Definitions and Theorems

Definition 1. *$D \subset \mathbb{C}$ is a domain it it is open and connected.*

Definition 2. *$f : D \longrightarrow \mathbb{C}$ is real differentiable at z_0 if there exists a good affine approximation. That is*

$$f(z) = f(z_0) + A(x - x_0) + B(y - y_0) + o(z - z_0).$$

We use the following notation:

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{df}{dx} - i\frac{df}{dy}\right)$$
$$\frac{df}{d\bar{z}} = \frac{1}{2}\left(\frac{df}{dx} + i\frac{df}{dy}\right).$$

Definition 3. *$f : D \longrightarrow \mathbb{C}$ is complex differentiable if $\frac{df}{d\bar{z}} = 0$.*

Lemma 1. *If $f = u + -iv$, where u and v are real functions, then complex differentiability is written as*

$$\frac{du}{dx} = \frac{dv}{dy} \tag{1}$$
$$\frac{du}{dy} = -\frac{dv}{dx} \tag{2}$$

Definition 4. *Let $f : D \longrightarrow \mathbb{C}$ be a bijection. We say f is conformal if*

$$\langle f(z_0 + hz_1) - f(z_0), f(z_0 + hz_2) - f(z_0)\rangle = c(z_0)h^2\langle z_0, z_1\rangle + o(h^2)$$

Theorem 1. *Let $f : D \longrightarrow \mathbb{C}$ be real differentiable function. The following conditions are equivalent.*

- *f is complex differentiable and bijective*

- *f is conformal*

Definition 5. *A differentiable mapping $\gamma : [0, 1] \longrightarrow \mathbb{C}$ is called a contour.*

Definition 6. *Let $f : D \longrightarrow \mathbb{C}$ be continuous and let $\gamma : [0, 1] \longrightarrow D$ be a contour. If $0 = t_0 < t_1 < \cdots < t_n = 1$ then the Riemann sum is defined by*

$$\sum_{k=0}^{n-1} f(\gamma(t_k))(\gamma(t_{k-1}) - \gamma(t_k)).$$

If this Riemann sum converges for all partitions as $\sup_{0\le k<n}(t_{k+1} - t_k) \longrightarrow 0$, then the result is called the integral over the contour γ.

Theorem 2. *Since f is continuous, the integral $\int_\gamma f(z)dz$* always *exists.*

Lemma 2. *If*

$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & 0 \le t < \frac{1}{2} \\ \gamma_2(2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

then

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_3} f(z)dz.$$

Lemma 3. *If $\gamma_2(t) = \gamma_1(1-t)$ then*

$$\int_{\gamma_1} f(z)dz = -\int_{\gamma_2} f(z)dz.$$

Theorem 3. *If D is a simply connected domain and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then $\int_\gamma f(z)dz = 0$ for every closed contour $\gamma : [0,1] \longrightarrow D$.*

Theorem 4. *If D is a simply connected domain and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz = f(w)$$

for every closed contour $\gamma : [0,1] \longrightarrow D$ and for every w in the interior *of γ.*

Theorem 5. *If D is a simply connected domain and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then*

$$f(w) = \sum_{k\geq 0} c_k(w-w_0)^k$$

where the series converges in some neighbourhood of w_0.

Corollary 1. *If D is a simply connected domain and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then*

$$\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{n+1}}dz = f^{(n+1)}(w)$$

for every closed contour $\gamma : [0,1] \longrightarrow D$ and for every w in the interior *of γ.*

Theorem 6. *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is bounded an complex differentiable, then $f = const$.*

Theorem 7. *Let $f_1, f_2 : D \longrightarrow \mathbb{C}$ be complex differentiable functions. If $A \subset D$ admits a limit point $a \in D$ and if $f_1 = f_2$ on A, then $f_1 = f_2$ on D.*

Theorem 8. *Let $f_n : D \longrightarrow \mathbb{C}$ be complex differentiable. Set*

$$f(z) = \sum_{n\geq 0} f_n(z)$$

where the series converges uniformly on compact subsets of D. We have that f is complex differentiable in D and

$$f^{(k)}(z) = \sum_{n\geq 0} f_n^{(k)}(z).$$

Theorem 9. *Let D be a simply connected domain and let $K \subset D$ be a compact set. Let $f : D \longrightarrow \mathbb{C}$ be complex differentiable. For a given $\varepsilon > 0$, there exists a polynomial P such that*

$$\sum_{z\in K} |f(z) - P(z)| \leq \varepsilon$$

Theorem 10. *Let D be a comply connected domain and let $f : D \longrightarrow \mathbb{C}$ be complex differentiable. We have*

$$f(z) = \sum_{n\geq 0} P_n(z)$$

where each P_n is a polynomial and the series converges uniformly on compact subsets in D.

