University of New South Wales

School of Mathematics and Statistics

Assignment 1

Complex Analysis

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Write $s = \sigma + it$. Then consider

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} n^{-s} - \frac{1}{s-1} \\ &= \sum_{n=1}^{\infty} \left[n^{-s} \int_{n}^{n+1} x^{-s} dx \right] \\ &= \sum_{n=1}^{\infty} \int_{n}^{n+1} (n^{-s} - x^{-s}) dx.\end{aligned}$$

Now note that

$$|n^{-s} - x^{-s}| = \left| s \int_{n}^{x} y^{-1-s} dy \right| \leq |s| n^{-1-\sigma}$$

for $x \in [n, n+1]$ and hence

$$\int_{n}^{n+1} (n^{-s} - x^{-s}) dx \leq |s| n^{-1-\sigma}$$

which means that

$$\sum_{n=1}^{\infty} \int_{n}^{n+1} (n^{-s} - x^{-s}) dx$$

converges absolutely on compact subsets of $\Re(s) > 0$. Now as each term in the sum is an analytic function then the sum is an analytic function for $\sigma > 0$. So

$$\zeta(s) = \sum_{n=1}^{\infty} \int_{n}^{n+1} (n^{-s} - x^{-s}) dx + \frac{1}{s-1}$$

defines an analytic continuation of $\zeta(s)$ for $\Re(s) > 0$ and $s \neq 1$.

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Since f_1 is complex differentiable on $\mathbb{C}$, f_1 cannot have any singularities in the unit ball and so all the of singularities of f_1/f_2 in the unit ball come from $1/f_2$. We therefore turn our attention to $1/f_2$. It is simple to see that if $1/f_2$ has infinitely many poles in the unit ball then f_2 has infinitely many zeros in the unit ball (at the same locations). Say these zeros are at $\{z_n\}_n$ then by the complex analogue of the Bolzano-Wierstrass theorem there exists a subsequence $\{x_{n_k}\}_k \subseteq \{x_n\}_n$ such that $\lim_{k \longrightarrow \infty} x_{n_k}$ exists.
Suppose x^* is the limit point of this sequence then by theorem 1 of lecture notes 4, we have that $f_2 \equiv 0$ for all z in the unit ball.
This means that if f_1/f_2 has infinitely many poles in the unit ball then $f_2 \equiv 0$ in the unit ball. So if we disregard this degenerate case then f_1/f_2 cannot have infinitely many poles in the unit ball.

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