UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment

Discrete Time Financial Modeling

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Question 1

Consider the model with two states of the world, one risky security $S_n, n = 1$ and one taking the interest rate $r = \frac{1}{9}$, along with the following security prices. What is the fair price of a European

n	$S_n(0)$	$S_n(1)(\omega_1)$	$S_n(1)(\omega_2)$
1	5	$\frac{20}{3}$	$\frac{40}{9}$

put option with strike price $K = 5$? What trading trategy generates this contingent claim?

Solution

In general, for a one step model,

$$V_{put} = \frac{1}{1+r}\left[\frac{1+r-d}{u-d}(K - S_0u)^+ + \frac{u-r-1}{u-d}(K - S_0d)^+\right]$$

and in this case we have $K = 5$, $u = \frac{4}{3}$, $d = \frac{8}{9}$, $S_0 = 5$, $r = \frac{1}{9}$ so

$$\begin{aligned} V_{put} &= \frac{1}{1+\frac{1}{9}}\left[\frac{1+\frac{1}{9}-\frac{8}{9}}{\frac{4}{3}-\frac{8}{9}}\left(5 - 5\frac{4}{3}\right)^+ + \frac{\frac{4}{3}-\frac{1}{9}-1}{\frac{4}{3}-\frac{8}{9}}\left(5 - 5\frac{8}{9}\right)^+\right] \\ &= \frac{9}{10}\left[0 + \frac{1}{2} \times \frac{5}{9}\right] \\ &= \frac{1}{4} \end{aligned}$$

This pricing corresponds to the following trading strategy:

at time 0

Sell 1 put for $\frac{1}{4}$.	Cash Flow: $+\frac{1}{4}$
Sell $\frac{1}{4}$ shares.	Cash Flow: $+\frac{5}{4}$
Invest $\frac{3}{2}$ at the risk free rate.	Cash Flow: $-\frac{3}{2}$
	Total: 0

at time 1 if $S_1 = \frac{20}{3}$

Payoff option.	Cashflow: 0
Liquidate risk free rate investment.	Cashflow: $+\frac{5}{3}$
Repurcahse shares.	Cashflow: $-\frac{5}{3}$
	Total: 0

at time 1 if $S_1 = \frac{40}{9}$

Payoff option.	Cashflow: $-\frac{5}{9}$
Liquidate risk free rate investment.	Cashflow: $+\frac{5}{3}$
Repurchase shares.	Cashflow: $-\frac{10}{9}$
	Total: 0

Question 2

Suppose the interest rate r is a scalar, and let c and p denote the prices of a call and put, respectively, both having the same strike price K. Show that either both are attainable or neither is attainable. Use risk neutral valuation to show that in the former case one has

$$c - p = S_0 - \frac{K}{1+r}.$$

Assume the put is attainable. We wish to show that the call is attainable, i.e that using a put we can replicate the payoff of the call.

Solution

Assume the put is attainable. We wish to show that the call is attainable, i.e that using a put we can replicate the payoff of the call.

time 0

Purchase 1 put. Cashflow: $-p$
Purchase 1 share. Cashflow: $-S_0$
Borrow $\frac{K}{1+r}$ at the risk free rate. Cashflow: $+\frac{K}{1+r}$

time 1 when $S_1 = S_1(up)$

Collect payoff from the put. Cashflow: $(K - S_1(up))^+$
Repay loan. Cashflow: $-K$
Sell share. Cashflow: $S_1(up)$
Total: $S_1(up) - K + (K - S_1(up))^+ = (S_1(up) - k)^+$

time 1 when $S_1 = S_1(down)$

Collect payoff from the put. Cashflow: $(K - S_1(down))^+$
Repay loan. Cashflow: $-K$
Sell share. Cashflow: $S_1(down)$
Total: $S_1(down) - K + (K - S_1(down))^+ = (S_1(down) - k)^+$
Notice that in either case this is just the payoff of a call option. So if the put is attainable the call is attainable.
Assume that the call is attainable.

