Introduction to Arbitrage Pricing

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Chapter 1

An Introduction to Financial Derivatives

We shall first review briefly the most important kinds of financial contracts, traded either on exchanges or over-the-counter (OTC), between financial institutions and their clients. For a detailed account of the fundamental features of *spot* (i.e., *cash*) and *futures* financial markets the reader is referred, for instance, to Cox and Rubinstein (1985), Ritchken (1987), Chance (1989), Duffie (1989), Merrick (1990), Kolb (1991), Dubofsky (1992), Edwards and Ma (1992), Sutcliffe (1993), Hull (1994, 1997) or Redhead (1996).

1.1 Derivative Markets

We start this section by describing the spot options market, after which we shall focus on futures contracts and futures options. Finally, the basic features of forward contracts will be discussed.

1.1.1 Options

Options are examples of exchange-traded *derivative securities* – that is, securities whose value depends on the prices of other more basic securities (so called *primary* securities or assets) such as stocks or bonds. By *stocks* we mean *common stocks* – that is, shares in the net asset value not bearing fixed interest. They give the right to dividends according to profits, after payments on *preferred stocks.* The preferred stocks give some special rights to the stockholder, typically a guaranteed fixed dividend. A *bond* is a certificate issued by a government or a public company promising to repay borrowed money at a fixed rate of interest at a specified time. Basically, a *call option* (a *put option,* respectively) is the right to buy (to sell, respectively) the option's underlying asset at some future date for a prespecified price. Options (in particular, warrants[1]) have been traded for centuries; unprecedented expansion of the options market started, however, quite recently with the introduction in 1973 of exchange-traded options on stocks in the United States.

It should be emphasized that most of the traded options are of American type – that is, the holder has the right to exercise an option at any instant before the option's expiry. When an investor notifies his broker of the intention to exercise an option, the broker in turn notifies the OCC[2] member who clears the investor's trade. This member then places an exercise order with the OCC. The OCC randomly selects a member with an outstanding short position in the same

[1] A *warrant* is a call option issued by a company or a financial institution. Warrants are frequently issued by companies on their own stocks; new shares are issued when warrants are exercised. In some cases, the warrants are subsequently traded on an exchange. Warrants are bought and sold in much the same way as stocks.

[2] OCC stands for the Options Clearing Corporation. The OCC keeps the record of all long and short positions. The OCC guarantees that the option writer will fulfil obligations under the terms of the option contract. The OCC has a number of *members,* and all option trades must be cleared through a member.

option. The chosen member, in turn, selects a particular investor who has written the option (such an investor is said to be *assigned*). If the option is a call, this investor is required to sell stock at the so-called *strike price* or *exercise price* (if it is a put, he is required to buy stock at the strike price). When the option is exercised, the *open interest* (that is, the number of options outstanding) goes down by one.

In addition to options on particular stocks, a large variety of other option contracts are traded nowadays on exchanges: foreign currency options (such as, e.g., British pound, German mark or Japanese yen option contracts traded on the Philadelphia Exchange), index options (e.g., those on S&P100 and S&P500 traded on the CBOE), and futures options (e.g., the Treasury bond futures option traded on the Chicago Board of Trade (CBOT)). Interest rate options are also implicit in several other interest rate instruments, such as *caps* or *floors* (these are, however, over-the-counter traded contracts). Derivative financial instruments involving options are also widely traded outside the exchanges by financial institutions and their clients. We may identify here such contracts as *swaptions* – that is, options on an *interest rate swap,* or a large variety of *exotic* options. Finally, options are implicit in several financial instruments, for example in some bond or stock issues (*callable bonds, savings bonds* or *convertible bonds,* to mention a few).

One of the most appealing features of options (apart from the obvious chance of making extraordinary returns) is the possibility of easy speculation on the future behavior of a stock price. Usually this is done by means of so called *combinations* – that is, combined positions in several options, and possibly the underlying asset. For instance, a *bull spread* is portfolio created by buying a call option on a stock with a certain strike price and selling a call option on the same stock with a higher strike price (both options have the same expiry date). Equivalently, bull spreads can be created by buying a put with a low strike price and selling a put with a high strike price. An investor entering a bull spread is hoping that the stock price will increase. Like a bull spread, a *bear spread* can be created by buying a call with one strike price and selling a call with another strike price. The strike price of the option purchased is now greater than the strike price of the option sold, however. An investor who enters a bear spread is hoping that the stock price will decline.

A *butterfly spread* involves positions in options with three different strike prices. It can be created by buying a call option with a relatively low strike price, buying another call option with a relatively high strike price, and selling two call options with a strike price halfway between the other two strike prices. The butterfly spread leads to a profit if the stock price stays close to the strike price of the call options sold, but gives rise to a small loss if there is a significant stock price move in either direction. A portfolio created by selling a call option with a certain strike price and buying a longer-maturity call option with the same strike price is commonly known as a *calendar spread.* A *straddle* involves buying a call and put with the same strike price and expiry date. If the stock price is close to this strike price at expiry of the option, the straddle leads to a loss. A straddle is appropriate when an investor is expecting a large move in stock price but does not know in which direction the move will be. Related types of trading strategies are commonly known as *strips, straps* and *strangles.*

1.1.2 Futures Contracts and Options

Another important class of exchange-traded derivative securities comprises *futures contracts,* and options on futures contracts, commonly known as *futures options.* Futures contracts apply to a wide range of commodities (e.g., sugar, wool, gold) and financial assets (e.g., currencies, bonds, stock indices); the largest exchanges on which futures contracts are traded are the Chicago Board of Trade and the Chicago Mercantile Exchange (CME). In what follows, we restrict our attention to financial futures (as opposed to commodity futures). To make trading possible, the exchange specifies certain standardized features of the contract. Futures prices are regularly reported in the financial press. They are determined on the floor in the same way as other prices – that is, by the law of supply and demand. If more investors want to go long than to go short, the price goes up; if the reverse is true, the price falls. Positions in futures contracts are governed by a specific daily settlement procedure commonly referred to as *marking to market.* An investor's initial deposit, known as the *initial margin,* is adjusted daily to reflect the gains or losses that are due to the futures price movements. Let us consider, for instance, a party assuming a long position (the party who agreed to buy). When there is a decrease in the futures price, her margin account is reduced by an appropriate amount of money, her broker has to pay this sum to the exchange and the exchange passes the money on to the broker of the party who assumes the short position. Similarly, when the futures price rises, brokers for parties with short positions pay money to the exchange, and brokers of parties with long positions receive money from the exchange. This way, the trade is marked to market at the close of each trading day. Finally, if the delivery period is reached and delivery is made by a party with a short position, the price received is generally the futures price at the time the contract was last marked to market.

In a *futures option,* the underlying asset is a futures contract. The futures contract normally matures shortly after the expiry of the option. When the holder of a call futures option exercises the option, she acquires from the writer a long position in the underlying futures contract plus a cash amount equal to the excess of the current futures price over the option's strike price. Since futures contracts have zero value and can be closed out immediately, the payoff from a futures option is the same as the payoff from a stock option, with the stock price replaced by the futures price. Futures options are now available for most of the instruments on which futures contracts are traded. The most actively traded futures option is the Treasury bond futures option traded on the Chicago Board of Trade. On some markets (for instance, on the Australian market), futures options have the same features as futures contracts themselves – that is, they are not paid up-front as classic options, but are traded at the margin. Unless otherwise stated, by a futures option we mean here a standard option written on a futures contract.

1.1.3 Forward Contracts

A *forward contract* is an agreement to buy or sell an asset at a certain future time for a certain price. One of the parties to a forward contract assumes a long position and agrees to buy the underlying asset on a certain specified future date for a *delivery price*; the other party assumes a short position and agrees to sell the asset on the same date for the same price. At the time the contract is entered into, the delivery price is determined so that the value of the forward contract to both parties is zero. It is thus clear that some features of forward contracts resemble those of futures contracts. However, unlike futures contracts, forward contracts do not trade on exchanges. Also, a forward contract is settled only once, at the maturity date. The holder of the short position delivers the asset to the holder of the long position in return for a cash amount equal to the delivery price. The following list summarizes the main differences between forward and futures contracts.

1. Contract specification and delivery
Futures contracts. The contract precisely specifies the underlying instrument and price. Delivery dates and delivery procedures are standardized to a limited number of specific dates per year, at approved locations. Delivery is not, however, the objective of the transaction, and less than 2% are delivered.
Forward contracts. There is an almost unlimited range of instruments, with individually negotiated prices. Delivery can take place on any individual negotiated date and location. Delivery is the object of the transaction, with over 90% of forward contracts settled by delivery.

2. Prices
Futures contracts. The price is the same for all participants, regardless of transaction size. Typically, there is a daily price limit (although, for instance, on the FT-SE 100 index, futures prices are unlimited). Trading is usually by open outcry auction on the trading floor of the exchange. Prices are disseminated publicly. Each transaction is conducted at the best price available at the time.
Forward contracts. The price varies with the size of the transaction, the credit risk, etc. There are no daily price limits. Trading takes place by telephone and fax between individual buyers and sellers. Prices are not disseminated publicly. Hence, there is no guarantee that the price is the best available.

3. Marketplace and trading hours
Futures contracts. Trading is centralized on the exchange floor, with worldwide communications, during hours fixed by the exchange.
Forward contracts. Trading takes place by telephone and telex between individual buyers and sellers. Trading is over-the-counter world-wide, 24 hours per day, with telephone and telex access. **4. Security deposit and margin**
Futures contracts. The exchange rules require an initial margin and the daily settlement of variation margins. A central clearing house is associated with each exchange to handle the daily revaluation of open positions, cash payments and delivery procedures. The clearing house assumes the credit risk.
Forward contracts. The collateral level is negotiable, with no adjustment for daily price fluctuations. There is no separate clearing house function. Thus, the market participant bears the risk of the counter-party defaulting.

5. Volume and market liquidity
Futures contracts. Volume (and open interest) information is published. There is very high liquidity and ease of offset with any other market participant due to standardized contracts.
Forward contracts. Volume information is not available. The limited liquidity and offset is due to the variable contract terms. Offset is usually with the original counter-party.

1.2 Call and Put Spot Options

Let us first describe briefly the set of general assumptions imposed on our models of financial markets. We consider throughout, unless explicitly stated otherwise, the case of a so-called *frictionless market,* meaning that: all investors are price-takers, all parties have the same access to the relevant information, there are no transaction costs or commissions, and all assets are assumed to be perfectly divisible and liquid. There is no restriction whatsoever on the size of a bank credit, and the lending and borrowing rates are equal. Finally, individuals are allowed to sell short any security and receive full use of the proceeds (of course, restitution is required for payoffs made to securities held short). Unless otherwise specified, by an *option* we shall mean throughout a European option, giving the right to exercise the option only at the expiry date. In mathematical terms, the problem of pricing of American options is closely related to *optimal stopping* problems. Unfortunately, closed-form expressions for the prices of American options are rarely available; for instance, no closed-form solution is available for the price of an American put option in the now classic framework of the Black-Scholes option pricing model.

A *European call option* written on a common stockis a financial security that gives its holder the right (but not the obligation) to buy the underlying stock on a prespecified date and for a prespecified price. The act of making this transaction is referred to as *exercising* the option. If an option is not exercised, we say it is *abandoned.* Another class of options comprises so-called *American options.* These may be exercised at any time on or before the prespecified date. The prespecified fixed price, say K, is termed the *strike* or *exercise* price; the terminal date, denoted by T in what follows, is called the *expiry date* or *maturity.* It should be emphasized that an option gives the holder the right to do something; however, the holder is not obliged to exercise this right. In order to purchase an option contract, an investor needs to pay an option's price (or *premium*) to a second party at the initial date when the contract is entered into.

Let us denote by S_T the stock price at the terminal date T. It is natural to assume that S_T is not known at time 0, hence S_T gives rise to uncertainty in our model. We argue that from the perspective of the option holder, the payoff g at expiry date T from a European call option is given by the formula

$$g(S_T) = (S_T - K)^+ \stackrel{\text{def}}{=} \max\{S_T - K, 0\}, \tag{1.1}$$

that is to say

$$g(S_T) = \begin{cases} S_T - K & \text{if } S_T > K \text{ (option is exercised)}, \\ 0 & \text{if } S_T \leq K \text{ (option is abandoned)}. \end{cases}$$

In fact, if at the expiry date T the stock price is lower than the strike price, the holder of the call option can purchase an underlying stock directly on a spot (i.e., cash) market, paying less than K. In other words, it would be irrational to exercise the option, at least for an investor who prefers more wealth to less. On the other hand, if at the expiry date the stock price is greater than K, an investor should exercise his right to buy the underlying stock at the strike price K. Indeed, by selling the stock immediately at the spot market, the holder of the call option is able to realize an instantaneous net profit $S_T - K$ (note that transaction costs and/or commissions are ignored here). In contrast to a call option, a *put option* gives its holder the right to sell the underlying asset by a certain date for a prespecified price. Using the same notation as above, we arrive at the following expression for the payoff h at maturity T from a European put option

$$h(S_T) = (K - S_T)^+ \stackrel{\text{def}}{=} \max\{K - S_T, 0\}, \tag{1.2}$$

or more explicitly

$$h(S_T) = \begin{cases} 0 & \text{if } S_T \geq K \text{ (option is abandoned)}, \\ K - S_T & \text{if } S_T < K \text{ (option is exercised)}. \end{cases}$$

It follows immediately that the payoffs of call and put options satisfy the following simple but useful equality

$$g(S_T) - h(S_T) = (S_T - K)^+ - (K - S_T)^+ = S_T - K. \tag{1.3}$$

The last equality can be used, in particular, to derive the so-called *put-call parity* relationship for option prices. Basically, put-call parity means that the price of a European put option is determined by the price of a European call option with the same strike and expiry date, the current price of the underlying asset, and the properly discounted value of the strike price.

1.2.1 One-period Spot Market

Let us start by considering an elementary example of an option contract.

Example 1.2.1 Assume that the current stock price is \$280, and after three months the stock price may either rise to \$320 or decline to \$260. We shall find the rational price of a 3-month European call option with strike price $K = \$280$, provided that the simple risk-free interest rate r for 3-month deposits and loansis $r = 5\%$.

Suppose that the subjective probability of the price rise is 0.2, and that of the fall is 0.8; these assumptions correspond, loosely, to a so-called *bear market.* Note that the word *subjective* means that we take the point of view of a particular individual. Generally speaking, the two parties involved in an option contract may have (and usually do have) differing assessments of these probabilities. To model a *bull market* one may assume, for example, that the first probability is 0.8, so that the second is 0.2.

Let us focus first on the bear market case. The terminal stock price S_T may be seen as a random variable on a probability space $\Omega = \{\omega_1, \omega_2\}$ with a probability measure $\mathbf{P}$ given by

$$\mathbf{P}\{\omega_1\} = 0.2 = 1 - \mathbf{P}\{\omega_2\}.$$

Formally, S_T is a function $S_T : \Omega \to R_+$ given by the following formula

$$S_T(\omega) = \begin{cases} S^u = 320, & \text{if } \omega = \omega_1, \\ S^d = 260, & \text{if } \omega = \omega_2. \end{cases}$$

Consequently, the terminal option's payoff $X = C_T = (S_T - K)^+$ satisfies

$$C_T(\omega) = \begin{cases} C^u = 40, & \text{if } \omega = \omega_1, \\ C^d = 0, & \text{if } \omega = \omega_2. \end{cases}$$

Note that the expected value under $\mathbf{P}$ of the discounted option's payoff equals

$$\mathbf{E}_{\mathbf{P}}\left((1+r)^{-1}C_T\right) = 0.2 \times 40 \times (1.05)^{-1} = 7.62.$$

It is clear that the above expectation depends on the choice of the probability measure $\mathbf{P}$; that is, it depends on the investor's assessment of the market. For a call option, the expectation corresponding to the case of a bull market would be greater than that which assumes a bear market. In our example, the expected value of the discounted payoff from the option under the bull market hypothesis equals 30.48. Still, to construct a reliable model of a financial market, one has to guarantee the uniqueness of the price of any derivative security. This can be done by applying the concept of the so-called replicating portfolio, which we will now introduce.

1.2.2 Replicating Portfolios

The two-state option pricing model presented below was developed independently by Sharpe (1978) and Rendleman and Bartter (1979) (a point worth mentioning is that the ground-breaking papers of Black and Scholes (1973) and Merton (1973), who examined the arbitrage pricing of options in a continuous-time framework, were published five years earlier). The idea is to construct a portfolio at time 0 which replicates exactly the option's terminal payoff at time T. Let $\phi = \phi_0 = (\alpha_0, \beta_0) \in \mathbf{R}^2$ denote a portfolio of an investor with a short position in one call option. More precisely, let α_0 stand for the number of shares of stock held at time 0, and β_0 be the amount of money deposited on a bank account or borrowed from a bank. By $V_t(\phi)$ we denote the wealth of this portfolio at dates $t = 0$ and $t = T$; that is, the payoff from the portfolio ϕ at given dates. It should be emphasized that once the portfolio is set up at time 0, it remains fixed until the terminal date T. Therefore, for its wealth process $V(\phi)$ we have

$$V_0(\phi) = \alpha_0 S_0 + \beta_0 \quad \text{and} \quad V_T(\phi) = \alpha_0 S_T + \beta_0(1+r). \tag{1.4}$$

We say that a portfolio ϕ *replicates* the option's terminal payoff whenever $V_T(\phi) = C_T$, that is, if

$$V_T(\phi)(\omega) = \begin{cases} V^u(\phi) = \alpha_0 S^u + (1+r)\beta_0 = C^u, & \text{if } \omega = \omega_1, \\ V^d(\phi) = \alpha_0 S^d + (1+r)\beta_0 = C^d, & \text{if } \omega = \omega_2. \end{cases}$$

For the data of Example 1.2.1, the portfolio ϕ is determined by the following system of linear equations

$$\begin{cases} 320\,\alpha_0 + 1.05\,\beta_0 = 40, \\ 260\,\alpha_0 + 1.05\,\beta_0 = 0, \end{cases}$$

with unique solution $\alpha_0 = 2/3$ and $\beta_0 = -165.08$. Observe that for every call we are short, we hold α_0 of stock[3] and the dollar amount β_0 in riskless bonds in the hedging portfolio. Put another way, by purchasing shares and borrowing against them in the right proportion, we are able to replicate an option position. (Actually, one can easily check that this property holds for any *contingent claim* X which settles at time T.) It is natural to define the *manufacturing cost* C_0 of a call option as the initial investment needed to construct a replicating portfolio, i.e.,

$$C_0 = V_0(\phi) = \alpha_0 S_0 + \beta_0 = (2/3) \times 280 - 165.08 = 21.59.$$

It should be stressed that in order to determine the manufacturing cost of a call we did not need to know the probability of the rise or fall of the stock price. In other words, it appears that the manufacturing cost is invariant with respect to individual assessments of market behavior. In particular, it is identical under the bull and bear market hypotheses. To determine the *rational* price of a call we have used the option's strike price, the current value of the stock price, the range of fluctuations in the stock price (that is, the future levels of the stock price), and the risk-free rate of

[3]We shall refer to the number of shares held for each call sold as the *hedge ratio*. Basically, to *hedge* means to reduce risk by making transactions that reduce exposure to market fluctuations.

interest. The investor's transactions and the corresponding cash flows may be summarized by the following two exhibits

$$\text{at time } t = 0 \quad \begin{cases} \text{one written call option} & C_0, \\ \alpha_0 \text{ shares purchased} & -\alpha_0 S_0, \\ \text{amount of cash borrowed} & \beta_0, \end{cases}$$

and

$$\text{at time } t = T \quad \begin{cases} \text{payoff from the call option} & -C_T, \\ \alpha_0 \text{ shares sold} & \alpha_0 S_T, \\ \text{loan paid back} & -\hat{r}\beta_0, \end{cases}$$

where $\hat{r} = 1+r$. It should be observed that no net initial investment is needed to establish the above portfolio; that is, the portfolio is costless. On the other hand, for each possible level of stock price at time T, the hedge exactly breaks even on the option's expiry date. Also, it is easy to verify that if the call were not priced at \$21.59, it would be possible for a sure profit to be gained, either by the option's writer (if the option's price were greater than its manufacturing cost) or by its buyer (in the opposite case). Still, the manufacturing cost cannot be seen as a fair price of a claim X, unless the market model is arbitrage-free, in a sense examined below. Indeed, it may happen that the manufacturing cost of a non-negative claim is a strictly negative number. Such a phenomenon contradicts the usual assumption that it is not possible to make riskless profits.

1.2.3 Martingale Measure for a Spot Market

Although, as shown above, subjective probabilities are useless when pricing an option, probabilistic methods play an important role in contingent claims valuation. They rely on the notion of a *martingale,* which is, intuitively, a probabilistic model of a fair game. In order to apply the so-called *martingale method* of derivative pricing, one has to find first a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$, and such that the *discounted* (or *relative*) stock price process S^*, which is defined by the formula

$$S_0^* = S_0, \quad S_T^* = (1+r)^{-1} S_T,$$

follows a $\mathbf{P}^*$-martingale; that is, the equality $S_0^* = \mathbf{E}_{\mathbf{P}^*}(S_T^*)$ holds. Such a probability measure $\mathbf{P}^*$ is called a *martingale measure* for the discounted stock price process S^*. In the case of a two-state model, the probability measure $\mathbf{P}^*$ is easily seen to be uniquely determined (provided it exists) by the following linear equation

$$S_0 = (1+r)^{-1}(p_* S^u + (1-p_*)S^d), \tag{1.5}$$

where $p_* = \mathbf{P}^*\{\omega_1\}$ and $1 - p_* = \mathbf{P}^*\{\omega_2\}$. Solving this equation for p_* yields

$$\mathbf{P}^*\{\omega_1\} = \frac{(1+r)S_0 - S^d}{S^u - S^d}, \quad \mathbf{P}^*\{\omega_2\} = \frac{S^u - (1+r)S_0}{S^u - S^d}. \tag{1.6}$$

Let us now check that the price C_0 coincides with C_0^*, where we write C_0^* to denote the expected value under $\mathbf{P}^*$ of an option's discounted terminal payoff – that is

$$C_0^* \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1} C_T\big) = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}(S_T - K)^+\big).$$

Indeed, using the data of Example 1.2.1 we find $p_* = 17/30$, so that

$$C_0^* = (1+r)^{-1}\big(p_* C^u + (1-p_*)C^d\big) = 21.59 = C_0.$$

Remarks. Observe that since the process S^* follows a $\mathbf{P}^*$-martingale, we may say that the discounted stock price process may be seen as a fair game model in a *risk-neutral economy* – that is, in the stochastic economy in which the probabilities of future stock price fluctuations are determined by the martingale measure $\mathbf{P}^*$. It should be stressed, however, that the fundamental idea of arbitrage pricing is based solely on the existence of a portfolio that hedges perfectly the risk exposure related to uncertain future prices of risky securities. Therefore, the probabilistic properties of the model are not essential. In particular, we do not assume that the real-world economy is actually risk-neutral. On the contrary, the notion of a risk-neutral economy should be seen rather as a technical tool. The aim of introducing the martingale measure is twofold: firstly, it simplifies the explicit evaluation of arbitrage prices of derivative securities; secondly, it describes the arbitrage-free property of a given pricing model for primary securities in terms of the behavior of relative prices. This approach is frequently referred to as the *partial equilibrium approach,* as opposed to the *general equilibrium approach.* Let us stress that in the latter theory the investors' preferences, usually described in stochastic models by means of their (expected) utility functions, play an important role.

To summarize, the notion of an arbitrage price for a derivative security does not depend on the choice of a probability measure in a particular pricing model for primary securities. More precisely, using standard probabilistic terminology, this means that the arbitrage price depends on the support of a subjective probability measure $\mathbf{P}$, but is invariant with respect to the choice of a particular probability measure from the class of mutually equivalent probability measures. In financial terminology, this can be restated as follows: all investors agree on the range of future price fluctuations of primary securities; they may have different assessments of the corresponding subjective probabilities, however.

1.2.4 Absence of Arbitrage

Let us consider a simple two-state, one-period, two-security market model defined on a probability space $\Omega = \{\omega_1, \omega_2\}$ equipped with the σ-fields $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = 2^\Omega$ (i.e., $\mathcal{F}_T$ contains all subsets of Ω), and a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbf{P}\{\omega_1\}$ and $\mathbf{P}\{\omega_2\}$ are strictly positive numbers. The first security is a stock whose price process is modelled as a strictly positive discrete-time process $S = (S_t)_{t\in\{0,T\}}$. We assume that the process S is $(\mathcal{F}_t)$-adapted, i.e., that the random variables S_t are $\mathcal{F}_t$-measurable for $t \in \{0, T\}$. This means that S_0 is a real number, and

$$S_T(\omega) = \begin{cases} S^u & \text{if } \omega = \omega_1, \\ S^d & \text{if } \omega = \omega_2, \end{cases}$$

where, without loss of generality, $S^u > S^d$. The second security is a riskless bond whose price process is $B_0 = 1$, $B_T = 1 + r$ for some real $r \geq 0$. Let Φ stand for the linear space of all stock-bond portfolios $\phi = \phi_0 = (\alpha_0, \beta_0)$, where α_0 and β_0 are real numbers (clearly, the class Φ may be thus identified with $\mathbf{R}^2$). We shall consider the pricing of contingent claims in a security market model $\mathcal{M} = (S, B, \Phi)$. We shall now check that an arbitrary *contingent claim* X which settles at time T (i.e., any $\mathcal{F}_T$-measurable real-valued random variable) admits a unique replicating portfolio in our market model. In other words, an arbitrary contingent claim X is *attainable* in the market model $\mathcal{M}$. Indeed, if

$$X(\omega) = \begin{cases} X^u & \text{if } \omega = \omega_1, \\ X^d & \text{if } \omega = \omega_2, \end{cases}$$

then the replicating portfolio ϕ is determined by the following system of linear equations

$$\begin{cases} \alpha_0 S^u + (1+r)\beta_0 = X^u, \\ \alpha_0 S^d + (1+r)\beta_0 = X^d, \end{cases} \tag{1.7}$$

which admits a unique solution

$$\alpha_0 = \frac{X^u - X^d}{S^u - S^d}, \quad \beta_0 = \frac{X^d S^u - X^u S^d}{(1+r)(S^u - S^d)}, \tag{1.8}$$

for arbitrary values of X^u and X^d. Consequently, an arbitrary contingent claim X admits a unique *manufacturing cost* $\pi_0(X)$ in $\mathcal{M}$ which is given by the formula

$$\pi_0(X) \stackrel{\text{def}}{=} V_0(\phi) = \alpha_0 S_0 + \beta_0 = \frac{X^u - X^d}{S^u - S^d} S_0 + \frac{X^d S^u - X^u S^d}{(1+r)(S^u - S^d)}. \tag{1.9}$$

As already mentioned, the manufacturing cost of a strictly positive contingent claim may appear to be a negative number, in general. If this were the case, there would be a profitable riskless trading strategy (so-called *arbitrage opportunity*) involving only the stock and riskless borrowing and lending. To exclude such situations, which are clearly inconsistent with any broad notion of a rational market equilibrium (as it is common to assume that investors are *non-satiated,* meaning that they prefer more wealth to less), we have to impose further essential restrictions on our market model.

