\tau \geq a\}$ such that $a = \inf\{\tau'\tilde{\mathcal{I}}\tau : h = (\tau, b) \in K_a\}$. If the functions $h \mapsto P_{n,h}\psi_{n,\theta}$ are asymptotically equicontinuous on K_a,

$$\lim_{n\to\infty} \inf_{h\in K_a} P_{n,h}\psi_{n,\theta} = 1 - \mathrm{P}\left(\chi^2_r(a) \leq c_r\right). \tag{23}$$

A sufficient condition for the asymptotic equicontinuity required for the second part of Corollary 3 was given in Lemma 1.[23]

3.3 The degenerate case

If the efficient information matrix $\tilde{\mathcal{I}}$ is zero, no test with correct asymptotic size has non – trivial asymptotic power against any sequence of local alternatives.

Proposition 4: *Suppose Assumptions 1 and 4 hold and $r := \operatorname{rank}(\tilde{\mathcal{I}}_\gamma) = 0$. Let $\phi_n : \mathcal{W}_n \to [0, 1]$ be a sequence of tests such that $\limsup_{n\to\infty} P_{n,h}\phi_n \leq \alpha$ for each $h = (0, b) \in H_0$ Then, for $h \in H$, $\limsup_{n\to\infty} P_{n,h}\phi_n \leq \alpha$.*

3.4 Discussion of the power bounds

There are a number of important aspects to highlight regarding the interpretation of the power bounds obtained in the preceding subsections.

[22] The pseudometric d in Corollary 3 need not be related to the seminorm $\|\cdot\|_K$.

[23] Lemma S2 provides some weaker sufficient conditions; in the present context, condition (iii) of Lemma S2 is not required (cf. Remark S3).

Optimality in multivariate testing problems Just as in the classical finite – dimensional regular case, the maximin optimality results just presented should not be taken in an absolute sense. Nevertheless they seem reasonable if the researcher does not have directions against which they wish to direct power a priori. If there are alternatives of particular interest, one could focus on these by weighting the moment conditions g_n differently (cf. Bickel, Ritov, and Stoker, 2006).

The intermediate case with $1 \leq \operatorname{rank}(\tilde{\mathcal{I}}) < d_\theta$ A key benefit of the multivariate power results is that they apply equally to non-regular models, i.e. cases where $\tilde{\mathcal{I}}$ is rank deficient. This scenario can occur for various reasons. Firstly the model may not identify all parameters of interest θ (i.e. underidentification). Secondly some of the elements of θ may be weakly identified (i.e. weak underidentification). The power results above apply in either of these cases.

There are a number of other papers which provide inference results in similarly rank deficient settings (e.g. Rotnitzky, Cox, Bottai, and Robins, 2000; Han and McCloskey, 2019; Andrews and Guggenberger, 2019; Amengual, Bei, and Sentana, 2023); none of these papers consider optimal testing.

Alternative approximations In the case where $\operatorname{rank}(\tilde{\mathcal{I}}) = 0$, Proposition 4 reveals that the limit experiment $\mathscr{E}$ based on the LAN approximation in Assumption 1 does not provide any useful way of (asymptotically) comparing tests. Other approximations might provide valuable comparisons. Alternative approximations have been explored in, for example, the IV model (e.g. Moreira, 2009) and semiparametric GMM models by Andrews and Mikusheva (2022, 2023). Whether such an approach can be developed for the class of models considered here is an interesting question for future work.

3.5 Attaining the power bounds

Provided that the L_2 distance between g_n and $\tilde{\ell}_n$ (as defined in (12)) vanishes, $\psi_{n,\theta}$ attains the power bounds established in the preceding subsections. For regular models this result is well known in two special cases: (i) the i.i.d. case (cf. Section 25.6 in van der Vaart (1998); Lemma S4 below) and (ii) when the information operator B in Remark 5 is positive – definite with B_{22}, the information operator for η, boundedly invertible (Choi et al., 1996). Here I provide a general version of this result which applies to both regular and non-regular models and does not require (i) or (ii). In particular, I show that $\Sigma_{21} V^{\dagger} \Sigma_{21}' = \tilde{\mathcal{I}}$, which suffices given

Theorem 1 and the power bounds in Theorems 2, 3 and S1.

THEOREM 4: *Suppose that Assumptions 1, 2, 3 and 4 hold and* g_n *is such that* $\lim_{n\to\infty} \int \|g_n - \tilde{\ell}_n\|^2 \,\mathrm{d}P_n = 0$. *Then* $\Sigma_{21} = V = \tilde{\mathcal{I}}$, *hence* $\Sigma_{21}V^{\dagger}\Sigma_{21}' = \tilde{\mathcal{I}}$.

4 The smooth i.i.d. case

In this section I give conditions which are sufficient for some of the foregoing Assumptions in the in the benchmark case for semiparametric theory: where the observations are i.i.d. and the model is "smooth".

ASSUMPTION 5 (Product measures): *Suppose* $W^{(n)} = (W_1, \ldots, W_n) \in \prod_{i=1}^n \mathcal{W} = \mathcal{W}_n$ *and that each* $P_{n,h}$ *is a product measure:* $P_{n,h} = P_h^n$. *Each probability measure in* $\mathcal{P}_n$ *is dominated by the n-fold product of a σ-finite measure* ν.

In the i.i.d. setting, it is well known that quadratic mean differentiability of the square root of the density $p = \frac{\mathrm{d}P}{\mathrm{d}\nu}$ of $P \coloneqq P_0$ is sufficient for LAN. In particular, if

$$\lim_{n\to\infty} \int \left[\sqrt{n}\left(\sqrt{p_{h_n}} - \sqrt{p}\right) - \frac{1}{2}Ah\sqrt{p}\right]^2 = 0, \tag{24}$$

for a measurable $Ah : \mathcal{W} \to \mathbb{R}$, then with $\Delta_n h \coloneqq \frac{1}{\sqrt{n}}\sum_{i=1}^n [Ah](W_i)$ the remainder term R_n in the LAN expansion satisfies $R_n(h_n) \xrightarrow{P} 0$ (e.g. van der Vaart and Wellner, 1996, Lemma 3.10.11). This can be used to establish either the LAN expansion required by Assumption 1 by taking $h_n = h$ for each $n \in \mathbb{N}$ or the ULAN expansion as in Assumption S1 by considering sequences $h_n \to h$. Sufficient conditions for (24) are well known (e.g. van der Vaart, 1998, Lemma 7.6).

In this case, the scores Ah typically take the form

$$[Ah](W_i) = \tau'\dot{\ell}(W_i) + [Db](W_i), \quad h = (\tau, b) \in H, \tag{25}$$

where $\dot{\ell}$ is a vector of functions in $L_2^0(P)$ (usually the partial derivatives of $\theta \mapsto \log p_\gamma$ at γ) and $D : \overline{\mathrm{lin}}\ B \to L_2^0(P)$ a bounded linear map. Showing that (24) holds (with $h_n = h$) is often the most straightforward way to verify the LAN expansion required by Assumption 1. If $A : \overline{\mathrm{lin}}\ H \to L_2(P)$ is a bounded linear map, then the remainder of Assumption 1 also follows directly.[24]

LEMMA 4: *Suppose that Assumption 5 holds and for each* $h \in H$ *equation* (24)

[24] A version of Lemma 4 for ULAN (Assumption S1) is Lemma S1 in the supplementary material.

holds (with $h_n = h$*) with* $A : \overline{\text{lin}}\ H \to L_2(P)$ *a bounded linear map. Then Assumption 1 holds with* $P_{n,h} = P^n_{h/\sqrt{n}}$ *and* $[\Delta_n h](W^{(n)}) = \mathbb{G}_n Ah$*.*

When the data are i.i.d., the the joint convergence of $(\Delta_n h, g_n')$ as in Assumption 2 is particularly straightforward to verify. As noted in the discussion around (11), it can be ensured that the orthogonality condition holds by performing an orthogonal projection. Assumption 2 then follows straightforwardly. In the i.i.d. setting typically g_n will have the form $g_n(W^{(n)}) = \mathbb{G}_n g$.

LEMMA 5: *Suppose that Assumptions 1 and 5 hold, with* $\Delta_n h = \frac{1}{\sqrt{n}} \sum_{i=1}^n Ah$*, where* Ah *is as in equation (25). Additionally suppose that* $g \in \{Db : b \in B\}^\perp \subset L_2^0(P)$*. Then Assumption 2 holds with* $g_n(W^{(n)}) := \mathbb{G}_n g$*.*

COROLLARY 4: *In the setting of Lemma 5, if* $f \in L_2^0(P)$ *and* g *is the orthogonal projection* $g = \Pi[f|\{Db : b \in B\}^\perp]$ *then Assumption 2 holds with* $g_n(W^{(n)}) := \mathbb{G}_n g$*.*

5 Examples

I now illustrate the application of the theoretical results to the single index and IV models and conduct simulation studies to investigate finite sample performance of the proposed approach. In this section I work under high level conditions to avoid repeating standard regularity conditions; lower level sufficient conditions are given in section S4 of the supplementary material.

5.1 Single index model

Consider the single index model of Example 1. I now formalise the development given in Section 2. The model parameters are $\gamma = (\theta, \eta)$ where $\eta = (f, \zeta)$ and the density of one observation with respect to a σ-finite measure $\tilde{\nu}$ is p_γ as in (2); P_γ denotes the corresponding probability measure. The parameters γ are restricted by the following Asssumption. Let $\mathscr{X}$ be the support of X, $\mathscr{D}$ a convex open set containing $\{x_1 + x_2'\theta : \theta \in \Theta, x \in \mathscr{X}\}$ and $C_b^1(\mathscr{D})$ the class of real functions which are bounded and continuously differentiable with bounded derivative on $\mathscr{D}$.

ASSUMPTION 6: *The parameters* $\gamma = (\theta, f, \zeta) \in \Gamma = \Theta \times \mathscr{F} \times \mathscr{Z}$ *where* Θ *is an open subset of* $\mathbb{R}^{d_\theta}$*,* $\mathscr{F} = C_b^1(\mathscr{D})$ *and* $\zeta \in \mathscr{Z}$*, for*

$$\mathscr{Z} := \left\{ \zeta \in L_1(\mathbb{R}^{1+K}, \nu) : \zeta \geq 0, \int_{\mathbb{R}\times\mathscr{X}} \zeta \, \mathrm{d}\nu = 1, \text{ if } (\epsilon, X) \sim \zeta \text{ then (26) holds} \right\},$$

with $L_1(A,\nu)$ is the space of ν – integrable functions on A and

$$\mathbb{E}[\epsilon|X]=0,\ \mathbb{E}[\epsilon^2]<\infty,\ \mathbb{E}[(|\epsilon|^{2+\rho}+|\phi(\epsilon,X)|^{2+\rho}+1)\|X\|^{2+\rho}]<\infty,\ \mathbb{E}[XX']\succ 0, \tag{26}$$

for $\phi(\epsilon,X)$ the derivative of $e\mapsto\log\zeta(e,X)$. Additionally, for each $\gamma\in\Gamma$, p_γ is a probability density with respect to some σ-finite measure $\tilde{\nu}$.

That p_γ is a valid probability density holds automatically (with $\tilde{\nu}=\nu$) when $\epsilon|X$ is continuously distributed, see Appendix section S4.1.2.

