University of New South Wales

School of Mathematics and Statistics

Assignment 1

Biomathematics

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Question 1

a

F is a linear combination of N and $\frac{dN}{dt}$. This means that the larger the population is the more demand for the food there is. It also means that the faster the populations is growing, the more demand for food there is which would make sense because cell division could require *more resources*.

b

$$\begin{aligned}
\frac{dN}{dt} &= rN\frac{T - \left(c_1 N + c_2 \frac{dN}{dt}\right)}{T} \\
\frac{dN}{dt}\left(1 + \frac{rc_2 N}{T}\right) &= rN\left(1 - \frac{c_1 N}{T}\right) \\
\frac{dN}{dt} &= rN\left(\frac{\frac{T}{c_1} - N}{\frac{T}{c_1} + \frac{rc_2 N}{c_1}}\right) \\
&= rN\left(\frac{K - N}{K + aN}\right)
\end{aligned}$$

where $K = \frac{T}{c_1}$ and $a = \frac{rc_2}{c_1}$.

c

Let $f(N) = rN\left(\frac{K-N}{K+aN}\right)$ thus $f(N) = 0 \Longrightarrow N^* = 0, N^* = K$.

$$f'(N) = r\left(\frac{K - N}{K + aN}\right) + rN\left(\frac{-(K + aN) - (K - N)a}{(K + aN)^2}\right)$$

and

$$\begin{aligned}
f'(0) &= r > 0 \\
f'(K) &= \frac{-r}{a + 1} < 0
\end{aligned}$$

thus the steady states are $N = 0$ and $N = k$ and they are unstable and stable respectively.

Question 2

a

$$\begin{aligned}
\frac{dx}{dt} = 0 &\Longrightarrow 0.2x\left(1 - \frac{x}{3}\right) = 0 \\
&\Longrightarrow x^* = 0, 3
\end{aligned}$$

So the steady states of the system are $x^* = 0$ and $x^* = 3$.

b

$$
\begin{aligned}
\frac{dx}{dt} &= 0.2x\left(1-\frac{x}{3}\right)\\
\int \frac{dx}{x(3-x)} &= \frac{1}{15}\int dt\\
\int \frac{dx}{3x} + \int \frac{dx}{3(3-x)} &= \frac{1}{15}\int dt\\
\frac{1}{3}\ln(x) - \frac{1}{3}\ln(3-x) &= \frac{t}{15} + C\\
\ln\left(\frac{x}{3-x}\right) &= \frac{t}{5} + B\\
\frac{x}{3-x} &= \exp(\frac{t}{5} + B)\\
x &= \frac{3\exp(\frac{t}{5}+B)}{1+\exp(\frac{t}{5}+B)}
\end{aligned}
$$

As $x(0) = x_0$

$$B = \ln\left(\frac{x_0}{3-x_0}\right)$$

and so

$$x = \frac{3x_0\exp(\frac{t}{5})}{3 + x_0(\exp(\frac{t}{5}) - 1)}.$$

c

The steady states of the population are $x^* = 0$ and $x^* = 3 - 15E$. Note that the second steady state is a function of E and for a sustainable yield we would require $E < \frac{1}{5}$ so that the steady state is positive.

d

As per the last section the maximum sustainable yeild is $E = \frac{1}{5}$.

Question 3

a

Let $H : [0,\infty) \longrightarrow \mathbb{R}$ be defined as

$$H(t) = \begin{cases} h & t \mod 12 \leq T \\ 0 & \text{otherwise} \end{cases}$$

where h is the harvesting quantity. Then assuming that there are no other population limiting factors, the governing equation would be

$$\frac{dF}{dt} = rF - H(t)$$

where $F(t)$ is the population of fish at time t (months) and r is the population growth rate in fish per month.

Question 4

a

The coefficient $2a$ steers the system in the sense that for $a = 0$ the system models a commensalistic symbiotic relationship. That is one party, x, benefits without harm to the other party, y. As a increases this scales into a mutualistic relationship where both parties benefit. For values of $a > 1$ we would actually have y benefiting *more* from the relationship, than y does.

b

The nontrivial steady state is at $x = 6/(1-4a)$, $y = -1 - 3/(4a-1)$. Now if we want the non-trivial steady state to be in the population quadrant we need $x > 0$ and $y > 0$. For $x > 0$ we need $a < 1/4$. For $y > 0$ we need $a > -1/2$. To prevent explosive growth we would need $a < 1/4$.