Definition 1.2.1 We say that a security pricing model $\mathcal{M}$ is *arbitrage-free* if there is no portfolio $\phi \in \Phi$ for which

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0 \quad \text{and} \quad \mathbf{P}\{V_T(\phi) > 0\} > 0. \tag{1.10}$$

A portfolio ϕ for which the set (1.10) of conditions is satisfied is called an *arbitrage opportunity.* A *strong arbitrage opportunity* is a portfolio ϕ for which

$$V_0(\phi) < 0 \quad \text{and} \quad V_T(\phi) \geq 0. \tag{1.11}$$

It is customary to take either (1.10) or (1.11) as the definition of an arbitrage opportunity. Note, however, that both notions are not necessarily equivalent. We are in a position to introduce the notion of an arbitrage price; that is, the price derived using the no-arbitrage arguments.

Definition 1.2.2 Suppose that the security market $\mathcal{M}$ is arbitrage-free. Then the manufacturing cost $\pi_0(X)$ is called the *arbitrage price* of X at time 0 in security market $\mathcal{M}$.

As the next result shows, under the absence of arbitrage in a market model, the manufacturing cost may be seen as the unique rational price of a given contingent claim – that is, the unique price compatible with any rational market equilibrium.

Proposition 1.2.1 *Suppose that the spot market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free. Let H stand for the rational price process of some attainable contingent claim X; more explicitly, $H_0 \in \mathbf{R}$ and $H_T = X$. Let Φ_H denote the class of all portfolios in stock, bond and derivative security H. Then the spot market (S, B, H, Φ_H) is arbitrage-free if and only if $H_0 = \pi_0(X)$.*

Proof. Since the proof is straightforward, it is left to the reader. □

1.2.5 Optimality of Replication

Let us show that replication is, in a sense, an optimal way of hedging. Firstly, we say that a portfolio ϕ *perfectly hedges* against X if $V_T(\phi) \geq X$, that is, whenever

$$\begin{cases} \alpha_0 S^u + (1+r)\beta_0 \geq X^u, \\ \alpha_0 S^d + (1+r)\beta_0 \geq X^d. \end{cases} \tag{1.12}$$

The minimal initial cost of a perfect hedging portfolio against X is called the *seller's price* of X, and it is denoted by $\pi_0^s(X)$. Let us check that $\pi_0^s(X) = \pi_0(X)$. By denoting $c = V_0(\phi)$, we may rewrite (1.12) as follows

$$\begin{cases} \alpha_0(S^u - S_0(1+r)) + c(1+r) \geq X^u, \\ \alpha_0(S^d - S_0(1+r)) + c(1+r) \geq X^d. \end{cases} \tag{1.13}$$

It is trivial to check that the minimal $c \in \mathbf{R}$ for which (1.13) holds is actually that value of c for which inequalities in (1.13) become equalities. This means that the replication appears to be the least expensive way of perfect hedging for the seller of X. Let us now consider the other party of the contract, i.e., the buyer of X. Since the buyer of X can be seen as the seller of $-X$, the associated problem is to minimize $c \in \mathbf{R}$, subject to the following constraints

$$\begin{cases} \alpha_0(S^u - S_0(1+r)) + c(1+r) \geq -X^u, \\ \alpha_0(S^d - S_0(1+r)) + c(1+r) \geq -X^d. \end{cases}$$

It is clear that the solution to this problem is $\pi^s(-X) = -\pi(X) = \pi(-X)$, so that replication appears to be optimal for the buyer also. We conclude that the least price the seller is ready to accept for X equals the maximal amount the buyer is ready to pay for it. If we define the *buyer's price* of X, denoted by $\pi_0^b(X)$, by setting $\pi_0^b(X) = -\pi_0^s(-X)$, then

$$\pi_0^s(X) = \pi_0^b(X) = \pi_0(X);$$

that is, all prices coincide. This shows that in a two-state, arbitrage-free model, the arbitrage price of any contingent claim can be defined using the optimality criterion. It appears that such an approach to arbitrage pricing can be extended to other models; we prefer, however, to define the arbitrage price as that value of the price which excludes arbitrage opportunities. Indeed, the fact that observed market prices are close to arbitrage prices predicted by a suitable stochastic model should be explained by the presence of the traders known as *arbitrageurs*[4] on financial markets, rather than by the clever investment decisions of most market participants.

The next proposition explains the role of the so-called *risk-neutral* economy in arbitrage pricing of derivative securities. Observe that the important role of risk preferences in classic equilibrium asset pricing theory is left aside in the present context. Notice, however, that the use of a martingale measure $\mathbf{P}^*$ in arbitrage pricing corresponds to the assumption that all investors are risk-neutral, meaning that they do not differentiate between all riskless and risky investments with the same expected rate of return. The arbitrage valuation of derivative securities is thus done as if an economy actually were *risk-neutral.* Formula (1.14) shows that the arbitrage price of a contingent claim X can be found by first modifying the model so that the stock earns at the riskless rate, and then computing the expected value of the discounted claim (to the best of our knowledge, this method of computing the price was discovered in Cox and Ross (1976)).

Proposition 1.2.2 *The spot market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free if and only if the discounted stock price process S^* admits a martingale measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$. In this case, the arbitrage price at time 0 of any contingent claim X which settles at time T is given by the risk-neutral valuation formula*

$$\pi_0(X) = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}X\big), \tag{1.14}$$

or explicitly

$$\pi_0(X) = \frac{S_0(1+r) - S^d}{S^u - S^d}\,\frac{X^u}{1+r} + \frac{S^u - S_0(1+r)}{S^u - S^d}\,\frac{X^d}{1+r}. \tag{1.15}$$

Proof. We know already that the martingale measure for S^* equivalent to $\mathbf{P}$ exists if and only if the unique solution p_* of equation (1.5) satisfies $0 < p_* < 1$. Suppose there is no equivalent martingale measure for S^*; for instance, assume that $p_* \geq 1$. Our aim is to construct explicitly an arbitrage opportunity in the market model (S, B, Φ). To this end, observe that the inequality $p_* \geq 1$ is equivalent to $(1+r)S_0 \geq S^u$ (recall that S^u is always greater than S^d). The portfolio $\phi = (-1, S_0)$ satisfies $V_0(\phi) = 0$ and

$$V_T(\phi) = \begin{cases} -S^u + (1+r)S_0 \geq 0 & \text{if } \omega = \omega_1, \\ -S^d + (1+r)S_0 > 0 & \text{if } \omega = \omega_2, \end{cases}$$

so that ϕ is indeed an arbitrage opportunity. On the other hand, if $p_* \leq 0$, then the inequality $S^d \geq (1+r)S_0$ holds, and it is easily seen that in this case the portfolio $\psi = (1, -S_0) = -\phi$ is an arbitrage opportunity. Finally, if $0 < p_* < 1$ for any portfolio ϕ satisfying $V_0(\phi) = 0$, then by virtue of (1.9) and (1.6) we get

$$p_* V^u(\phi) + (1 - p_*)V^d(\phi) = 0$$

so that $V^d(\phi) < 0$ when $V^u(\phi) > 0$ and $V^d(\phi) > 0$ if $V^u(\phi) < 0$. This shows that there are no arbitrage opportunities in $\mathcal{M}$ when $0 < p_* < 1$. To prove formula (1.14) it is enough to compare it

[4] An *arbitrageur* is that market participant who consistently uses the price discrepancies to make (almost) risk-free profits. Arbitrageurs are relatively few, but they are far more active than most long-term investors.

with (1.9). Alternatively, we may observe that for the unique portfolio $\phi = (\alpha_0, \beta_0)$ which replicates the claim X, we have

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}X\big) &= \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}V_T(\phi)\big) = \mathbf{E}_{\mathbf{P}^*}(\alpha_0 S_T^* + \beta_0) \\ &= \alpha_0 S_0^* + \beta_0 = V_0(\phi) = \pi_0(X), \end{aligned}$$

so that we are done. □

Remarks. The choice of the bond price process as a discount factor is not essential. Suppose, on the contrary, that we have chosen the stock price S as a *numeraire.* In other words, we now consider the bond price B discounted by the stock price S

$$B_t^* = B_t / S_t$$

for $t \in \{0, T\}$. The martingale measure $\bar{\mathbf{P}}$ for the process B^* is determined by the equality $B_0 = \mathbf{E}_{\bar{\mathbf{P}}}(B_T^*)$, or explicitly

$$\bar{p}\,\frac{1+r}{S^u} + \bar{q}\,\frac{1+r}{S^d} = \frac{1}{S_0}\,, \tag{1.16}$$

where $\bar{q} = 1 - \bar{p}$. One finds that

$$\bar{\mathbf{P}}\{\omega_1\} = \bar{p} = \Big(\frac{1}{S^d} - \frac{1}{(1+r)S_0}\Big)\,\frac{S^u S^d}{S^u - S^d} \tag{1.17}$$

and

$$\bar{\mathbf{P}}\{\omega_2\} = \bar{q} = \Big(\frac{1}{S^u} - \frac{1}{(1+r)S_0}\Big)\,\frac{S^u S^d}{S^d - S^u}\,. \tag{1.18}$$

It is easy to show that the properly modified version of the risk-neutral valuation formula has the following form

$$\pi_0(X) = S_0\,\mathbf{E}_{\bar{\mathbf{P}}}\big(S_T^{-1}X\big), \tag{1.19}$$

where X is a contingent claim which settles at time T. It appears that in some circumstances the choice of the stock price as a numeraire is more convenient than that of the savings account.

Let us apply this approach to the call option of Example 1.2.1. One finds easily that $\bar{p} = 0.62$, and thus formula (1.19) gives

$$\hat{C}_0 = S_0\,\mathbf{E}_{\bar{\mathbf{P}}}\big(S_T^{-1}\,(S_T - K)^+\big) = 21.59 = C_0,$$

as expected.

1.2.6 Put Option

We refer once again to Example 1.2.1. However, we shall now focus on a European put option instead of a call option. Since the buyer of a put option has the right to sell a stock at a given date T, the terminal payoff from the option is now $P_T = (K - S_T)^+$, i.e.,

$$P_T(\omega) = \begin{cases} P^u = 0, & \text{if } \omega = \omega_1, \\ P^d = 20, & \text{if } \omega = \omega_2, \end{cases}$$

where we have taken, as before, $K=\$280$. The portfolio $\phi = (\alpha_0, \beta_0)$ which replicates the European put option is thus determined by the following system of linear equations

$$\begin{cases} 320\,\alpha_0 + 1.05\,\beta_0 = 0, \\ 260\,\alpha_0 + 1.05\,\beta_0 = 20, \end{cases}$$

so that $\alpha_0 = -1/3$ and $\beta_0 = 101.59$. Consequently, the arbitrage price P_0 of the European put option equals

$$P_0 = -(1/3) \times 280 + 101.59 = 8.25.$$

Notice that the number of shares in a replicating portfolio is negative. This means that an option writer who wishes to hedge risk exposure should sell short at time 0 the number $-\alpha_0 = 1/3$ shares of stock for each sold put option. The proceeds from the short-selling of shares, as well as the option's premium, are invested in an interest-earning account. To find the arbitrage price of the put option we may alternatively apply Proposition 1.2.2. By virtue of (1.14), with $X = P_T$, we get

$$P_0 = \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-1}P_T\big) = 8.25.$$

Finally, the put option value can also be found by applying the following relationship between the prices of call and put options.

Corollary 1.2.1 *The following put-call parity relationship is valid*

$$C_0 - P_0 = S_0 - (1+r)^{-1}K. \tag{1.20}$$

Proof. The formula is an immediate consequence of equality (1.3) and the pricing formula (1.14) applied to the claim $S_T - K$. □

It is worthwhile to mention that relationship (1.20) is universal – that is, it does not depend on the choice of the model (the only assumption we need to make is the additivity of the price). Using the put-call parity, we can calculate once again the arbitrage price of the put option. Formula (1.20) yields immediately

$$P_0 = C_0 - S_0 + (1+r)^{-1}K = 8.25.$$

For ease of further reference, we shall write down explicit formulae for the call and put price in the one-period, two-state model. We assume, as usual, that $S^u > K > S^d$. Then

$$C_0 = \frac{S_0(1+r) - S^d}{S^u - S^d}\,\frac{S^u - K}{1+r}, \tag{1.21}$$

and

$$P_0 = \frac{S^u - S_0(1+r)}{S^u - S^d}\,\frac{K - S^d}{1+r}. \tag{1.22}$$

1.3 Futures Call and Put Options

We will first describe very succinctly the main features of futures contracts, which are reflected in stochastic models of futures markets to be developed later. As in the previous section, we will focus mainly on the arbitrage pricing of European call and put options; clearly, instead of the spot price of the underlying asset, we will now consider its futures price. The model of futures prices we adopt here is quite similar to the one used to describe spot prices. Still, due to the specific features of futures contracts used to set up a replicating strategy, one has to modify significantly the way in which the payoff from a portfolio is defined.

1.3.1 Futures Contracts and Futures Prices

A *futures contract* is an agreement to buy or sell an asset at a certain date in the future for a certain price. The important feature of these contracts is that they are traded on exchanges. Consequently, the authorities need to define precisely all the characteristics of each futures contract in order to make trading possible. More importantly, the *futures price* – the price at which a given futures contract is entered into – is determined on a given futures exchange by the usual law of demand and supply (in a similar way as for spot prices of listed stocks). Futures prices are therefore settled daily and the quotations are reported in the financial press. A futures contract is referred to by its delivery month, however an exchange specifies the period within that month when delivery must be made. The exchange specifies the amount of the asset to be delivered for one contract, as well as some additional details when necessary (e.g., the quality of a given commodity or the maturity of a bond). From our perspective, the most fundamental feature of a futures contract is the way the contract is settled. The procedure of daily settlement of futures contracts is called *marking to market.* A futures contract is worth zero when it is entered into; however, each investor is required to deposit funds into a *margin account.* The amount that should be deposited when the contract is entered into is known as the *initial margin.* At the end of each trading day, the balance of the investor's margin account is adjusted in a way that reflects daily movements of futures prices. To be more specific, if an investor assumes a long position, and on a given day the futures price rises, the balance of the margin account will also increase. Conversely, the balance of the margin account of any party with a short position in this futures contract will be properly reduced. Intuitively, it is thus possible to argue that futures contracts are actually closed out after each trading day, and then start afresh the next trading day. Obviously, to offset a position in a futures contract, an investor enters into the opposite trade to the original one. Finally, if the delivery period is reached, the delivery is made by the party with a short position.

1.3.2 One-period Futures Market

It will be convenient to start this section with a simple example which, in fact, is a straightforward modification of Example 1.2.1 to a futures market.

Example 1.3.1 Let $f_t = f_S(t, T^*)$ be a one-period process which models the futures price of a certain asset S, for the settlement date $T^* \geq T$. We assume that $f_0 = 280$, and

$$f_T(\omega) = \begin{cases} f^u = 320, & \text{if } \omega = \omega_1, \\ f^d = 260, & \text{if } \omega = \omega_2, \end{cases}$$

where $T = 3$ months.[5] We consider a 3-month European futures call option with strike price $K = \$280$. As before, we assume that the simple risk-free interest rate for 3-month deposits and loans is $r = 5\%$.

The payoff from the futures call option $C_T^f = (f_T - K)^+$ equals

$$C_T^f(\omega) = \begin{cases} C^{fu} = 40, & \text{if } \omega = \omega_1, \\ C^{fd} = 0, & \text{if } \omega = \omega_2. \end{cases}$$

A portfolio ϕ which replicates the option is composed of α_0 futures contracts and β_0 units of cash invested in riskless bonds (or borrowed). The wealth process $V_t^f(\phi)$, $t \in \{0, T\}$, of this portfolio equals $V_0^f(\phi) = \beta_0$, since futures contracts are worthless when they are first entered into. Furthermore, the terminal wealth of ϕ is

$$V_T^f(\phi) = \alpha_0\,(f_T - f_0) + (1+r)\beta_0, \tag{1.23}$$

where the first term on the right-hand side represents gains (or losses) from the futures contract, and the second corresponds to a savings account (or loan). Note that (1.23) reflects the fact that futures contracts are marked to market daily (that is, after each period in our model). A portfolio $\phi = (\alpha_0, \beta_0)$ is said to replicate the option when $V_T^f = C_T^f$, or more explicitly, if the equalities

$$V_T^f(\omega) = \begin{cases} \alpha_0(f^u - f_0) + (1+r)\beta_0 = C^{fu}, & \text{if } \omega = \omega_1, \\ \alpha_0(f^d - f_0) + (1+r)\beta_0 = C^{fd}, & \text{if } \omega = \omega_2 \end{cases}$$

are satisfied. For Example 1.3.1, this gives the following system of linear equations

$$\begin{cases} 40\,\alpha_0 + 1.05\,\beta_0 = 40, \\ -20\,\alpha_0 + 1.05\,\beta_0 = 0, \end{cases}$$

[5] Notice that in the present context, the knowledge of the settlement date T^* of a futures contract is not essential. It is sufficient to assume that $T^* \geq T$.

yielding $\alpha_0 = 2/3$ and $\beta_0 = 12.70$. The manufacturing cost of a futures call option is thus $C_0^f = V_0^f(\phi) = \beta_0 = 12.70$. Similarly, the unique portfolio replicating a sold put option is determined by the following conditions

$$\begin{cases} \;\;\, 40\,\alpha_0 + 1.05\,\beta_0 = 0, \\ -20\,\alpha_0 + 1.05\,\beta_0 = 20, \end{cases}$$

so that $\alpha_0 = -1/3$ and $\beta_0 = 12.70$ in this case. Consequently, the manufacturing costs of put and call futures options are equal in our example. As we shall see soon, this is not a pure coincidence; in fact, by virtue of formula (1.29) below, the prices of call and put futures options are equal when the option's strike price coincides with the initial futures price of the underlying asset. The above considerations may be summarized by means of the following exhibits (note that β_0 is a positive number)

$$\text{at time } t = 0 \quad \begin{cases} \text{one sold futures option} & C_0^f, \\ \text{futures contracts} & 0, \\ \text{cash deposited in a bank} & -\beta_0 = -C_0^f, \end{cases}$$

and

$$\text{at time } t = T \quad \begin{cases} \text{option's payoff} & -C_T^f, \\ \text{profits/losses from futures} & \alpha_0\,(f_T - f_0), \\ \text{cash withdrawal} & \hat{r}\beta_0, \end{cases}$$

where, as before, $\hat{r} = 1 + r$.

1.3.3 Martingale Measure for a Futures Market

We are looking now for a probability measure $\tilde{\mathbf{P}}$ which makes the futures price process (with no discounting) follow a $\tilde{\mathbf{P}}$-martingale. A probability $\tilde{\mathbf{P}}$, if it exists, is thus determined by the equality

$$f_0 = \mathbf{E}_{\tilde{\mathbf{P}}}\big(f_T\big) = \tilde{p}\, f^u + (1 - \tilde{p})\, f^d. \tag{1.24}$$

It is easily seen that

$$\tilde{\mathbf{P}}\{\omega_1\} = \tilde{p} = \frac{f_0 - f^d}{f^u - f^d}, \quad \tilde{\mathbf{P}}\{\omega_2\} = 1 - \tilde{p} = \frac{f_u - f_0}{f^u - f^d}\,. \tag{1.25}$$

Using the data of Example 1.3.1, one finds easily that $\tilde{p} = 1/3$. Consequently, the expected value under the probability $\tilde{\mathbf{P}}$ of the discounted payoff from the futures call option equals

$$\tilde{C}_0^f = \mathbf{E}_{\tilde{\mathbf{P}}}\big((1 + r)^{-1}(f_T - K)^+\big) = 12.70 = C_0^f\,.$$

This illustrates the fact that the martingale approach may be used also in the case of futures markets, with a suitable modification of the notion of a martingale measure.

Remarks. Using the traditional terminology of mathematical finance, we may conclude that the risk-neutral futures economy is characterized by the fair-game property of the process of a futures price. Remember that the risk-neutral spot economy is the one in which the discounted stock price (as opposed to the stock price itself) models a fair game.

1.3.4 Absence of Arbitrage

In this subsection, we shall study a general two-state, one-period model of a futures price. We consider the filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \{0,T\}}, \mathbf{P})$ introduced in Sect. 1.2.4. The first process, which intends to model the dynamics of the futures price of a certain asset for the fixed settlement date $T^* \geq T$, is an adapted and strictly positive process $f_t = f_S(t, T^*)$, $t = 0, T$. More specifically, f_0 is assumed to be a real number, and f_T is the following random variable

$$f_T(\omega) = \begin{cases} f^u, & \text{if } \omega = \omega_1, \\ f^d, & \text{if } \omega = \omega_2, \end{cases}$$

where, by convention, $f^u > f^d$. The second security is, as in the case of a spot market, a riskless bond whose price process is $B_0 = 1$, $B_T = 1 + r$ for some real $r \geq 0$. Let Φ^f stand for the linear space of all futures contracts-bonds portfolios $\phi = \phi_0 = (\alpha_0, \beta_0)$; it may be, of course, identified with the linear space $\mathbf{R}^2$. The wealth process $V^f(\phi)$ of any portfolio equals

$$V_0(\phi) = \beta_0, \quad \text{and} \quad V_T^f(\phi) = \alpha_0(f_T - f_0) + (1+r)\beta_0 \tag{1.26}$$

(it is useful to compare these formulae with (1.4)). We shall study the valuation of derivatives in the futures market model $\mathcal{M}^f = (f, B, \Phi^f)$. It is easily seen that an arbitrary contingent claim X which settles at time T admits a unique replicating portfolio $\phi \in \Phi^f$. Put another way, all contingent claims which settle at time T are *attainable* in the market model $\mathcal{M}^f$. In fact, if X is given by the formula

$$X(\omega) = \begin{cases} X^u & \text{if } \omega = \omega_1, \\ X^d & \text{if } \omega = \omega_2, \end{cases}$$

then its replicating portfolio $\phi \in \Phi^f$ may be found by solving the following system of linear equations

$$\begin{cases} \alpha_0(f^u - f_0) + (1+r)\beta_0 = X^u, \\ \alpha_0(f^d - f_0) + (1+r)\beta_0 = X^d. \end{cases} \tag{1.27}$$

The unique solution of (1.27) is

$$\alpha_0 = \frac{X^u - X^d}{f^u - f^d}, \quad \beta_0 = \frac{X^u(f_0 - f^d) + X^d(f^u - f_0)}{(1+r)(f^u - f^d)}. \tag{1.28}$$

Consequently, the *manufacturing cost* $\pi_0^f(X)$ in $\mathcal{M}^f$ equals

$$\pi_0^f(X) \stackrel{\text{def}}{=} V_0^f(\phi) = \beta_0 = \frac{X^u(f_0 - f^d) + X^d(f^u - f_0)}{(1+r)(f^u - f^d)}. \tag{1.29}$$

We say that a model $\mathcal{M}^f$ of the futures market is *arbitrage-free* if there are no arbitrage opportunities in the class Φ^f of trading strategies. The following simple result provides necessary and sufficient conditions for the arbitrage-free property of $\mathcal{M}^f$.

Proposition 1.3.1 *The futures market $\mathcal{M}^f = (f, B, \Phi^f)$ is arbitrage-free if and only if the process f that models the futures price admits a (unique) martingale measure $\tilde{\mathbf{P}}$ equivalent to $\mathbf{P}$. In this case, the arbitrage price at time 0 of any contingent claim X which settles at time T equals*

$$\pi_0^f(X) = \mathbf{E}_{\tilde{\mathbf{P}}}\big((1+r)^{-1}X\big), \tag{1.30}$$

or explicitly

$$\pi_0^f(X) = \frac{f_0 - f^d}{f^u - f^d}\,\frac{X^u}{1+r} + \frac{f^u - f_0}{f^u - f^d}\,\frac{X^d}{1+r}. \tag{1.31}$$

Proof. If there is no martingale measure for f which is equivalent to $\mathbf{P}$, we have either $\tilde{p} \geq 1$ or $\tilde{p} \leq 0$. In the first case, we have $f_0 - f^d \geq f^u - f^d$ and thus $f_0 \geq f^u > f^d$. Consequently, a portfolio $\phi = (-1, 0)$ is an arbitrage opportunity. Similarly, when $\tilde{p} \leq 0$ the inequalities $f_0 \leq f^d < f^u$ are valid. Therefore the portfolio $\phi = (1, 0)$ is an arbitrage opportunity. Finally, if $0 < \tilde{p} < 1$ and for some $\phi \in \Phi^f$ we have $V_0^f(\phi) = 0$, then it follows from (1.29) that

$$\frac{f_0 - f^d}{f^u - f^d} V^{fu} + \frac{f^u - f_0}{f^u - f^d} V^{fd} = 0$$

so that $V^{fd} < 0$ if $V^{fu} > 0$, and $V^{fu} < 0$ when $V^{fd} > 0$. This shows that the market model $\mathcal{M}^f$ is arbitrage-free if and only if the process f admits a martingale measure equivalent to $\mathbf{P}$. The valuation formula (1.30) now follows by (1.25)–(1.29). □

When the price of the futures call option is already known, in order to find the price of the corresponding put option one may use the following relation, which is an immediate consequence of equality (1.3) and the pricing formula (1.30)

$$C_0^f - P_0^f = (1+r)^{-1}(f_0 - K). \tag{1.32}$$

It is now obvious that the equality $C_0^f = P_0^f$ is valid if and only if $f_0 = K$; that is, when the current futures price and the strike price of the option are equal. Equality (1.32) is referred to as the *put-call parity relationship* for futures options.

1.3.5 One-period Spot/Futures Market

Consider an arbitrage-free, one-period spot market (S, B, Φ) described in Sect. 1.2. Moreover, let $f_t = f_S(t,T), t \in \{0, T\}$ be the process of futures prices with the underlying asset S and for the maturity date T. In order to preserve consistency with the financial interpretation of the futures price, we have to assume that $f_T = S_T$. Our aim is to find the right value f_0 of the futures price at time 0; that is, that level of the price f_0 which excludes arbitrage opportunities in the combined spot/futures market. In such a market, trading in stocks, bonds, as well as entering into futures contracts is allowed.

