University of New South Wales

School of Mathematics and Statistics

Assignment 1

Complex Analysis

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1

2

3

Let G_1 be the group of all conformal mappings from the unit ball into itself and let G_2 be the set of all conformal mappings from the upper half plane into itself. The group operation in both cases is function composition. We claim that these two groups are isomorphic.
Let $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $U = \{z : \Im(z) > 1\}$.
Define the mapping $M : D_1 \longrightarrow U$ by

$$M(z) = \frac{(z+1)(1-i)}{2(z-i)}.$$

This is a Mobius transformation, so it is a conformal mapping and importantly bijective.
We can produce an *induced* transformation $M_G : G_1 \longrightarrow G_2$ where for $f \in G_1$

$$(M_G f)(x) = (M \circ f \circ M^{-1})(x).$$

Obviously the inverse transformation $M_G^{-1} : G_2 \longrightarrow G_1$ is given by

$$(M_G^{-1} f)(x) = (M^{-1} \circ f \circ M)(x)$$

for $f \in G_2$.
We just have to show that this is a group homomorphism.
For f_1, f_2 in G_1

$$M_G(f_1 \circ f_2) = M \circ f_1 \circ f_2 \circ M^{-1} = M \circ f_1 \circ M^{-1} \circ M \circ f_2 \circ M^{-1} = M_G(f_1) \circ M_G(f_2)$$

and so M_G is a group homomorphism.
Thus the groups G_1 and G_2 are isomorphic.

4

Let D be a multiply connected domain. Let D_0 be its convex hull. Then pick an $a \in D_0 \setminus D$. Such an a must exist as D is multiply connected.
Then the function $f : D \longrightarrow \mathbb{C}$,

$$f(z) = \frac{1}{z-a}$$

cannot be analytically extended to all of D_0.

5

We can identify any Mobius transformation

$$f(z) = \frac{az+b}{cz+d} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

with a matrix in $GL_2(\mathbb{C})$.
Also see that if

$$f \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

$$g \sim \begin{bmatrix} e & f \\ g & h \end{bmatrix} \tag{3}$$

then

$$(g \circ f) \sim \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4}$$

$$\sim \begin{bmatrix} ea+fc & eb+fd \\ ag+ch & gb+dh \end{bmatrix}. \tag{5}$$

Noice that this identificaiton is not unqiue as

$$\frac{az+b}{cz+d} \sim \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6}$$

for $\lambda \in \mathbb{C}$. It turns out that the group of automorphisms on $\hat{\mathbb{C}}$ (the projective line) is isomorphic the the projective linear group, that is

$$\mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathrm{PGL}(2, \mathbb{C}). \tag{7}$$

We are interested finite subgroups of $\mathrm{Aut}(\hat{\mathbb{C}})$ and hence we are interested in finite subgroups of $\mathrm{PGL}(2, \mathbb{C})$.