Question 5

a

Clearly there is a steady state at $x = 0 = y$. Also there is a steady state at $y = 0, x = K$.
There is also a steady state at $x = c/d$ and $y = a/b - ac/bdK$. This is the non-trivial steady state. Note that

$$\begin{aligned} J &= \begin{bmatrix} a - 2ax/K - by & -bx \\ dy & -c + dx \end{bmatrix} \\ J\big|_{x=c/d, y=a(1/b-c/bdK} &= \begin{bmatrix} -ac/dK & -bc/d \\ ad/b - ac/bK & 0 \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{Tr}\left(J\big|_{x=c/d, y=a(1/b-c/bdK}\right) &= \frac{-ac}{dK} \\ \det\left(J\big|_{x=c/d, y=a(1/b-c/bdK}\right) &= ac\left(1 - c/dK\right). \end{aligned}$$

Assuming that all constants are positive this means that the non-trivial steady state is stable if $c/dK < 1$.

Question 6

a

$r_M S_M I_F$ is the term which describes the rate at which males leave the susceptible state because they have become infected due to a male - female interaction.
$a_M I_M$ is the tern which describes how quickly males recover from the infected state and become susceptible again. Note that in this model there is no period of confered immunity and infected individuals simply become susceptible again.

b

Clearly adding equations 1 and 3 and 2 and 4 yeilds 0 in each case.

As we can write $S_F = N_F - I_F$ and $S_M = N_M - I_M$ we can substitute these into equations 3 and 4 to get

$$\frac{dI_M}{dt} = r_M I_F (N_M - I_M) - a_M I_M$$
$$\frac{dI_F}{dt} = r_F I_M (N_F - I_F) - a_F I_F.$$

Note that this completely determines the system because $\frac{dS_M}{dt} = -\frac{dI_M}{dt}$ and $\frac{dS_F}{dt} = -\frac{dI_F}{dt}$.

c

The threshold population must exist at an equilibrium. Solving for the equilibriums we get that

$$N_F = I_F - \frac{a_F I_F}{r_F I_M}$$
$$N_M = I_M - \frac{a_M I_M}{r_M I_F}$$

and taking the product of these we get that

$$\begin{aligned} N_M N_F &= \left(\frac{I_F + a_F I_F}{r_F I_M}\right)\left(I_M + \frac{a_M I_M}{r_M I_F}\right) \\ &= I_F I_M + \frac{a_M a_F}{r_M r_F} + \frac{a_M I_M}{r_M} + \frac{a_f I_F}{r_F} \\ &\geq \frac{a_M a_F}{r_M r_F} \end{aligned}$$

because all the other terms must be positive if we are at (or above) the threshold population.

Question 8

a

$$\begin{aligned} \underbrace{\frac{dx}{dt}}_{animals \cdot km^{-2} \cdot day^{-1}} &= \underbrace{a}_{day^{-1}} \underbrace{x}_{animals \cdot km^{-2}} - \underbrace{b}_{animals^{-1} \cdot km^{2} \cdot day^{-1}} \underbrace{xy}_{animals^{2} \cdot km^{-4}} \\ \underbrace{\frac{dy}{dt}}_{animals \cdot km^{-2} \cdot day^{-1}} &= -\underbrace{c}_{day^{-1}} \underbrace{y}_{animals \cdot km^{-2}} + \underbrace{d}_{animals^{-1} \cdot km^{2} \cdot day^{-1}} \underbrace{xy}_{animals^{2} \cdot km^{-4}} \end{aligned}$$

b

$$\begin{aligned} \frac{dx(t)}{c} &\sim \frac{animals^{-1} \cdot km^2 \cdot days^{-1} animals \cdot km^{-2}}{days^{-1}} && \sim unitless \\ \frac{by(t)}{a} &\sim \frac{animals^{-1} \cdot km^2 \cdot days^{-1} animals \cdot km^{-2}}{days^{-1}} && \sim unitless \\ at &\sim days^{-1} \cdot days && \sim unitless \\ \frac{c}{a} &\sim \frac{days^{-1}}{days^{-1}} && \sim unitless \end{aligned}$$

c

Chain rule:

$$\begin{aligned}\frac{du}{d\tau} &= \frac{du}{dt}\frac{dt}{d\tau} = \frac{d}{ca}\frac{dx}{dt} \\ \frac{dv}{d\tau} &= \frac{dv}{dt}\frac{dt}{d\tau} = \frac{b}{a^2}\frac{dy}{dt}\end{aligned}$$