Corollary 1.3.1 *The futures price at time 0 for the delivery date T of the underlying asset S which makes the spot/futures market arbitrage-free equals $f_0 = (1+r)S_0$.*

Proof. Suppose an investor enters at time 0 into one futures contract. The payoff of his position at time T corresponds to a time T contingent claim $X = f_T - f_0 = S_T - f_0$. Since it costs nothing to enter a futures contract we should have

$$\pi_0(X) = \pi_0(S_T - f_0) = 0,$$

or equivalently

$$\pi_0(X) = S_t - (1+r)^{-1} f_0 = 0.$$

This proves the asserted formula. Alternatively, one can check that if the futures price f_0 were different from $(1+r)S_0$, this would lead to arbitrage opportunities in the spot/futures market. □

1.4 Forward Contracts

A *forward contract* is an agreement, signed at the initial date 0, to buy or sell an asset at a certain future time T (called *delivery date* or *maturity* in what follows) for a prespecified price K, referred to as the *delivery price.* In contrast to stock options and futures contracts, forward contracts are not traded on exchanges. By convention, the party who agrees to buy the underlying asset at time T for the delivery price K is said to assume a *long position* in a given contract. Consequently, the other party, who is obliged to sell the asset at the same date for the price K, is said to assume a *short position.* Since a forward contract is settled at maturity and a party in a long position is obliged to buy an asset worth S_T at maturity for K, it is clear that the payoff from the long position (from the short position, respectively) in a given forward contract with a stock S being the underlying asset corresponds to the time T contingent claim X ($-X$, respectively), where

$$X = S_T - K. \tag{1.33}$$

It should be emphasized that there is no cash flow at the time the forward contract is entered into. In other words, the price (or value) of a forward contract at its initiation is zero. Notice, however, that for $t > 0$, the value of a forward contract may be negative or positive. As we shall now see, a forward contract is worthless at time 0 provided that a judicious choice of the delivery price K is made.

Before we end this section, we shall find the rational delivery price for a forward contract. To this end, let us introduce first the following definition which is, of course, consistent with typical features of a forward contract. Recall that, typically, there is no cash flow at the initiation of a forward contract.

Definition 1.4.1 The delivery price K that makes a forward contract worthless at initiation is called the *forward price* of an underlying financial asset S for the settlement date T.

Note that we use here the adjective *financial* in order to emphasize that the storage costs, which have to be taken into account when studying forward contracts on commodities, are neglected. In the case of a dividend-paying stock, in the calculation of the forward price, it is enough to substitute S_0 with $S_0 - \hat{I}_0$, where $\hat{I}_0$ is the present value of all future dividend payments during the contract's lifetime (cf. Sect. 3.2.7).

Proposition 1.4.1 *Assume that the one-period, two-state security market model* (S, B, Φ) *is arbitrage-free. Then the forward price at time 0 for the settlement date* T *of one share of stock* S *equals* $F_S(0,T) = (1+r)S_0$.

Proof. We shall apply the martingale method of Proposition 1.2.2. By applying formulae (1.14) and (1.33), we get

$$\pi_0(X) = \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-1}X\big) = \mathbf{E}_{\mathbf{P}^*}(S_T^*) - \hat{r}^{-1}K = S_0 - \hat{r}^{-1}K = 0, \tag{1.34}$$

where $\hat{r} = 1 + r$. It is now apparent that $F_S(0,T) = (1+r)S_0$. □

By combining Corollary 1.3.1 with the above proposition, we conclude that in a one-period model of a spot market, the futures and forward prices of financial assets for the same settlement date are equal.

1.5 Options of American Style

An option of *American style* (or briefly, an *American option*) is an option contract in which not only the decision whether to exercise the option or not, but also the choice of the exercise time, is at the discretion of the option's holder. The exercise time cannot be chosen after the option's expiry date T. Hence, in our simple one-period model, the strike price can either coincide with the initial date 0, or with the terminal date T. Notice that the value (or the price) at the terminal date of the American call or put option written on any asset equals the value of the corresponding European option with the same strike price K. Therefore, the only unknown quantity is the price of the American option at time 0. In view of the early exercise feature of the American option, the concept of perfect replication of the terminal option's payoff is not adequate for valuation purposes. To determine this value, we shall make use of the general rule of absence of arbitrage in the market model. By definition, the arbitrage price at time 0 of the American option should be set in such a way that trading in American options would not destroy the arbitrage-free feature the market. We will first show that the American call written on a stock that pays no dividends during the option's lifetime is always equivalent to the European call; that is, that both options necessarily have identical prices at time 0. As we shall see in what follows, such a property is not always true in the case of American put options; that is, American and European puts are not equivalent, in general.

We place ourselves once again within the framework of a one-period spot market $\mathcal{M} = (S, B, \Phi)$, as specified in Sect. 1.2.1. It will be convenient to assume that European options are traded securities in our market. For $t = 0, T$, let us denote by C_t^a and P_t^a the arbitrage price at time t of the American call and put, respectively. It is obvious that $C_T^a = C_T$ and $P_T^a = P_T$. As mentioned earlier, both arbitrage prices C_0^a and P_0^a will be determined using the following property: if the market $\mathcal{M} = (S, B, \Phi)$ is arbitrage-free, then the market with trading in stocks, bonds and American options should remain arbitrage-free. It should be noted that it is not evident a priori that the last property determines in a unique way the values of C_0^a and P_0^a. We assume throughout that the inequalities $S^d < S_0(1+r) < S^u$ hold and the strike price satisfies $S^d < K < S^u$. Otherwise, either the market model would not be arbitrage-free, or valuation of the option would be a trivial matter.

Proposition 1.5.1 *Assume that the risk-free interest rate r is a non-negative real number. Then the arbitrage price C_0^a of an American call option in the arbitrage-free market model $\mathcal{M} = (S, B, \Phi)$ coincides with the price C_0 of the European call option with the same strike price K.*

Proof. Assume, on the contrary, that $C_0^a \neq C_0$. Suppose first that $C_0^a > C_0$. Notice that the arbitrage price C_0 satisfies

$$C_0 = p_* \frac{S^u - K}{1+r} = \frac{(1+r)S_0 - S^d}{S^u - S^d} \, \frac{S^u - K}{1+r} > S_0 - K, \tag{1.35}$$

if $r \geq 0$. It is now straightforward to check that there exists an arbitrage opportunity in the market. In fact, to create a riskless profit, it is sufficient to sell the American call option at C_0^a, and simultaneously buy the European call option at C_0. If European options are not traded, one may, of course, create a replicating portfolio for the European call at initial investment C_0. The above portfolio is easily seen to lead to a riskless profit, independently from the decision regarding the exercise time made by the holder of the American call. If, on the contrary, the price C_0^a were strictly smaller than C_0, then by selling European calls and buying American calls, one would be able to create a profitable riskless portfolio. □

It is worthwhile to observe that inequality (1.35) is valid in a more general setup. Indeed, if $r \geq 0$, $S_0 > K$, and S_T is a $\mathbf{P}^*$-integrable random variable, then we have always

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(S_T - K)^+) &\geq \Big(\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}S_T) - (1+r)^{-1}K\Big)^+ \\ &= (S_0 - (1+r)^{-1}K)^+ \geq S_0 - K, \end{aligned}$$

where the first inequality follows by Jensen's inequality. Notice that in the case of the put option we get merely

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(K - S_T)^+) &\geq \Big(\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}K - (1+r)^{-1}S_T)\Big)^+ \\ &= ((1+r)^{-1}K - S_0)^+ > K - S_0, \end{aligned}$$

where the last inequality holds provided that $-1 < r < 0$. If $r = 0$, we obtain

$$\mathbf{E}_{\mathbf{P}^*}((1+r)^{-1}(K - S_T)^+) = K - S_0.$$

Finally, if $r > 0$, no obvious relationship between P_0 and $S_0 - K$ is available. This feature suggests that the counterpart of Proposition 1.5.1 – the case of American put – should be more interesting.

Proposition 1.5.2 *Assume that $r > 0$. Then $P_0^a = P_0$ if and only if the inequality*

$$K - S_0 \leq \frac{S^u - (1+r)S_0}{S^u - S^d} \, \frac{K - S^d}{1+r} \tag{1.36}$$

is valid. Otherwise, $P_0^a = K - S_0 > P_0$. If $r = 0$, then invariably $P_0^a = P_0$.

Proof. In view of (1.22), it is clear that inequality (1.36) is equivalent to $P_0 \geq K - S_0$. Suppose first that the last inequality holds. If, in addition, $P_0^a > P_0$ ($P_0^a < P_0$, respectively), by selling the American put and buying the European put (by buying the American put and selling the European put, respectively) one creates a profitable riskless strategy. Hence, $P_0^a = P_0$ in this case.[6] Suppose

[6]To be formal, we need to check that no arbitrage opportunities are present if $P_0^a = P_0$ and (1.36) holds.

now that (1.36) fails to hold – that is, $P_0 < K - S_0$, and assume that $P_0^a \neq K - S_0$. We wish to show that P_0^a should be set to be $K - S_0$, otherwise arbitrage opportunities arise. Actually, if P_0^a were strictly greater that $K - S_0$, the seller of an American put would be able to lock in a profit by perfectly hedging exposure using the European put acquired at a strictly lower cost P_0. If, on the contrary, inequality $P_0^a < K - S_0$ were true, it would be profitable to buy the American put and exercise it immediately. Summarizing, if (1.36) fails to hold, the arbitrage price of the American put is strictly greater than the price of the European put. Finally, one verifies easily that if the holder of the American put fails to exercise it at time 0, the option's writer is still able to lock in a profit. Hence, if (1.36) fails to hold, the American put should be exercised immediately, otherwise arbitrage opportunities would arise in the market. For the last statement, observe that if $r = 0$, then inequality (1.36), which now reads

$$K - S_0 \leq \frac{S^u - S_0}{S^u - S^d}\,(K - S^d),$$

is easily seen to be valid (it is enough to take $K = S^d$ and $K = S^u$). □

The above results suggest the following general "rational" exercise rule in a discrete-time framework: at any time t before the option's expiry, find the maximal expected payoff over all admissible exercise rules and compare the outcome with the payoff obtained by exercising the option immediately. If the latter value is greater, exercise the option immediately, otherwise go one step further. In fact, one checks easily that the price at time 0 of an American call or put option may be computed as the maximum expected value of the payoff over all exercises, provided that the expectation in question is taken under the martingale probability measure. The last feature distinguishes arbitrage pricing of American options from the typical optimal stopping problems, in which maximization of expected payoffs takes place under a subjective (or actual) probability measure rather than under an artificial martingale measure. We conclude that a simple argument that the rational option's holder will always try to maximize the expected payoff of the option at exercise is not sufficient to determine arbitrage prices of American claims. A more precise statement would read: the American put option should be exercised by its holder at the same date as it is exercised by a risk-neutral individual whose objective is to maximize the discounted expected payoff of the option; otherwise arbitrage opportunities would arise in the market. It will be useful to formalize the concept of an *American contingent claim.*

Definition 1.5.1 A contingent claim of American style (or shortly, *American claim*) is a pair $X^a = (X_0, X_T)$, where X_0 is a real number and X_T is a random variable. We interpret X_0 and X_T as the payoffs received by the holder of the American claim X^a if he chooses to exercise it at time 0 and at time T, respectively.

Notice that in our present setup, the only admissible *exercise times* are the initial date and the expiry date, say $\tau_0 = 0$ and $\tau_1 = T$. By convention, we say that an option is exercised at expiry date T if it is not exercised prior to that date, even when its terminal payoff equals zero (so that in fact the option is abandoned). We assume also, for simplicity, that $T = 1$. Then we may formulate the following corollary to Propositions 1.5.1–1.5.2, whose proof is left as exercise.

Corollary 1.5.1 *The arbitrage prices of an American call and an American put option in the arbitrage-free market model $\mathcal{M} = (S, B, \Phi)$ are given by*

$$C_0^a = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau}(S_\tau - K)^+\big)$$

and

$$P_0^a = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau}(K - S_\tau)^+\big)$$

respectively, where $\mathcal{T}$ denotes the class of all exercise times. More generally, if $X^a = (X_0, X_T)$ is an arbitrary contingent claim of American style, then its arbitrage price $\pi(X^a)$ in $\mathcal{M} = (S, B, \Phi)$ equals

$$\pi_0(X^a) = \max_{\tau \in \mathcal{T}} \mathbf{E}_{\mathbf{P}^*}\big((1+r)^{-\tau} X_\tau\big), \quad \pi_T(X^a) = X_T.$$

1.5.1 General No-arbitrage Inequalities

It is clear that the following property is valid in any discrete- or continuous-time, arbitrage-free market.

Price monotonicity rule. In any model of an arbitrage-free market, if X_T and Y_T are two European contingent claims, where $X_T \geq Y_T$, then $\pi_t(X_T) \geq \pi_t(Y_T)$ for every $t \in [0,T]$, where $\pi_t(X_T)$ and $\pi_t(Y_T)$ denote the arbitrage prices at time t of X_T and Y_T, respectively. Moreover, if $X_T > Y_T$, then $\pi_t(X_T) > \pi_t(Y_T)$ for every $t \in [0,T]$.

For the sake of notational convenience, a constant (non-negative) rate r will now be interpreted as a continuously compounded rate of interest. Hence, the price at time t of one dollar to be received at time $T \geq t$ equals $e^{-r(T-t)}$; in other words, the savings account process equals $B_t = e^{rt}$ for every $t \in [0,T]$.

Proposition 1.5.3 *Let C_t and P_t (C_t^a and P_t^a, respectively) stand for the arbitrage prices at time t of European (American, respectively) call and put options, with strike price K and expiry date T. Then the following inequalities are valid for every $t \in [0,T]$*

$$(S_t - Ke^{-r(T-t)})^+ \leq C_t = C_t^a \leq S_t, \tag{1.37}$$

$$(Ke^{-r(T-t)} - S_t)^+ \leq P_t \leq K, \tag{1.38}$$

and

$$(K - S_t)^+ \leq P_t^a \leq K. \tag{1.39}$$

The put-call parity relationship, which in the case of European options reads

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \tag{1.40}$$

takes, in the case of American options, the form of the following inequalities

$$S_t - K \leq C_t^a - P_t^a \leq S_t - Ke^{-r(T-t)}. \tag{1.41}$$

Proof. All inequalities may be derived by constructing appropriate portfolios at time t and holding them to the terminal date. Let us consider, for instance, the first one. Consider the following portfolios, A and B. Portfolio A consists of one European call and $Ke^{-r(T-t)}$ of cash; portfolio B contains only one share of stock. The value of the first portfolio at time T equals

$$C_T + K = (S_T - K)^+ + K = \max\{S_T, K\} \geq S_T,$$

while the value of portfolio B is exactly S_T. Hence, the arbitrage price of portfolio A at time t dominates the price of portfolio B – that is, $C_t + Ke^{-r(T-t)} \geq S_t$, for every $t \in [0,T]$. Since the price of the option is non-negative, this proves the first inequality in (1.37). All remaining inequalities in (1.37)–(1.39) may be verified by means of similar arguments. To check that $C_t^a = C_t$, we consider the following portfolios: portfolio A – one American call option and $Ke^{-r(T-t)}$ of cash; and portfolio B – one share of stock. If the call option is exercised at some date $t^* \in [t,T]$, then the value of portfolio A at time t^* equals $S_{t^*} - K + Ke^{-r(T-t^*)} < S_{t^*}$, while the value of B is S_{t^*}. On the other hand, the value of portfolio A at the terminal date T is $\max\{S_T, K\}$, hence it dominates the value of portfolio B, which is S_T. This means that early exercise of the call option would contradict our general price monotonicity rule. A justification of relationship (1.40) is straightforward, as $C_T - P_T = S_T - K$. To justify the second inequality in (1.41), notice that in view of (1.40) and the obvious inequality $P_t^a \geq P_t$, we get

$$P_t^a \geq P_t = C_t^a + Ke^{-r(T-t)} - S_t, \quad \forall\, t \in [0,T].$$

The proof of the first inequality in (1.41) goes as follows. Take the two following portfolios: portfolio A – one American call and K units of cash; and portfolio B – one American put and one share of stock. If the put option is exercised at time $t^* \in [t,T]$, then the value of portfolio B at time t^* is K. On the other hand, the value of portfolio A at this date equals $C_t + Ke^{r(t^*-t)} \geq K$. Therefore, portfolio A is more valuable at time t than portfolio B; that is $C_t^a + K \geq P_t^a + S_t$ for every $t \in [0,T]$. □

Chapter 2

The Cox-Ross-Rubinstein Model

A European call option written on one share of a stock S, which pays no dividends during the option's lifetime, is formally equivalent to the claim X whose payoff at time T is contingent on the stock price S_T, and equals

$$X = (S_T - K)^+ \stackrel{\text{def}}{=} \max\{S_T - K, 0\}. \tag{2.1}$$

Therefore, the call option value (or price) C_T at the expiry date T equals simply $C_T = (S_T - K)^+$. We assume here that the terminal date T is represented by a natural number. Our first aim is to evaluate the option price C_t at any instant $t = 0, \dots, T$, when the price of a risky asset (a stock) is modelled by the Cox et al. (1979) multiplicative binomial lattice (this will be referred to as the CRR model of a stock price hereafter).

2.1 The CRR Model of a Stock Price

We consider a discrete-time model of a financial market with the set of dates $0, 1, \dots, T^*$, and with two primary traded securities: a risky asset, referred to as a *stock,* and a risk-free investment, called a *savings account* (or a *bond*). The savings account yields a constant rate of return $r \geq 0$ over each time period $[t, t+1]$, meaning that its price process B equals (by convention $B_0 = 1$)

$$B_t = (1 + r)^t = \hat{r}^t, \quad \forall\, t \leq T^*, \tag{2.2}$$

where $\hat{r} = 1 + r$. The stock price process S satisfies

$$\xi_{t+1} = S_{t+1}/S_t \in \{u, d\} \tag{2.3}$$

for $t = 0, \dots, T^* - 1$, where $d < 1 + r < u$ are given real numbers and S_0 is a strictly positive constant.

To provide a simple probabilistic model of the stock price, we assume that ξ_t, $t = 1, \dots, T^*$ are mutually independent random variables on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with identical probability law

$$\mathbf{P}\{\xi_t = u\} = p = 1 - \mathbf{P}\{\xi_t = d\}$$

for every $t = 1, \dots, T^*$. The stock price process S can be modelled by setting

$$S_t = S_0 \prod_{j=1}^{t} \xi_j, \quad \forall\, t \leq T^*. \tag{2.4}$$

Equivalently,

$$S_t = S_0 \exp\Big(\sum_{j=1}^{t} \zeta_j\Big), \quad \forall\, t \leq T^*, \tag{2.5}$$

where ζ_j's are independent, identically distributed random variables such that

$$\mathbf{P}\{\zeta_j = \ln u\} = p = 1 - \mathbf{P}\{\zeta_j = \ln d\}, \quad \forall\, t \le T^*.$$

Due to representation (2.5), the stock price S given by (2.4) is frequently referred to as an *exponential random walk.* It should be stressed, however, that the assumption that the random variables ξ_t, $t = 1, \ldots, T^*$, are mutually independent is not essential for our further purposes; we may make this assumption, without loss of generality, for mathematical convenience. As we will see in what follows, the arbitrage price of any European or American contingent claim in the binomial model of a financial market is independent of the choice of the probability of upward and downward movements of the stock price at any particular node.

2.1.1 The CRR Option Pricing Formula

Let us introduce some notation. For any $t \le T$, we write α_t to denote the number of shares held during the period $[t, t+1)$, while β_t stands for the dollar investment in the savings account during this period. To determine the arbitrage price of a call option we shall show, using backward induction, that by adjusting his or her portfolio $\phi_t = (\alpha_t, \beta_t)$, $t = 0, \ldots, T-1$, at the beginning of each period, an investor is able to mimic the payoff of an option at time T for every state. We shall refer to this fact by saying that the contingent claim $X = (S_T - K)^+$ admits a unique, dynamic, replicating, self-financing strategy. Though for concreteness we concentrate on European options, it will soon become clear that this property remains valid for any European contingent claim X of the form $X = g(S_T)$. Let us mention that it is common to refer to such claims as *path-independent* (as opposed to *path-dependent* claims, which have the form $X = h(S_0, S_1, \ldots, S_T)$; that is, they may depend on the whole sample path). The valuation of a put option can be reduced, through the put-call parity relationship, to that of a call option.

Given a fixed maturity date $1 \le T \le T^*$, we start our analysis by considering the last period before the expiry date, $[T-1, T]$. We assume that a portfolio which replicates the terminal payoff of a call option is established at time $T-1$, and remains fixed until the expiry date T. In other words, we need to find the composition of a portfolio $\phi_{T-1} = (\alpha_{T-1}, \beta_{T-1})$ at the beginning of the last period in such a way that its terminal wealth $V_T(\phi)$, which satisfies

$$V_T(\phi) = \alpha_{T-1} S_T + \beta_{T-1} \hat{r}, \tag{2.6}$$

replicates the option payoff C_T; that is, $V_T(\phi) = C_T$. Combining (2.1) with (2.6), we get the following equality

$$\alpha_{T-1} S_T + \beta_{T-1} \hat{r} = (S_T - K)^+. \tag{2.7}$$

By virtue of our assumptions, we have $S_T = S_{T-1}\xi_{T-1}$; therefore, we may rewrite (2.7) in a more explicit form

$$\begin{cases} \alpha_{T-1} u S_{T-1} + \beta_{T-1} \hat{r} = (u S_{T-1} - K)^+, \\ \alpha_{T-1} d S_{T-1} + \beta_{T-1} \hat{r} = (d S_{T-1} - K)^+. \end{cases}$$

Such a system of linear equations can be solved easily, yielding

$$\alpha_{T-1} = \frac{(uS_{T-1} - K)^+ - (dS_{T-1} - K)^+}{S_{T-1}(u-d)}, \tag{2.8}$$

and

$$\beta_{T-1} = \frac{u(dS_{T-1} - K)^+ - d(uS_{T-1} - K)^+}{\hat{r}(u-d)}. \tag{2.9}$$

Furthermore, the wealth $V_{T-1}(\phi)$ of this portfolio at time $T-1$ equals

$$\begin{aligned} V_{T-1}(\phi) &= \alpha_{T-1} S_{T-1} + \beta_{T-1} \\ &= \hat{r}^{-1}\Big(p_*(uS_{T-1} - K)^+ + (1-p_*)(dS_{T-1} - K)^+\Big), \end{aligned}$$

where $p_* = (\hat{r} - d)/(u - d)$. Assuming the absence of arbitrage in the market model,[1] the wealth $V_{T-1}(\phi)$ agrees with the value (that is, the *arbitrage price*) of a call option at time $T-1$. Put another way, the equality $C_{T-1} = V_{T-1}(\phi)$ is valid. We will continue the above procedure by considering the time-period $[T-2, T-1]$. In this step, we are searching for a portfolio $\phi_{T-2} = (\alpha_{T-2}, \beta_{T-2})$ which is created at time $T-2$ in such a way that its wealth at time $T-1$ replicates option value C_{T-1}; that is

$$\alpha_{T-2} S_{T-1} + \beta_{T-2} \hat{r} = C_{T-1}. \tag{2.10}$$

Notice that since $C_{T-1} = V_{T-1}(\phi)$, the dynamic trading strategy ϕ constructed in this way will possess the *self-financing property* at time $T-1$

$$\alpha_{T-2} S_{T-1} + \beta_{T-2} \hat{r} = \alpha_{T-1} S_{T-1} + \beta_{T-1}.$$

Basically, the self-financing feature means that the portfolio is adjusted at time $T-1$ (and more generally, at any trading date) in such a way that no withdrawals or inputs of funds take place. Since $S_{T-1} = S_{T-2}\xi_{T-2}$ and $\xi_{T-2} \in \{u, d\}$, we get the following equivalent form of equality (2.10)

$$\begin{cases} \alpha_{T-2} u S_{T-2} + \beta_{T-2} \hat{r} = C^u_{T-1}, \\ \alpha_{T-2} d S_{T-2} + \beta_{T-2} \hat{r} = C^d_{T-1}, \end{cases} \tag{2.11}$$

where

$$C^u_{T-1} = \frac{1}{\hat{r}} \Big(p_* (u^2 S_{T-2} - K)^+ + (1 - p_*)(ud S_{T-2} - K)^+ \Big)$$

and

$$C^d_{T-1} = \frac{1}{\hat{r}} \Big(p_* (ud S_{T-2} - K)^+ + (1 - p_*)(d^2 S_{T-2} - K)^+ \Big).$$

In view of (2.11), it is evident that

$$\alpha_{T-2} = \frac{C^u_{T-1} - C^d_{T-1}}{S_{T-2}(u - d)}, \quad \beta_{T-2} = \frac{u C^d_{T-1} - d C^u_{T-1}}{\hat{r}(u - d)}.$$

Consequently, the wealth $V_{T-2}(\phi)$ of the portfolio $\phi_{T-2} = (\alpha_{T-2}, \beta_{T-2})$ at time $T-2$ equals

$$\begin{aligned} V_{T-2}(\phi) = \alpha_{T-2} S_{T-2} + \beta_{T-2} &= \frac{1}{\hat{r}} \left(p_* C^u_{T-1} + (1 - p_*) C^d_{T-1} \right) \\ &= \frac{1}{\hat{r}^2} \left(p_*^2 (u^2 S_{T-2} - K)^+ + 2 p_* q_* (ud S_{T-2} - K)^+ + q_*^2 (d^2 S_{T-2} - K)^+ \right). \end{aligned}$$

Using the same arbitrage arguments as in the first step, we argue that the wealth $V_{T-2}(\phi)$ of the portfolio ϕ at time $T-2$ gives the arbitrage price at time $T-2$, i.e., $C_{T-2} = V_{T-2}(\phi)$. It is evident that by repeating the above procedure, one can completely determine the option price at any date $t \le T$, as well as the (unique) trading strategy ϕ that replicates the option. Summarizing, the above reasoning provides a recursive procedure for finding the value of a call with any number of periods to go (note that it extends to the case of any claim X of the form $X = g(S_T)$). It is worthwhile to note that in order to value the option at a given date t and for a given level of the current stock price S_t, it is enough to consider a sub-lattice of the CRR binomial lattice, which starts from S_t and involves $T - t$ periods.