Local Asymptotic Normality Consider local perturbations $P_{\gamma+\varphi_n(h)}$ for

$$\varphi_n(h)=\left(\frac{\tau}{\sqrt{n}},\ \varphi_{n,2}(b_1,b_2)\right),\qquad h=(\tau,b_1,b_2)\in H=\mathbb{R}^{d_\theta}\times B_1\times B_2. \tag{27}$$

B_1 is the set which indexes the perturbations to f and consists of a subset of the continuously differentiable functions $b_1:\mathscr{D}\to\mathbb{R}$. B_2 indexes the perturbations to ζ and consists of a subset of the functions $b_2:\mathbb{R}^{1+K}\to\mathbb{R}$ which are continuously differentiable in their first argument and satisfy

$$\mathbb{E}[b_2(\epsilon,X)]=0,\ \mathbb{E}[\epsilon b_2(\epsilon,X)|X]=0,\ \mathbb{E}[b_2(\epsilon,X)^2]<\infty\quad\text{for }(\epsilon,X)\sim\zeta. \tag{28}$$

The precise form of $\varphi_{n,2}$ is left unspecified. It is required only that local perturbations satisfy the LAN property below.

Assumption 7: *Suppose that $\mathcal{W}_n=\prod_{i=1}^n\mathbb{R}^{1+K}$ and $P_{n,h}:=P^n_{\gamma+\varphi_n(h)}\ll\nu_n$ for all $\gamma\in\Gamma$ and $h\in H$ and are such that Assumption 1 holds with*

$$\log\frac{p_{n,h}}{p_{n,0}}=\frac{1}{\sqrt{n}}\sum_{i=1}^n[Ah](W_i)-\frac{1}{2}\sigma(h)+o_{P_{n,0}}(1),\quad h\in H, \tag{29}$$

where $\sigma(h)=\int[Ah]^2\,\mathrm{d}P$ and A is as in equation (25) with

$$\begin{aligned}\dot{\ell}(W)&:=-\phi(Y-f(V_\theta),X)f'(V_\theta)X_2\\ [Db](W)&:=-\phi(Y-f(V_\theta),X)b_1(V_\theta)+b_2(Y-f(V_\theta),X).\end{aligned}$$

The moment conditions The test statistic is based on $g_n:=\mathbb{G}_ng$ for g given in (13). This satisfies Assumption 2 under Assumptions 6 & 7 and (30) below.

PROPOSITION 5: *Suppose Assumptions 6 & 7 hold and under P,*

$$\mathbb{E}[\epsilon^2|X] \leq C < \infty, \quad \mathbb{E}\left[\epsilon\phi(\epsilon, X)|X\right] = -1, \quad a.s. \; . \tag{30}$$

Then Assumption 2 holds with $g_n := \mathbb{G}_n g$ for g given in (13).

A feasible test To form a feasible test g_n must be replaced by an estimator $\hat{g}_{n,\theta}$. Let $\hat{g}_{n,\theta}(W^{(n)}) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \hat{g}_{n,\theta,i}$, for $\hat{g}_{n,\theta,i}$ defined as in (14). To keep the notation concise let $Z_3 := f$, $Z_4 := f'$, $Z_0 := Z_1/Z_2$ and correspondingly $\hat{Z}_{0,n,i} := \hat{Z}_{1,n,i}/\hat{Z}_{2,n,i}$. Let $\check{V}_{n,\theta} := \frac{1}{n}\sum_{i=1}^{n} \hat{g}_{n,\theta,i}\hat{g}_{n,\theta,i}'$ and If V_γ is known to have full rank then let $\hat{V}_{n,\theta} := \check{V}_{n,\theta}$, $\hat{\Lambda}_{n,\theta} := \hat{V}_{n,\theta}^{-1}$ and $\hat{r}_{n,\theta} = \operatorname{rank}(V_\gamma)$. Otherwise form the estimator $\hat{V}_{n,\theta}$ according to the construction in Section S5 of Lee and Mesters (2024b) using a truncation rate ν_n, set $\hat{\Lambda}_{n,\theta} := \hat{V}_{n,\theta}^{\dagger}$ and $\hat{r}_{n,\theta} := \operatorname{rank}(\hat{V}_{n,\theta})$. Under the following condition, these estimators satisfy the conditions of Assumption 3.

ASSUMPTION 8: *Suppose that equation (30) holds (under P), X has compact support, $\mathbb{E}[\epsilon^4] < \infty$, and with P probability approaching one $\mathsf{R}_{l,n,i} \leq r_n = o(n^{-1/4})$,*

$$\mathsf{R}_{l,n,i} := \left[\int \left\|\hat{Z}_{l,n,i}(v) - Z_l(v)\right\|^2 \mathrm{d}\mathcal{V}(v)\right]^{1/2}, \quad l = 1, \ldots 4,$$

where $\mathcal{V}$ is the law of V_θ under P and where $\hat{Z}_{l,n,i}(V_{\theta,i})$ is $\sigma(\{V_{\theta,i}\} \cup \mathcal{C}_{n,j})$ measurable with $j = 1$ if $i > \lfloor n/2 \rfloor$ and 2 otherwise, with $\mathcal{C}_{n,1} := \{W_j : j \in \{1, \ldots, \lfloor n/2 \rfloor\}\}$ and $\mathcal{C}_{n,2} := \{W_j : j \in \{\lfloor n/2 \rfloor + 1, \ldots, n\}\}$.

The rate conditions in Assumption 8 can be satisfied by e.g. (sample – split) series estimators under standard conditions; see e.g. Belloni, Chernozhukov, Chetverikov, and Kato (2015).

PROPOSITION 6: *Suppose Assumptions 6, 7 and 8 hold and ν_n is such that $r_n = o(\nu_n)$. Then Assumption 3 holds with $V := \int gg' \,\mathrm{d}P$.*

Since Assumption 7 and Propositions 5 and 6 verify the conditions of Theorem 1, the test $\psi_{n,\theta}$ formed as in (16) is locally regular.

Simulation study I take $K = 1$ and test $H_0 : \theta = \theta_0 = 1$ at a nominal level of 5%. Each study reports the results of 5000 monte carlo replications with a sample size $n \in \{400, 600, 800\}$. I report empirical rejection frequencies for $\psi_{n,\theta}$ along with a Wald test based on an estimator in the style of Ichimura (1993).

I consider two different classes of link function. The first sets $f(v) = f_j(v) =$

$5\exp(-v^2/2c_j^2)$ ("exponential"); the second $f(v) = f_j(v) = 25\,(1 + \exp(-v/c_j))^{-1}$ ("logistic"). The values of c_j considered are recorded in Table 1.

Table 1: Index functions used in the simulation exeriments

name	expression	c_1	c_2	c_3
Exponential	$f_j(v) = 5\exp(-v^2/2c_j^2)$	1.25	2	4
Logistic	$f_j(v) = 25(1 + \exp(-v/c_j))^{-1}$	0.75	3	12

In each case, as c_j increases, the derivative of f flattens out, moving towards a point with $f' = 0$, at which θ is unidentified.[25] I draw covariates as $X = (Z_1, 0.2Z_1 + 0.4Z_2 + 0.8)$, where each $Z_k \sim U(-1, 1)$ is independent. The error term is drawn either as $\epsilon = v/\sqrt{3/2}$ with $v \sim t(6)$ ("homoskedastic") or $\epsilon \sim \mathcal{N}(0, 1 + \sin(X_1)^2)$ ("heteroskedastic").

I compute $\psi_{n,\theta}$ as described on p. 20, with $\omega(X) = 1$. The functions f, f' and Z_1 are estimated via sample split smoothing splines.[26] The truncation parameter ν is set to 10^{-4}. I additionally compute a Wald test in the style of Ichimura (1993), using the same non-parametric estimates as for $\hat{g}_{n,\theta}$.[27]

Table 2 displays the empirical rejection frequencies of these procedures: $\psi_{n,\theta}$ rejects at close to the nominal 5% for all simulation designs considered whilst the Wald test over – rejects in most of the simulation designs considered.[28] Figure 1 contains power plots of the $\psi_{n,\theta}$ test ($n = 800$). For almost flat index functions there is very identifying information and hence very little power available. As the index function moves away from the point of identification failure ($f' = 0$), the available power increases and is provided by $\psi_{n,\theta}$.

5.2 IV model

Consider the instrumental variables model in Example 2: n i.i.d. copies of $W = (Y, X, Z)$ are observed where

$$Y = X'\theta + Z_1'\beta + \epsilon, \qquad \mathbb{E}[\epsilon|Z] = 0, \qquad Z = (Z_1', Z_2')'. \tag{31}$$

[25]The functions f and f' are plotted in Figures S1 and S2.

[26]I use the base R function `smooth.spline` with 20 knots. In this setting $Z_2(V_\theta) = 1$ is known.

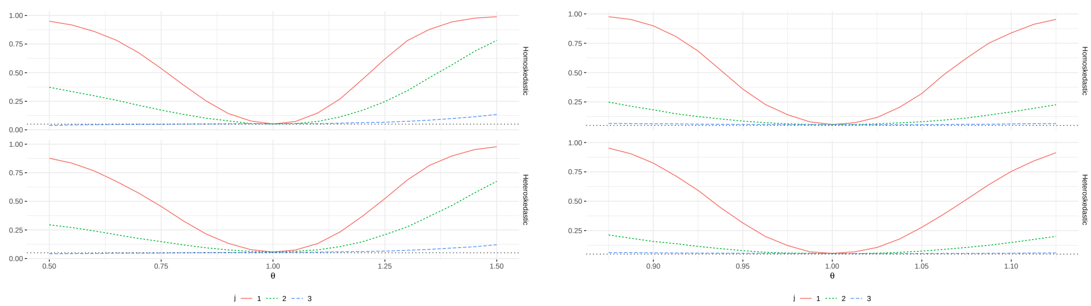
[27]Given $\hat{f}$, $\hat{\theta} = \arg\min_{\theta \in \Theta_\star} \frac{1}{n}\sum_{i=1}^n (Y_i - \hat{f}(V_{\theta,i}))^2$, for $\Theta_\star = [-10, 10]$. The asymptotic variance is estimated by $\hat{\sigma}^2 / \frac{1}{n}\sum_{i=1}^n \left(\widehat{f}'(V_{\hat{\theta},i})\left[X_2 - \hat{Z}(V_{\hat{\theta},i})\right]\right)^2$, for $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \hat{f}(V_{\hat{\theta},i}))^2$.

[28]This is not surprising: Wald tests are known to perform poorly in cases of weak identification.

