University of New South Wales

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Assignment 1

Complex Analysis

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We recall Runge's theorem.

Theorem 1. *Let K be a compact subset of $\mathbb{C}$ and let f be a function which is holomorphic on an open subset containing K. If A is a set containing at least one complex number from every bounded component of $\mathbb{C} \setminus K$ then there exists a sequence $\{r_n\}_{n\in\mathbb{Z}}$ of rational functions which converges uniformly to f on K and such that all the poles of the functions $\{r_n\}_{n\in\mathbb{Z}}$ are in A.*

An important corollary of Runge's theorem is the following

Corollary 1. *If K is a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is a connected set and f is a holomorphic function on K, then there exists a sequence of polynomials which approaches f uniformly on K.*

This corollary is important as it will allow us to construct a sequence of polynomials whose limit is discontinuous.
Consider the sequence of sets K_n and L_n as described in figure $1 K_n = \{z \in \mathbb{C} : |z| \leq 1, \mathrm{dist}(z, \mathbb{R}_+) \geq 1/n\} \cup [2/n, 1] \cup \{0\}$ and $L_n = K_n \cup \{1/n\}$figure.1. Importantly note that $\mathbb{C} \setminus L_n$ is connected. This means we can apply the corollary to Runge's theorem.

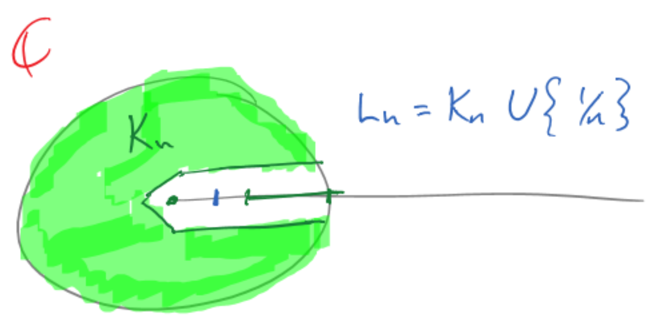

Figure 1: $K_n = \{z \in \mathbb{C} : |z| \leq 1, \mathrm{dist}(z, \mathbb{R}_+) \geq 1/n\} \cup [2/n, 1] \cup \{0\}$ and $L_n = K_n \cup \{1/n\}$

Let f_n be a holomorphic function on a (non-connected) neighbourhood of L_n such that $f_n(z) = 0$ for any $z \in K_n$ but $f_n(1/n) = 1$. By Runge's theorem, there must exist a polynomial p_n such that $|p_n - f_n| < 1/n$ on L_n. Now we can see that $p_n \longrightarrow 0$ pointwise on the unit disk. The convergence is not uniform however, as on some neighbourhood of 0 we have $|p_n(1/n) - 1| \leq 1/n$.
Although we have constructed a counter example there is a remarkable theorem that almost guarantees the the limit function will be analytic.
Osgood, Montel and Lavrentiev, [1] and Hartogs and Rosenthal [2] go into this subject quite deeply.

Osgood showed that if a sequence of polynomials converges pointwise in a region D, then the limit function is analytic except for a closed nowhere dense set.
Note this solution is based of ideas from
`http://mathoverflow.net/questions/117633/a-question-about-the-limit-of-a-sequence-of-pointwise-convergent-analytic-funtio`
and
`http://math.stackexchange.com/questions/113240/pointwise-convergence-of-sequences-of-holomorphic-functions-to-holomorphic-funct`