Before we proceed further, let us comment briefly on the *information structure* of the CRR model. Let us denote by $\mathcal{F}^S_t$ the σ-field of all events of $\mathcal{F}$ generated by the observations of the stock price S up to the date t, formally $\mathcal{F}^S_t = \sigma(S_0, \dots, S_t)$ for every $t \le T$, where $\sigma(S_0, \dots, S_t)$ denotes the least σ-field with respect to which the random variables $S_0, \dots, S_t$ are measurable. By construction of the replicating strategy, it is evident that for any fixed t the random variables α_t, β_t which define the portfolio at time t, as well as the wealth $V_t(\phi)$ of this portfolio, are measurable with respect to the σ-field $\mathcal{F}^S_t$.

[1] We will return to this point later in this chapter. Let us only mention here that the necessary and sufficient condition for the absence of arbitrage has the same form as in the case of the one-period model; that is, $d < 1 + r < u$.

For any fixed m, let the function $a_m : \mathbf{R} \to \mathbf{N}^*$ be given by the formula ($\mathbf{N}^*$ stands hereafter for the set of all non-negative integers) $a_m(x) = \min\{\, j \in \mathbf{N}^* \,|\, xu^j d^{m-j} > K \,\}$, where, by convention, $\min \emptyset = \infty$. To simplify the notation, we write

$$\Delta_m(x,j) = \binom{m}{j} p_*^j (1-p_*)^{m-j} (u^j d^{m-j} x - K). \tag{2.12}$$

Proposition 2.1.1 *The arbitrage price of a European call option at time $t = T - m$ is given by the Cox-Ross-Rubinstein valuation formula*

$$C_{T-m} = S_{T-m} \sum_{j=a}^{m} \binom{m}{j} \bar{p}^j (1-\bar{p})^{m-j} - \frac{K}{\hat{r}^m} \sum_{j=a}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \tag{2.13}$$

for $m = 1, \ldots, T$, where $a = a_m(S_{T-m})$, $p_* = (\hat{r} - d)/(u-d)$ and $\bar{p} = p_* u/\hat{r}$. At time $t = T - m - 1$, the unique replicating strategy has the form $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$, where

$$\begin{aligned}
\alpha_{T-m-1} &= \sum_{j=a^d}^{m} \binom{m}{j} \bar{p}^j (1-\bar{p})^{m-j} + \frac{\delta \Delta_m(uS_{T-m-1}, a^u)}{S_{T-m-1}(u-d)}, \\
\beta_{T-m-1} &= -\frac{K}{\hat{r}^{m+1}} \sum_{j=a^d}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} - \frac{d\delta \Delta_m(uS_{T-m-1}, a^u)}{\hat{r}(u-d)},
\end{aligned}$$

where $a^d = a_m(dS_{T-m-1})$, $a^u = a_m(uS_{T-m-1})$ and $\delta = 0$ if $a^d = a^u$ (otherwise, $\delta = 1$).

Proof. Straightforward calculations yield $1 - \bar{p} = d(1-p_*)/\hat{r}$, and thus

$$\bar{p}^j (1-\bar{p})^{m-j} = p_*^j (1-p_*)^{m-j} u^j d^{m-j} / \hat{r}^m.$$

Therefore, formula (2.13) is equivalent to the following

$$\begin{aligned}
C_{T-m} &= \frac{1}{\hat{r}^m} \sum_{j=a}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \Big(u^j d^{m-j} S_{T-m} - K \Big) \\
&= \frac{1}{\hat{r}^m} \sum_{j=0}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \Big(u^j d^{m-j} S_{T-m} - K \Big)^+.
\end{aligned}$$

We will now proceed by induction with respect to m. For $m = 0$, the above formula is manifestly true. Assume now that C_{T-m} is the arbitrage price of a European call option at time $T - m$. We have to select a portfolio $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$ for the period $[T - m - 1, T - m)$ (that is, established at time $T - m - 1$ at each node of the binomial lattice) in such a way that the portfolio's wealth at time $T - m$ replicates the value C_{T-m} of the option. Formally, the wealth of the portfolio $(\alpha_{T-m-1}, \beta_{T-m-1})$ needs to satisfy the relationship

$$\alpha_{T-m-1} S_{T-m} + \beta_{T-m-1} \hat{r} = C_{T-m}, \tag{2.14}$$

which in turn is equivalent to the following pair of equations

$$\begin{cases} \alpha_{T-m-1} u S_{T-m-1} + \beta_{T-m-1} \hat{r} = C^u_{T-m}, \\ \alpha_{T-m-1} d S_{T-m-1} + \beta_{T-m-1} \hat{r} = C^d_{T-m}, \end{cases}$$

where

$$\begin{aligned}
C^u_{T-m} &= \frac{1}{\hat{r}^m} \sum_{j=0}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \Big(u^{j+1} d^{m-j} S_{T-m-1} - K \Big)^+ \\
&= \frac{1}{\hat{r}^m} \sum_{j=a^u}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \Big(u^{j+1} d^{m-j} S_{T-m-1} - K \Big)
\end{aligned}$$

and

$$\begin{aligned}
C^d_{T-m} &= \frac{1}{\hat{r}^m}\sum_{j=0}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}\Big(u^j d^{m-j+1}S_{T-m-1}-K\Big)^+ \\
&= \frac{1}{\hat{r}^m}\sum_{j=a^d}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}\Big(u^j d^{m-j+1}S_{T-m-1}-K\Big).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\alpha_{T-m-1} &= \frac{C^u_{T-m}-C^d_{T-m}}{S_{T-m-1}(u-d)} \\
&= \frac{1}{\hat{r}^m(u-d)}\sum_{j=a^d}^{m}\binom{m}{j}p_*^j q_*^{m-j}\big(u^{j+1}d^{m-j}-u^j d^{m-j+1}\big)+\frac{\delta\Delta_m(uS_{T-m-1},a^u)}{S_{T-m-1}(u-d)} \\
&= \sum_{j=a^d}^{m}\binom{m}{j}\bar{p}^j(1-\bar{p})^{m-j}+\frac{\delta\Delta_m(uS_{T-m-1},a^u)}{S_{T-m-1}(u-d)}\,.
\end{aligned}$$

where we write $q_* = 1-p_*$. Similarly,

$$\begin{aligned}
\beta_{T-m-1} &= \frac{uC^d_{T-m}-dC^u_{T-m}}{\hat{r}(u-d)} \\
&= \frac{1}{\hat{r}^{m+1}(u-d)}\sum_{j=a^d}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}(dK-uK)-\frac{\delta\Delta_m(uS_{T-m-1},a^u)}{\hat{r}(u-d)} \\
&= -\frac{K}{\hat{r}^{m+1}}\sum_{j=a^d}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}-\frac{d\delta\Delta_m(uS_{T-m-1},a^u)}{\hat{r}(u-d)}\,.
\end{aligned}$$

The wealth of this portfolio at time $T-m-1$ equals (note that just established explicit formulas for the replicating portfolio are not employed here)

$$\begin{aligned}
C_{T-m-1} &= \alpha_{T-m-1}S_{T-m-1}+\beta_{T-m-1} \\
&= (u-d)^{-1}\Big(C^u_{T-m}-C^d_{T-m}+\hat{r}^{-1}(uC^d_{T-m}-dC^u_{T-m})\Big) \\
&= \hat{r}^{-1}\left(p_*C^u_{T-m}+(1-p_*)C^d_{T-m}\right) \\
&= \frac{1}{\hat{r}^{m+1}}\Big\{\sum_{j=0}^{m}\binom{m}{j}p_*^{j+1}q_*^{m-j}\big(u^{j+1}d^{m-j}S_{T-m-1}-K\big)^+ \\
&\quad+\sum_{j=0}^{m}\binom{m}{j}p_*^j q_*^{m+1-j}\big(u^j d^{m+1-j}S_{T-m-1}-K\big)^+\Big\} \\
&= \frac{1}{\hat{r}^{m+1}}\Big\{\sum_{j=1}^{m+1}\binom{m}{j-1}p_*^j q_*^{m+1-j}\big(u^j d^{m+1-j}S_{T-m-1}-K\big)^+ \\
&\quad+\sum_{j=0}^{m}\binom{m}{j}p_*^j q_*^{m+1-j}\big(u^j d^{m+1-j}S_{T-m-1}-K\big)^+\Big\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
C_{T-m-1} &= \frac{1}{\hat{r}^{m+1}}\Big\{\,p_*^{m+1}\big(u^{m+1}S_{T-m-1}-K\big)^+ \\
&\quad+\sum_{j=1}^{m}\binom{m}{j}\,p_*^j q_*^{m+1-j}\big(u^j d^{m+1-j}S_{T-m-1}-K\big)^+
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{m} \binom{m}{j-1} p_*^j q_*^{m+1-j} \big(u^j d^{m+1-j} S_{T-m-1} - K\big)^+ \\
&+ q_*^{m+1} \big(d^{m+1} S_{T-m-1} - K\big)^+ \Big\} \\
= \;& \frac{1}{\hat{r}^{m+1}} \sum_{j=0}^{m+1} \binom{m+1}{j} p_*^j q_*^{m+1-j} \big(u^j d^{m+1-j} S_{T-m-1} - K\big)^+,
\end{aligned}
$$

since $\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j}$. This ends the proof of Proposition 2.1.1. □

It is important to notice that the CRR valuation formula (2.13) makes no reference to the subjective probability p. Intuitively, the pricing formula does not depend on the investor's attitudes toward risk. The only assumption made about the behavior of an individual is that all investors prefer more wealth to less wealth, and thus have an incentive to take advantage of riskless profitable investments. Consequently, if arbitrage opportunities were present in the market, no market equilibrium would be possible. This feature of arbitrage-free markets explains the term *partial equilibrium approach,* frequently used in economic literature in relation to arbitrage pricing of derivative securities. To summarize, no matter whether investors are risk-averse or risk-preferring,[2] we obtain identical arbitrage prices for derivative securities. In this context, it is worthwhile to point out that the value $p_* = (\hat{r} - d)/(u - d)$ corresponds to the risk-neutral world – that is, a model of an economy in which all investors are indifferent with respect to risky investments whose discounted wealth is represented by *martingales* (at the intuitive level, a *martingale* may be seen as a formalization of the concept of a fair game).

2.1.2 The Black-Scholes Option Pricing Formula

We will now show that the classic Black-Scholes option valuation formula (2.22) can be obtained from the CRR option valuation result by an asymptotic procedure, using a properly chosen sequence of binomial models. To this end, we need to examine the asymptotic properties of the CRR model when the number of steps goes to infinity and, simultaneously, the size of time and space steps tends to zero in an appropriate way. Let $T > 0$ be a fixed, but arbitrary, real number. For any n of the form $n = 2^k$, we divide the interval $[0, T]$ into n equal subintervals I_j of length $\Delta_n = T/n$, namely $I_j = [j\Delta_n, (j+1)\Delta_n]$ for $j = 0, \ldots, n-1$ (note that n corresponds to T^*). Let us first introduce the modified accumulation factor. We write r_n to denote the riskless rate of return over each interval $I_j = [j\Delta_n, (j+1)\Delta_n]$, hence

$$B_{j\Delta_n} = (1 + r_n)^j, \quad \forall\, j = 0, \ldots, n.$$

It is clear that we deal in fact with a sequence of processes, B^n say; however, for the sake of notational simplicity, we shall usually omit the superscript n in what follows. The same remark applies to the sequence $S = S^n$ of binomial lattices and random variables $\xi_{n,j}$ introduced below. For every n, we assume that the stock price can appreciate over the period I_j by u_n or decline by d_n; that is

$$S_{(j+1)\Delta_n} = \xi_{n,j+1} S_{j\Delta_n} \tag{2.15}$$

for $j = 1, \ldots, n-1$, where for every fixed n, $\xi_{n,j}$'s are random variables with values in the two-element set $\{u_n, d_n\}$. In view of Proposition 2.1.1, we may assume, without loss of generality, that for any n the random variables $\xi_{n,j}$, $j = 1, \ldots, n$ are defined on a common probability space $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$, are mutually independent, and

$$\mathbf{P}_n\{\xi_{n,j} = u_n\} = p = 1 - \mathbf{P}_n\{\xi_{n,j} = d_n\}, \quad \forall\, j = 1, \ldots, n,$$

[2] An interested reader may consult, for instance, Huang and Litzenberger (1988) for the study of the notion of risk preferences under uncertainty.

for some $p \in (0,1)$. Note that the choice of the parameter $p \in (0,1)$ is arbitrary; for instance, we may assume that $p = 1/2$ for every n. In order to guarantee the convergence of the CRR option valuation formula to the Black-Scholes one, we need to impose, in addition, specific restrictions on the asymptotic behavior of the quantities r_n, u_n and d_n. Let us put

$$1 + r_n = e^{r\Delta_n}, \quad u_n = e^{\sigma\sqrt{\Delta_n}}, \quad d_n = u_n^{-1}, \tag{2.16}$$

where $r \geq 0$ and $\sigma > 0$ are given real numbers. As mentioned earlier, we wish to calculate the asymptotic value of the call option price when the number of steps, $T^*(n) = n$, tends to infinity. Assume that $t = j\Delta_n = jT/2^k$ for some natural j and k; that is, t is an arbitrary dyadic number from the interval $[0,T]$. Given any such number, we introduce the sequence $m_n(t)$ by setting

$$m_n(t) = n(T-t)/T, \quad \forall\, n \in \mathbf{N}. \tag{2.17}$$

It is apparent that the sequence $m_n(t)$ has natural values in the set for n sufficiently large. On the other hand, $T - t = m_n(t)\Delta_n$ for every $n \in \mathbf{N}$. Notice also that

$$\lim_{n\to+\infty} (1+r_n)^{-m_n(t)} = \lim_{n\to+\infty} e^{-r\Delta_n m_n(t)} = e^{-r(T-t)}. \tag{2.18}$$

Furthermore, for every $n > r^2\sigma^{-2}T$ we have

$$d_n = u_n^{-1} < \hat{r}_n^{-1} \leq \hat{r}_n < u_n,$$

where $\hat{r}_n = 1 + r_n$. Also, it is not difficult to check that

$$\lim_{n\to+\infty} p_{*,n} = \lim_{n\to+\infty} \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} = 1/2, \tag{2.19}$$

and

$$\lim_{n\to+\infty} \bar{p}_n = \lim_{n\to+\infty} \hat{r}_n^{-1} p_{*,n} u_n = 1/2. \tag{2.20}$$

For a generic value of stock price at time t, $S_t = S_{T-m_n(t)\Delta_n}$, we define

$$a_n(t) = \min\{\, j \in \mathbf{N}^* \mid S_t u_n^j d_n^{m_n(t)-j} > K \,\}. \tag{2.21}$$

The next proposition provides the derivation of the classic Black-Scholes option valuation formula by means of an asymptotic procedure. It should be emphasized that the limit of the CRR option price depends essentially on the choice of sequences u_n and d_n. For the choice of u_n's and d_n's that we have made here, the asymptotic dynamic of the stock price is that of the *geometric Brownian motion* (known also as the geometric Wiener process). This means, in particular, that the asymptotic evolution of the stock price may be described by a stochastic process whose sample paths almost all follow continuous functions; furthermore, the probability law of the continuous-time stock price at any time t is lognormal.

A straightforward analysis of the continuous-time Black-Scholes model, based on the Itô stochastic calculus, is presented in Sect. 3.2.1. The proof of the next result is left to the reader.

Proposition 2.1.2 *The following convergence is valid for any dyadic $t \in [0,T]$*

$$\lim_{n\to+\infty} \sum_{j=a_n(t)}^{m_n(t)} \binom{m_n(t)}{j} \Big\{ S_t\, \bar{p}_n^j \bar{q}_n^{m_n(t)-j} - K\hat{r}_n^{-m_n(t)} p_{*,n}^j q_{*,n}^{m_n(t)-j} \Big\} = C_t,$$

where $\bar{q}_n = 1 - \bar{p}_n$, and C_t is given by the Black-Scholes formula

$$C_t = S_t\, N(d_1(S_t, T-t)) - Ke^{-r(T-t)}\, N(d_2(S_t, T-t)), \tag{2.22}$$

where

$$d_1(s,t) = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, \tag{2.23}$$

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{t} = \frac{\ln(s/K) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, \tag{2.24}$$

and N stands for the standard Gaussian cumulative distribution function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad \forall\, x \in \mathbf{R}.$$

Remarks. Let us stress that for a different choice of the sequences u_n, d_n and r_n, the stock price may asymptotically follow a stochastic process with discontinuous sample paths. For instance, if we put $u_n = u$ and $d_n = e^{ct/n}$ then the stock process will follow asymptotically a log-Poisson process, examined by Cox and Ross (1975). Essentially, this interesting feature is due to the fact that we deal here with a triangular array of random variables as opposed to an infinite sequence of independent random variables. For this reason, the class of asymptotic probability laws is much larger than in the case of the classic central limit theorem.

2.2 Probabilistic Approach

The aim of this section is to give a purely probabilistic interpretation of the CRR option valuation formula. We will proceed along the lines suggested in remarks preceding Proposition 2.1.1. We start by introducing a finite probability space Ω; namely, for a fixed natural number T^*, we take

$$\Omega = \{\, \omega = (a_1, \ldots, a_{T^*}) \,|\, a_j = 1 \text{ or } a_j = 0 \,\}.$$

In the present context, it will be sufficient to consider a specific class $\mathcal{P}$ of probability measures on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the σ-field of all subsets of Ω, i.e., $\mathcal{F} = 2^{\Omega}$. For any elementary event $\omega = (a_1, \ldots, a_T^*)$, we define its probability $\mathbf{P}(\omega)$ by setting

$$\mathbf{P}\{\omega\} = p^{\sum_{j=1}^{T^*} a_j} (1-p)^{T^* - \sum_{j=1}^{T^*} a_j},$$

where $0 < p < 1$ is a fixed real number. We write $\mathcal{P}$ to denote the class of all probability measures of this form on $(\Omega, \mathcal{F})$. It is clear that any element $\mathbf{P} \in \mathcal{P}$ is uniquely determined by the value of the parameter p. Note that for every $\mathbf{P} \in \mathcal{P}$, the probability of any event $A \in \mathcal{F}$ equals $\mathbf{P}\{A\} = \sum_{\omega \in A} \mathbf{P}\{\omega\}$. For any $j = 1, \ldots, T^*$, let us denote by A_j the event $A_j = \{\, \omega \in \Omega \,|\, a_j = 1\}$. It is easily seen that the events A_j, $j = 1, \ldots, T^*$, are mutually independent; moreover, $\mathbf{P}(A_j) = p$ for every j. We are in a position to define a sequence of random variables ξ_j, $j = 1, \ldots, T^*$ by setting

$$\xi_j(\omega) = u a_j + d(1 - a_j), \quad \forall\, \omega \in \Omega, \tag{2.25}$$

where, without loss of generality, $0 < d < u$. The random variables ξ_j are easily seen to be independent and identically distributed, with the following probability law under $\mathbf{P}$

$$\mathbf{P}\{\xi_j = u\} = p = 1 - \mathbf{P}\{\xi_j = d\}, \quad \forall\, t \le T^*. \tag{2.26}$$

As anticipated, the sequence ξ_j will be used to model stock price fluctuations in a probabilistic version of the CRR binomial lattice.

2.2.1 Conditional Expectations

Let us start by considering a finite decomposition of the underlying probability space. We say that a finite collection $\mathcal{D} = \{D_1, \dots, D_k\}$ of non-empty subsets of Ω is a *decomposition* of Ω if the sets $D_1, \dots, D_k$ are pairwise disjoint; that is, if $D_i \cap D_j = \emptyset$ for every $i \neq j$, and the equality $D_1 \cup D_2 \cup \ldots \cup D_k = \Omega$ holds. A random variable ψ on Ω is called *simple* if it admits a representation

$$\psi(\omega) = \sum_{i=1}^{m} x_i \, \mathrm{I}_{D_i(\psi)}(\omega), \tag{2.27}$$

where $D_i(\psi) = \{\, \omega \in \Omega \,|\, \psi(\omega) = x_i\}$ and x_i, $i = 1, \dots, m$ are real numbers satisfying $x_i \neq x_j$ for $i \neq j$. For a simple random variable ψ, we denote by $\mathcal{D}(\psi)$ the decomposition $\{D_1(\psi), \dots, D_m(\psi)\}$ generated by ψ. It is clear that if Ω is a finite set, then any random variable $\psi : \Omega \to \mathbf{R}$ is simple.

Definition 2.2.1 For any decomposition $\mathcal{D}$ of Ω and any event $A \in \mathcal{F}$, the *conditional probability* of A with respect to $\mathcal{D}$ is defined by the formula

$$\mathbf{P}(A \,|\, \mathcal{D}) = \sum_{j=1}^{k} \mathbf{P}(A \,|\, D_j) \, \mathrm{I}_{D_j}. \tag{2.28}$$

Moreover, if ψ is a simple random variable with the representation (2.27), its *conditional expectation* given $\mathcal{D}$ equals

$$\mathbf{E}_{\mathbf{P}}(\psi \,|\, \mathcal{D}) = \sum_{i=1}^{m} x_i \, \mathbf{E}_{\mathbf{P}}(I_{D_i(\psi)} \,|\, \mathcal{D}) = \sum_{i=1}^{m} \sum_{j=1}^{k} x_i \, \mathbf{P}(D_i(\psi) \,|\, D_j) \, \mathrm{I}_{D_j}. \tag{2.29}$$

Observe that the conditional expectation $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \mathcal{D})$ is constant on each set D_j from $\mathcal{D}$. Let η be another simple random variable on Ω, i.e.,

$$\eta(\omega) = \sum_{l=1}^{r} y_l \, \mathrm{I}_{D_l}(\eta)(\omega), \tag{2.30}$$

where $D_l(\eta) = \{\, \omega \in \Omega \,|\, \eta(\omega) = y_l \,\}$ and $y_i \neq y_j$ for $i \neq j$. Suppose that η and ψ are simple random variables given from (2.27) and (2.30), respectively. Then, by the definition of conditional expectation, $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta)$ coincides with $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \mathcal{D}(\eta))$, and thus

$$\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta) = \mathbf{E}_{\mathbf{P}}(\psi \,|\, \mathcal{D}(\eta)) = \sum_{i=1}^{m} x_i \, \mathbf{E}_{\mathbf{P}}(I_{D_k(\psi)} \,|\, \mathcal{D}(\eta)), \tag{2.31}$$

where the second equality follows by (2.29). Consequently,

$$\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta) = \sum_{i=1}^{m} \sum_{l=1}^{r} x_i \, \mathbf{P}(D_i(\psi) \,|\, D_l(\eta)) \, \mathrm{I}_{D_l(\eta)} = \sum_{l=1}^{r} c_l \, \mathrm{I}_{D_l(\eta)}, \tag{2.32}$$

where $c_l = \sum_{i=1}^{m} x_i \, \mathbf{P}(D_i(\psi) \,|\, D_l(\eta))$ for $l = 1, \dots, r$. Note that $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta)$ does not depend on the particular values of η. More precisely, if for two random variables η_1 and η_2 we have $\mathcal{D}(\eta_1) = \mathcal{D}(\eta_2)$, then $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta_1) = \mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta_2)$. On the other hand, however, by virtue of (2.32) it is clear that $\mathbf{E}_{\mathbf{P}}(\psi \,|\, \eta) = g(\eta)$, where the function $g : \{y_1, \dots, y_r\} \to \mathbf{R}$ is given by the formula $g(y_l) = c_l$ for $l = 1, 2, \dots, r$. We shall now define the conditional expectation of a random variable with respect to an arbitrary σ-field $\mathcal{G}$.

Definition 2.2.2 Suppose that ψ is an integrable random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ (i.e., $\mathbf{E}_{\mathbf{P}}(|\psi|) < \infty$). For an arbitrary σ-field $\mathcal{G}$ of subsets of Ω satisfying $\mathcal{G} \subseteq \mathcal{F}$ (i.e., a *sub-σ-field* of $\mathcal{F}$), the *conditional* *expectation* $\mathbf{E_P}(\psi \,|\, \mathcal{G})$ of ψ with respect to $\mathcal{G}$ is defined by the following conditions: (i) $\mathbf{E_P}(\psi \,|\, \mathcal{G})$ is $\mathcal{G}$-measurable, i.e., $\{\omega \in \Omega \mid \mathbf{E_P}(\psi \,|\, \mathcal{G})(\omega) \le a\} \in \mathcal{G}$ for any $a \in \mathbf{R}$; (ii) for an arbitrary event $A \in \mathcal{G}$, we have

$$\int_A \mathbf{E_P}(\psi \,|\, \mathcal{G})\, d\mathbf{P} = \int_A \psi\, d\mathbf{P}. \tag{2.33}$$

It is well known that the conditional expectation exists and is unique (up to the $\mathbf{P}$-a.s. equivalence of random variables).

Remarks. If $\mathcal{D}$ is a (finite) decomposition of Ω, the family of all unions of sets from $\mathcal{D}$, together with an empty set, forms a σ-field of subsets of Ω. We denote it by $\sigma(\mathcal{D})$ and we call it the σ-field *generated* by the decomposition[3] $\mathcal{D}$. It can be easily checked that if $\mathcal{G} = \sigma(\mathcal{D})$, then $\mathbf{P}(A \,|\, \mathcal{G}) = \mathbf{P}(A \,|\, \mathcal{D})$, so that the conditional expectation with respect to the σ-field $\sigma(\mathcal{D})$ coincides with the conditional expectation with respect to the decomposition $\mathcal{D}$.

Definition 2.2.3 For an arbitrary simple random variable η, the σ-field $\mathcal{F}(\eta) = \sigma(\mathcal{D}(\eta))$ is called the *σ-field generated by* η, or briefly, the *natural σ-field of* η. More generally, for any random variable η, the *σ-field generated by* η is the least σ-field of subsets of Ω with respect to which η is measurable. It is denoted by either $\sigma(\eta)$ or $\mathcal{F}(\eta)$.

