Lecture 2: Transformations

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Contents

1 Conventions

- Matrices written in uppercase bold eg: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- Matrices are also indexed (row, column) from top left starting at 1.
- Individual elements are written as lowercase:
$$a_{11} = 1, a_{12} = 2, a_{21} = 3, a_{22} = 4$$
- Vectors are simply matrices with a single row or column represented by a lower-case bold symbol:
$$\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; OR \;\; \mathbf{a} = \begin{bmatrix} x & y & z \end{bmatrix}$$

2 Transformations

2.1 Translation

Translation involves ***component-wise addition***.

For example:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

2.2 Scaling

Scaling involves ***multiplication by a scalar***.

For example:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} x \\ y \end{bmatrix}$$

Or to have a separate dimensional scalars:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

2.3 Rotation

Rotation involves ***matrix multiplication***. Using the formula below to give a 2D rotation counter clockwise by angle θ:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation preserves distances:

$$||\mathbf{x}|| = ||\mathbf{Rx}||$$
$$||\mathbf{x}||^2 = x_1^2 + x_2^2 + ... + x_n^2 = \mathbf{x}^T\mathbf{x}$$
$$\mathbf{x}^T\mathbf{x} = (\mathbf{Rx})^T(\mathbf{Rx})$$

$$\begin{aligned}\mathbf{x}^T\mathbf{I}\mathbf{x} &= (\mathbf{x})^T(\mathbf{R}^T)(\mathbf{R}\mathbf{x})\\ &= \mathbf{x}^T(\mathbf{R}^T\mathbf{R})\mathbf{x}\end{aligned}$$

And hence:

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}$$
$$\mathbf{R}^T\mathbf{R}\mathbf{R}^{-1} = \mathbf{I}\mathbf{R}^{-1}$$
$$\mathbf{R}^T = \mathbf{R}^{-1}$$

So to get the inverse of a rotation you only need to transpose it.

2.4 Affine Transformations

All of the previously mentioned transformations have been **affine** transformations. These are transformations which can be expressed as matric multiplication and addition:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

For 2D, $\mathbf{A}$ is a 2x2 matrix and $\mathbf{b}$ a 2x1 vector.
For 3D, $\mathbf{A}$ is a 3x3 matrix and $\mathbf{b}$ a 3x1 vector.

3 Homogeneous Coordinates

3.1 Defining Homogenous Coordinates

Representing the affine transformations can be done by moving from Cartesian to homogeneous coordinates.

All 2D affine transformations can be then represented by a 3x3 matrix (or 4x4 for 3D)

$$\begin{bmatrix} x \\ y \end{bmatrix} \textit{becomes} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3.2 Affine Transformations as Homogeneous Coordinates

The extra dimension allows uniform treatment of transformations:

Translation: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Scaling: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & 0 & s_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Rotation: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

In general, any affine transformation:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

Can be expressed as:

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & \mathbf{A} & & \mathbf{b} \\ 0 & \ldots & 0 & 1 \end{array} \right] \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

Note: Two homogeneous points are equal if they differ only by a scale factor.