



UNSW
A U S T R A L I A



UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 1

Complex Analysis

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1

Write $s = \sigma + it$. Then consider

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} n^{-s} - \frac{1}{s-1} \\ &= \sum_{n=1}^{\infty} \left[n^{-s} \int_n^{n+1} x^{-s} dx \right] \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx.\end{aligned}$$

Now note that

$$|n^{-s} - x^{-s}| = \left| s \int_n^x y^{-1-s} dy \right| \leq |s| n^{-1-\sigma}$$

for $x \in [n, n+1]$ and hence

$$\int_n^{n+1} (n^{-s} - x^{-s}) dx \leq |s| n^{-1-\sigma}$$

which means that

$$\sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx$$

converges absolutely on compact subsets of $\Re(s) > 0$. Now as each term in the sum is an analytic function then the sum is an analytic function for $\sigma > 0$. So

$$\zeta(s) = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx + \frac{1}{s-1}$$

defines an analytic continuation of $\zeta(s)$ for $\Re(s) > 0$ and $s \neq 1$.

2

Since f_1 is complex differentiable on \mathbb{C} , f_1 cannot have any singularities in the unit ball and so all the singularities of f_1/f_2 in the unit ball come from $1/f_2$. We therefore turn our attention to $1/f_2$. It is simple to see that if $1/f_2$ has infinitely many poles in the unit ball then f_2 has infinitely many zeros in the unit ball (at the same locations). Say these zeros are at $\{z_n\}_n$ then by the complex analogue of the Bolzano-Weierstrass theorem there exists a subsequence $\{x_{n_k}\}_k \subseteq \{x_n\}_n$ such that $\lim_{k \rightarrow \infty} x_{n_k}$ exists.

Suppose x^* is the limit point of this sequence then by theorem 1 of lecture notes 4, we have that $f_2 \equiv 0$ for all z in the unit ball.

This means that if f_1/f_2 has infinitely many poles in the unit ball then $f_2 \equiv 0$ in the unit ball. So if we disregard this degenerate case then f_1/f_2 cannot have infinitely many poles in the unit ball.

3

$$w(z) := \sqrt{k(\rho)} \operatorname{sn} \left(\frac{2K}{\pi} \sin^{-1} z; \rho \right)$$

where

$$\rho = \left(\frac{a-b}{a+b} \right)^2$$

and where sn corresponds to the inversion of the Jacobi Elliptic function of the first kind, is the conformal mapping of the ellipse with foci $\pm \sqrt{k(\rho)}$. A full derivation of this mapping, can be found in *Conformal Mapping* by Zeev Nehari.

4

To do this we just have to prove equivalence of the norm to the sup norm.

$$\int \int_{B_R(z)} f dA = \pi R^2 f(z)$$

where $B_R(z)$ is a unit ball centered at z of radius R because $f = u + iv$ has u and v harmonic. Then

$$|f(z)| \leq \frac{1}{\pi R^2} \int \int_{B_R(z)} |f(z)| dA \leq \frac{1}{\pi R^2} \sqrt{\int \int_{B_R(z)} |f(z)|^2 dA}$$

Thus

$$\sup_{|z| \leq R} |f(z)| \leq \frac{1}{\pi R^2} \sqrt{\int \int_{B_R(0)} |f(z)| dA}.$$

Next see that by the integral estimation lemma we have the opposite

$$\sup_{|z| \leq R} |f(z)| \geq \frac{1}{\pi R^2} \sqrt{\int \int_{B_R(0)} |f(z)| dA}$$

and hence the norms are equivalent. Note that by setting $R = 1$ we have a result specific to the question.

5

$$f(z) := \sum_{n=1}^{\infty} z^{2^n}$$

Firstly we need to show that $f(z)$ is complex differentiable inside the unit ball. To do that note firstly

$$\sum_{n=1}^{\infty} |z^{2^n}| \leq \sum_{n=1}^{\infty} |z^n|$$

so long as $|z| < 1$, and so by the comparison test $f(z)$ must converge inside the unit ball.

Next note that z^{2^n} is clearly complex differentiable because z is complex differentiable. So in the unit ball $f(z)$ is the sum of complex differentiable functions, and the sum converges absolutely. Therefore, $f(z)$ is complex differentiable inside the unit ball.

Let us next show that for $z = e^{\frac{k}{2^j}\pi}$, $f(z)$ does not converge. See that

$$\begin{aligned} f(e^{\frac{k}{2^j}\pi}) &= \sum_{n=1}^{\infty} e^{2^{n-j}k\pi} \\ &= \sum_{n=1}^{n=j} e^{2^{(n-j)}k\pi} + \sum_{n=1}^{\infty} 1 \end{aligned}$$

and so $f(e^{\frac{k}{2^j}\pi})$ does not converge. It is well known that numbers of the form $\mathbb{Q}_\pi := e^{\frac{k\pi}{2^j}}$ are dense on the boundary of the unit ball. So for some z^* such that $|z^*| = 1$

$$\lim_{z \rightarrow z^*} f(z) = \underbrace{\lim_{z \rightarrow z^*, z \in \mathbb{Q}_\pi} f(z)}_{\text{does not exist}}$$

and hence $\lim_{z \rightarrow z^*} f(z)$ does not exist.