Table 2: ERF (%), Single-index model

	Exponential						Logistic					
	Homoskedastic			Heteroskedastic			Homoskedastic			Heteroskedastic		
n	f_1	f_2	f_3	f_1	f_2	f_3	f_1	f_2	f_3	f_1	f_2	f_3
$\psi_{n,\theta}$												
400	6.04	5.86	5.28	5.38	5.42	4.80	6.24	5.72	5.62	6.06	5.68	5.24
600	5.50	5.38	5.10	5.82	5.48	5.32	5.70	5.54	5.62	5.44	5.32	5.24
800	5.10	5.14	5.40	5.62	5.82	5.28	5.82	5.60	5.68	5.66	5.42	5.46
Wald												
400	15.10	19.24	13.62	17.68	20.38	14.60	8.26	11.44	10.26	8.68	13.52	9.98
600	12.20	16.64	13.00	14.46	19.38	14.54	7.58	9.12	9.74	6.82	11.68	9.94
800	10.16	15.22	12.90	12.06	19.48	13.52	7.20	8.96	12.54	6.16	9.60	11.48

Figure 1: Single-index model, $\psi_{n,\theta}$ power



(a) Exponential (b) Logistic

Let $d_W := d_\theta + d_Z + 1$. With $\pi(Z) := \mathbb{E}[X|Z]$ and $\upsilon = X - \pi(Z)$,

$$\begin{aligned} Y &= X'\theta + Z_1'\beta + \epsilon \\ X &= \pi(Z) + \upsilon \end{aligned}, \qquad \mathbb{E}[U|Z] = 0, \quad U := (\epsilon, \upsilon')'. \tag{32}$$

If $\pi(Z) = 0$ the instruments Z provide no information about θ. Lack of identification (or weak identification) in this model can be very different from in the IV model with a linear first stage: there are many data configurations in which $\mathbb{E}[X|Z]$ provides substantial identifying information about θ whilst $\mathbb{E}[XZ']\mathbb{E}[ZZ']^{-1}Z$ is uniformative. In such situations, tests which can exploit such non-linear identifying information can provide substantially more power than tests which (explicitly or implicitly) use a linear first stage. In this section I develop a test which can capture such identifying information whilst remaining robust to weak identification. This is illustrated in the simulation study below.[29, 30]

Let ζ denote the density of $\xi := (\epsilon, \upsilon', Z')$ with respect to a σ-finite measure ν. The parameters of the IV model are $\gamma = (\theta, \eta)$ with the nuisance parameters collected in $\eta = (\beta, \pi, \zeta)$. The density of one observation is

$$p_\gamma(W) = \zeta(Y - X'\theta - Z_1'\beta, X - \pi(Z), Z), \tag{33}$$

with respect to a σ-finite measure $\tilde{\nu}$ and P_γ denotes the corresponding measure. The model parameters are restricted as follows.

Assumption 9: *The parameters $\gamma = (\theta, \beta, \pi, \zeta) \in \Gamma = \Theta \times \mathcal{B} \times \mathscr{P} \times \mathscr{Z}$ where*

(i) *Θ is an open subset of $\mathbb{R}^{d_\theta}$ and $\mathcal{B}$ is an open subset of $\mathbb{R}^{d_\beta}$;*

(ii) *$\mathscr{Z}$ is a subset of the set of density functions on $\mathbb{R}^{d_W}$ with respect to ν;*

(iii) *For $(\pi, \zeta) \in \mathscr{P} \times \mathscr{Z}$, if $\xi := (U', Z')'$, then*

$$\mathbb{E}[U|Z] = 0, \qquad \mathbb{E}\|\xi\|^4 < \infty, \qquad \mathbb{E}\|\pi(Z)\|^4 < \infty, \qquad \mathbb{E}\|\phi(\xi)\|^4 < \infty,$$

where $\phi_1 := \nabla_\epsilon \log \zeta(\epsilon, \upsilon, Z)$, $\phi_2 := \nabla_\upsilon \log \zeta(\epsilon, \upsilon, Z)$ and $\phi := (\phi_1, \phi_2')'$.

Additionally, p_γ is a probability density for each $\gamma \in \Gamma$ with respect to a σ-finite measure $\tilde{\nu}$.

[29]This does not contradict optimality results that are known for, e.g., the AR test (Moreira, 2009; Chernozhukov, Hansen, and Jansson, 2009) as these results assume a linear first stage.

[30]An alternative approach to capturing this non-linear identifying information (whilst remaining robust to weak instruments) is to use a large number of transformations of the instruments, $f_1(Z), \ldots, f_M(Z)$, in a linear first stage, combined with a testing procedure which remains robust in the presence of many weak instruments. In the simulation study below, I compare the $\psi_{n,\theta}$ test proposed here to this approach, using the test of Mikusheva and Sun (2022).

Assumption 9 imposes the existence of certain moments and the (IV) conditional mean restriction. That p_γ is a valid probability density holds automatically (with $\nu = \tilde{\nu}$) when $U|Z$ is continuously distributed.

Local Asymptotic Normality Consider local perturbations $P_{\gamma+\varphi_n(h)}$ for

$$\varphi_n(h) := \left(\frac{\tau}{\sqrt{n}}, \frac{b_0}{\sqrt{n}}, \varphi_{n,1}(b_1), \varphi_{n,2}(b_2)\right), \quad h = (\tau, b) \in H := \mathbb{R}^{d_\theta} \times B, \tag{34}$$

with $B := \mathbb{R}^{d_\beta} \times B_1 \times B_2$. B_1 is a subset of the bounded functions $b_1 : \mathbb{R}^{d_Z} \to \mathbb{R}^{d_\theta}$ and B_2 a subset of the functions $b_2 : \mathbb{R}^{d_W} \to \mathbb{R}$ which are bounded and continuously differentiable in its first $1 + d_\theta$ components with bounded derivative and such that

$$\mathbb{E}\left[b_2(U, Z)\right] = 0, \quad \mathbb{E}\left[U b_2(U, Z)|Z\right] = 0, \qquad \text{for } (U', Z')' \sim \zeta. \tag{35}$$

The precise forms of $\varphi_{n,1}, \varphi_{n,2}$ are left unspecified. It is required only that the local perturbations satisfy LAN.[31]

Assumption 10: *Suppose that $\mathcal{W}_n = \prod_{i=1}^n \mathbb{R}^{d_W}$, $P_{n,h} := P^n_{\gamma+\varphi_n(h)} \ll \nu_n$ for all $\gamma \in \Gamma$ and $h \in H$ and are such that Assumption 1 holds with*

$$\log \frac{p_{n,h}}{p_{n,0}} = \frac{1}{\sqrt{n}} \sum_{i=1}^n [Ah](W_i) - \frac{1}{2}\sigma(h) + o_{P_{n,0}}(1), \quad h \in H, \tag{36}$$

where $\sigma(h) = \int [Ah]^2 \,\mathrm{d}P$ and A is as in equation (25) with

$$\begin{aligned}
\dot{\ell}(W) &:= -\phi_1(\epsilon(\theta, \beta), \upsilon(\pi), Z) X_1 \\
[Db](W) &:= -\phi(\epsilon(\theta, \beta), \upsilon(\pi), Z)' \left[\begin{smallmatrix} b_0' Z_1 & b_1(Z) \end{smallmatrix}\right] + b_2(\epsilon(\theta, \beta), \upsilon(\pi), Z),
\end{aligned}$$

where $\epsilon(\theta, \beta) := Y - X'\theta - Z_1'\beta$ and $\upsilon(\pi) := X - \pi(Z)$.

The moment conditions The test will be based on moment conditions related to the efficient score function for θ, $\tilde{\ell}$. This is given in the following Lemma.[32]

Lemma 6: *Suppose Assumptions 9, 10 hold, for $J(Z) := \mathbb{E}[UU'|Z]$,*

$$\begin{aligned}
0 < c \le \lambda_{\min}(J(Z)) \le \lambda_{\max}(J(Z)) \le C < \infty, \qquad & \lambda_{\min}(\mathbb{E}[Z_1 Z_1']) > 0, \\
\mathbb{E}\left[\phi(\epsilon, \upsilon, Z) U' | Z\right] = -I, \qquad & \mathbb{E}[\phi_1(\epsilon, \upsilon, Z) \upsilon U'] = 0,
\end{aligned} \tag{37}$$

[31] Examples of $\varphi_{n,1}, \varphi_{n,2}$ and B for which Assumption 10 holds are given in Section S4.2.2.

[32] A sufficient condition for the last two conditions in (37) is $\lim_{|u_i| \to \infty} |u_i| \zeta(u, z) = 0$ for $i = 1, \ldots, d_\alpha$. This can be seen by integrating by parts.

and B_1 is dense in L_2. Define $\omega(Z) := \mathbb{E}[\epsilon^2|Z]^{-1}$. The efficient score for θ is

$$\tilde{\ell}(W) = \omega(Z)(Y - X'\theta - Z_1'\beta)\left[\pi(Z) - \mathbb{E}[\omega(Z)XZ_1']\mathbb{E}[\omega(Z)Z_1Z_1']^{-1}Z_1\right]. \quad (38)$$

For simplicity, I will use the moment functions

$$g(W) := \mathbb{E}[\epsilon^2]^{-1}(Y - X'\theta - Z_1'\beta)\left[\pi(Z) - \mathbb{E}[XZ_1']\mathbb{E}[Z_1Z_1']^{-1}Z_1\right]. \quad (39)$$

g belongs to the orthocomplement of $\{Db : b \in B\}$ and coincides with the efficient score function when $\mathbb{E}[\epsilon^2|Z] = \mathbb{E}[\epsilon^2]$ a.s. (i.e. under homoskedasticity).[33, 34]

LEMMA 7: *Suppose that Assumptions 9, 10 and equation (37) hold. Then, the moment conditions $g \in \{Db : b \in B\}^\perp$. If $\mathbb{E}[\epsilon^2|Z] = \mathbb{E}[\epsilon^2]$ a.s., then $g = \tilde{\ell}$ a.s..*

As $g \in \{Db : b \in B\}^\perp$, Assumption 2 is satisfied with $g_n = \mathbb{G}_n g$ by Lemma 5.

PROPOSITION 7: *Suppose that Assumptions 9, 10 and equation (37) hold. Then Assumption 2 is satisfied with $g_n = \mathbb{G}_n g$.*

A feasible test Suppose that $\hat{\beta}_n$ and $\hat{\pi}_{n,i}(Z_i)$ are estimators of β and $\pi(Z_i)$ respectively. Let the i-th residual in (31) based on $\theta = \theta_0$ and $\hat{\beta}_n$ be $\hat{\epsilon}_{n,i} := Y_i - X_i'\theta - Z_{1,i}'\hat{\beta}_n$. Let $\hat{s}_n := \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_{n,i}^2$ and define

$$\hat{g}_{n,\theta,i} := \hat{s}_n^{-1}\hat{\epsilon}_{n,i}\left[\hat{\pi}_n(Z_i) - \left[\frac{1}{n}\sum_{i=1}^n X_i Z_{1,i}'\right]\left[\frac{1}{n}\sum_{i=1}^n Z_{1,i}Z_{1,i}'\right]^{-1} Z_i\right], \quad (40)$$

and $\check{V}_{n,\theta} := \frac{1}{n}\sum_{i=1}^n \hat{g}_{n,\theta,i}\hat{g}_{n,\theta,i}'$. Based on $\check{V}_{n,\theta}$, form $\hat{V}_{n,\theta}$ according to the construction in Section S5 of Lee and Mesters (2024b) using a truncation rate ν_n, set $\hat{\Lambda}_{n,\theta} := \hat{V}_{n,\theta}^\dagger$ and $\hat{r}_{n,\theta} := \operatorname{rank}(\hat{V}_{n,\theta})$.

The following assumption provides sufficient high-level conditions on the estimators $\hat{\beta}_n$ and $\hat{\pi}_{n,i}(Z_i)$ such that Assumption 3 holds. These conditions are compatible with $\hat{\pi}_{n,i}$ being a leave-one-out series estimator.[35, 36]

[33] Nevertheless, homoskedasticity is *not* assumed and the results below hold under heteroskedasticity. For full efficiency one could base the test on (38). This is left for future work.

[34] g is very similar to the orthogonalised score derived for the partially linear IV model in Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, and Robins (2018).

[35] The discretisation of $\hat{\beta}_n$ is a technical device which permits the proof to go through under weaker conditions (cf. Le Cam and Yang, 2000, Chapter 6). This can be arranged given a $\sqrt{n}$ – consistent initial estimator, by replacing its value with the closest point in the set $\mathscr{S}_n$.

[36] See e.g. Belloni et al. (2015) for sufficient conditions for (41) and Section S4.2 for a discussion of (42).

