University of New South Wales

School of Mathematics and Statistics

Assignment 1

Complex Analysis

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1

Write $s = \sigma + it$. Then consider

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} n^{-s} - \frac{1}{s-1} \\ &= \sum_{n=1}^{\infty}\left[n^{-s} \int_n^{n+1} x^{-s}dx\right] \\ &= \sum_{n=1}^{\infty}\int_n^{n+1}(n^{-s} - x^{-s})dx.\end{aligned}$$

Now note that

$$|n^{-s} - x^{-s}| = \left|s\int_n^x y^{-1-s}dy\right| \leq |s|n^{-1-\sigma}$$

for $x \in [n, n+1]$ and hence

$$\int_n^{n+1}(n^{-s} - x^{-s})dx \leq |s|n^{-1-\sigma}$$

which means that

$$\sum_{n=1}^{\infty}\int_n^{n+1}(n^{-s} - x^{-s})dx$$

converges absolutely on compact subsets of $\Re(s) > 0$. Now as each term in the sum is an analytic function then the sum is an analytic function for $\sigma > 0$. So

$$\zeta(s) = \sum_{n=1}^{\infty}\int_n^{n+1}(n^{-s} - x^{-s})dx + \frac{1}{s-1}$$

defines an analytic continuation of $\zeta(s)$ for $\Re(s) > 0$ and $s \neq 1$.

2

Since f_1 is complex differentiable on $\mathbb{C}$, f_1 cannot have any singularities in the unit ball and so all the of singularities of f_1/f_2 in the unit ball come from $1/f_2$. We therefore turn our attention to $1/f_2$. It is simple to see that if $1/f_2$ has infinitely many poles in the unit ball then f_2 has infinitely many zeros in the unit ball (at the same locations). Say these zeros are at $\{z_n\}_n$ then by the complex analogue of the Bolzano-Wierstrass theorem there exists a subsequence $\{x_{n_k}\}_k \subseteq \{x_n\}_n$ such that $\lim_{k\longrightarrow\infty} x_{n_k}$ exists.
Suppose x^* is the limit point of this sequence then by theorem 1 of lecture notes 4, we have that $f_2 \equiv 0$ for all z in the unit ball.
This means that if f_1/f_2 has infinitely many poles in the unit ball then $f_2 \equiv 0$ in the unit ball. So if we disregard this degenerate case then f_1/f_2 cannot have infinitely many poles in the unit ball.

3

$$w(z) := \sqrt{k(\rho)}\,\mathrm{sn}\left(\frac{2K}{\pi}\sin^{-1} z; \rho\right)$$

where

$$\rho = \left(\frac{a-b}{a+b}\right)^2$$

and where *sn* corresponds the the inversion of the Jacobi Elliptic function of the first kind, is the conformal mapping of the ellipse with foci $\pm\sqrt{k(\rho)}$. A full derivation of this mapping, can be found in *Conformal Mapping* by Zeev Nehari.

4

To do this we just have to prove equivelence of the norm to the sup norm.

$$\int\int_{B_R(z)} f dA = \pi R^2 f(z)$$

where $B_R(z)$ is a unit ball centered at z of radius R because $f = u + iv$ has u and v harmonic. Then

$$|f(z)| \leq \frac{1}{\pi R^2}\int\int_{B_R(z)} |f(z)| dA \leq \frac{1}{\pi R^2}\sqrt{\int\int_{B_R(z)} |f(z)|^2 dA}$$

Thus

$$\sup_{|z|\leq R} |f(z)| \leq \frac{1}{\pi R^2}\sqrt{\int\int_{B_R(0)} |f(z)| dA}.$$

Next see that by the integral estimation lemma we have the opposite

$$\sup_{|z|\leq R} |f(z)| \geq \frac{1}{\pi R^2}\sqrt{\int\int_{B_R(0)} |f(z)| dA}$$

and hence the norms are equivelent. Note that by setting $R = 1$ we have a result specific to the question.

5

$$f(z) := \sum_{n=1}^{\infty} z^{2^n}$$

Firstly we need to show that $f(z)$ is complex differentiable inside the unit ball. To do that note firstly

$$\sum_{n=1}^{\infty}\left|z^{2^n}\right| \leq \sum_{n=1}^{\infty}|z^n|$$

so long as $|z| < 1$, and so by the comparison test $f(z)$ must converge inside the unit ball.
Next note that z^{2^n} is clearly complex differentiable because z is complex differentiable. So in the unit ball $f(z)$ is the sum of complex differentiable functions, and the sum converges absolutely. Therefore, $f(z)$ is complex differentaible inside the unit ball.
Let us next show that for $z = e^{\frac{k}{2^j}\pi}$, $f(z)$ does not converge. See that

$$\begin{aligned}
f(e^{\frac{k}{2^j}\pi}) &= \sum_{n=1}^{\infty} e^{2^{n-j}k\pi} \\
&= \sum_{n=1}^{n=j} e^{2^(n-j)k\pi} + \sum_{n=1}^{\infty} 1
\end{aligned}$$

and so $f(e\frac{k\pi}{2^j})$ does not converge. It is well known that numbers of the form $\mathbb{Q}_\pi := e\frac{k\pi}{2^j}$ are dense on the boundry of the unit ball.
So for some z^* such that $|z^*| = 1$

$$\lim_{z \longrightarrow z^*} f(z) = \underbrace{\lim_{z \longrightarrow z^*, z \in \mathbb{Q}_\pi} f(z)}_{\text{does not exist}}$$

and hence $\lim_{z \longrightarrow z^*} f(z)$ does not exist.