In the case of a real-valued random variable η, it can be shown that the σ-field $\mathcal{F}(\eta)$ is the smallest σ-field that contains all events of the form $\{\omega \in \Omega \,|\, \eta(\omega) \le x\}$, where x is an arbitrary real number. More generally, if $\eta = (\eta^1, \dots, \eta^d)$ is a d-dimensional random variable, then

$$\mathcal{F}(\eta) = \sigma\big(\{\omega \in \Omega \,|\, \eta^1 \le x_1, \dots, \eta^d \le x_d\} \,|\, x_1, \dots, x_d \in \mathbf{R}\big).$$

For any $\mathbf{P}$-integrable random variable ψ and any random variable η, we define the conditional expectation of ψ with respect to η by setting $\mathbf{E_P}(\psi \,|\, \eta) = \mathbf{E_P}(\psi \,|\, \mathcal{F}(\eta))$. It is possible to show that for arbitrary real-valued random variables ψ and η, there exists a Borel measurable function $g : \mathbf{R} \to \mathbf{R}$ such that $\mathbf{E_P}(\psi \,|\, \eta) = g(\eta)$. The following result is standard.

Lemma 2.2.1 *Let ψ and η be integrable random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Also let $\mathcal{G}$ and $\mathcal{H}$ be some sub-σ-fields of $\mathcal{F}$. Then:* (i) *if ψ is $\mathcal{G}$-measurable (or equivalently, if $\mathcal{F}(\psi) \subseteq \mathcal{G}$), then $\mathbf{E_P}(\psi \,|\, \mathcal{G}) = \psi$;* (ii) *for arbitrary real numbers c, d, we have*

$$\mathbf{E_P}(c\psi + d\eta \,|\, \mathcal{G}) = c\mathbf{E_P}(\psi \,|\, \mathcal{G}) + d\mathbf{E_P}(\eta \,|\, \mathcal{G});$$

(iii) *if $\mathcal{H} \subseteq \mathcal{G}$, then*

$$\mathbf{E_P}\big(\mathbf{E_P}(\psi \,|\, \mathcal{G}) \,|\, \mathcal{H}\big) = \mathbf{E_P}\big(\mathbf{E_P}(\psi \,|\, \mathcal{H}) \,|\, \mathcal{G}\big) = \mathbf{E_P}(\psi \,|\, \mathcal{H});$$

(iv) *if ψ is independent of $\mathcal{G}$, i.e., the σ-fields $\mathcal{F}(\psi)$ and $\mathcal{G}$ are independent*[4] *under $\mathbf{P}$, then $\mathbf{E_P}(\psi \,|\, \mathcal{G}) = \mathbf{E_P}(\psi)$;* (v) *if ψ is $\mathcal{G}$-measurable and η is independent of $\mathcal{G}$, then for any Borel measurable function $h : \mathbf{R}^2 \to \mathbf{R}$ we have $\mathbf{E_P}(h(\psi, \eta) \,|\, \mathcal{G}) = H(\psi)$, where $H(x) = \mathbf{E_P} h(x, \eta)$, provided that the inequality $\mathbf{E_P}|h(\psi, \eta)| < \infty$ holds.*

Abstract Bayes formula. The last result refers to the situation where two mutually equivalent probability measures, $\mathbf{P}$ and $\mathbf{Q}$ say, are defined on a common measurable space $(\Omega, \mathcal{F})$. Suppose that the Radon-Nikodým derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$ equals

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \eta, \quad \mathbf{P}\text{-a.s.} \tag{2.34}$$

Note that the random variable η is strictly positive $\mathbf{P}$-a.s. , moreover η is $\mathbf{P}$-integrable, with $\mathbf{E_P}\eta = 1$. Finally, by virtue of (2.34), it is clear that equality $\mathbf{E_Q}\psi = \mathbf{E_P}(\psi\eta)$ holds for any $\mathbf{Q}$-integrable random variable ψ.

[3]In the case of a finite Ω, any σ-field $\mathcal{G}$ of subsets of Ω is of this form; that is, $\mathcal{G} = \sigma(\mathcal{D})$ for some decomposition $\mathcal{D}$ of Ω.

[4]We say that two σ-fields, $\mathcal{G}$ and $\mathcal{H}$, are independent under $\mathbf{P}$ whenever $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$ for every $A \in \mathcal{G}$ and every $B \in \mathcal{H}$.

Lemma 2.2.2 *Let $\mathcal{G}$ be a sub-σ-field of the σ-field $\mathcal{F}$, and let ψ be a random variable integrable with respect to* $\mathbf{Q}$. *Then the following abstract version of the Bayes formula holds*

$$\mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G}) = \frac{\mathbf{E}_{\mathbf{P}}(\psi\eta \,|\, \mathcal{G})}{\mathbf{E}_{\mathbf{P}}(\eta \,|\, \mathcal{G})}. \tag{2.35}$$

Proof. It can be easily checked that $\mathbf{E}_{\mathbf{P}}(\eta \,|\, \mathcal{G})$ is strictly positive $\mathbf{P}$-a.s. so that the right-hand side of (2.35) is well-defined. By our assumption, the random variable $\xi = \psi\eta$ is $\mathbf{P}$-integrable, therefore it is enough to show that

$$\mathbf{E}_{\mathbf{P}}(\xi \,|\, \mathcal{G}) = \mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G})\mathbf{E}_{\mathbf{P}}(\eta \,|\, \mathcal{G}).$$

Since the right-hand side of the last formula defines a $\mathcal{G}$-measurable random variable, we need to verify that for any set $A \in \mathcal{G}$, we have

$$\int_A \psi\eta \, d\mathbf{P} = \int_A \mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G})\mathbf{E}_{\mathbf{P}}(\eta \,|\, \mathcal{G}) \, d\mathbf{P}.$$

But for every $A \in \mathcal{G}$, we get

$$\begin{aligned} \int_A \psi\eta \, d\mathbf{P} &= \int_A \psi \, d\mathbf{Q} = \int_A \mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G}) \, d\mathbf{Q} = \int_A \mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G})\eta \, d\mathbf{P} \\ &= \int_A \mathbf{E}_{\mathbf{P}}\Big(\mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G})\eta \,\Big|\, \mathcal{G}\Big) \, d\mathbf{P} = \int_A \mathbf{E}_{\mathbf{Q}}(\psi \,|\, \mathcal{G}) \, \mathbf{E}_{\mathbf{P}}(\eta \,|\, \mathcal{G}) \, d\mathbf{P}. \qquad \square \end{aligned}$$

2.2.2 Martingale Measure

Let us return to the multiplicative binomial lattice modelling the stock price. In the present framework, the process S is determined by the initial stock price S_0 and the sequence $\xi_j, \ j = 1, \dots, T^*$, of independent random variables given by (2.25). More precisely, S_t, $t = 0, \dots, T^*$, is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ by means of (2.4), or equivalently, by the relation

$$S_{t+1} = \xi_{t+1} S_t, \quad \forall \, t < T^*, \tag{2.36}$$

with $S_0 \in \mathbf{R}_+$. Let us introduce the process S^* of the *discounted* stock price by setting

$$S_t^* = S_t / B_t = S_t / \hat{r}^t, \quad \forall \, t \leq T^*. \tag{2.37}$$

Let $\mathcal{D}_t^S$ be the family of decompositions[5] of Ω generated by random variables S_u, $u \leq t$; that is, $\mathcal{D}_t^S = \mathcal{D}(S_0, \dots, S_t)$ for every $t \leq T^*$. It is clear that the family $\mathcal{D}_t^S$, $t \leq T^*$, of decompositions is an increasing family of σ-fields, meaning that $\mathcal{D}_t^S \subset \mathcal{D}_{t+1}^S$ for every $t \leq T^* - 1$. Notice that the family $\mathcal{D}_t^S$ is also generated by the family $\xi_1, \dots, \xi_T^*$ of random variables, more precisely

$$\mathcal{D}_t^S = \mathcal{D}(\xi_0, \xi_1, \dots, \xi_t), \quad \forall \, t \leq T^*,$$

where by convention $\xi_0 = 1$. The family $\mathcal{D}_t^S$, $t \leq T^*$, models a discrete-time flow of information generated by the observations of stock prices. In financial interpretation, the decomposition $\mathcal{D}_t^S$ represents the market information available to all investors at time t. Let us denote $\mathcal{F}_t^S = \sigma(\mathcal{D}_t^S)$ for every $t \leq T^*$, where $\sigma(\mathcal{D}_t^S)$ is the σ-field generated by the decomposition $\mathcal{D}_t^S$. It is clear that

$$\mathcal{F}_t^S = \sigma(S_0, S_1, \dots, S_t), \quad \forall \, t \leq T^*.$$

Finally, we write $\mathbf{F}^S = (\mathcal{F}_t^S)_{t \leq T^*}$ to denote the family of natural σ-fields of the process S, or briefly, the *natural filtration* of the process S.

[5]We say that a finite collection $\mathcal{D} = \{D_1, \dots, D_k\}$ of non-empty events is a *decomposition* of Ω if the events $D_1, \dots, D_k$ are pairwise disjoint; that is, if $D_i \cap D_j = \emptyset$ for every $i \neq j$, and the equality $D_1 \cup D_2 \cup \ldots \cup D_k = \Omega$ holds.

The next definition introduces a probability measure, $\mathbf{P}^*$, which is equivalent to any probability measure $\mathbf{P}$ from $\mathcal{P}$, such that the discounted stock price S^* behaves under $\mathbf{P}^*$ like a *fair game* with respect to its natural filtration. We will assume, in addition, that $\mathbf{P}^*$ belongs to the class $\mathcal{P}$. In view of the definition of the class $\mathcal{P}$, any probability measure from $\mathcal{P}$ depends on the choice of the underlying parameters u, d and r through the value of p only. Furthermore, no generality is lost by this assumption; it is not hard to check that there is no probability measure $\mathbf{P}^*$ outside $\mathcal{P}$ which would be equivalent to any probability measure from the class $\mathcal{P}$, and such that the discounted stock price S^* in the CRR model would follow a $\mathbf{P}^*$-martingale. Therefore, we may adopt, without loss of generality, the following definition of the martingale measure.

Definition 2.2.4 A probability measure $\mathbf{P}^* \in \mathcal{P}$ is called a *martingale measure* for the discounted stock price process S^* if

$$\mathbf{E}_{\mathbf{P}^*}(S^*_{t+1} \,|\, \mathcal{F}^S_t) = S^*_t, \quad \forall\, t \le T^* - 1; \tag{2.38}$$

that is, if the process S^* follows a *martingale* under $\mathbf{P}^*$ with respect to the filtration $\mathbf{F}^S$. In this case, we say that the discounted stock price S^* is a $(\mathbf{P}^*, \mathbf{F}^S)$*-martingale,* or briefly, a $\mathbf{P}^*$*-martingale,* if no confusion may arise.

In some circumstances, we shall assume that the stock price is given on an underlying filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where the underlying filtration $\mathbf{F}$ is strictly greater than the natural filtration $\mathbf{F}^S$ generated by the stock price process. In the present setting, however, it is convenient to take $\mathbf{F} = \mathbf{F}^S$, as is implicit in the definition above.

Lemma 2.2.3 *A martingale measure $\mathbf{P}^* \in \mathcal{P}$ for the discounted stock price S^* exists if and only if $d < 1 + r = \hat{r} < u$. In this case, the martingale measure $\mathbf{P}^*$ for S^* is the unique element from the class $\mathcal{P}$ that corresponds to $p = p_* = (1 + r - d)/(u - d)$.*

Proof. Using (2.36)–(2.37), we may re-express equality (2.38) in the following way

$$\mathbf{E}_{\mathbf{P}^*}(\hat{r}^{-(t+1)}\xi_{t+1} S_t \,|\, \mathcal{F}^S_t) = \hat{r}^{-t} S_t, \quad \forall\, t \le T^* - 1, \tag{2.39}$$

or equivalently

$$\hat{r}^{-(t+1)} S_t\, \mathbf{E}_{\mathbf{P}^*}(\xi_{t+1} \,|\, \mathcal{F}^S_t) = \hat{r}^{-t} S_t, \quad \forall\, t \le T^* - 1.$$

Since the random variable ξ_{t+1} is independent of the σ-field $\mathcal{F}^S_t$, for the last equality to hold it is necessary and sufficient that $\mathbf{E}_{\mathbf{P}^*}(\xi_{t+1}) = \hat{r}$, or explicitly, that the equality $up + d(1-p) = \hat{r}$ is satisfied. By solving the last equation for $p,$ we find the unique value of p_* for which (2.39) is valid. □

Notice that the stock price follows under the (unique) martingale measure $\mathbf{P}^*$, an exponential random walk (cf. formula (2.5)), with the probability of upward movement equal to p_*. This feature explains why it was possible, with no loss of generality, to restrict attention to the special class of probability measures on the underlying canonical space Ω. It should be stressed, however, that the martingale probability measure $\mathbf{P}^*$ does not model the observed real-world fluctuations of stock prices. On the contrary, it is merely a technical tool which proves to be very useful in the arbitrage valuation.

Remarks. The notion of a martingale measure (or *risk-neutral probability*) depends essentially on the choice of a numeraire asset. It can be checked that the unique martingale measure for the relative bond price $B^* = B/S$ is the unique element $\bar{\mathbf{P}}$ from the class $\mathcal{P}$ that corresponds to the following value of p (see formula (1.17))

$$p = \bar{p} = \Big(\frac{1}{d} - \frac{1}{\hat{r}}\Big)\,\frac{ud}{u-d}\,.$$

2.2.3 Risk-neutral Valuation Formula

For our further purposes, we find it convenient to re-examine the problem of arbitrage option pricing within the framework of the CRR binomial model by means of probabilistic methods. Observe that in m periods to the option expiry date T, the discounted payoff of a call option equals $\hat{r}^{-m}(S_T - K)^+$, hence it depends on the terminal stock price S_T (which, of course, is not known at time $T-m$). We shall show that the CRR option valuation formula established in Proposition 2.1.1 may be alternatively derived by the direct evaluation of the conditional expectation, under the martingale measure $\mathbf{P}^*$, of the discounted option's payoff.

Proposition 2.2.1 *Consider a European call option, with expiry date T and strike price K, written on one share of a common stock whose price S is assumed to follow the CRR multiplicative binomial process* (2.4)*. Then the arbitrage price C_{T-m}, given by formula* (2.13)*, coincides with the conditional expectation C^*_{T-m}, which equals*

$$C^*_{T-m} = \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-m}(S_T - K)^+ \,\big|\, \mathcal{F}^S_{T-m}\big), \quad \forall\, m \le T. \tag{2.40}$$

Proof. It is sufficient to find explicitly the conditional expectation (2.40). Recall that $S_T = S_{T-m}\,\xi_{T-m+1}\ldots\xi_T = S_{T-m}\,\eta_m$, where the stock price S_{T-m} is a $\mathcal{F}^S_{T-m}$-measurable random variable, and the random variable $\eta_m = \xi_{T-m+1}\ldots\xi_T$ is independent of the σ-field $\mathcal{F}^S_{T-m}$. By applying the well-known property of conditional expectations (see Lemma 2.2.1) to the random variables $\psi = S_{T-m}$, $\eta = \eta_m$ and to the function $h(x,y) = \hat{r}^{-m}(xy - K)^+$, one finds that

$$C^*_{T-m} = \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-m}(S_T - K)^+ \,\big|\, \mathcal{F}^S_{T-m}\big) = H(S_{T-m}),$$

where the function $H : \mathbf{R} \to \mathbf{R}$ equals

$$H(x) = \mathbf{E}_{\mathbf{P}^*}\big(h(x,\eta_m)\big) = \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-m}(x\eta_m - K)^+\big), \quad \forall\, x \in \mathbf{R}.$$

Since the random variables $\xi_{T-m+1},\ldots,\xi_T$ are mutually independent and identically distributed under $\mathbf{P}^*$, with $\mathbf{P}^*\{\xi_j = u\} = p_* = 1 - \mathbf{P}^*\{\xi_j = d\}$, it is also clear that

$$H(x) = \hat{r}^{-m}\sum_{j=0}^{m}\binom{m}{j} p_*^j (1-p_*)^{m-j}\big(xu^j d^{m-j} - K\big)^+.$$

Using equalities $\bar{p} = p_* u/\hat{r}$ and $1 - \bar{p} = (1-p_*)d/\hat{r}$, we conclude that

$$C^*_{T-m} = \sum_{j=a}^{m}\binom{m}{j}\Big(S_{T-m}\bar{p}^j(1-\bar{p})^{m-j} - K\hat{r}^{-m}p_*^j(1-p_*)^{m-j}\Big),$$

where $a = \min\{\, j \in \mathbf{N}^* \mid S_{T-m}u^j d^{m-j} > K\}$. □

We may rewrite (2.40) in the following way

$$C^*_t = B_t\,\mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1}(S_T - K)^+ \,\big|\, \mathcal{F}^S_t\big) = B_t\,\mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1}X \,\big|\, \mathcal{F}^S_t\big), \tag{2.41}$$

where $X = (S_T - K)^+$. One might wonder if the valuation formula (2.41) remains in force for a larger class of financial models and European contingent claims X. Generally speaking, the answer to this question is positive, even if the interest rate is assumed to follow a stochastic process.

Remarks. It is interesting to notice that the CRR valuation formula (2.13) may be rewritten as follows (for simplicity, we focus on the case $t = 0$)

$$C_0 = S_0\,\bar{\mathbf{P}}\,\{S_T > K\} - K\hat{r}^{-T}\mathbf{P}^*\{S_T > K\}, \tag{2.42}$$

where $\bar{\mathbf{P}}$ and $\mathbf{P}^*$ are martingale measures corresponding to the choice of the stock price and the bond price as a numeraire, respectively.

2.3 Valuation of American Options

Let us first consider the case of the American call option – that is, the option to buy a specified number of shares, which may be exercised at any time before the option expiry date T, or on that date. The exercise policy of the option holder is necessarily based on the information accumulated to date and not on the future prices of the stock. As in the previous chapter, we will write C_t^a to denote the arbitrage price at time t of an American call option written on one share of a stock. By arbitrage price of the American call we mean such price process C_t^a, $t \leq T$, that an extended financial market model – that is, a market with trading in riskless bonds, stocks and American call options – remains arbitrage-free. Our first goal is to show that the price of an American call option in the CRR arbitrage-free market model coincides with the arbitrage price of a European call option with the same expiry date and strike price. For this purpose, it is sufficient to show that the American call option should never be exercised before maturity, since otherwise the option writer would be able to make riskless profit. The argument hinges on the following simple inequality

$$C_t \geq (S_t - K)^+, \quad \forall\, t \leq T, \tag{2.43}$$

which can be justified in several ways. An intuitive way of deriving (2.43) is based on no-arbitrage arguments. Notice that since the option's price C_t is always non-negative, it is sufficient to consider the case when the current stock price is greater than the exercise price – that is, when $S_t - K > 0$. Suppose, on the contrary, that $C_t < S_t - K$ for some t, i.e., $S_t - C_t > K$. Then it would be possible, with zero net initial investment, to buy at time t a call option, short a stock, and invest the sum $S_t - C_t$ in a savings account. By holding this portfolio unchanged up to the maturity date T, we would be able to lock in a riskless profit. Indeed, the value of our portfolio at time T would satisfy (recall that $r \geq 0$)

$$\hat{r}^{T-t}(S_t - C_t) + C_T - S_T > \hat{r}^{T-t}K + (S_T - K)^+ - S_T \geq 0.$$

We conclude once again that inequality (2.43) is necessary for the absence of arbitrage opportunities.

Taking (2.43) for granted, we may deduce the property $C_t^a = C_t$ by simple no-arbitrage arguments. Suppose, on the contrary, that the writer of an American call is able to sell the option at time 0 at the price $C_0^a > C_0$ (it is evident that, at any time, an American option is worth at least as much as a European option with the same contractual features; in particular, $C_0^a \geq C_0$). In order to profit from this transaction, the option writer establishes a dynamic portfolio which replicates the value process of the European call, and invests the remaining funds in riskless bonds. Suppose that the holder of the option decides to exercise it at instant t before the expiry date T. Then the option's writer locks in a riskless profit, since the value of portfolio satisfies

$$C_t - (S_t - K)^+ + \hat{r}^{T-t}(C_0^a - C_0) > 0, \quad \forall\, t \leq T.$$

The above reasoning implies that the European and American call options are equivalent from the point of view of arbitrage pricing theory; that is, both options have the same price, and an American call should never be exercised by its holder before expiry. The last statement means also that a risk-neutral investor who is long an American call should be indifferent between selling it before, and holding it to, the option's expiry date (provided that the market is efficient – that is, options are neither underpriced nor overpriced).

2.3.1 American Put Option

Since the early exercise feature of American put options was examined in Sect. 1.5, we will focus on the justification of the valuation formula. Let us denote by $\mathcal{T}$ the class of all *stopping times* defined on the filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where $\mathcal{F}_t = \mathcal{F}_t^S$ for every $t = 0, \dots, T^*$. By a *stopping time* we mean an arbitrary function $\tau : \Omega \to \{0, \dots, T^*\}$ such that for any $t = 0, \dots, T^*$, a random event $\{\omega \in \Omega \,|\, \tau(\omega) = t\}$ belongs to the σ-field $\mathcal{F}_t$. Intuitively, this property means that the decision whether to stop a process at time t (that is, whether to exercise an option at time t or not) depends on the stock price fluctuations up to time t only. Also, let $\mathcal{T}_{[t,T]}$ stand for the subclass of those stopping times τ which satisfy $t \le \tau \le T$. Corollary 1.5.1 and the preceding discussion suggest the following result.

Proposition 2.3.1 *The arbitrage price P_t^a of an American put option equals*

$$P_t^a = \max{}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-(\tau-t)}(K - S_\tau)^+ \,|\, \mathcal{F}_t\big), \quad \forall\, t \le T. \tag{2.44}$$

Moreover, the stopping time τ_t^* which realizes the maximum in (2.44) *is given by the expression*

$$\tau_t^* = \min\{u \ge t \,|\, (K - S_u)^+ \ge P_u^a\}, \quad \forall\, t \le T. \tag{2.45}$$

Proof. The problem of arbitrage pricing of American contingent claims within a continuous-time setup is examined in detail in Chap. 5 below. In particular, a continuous-time counterpart of formula (2.44) is proved in Theorem 5.1.1. The verification of (2.44) is, of course, much simpler, but it is based on similar arguments. For these reasons, the proof is left to the reader. □

The stopping time τ_t^* will be referred to as the *rational exercise time* of an American put option that is still alive at time t. It should be pointed out that τ_t^* does not solve the optimal stopping time for any individual, but only for those investors who are risk-neutral. A straightforward application of the classic Bellman principle[6] reduces the optimal stopping problem (2.44) to an explicit recursive procedure which allows us to find the value function V^p. These observations lead to the following corollary to Proposition 2.3.1.

Corollary 2.3.1 *Let the non-negative adapted process V_t^p, $t \le T$, be defined recursively by setting $V_T^p = (K - S_T)^+$, and*

$$V_t^p = \max\Big\{K - S_t,\, \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-1} V_{t+1}^p \,|\, \mathcal{F}_t\big)\Big\}, \quad \forall\, t \le T-1. \tag{2.46}$$

Then the arbitrage price P_t^a of an American put option at time t equals V_t^p. Moreover, the rational exercise time after time t equals $\tau_t^* = \min\{u \ge t \,|\, K - S_u \ge V_u^p\}$.

Remarks. It is also possible to go the other way around – that is, to first show directly that the price P_t^a needs necessarily to satisfy the recursive relationship

$$P_t^a = \max\Big\{K - S_t,\, \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-1} P_{t+1}^a \,|\, \mathcal{F}_t\big)\Big\}, \quad \forall\, t \le T-1, \tag{2.47}$$

subject to the terminal condition $P_T^a = (K - S_T)^+$, and subsequently derive the equivalent representation (2.44). In the case of the CRR model (indeed, in the case of any discrete-time security pricing model), the latter approach appears to be the simplest way to value American options. The main reason for this is that an apparently difficult valuation problem is thus reduced to the simple one-period case. To show this we shall argue, as usual, by contradiction. To start with, we assume that (2.47) fails to hold for $t = T-1$. If this is the case, by reasoning along the same lines as in Sect. 1.5, one may easily construct at time $T-1$ a portfolio which produces riskless profit at time T. Hence, we conclude that necessarily

$$P_{T-1}^a = \max\Big\{K - S_{T-1},\, \mathbf{E}_{\mathbf{P}^*}\big(\hat{r}^{-1}(K - S_T)^+ \,|\, \mathcal{F}_T\big)\Big\}.$$

The next step is to consider the time period $[T-2, T-1]$, with $T-1$ now playing the role of the terminal date, and P_{T-1}^a being the terminal payoff. This procedure may be repeated as many times as needed. Summarizing, in the case of the CRR model, the arbitrage pricing of an American put reduces to the following simple recursive recipe

$$P_t^a = \max\Big\{K - S_t\,,\, \hat{r}^{-1}\big(p_* P_{t+1}^{au} + (1 - p_*) P_{t+1}^{ad}\big)\Big\}, \quad \forall\, t \le T-1, \tag{2.48}$$

[6] It should be observed that the process S is Markovian under $\mathbf{P}^*$. This is an immediate consequence of the independence of random variables $\xi_1, \dots, \xi_{T^*}$.

with $P_T^a = (K - S_T)^+$. Notice that P_{t+1}^{au} and P_{t+1}^{ad} represent the value of the American put in the next step corresponding to the upward and downward movement of the stock price starting from a given node on the lattice. The above results may be easily generalized to the case of an arbitrary contingent claim of American style.