Assumption 11: *Suppose that, given θ, (i) $\hat{\beta}_n$ is an estimator valued in $\mathscr{S}_n := \{CZ/\sqrt{n} : Z \in \mathbb{Z}^{d_\beta}\}$ for some $C \in \mathbb{R}^{d_\beta \times d_\beta}$ and satisfying $\sqrt{n}(\hat{\beta}_n - \beta) = O_{P_{n,0}}(1)$ and (ii) $\hat{\pi}_{n,i}(Z_i)$ are estimators such that $\hat{\pi}_{n,i}(Z_i)$ is $\sigma(Z_i, \mathcal{C}_{n,-i})$ measurable for $\mathcal{C}_{n,-i} := \{W_j : j = 1, \ldots, n,\ j \neq i\}$, on events F_n with $P_{n,0}(F_n) \to 1$,*

$$\left[\int \|\hat{\pi}_{n,i}(z) - \pi(z)\|^2 \,\mathrm{d}\zeta_Z(z)\right]^{1/2} \leq \delta_n = o(1), \tag{41}$$

where ζ_Z is the marginal distribution of Z and for each $k = 1, \ldots, d_\theta$, $i \neq j$,

$$\mathbb{E}\left[\mathbf{1}_{F_n}\mathbf{1}_{G_n}(\hat{\pi}_{n,i,k}(Z_i) - \pi_k(Z_i))(\hat{\pi}_{n,j,k}(Z_j) - \pi_k(Z_j))'\epsilon_i\epsilon_j\right] \lesssim \delta_n^2/n,\ \ P_{n,0}(G_n) \to 1. \tag{42}$$

Suppose also $\delta_n^2 + n^{-1/2} = o(\nu_n)$, (37) holds and $\mathbb{E}\left[\epsilon^4(\|\pi(Z)\| + \|Z_1\|)^4\right] < \infty$.

There is no requirement on the rate δ_n in (41), (42) beyond $\delta_n = o(1)$.

Proposition 8: *Suppose that Assumptions 9, 10, & 11 hold. Then Assumption 3 holds with $\hat{g}_{n,\theta} := \frac{1}{\sqrt{n}}\sum_{i=1}^n g_{n,\theta,i}$, $g_n := \mathbb{G}_n g$ and $\hat{\Lambda}_{n,\theta}$ defined below equation (40).*

A consequence of Assumption 10 and Propositions 7 and 8 is that the test $\psi_{n,\theta}$ formed as in (16) is locally regular by Theorem 1.

Simulation study I test $H_0 : \theta = 0$ at a nominal level of 5%. Each study reports the results of 5000 monte carlo replications with a sample size $n \in \{200, 400, 600\}$.[37] Two simulation designs are considered.

Design 1 is a bivariate, just identified design. Here $d_\theta = 2$ and Z_2 is drawn from a zero - mean multivariate normal distribution with covariance matrix $\mathrm{Var}(Z_2) = \left[\begin{smallmatrix} 1 & 0.4 \\ 0.4 & 1 \end{smallmatrix}\right]$. The error terms ϵ, υ are drawn from a zero-mean multivariate normal such that each has variance 1 and the covariances are $\mathrm{Cov}(\epsilon, \upsilon_i) = 0.9$ and $\mathrm{Cov}(\upsilon_1, \upsilon_2) = 0.7$. $Z_1 = 1$ with $\beta = 1$ and $\pi(Z) = \pi(Z_2) = (\pi_1(Z_{2,1}), \pi_2(Z_{2,2}))'$ with each π_i $(i = 1, 2)$ being one of the exponential or logistic functions f_j in Table 1.[38]

I consider the $\psi_{n,\theta}$ test developed above, with a sample split series estimator of π based on (tensor product) Legendre polynomials. I consider both fixing the number of polynomials at $k = 3$ in each of the univariate series which form the tensor product basis and choosing $k \in \{3, 4, 5, 6, 7\}$ using information criteria. ν is

[37] The power surfaces in Design 1 are computed with 2500 replications.

[38] These functions are plotted in Figures S1 and S2. The separation $\pi(Z_2) = (\pi_1(Z_{2,1}), \pi_2(Z_{2,2}))'$ is assumed unknown and is not imposed in the estimation of π.

set to 0.1. I additionally consider the Anderson and Rubin (1949) (AR) test.[39, 40]

The null rejection frequencies are shown in Tables 3 and 4. The parameter j controls the level of identification: the larger is j the closer π_j is to the zero function. In each specfication all the $\psi_{n,\theta}$ tests and the AR test are approximately bounded above by the nominal level. Power surfaces for the $\psi_{n,\theta}$ and AR tests are shown in figures 2 – 7. These show that the $\psi_{n,\theta}$ test is able to detect deviations from the null when π_j has the exponential form, in contrast to the AR test.[41] For the logistic form, the power of the two tests is similar. Unsurpringly, neither test has non-trivial power when identification is very weak (i.e. $j = 3$).

Table 3: Empirical rejection frequencies, IV, Design 1

		Exponential - Exponential				Logistic - Logistic				Exponential - Logistic			
		AR	$\psi_{n,\theta}$			AR	$\psi_{n,\theta}$			AR	$\psi_{n,\theta}$		
n	j		$k=3$	AIC	BIC		$k=3$	AIC	BIC		$k=3$	AIC	BIC
200	1	5.28	4.46	4.08	4.50	5.28	5.06	4.64	5.04	5.28	4.56	4.36	4.56
200	2	5.28	4.08	4.08	4.08	5.28	4.84	4.72	4.84	5.28	4.38	4.40	4.38
200	3	5.28	3.12	3.08	3.12	5.28	4.18	4.12	4.18	5.28	3.98	3.96	3.98
400	1	5.20	4.70	4.98	4.82	5.20	5.34	5.04	5.38	5.20	5.24	4.58	5.22
400	2	5.20	4.44	4.58	4.44	5.20	4.86	4.92	4.86	5.20	4.68	4.82	4.68
400	3	5.20	3.20	3.30	3.20	5.20	4.84	4.88	4.84	5.20	4.58	4.60	4.58
600	1	5.34	4.38	4.60	4.28	5.34	4.98	4.34	5.04	5.34	4.96	5.00	4.96
600	2	5.34	4.36	4.42	4.36	5.34	4.94	4.80	4.94	5.34	4.68	4.78	4.68
600	3	5.34	1.88	1.94	1.88	5.34	4.66	4.54	4.66	5.34	4.72	4.62	4.72

Notes: The column headings, e.g. "Exponential - Logistic" indicates that π_1, π_2 have the exponential and logistic form in Table 1 respectively, with c_j corresponding to column j.

Design 2 is a univariate, over identified model with heteroskedastic errors. $d_\theta = 1$, Z_1, β and Z_2 are as in Design 1, whilst $\pi(Z_2) = (\pi_1(Z_{2,1}) + \pi_2(Z_{2,2}))/2$ where the π_i have one of the exponential or logistic forms of Table 1.[42] I draw $(\tilde{\epsilon}, \tilde{v})$ from a multivariate normal distribution with unit variances and covariance 0.95 and $(\epsilon, v)' = \begin{bmatrix} \sqrt{1+\sin(Z_{2,1})^2} & 0 \\ 0 & \sqrt{1+\cos(Z_{2,2})^2} \end{bmatrix} (\tilde{\epsilon}, \tilde{v})'$.

The $\psi_{n,\theta}$ tests are computed in the same manner as in Design 1. I also compute the AR, LM and CLR tests based on Z_2 (with Z_1 partialled out) as well as the

[39] The AR test is computed with Z_2 as instruments, after partialling out Z_1.

[40] I do not consider alternative weak instrument robust tests based on a linear first stage (e.g. LM, CLR) in this design as the AR test is known to be optimal when the model is just-identified.

[41] One could consider an AR test using e.g. some basis functions $f_1(Z_2), \ldots, f_K(Z_2)$ however as noted in Mikusheva and Sun (2022, p. 2669), the AR statistic is not well behaved for large K. The jackknife AR test of Mikusheva and Sun (2022) applies only to the case where $d_\theta = 1$.

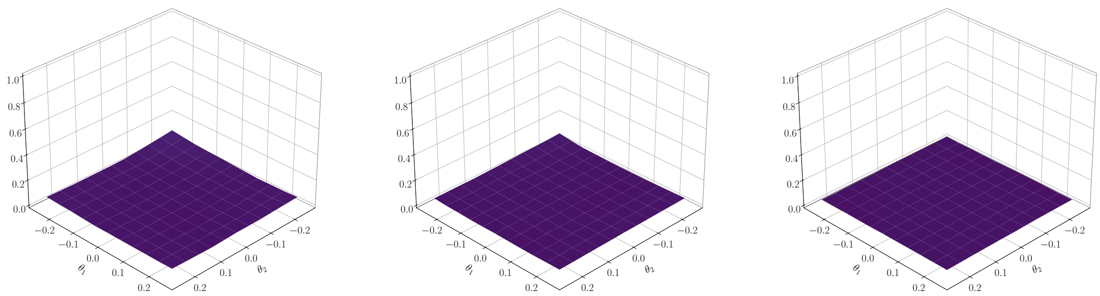
[42] This functional form is treated as unknown and not imposed in the estimation of π.

Table 4: Empirical rejection frequencies, IV, Design 1

		Exponential - Exponential				Logistic - Logistic				Exponential - Logistic			
		AR	$\psi_{n,\theta}$			AR	$\psi_{n,\theta}$			AR	$\psi_{n,\theta}$		
n	$j_1 - j_2$		$k=3$	AIC	BIC		$k=3$	AIC	BIC		$k=3$	AIC	BIC
200	1 - 3	5.28	3.68	3.66	3.72	5.28	4.26	4.18	4.26	5.28	4.26	4.56	4.26
200	2 - 3	5.28	3.84	3.80	3.84	5.28	4.16	4.16	4.16	5.28	4.28	4.26	4.28
200	3 - 3	5.28	3.12	3.08	3.12	5.28	4.18	4.12	4.18	5.28	3.98	3.96	3.98
400	1 - 3	5.20	4.80	4.38	4.82	5.20	4.76	4.70	4.76	5.20	4.92	4.62	4.92
400	2 - 3	5.20	4.84	4.82	4.84	5.20	4.56	4.60	4.56	5.20	4.56	4.72	4.56
400	3 - 3	5.20	3.20	3.30	3.20	5.20	4.84	4.88	4.84	5.20	4.58	4.60	4.58
600	1 - 3	5.34	4.96	4.74	4.88	5.34	4.70	4.26	4.72	5.34	4.66	4.80	4.60
600	2 - 3	5.34	4.78	4.62	4.78	5.34	4.80	4.74	4.80	5.34	4.56	4.60	4.56
600	3 - 3	5.34	1.88	1.94	1.88	5.34	4.66	4.54	4.66	5.34	4.72	4.62	4.72

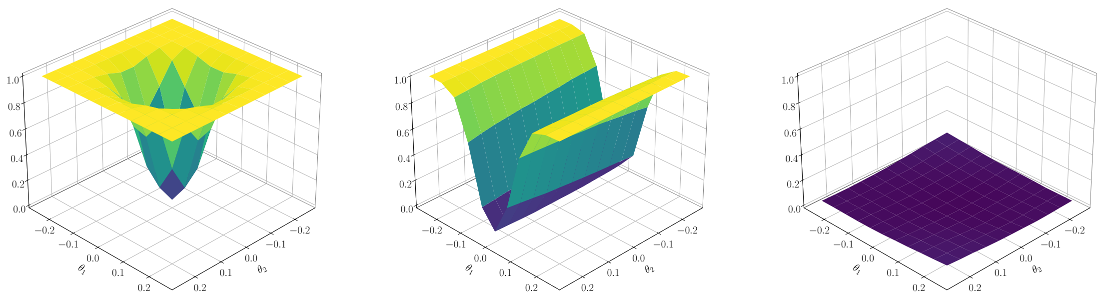
Notes: The column headings, e.g. "Exponential - Logistic" indicates that π_1, π_2 have the exponential and logistic form in Table 1 respectively, with c_{j_1} and c_{j_2} corresponding to column j_1 - j_2.

