University of New South Wales

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# Assignment

Number Theory

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# 1 Question 4

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j) \qquad\qquad n = 1, \ldots, N$$

See that

$$\begin{aligned}
\mathbb{E}\left[I_{n+1} | F_n\right] &= \mathbb{E}\left[\sum_{j=0}^{n} \Delta_j (M_{j+1} - M_j) | \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j) + \Delta_n (M_{n+1} - M_n) | \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j) | \mathcal{F}_n\right] + \mathbb{E}\left[\Delta_n (M_{n+1} - M_n) | \mathcal{F}_n\right] \\
&\text{by linearity of conditional expectation} \\
&= I_n + \Delta_n \mathbb{E}\left[M_{n+1} - \mathbb{E}\left[M_n\right] | \mathcal{F}\right] \\
&\text{because } \Delta_n \text{ is adapted to } \mathcal{F}_n \\
&= I_n + \Delta_n (\mathbb{E}\left[M_{n+1} | \mathcal{F}\right] - M_n) \\
&\text{because } M_n \text{ is completely determined by } \mathcal{F}_n \\
&= I_n + \Delta_n (M_n - M_n) \\
&\text{because } M_n \text{ is a martingale} \\
&= I_n
\end{aligned}$$

and so $\{I_n\}_{n=0}^{N}$ is a martingale.