2.4 Options on a Dividend-paying Stock

So far we have assumed that a stock pays no dividend during an option's lifetime. Suppose now that the stock pays dividends, and the dividend policy is of the following specific form: the stock maintains a constant yield, κ, on each ex-dividend date. We shall restrict ourselves to the last period before the option's expiry. However, the analysis we present below may be easily extended to the case of multi-period trading. We assume that the option's expiry date T is an ex-dividend date. This means that the shareholder will receive at that time a dividend payment d_T which amounts to $\kappa u S_{T-1}$ or $\kappa d S_{T-1}$, according to the actual stock price fluctuation. On the other hand, we postulate that the ex-dividend stock price at the end of the period will be either $u(1-\kappa)S_{T-1}$ or $d(1-\kappa)S_{T-1}$. This corresponds to the traditional assumption that the stock price declines on the ex-dividend date by exactly the dividend amount. Therefore, the option's payoff C_T^κ is either $C_T^u = \big(u(1-\kappa)S_{T-1} - K\big)^+$ or $C_T^d = \big(d(1-\kappa)S_{T-1} - K\big)^+$, depending on the stock price fluctuation during the last period. If someone is long a stock, he or she receives the dividend at the end of the period; a party in a short position has to make restitution for the dividend to the party from whom the stock was borrowed. Under these assumptions, the replicating strategy of a call option is determined by the following system of equations (independently of the sign of α_{T-1}; that is, whether the position is long or short)

$$\begin{cases} \alpha_{T-1} u S_{T-1} + \beta_{T-1}\hat{r} = \big(u(1-\kappa)S_{T-1} - K\big)^+, \\ \alpha_{T-1} d S_{T-1} + \beta_{T-1}\hat{r} = \big(d(1-\kappa)S_{T-1} - K\big)^+. \end{cases}$$

Note that, in contrast to the option payoff, the terminal value of the portfolio $(\alpha_{T-1}, \beta_{T-1})$ is not influenced by the fact that T is the ex-dividend date. This nice feature of portfolio's wealth depends essentially on our assumption that the ex-dividend drop of the stock price coincides with the dividend payment. Solving the above equations for α_{T-1} and β_{T-1}, we find

$$\alpha_{T-1} = \frac{(u_\kappa S_{T-1} - K)^+ - (d_\kappa S_{T-1} - K)^+}{S_{T-1}(u-d)} = \frac{C_T^u - C_T^d}{S_T^u - S_T^d} \tag{2.49}$$

and

$$\beta_{T-1} = \frac{u(d_\kappa S_{T-1} - K)^+ - d(u_\kappa S_{T-1} - K)^+}{\hat{r}(u-d)} = \frac{uC_T^d - dC_T^u}{\hat{r}(u-d)}, \tag{2.50}$$

where $u_\kappa = (1-\kappa)u$ and $d_\kappa = (1-\kappa)d$. By standard arguments, we conclude that the price C_{T-1}^κ of the option at the beginning of the period equals

$$C_{T-1}^\kappa = \alpha_{T-1} S_{T-1} + \beta_{T-1} = \hat{r}^{-1}\big(p_* C_T^u + (1-p_*)C_T^d\big),$$

or explicitly

$$C_{T-1}^\kappa = \hat{r}^{-1}\Big(p_*(u_\kappa S_{T-1} - K)^+ + (1-p_*)(d_\kappa S_{T-1} - K)^+\Big), \tag{2.51}$$

where $p_* = (\hat{r} - d)/(u-d)$. Working backwards in time from the expiry date, one finds the general formula for the arbitrage price of a European call option, provided that the ex-dividend dates and the dividend ratio $\kappa \in (0,1)$ are known in advance. If we price a put option, the corresponding hedging portfolio at time $T-1$ satisfies

$$\begin{cases} \alpha_{T-1} u S_{T-1} + \beta_{T-1}\hat{r} = \big(K - u_\kappa S_{T-1}\big)^+, \\ \alpha_{T-1} d S_{T-1} + \beta_{T-1}\hat{r} = \big(K - d_\kappa S_{T-1}\big)^+. \end{cases}$$

Chapter 3

The Black-Scholes Model

The option pricing model developed by Black and Scholes (1973), formalized and extended by Merton (1973), enjoys great popularity. It is computationally simple and, like all arbitrage-based pricing models, does not require the knowledge of an investor's risk preferences. Option valuation within the Black-Scholes framework is based on the already familiar concept of perfect replication of contingent claims. More specifically, we will show that an investor can replicate an option's return stream by continuously rebalancing a self-financing portfolio involving stocks and risk-free bonds. For instance, a replicating portfolio for a call option involves, at any date t before the option's expiry date, a long position in stock, and a short position in risk-free bonds. By definition, the wealth at time t of a replicating portfolio equals the arbitrage price of an option. Our main goal is to derive closed-form expressions for both the option's price and the replicating strategy in the Black-Scholes setting. To do this in a formal way, we need first to construct an arbitrage-free market model with continuous trading. This can be done relatively easily if we take for granted certain results from the theory of stochastic processes, more precisely, from the *Itô stochastic calculus.* The theory of Itô stochastic integration is presented in several monographs; to cite a few, Elliott (1982), Karatzas and Shreve (1988), Protter (1990), Revuz and Yor (1991), and Durrett (1996).

In financial literature, it is not uncommon to derive the Black-Scholes formula by introducing a continuously rebalanced *risk-free portfolio* containing an option and underlying stocks. In the absence of arbitrage, the return from such a portfolio needs to match the returns on risk-free bonds. This property leads to the Black-Scholes partial differential equation satisfied by the arbitrage price of an option. It appears, however, that the risk-free portfolio does not satisfy the formal definition of a self-financing strategy.

We make throughout the following basic assumptions concerning market activities: trading takes place continuously in time, and unrestricted borrowing and lending of funds is possible at the same constant interest rate. Furthermore, the market is frictionless, meaning that there are no transaction costs or taxes, and no discrimination against the short sale. Finally, unless explicitly stated otherwise, we will assume that a stock which underlies an option does not pay dividends (at least during the option's lifetime).

3.1 Itô Stochastic Calculus

This section provides a very brief account of the Itô stochastic integration theory. For more details we refer the reader, for instance, to monographs Durrett (1996), Karatzas and Shreve (1988, Chap. 2–3) or Revuz and Yor (1991, Chap. 4–5 and 8–9).

Let us consider a probability space $(\Omega, \mathbf{F}, \mathbf{P})$, equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ (a *filtration* is an increasing family of σ-fields).

Definition 3.1.1 A sample-paths continuous, $\mathbf{F}$-adapted process W, with $W_0 = 0$, defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, is called a *one-dimensional standard Brownian motion* with respect to the filtration $\mathbf{F}$ if, for every $u \le t \le T$, the increment $W_t - W_u$ is independent of the σ-field $\mathcal{F}_u$, and the probability distribution of $W_t - W_u$ is Gaussian, with mean 0 and variance $t-u$.

We shall describe only the most important properties of a Brownian motion. Firstly, it can easily be seen that a Brownian motion is a continuous martingale with respect to the underlying filtration $\mathbf{F}$, since $\mathbf{E}_{\mathbf{P}}|W_t| < \infty$ and

$$\mathbf{E}_{\mathbf{P}}(W_t \,|\, \mathcal{F}_u) = \mathbf{E}_{\mathbf{P}}(W_t - W_u \,|\, \mathcal{F}_u) + \mathbf{E}_{\mathbf{P}}(W_u \,|\, \mathcal{F}_u) = W_u, \tag{3.1}$$

if $u \le t \le T$. It is well known that almost all sample paths of a Brownian motion have infinite variation on every open interval, so classical Lebesgue-Stieltjes integration theory cannot be applied to define an integral of a stochastic process with respect to a Brownian motion. Finally, W is a process of finite *quadratic variation,* as the following result shows.

Proposition 3.1.1 *For every $0 \le u < t \le T$ and an arbitrary sequence $\{\mathcal{T}^n\}$ of finite partitions $\mathcal{T}^n = \{t_0^n = u < t_1^n < \ldots < t_n^n = t\}$ of the interval $[u,t]$ satisfying $\lim_{n\to\infty} \delta(\mathcal{T}^n) = 0$, where*

$$\delta(\mathcal{T}^n) \stackrel{\text{def}}{=} \max_{k=0,\ldots,n-1} (t_{k+1}^n - t_k^n),$$

we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (W_{t_{k+1}^n} - W_{t_k^n})^2 = t - u, \tag{3.2}$$

where the convergence in (3.2) *is in* $L^2(\Omega, \mathcal{F}_T, \mathbf{P})$.

Let W be a standard one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. For simplicity, the horizon date $T > 0$ will be fixed throughout. We shall introduce the Itô stochastic integral as an isometry I from a certain space $\mathcal{L}^2_{\mathbf{P}}(W)$ of stochastic processes into the space $L^2 = L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ of square-integrable, $\mathcal{F}_T$-measurable random variables. To start with, let us denote by $\mathcal{L}^2_{\mathbf{P}}(W)$ the class of those *progressively measurable* (for the definition of a progressively measurable process we refer to, e.g., Karatzas and Shreve (1988)) processes γ defined on $(\Omega, \mathbf{F}, \mathbf{P})$ for which

$$\|\,\gamma\,\|_W^2 \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}}\Big(\int_0^T \gamma_u^2 \, du\Big) < \infty. \tag{3.3}$$

Also, let $\mathcal{K}$ stand for the space of *elementary processes,* that is, processes of the form

$$\gamma(t) = \gamma_{-1}\,\mathrm{I}_0 + \sum_{j=0}^{m-1} \gamma_j\,\mathrm{I}_{(t_j, t_{j+1}]}(t), \quad \forall\, t \in [0,T], \tag{3.4}$$

where $t_0 = 0 < t_1 < \ldots < t_m = T$, the random variables γ_j, $j = 0, \ldots, m-1$, are uniformly bounded and $\mathcal{F}_{t_j}$-measurable, and, finally, the random variable γ_{-1} is $\mathcal{F}_0$-measurable. For any process $\gamma \in \mathcal{K}$, the *Itô stochastic integral* $\hat{I}_T(\gamma)$ with respect to W over the time interval $[0,T]$ is defined by the formula

$$\hat{I}_T(\gamma) = \int_0^T \gamma_u \, dW_u \stackrel{\text{def}}{=} \sum_{j=0}^{m-1} \gamma_j (W_{t_{j+1}} - W_{t_j}). \tag{3.5}$$

Similarly, the Itô stochastic integral of γ with respect to W over any subinterval $[0,t]$, where $t \le T$, is defined by setting

$$\hat{I}_t(\gamma) = \int_0^t \gamma_u \, dW_u \stackrel{\text{def}}{=} \hat{I}_T(\gamma\,\mathrm{I}_{[0,t]}) = \sum_{j=0}^{m-1} \gamma_j \big(W_{t_{j+1}\wedge t} - W_{t_j \wedge t}\big), \tag{3.6}$$

where $x \wedge y = \min\{x, y\}$. It is easily seen that for any process $\gamma \in \mathcal{K}$, the Itô integral $I_t(\gamma)$, $t \in [0,T]$, follows a continuous martingale on the space $(\Omega, \mathbf{F}, \mathbf{P})$; that is, $\mathbf{E}_{\mathbf{P}}(I_t(\gamma) \,|\, \mathcal{F}_u) = I_u(\gamma)$ for $u \le t \le T$.

Lemma 3.1.1 *The class $\mathcal{K}$ is a subset of $\mathcal{L}^2_{\mathbf{P}}(W)$, and*

$$\mathbf{E}_{\mathbf{P}}\Big(\int_0^T \gamma_u\, dW_u\Big)^2 = \| I_T(\gamma) \|^2_{L^2} = \| \gamma \|^2_W \tag{3.7}$$

for any process γ from $\mathcal{K}$. The space $\mathcal{L}^2_{\mathbf{P}}(W)$ of progressively measurable stochastic processes, equipped with the norm $\|\cdot\|_W$, is a complete normed linear space – that is, a Banach space. Moreover, the class $\mathcal{K}$ of elementary stochastic processes is a dense linear subspace of $\mathcal{L}^2_{\mathbf{P}}(W)$.

By virtue of Lemma 3.1.1, the isometry $\hat{I}_T : (\mathcal{K}, \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ can be extended to an isometry $I_T : (\mathcal{L}^2_{\mathbf{P}}(W), \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbf{P})$. This leads to the following definition.

Definition 3.1.2 For any process $\gamma \in \mathcal{L}^2_{\mathbf{P}}(W)$, the random variable $I_T(\gamma)$ is called the *Itô stochastic integral* of γ with respect to W over $[0,T]$, and it is denoted by $\int_0^T \gamma_u\, dW_u$.

More generally, for every $\gamma \in \mathcal{L}^2_{\mathbf{P}}(W)$ and every $t \in [0,T]$, we set

$$I_t(\gamma) = \int_0^t \gamma_u\, dW_u \stackrel{\text{def}}{=} I_T(\gamma \mathrm{I}_{[0,t]}), \tag{3.8}$$

so that the Itô stochastic integral $I_t(\gamma)$ is a well-defined stochastic process. The next result summarizes the most important properties of this process. By $\langle I(\gamma)\rangle$ we denote the stochastic process given by the formula

$$\langle I(\gamma)\rangle_t \stackrel{\text{def}}{=} \int_0^t \gamma_u^2\, du, \quad \forall\, t \in [0,T]. \tag{3.9}$$

Proposition 3.1.2 *For any process $\gamma \in \mathcal{L}^2_{\mathbf{P}}(W)$, the Itô stochastic integral $I_t(\gamma)$ follows a square-integrable continuous martingale on $(\Omega, \mathbf{F}, \mathbf{P})$. Moreover, the process*

$$(I_t(\gamma))^2 - \langle I(\gamma)\rangle_t, \quad \forall\, t \in [0,T], \tag{3.10}$$

is a continuous martingale on $(\Omega, \mathbf{F}, \mathbf{P})$.

In a more general framework, if M is a continuous local martingale, then we denote by $\langle M\rangle$ the unique, continuous, increasing adapted process vanishing at zero such that $M^2 - \langle M\rangle$ is a local martingale. The process $\langle M\rangle$ is referred to as the *quadratic variation* of M. In view of Lemma 3.1.2, it is clear that formula (3.9) is consistent with this more general definition.

By applying the optional stopping technique (known also as a *localization*), it is possible to extend the definition of Itô stochastic integral to the class of all progressively measurable processes γ for which

$$\mathbf{P}\Big\{\int_0^T \gamma_u^2\, du < \infty\Big\} = 1. \tag{3.11}$$

In this case, the Itô integral $I(\gamma)$ is known to follow a continuous local martingale on $(\Omega, \mathbf{F}, \mathbf{P})$, in general. Recall that a process M is said to be a *local martingale* if there exists an increasing sequence τ_n of stopping times such that τ_n tends to T a.s., and for every n the process M^n, given by the formula

$$M^n_t = \begin{cases} M_{t\wedge\tau_n(\omega)}(\omega) & \text{if } \tau_n(\omega) > 0, \\ 0 & \text{if } \tau_n(\omega) = 0, \end{cases}$$

follows a uniformly integrable martingale.

Remarks. A random variable $\tau : \Omega \to [0,T]$ is called a *stopping time* with respect to the filtration $\mathbf{F}$ if, for every $t \in [0,T]$, the event $\{\tau \le t\}$ belongs to the σ-field $\mathcal{F}_t$. For any progressively measurable process γ, and any stopping time τ, the *stopped process* γ^τ, which is defined by $\gamma^\tau_t = \gamma_{\tau\wedge t}$, is also progressively measurable.

Let us denote by $\mathcal{L}_{\mathbf{P}}(W)$ the class of all progressively measurable processes γ satisfying the integrability condition (3.11). It is clear that this space of stochastic processes is invariant with respect to the equivalent change of probability measure; that is, $\mathcal{L}_{\mathbf{P}}(W) = \mathcal{L}_{\tilde{\mathbf{P}}}(\tilde{W})$ whenever $\mathbf{P}$ and $\tilde{\mathbf{P}}$ are mutually equivalent probability measures on $(\Omega, \mathcal{F}_T)$, and processes W and $\tilde{W}$ are Brownian motions under $\mathbf{P}$ and under $\tilde{\mathbf{P}}$, respectively. Since we restrict ourselves to equivalent changes of probability measures, given a fixed underlying probability space $(\Omega, \mathcal{F}_T, \mathbf{P})$, we shall write shortly $\mathcal{L}(W)$ instead of $\mathcal{L}_{\mathbf{P}}(W)$ in what follows. Thus, a process γ is called *integrable with respect to* W if it belongs to the class $\mathcal{L}(W)$.

3.1.1 Itô's Lemma

In this section, we shall deal with the following problem: does the process $g(X_t)$ follow a semimartingale if X is a semimartingale and g is a sufficiently regular function? It turns out that the class of continuous semimartingales is invariant with respect to compositions with C^2-functions (of course, much more general results are also available). We start by introducing a particular class of continuous semimartingales, referred to as *Itô processes.*

Definition 3.1.3 An adapted continuous process X is called an *Itô process* if it admits a representation

$$X_t = X_0 + \int_0^t \alpha_u \, du + \int_0^t \beta_u \, dW_u, \quad \forall\, t \in [0,T], \tag{3.12}$$

for some adapted processes $\alpha,\ \beta$ defined on $(\Omega, \mathbf{F}, \mathbf{P})$, which are integrable in a suitable sense.

For the sake of notational simplicity, it is customary to use a more condensed differential notation in which (3.12) takes the following form

$$dX_t = \alpha_t \, dt + \beta_t \, dW_t.$$

A continuous adapted process X is called a *continuous semimartingale* if it admits a decomposition $X = X_0 + M + A$, where X_0 is a $\mathcal{F}_0$-measurable random variable, M is a continuous local martingale, and A is a continuous process whose sample paths are almost all of finite variation on $[0,T]$. It is clear that an Itô process follows a continuous semimartingale and (3.12) gives its canonical decomposition. We denote by $\mathcal{S}^c(\mathbf{P})$ the class of all real-valued continuous semimartingales on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$.

One-dimensional case. Let us consider a function $g = g(x,t)$, where $x \in \mathbf{R}$ is the space variable, and $t \in [0,T]$ is the time variable. It is evident that if X is a continuous semimartingale and $g : \mathbf{R} \times [0,T] \to \mathbf{R}$ is a jointly continuous function, then the process $Y_t = g(X_t, t)$ is $\mathcal{F}_t$-adapted and has almost all sample paths continuous. The next result, which is a special case of *Itô's lemma,* states that Y follows a semimartingale, provided that the function g is sufficiently smooth.

Theorem 3.1.1 *Suppose that $g : \mathbf{R} \times [0,T] \to \mathbf{R}$ is a function of class $C^{2,1}(\mathbf{R} \times [0,T], \mathbf{R})$. Then for any Itô process X, the process $Y_t = g(X_t, t)$, $t \in [0,T]$, follows an Itô process. Moreover, its canonical decomposition is given by the Itô formula*

$$dY_t = g_t(X_t,t)\, dt + g_x(X_t,t)\alpha_t \, dt + g_x(X_t,t)\beta_t \, dW_t + \tfrac{1}{2} g_{xx}(X_t,t)\beta_t^2 \, dt.$$

More generally, if $X = X_0 + M + A$ is a real-valued continuous semimartingale, and g is a function of class $C^{2,1}(\mathbf{R} \times [0,T], \mathbf{R})$, then $Y_t = g(X_t, t)$ follows a continuous semimartingale with the following canonical decomposition

$$dY_t = g_t(X_t,t)\, dt + g_x(X_t,t)\, dX_t + \tfrac{1}{2}\, g_{xx}(X_t,t)\, d\langle M \rangle_t. \tag{3.13}$$

Multidimensional case. Let us start by defining a d-dimensional Brownian motion. A $\mathbf{R}^d$-valued stochastic process $W = (W^1, \dots, W^d)$ defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$ is called a *d-dimensional standard Brownian motion* if $W^1, W^2, \dots, W^d$ are mutually independent one-dimensional standard Brownian motions. In this paragraph, W denotes a d-dimensional standard Brownian motion. Let γ be an adapted $\mathbf{R}^d$-valued process satisfying the following condition

$$\mathbf{P}\Big\{\int_0^T |\gamma_u|^2\,du < \infty\Big\} = 1, \tag{3.14}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^d$. Then the Itô stochastic integral of γ with respect to W equals

$$I_t(\gamma) = \int_0^t \gamma_u \cdot dW_u = \sum_{i=1}^d \int_0^t \gamma_u^i\,dW_u^i, \quad \forall\, t \in [0,T]. \tag{3.15}$$

Let $X = (X^1, X^2, \dots, X^k)$ be a k-dimensional process such that

$$X_t^i = X_0^i + \int_0^t \alpha_u^i\,du + \int_0^t \beta_u^i \cdot dW_u, \tag{3.16}$$

where α^i are real-valued adapted processes, and β^i are $\mathbf{R}^d$-valued processes for $i = 1,2,\dots,k$, integrable in a suitable sense. Let $g = g(x,t)$ be a function $g : \mathbf{R}^k \times [0,T] \to \mathbf{R}$. Before stating the next result, it will be convenient to introduce the notion of the *cross-variation* (or *quadratic covariation*) of two continuous semimartingales. If $X^i = X_0^i + M^i + A^i$ are in $\mathcal{S}^c(\mathbf{P})$ for $i = 1,2$, then $\langle X^1, X^2\rangle_t \stackrel{\text{def}}{=} \langle M^1, M^2\rangle_t$, where in turn

$$\langle M^1, M^2\rangle_t \stackrel{\text{def}}{=} \tfrac{1}{4}\big(\langle M^1+M^2, M^1+M^2\rangle_t - \langle M^1,M^1\rangle_t - \langle M^2,M^2\rangle_t\big).$$

For instance, if X^1 and X^2 are the Itô processes given by (3.16), then it is easily seen that

$$\langle X^1, X^2\rangle_t = \int_0^t \beta_u^1 \cdot \beta_u^2\,du, \quad \forall\, t \in [0,T].$$

Proposition 3.1.3 *Suppose that g is a function of class $C^2(\mathbf{R}^k, \mathbf{R})$. Then the following form of Itô's formula is valid*

$$dg(X_t) = \sum_{i=1}^k g_{x_i}(X_t)\alpha_t^i\,dt + \sum_{i=1}^k g_{x_i}(X_t)\beta_t^i \cdot dW_t + \frac{1}{2}\sum_{i,j=1}^k g_{x_i x_j}(X_t)\,\beta_t^i \cdot \beta_t^j\,dt.$$

More generally, if processes X^i are in $\mathcal{S}^c(\mathbf{P})$ for $i = 1,2,\dots,k$, then

$$g(X_t) = g(X_0) + \sum_{i=1}^k \int_0^t g_{x_i}(X_u)\,dX_u^i + \frac{1}{2}\sum_{i,j=1}^k \int_0^t g_{x_i x_j}(X_u)\,d\langle X^i, X^j\rangle_u.$$

A special case of the Itô formula, known as the *integration by parts formula,* is obtained by taking the function $g(x_1,x_2) = x_1 x_2$.

Corollary 3.1.1 *Suppose that X^1, X^2 are real-valued continuous semimartingales. Then the following integration by parts formula is valid*

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_u^1\,dX_u^2 + \int_0^t X_u^2\,dX_u^1 + \langle X^1, X^2\rangle_t. \tag{3.17}$$

3.1.2 Predictable Representation Property

In this section, we shall assume that the filtration $\mathbf{F} = \mathbf{F}^W$ is the standard augmentation[1] of the natural filtration $\sigma\{W_u \,|\, u \leq t\}$ of the Brownian motion W. In other words, we assume here that the underlying probability space is $(\Omega, \mathbf{F}^W, \mathbf{P})$, where W is a one-dimensional Brownian motion.

Theorem 3.1.2 *For any random variable $X \in L^2(\Omega, \mathcal{F}^W, \mathbf{P})$, there exists a unique predictable process γ from the class $\mathcal{L}_{\mathbf{P}}(W)$ such that*

$$\mathbf{E}_{\mathbf{P}}\Big(\int_0^T \gamma_u^2 \, du\Big) < \infty \tag{3.18}$$

and the following equality is valid

$$X = \mathbf{E}_{\mathbf{P}}(X) + \int_0^T \gamma_u \, dW_u. \tag{3.19}$$

Basically, it can be deduced from Proposition 3.1.2 that any local martingale on the filtered probability space $(\Omega, \mathbf{F}^W, \mathbf{P})$ admits a modification with continuous sample paths.

3.1.3 Girsanov's theorem

Let W be a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. For an adapted $\mathbf{R}^d$-valued process $\gamma \in \mathcal{L}(W)$, we define the process U by setting

$$U_t = I_t(\gamma) = \int_0^t \gamma_u \cdot dW_u, \quad \forall\, t \in [0, T]. \tag{3.20}$$

The process U defined in this way follows, of course, a continuous local martingale under $\mathbf{P}$. One may check, using Itô's formula, that the *Doléans exponential of* U – that is, the unique solution $\mathcal{E}(U)$ of the stochastic differential equation

$$d\mathcal{E}_t(U) = \mathcal{E}_t(U)\, \gamma_t \cdot dW_t = \mathcal{E}_t(U)\, dU_t, \tag{3.21}$$

with the initial condition $\mathcal{E}_0(U) = 1$ – is given by the formula

$$\mathcal{E}_t(U) = \mathcal{E}_t\Big(\int_0^\cdot \gamma_u \cdot dW_u\Big) = \exp\Big(\int_0^t \gamma_u \cdot dW_u - \frac{1}{2}\int_0^t |\gamma_u|^2 \, du\Big),$$

i.e., $\mathcal{E}_t(U) = \exp(U_t - \langle U \rangle_t / 2)$. Note that $\mathcal{E}(U)$ follows a strictly positive continuous local martingale under $\mathbf{P}$. For any probability measure $\tilde{\mathbf{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbf{P}$, we define the *density process* η by setting

$$\eta_t \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}}\Big(\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} \,\Big|\, \mathcal{F}_t\Big), \quad \forall\, t \in [0, T]. \tag{3.22}$$

It is clear that η follows a strictly positive, uniformly integrable martingale under $\mathbf{P}$; in particular, $\eta_t = \mathbf{E}_{\mathbf{P}}(\eta_T \,|\, \mathcal{F}_t)$ for every $t \in [0, T]$. Observe that an adapted process X follows a martingale under $\tilde{\mathbf{P}}$ if and only if the product ηX follows a martingale under $\mathbf{P}$. We are in a position to state a classical version of Girsanov's theorem.

Theorem 3.1.3 *Let W be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Suppose that γ is an adapted real-valued process such that*

$$\mathbf{E}_{\mathbf{P}}\Big\{\mathcal{E}_T\Big(\int_0^\cdot \gamma_u \cdot dW_u\Big)\Big\} = 1. \tag{3.23}$$

[1] If a filtration $\mathbf{F}$ is not $\mathbf{P}$-complete, its $\mathbf{P}$-completion runs as follows. First, we put $\tilde{F}_t = \sigma(\mathcal{F}_t \cup \mathcal{N})$, where $\mathcal{N}$ is the class of all $\mathbf{P}$-negligible sets from $\mathcal{F}_T$. Second, for any t we define $\hat{\mathcal{F}}_t = \tilde{\mathcal{F}}_{t+}$, where $\tilde{\mathcal{F}}_{t+} = \bigcap_{\epsilon > 0} \tilde{\mathcal{F}}_{t+\epsilon}$. Filtration $\hat{\mathbf{F}}$ is then $\mathbf{P}$-complete and right-continuous; it is referred to as the $\mathbf{P}$-*augmentation* of $\mathbf{F}$.