Figure 2: IV Design 1, AR power, π_i exponential



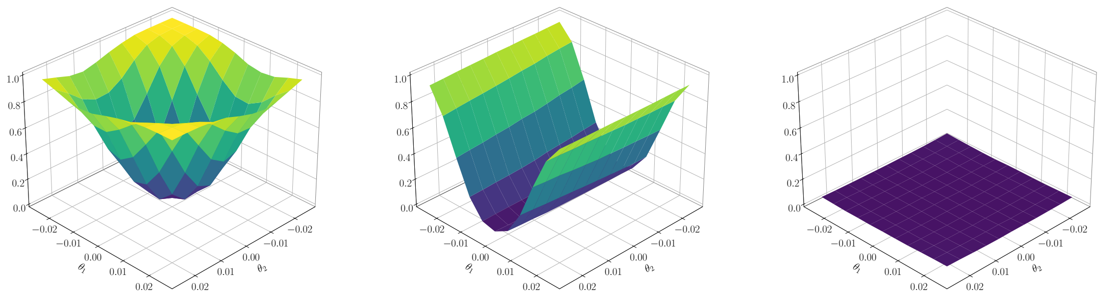
(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

Figure 3: IV Design 1, $\psi_{n,\theta}$ ($k = 3$) power, π_i exponential



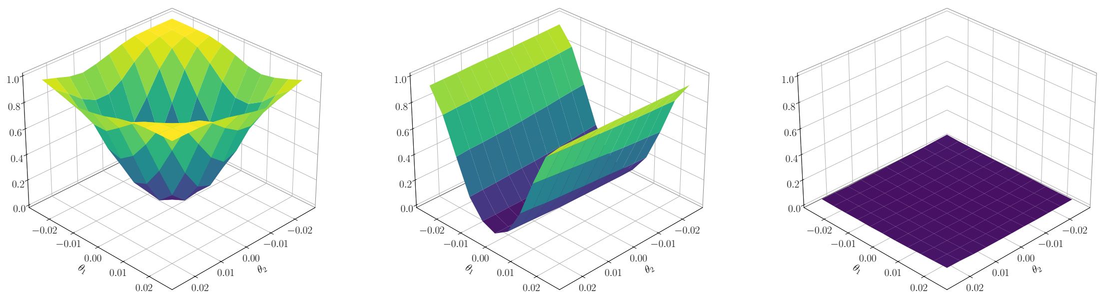
(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

Figure 4: IV Design 1, AR power, π_i logistic



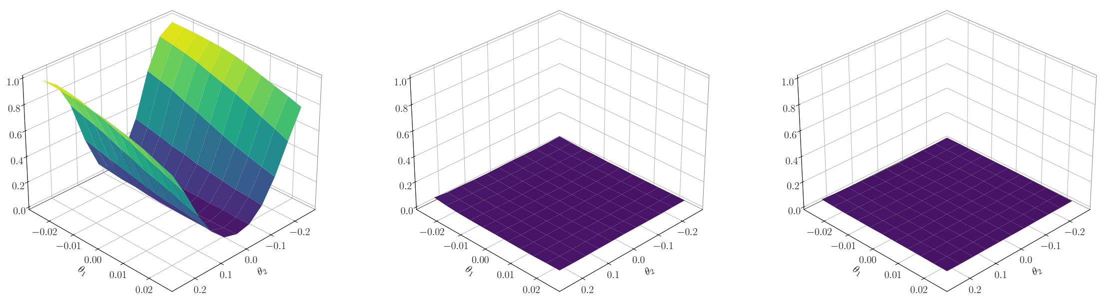
(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

Figure 5: IV Design 1, $\psi_{n,\theta}$ $(k = 3)$ power, π_i logistic



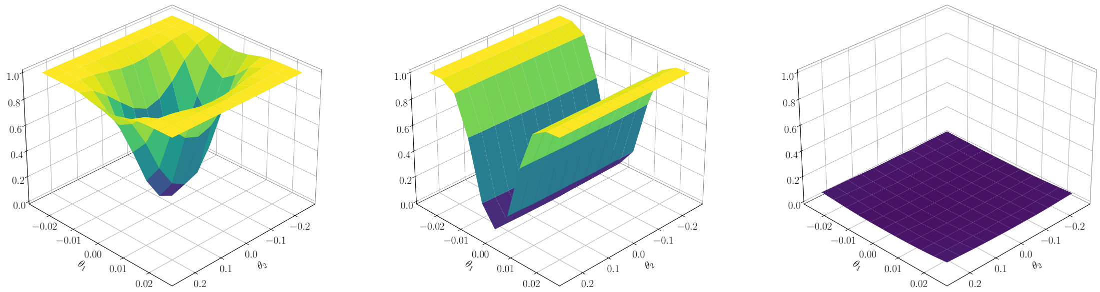
(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

Figure 6: IV Design 1, AR power, π_1 exponential, π_2 logistic



(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

Figure 7: IV Design 1, $\psi_{n,\theta}$ ($k = 3$) power, π_1 exponential, π_2 logistic



(a) $j_1 = 1$, $j_2 = 1$ (b) $j_1 = 1$, $j_2 = 3$ (c) $j_1 = 3$, $j_2 = 3$

many weak instrument robust jacknife AR test of Mikusheva and Sun (2022). MS_1 uses Z_2 as instruments; MS_2 uses the (tensor product) of Legendre polynomials used to estimate π as instruments.

The empirical rejection frequencies under the null are shown in Tables 5 – 7. As in Design 1, the parameter j controls the level of identification: the larger is j the closer π_j is to the zero function and hence θ unidentified. In each specfication all the $\psi_{n,\theta}$ tests and the AR, LM tests are approximately bounded above by the nominal level. The CLR, MS_1 and MS_2 tests appear to overreject slightly for smaller n in some designs. The power of these tests is plotted in Figures 8 – 10; the ψ_{n,θ_0} tests are denoted by $k = 3$, AIC and BIC, corresponding to how π is estimated. For the design with both π_i exponential, the $\psi_{n,\theta}$ test clearly delivers the highest power whenever there is non-trivial power available; of the other tests, only MS_2 delivers non-trivial power in this specification. For the case with both π_i logistic, all tests except MS_2 perform similarly. In the final specification, with π_1 exponential and π_2 logistic, in the $j = 1, j = 2$ cases, the $\psi_{n,\theta}$ tests provide the highest power, whilst those based on a linear first stage provide more power in the very weakly identified case with $j = 3$.

6 Empirical applications

In this section I re-analyse two instrumental variables models with potentially weak instruments by inverting the weak instrument robust test $\psi_{n,\theta}$ developed in section 5.2. In each case this procedure is able to exploit non-linearities to yield substantial reductions in confidence interval (CI) length relative to AR CIs.

Table 5: Empirical rejection frequencies, IV, Design 2, Exponential - Exponential

n	j	AR	LM	CLR	MS$_1$	MS$_2$	$\psi_{n,\theta}$ $k=3$	AIC	BIC
200	1	6.48	6.34	6.80	7.54	9.78	4.50	4.42	4.50
200	2	6.48	6.44	6.74	7.54	9.78	4.00	3.90	4.00
200	3	6.48	6.82	6.70	7.54	9.78	3.08	2.98	3.08
400	1	6.94	6.32	6.86	7.50	8.36	5.00	4.50	5.00
400	2	6.94	6.46	6.70	7.50	8.36	4.66	4.70	4.66
400	3	6.94	6.36	6.70	7.50	8.36	0.96	1.06	0.96
600	1	6.58	6.62	6.82	7.12	7.80	5.00	4.70	5.00
600	2	6.58	6.58	6.64	7.12	7.80	5.08	5.14	5.08
600	3	6.58	6.26	6.48	7.12	7.80	0.48	0.56	0.48

Notes: The functions π_i have the exponential form in Table 1 with c_j corresponding to column j.

Table 6: Empirical rejection frequencies, IV, Design 2, Logistic - Logistic

n	j	AR	LM	CLR	MS$_1$	MS$_2$	$\psi_{n,\theta}$ $k=3$	AIC	BIC
200	1	6.48	6.84	8.30	7.54	9.78	4.92	4.78	4.92
200	2	6.48	6.84	8.22	7.54	9.78	4.94	4.82	4.94
200	3	6.48	6.90	7.58	7.54	9.78	4.12	4.02	4.12
400	1	6.94	6.94	8.28	7.50	8.36	5.32	5.24	5.32
400	2	6.94	7.00	8.34	7.50	8.36	5.06	5.10	5.06
400	3	6.94	6.84	8.04	7.50	8.36	4.30	4.32	4.30
600	1	6.58	6.92	8.20	7.12	7.80	5.12	5.10	5.12
600	2	6.58	6.90	8.20	7.12	7.80	5.04	5.12	5.04
600	3	6.58	6.78	8.14	7.12	7.80	4.64	4.70	4.64

Notes: The functions π_i have the logistic form in Table 1 with c_j corresponding to column j.

Figure 8: IV design 2, Power curves, exponential π_i

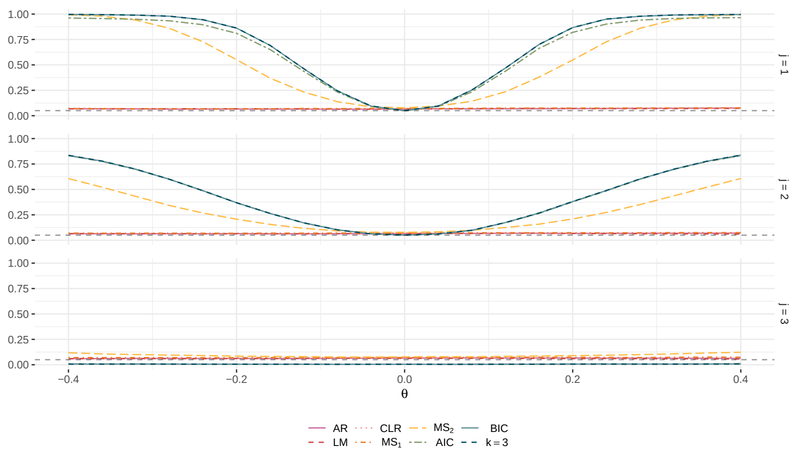


Table 7: Empirical rejection frequencies, IV, Design 2, Exponential - Logistic

		AR	LM	CLR	MS_1	MS_2	$\psi_{n,\theta}$		
n	j						$k=3$	AIC	BIC
200	1	6.48	5.10	6.58	7.54	9.78	4.58	4.54	4.58
200	2	6.48	5.32	6.54	7.54	9.78	4.80	4.72	4.80
200	3	6.48	5.60	6.50	7.54	9.78	4.28	4.12	4.28
400	1	6.94	5.48	6.40	7.50	8.36	4.60	4.68	4.60
400	2	6.94	5.52	6.54	7.50	8.36	4.64	4.66	4.64
400	3	6.94	5.92	6.50	7.50	8.36	2.88	2.94	2.88
600	1	6.58	5.04	6.50	7.12	7.80	4.68	4.88	4.68
600	2	6.58	5.10	6.44	7.12	7.80	4.72	4.68	4.72
600	3	6.58	5.10	6.10	7.12	7.80	2.08	2.14	2.08

Notes: π_1, π_2 have the exponential and logistic form in Table 1 respectively with c_{j_i} corresponding to column j.