Theorem 11. *If $D = \{f < |z| < R\}$ and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where the series converges in D. The coefficients c_n, $n \in \mathbb{Z}$ are given by

$$c_n = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w)dw}{w^{n+1}}$$

when $r < \rho < R$.

Corollary 2. *If $D = \{r < |z| < R\}$ and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then $f = f_1 + f_2$ where f_1 is complex differentiable on $\{|z| < R\}, f_2$ is complex differentiable on $\{|z| > r\}$. This representation is* not *unique.*

Theorem 12. *Let $D = \{f < |z| < R\}$ and let*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where the series converges in D. We have

$$c_n = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{f(w)dw}{w^{n+1}}$$

when $r < \rho < R$.

Theorem 13. *If $D = \{r < |z| < R\}$ and if $f : D \longrightarrow \mathbb{C}$ is complex differentiable, then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where

$$|c_n| \leq \frac{1}{\rho^n} \sum_{|z|=\rho} |f(z)|.$$

Theorem 14. *An isolated singularity is removable if and only if $c_n = 0$ for all $n < 0$. An isolated singularity is a pole if and only if $c_n = 0$ for all $n < -N$ for some $N \in \mathbb{N}$.*

Theorem 15. *If a is an essential singularity of a complex differentiable function f, then for every $b \in \mathbb{C}$, there exists a sequence $z_n \longrightarrow a$ such that $f(z_n) \longrightarrow b$.*

Theorem 16. *If a meromorphic function f does not have an essential singularity at ∞, then f is rational.*

Definition 7. *Let a complex differentiable function f have an isolated singularity at a. Its residue at a is defined by the formula*

$$\mathrm{Res}_{z=\infty} f(z) = 1\frac{1}{2\pi i} \int_{|z|=r} f(z)dz$$

where r is sufficiently large.

Theorem 17. *Let f be a complex differentiable function on a simply connected domain D except at the points a_k, $1 \leq k \leq N$. For every contour γ in D, we have*

$$\int_\gamma f(z)dz = 2\pi i \sum a_k \textit{ is inside } \gamma \, \mathrm{Res}_{z=a_k} f(z).$$

In particular, for a meromorphic function f with an isolated singularity at ∞ we have

$$\mathrm{Res}_{z=\infty} f(z) = \sum_{k=1}^{N} \mathrm{Res}_{z=a_k} f(z).$$

Theorem 18. *For every sequence $a_n \longrightarrow \infty$, there exists a meromorphic function f with given principal parts f_{a_n}. In particular, every meromorphic function is written as follows*

$$f(z) = h(z) + \sum_{n=1}^{\infty} (f_{a_n} - P_n)$$

where P_n are polynomials and h is an entire function.

Theorem 19. *If P is a polynomial, then*

$$P(z) = K \cdot \prod_{k=0}^{n-1} (z - a_k)$$

where K is a constant and a_k are the roots of P.

Theorem 20. *If $a_n \longrightarrow \infty$, there there exists an entire function f which has zeros exactly at the points a_n.*

Corollary 3. *Every entire function admits a decomposition*

$$f(z) = z^m e^{g(z)} \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{k=1}^{n} \frac{z^k}{k a_n^k}\right)$$

where g is also entire.

Definition 8. *An entire function is said to be of type p if $|f(z)| \leq \exp(K|z|^p)$ for some constant K and $|z| > R$.*

Theorem 21. *Let f be an entire function of order $p \in [m, m+1)$. We have*

$$f(z) = z^m \exp(g(z)) \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{k=1}^{m} \frac{z^k}{k a_n^k}\right)$$

Definition 9. *Let $f : D_1 \longrightarrow \mathbb{C}$ be complex differentiable. If $D_1 \cap D_2 \neq \emptyset$, then the complex differentiable function $g : D_2 \longrightarrow \mathbb{C}$ is called an analytic continuation of f if $g|_{D_1 \cap D_2} = f|_{D_1 \cap D_2}$.*

Definition 10. *Let $\emptyset \neq \Delta \subset D_1 \cap D_2$. We say that $f_2 : D_2 \longrightarrow \mathbb{C}$ is an immediate analytic continuation of $f_1 : D_1 \longrightarrow \mathbb{C}$ through Δ if $f_1|_\Delta = f_2|_\Delta$.*