Define a probability measure $\tilde{\mathbf{P}}$ *on* $(\Omega, \mathcal{F}_T)$ *equivalent to* $\mathbf{P}$ *by means of the Radon-Nikodým derivative*

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \mathcal{E}_T\Big(\int_0^{\cdot} \gamma_u \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.} \tag{3.24}$$

Then the process $\tilde{W}$, *which is given by the formula*

$$\tilde{W}_t = W_t - \int_0^t \gamma_u \, du, \quad \forall\, t \in [0,T], \tag{3.25}$$

follows a standard d-dimensional Brownian motion on the space $(\Omega, \mathbf{F}, \tilde{\mathbf{P}})$.

Obviously, we always have $\mathcal{F}_t^{\tilde{W}} \subseteq \mathcal{F}_t$, $t \in [0,T]$. The filtrations generated by W and of $\tilde{W}$ do not coincide, in general. In particular, if the underlying filtration $\mathbf{F}$ is the $\mathbf{P}$-augmentation of the natural filtration of W, then we obtain $\mathcal{F}_t^{\tilde{W}} \subseteq \mathcal{F}_t^W$, $t \in [0,T]$. The next well-known result shows that, if the underlying filtration is the natural filtration of a Brownian motion, the density process of any probability measure equivalent to $\mathbf{P}$ has an exponential form.

Proposition 3.1.4 *Assume that the filtration* $\mathbf{F}$ *is the usual augmentation of the natural filtration of* W*; that is,* $\mathbf{F} = \mathbf{F}^W$. *Then for any probability measure* $\tilde{\mathbf{P}}$ *on* $(\Omega, \mathcal{F}_T)$ *equivalent to* $\mathbf{P}$, *there exists a d-dimensional process* γ, *adapted to the filtration* $\mathbf{F}^W$, *and such that the Radon-Nikodým derivative of* $\tilde{\mathbf{P}}$ *with respect to* $\mathbf{P}$ *equals*

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \mathcal{E}_t\,\Big(\int_0^{\cdot} \gamma_u \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.} \tag{3.26}$$

3.2 The Black-Scholes Option Valuation Formula

In Black and Scholes (1973), two alternative justifications of the option valuation formula are provided. The first relies on the fact that the risk-free return can be replicated by holding a continuously revised position in the underlying stock and an option. In other words, if an option is not priced according to the Black-Scholes formula, there is a sure profit to be made by some combination of either short or long sales of the option and the underlying asset. The second method of derivation is based on equilibrium arguments which require, in particular, that the option earns an expected rate of return commensurate with the risk involved in holding the option as an asset. The first approach is usually referred to as the *risk-free portfolio method,* while the second is known as the *equilibrium derivation* of the Black-Scholes formula.

The replication approach presented below is based on the observation that in the Black-Scholes setting the option value can be mimicked by holding a continuously revised position in the underlying stock and risk-free bonds. It should be stressed that, unless otherwise stated, we assume throughout that the financial market we are dealing with is perfect (partially, this was already implicit in the definition of a self-financing trading strategy).

3.2.1 Stock Price

Let us first describe stochastic processes which model the prices of primary securities, a common stock and a risk-free bond. Following Samuelson (1965) and, of course, Black and Scholes (1973), we take a *geometric* (or *exponential*) *Brownian motion* as a stochastic process which models the stock price. More specifically, the evolution of the stock price process S is assumed to be described by the following linear stochastic differential equation (SDE)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{3.27}$$

where $\mu \in \mathbf{R}$ is a constant *appreciation rate* of the stock price, $\sigma > 0$ is a constant *volatility* coefficient, and $S_0 \in \mathbf{R}_+$ is the initial stock price. (we write $\mathbf{R}_+$ to denote the set of all strictly positive real numbers). Finally, W_t, $t \in [0,T]$, stands for a one-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Let us emphasise that (3.27) is merely a shorthand notation for the following Itô integral equation

$$S_t = S_0 + \int_0^t \mu S_u \, du + \int_0^t \sigma S_u \, dW_u, \quad \forall \, t \in [0, T^*].$$

Actually, the sample paths of a Brownian motion are known to be almost everywhere non-differentiable functions, with probability 1, so that (3.27) can not be seen as a family of ordinary differential equations (for each fixed elementary event $\omega \in \Omega$). In the present context, it is convenient to assume that the underlying filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T^*]}$ is the standard augmentation of the natural filtration $\mathbf{F}^W$ of the underlying Brownian motion, i.e., that the equality $\mathcal{F}_t = \mathcal{F}_t^W$ holds for every $t \in [0, T^*]$.

Remarks. This assumption is not essential if our aim is to value European options written on a stock S. If this condition were not satisfied, the uniqueness of the martingale measure, and thus also the completeness of the market, would fail to hold, in general. This would not affect the arbitrage valuation of standard options written on a stock S, however.

It is elementary to check, using Itô's formula, that the process S which equals

$$S_t = S_0 \, \exp\big(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)\, t\big), \quad \forall \, t \in [0, T^*], \tag{3.28}$$

is indeed a solution of (3.27), starting from S_0 at time 0. The uniqueness of a solution is an immediate consequence of a general result due to Itô, which states that a SDE with Lipschitz continuous coefficients has a unique solution. It is apparent from (3.28) that the stock returns are lognormal, meaning that the random variable $\ln(S_t/S_u)$ has under $\mathbf{P}$ a Gaussian probability distribution for any choice of dates $u \le t \le T^*$. Since for any fixed t, the random variable $S_t = f(W_t)$ for some invertible function $f : \mathbf{R} \to \mathbf{R}_+$, it is clear that we have

$$\mathcal{F}_t^W = \sigma\{W_u \,|\, u \le t\} = \sigma\{S_u \,|\, u \le t\} = \mathcal{F}_t^S.$$

Therefore, the filtration generated by the stock price coincides with the natural filtration of the underlying noise process W, and thus $\mathbf{F}^S = \mathbf{F}^W = \mathbf{F}$. This means that the information structure of the model is based on observations of the stock price process only. Moreover, it is worthwhile to observe that the stock price S follows a time-homogeneous Markov process under $\mathbf{P}$ with respect to the filtration $\mathbf{F}$. In particular, we have

$$\mathbf{E}_{\mathbf{P}}(S_u \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}}(S_u \,|\, \mathcal{F}_t^S) = \mathbf{E}_{\mathbf{P}}(S_u \,|\, S_t) = S_t \, e^{\mu(u-t)}$$

for every $t \le u \le T^*$. This follows from the fact that

$$S_u = S_t \, \exp\big(\sigma(W_u - W_t) + (\mu - \tfrac{1}{2}\sigma^2)(u - t)\big), \tag{3.29}$$

and the increment $W_u - W_t$ of the Brownian motion W is independent of the σ-field $\mathcal{F}_t$, with the Gaussian law $N(0, \sqrt{u-t})$. The second security, whose price process is denoted by B in what follows, represents in our model an accumulation factor corresponding to a *savings account* (known also as a *money market account*). We assume throughout that the so-called short-term interest rate r is constant over the trading interval $[0, T^*]$. The risk-free security is assumed to continuously compound in value at the rate r; that is, $dB_t = rB_t \, dt$, or equivalently

$$B_t = e^{rt}, \quad \forall \, t \in [0, T^*], \tag{3.30}$$

as, by convention, we take $B_0 = 1$. Let us observe that we could have assumed instead that the interest rate r, and the stock price volatility σ, are deterministic functions of time. Also, the appreciation rate μ could be an adapted stochastic process, satisfying mild regularity conditions. Extensions of most results presented in this chapter to such a case are rather straightforward. It should be made clear that we are not concerned here with the question of whether the maintained model is the "correct" model of asset price fluctuations. Let us agree that (3.27) can hardly be seen as a realistic model of the real-world behavior of a stock price. On the other hand, the option prices obtained within the Black-Scholes framework are reasonably close to those observed on the option exchanges – at least for short-maturity options on liquid instruments. It is a puzzling question as to what extent this coincidence is a consequence of the notoriety of the Black-Scholes formula among market practitioners.

3.2.2 Self-financing Strategies

By a *trading strategy* we mean a pair $\phi = (\phi^1, \phi^2)$ of *progressively measurable* stochastic processes on the underlying probability space $(\Omega, \mathbf{F}, \mathbf{P})$. The concept of a self-financing trading strategy in the Black-Scholes market is formally based on the notion of the Itô integral. Intuitively, such a choice of stochastic integral is supported by the fact that, in the case of the Itô integral (as opposed to the Fisk-Stratonovich integral), the underlying process is integrated in a predictable way, meaning that we take its values on the left-hand end of each (infinitesimal) time interval. Formally, we say that a trading strategy $\phi = (\phi^1, \phi^2)$ over the time interval $[0, T]$ is *self-financing* if its wealth process $V(\phi)$, which is set to equal

$$V_t(\phi) = \phi^1_t S_t + \phi^2_t B_t, \quad \forall\, t \in [0, T], \tag{3.31}$$

satisfies the following condition

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi^1_u \, dS_u + \int_0^t \phi^2_u \, dB_u, \quad \forall\, t \in [0, T], \tag{3.32}$$

where the first integral is understood in the Itô sense. It is, of course, implicitly assumed that both integrals on the right-hand side of (3.32) are well-defined. It is well known that a sufficient condition for this is that[2]

$$\mathbf{P}\Big\{ \int_0^T (\phi^1_u)^2 \, du < \infty \Big\} = 1 \quad \text{and} \quad \mathbf{P}\Big\{ \int_0^T |\phi^2_u| \, du < \infty \Big\} = 1. \tag{3.33}$$

We denote by Φ the class of all self-financing trading strategies. It is well known that arbitrage opportunities are not excluded a priori from the class of self-financing trading strategies.

3.2.3 Martingale Measure for the Spot Market

We find it convenient to introduce the concept of the admissibility of a trading strategy directly in terms of a martingale measure (see Chap. 8 for an alternative approach). Let us denote $S^* = S/B$. By definition, a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F}_{T^*})$, equivalent to $\mathbf{P}$, is called a *martingale measure for* the process S^* if S^* is a local martingale under $\mathbf{Q}$. Similarly, a probability measure $\mathbf{P}^*$ is said to be a *martingale measure for the spot market* (or briefly, a *spot martingale measure*) if the discounted wealth of any self-financing trading strategy follows a local martingale under $\mathbf{P}^*$. The following result shows that both these notions coincide.

Lemma 3.2.1 *A probability measure is a spot martingale measure if and only if it is a martingale measure for the discounted stock price S^*.*

Proof. The proof relies on the following equality, which easily follows from the Itô formula

$$V^*_t(\phi) = V^*_0(\phi) + \int_0^t \phi^1_u \, dS^*_u, \quad \forall\, t \in [0, T^*],$$

where $V^*_t(\phi) = V_t(\phi)/B_t$ and ϕ is a self-financing strategy up to time T^*. It is now sufficient to make use of the local martingale property of the Itô stochastic integral. □

In the Black-Scholes setting, the martingale measure for the discounted stock price process is unique, and is explicitly known, as the following result shows.

[2] Note that condition (3.33) is invariant with respect to an equivalent change of a probability measure.

Lemma 3.2.2 *The unique martingale measure* $\mathbf{Q}$ *for the discounted stock price process* S^* *is given by the Radon-Nikodým derivative*

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\Big(\frac{r-\mu}{\sigma}\,W_{T^*} - \frac{1}{2}\,\frac{(r-\mu)^2}{\sigma^2}\,T^*\Big), \quad \mathbf{P}\text{-a.s.} \tag{3.34}$$

The dynamics of the discounted stock price S^* *under the martingale measure* $\mathbf{Q}$ *are*

$$dS^*_t = \sigma S^*_t\,dW^*_t, \tag{3.35}$$

and the process W^* *which equals*

$$W^*_t = W_t - \frac{r-\mu}{\sigma}\,t, \quad \forall\, t \in [0,T^*],$$

follows a standard Brownian motion on a probability space $(\Omega, \mathbf{F}, \mathbf{Q})$.

Proof. Essentially, all statements are direct consequences of Girsanov's theorem (cf. Theorem 3.1.3 and Proposition 3.1.4). □

Combining the two lemmas, we conclude that the unique spot martingale measure $\mathbf{P}^*$ is given on $(\Omega, \mathcal{F}_{T^*})$ by means of the Radon-Nikodým derivative

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big(\frac{r-\mu}{\sigma}W_{T^*} - \frac{1}{2}\,\frac{(r-\mu)^2}{\sigma^2}T^*\Big), \quad \mathbf{P}\text{-a.s.} \tag{3.36}$$

The discounted stock price S^* follows under $\mathbf{P}^*$ a strictly positive martingale, since (clearly $S_0 = S^*_0$)

$$S^*_t = S_0 \exp\big(\sigma W^*_t - \tfrac{1}{2}\sigma^2 t\big) \tag{3.37}$$

for every $t \in [0,T^*]$. Notice also that in view of (3.35), the dynamics of the stock price S under $\mathbf{P}^*$ are

$$dS_t = rS_t\,dt + \sigma S_t\,dW^*_t, \quad S_0 > 0, \tag{3.38}$$

and thus the stock price at time t equals

$$S_t = S_0 \exp\big(\sigma W^*_t + (r - \tfrac{1}{2}\sigma^2)t\big). \tag{3.39}$$

Finally, it is useful to observe that all filtrations involved in the model coincide; that is, $\mathbf{F} = \mathbf{F}^W = \mathbf{F}^{W^*} = \mathbf{F}^S = \mathbf{F}^{S^*}$. We are in a position to introduce the class of admissible trading strategies. An unconstrained Black-Scholes market model would involve arbitrage opportunities, so that reliable valuation of derivative instruments would not be possible.

Definition 3.2.1 A trading strategy $\phi \in \Phi$ is called $\mathbf{P}^*$-*admissible* if the discounted wealth process

$$V^*_t(\phi) = B_t^{-1} V_t(\phi), \quad \forall\, t \in [0,T], \tag{3.40}$$

follows a martingale under $\mathbf{P}^*$. We write $\Phi(\mathbf{P}^*)$ to denote the class of all $\mathbf{P}^*$-admissible trading strategies. The triple $\mathcal{M}_{BS} = (S, B, \Phi(\mathbf{P}^*))$ is called the arbitrage-free Black-Scholes model of a financial market, or briefly, the *Black-Scholes market.*

It is not hard to check that by restricting our attention to the class of $\mathbf{P}^*$-admissible strategies, we have guaranteed the absence of arbitrage opportunities in the Black-Scholes market. Consequently, given a contingent claim X which settles at time $T \le T^*$ and is *attainable* (i.e., can be replicated by means of a $\mathbf{P}^*$-admissible strategy) we can uniquely define its *arbitrage price,* $\pi_t(X)$, as the wealth $V_t(\phi)$ at time t of any $\mathbf{P}^*$-admissible trading strategy ϕ which replicates X – that is, satisfies $V_T(\phi) = X$. If no replicating $\mathbf{P}^*$-admissible strategy exists,[3] the arbitrage price of such a claim is not defined.

Conforming with the definition of an arbitrage price, to value a derivative security we will usually search first for its replicating strategy. Another approach to the valuation problem is also possible, as the following simple result shows. Since all statements are immediate consequences of definitions above, the proof is left to the reader.

[3] One can show that this happens only if a claim is not integrable under the martingale measure $\mathbf{P}^*$.

Corollary 3.2.1 *Let X be a $\mathbf{P}^*$-attainable European contingent claim which settles at time T. Then the arbitrage price $\pi_t(X)$ at time $t \in [0,T]$ in $\mathcal{M}_{BS}$ is given by the risk-neutral valuation formula*

$$\pi_t(X) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1} X \,\big|\, \mathcal{F}_t\big), \quad \forall\, t \in [0,T]. \tag{3.41}$$

In particular, the price of X at time 0 equals $\pi_0(X) = \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1} X\big)$.

For the sake of concreteness, we shall first consider a European call option written on a stock S, with expiry date T and strike price K. Let the function $c : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ be given by the formula

$$c(s,t) = sN\big(d_1(s,t)\big) - Ke^{-rt}N\big(d_2(s,t)\big), \tag{3.42}$$

where

$$d_1(s,t) = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \tag{3.43}$$

and

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{t}. \tag{3.44}$$

Furthermore, N stands for the standard Gaussian cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz, \quad \forall\, x \in \mathbf{R}.$$

We adopt the following notational convention

$$d_{1,2}(s,t) = \frac{\ln(s/K) + (r \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\,.$$

Let us denote by C_t the arbitrage price of a European call option at time t in the Black-Scholes market. We are in a position to state the main result of this chapter.

Theorem 3.2.1 *The arbitrage price at time $t \in [0,T]$ of the European call option with expiry date T and strike price K in the Black-Scholes market is given by the formula*

$$C_t = c(S_t, T-t), \quad \forall\, t \in [0,T], \tag{3.45}$$

where the function $c : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ is given by (3.42)–(3.44). *Moreover, the unique $\mathbf{P}^*$-admissible replicating strategy ϕ of the call option satisfies*

$$\phi_t^1 = \frac{\partial c}{\partial s}(S_t, T-t), \quad \phi_t^2 = e^{-rt}\left(c(S_t, T-t) - \phi_t^1 S_t\right) \tag{3.46}$$

for every $t \in [0,T]$.

Proof. We provide two alternative proofs of the Black-Scholes result. The first relies on the calculation of the replicating strategy. It thus gives not only the valuation formula (this, however, requires solving the Black-Scholes PDE (3.52)), but also explicit formulae for the replicating strategy. The second method makes direct use of the risk-neutral valuation formula (3.41) of Corollary 3.2.1. It focuses on the explicit computation of the arbitrage price of the option, rather than on the derivation of the hedging strategy.

First method. We start by assuming that the option price, C_t, satisfies the equality $C_t = v(S_t, t)$ for some function $v : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$. We may thus assume that the replicating strategy ϕ we are looking for has the following form

$$\phi_t = (\phi_t^1, \phi_t^2) = (g(S_t,t), h(S_t,t)) \tag{3.47}$$

for $t \in [0,T]$, where $g, h : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ are unknown functions. Since ϕ is assumed to be self-financing, the wealth process $V(\phi)$, which equals

$$V_t(\phi) = g(S_t,t)S_t + h(S_t,t)B_t = v(S_t,t), \tag{3.48}$$

needs to satisfy the following: $dV_t(\phi) = g(S_t,t)\,dS_t + h(S_t,t)\,dB_t$. Under the present assumptions, the last equality can be given the following form

$$dV_t(\phi) = (\mu - r)S_t g(S_t,t)\,dt + \sigma S_t g(S_t,t)\,dW_t + rv(S_t,t)dt, \tag{3.49}$$

since from the second equality in (3.48) we obtain

$$\phi_t^2 = h(S_t,t) = B_t^{-1}\big(v(S_t,t) - g(S_t,t)S_t\big).$$

We shall search for the wealth function v in the class of smooth functions on the open domain $\mathcal{D} = (0,+\infty) \times (0,T)$; more exactly, we assume that $v \in C^{2,1}(\mathcal{D})$. An application of Itô's formula yields[4]

$$dv(S_t,t) = \big(v_t(S_t,t) + \mu S_t\, v_s(S_t,t) + \tfrac{1}{2}\sigma^2 S_t^2\, v_{ss}(S_t,t)\big)\,dt + \sigma S_t\, v_s(S_t,t)\,dW_t.$$

Combining the expression above with (3.49), we arrive at the following expression for the Itô differential of the process Y, which equals $Y_t = v(S_t,t) - V_t(\phi)$

$$\begin{aligned} dY_t \;\; = \;\; & \big(v_t(S_t,t) + \mu S_t\, v_s(S_t,t) + \tfrac{1}{2}\sigma^2 S_t^2\, v_{ss}(S_t,t)\big)\,dt + \sigma S_t\, v_s(S_t,t)\,dW_t \\ & + (r-\mu)S_t g(S_t,t)\,dt - \sigma S_t g(S_t,t)\,dW_t - rv(S_t,t)\,dt. \end{aligned}$$

On the other hand, in view of (3.48), Y vanishes identically, thus $dY_t = 0$. By virtue of the uniqueness of canonical decomposition of continuous semimartingales, the diffusion term in the above decomposition of Y vanishes. In our case, this means that for every $t \in [0,T]$ we have

$$\int_0^t \sigma S_u \left(g(S_u,u) - v_s(S_u,u)\right) dW_u = 0,$$

or equivalently

$$\int_0^T S_u^2 \left(g(S_u,u) - v_s(S_u,u)\right)^2 du = 0. \tag{3.50}$$

For (3.50) to hold, it is sufficient and necessary that the function g satisfies

$$g(s,t) = v_s(s,t), \quad \forall\,(s,t) \in \mathbf{R}_+ \times [0,T]. \tag{3.51}$$

We shall assume from now on that (3.51) holds. Then, using (3.51), we get still another representation for Y, namely

$$Y_t = \int_0^t \Big\{ v_t(S_u,u) + \tfrac{1}{2}\sigma^2 S_u^2\, v_{ss}(S_u,u) + rS_u v_s(S_u,u) - rv(S_u,u)\Big\}\,du.$$

It is thus apparent that Y vanishes whenever v satisfies the following partial differential equation (PDE), referred to as the *Black-Scholes PDE*

$$v_t(s,t) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(s,t) + rsv_s(s,t) - rv(s,t) = 0. \tag{3.52}$$

Since $C_T = v(S_T,T) = (S_T - K)^+$, we need to impose also the terminal condition $v(s,T) = (s-K)^+$ for $s \in \mathbf{R}_+$. It is not hard to check by direct calculation that the function $v(s,t) = c(s,T-t)$, where c

[4]Subscripts on v denote partial derivatives with respect to the corresponding variables.

is given by (3.42)–(3.44), actually solves this problem.[5] It thus remains to check that the replicating strategy ϕ, which equals

$$\phi_t^1 = g(S_t,t) = v_s(S_t,t), \quad \phi_t^2 = h(S_t,t) = B_t^{-1}\big(v(S_t,t) - g(S_t,t)S_t\big),$$

is $\mathbf{P}^*$-admissible. Let us first check that ϕ is indeed self-financing. Though this property is here almost trivial by the construction of ϕ, it is nevertheless always preferable to check directly the self-financing property of a given strategy. We need to check that

$$dV_t(\phi) = \phi_t^1\, dS_t + \phi_t^2\, dB_t.$$

Since $V_t(\phi) = \phi_t^1\, S_t + \phi_t^2\, B_t = v(S_t,t),$ by applying Itô's formula, we get

$$dV_t(\phi) = v_s(S_t,t)\, dS_t + \tfrac{1}{2}\sigma^2 S_t^2\; v_{ss}(S_t,t)\, dt + v_t(S_t,t)\, dt.$$

In view of (3.52), the last equality can also be given the following form

$$dV_t(\phi) = v_s(S_t,t)\, dS_t + rv(S_t,t)\, dt - \, rS_t\, v_s(S_t,t)\, dt,$$

and thus

$$dV_t(\phi) = \phi_t^1\, dS_t + rB_t\, \frac{v(S_t,t) - S_t\phi_t^1}{B_t}\; dt = \phi_t^1\, dS_t + \phi_t^2\, dB_t.$$

This ends the verification of the self-financing property. In view of our definition of admissibility of trading strategies, we need to verify that the discounted wealth process $V^*(\phi),$ which satisfies

$$V_t^*(\phi) = V_0^*(\phi) + \int_0^t v_s(S_u,u)\, dS_u^*, \tag{3.53}$$

follows a martingale under the martingale measure $\mathbf{P}^*$. By direct calculation we obtain $v_s(s,t) = N\big(d_1(s,T-t)\big)$ for every $(s,t) \in \mathbf{R}_+ \times [0,T],$ and thus, using also (3.35), we find that

$$V_t^*(\phi) = V_0^*(\phi) + \int_0^t \sigma S_u N\big(d_1(S_u,T-u)\big)\, dW_u^* = V_0^*(\phi) + \int_0^t \zeta_u\, dW_u^*,$$

where $\zeta_u = \sigma S_u N\big(d_1(S_u,T-u)\big)$. The existence of the stochastic integral is an immediate consequence of the sample path continuity of the process η. From the general properties of the Itô stochastic integral, it is thus clear that the discounted wealth $V^*(\phi)$ follows a local martingale under $\mathbf{P}^*$. To show that $V^*(\phi)$ is a genuine martingale (even a square-integrable martingale), it is enough to observe that

$$\mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T \zeta_u\, dW_u^*\Big)^2 = \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T \zeta_u^2\, du\Big) \le \sigma^2 \int_0^T \mathbf{E}_{\mathbf{P}^*} S_u^2\, du < \infty,$$

where the second inequality follows easily from the existence of the exponential moments of a Gaussian random variable. □

Second method. The second method of the proof puts more emphasis on the explicit calculation of the price function c. The form of the replicating strategy will not be examined here. Since we wish to apply Corollary 3.2.1, we need to check first that the contingent claim $X = (S_T - K)^+$ is attainable in the Black-Scholes market model, however. This follows easily from the general results – more specifically, from the predictable representation property (see Theorem 3.1.2) combined with the square-integrability of the random variable $X^* = B_T^{-1}(S_T - K)^+$ under the martingale measure $\mathbf{P}^*$. We conclude that there exists a predictable process θ such that the stochastic integral

$$V_t^* = V_0^* + \int_0^t \theta_u\, dW_u^*, \quad \forall\, t \in [0,T], \tag{3.54}$$

[5] More precisely, the function v solves the final value problem for the backward PDE (3.52), while the function c solves the associated initial value problem for the forward PDE. To get the forward PDE, it is enough to substitute $v_t(s,t)$ with $-v_t(s,t)$ in (3.52). Both of these PDEs are of parabolic type on $\mathcal{D}$.

follows a (square-integrable) continuous martingale under $\mathbf{P}^*$, and

$$X^* = B_T^{-1}(S_T - K)^+ = \mathbf{E}_{\mathbf{P}^*} X^* + \int_0^T \theta_u \, dW_u^* = \mathbf{E}_{\mathbf{P}^*} X^* + \int_0^T h_u dS_u^*, \tag{3.55}$$

where we have put $h_t = \theta_t / (\sigma S_t^*)$. Let us consider a trading strategy ϕ that is given by

$$\phi_t^1 = h_t, \quad \phi_t^2 = V_t^* - h_t S_t^* = B_t^{-1}(V_t - h_t S_t), \tag{3.56}$$

where $V_t = B_t V_t^*$. Let us check first that the strategy ϕ is self-financing. Observe that the wealth process $V(\phi)$ coincides with V, and thus

$$\begin{aligned} dV_t(\phi) &= d(B_t V_t^*) = B_t \, dV_t^* + r V_t^* B_t \, dt \\ &= B_t h_t \, dS_t^* + r V_t \, dt = B_t h_t (B_t^{-1} \, dS_t - r B_t^{-1} S_t \, dt) + r V_t \, dt. \end{aligned}$$