Figure 9: IV Design 2, Power curves, logistic π_i

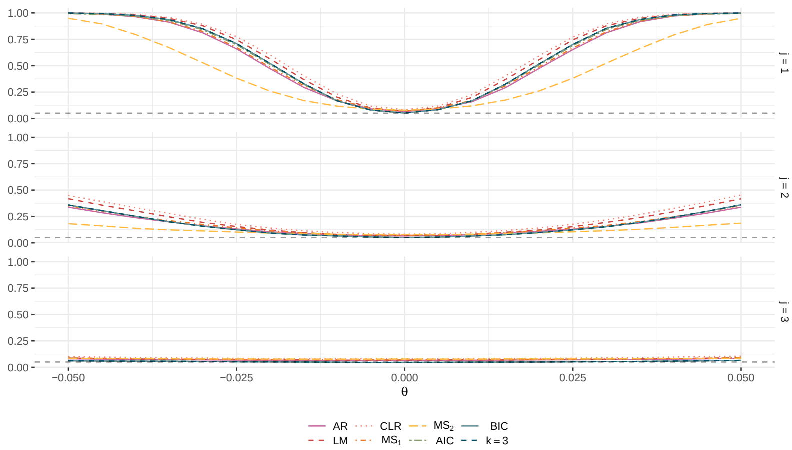
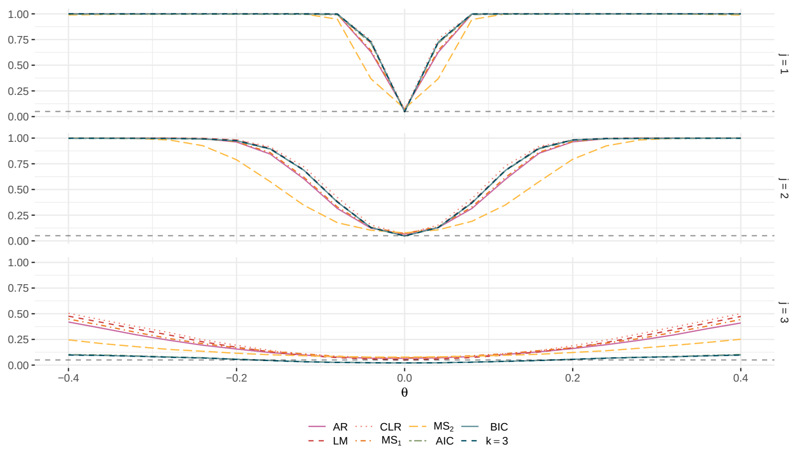


Figure 10: IV Design 2, Power curves, exponential π_1, logistic π_2



6.1 The effect of skilled immigration on productivity

Hornung (2014) studies the long term effect of skilled immigration on productivity using a natural experiment in which the skilled Hugenots (French protestants) fled religious persecution and settled in Prussia.[43] In the notation of Example 2, Y is log output in textile manufacturing, X is the proportion of Hugenots in each town and Z_1 contains various control variables, see Hornung (2014) for details. Hornung (2014) argues that "by the order of centralized ruling by the king and his agents Huguenots were channeled into Prussian towns in order to compensate for severe population losses during the Thirty Years' War", motivating the instrument Z_2: the percentage population losses during the war. In particular, three different measurements of this population loss are used and refered to as specifications (1), (2) and (3) hereafter.[44]

As noted in Hornung (2014), this instrument may be weak: in each case the first stage F statistic is "small".[45] I implement the $\psi_{n,\theta}$ test using a leave-one-out series estimator of π of the form

$$\hat{\pi}_i(Z_i) = \hat{\pi}_i' p_K(Z_{2,i}) + \hat{\beta}_i Z_{1,i}, \tag{43}$$

where the i subscript on the estimated coefficients indicates they have been estimated on all observations except for the i-th. p_K is a vector of a constant and the first K Legendre polynomials. I choose $K \in \{1, 2, 3, 4\}$ and whether to include Z_1 in the model for π based on the BIC: all specifications include Z_1 and $K = 4$.

Table 8 reports 2SLS estimates of θ along with 2SLS CIs, AR CIs and CIs found by inverting the $\psi_{n,\theta}$ test.[46] The resulting CIs provide a similar interpretation as that based on the AR CIs: the effect of (skilled) Hugenot immigration was positive on textile output. However, the $\psi_{n,\theta}$ based CIs are smaller than the AR CIs, achieving approximately a 40% - 50% reduction in length.

6.2 The effect of racial segregation on inequality

Ananat (2011) estimates the effect of racial segregation (X) on poverty and inequality (Y, measured respectively by the poverty rate and log gini coefficient for black / white city residents), instrumenting segregation by a "railroad division

[43] That the Hugenots were more skilled (on average) than the Prussian population seems to be broadly accepted cf. pp. 85-86, 93-95 in Hornung (2014).

[44] Specifications (1) & (2) are those considered in the left and and right hand parts of Table 4 of Hornung (2014); Specification (3) is that considered in the left hand part of Table 5.

[45] It is less than the cutoff of 10 Staiger and Stock (1997) suggest for the homoskedastic case.

[46] The inversion is performed over a grid of 5000 equally spaced points from -1 to 7.

Table 8: Point estimates and confidence intervals

	(1)	(2)	(3)
n	150	150	186
F	3.668	4.791	5.736
Point estimate			
OLS	1.741	1.741	1.592
2SLS	3.475	3.38	1.671
Confidence intervals			
OLS	[0.938, 2.544]	[0.938, 2.544]	[0.788, 2.396]
2SLS	[1.27, 5.68]	[1.294, 5.467]	[0.032, 3.31]
AR	[1.427, 6.303]	[1.43, 5.985]	[-0.022, 3.379]
$\psi_{n,\theta}$	[1.626, 4.099]	[1.637, 4.073]	[1.136, 3.228]
Relative length of confidence intervals to AR			
OLS	0.329	0.353	0.473
2SLS	0.904	0.916	0.964
$\psi_{n,\theta}$	0.507	0.535	0.615

Notes: F is the first stage F statistic. All confidence intervals have nominal coverage of 95%.

index" (RDI, Z_2), "a variation on a Herfindahl index that measures the dispersion of a city's land into subunits" via the layout of railroad tracks.[47] Both stages also include an intercept and control for the length of the railroad track (Z_1).

The instrument may be weak: I calculate the first stage F statistic to be 2.307.[48] I implement the $\psi_{n,\theta}$ test using a leave-one-out series estimator of π of the form (43). I choose $K \in \{1, 2, 3, 4\}$ and whether to include Z_1 in the model for π based on the BIC: this excludes Z_1 and chooses $K = 2$.

Table 9 reports 2SLS estimates of θ along with 2SLS confidence intervals (CIs), AR CIs and CIs found by inverting the $\psi_{n,\theta}$ test.[49] The resulting CIs provide a similar interpretation as that based on the AR CIs: racial segregation increases poverty and inequality within the Black community and decreases poverty and inequality within the White community. However, the $\psi_{n,\theta}$ based CIs are smaller than the AR CIs, achieving a reduction in length varying from 5% to around 38%.

[47] Section 3 and Appendix A of Ananat (2011) provides evidence that the choice of railroad placement was not related to local social or economic concerns.

[48] Ananat (2011) refers to Column 1 of Table 1 when discussing the first stage F statistic. The values in this table imply a first stage F of approximately 16.458. This discrepancy is due to different (default) choices of robust covariance estimate in R's `sandwich` package (HC3) [my calculation] and STATA's `robust` command (HC1) [Ananat (2011)].

[49] The inversion is performed over a grid of 5000 equally spaced points from -1 to 2.

Table 9: Point estimates and confidence intervals

	Poverty rate		Gini coefficient	
	White	Black	White	Black
Point estimate				
OLS	0.189	-0.065	0.449	-0.075
2SLS	0.258	-0.196	0.875	-0.334
Confidence intervals				
OLS	[0.083, 0.295]	[-0.111, -0.019]	[0.238, 0.66]	[-0.144, -0.005]
2SLS	[-0.026, 0.543]	[-0.334, -0.058]	[0.277, 1.474]	[-0.558, -0.111]
AR	[-0.04, 0.598]	[-0.394, -0.075]	[0.319, 1.684]	[-0.674, -0.149]
$\psi_{n,\theta}$	[0.072, 0.568]	[-0.376, -0.097]	[0.29, 1.138]	[-0.608, -0.106]
Relative length of confidence intervals to AR				
OLS	0.332	0.288	0.309	0.264
2SLS	0.891	0.866	0.877	0.853
$\psi_{n,\theta}$	0.775	0.877	0.622	0.955

Notes: All confidence intervals have nominal coverage of 95%.

7 Conclusion

In this paper I establish that C(α)-style tests are locally regular under mild conditions, including in non-regular cases where locally regular estimators do not exist. As a consequence, these tests do not overreject under semiparametric weak identification asymptotics. Additionally I generalise the classical local asymptotic power bounds for LAN models to the case where the efficient information matrix has positive, but potentially deficient, rank, such that these results also apply in cases of underidentification (or weak underidentification). Moreover, I show that, if the C(α) test is based on the efficient score function, it attains these power bounds. This (attainment) result improves on results known in the literature in two ways: (i) it applies also to non-regular models and (ii) it does not require the data to be i.i.d. nor the information operator to be boundedly invertible. A simulation study based on two examples shows that the asymptotic theory provides a good approximation to the finite sample performance of the proposed tests.

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A Proofs of the main results

Proof of Proposition 1. Combination of Assumptions 1 and 2 yields

$$(g_n', L_n(h))' \overset{P_n}{\rightsquigarrow} \mathcal{N}\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma(h)\end{pmatrix}, \begin{pmatrix} V & \tau'\Sigma_{21}' \\ \Sigma_{21}\tau & \sigma(h)\end{pmatrix}\right).$$

By Le Cam's third Lemma $g_n \overset{P_{n,h}}{\rightsquigarrow} Z_\tau \sim \mathcal{N}(\Sigma_{21}\tau, V)$. The second claim follows from the first with Assumption 3(i), Remark 1 and Slutsky's Theorem. By the second claim, Assumption 3(ii) and standard arguments, $\hat{S}_{n,\theta_0} \overset{P_{n,h}}{\rightsquigarrow} Z_\tau' V^\dagger Z_\tau$. □

Proof of Theorem 1. If $r \geq 1$, as $\hat{r}_n \xrightarrow{P_n} r$, $P_n\{c_n = c_\alpha\} \to 1$. By Proposition 1, Remark 1 and Slutsky's Theorem, $\hat{S}_{n,\theta_0} - c_n \rightsquigarrow S - c$ under $P_{n,h}$ where $S \sim \chi_r^2(a)$. Since the χ_r^2 distribution is continuous, by the Portmanteau Theorem,

$$\lim_{n\to\infty} P_{n,h}\psi_{n,\theta_0} = \lim_{n\to\infty} P_{n,h}\left(\hat{S}_{n,\theta_0} > c_n\right) = L\{S - c > 0\} = 1 - \mathrm{P}(\chi_r^2(a) \leq c_\alpha),$$

for L the law of S. If $r = 0$, $\operatorname{rank}(\hat{\Lambda}_{n,\theta_0}) \xrightarrow{P_n} 0 \implies P_n R_n \to 1$ for $R_n := \{\hat{\Lambda}_{n,\theta_0} = 0\}$. On R_n $\hat{S}_{n,\theta_0} = 0 \implies \psi_{n,\theta_0} = 0$. By Remark 1, $P_{n,h}\psi_{n,\theta_0} \leq 1 - P_{n,h}R_n \to 0$. □

