Logic practical
Answer: Solutions

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1 Logic exercises

1. Consider a vocabulary (sometimes called a *propositional language*) with 4 propositional symbols: A, B, C and D.

   (a) How many models are there for this language? **Answer:** $2^4 = 16$

   (b) How many models for the formula $A \vee \neg C$? **Answer: Consider how many models there are for the negation: $\neg(A \vee \neg C)$. This is equivalent to $\neg A \wedge C$ which has 4 models. So $A \vee \neg C$ has $16 - 4 = 12$ models.**

   (c) How many models for the formula $A \wedge \neg C$? **Answer:** $2^2 = 4$. **We need A true and C false. It doesn't matter about B and D so consider all choices for B and D.**

   (d) How many models for the formula $A \vee \neg A$? **Answer: 16, it is true in all models.**

   (e) How many models for the formula $A \wedge \neg A$? **Answer: 0, this is true in no model.**

   (f) How many *literals* are there for this language? **Answer: There are 8 possible literals: $A, B, C, D, \neg A, \neg B, \neg C$ and $\neg D$.**

2. Which of the following are correct?

   (a) $\models A \vee \neg A$ (This statement is equivalent to True $\models A \vee \neg A$, i.e. it asserts that $A \vee \neg A$ is valid). **Answer: Correct. In any model A must have a truth-value: either true or false.**

   (b) $\models A$ **Answer: Wrong. Consider a model where A is false.**

   (c) $A \models B$ **Answer: Wrong. Consider a model where A is true, but B is false.**

   (d) $A \wedge B \models B$ **Answer: Correct. In any model where both A and B are true, B is obviously true.**

(e) $A \wedge \neg A \models B$ **Answer: Correct. There are no models where $A \wedge \neg A$, so we can say 'in any model where $A \wedge \neg A$ is true so is B (because there aren't any!)'. Everything follows from a contradition. Weird eh?**

(f) $A \Leftrightarrow B \models A \vee B$ **Answer: Wrong. The LHS is true in any model where $A = B =$ false, but the RHS would be false in any such model.**

(g) $A \Leftrightarrow B \models \neg A \vee B$ **Answer: Correct. The LHS is true if and only if $A = B =$ false or $A = B =$ true. Either way $\neg A \vee B$ will be true.**

3. (*) (The Deduction theorem). Prove that for any two propositional formulae α and β:

   - $\alpha \models \beta$ if and only if $\models \alpha \Rightarrow \beta$

   Answer: 'Only if' direction: Suppose $\alpha \models \beta$. Now consider all possible models. If $\alpha = false$ in a model M then M satisfies $\alpha \Rightarrow \beta$. If $\alpha = true$ in a model M' then β is also true in M' since we are assuming $\alpha \models \beta$. So $\alpha \Rightarrow \beta$ is true in M'. Since either $\alpha = false$ or $\alpha = true$ in all models we are done. 'If' direction: Suppose $\models \alpha \Rightarrow \beta$. Then in any model either $a =$ false or $b =$ true. So whenever $a =$ true, we must have $b =$ true. This is enough to prove that $\alpha \models \beta$.

4. Convert each of the following formulae into CNF:

   (a) $A \wedge B$ **Answer: Already in CNF! Two unit clauses.**

   (b) $A \Leftrightarrow B$ **Answer:** $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A) \equiv (\neg A \vee B) \wedge (\neg B \vee A)$

   (c) $A \wedge \neg B \Rightarrow C$ **Answer:** $\neg A \vee B \vee C$

5. For each of the following inference procedures for propositional logic, decide (i) whether they are sound and (ii) whether they are complete.

   (a) For any KB and any α: $KB \vdash \alpha$ (i.e. any formula can be derived from any knowledge base.) **Answer: Unsound, but complete**

   (b) For any KB and any α: $KB \wedge (\beta \wedge \neg\beta) \vdash \alpha$ **Answer: Sound, but incomplete. Inference says that if you are prepared to add a contradiction to your KB (i.e. 'believe the impossible') you can infer anything. This is true, but doesn't help you infer formula from consistent KBs.**

   (c) For any KB and any α construct all possible models of KB. Derive α from KB if and only if α is true in at least one of these models. **Answer: Unsound. An empty KB is satisfied by any model (since it rules nothing out) but α will be false in exactly**

half of them. Incomplete. Suppose $KB = \beta \wedge \neg\beta$. This has no models and so, using this inference rule nothing can be inferred. In fact everything follows from this KB.

(d) For any KB and any α construct all possible models of KB. Derive α from KB if and only if α is true in all of these models. **Answer: Sound and complete. This just basically follows from the definitions of soundness and completeness.**

2 Graph colouring

Here's the obvious approach which is easily extended to many colours. Let the two 'colours' be called 1 and 2. Create 6 propositional symbols: A_1, A_2, B_1, B_2, C_1, C_2, where e.g. A_1 says that vertex A has colour 1. Each vertex has exactly one colour so we have these clauses:

1. $A_1 \vee A_2$
2. $\neg A_1 \vee \neg A_2$
3. $B_1 \vee B_2$
4. $\neg B_1 \vee \neg B_2$
5. $C_1 \vee C_2$
6. $\neg C_1 \vee \neg C_2$

A and B cannot be the same colour so, we have:

1. $\neg A_1 \vee \neg B_1$
2. $\neg A_2 \vee \neg B_2$

Same deal for B and C

1. $\neg B_1 \vee \neg C_1$
2. $\neg B_2 \vee \neg C_2$

However, *since there are only 2 colours* we can get away with just 3 symbols: A_1, B_1 and C_1 and think of $\neg A_1$ as standing for A_2. In this encoding we just need the following clauses:

1. $\neg A_1 \vee \neg B_1$
2. $A_1 \vee B_1$
3. $\neg B_1 \vee \neg C_1$
4. $B_1 \vee C_1$

Sticking with the compact encoding, if we add the $A - C$ edge we need to add these clauses:

1. $\neg A_1 \vee \neg C_1$
2. $A_1 \vee C_1$