Subbing in:

$$\begin{aligned}\frac{du}{d\tau} &= \frac{d}{ca}\left(ax + bxy\right) \\ &= \frac{d}{ca}\left(\frac{ac}{d}u + \frac{bca}{db}v\right)u \\ &= u(1-v) \\ \frac{dv}{d\tau} &= \frac{d}{a^2}\left(-cy + dxy\right) \\ &= \frac{d}{a^2}\left(\frac{-ca}{b}v + d\frac{bca}{db}vu\right) \\ &= \frac{c}{a}v\left(u-1\right) \\ &= \alpha v(u-1)\end{aligned}$$

Equilibrium points:

$$\begin{aligned}\frac{du}{d\tau} &= u(1-v) = 0 \implies u = 0 \text{ or } v = 0 \\ \frac{dv}{d\tau}&\alpha v(u-1) = 0 \implies v = 0 \text{ or } u = 1\end{aligned}$$

For equilibriums $u = 0 \implies v = 0$ and $v = 1 \implies u = 0$. Thus equilibriums are at $(0, 0)$ and $(1, 1)$.

d

$$\begin{aligned}\frac{dv}{du} &= \frac{dv}{dt}\frac{dt}{du} \\ &= \frac{\alpha v(u-1)}{u(1-v)}\end{aligned}$$

Integrating:

$$\begin{aligned}\int \frac{(1-v)}{v}dv &= \alpha \int \frac{(1-u)}{u}du \\ \ln(v) - v &= \alpha\ln(u) - \alpha u + C \\ \ln(v) - v &= -\ln(u^\alpha) + \alpha u + C \\ \ln(u^\alpha v) - v - \alpha u = C\end{aligned}$$

f

Differentiating:

$$\begin{aligned}\frac{dC}{du} &= \frac{1}{u^\alpha v}(\alpha u^{\alpha-1}v) - \alpha \\ \frac{dC}{dv} &= \frac{1}{u^\alpha v} - 1\end{aligned}$$

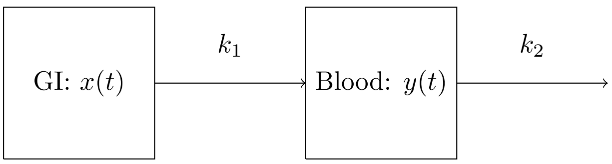
Stationary points:

$$\begin{aligned}\frac{1}{u^\alpha v}(\alpha u^{\alpha-1}v) - \alpha = 0 &\implies u = 1 \\ \frac{1}{u^\alpha v} - 1 = 0 \implies u^\alpha v = 1 &\implies v = 1\end{aligned}$$

Only one stationary point so this *must* be the maximum.
Maximum at $(1, 1)$ with value $1 + \alpha$ (by simple subsitituion).

Question 9

a



The system of differential equations is given by

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

along with the inital conditions

$$\begin{aligned}x(0) &= A \\ y(0) &= 0\end{aligned}$$

b

Let

$$B := \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix}.$$

It is clear to see that this has eigenvalues $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$. Simple computations show that the corresponding eigenvectors are

$$\begin{aligned}\mathbf{v}_1 &= \begin{pmatrix} 1/k_1 \\ 1/(k_2 - k_1) \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

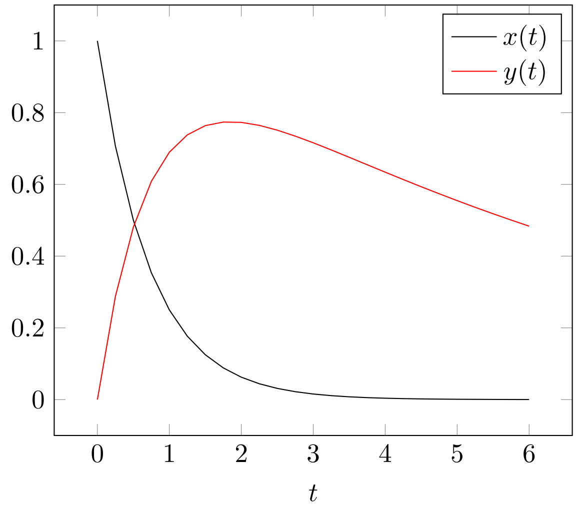
and so the general solution to the system of differential equations is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha \begin{pmatrix} 1/k_1 \\ 1/(k_2 - k_1) \end{pmatrix} e^{-k_1 t} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-k_2 t}.$$

By applying the initial conditions we get that system of equations

$$\alpha \begin{pmatrix} 1/k_1 \\ 1/(k_2 - k_1) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix} \tag{1}$$

which gives us that $\alpha = k_1 A$ and $\beta = -k_1 A/(k_2 - k_1)$. So the solutions look like



Decongestant levels quickly drop in the GI tract but they build up quickly in the blood and are slow to clear from the blood.