time 0

Purcahse 1 call. Cashflow $-c$
Sell 1 share. Cashflow $-S_0$
Invest $\frac{K}{1+r}$ at the risk free rate. Cashflow: $-\frac{K}{1+r}$

time 1 when $S_1 = S_1(up)$

Collect payoff from the call. Cashflow: $(S_1(up) - K)^+$
Liquidate risk free investment. Cashflow: K
Repurcahse the share. Cashflow: $-S_1(up)$ Total: $(S_1(up) - K)^+ + K - S_1(up) = (K - S_1(up))^+$

time 1 when $S_1 = S_1(down)$

Collect payoff from the call. Cashflow: $(S_1(down) - K)^+$
Liquidate risk free investment. Cashflow: K
Repurcahse the share. Cashflow: $-S_1(down)$ Total: $(S_1(down) - K)^+ + K - S_1(down) = (K - S_1(down))^+$
So in either case we just have the payoff of a put option. So if the call is attainable, then the put is.
This is to say that either the put *and* call are attainable, or neither are.
This demonstration hints that we should have $c = p + S_0 - \frac{K}{1+r}$ because the if two portfolios have the same future payoffs that must have the same current value. The following risk neutral valuation formalises this.
Observe that

$$(S_1 - K)^+ - (K - S_1^+ = S_1 - K$$

which is equivelent to

$$\frac{(S_1 - K)^+ + K}{1+r} = \frac{(K - S_1)^+ + S_1}{1+r}.$$

Now taking conditional expectations of each side with respect to the risk neutral measure yields

$$\mathbb{E}_{\mathbb{P}}\left[(1+r)^{-1}(S_1 - K)^+ | S_0\right] + \frac{K}{1+r} = \mathbb{E}_{\mathbb{P}}\left[(1+r)^{-1}(K - S_1)^+ | S_0\right] + \mathbb{E}_{\mathbb{P}}\left[\frac{S_1}{1+r} | S_0\right]$$

Now using the fact that under the risk free measure the discounted value of a risky asset follow a martingale process we must have

$$\underbrace{\mathbb{E}_{\mathbb{P}}\left[(1+r)^{-1}(S_1 - K)^+ | S_0\right]}_{\text{value of call option at time 0}} + \frac{K}{1+r} = c + \frac{K}{1+r}$$

$$\underbrace{\mathbb{E}_{\mathbb{P}}\left[(1+r)^{-1}(K - S_1)^+ | S_0\right]}_{\text{value of put option at time 0}} + \underbrace{\mathbb{E}_{\mathbb{P}}\left[\frac{S_1}{1+r} | S_0\right]}_{\text{value of stock at time at time 0}} = p + S_0$$

which leaves us with [1]

$$c = p + S_0 - \frac{K}{1+r}.$$

□

Question 3

Under the risk neutral measure we have $\tilde{p} = \tilde{q} = \frac{1}{p}$ and in this case the distribution of S_3 is

S_3	32	8	2	$\frac{1}{2}$
$\mathbb{P}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

We also have $\tilde{\mathbb{E}}[S_1] = \frac{8}{2} + \frac{2}{2} = 5$, $\tilde{\mathbb{E}}[S_2] = \frac{16}{4} + \frac{8}{4} + \frac{1}{4} = \frac{25}{4}$ and $\tilde{\mathbb{E}}[S_3] = \frac{32}{8} + \frac{24}{8} + \frac{6}{8} + \frac{0.5}{8} = \frac{125}{16}$
By taking the geometric mean the average rate of growth is $\frac{1}{4}$ which is consistent with the risk free rate implied in $\tilde{p} = \frac{1+r-d}{u-d}$.
In the case when we have $p = \frac{2}{3}$, $q = \frac{1}{3}$ we have

[1] This proof was modeled off a proof by Ophir Gottlieb available at `http://www.soarcorp.com/research/put_call_parity.pdf`

S_3	32	8	2	$\frac{1}{2}$
$\mathbb{P}$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

with the following expectations: $\mathbb{E}[S_1] = \frac{16}{3} + \frac{2}{3} = 6$ and $\mathbb{E}[S_2] = \frac{64}{9} + \frac{16}{9} + \frac{1}{9} = 9$ and $\mathbb{E}[S_3] = \frac{256}{27} + \frac{96}{27} + \frac{12}{27} + \frac{0.5}{27} = 13.5$
The average rate of growth is 0.5.