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Let G_1 be the group of all conformal mappings from the unit ball into itself and let G_2 be the set of all conformal mappings from the upper half plane into itself. The group operation in both cases is function composition. We claim that these two groups are isomorphic.
Let $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $U = \{z : \Im(z) > 1\}$.
Define the mapping $M : D_1 \longrightarrow U$ by

$$M(z) = \frac{(z+1)(1-i)}{2(z-i)}.$$

This is a Mobius transformation, so it is a conformal mapping and importantly bijective.
We can produce an *induced* transformation $M_G : G_1 \longrightarrow G_2$ where for $f \in G_1$

$$(M_G f)(x) = (M \circ f \circ M^{-1})(x).$$

Obviously the inverse transformation $M_G^{-1} : G_2 \longrightarrow G_1$ is given by

$$(M_G^{-1} f)(x) = (M^{-1} \circ f \circ M)(x)$$

for $f \in G_2$.
We just have to show that this is a group homomorphism.
For f_1, f_2 in G_1

$$M_G(f_1 \circ f_2) = M \circ f_1 \circ f_2 \circ M^{-1} = M \circ f_1 \circ M^{-1} \circ M \circ f_2 \circ M^{-1} = M_G(f_1) \circ M_G(f_2)$$

and so M_G is a group homomorphism.
Thus the groups G_1 and G_2 are isomorphic.

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Let D be a multiply connected domain. Let D_0 be its convex hull. Then pick an $a \in \partial D$. Such an a must exist in D_0 as D is multiply connected.
Then the function $f : D \longrightarrow \mathbb{C}$,

$$f(z) = \frac{1}{z-a}$$

cannot be analytically extended to all of D_0. The reason being that no only an immediate continuation could be applied as it is in the boundary and clearly no immediate continuation would exist.

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We can identify any Mobius transformation

$$f(z) = \frac{az+b}{cz+d} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

with a matrix in $GL_2(\mathbb{C})$.
Also see that if

$$f \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

$$g \sim \begin{bmatrix} e & f \\ g & h \end{bmatrix} \tag{3}$$

then

$$(g \circ f) \sim \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4}$$

$$\sim \begin{bmatrix} ea+fc & eb+fd \\ ag+ch & gb+dh \end{bmatrix}. \tag{5}$$

Notice that this identification is not unique as

$$\frac{az+b}{cz+d} \sim \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6}$$

for $\lambda \in \mathbb{C}$.
Either way we are interested in Mobius transformations which map the unit disk D_1 to itself. We claim that this consists of all the Mobius transformations of the form

$$f(z) = e^{i\theta} \frac{z+b}{\bar{b}z+1} \sim e^{i\theta} \begin{bmatrix} 1 & b \\ \bar{b} & 1 \end{bmatrix} \tag{7}$$

where $\theta \in \mathbb{R}$ and $b \in \mathbb{C}$ with $|b| < 1$.
This is possible to see from the three point construction of a Mobius transformation. This guarantees that the transformation specified by three points is unique.
We now want to break this generic transformation up into its finite subgroups.

Suppose firstly that $\theta = [0]$. Then the fixed points of the transformation are found by

$$z = \frac{z+b}{\bar{b}z+1} \tag{8}$$

$$z(\bar{b}z+1) = z+b \tag{9}$$

$$\bar{b}z^2 = b \tag{10}$$

$$z = \sqrt{\frac{b}{\bar{b}}}. \tag{11}$$

So the group of transformations generated (through repeated application) by

$$\frac{z+b}{\bar{b}z+1} \tag{12}$$

has a global fixed point (actually 2) given by $z = \sqrt{b/\bar{b}}$.
If we allow $\theta \neq [0]$ then the fixed points correspond to the solutions of the quadratic

$$z(\bar{z}+1) = e^{i\theta}(z+b). \tag{13}$$

We require the group to be finite. This corresponds to requiring that $\theta \in 2\pi\mathbb{Q}$ and to

$$\det \begin{bmatrix} 1 & b \\ \bar{b} & 1 \end{bmatrix} = 1 \tag{14}$$

but this last requirement means that $b = 0$ which just leaves us with transformations of the form $f(z) = e^{i\theta}z$ which are just rotations. So long as $\theta \in 2\pi\mathbb{Q}$ the group generated by any number of rotations will be finite, with a singled global fixed point of 0.

References

[1] Osgood, Montel and Lavrentiev, *Sur les fonctions d'une variable complexe representable par des series de polynomes*, Paris, 1936

[2] Hartogs and Rosenthal, *Uber Folgen analytischer Funktionen*, Math Ann., 1928, 100, 212-263