Definition 11. *We say that $f_2 : D_2 \longrightarrow \mathbb{C}$ is an analytic continuation of $f_1 : D_1 \longrightarrow \mathbb{C}$ if*

1. *there exists a domain $D_1 = G_0, G_1, \ldots, G_n = D_2$ such that $\emptyset \neq \Delta_k \subset G_k \cap G_{k+1}$, $0 < k < n$.*

2. *there exists functions $g_k : G_k \longrightarrow \mathbb{C}$, $0 \leq k \leq n$, such that $g_0 = f_1$ and $g_n = f_2$.*

3. *for every $0 \leq k < n$, $g_{k+1} : G_{k+1} \longrightarrow \mathbb{C}$ is an immediate analytic continuation of $g_k : G_k \longrightarrow \mathbb{C}$ through Δ_k.*

Definition 12. *Suppose $\gamma : [0,1] \longrightarrow$ Cplx is a contour. We say that $f_0 : \{|z - \gamma(0)| < r\}$ extends along γ if there exists a mapping $R : [0,1] \longrightarrow (0, \infty)$, $R(0) = r$ and complex differentiable functions $f_t : \{|z - \gamma(t)| < R(t)\}$, $t \in [0,1]$ such that*

1. *For every $t \in [0,1]$ let $(a(t), b(t))$ be the maximal interval containing t such that $|\gamma(t) - \gamma(s)| < R(t)$ for every $s \in (a(t), b(t))$.*

2. *For every s $in(a(t), b(t))$, the function $f_s : \{|z - \gamma(s)| < R(s)\}$ is an immediate analytic continuation of f_t :]$\{|z - \gamma(t)| < R(t)\}$.*

Theorem 22. *If $f_0 : \{|z - \gamma(0)| < r\}$ extends along γ, then f_1 does not depend on the particular choice of f_t.*

Theorem 23. *Analytic continuation along the path is an analytic continuation.*

Theorem 24. *Let γ_s with $s \in [0,1]$ be the continuous deformation of γ_0 into γ_1 (with common endpoints). If f_0 extends along every γ_s then the analytic continuation of f_0 along γ_1 is identical to that along γ_0.*

Theorem 25. *The set $\mathcal{B}$ of all complex differentiable function $f : D \longrightarrow \mathbb{C}$ such that $|f| \leq 1$ in D is compact.*

Theorem 26. *Let D be a comply connected domain and let $h : D \longrightarrow \mathbb{C}$ be complex differentiable. If $h(z) \neq 0$ for every $z \in D$, then there exists a complex differentiable function $g : D \longrightarrow \mathbb{D}$ such that $g^2 = h$.*

Lemma 4. *If $f_n : D \longrightarrow \mathbb{C}$ is a sequence of injective complex differentiable functions. If $f_n \longrightarrow F$ uniformly con compact subsets, then either f is injective or $f = const$.*

Theorem 27. *Every simply connected domain $D \subset \mathbb{C}$ is conformally equivalent to a unit ball.*

Theorem 28. *If $f : D \longrightarrow \mathbb{C}$ is complex differentiable and if $f : \hat{D} \longrightarrow \mathbb{C}$ is continuous then $|f|$ attains its maximum on the boundary.*

Lemma 5. *If $f : \{|z| < 1\} \longrightarrow \{|z| < 1\}$ and if $f(0) = 0$ then $|f(z)| \leq |z|$.*

Theorem 29. *The group of conformal automorphisms of a unit ball consists of the fractional linear transforms.*

Theorem 30. *The group of conformal automorphisms of the complex plane consists of affine transformations.*

Corollary 4. *The group of conformal automorphisms of the extended complex plane consists of fractional linear transforms.*

Theorem 31. *The group of conformal automorphisms of an annulus consists of fractional linear transforms.*

Theorem 32. *If D is a multiply connected domain (and is not conformally equivalent to an annulus), then the group of its conformal automorphisms is finite.*

Theorem 33. *If f has partial complex derivatives in the domain D, then it is complex differentiable.*

Lemma 6. *If f is bounded an has complex derivatives in the domain $\{|z| < r, |w| < R\}$ then f is continuous.*

Lemma 7. *If f is continuous an has complex derivatives, then*

$$F(z_0, w_0) = \frac{1}{(2\pi i)^2} \int_{|z-z_0|=\epsilon_1, |w-w_0|=\epsilon_2} \frac{f(z,w)dwdz}{(w-w_0)(z-z_0)}.$$