Proof of Corollary 1. By Theorem 1, $\pi_n(h) \to \pi(h)$ $(h \in H)$, pointwise. Since the π_n are asymptotically equicontinuous on K, the convergence is uniform on K. □

Proof of Lemma 1. Immediate from $|P_{n,h}\psi_{n,\theta_0} - P_{n,h'}\psi_{n,\theta_0}| \leq d_{TV}(P_{n,h}, P_{n,h'})$. □

Proof of Lemma 2. For $a, b \in \mathbb{R}$, $h_1, h_2 \in H$, $\Delta_n(a_1h_1 + a_2h_2) = a_1\Delta_n h_1 + a_2\Delta_n h_2$ and so $\Delta(a_1h_1 + a_2h_2) = a_1\Delta(h_1) + a_2\Delta(h_2)$, hence Δ is linear. We now establish K is a well-defined covariance kernel. For $h \in H$, $(\|\Delta_n h\|^2)_{n\in\mathbb{N}}$ is Cauchy. Letting $K_n(h, g) := P_n[\Delta_n h \Delta_n g]$ and using Cauchy – Schwarz

$$|K_n(h, g) - K_m(h, g)| \leq \|\Delta_n h - \Delta_m h\|\|\Delta_n g\| + \|\Delta_m h\|\|\Delta_n g - \Delta_m g\|,$$

hence $(K_n(h, g))_{n\in\mathbb{N}}$ is also Cauchy and thus has a limit. Bilinearity and symmetry are straightforward to check. For positive semi-definiteness, let $h_1, \ldots, h_K \in H$, $a \in \mathbb{R}^K$. As $\Delta_n h \in L_2^0(P_n)$, $\mathcal{K}_n := [K_n(h_k, h_j)]_{k,j=1}^K$ is a covariance matrix, hence

$\sum_{k=1}^{K}\sum_{j=1}^{K} a_k a_j K_n(h_k, h_j) = a'\mathcal{K}_n a \geq 0$ for each $n \in \mathbb{N}$ and hence the same holds with K_n and $\mathcal{K}_n$ replaced by K and $\mathcal{K} := [K(h_k, h_j)]_{k,j=1}^{K}$.

By Assumption 1 and the fact that $K(h, h) = \sigma(h)$, $\Delta h \sim \mathcal{N}(0, K(h, h))$. That Δ is a mean-zero Gaussian process with covariance kernel K then follows from the Cramér – Wold Theorem as $\sum_{k=1}^{K} a_k \Delta h_k \sim \mathcal{N}(0, a'\mathcal{K}a)$ and

$$\sum_{k=1}^{K} a_k \Delta_n(h_k) = \Delta_n\left(\sum_{k=1}^{K} a_k h_k\right) \overset{P_n}{\rightsquigarrow} \Delta\left(\sum_{k=1}^{K} a_k h_k\right) = \sum_{k=1}^{K} a_k \Delta h_k. \qquad \square$$

Proof of Proposition 2. Remark 1 and the transitivity of (mutual) contiguity ensures that the experiments $\mathscr{E}_n$ are contiguous. By Theorem 61.6 of Strasser (1985) it suffices to show that the finite dimensional marginal distributions (fdds) of L_n converge (under P_n) to those of L, where $L(h) := \Delta h - \frac{1}{2}\|h\|^2$. This follows as the fdds of Δ_n converge to those of Δ (under P_n), by the Cramér – Wold Theorem. $\square$

Proof of Proposition 3. Let $G_{[0]} := P_0$. Define $Z : \mathbb{H} \to L_2(\Omega, \mathcal{F}, G_{[0]})$ by $Z[h] := \Delta(h)$ for an arbitrary $h \in \pi_V^{-1}([h])$, where π_V is the quotient map from $H \to \mathbb{H}$.[50] This is a standard Gaussian process for $\mathbb{H}$. Define $G_{[h]}$ by $\frac{\mathrm{d}G_{[h]}}{\mathrm{d}G_{[0]}} = \exp\left(Z[h] - \frac{1}{2}\|[h]\|_K^2\right)$. $\mathscr{G}$ is a Gaussian shift on $(\mathbb{H}, \langle\cdot\,,\,\cdot\rangle_K)$ (Strasser, 1985, Theorem 69.4). For any $h \in H$ we have that $Z[h] = \Delta g$ for some $g \in \pi_V^{-1}([h])$ and $\Delta h = \Delta g$ P_0–almost surely (see footnote 50). Since $\|h\|_K = \|[h]\|_K$, P_0-a.s., $\frac{\mathrm{d}G_{[h]}}{\mathrm{d}G_{[0]}} = \exp\left(Z[h] - \frac{1}{2}\|[h]\|_K^2\right) = \exp\left(\Delta h - \frac{1}{2}\|h\|_K^2\right) = \frac{\mathrm{d}P_h}{\mathrm{d}P_0}$. As each $P_h \ll P_0$ and $G_{[h]} \ll G_{[0]}$, and $P_0 = G_{[0]}$, $d_{TV}(P_h, G_{[h]}) = \frac{1}{2}\int\left|\frac{\mathrm{d}P_h}{\mathrm{d}P_0} - \frac{\mathrm{d}G_{[h]}}{\mathrm{d}P_0}\right| \mathrm{d}P_0 = \frac{1}{2}\int\left|\frac{\mathrm{d}P_h}{\mathrm{d}P_0} - \frac{\mathrm{d}G_{[h]}}{\mathrm{d}G_{[0]}}\right| \mathrm{d}P_0 = 0$. $\square$

Proof of Lemma 3. By straightforward calculation

$$\langle[\tau, b]\,,\,[t, g]\rangle_K = \tau'\tilde{\mathcal{I}}t + \langle\Pi[\tau, 0] + [0, b]\,,\,\Pi[t, 0] + [0, g]\rangle_K \tag{44}$$

This and $\Pi[(\tau, 0)] \in \ker \pi_1$ imply $\|[\tau]\|^2 = \inf_{b\in B}\|[\tau, b]\|_K^2 = \tau'\tilde{\mathcal{I}}\tau + \inf_{[h]\in\ker\pi_1}\|\Pi[\tau, 0] - [h]\|_K^2 = \tau'\tilde{\mathcal{I}}_\gamma\tau$. Hence, $\|\tau\| = \|[\tau]\| = 0 \implies \tau'\tilde{\mathcal{I}}_\gamma\tau = 0 \implies \tilde{\mathcal{I}}_\gamma^{1/2}\tau = 0$, and so $\tilde{\mathcal{I}}_\gamma\tau = 0$. Conversely $\tau \in \ker\tilde{\mathcal{I}}_\gamma \implies \|\tau\|^2 = 0 \implies \|\tau\| = 0$. $\square$

Proof of Theorem 2. Define the bounded linear map $T : \overline{\mathbb{H}} \to \mathbb{R}$ according to $T[h] := \left\langle\Pi^\perp[1, 0]\,,\,\Pi^\perp[h]\right\rangle_K = \left\langle\Pi^\perp[1, 0]\,,\,[h]\right\rangle_K$. For any $[h] = [\tau, b] \in \mathbb{H}$,

$$T[h] = \left\langle\Pi^\perp[1, 0]\,,\,\Pi^\perp[\tau, b]\right\rangle_K = \tilde{\mathcal{I}}\tau. \tag{45}$$

[50]This is well-defined: for any other $g \in H$ with $\pi_V(g) = [h]$, one has $\Delta(g) = \Delta(h) + \Delta(v)$ where $\|v\| = K(v, v) = 0$ and hence $\Delta(v) = 0$ P_0 - a.s..

If $\tilde{\mathcal{I}} = 0$, (19) follows from Proposition 4, so assume $\tilde{\mathcal{I}} \neq 0$. Any unbiased level α test ϕ of $T[h] = 0$ against $T[h] \neq 0$ in the (restricted) Gaussian shift $\mathscr{G}$ satisfies

$$G_{[h]}\phi \leq 1 - \Phi\left(z_{\alpha/2} - \tilde{\mathcal{I}}^{1/2}\tau\right) + 1 - \Phi\left(z_{\alpha/2} + \tilde{\mathcal{I}}^{1/2}\tau\right), \tag{46}$$

(Strasser, 1985, Lemma 71.5). By Proposition 2, $\mathscr{E}_n \rightsquigarrow \mathscr{E}$; $\mathscr{E}$ is dominated. Let $\pi_n(h) := P_{n,h}\phi_n$ and fix an arbitrary $h^\star$. There is a subsequence $(\pi_{n_m})_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} \pi_{n_m}(h^\star) = \limsup_{n\to\infty} \pi_n(h^\star)$. Since $[0,1]^H$ is compact in the product topology, there is a subnet $(\pi_{n_{m(s)}})_{s\in S}$ and a function $\pi : H \to [0,1]$ such that $\lim_{s\in S} \pi_{n_{m(s)}}(h) = \pi(h)$ for all $h \in H$. By our hypotheses and equation (45) for any h_0 such that $[h_0] \in \ker T \cap \mathbb{H}$ and any h_1 such that $[h_1] \in \mathbb{H} \setminus (\ker T \cap \mathbb{H})$

$$\pi(h_0) = \lim_{s\in S} \pi_{n_{m(s)}}(h_0) \leq \alpha \leq \lim_{s\in S} \pi_{n_{m(s)}}(h_1) = \pi(h_1). \tag{47}$$

There exists a test ϕ in $\mathscr{E}$ with power function π (van der Vaart, 1991, Theorem 7.1). (47) and Proposition 3 ensure that ϕ is an unbiased, level α test of $\ker T \cap \mathbb{H}$ against $\mathbb{H} \setminus (\ker T \cap \mathbb{H})$ in $\mathscr{G}$. Conclude by combining (46) and (by Proposition 3)

$$\limsup_{n\to\infty} P_{n,h^\star}\phi_n = \lim_{m\to\infty} \pi_{n_m}(h^\star) = \pi(h^\star) = P_{h^\star}\phi = G_{[h^\star]}\phi. \qquad \square$$

Proof of Corollary 2. By Theorem 1, $\lim_{n\to\infty} P_{n,h_n}\psi_{n,\theta} = 1 - \mathrm{P}(Z^2 > c_\alpha)$ where $Z \sim \mathcal{N}\left(\tilde{\mathcal{I}}^{1/2}\tau, 1\right)$. $1 - \mathrm{P}(Z^2 > c_\alpha)$ is equal to the RHS of (20). $\square$

Proof of Theorem 3. By Lemma 3, $\mathbb{H}_1 = \mathbb{R}^{d_\theta} / \ker \tilde{\mathcal{I}}$. $\pi_1 : \mathbb{H} \to \mathbb{H}_1$ is surjective: for any $[\tau] \in \mathbb{H}_1$ let $t \in \pi^{-1}_{\ker \tilde{\mathcal{I}}}(\{[\tau]\})$ where $\pi_{\ker \tilde{\mathcal{I}}}$ is the quotient map from $\mathbb{R}^{d_\theta}$ to $\mathbb{H}_1$. Then $\pi_1[t, 0] = [t] = [\tau]$. It follows that $\dim \operatorname{ran} \pi_1 = \operatorname{codim} \ker \pi_1 = r$.[51] By linearity and $[0, b] \in \ker \pi_1$, $\Pi[\tau, b] = \Pi[\tau, 0] + [0, b]$. This with Lemma 3 yields $\|[\tau, b] - \Pi[\tau, b]\|_K^2 = \|[\tau, 0] - \Pi[\tau, 0]\|_K^2 = \|[\tau]\|^2 = \tau'\tilde{\mathcal{I}}\tau$. Define the sets

$$M_a := \left\{[\tau, b] \in \mathbb{H} : \tau'\tilde{\mathcal{I}}\tau = a\right\}, \qquad \overline{M}_a := \left\{[\tau, b] \in \overline{\mathbb{H}} : \tau'\tilde{\mathcal{I}}\tau = a\right\}.$$

It is straightforward to check that $\operatorname{cl} M_a = \overline{M}_a$. Suppose that ϕ is a test on $\mathscr{G}$ with $G_{[0]}\phi \leq \alpha$. First suppose $a > 0$. ϕ is a level α test of $K_0 : \{[0]\}$ against $K_1 : [\ker \pi_1]^\perp \setminus \{[0]\}$ in the restriction of the standard Gaussian shift experiment

[51] $\mathbb{H}_1 = \mathbb{R}^{d_\theta} / \ker \tilde{\mathcal{I}} \approx \operatorname{ran} \tilde{\mathcal{I}}$ and $\ker \pi_1 \approx \operatorname{ran} \pi_1$.

on $[\ker \pi_1]^\perp$.[52] By Theorem 30.2 in Strasser (1985)

$$\inf_{[h]\in M_a} G_{[h]}\phi = \inf_{[h]\in \bar{M}_a} G_{[h]}\phi \le \inf_{[h]\in \bar{M}_a \cap [\ker \pi_1]^\perp} G_{[h]}\phi \le 1 - \mathrm{P}(\chi^2_r(a) \le c_r), \tag{48}$$

since $[h] \mapsto G_{[h]}\phi$ is continuous. If, instead, $a = 0$, note that $[0] \in M_0$ and so,

$$\inf_{[h]\in M_0} G_{[h]}\phi \le G_{[0]}\phi \le \alpha = 1 - \mathrm{P}(\chi^2_r(0) \le c_r). \tag{49}$$

Fix $a \ge 0$ and let $\mathcal{R} := 1 - \mathrm{P}(\chi^2_r(a) \le c_r)$. Let $\pi_n(h) := P_{n,h}\phi_n$ and define $\beta_n := \inf\left\{P_{n,h}\phi_n : h = (\tau, b) \in H,\ \tau'\tilde{\mathcal{I}}\tau = a\right\}$. Suppose that $\limsup_{n\to\infty} \beta_n \ge \mathcal{R} + \varepsilon$ for some $\varepsilon > 0$. Hence, for some subsequence $(n_m)_{m\in\mathbb{N}}$, $\lim_{m\to\infty} \beta_{n_m} \ge \mathcal{R} + \varepsilon$. Since $[0, 1]^H$ is compact in the product topology, there is a subnet $(\pi_{n_{m(s)}})_{s\in S}$ and a function $\pi : H \to [0, 1]$ such that $\lim_{s\in S} \pi_{n_{m(s)}}(h) = \pi(h)$ for all $h \in H$. Take any h such that $[h] \in M_a$. The preceding display implies

$$\pi(h) = \lim_{s\in S} \pi_{n_{m(s)}}(h) \ge \lim_{s\in S} \inf\left\{\pi_{n_{m(s)}}(h) : h = (\tau, b) \in H,\ \tau'\tilde{\mathcal{I}}\tau = a\right\} \ge \mathcal{R} + \varepsilon. \tag{50}$$

By Proposition 2, $\mathscr{E}_n \rightsquigarrow \mathscr{E}$; $\mathscr{E}$ is dominated. There is a test ϕ in $\mathscr{E}$ with power function π (van der Vaart, 1991, Theorem 7.1). Consider the restriction of $\mathscr{G}$ to $[\ker \pi_1]^\perp$. By hypothesis, Corollary S2 and Proposition 3 $G_{[0]}\phi = P_0\phi = \pi(0) = \lim_{s\in S} \pi_{n_m(s)}(0) \le \limsup \pi_n(0) \le \alpha$, hence ϕ is a test of level α of K_0 against K_1 in this experiment, and $\inf_{[h]\in M_a} G_{[h]}\phi = \inf_{h:[h]\in M_a} P_h\phi = \inf_{h:[h]\in M_a} \pi(h) \ge \mathcal{R}+\varepsilon$, by (50) and Proposition 3, but this contradicts (48) if $a > 0$ or (49) if $a = 0$. □

Proof of Corollary 3. Equation (22) follows from Theorem 1. For equation (23), let $f_n(h) := P_{n,h}\psi_{n,\theta_0}$. By (22) and the asymptotic equicontinuity, $\lim_{n\to\infty} f_n(h) = 1 - \mathrm{P}\left(\chi^2_r\left(\tau'\tilde{\mathcal{I}}\tau\right) \le c_r\right) =: f(h)$, uniformly on K_a. Conclude that if $h_n \to h \in K_a$,

$$\lim_{n\to\infty} f_n(h_n) = f(h) \ge f_\star := 1 - \mathrm{P}\left(\chi^2_r(a) \le c_r\right). \tag{51}$$

If (23) fails there is a sequence $h_n \in K_a$ with $\limsup_{n\to\infty} f_n(h_n) < f_\star$. Extract a subsequence $h_{n_m} \to h \in K_a$. Let $h^*_m := h_{n_1}$ for $m = 1, \dots, n_1$ and $h^*_m := h_{n_k}$ for $n_k \le m < n_{k+1}$. $f_{n_m}(h_{n_m})$ is a subsequence of $f_m(h^*_m)$ and $h^*_m \to h$, so by (51) $\lim_{m\to\infty} f_{n_m}(h_{n_m}) = \lim_{m\to\infty} f_m(h^*_m) = f(h) \ge f_\star > \limsup_{n\to\infty} f_n(h_n)$. □

Proof of Proposition 4. By (44), $r = 0$ implies $\|[h] - \Pi[h]\|_K = 0$ and so $[h] =$

[52] $[\ker \pi_1]^\perp$ is finite dimensional since it is isomorphic to $\overline{\mathbb{H}} / \ker \pi_1$ which is of dimension r.

$\Pi[h] \in \ker \pi_1$. Hence there is a $h^* \in H_0$ with $\|h - h^*\|_K = 0$. By Corollary S2,

$$\limsup P_{n,\gamma,h}\phi_n \leq \limsup_{n\to\infty} P_{n,\gamma,h^*}\phi_n + \limsup_{n\to\infty} |P_{n,\gamma,h^*}\phi_n - P_{n,\gamma,h}\phi_n| \leq \alpha. \qquad \square$$

Proof of Theorem 4. Since $\lim_{n\to\infty} P_n[\dot{\ell}_n g_n'] = \lim_{n\to\infty} P_n[\dot{\ell}_n \tilde{\ell}_n']$ we may assume $g_n = \tilde{\ell}_n$. By Theorem 12.14 in Rudin (1991), $\tilde{\mathcal{I}}_n \coloneqq P_n[\tilde{\ell}_n\tilde{\ell}_n'] = P_n[\dot{\ell}_n\tilde{\ell}_n']$. Set $K_n(h,g) \coloneqq P_n[\Delta_n h \Delta_n g]$ and let G_n be a zero-mean Gaussian process with covariance kernel K_n. There exists a Hilbert space isomorphism, $Z_n : \mathrm{cl}\{\Delta_n h : h \in H\} \to \mathrm{cl}\{\mathsf{G}_n h : h \in H\}$ (Janson, 1997, Theorem 1.23). Let $R \coloneqq \Pi\left[\cdot\middle|\{\mathsf{G}_n(0,b) : b \in B\}^\perp\right]$ and $Q \coloneqq \Pi\left[\cdot\middle|\{\Delta_n(0,b) : b \in B\}^\perp\right]$. $R\mathsf{G}_n h = RZ_n(\Delta_n h) = Z_n Q Z_n^{-1} Z_n(\Delta_n h) = Z_n Q \Delta_n h$ for $h \in H$ and extends to elements in the closure by continuity. Hence

$$\tilde{\mathcal{I}}_{n,ij} = P_n\left[\Delta_n(e_i,0)Q\Delta_n(e_j,0)\right] = \mathbb{E}\left[\mathsf{G}_n(e_i,0)R\mathsf{G}_n(e_j,0)\right]. \tag{52}$$

By Theorem 9.1 in Janson (1997),

$$\mathbb{E}\left[\mathsf{G}_n(e_j,0)|\{\mathsf{G}_n(0,b) : b \in B\}\right] = \Pi\left[\mathsf{G}_n(e_j,0)|\mathrm{cl}\left\{\mathsf{G}_n(0,b) : b \in B\right\}\right], \tag{53}$$

and so $\tilde{\mathsf{G}}_n(e_j,0) \coloneqq \mathsf{G}_n(e_j,0) - \mathbb{E}\left[\mathsf{G}_n(e_j,0)|\{\mathsf{G}_n(0,b) : b \in B\}\right] = R\mathsf{G}_n(e_j,0)$. Then, $\tilde{\mathcal{I}}_{n,ij} = \mathbb{E}\left[\mathsf{G}_n(e_i,0)\tilde{\mathsf{G}}_n(e_j,0)\right]$ by (52). Set $\mathscr{G}_n \coloneqq \sigma(\{\mathsf{G}_n(0,b) : b \in B\})$, $\mathscr{G}_n \coloneqq \sigma(\{\mathsf{G}(0,b) : b \in B\})$ and define $X_n \coloneqq (\mathsf{G}_n(e_i,0), \mathbb{E}[\mathsf{G}_n(e_j,0)|\mathscr{G}_n])$ and $X \coloneqq (\mathsf{G}(e_i,0), \mathbb{E}[\mathsf{G}(e_j,0)|\mathscr{G}])$, where $\mathsf{G} \coloneqq \Delta$. By (53) and $K_n(h,h) \to K(h,h)$ (Lemma 2), $(X_n)_{n\in\mathbb{N}}$ are uniformly square integrable Gaussian random vectors and $X_n \rightsquigarrow X$ (by Theorem S3). Combine with Lemma S3 and Theorem 9.1 of Janson (1997). $\square$

Proof of Lemma 4. $R_n(h) \xrightarrow{P_n} 0$ in (5) and $Ah \in L_2^0(P)$ follows from (24) (van der Vaart and Wellner, 1996, Lemma 3.10.11). Hence $\Delta_n h$ is uniformly square integrable (i.i.d) and $[\Delta_n h](W^{(n)}) = \mathbb{G}_n Ah \rightsquigarrow \mathcal{N}(0, \int (Ah)^2 \,\mathrm{d}P)$ (CLT). $\square$

Proof of Lemma 5. $P^n\left(\Delta_n h, g_n'\right) = 0$. By $g \in \{Db : b \in B\}^\perp$ and Assumption 5, the covariance matrix of $(\Delta_n h, g_n')$ (under P^n) is $\Sigma(h) = P\left[\begin{smallmatrix} [Ah]^2 & \tau'\dot{\ell}g' \\ g\dot{\ell}'\tau & gg' \end{smallmatrix}\right]$. For each $h \in H$, the central limit theorem gives $(\Delta_n h, g_n') \overset{P^n}{\rightsquigarrow} \mathcal{N}(0, \Sigma(h))$. $\square$

Proof of Corollary 4. $g_\gamma \in \{D_\gamma b : b \in B_\eta\}^\perp$. Apply Lemma 5. $\square$