This in turn yields

$$dV_t(\phi) = h_t \, dS_t + r(V_t - h_t S_t) \, dt = \psi_t^1 \, dS_t + \psi_t^2 \, dB_t,$$

as expected. Finally, it is clear that $V_T(\phi) = V_T = (S_T - K)^+$, so that ϕ is in fact a $\mathbf{P}^*$-admissible replicating strategy for X. So far, we have shown that the call option is represented by a contingent claim that is attainable in the Black-Scholes market $\mathcal{M}_{BS}$. Our goal is now to evaluate the arbitrage price of X using the risk-neutral valuation formula. Since $\mathcal{F}_t^W = \mathcal{F}_t^S$ for every $t \in [0, T]$, the risk-neutral valuation formula (3.41) can be rewritten as follows

$$C_t = B_t \, \mathbf{E}_{\mathbf{P}^*} \big((S_T - K)^+ B_T^{-1} \big| \mathcal{F}_t^S \big) = c(S_t, T - t) \tag{3.57}$$

for some function $c : \mathbf{R}_+ \times [0, T] \to \mathbf{R}$. The second equality in (3.57) can be inferred, for instance, from the Markovian property of S (it is easily seen that S follows a time-homogeneous Markov process under $\mathbf{P}^*$). Alternatively, we can make use of equality (3.29). The increment $W_T^* - W_t^*$ of the Brownian motion is independent of the σ-field $\mathcal{F}_t^W = \mathcal{F}_t^{W^*}$; on the other hand, the stock price S_t is manifestly $\mathcal{F}_t^W$-measurable. By virtue of the well-known properties of conditional expectation (see Lemma 2.2.1), we get

$$\mathbf{E}_{\mathbf{P}^*} \big((S_T - K)^+ \big| \mathcal{F}_t^S \big) = H(S_t, T - t), \tag{3.58}$$

where the function $H(s, T - t)$ is defined as follows

$$H(s, T - t) = \mathbf{E}_{\mathbf{P}^*} \left\{ \left(s e^{\sigma (W_T^* - W_t^*) + (r - \sigma^2/2)(T - t)} - K \right)^+ \right\}$$

for $(s, t) \in \mathbf{R}_+ \times [0, T]$. Therefore, it is enough to find the unconditional expectation

$$\mathbf{E}_{\mathbf{P}^*} \big((S_T - K)^+ B_T^{-1} \big) = \mathbf{E}_{\mathbf{P}^*} \big(S_T B_T^{-1} \, \mathrm{I}_D \big) - \mathbf{E}_{\mathbf{P}^*} \big(K B_T^{-1} \, \mathrm{I}_D \big) = J_1 - J_2,$$

where D stands for the set $\{S_T > K\}$. For J_2 we have

$$\begin{aligned} J_2 &= e^{-rT} K \, \mathbf{P}^* \{ S_T > K \} \\ &= e^{-rT} K \, \mathbf{P}^* \Big\{ S_0 \exp\big(\sigma W_T^* + (r - \tfrac{1}{2}\sigma^2) \, T \big) > K \Big\} \\ &= e^{-rT} K \, \mathbf{P}^* \Big\{ -\sigma W_T^* < \ln(S_0/K) + (r - \tfrac{1}{2}\sigma^2) \, T \Big\} \\ &= e^{-rT} K \, \mathbf{P}^* \Big\{ \xi < \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)}{\sigma \sqrt{T}} \Big\} \\ &= e^{-rT} K N(d_2(S_0, T)), \end{aligned}$$

since the random variable $\xi = -W_T / \sqrt{T}$ has a standard Gaussian law $N(0, 1)$ under the martingale measure $\mathbf{P}^*$. For the second integral, note first that

$$J_1 = \mathbf{E}_{\mathbf{P}^*} (S_T B_T^{-1} \, \mathrm{I}_D) = \mathbf{E}_{\mathbf{P}^*} (S_T^* \, \mathrm{I}_D). \tag{3.59}$$

It is convenient to introduce an auxiliary probability measure $\bar{\mathbf{P}}$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\bar{\mathbf{P}}}{d\mathbf{P}^*} = \exp\big(\sigma W_T^* - \tfrac{1}{2}\sigma^2 T\big), \quad \mathbf{P}^*\text{-a.s.}$$

By virtue of Proposition 3.1.3, the process $\bar{W}_t = W_t^* - \sigma t$ follows a standard Brownian motion on the space $(\Omega, \mathbf{F}, \bar{\mathbf{P}})$. Moreover, using (3.37) we obtain

$$S_T^* = S_0 \exp\big(\sigma \bar{W}_T + \tfrac{1}{2}\sigma^2 T\big). \tag{3.60}$$

Combining (3.59) with (3.60), we find that

$$\begin{aligned} J_1 &= S_0\, \bar{\mathbf{P}}\,\{S_T^* > K B_T^{-1}\} \\ &= S_0\, \bar{\mathbf{P}}\,\Big\{S_0 \exp\big(\sigma \bar{W}_T + \tfrac{1}{2}\sigma^2 T\big) > K e^{-rT}\Big\} \\ &= S_0\, \bar{\mathbf{P}}\,\Big\{-\sigma \bar{W}_T < \ln(S_0/K) + (r + \tfrac{1}{2}\sigma^2)\,T\Big\}. \end{aligned}$$

Using similar arguments as for J_2, we find that $J_1 = S_0 N(d_1(S_0, T))$. Summarizing, we have shown that the price at time 0 of a call option equals

$$C_0 = c(S_0, T) = S_0\, N\big(d_1(S_0, T)\big) - K e^{-rT}\, N\big(d_2(S_0, T)\big),$$

where

$$d_{1,2}(S_0, T) = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.$$

This ends the proof for the special case of $t = 0$. The valuation formula for $t > 0$ can be easily deduced from (3.58). □

It can be checked that the probability measure $\bar{\mathbf{P}}$ is the martingale measure corresponding to the choice of the stock price as a numeraire asset, that is, the unique probability measure, equivalent to $\mathbf{P}$, under which the process $B^* = B/S$ follows a martingale. Notice that we have shown that (cf. formula (2.42))

$$C_0 = S_0\, \bar{\mathbf{P}}\,\{S_T > K\} - e^{-rT} K \mathbf{P}^*\{S_T > K\}. \tag{3.61}$$

Undoubtedly, the most striking feature of the Black-Scholes result is the fact that the appreciation rate μ does not enter the valuation formula. This is not surprising, however, as expression (3.39), which describes the dynamics of the stock price under the martingale measure $\mathbf{P}^*$, does not involve the stock appreciation rate μ. More generally, we could have assumed that the appreciation rate is not constant, but is varying in time, or even follows a stochastic process (adapted to the underlying filtration). Assume, for instance, that the stock price process is determined by the stochastic differential equation (it is implicitly assumed that SDE (3.62) admits a unique strong solution S, which follows a continuous, strictly positive process)

$$dS_t = \mu(t, S_t) S_t\, dt + \sigma(t) S_t\, dW_t\,, \quad S_0 > 0, \tag{3.62}$$

where $\mu : [0, T^*] \times \mathbf{R} \to \mathbf{R}$ is a deterministic function satisfying certain regularity conditions, and $\sigma : [0, T^*] \to \mathbf{R}$ is also deterministic, with $\sigma(t) > \epsilon > 0$ for some constant ϵ. We introduce the accumulation factor B by setting

$$B_t = \exp\Big(\int_0^t r(u)\, du\Big), \quad \forall\, t \in [0, T], \tag{3.63}$$

for a deterministic function $r : [0, T^*] \to \mathbf{R}_+$. In view of (3.63), we have

$$dB_t = r(t) B_t dt, \quad B_0 = 1,$$

so that it is clear that $r(t)$ represents the instantaneous, continuously compounded interest rate prevailing at the market at time t. It is easily seen that, under the present hypotheses, the martingale measure $\mathbf{P}^*$ is unique, and the risk-neutral valuation formula (3.41) is valid. In particular, the price of a European call option equals

$$C_t = e^{-\int_t^T r(u)\,du}\,\mathbf{E}_{\mathbf{P}^*}\big((S_T - K)^+ \,\big|\, \mathcal{F}_t\big) \tag{3.64}$$

for every $t \in [0, T]$. Notice that under the martingale measure $\mathbf{P}^*$, the dynamics of S are

$$dS_t = r(t)S_t\,dt + \sigma(t)S_t\,dW_t^*. \tag{3.65}$$

If r and σ are, for instance, continuous functions, the unique solution to (3.65) is known to be

$$S_t = S_0\,\exp\Big(\int_0^t \sigma(u)\,dW_u^* + \int_0^t \big(r(u) - \tfrac{1}{2}\sigma^2(u)\big)\,du\Big).$$

It is now an easy task to derive a suitable generalization of the Black-Scholes formula using (3.64). Indeed, it appears that it is enough to substitute the quantities $r(T-t)$ and $\sigma^2(T-t)$ in the standard Black-Scholes formula by

$$\int_t^T r(u)\,du \quad \text{and} \quad \int_t^T \sigma^2(u)\,du,$$

respectively. The function obtained in such a way solves the Black-Scholes PDE with time-dependent coefficients.

Remarks. Let us stress that we have worked within a fully continuous-time setup – that is, with continuously rebalanced portfolios. For obvious reasons, such an assumption is not justified from the practical viewpoint. It is thus interesting to note that the Black-Scholes result can be derived in a discrete-time setup, by making use of the general equilibrium arguments (see Rubinstein (1976), Brennan (1979) or Huang and Litzenberger (1988)).

3.2.4 The Put-Call Parity for Spot Options

If there are to be no arbitrage opportunities, otherwise identical puts and calls must at all times during their lives obey, at least theoretically, the put-call parity relationship. A point worth stressing is that equality (3.66) does not rely on specific assumptions imposed on the stock price model. Indeed, it is satisfied in any arbitrage-free, continuous-time model of a security market, provided that the savings account B is modelled by (3.30).

Proposition 3.2.1 *The arbitrage prices of European call and put options with the same expiry date T and strike price K satisfy the put-call parity relationship*

$$C_t - P_t = S_t - K\,e^{-r(T-t)} \tag{3.66}$$

for every $t \in [0, T]$.

Proof. It is sufficient to observe that the payoffs of the call and put options at expiry satisfy the equality

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

Relationship (3.66) now follows from the risk-neutral valuation formula. Alternatively, one may derive (3.66) using simple no-arbitrage arguments. □

The put-call parity can be used to derive a closed-form expression for the arbitrage price of a European put option. Let us denote by $p : \mathbf{R}_+ \times [0, T] \to \mathbf{R}$ the function

$$p(s, t) = Ke^{-rt}N\big(-d_2(s, t)\big) - sN\big(-d_1(s, t)\big), \tag{3.67}$$

with $d_1(s, t)$ and $d_2(s, t)$ given by (3.43)–(3.44). The following result is an immediate consequence of Proposition 3.2.1 combined with Theorem 3.2.1.

Corollary 3.2.2 *The Black-Scholes price at time* $t \in [0,T]$ *of a European put option with strike price* K *equals* $P_t = p(S_t, T-t)$, *where the function* $p : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ *is given by* (3.67).

In particular, the price at time 0 of a European put option equals

$$P_0 = K\, e^{-rT}\, N\big(-d_2(S_0,T)\big) - S_0\, N\big(-d_1(S_0,T)\big).$$

Since in typical situations it is not difficult to find a proper form of the call-put parity, we shall usually restrict our attention to the case of a call option. In some circumstances, it will be convenient to explicitly account for the dependence of the option's price on its strike price K, as well as on the parameters r and σ of the model. For this reason, we shall sometimes write $C_t = c(S_t, T-t, K, r, \sigma)$ and $P_t = p(S_t, T-t, K, r, \sigma)$ in what follows.

3.2.5 The Black-Scholes PDE

Suppose we are given a Borel-measurable function $g : \mathbf{R} \to \mathbf{R}$. Then we have the following result, which generalizes Theorem 3.2.1. Let us observe that the problem of attainability of any $\mathbf{P}^*$-integrable European contingent claim can be resolved by invoking the predictable representation property (completeness of the multidimensional Black-Scholes model is examined later in this chapter.

Corollary 3.2.3 *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a Borel-measurable function, such that the random variable* $X = g(S_T)$ *is integrable under* $\mathbf{P}^*$. *Then the arbitrage price in* $\mathcal{M}_{BS}$ *of the claim* X *which settles at time* T *is given by the equality* $\pi_t(X) = v(S_t, t)$, *where the function* $v : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ *solves the Black-Scholes partial differential equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0, \quad \forall\, (s,t) \in (0,\infty) \times (0,T), \tag{3.68}$$

subject to the terminal condition $v(s,T) = g(s)$.

3.2.6 Sensitivity Analysis

We will now examine the basic features of trading portfolios involving options. Let us start by introducing the terminology widely used in relation to option contracts. We say that at a given instant t before or at expiry, a call option is *in-the-money* and *out-of-the-money* if $S_t > K$ and $S_t < K$, respectively. Similarly, a put option is said to be in-the-money and out-of-the-money at time t when $S_t < K$ and $S_t > K$, respectively. Finally, when $S_t = K$ both options are said to be *at-the-money.* The *intrinsic values* of a call and a put options are defined by the formulae

$$I_t^C = (S_t - K)^+, \quad I_t^P = (K - S_t)^+, \tag{3.69}$$

respectively, and the *time values* equal

$$J_t^C = C_t - (S_t - K)^+, \quad J_t^P = P_t - (K - S_t)^+, \tag{3.70}$$

for $t \in [0,T]$. It is thus evident that an option is in-the-money if and only if its intrinsic value is strictly positive. A short position in a call option is referred to as a *covered call* if the writer of the option hedges his or her risk exposure by holding the underlying stock; in the opposite case, the position is known as a *naked call.* When an investor who holds a stock also purchases a put option on this stock as a protection against stock price decline, the position is referred to as a *protective put.* While writing covered calls truncates, roughly speaking, the right-hand side of the return distribution and simultaneously shifts it to the right, buying protective puts truncates the left-hand side of the return distribution and at the same time shifts the distribution to the left. The last effect is due to the fact that the cost of a put increases the initial investment of a portfolio. Note that the traditional mean-variance analysis pioneered in Markowitz (1952)[6] is not an appropriate performance measure for portfolios containing options because of the skewness that may be introduced into portfolio returns. For more details on the effectiveness of option portfolio management, the interested reader may consult Leland (1980), Bookstaber (1981), and Bookstaber and Clarke (1984, 1985).

To measure quantitatively the influence of an option's position on a given portfolio of financial assets, we will now examine the dependence of its price on the fluctuations of the current stock price, time to expiry, strike price, and other relevant parameters. For a fixed expiry date T and arbitrary $t \leq T$, we denote by τ the time to option expiry – that is, we put $\tau = T - t$. We write $p(S_t, \tau, K, r, \sigma)$ and $c(S_t, \tau, K, r, \sigma)$ to denote the price of a call and a put option, respectively. The functions c and p are thus given by the formulae

$$c(s, \tau, K, r, \sigma) = sN(d_1) - Ke^{-r\tau}N(d_2) \tag{3.71}$$

and

$$p(s, \tau, K, r, \sigma) = Ke^{-r\tau}N(-d_2) - sN(-d_1), \tag{3.72}$$

where

$$d_{1,2} = d_{1,2}(s, \tau, K, \sigma, r) = \frac{\ln(s/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Recall that at any time $t \in [0, T]$, the replicating portfolio of a call option involves α_t shares of stock and β_t units of borrowed funds, where

$$\alpha_t = c_s(S_t, \tau) = N\big(d_1(S_t, \tau)\big), \quad \beta_t = c(S_t, \tau) - \alpha_t S_t. \tag{3.73}$$

The strictly positive number α_t, which determines the number of shares in the replicating portfolio, is commonly referred to as the *hedge ratio* or, briefly, the *delta* of the option.

It is not hard to verify by straightforward calculations that

$$\begin{aligned}
c_s &= N(d_1) = \delta > 0, \\
c_{ss} &= \frac{n(d_1)}{s\sigma\sqrt{\tau}} = \gamma > 0, \\
c_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_1) + Kre^{-r\tau}N(d_2) = \theta > 0, \\
c_\sigma &= s\sqrt{\tau}n(d_1) = \lambda > 0, \\
c_r &= \tau Ke^{-r\tau}N(d_2) = \rho > 0, \\
c_K &= -e^{-r\tau}N(d_2) < 0,
\end{aligned}$$

where n stands for the standard Gaussian probability density function – that is

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \forall\, x \in \mathbf{R}.$$

Similarly, in the case of a put option we get

$$\begin{aligned}
p_s &= N(d_1) - 1 = -N(-d_1) = \delta < 0, \\
p_{ss} &= \frac{n(d_1)}{s\sigma\sqrt{\tau}} = \gamma > 0, \\
p_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_1) + Kre^{-r\tau}(N(d_2) - 1) = \theta, \\
p_\sigma &= s\sqrt{\tau}n(d_1) = \lambda > 0, \\
p_r &= \tau Ke^{-r\tau}(N(d_2) - 1) = \rho < 0, \\
p_K &= e^{-r\tau}(1 - N(d_2)) > 0.
\end{aligned}$$

[6] The more recent literature include Markowitz (1987), Huang and Litzenberger (1988), and Elton and Gruber (1995).

Consequently, the delta of a long position in a put option is a strictly negative number (equivalently, the price of a put option is a strictly decreasing function of a stock price). Generally speaking, the price of a put moves in the same direction as a short position in the asset. In particular, in order to hedge a written put option, an investor needs to short a certain number of shares of the underlying stock. Another useful coefficient which measures the relative change of an option's price as the stock price moves is the *elasticity.* For any date $t \le T$, the elasticity of a call option is given by the equality

$$\eta_t^c = c_s(S_t, \tau) S_t / C_t = N\big(d_1(S_t, \tau)\big) S_t / C_t \,,$$

and for a put option it equals

$$\eta_t^p = p_s(S_t, \tau) S_t / P_t = -N\big(-d_1(S_t, \tau)\big) S_t / P_t.$$

Let us check that the elasticity of a call option price is always greater than 1. Indeed, for every $t \in [0, T]$, we have

$$\eta_t^c = 1 + e^{-r\tau} K C_t^{-1} N\big(d_2(S_t, \tau)\big) > 1.$$

This implies also that $C_t - c_s(S_t, \tau) S_t < 0$, so that the replicating portfolio of a call option always involves the borrowing of funds. Similarly, the elasticity of a put option satisfies

$$\eta_t^p = 1 - K e^{-r\tau} P_t^{-1} N\big(-d_2(S_t, \tau)\big) < 1.$$

This in turn implies that $P_t - S_t p_s(S_t, \tau) > 0$ (this inequality is obvious anyway) and thus the replicating portfolio of a short put option generates funds which are invested in risk-free bonds. These properties of replicating portfolios have special consequences when the assumption that the borrowing and lending rates coincide is relaxed. It is instructive to determine the dynamics of the option price C. Using Itô's formula, one finds easily that under the martingale measure $\mathbf{P}^*$ we have

$$dC_t = rC_t \, dt + \sigma C_t \eta_t^c \, dW_t^*.$$

This shows that the appreciation rate of the option price in a risk-neutral economy equals the risk-free rate r; however, the volatility coefficient equals $\sigma \eta_t^c$, so that, in contrast to the stock price volatility, the volatility of the option price follows a stochastic process.

The *position delta* is obtained by multiplying the *face value*[7] of the option position by its delta. Clearly, the position delta of a long call option (or a short put option) is positive; on the contrary, the position delta of a short call option (and of a long put option) is a negative number. The position delta of a portfolio is obtained by summing up the position deltas of its components. In this context, let us make the trivial observation that the position delta of a long stock equals 1, and that of a short stock is -1. It should be stressed that the option's (or option portfolio's) position delta measures only the market exposure at the current price levels of underlying assets. More precisely, it gives the first order approximation of the change in option price, which is sufficiently accurate only for a small move in the underlying asset price. To measure the change in the option delta as the underlying asset price moves, one should use the second derivative with respect to s of the option's price – that is, the option's *gamma.* The *gamma effect* means that position deltas also move as asset prices fluctuate, so that predictions of revaluation profit and loss based on position deltas are not sufficiently accurate, except for small moves. It is easily seen that bought options have positive gammas, while sold options have negative gammas. A portfolio's gamma is the weighted sum of its options' gammas, and the resulting gamma is determined by the dominant options in the portfolio. In this regard, options close to the money with a short time to expiry have a dominant influence on the portfolio's gamma. Generally speaking, a portfolio with a positive gamma is more attractive than a negative gamma portfolio. Recall that by *theta* we have denoted the derivative of the option price with respect to time to expiry. Generally, a portfolio dominated by bought options will have a negative theta, meaning that the portfolio will lose value as time passes (other variables

[7]The *face value* equals the number of underlying assets, e.g., the face value of an option on a lot of 100 shares of stock equals 100.

held constant). In contrast, short options generally have positive thetas. The derivative of the option price with respect to volatility is known as the *vega* of an option. A positive vega position will result in profits from increases in volatility; similarly, a negative vega means a strategy will profit from falling volatility.

Example 3.2.1 Consider a call option on a stock S, with strike price \$30 and with 3 months to expiry. Suppose, in addition, that the current stock price equals \$31, the stock price volatility is $\sigma = 10\%$ per annum, and the risk-free interest rate is $r = 5\%$ per annum with continuous compounding. We may assume, without loss of generality, that $t = 0$ and $T = 0.25$. Using (3.43), we obtain (approximately) $d_1(S_0, T) = 0.93$, and thus $d_2(S_0, T) = d_1(S_0, T) - \sigma\sqrt{T} = 0.88$. Consequently, using formula (3.42) and the following values of the standard Gaussian probability distribution function: $N(0.93) = 0.8238$ and $N(0.88) = 0.8106$, we find that (approximately) $C_0 = 1.52$, $\phi_0^1 = 0.82$ and $\phi_0^2 = -23.9$. This means that to hedge a short position in the call option, which was sold at the arbitrage price $C_0 = \$1.52$, an investor needs to purchase at time 0 the number $\delta = 0.82$ shares of stock (this transaction requires an additional borrowing of 23.9 units of cash). The elasticity at time 0 of the call option price with respect to the stock price equals

$$\eta_0^c = \frac{N\big(d_1(S_0, T)\big) S_0}{C_0} = 16.72.$$

Suppose that the stock price rises immediately from \$31 to \$31.2, yielding a return rate of 0.65% flat. Then the option price will move by approximately 16.5 cents from \$1.52 to \$1.685, giving a return rate of 10.86% flat. Roughly speaking, the option has nearly 17 times the return rate of the stock; of course, this also means that it will drop 17 times as fast. If an investor's portfolio involves 5 long call options (each on a round lot of 100 shares of stock), the position delta equals $500 \times 0.82 = 410$, so that it is the same as for a portfolio involving 410 shares of the underlying stock. Let us now assume that an option is a put. The price of a put option at time 0 equals (alternatively, P_0 can be found from the put-call parity (3.66))

$$P_0 = 30\, e^{-0.05/4}\, N(-0.88) - 31\, N(-0.93) = 0.15.$$

The *hedge ratio* corresponding to a short position in the put option equals approximately $\delta = -0.18$ (since $N(-0.93) = 0.18$), therefore to hedge the exposure, using the Black-Scholes recipe, an investor needs to short 0.18 shares of stock for one put option. The proceeds from the option and share-selling transactions, which amount to \$5.73, should be invested in risk-free bonds. Notice that the elasticity of the put option is several times larger than the elasticity of the call option. If the stock price rises immediately from \$31 to \$31.2, the price of the put option will drop to less than 12 cents.

3.2.7 Option on a Dividend-paying Stock

We consider the case when the dividend yield, rather than the dividend payoffs, is assumed to be known. More specifically, we assume that the stock S continuously pays dividends at some fixed rate κ. Following Samuelson (1965), we assume that the effective dividend rate is proportional to the level of the stock price. Although this is rather impractical as a realistic dividend policy associated with a particular stock, Samuelson's model fits the case of a stock index option reasonably well. The dividend payments should be used in full, either to purchase additional shares of stock, or to invest in risk-free bonds (however, intertemporal consumption or infusion of funds is not allowed). Consequently, a trading strategy $\phi = (\phi^1, \phi^2)$ is said to be self-financing when its wealth process $V(\phi)$, which equals, as usual, $V_t(\phi) = \phi_t^1 S_t + \phi_t^2 B_t$, satisfies

$$dV_t(\phi) = \phi_t^1\, dS_t + \kappa \phi_t^1 S_t\, dt + \phi_t^2\, dB_t,$$

or equivalently

$$dV_t(\phi) = \phi_t^1 (\mu + \kappa) S_t\, dt + \phi_t^1 \sigma S_t\, dW_t + \phi_t^2\, dB_t.$$

We find it convenient to introduce an auxiliary process $\tilde{S}_t = e^{\kappa t} S_t$, whose dynamics are given by the stochastic differential equation

$$d\tilde{S}_t = \mu_\kappa \tilde{S}_t \, dt + \sigma \tilde{S}_t \, dW_t,$$

where $\mu_\kappa = \mu + \kappa$. In terms of this process we have

$$V_t(\phi) = \phi_t^1 e^{-\kappa t} \, \tilde{S}_t + \phi_t^2 B_t$$

and

$$dV_t(\phi) = \phi_t^1 e^{-\kappa t} d\tilde{S}_t + \phi_t^2 \, dB_t.$$

Also, it is not difficult to check that the discounted wealth $V^*(\phi)$ satisfies

$$dV_t^*(\phi) = \phi_t^1 e^{-\kappa t} d\tilde{S}_t^*,$$

where $\tilde{S}_t^* = \tilde{S}_t B_t^{-1}$. Put another way, we have

$$dV_t^*(\phi) = \sigma \phi_t^1 \tilde{S}_t^* \left(dW_t + \sigma^{-1}(\mu_\kappa - r) \, dt \right).$$

In view of the last equality, the unique martingale measure $\mathbf{Q}$ for our model is given by (3.34), but with μ replaced by μ_κ. The dynamics of $V^*(\phi)$ under $\mathbf{Q}$ are given by the expression

$$dV_t^*(\phi) = \sigma \phi_t^1 \tilde{S}_t^* \, d\tilde{W}_t,$$

while those of $\tilde{S}^*$ are

$$d\tilde{S}_t^* = \sigma \tilde{S}_t^* \, d\tilde{W}_t, \tag{3.74}$$

and the process $\tilde{W}_t = W_t - (r - \mu_\kappa)t/\sigma$ follows a standard Brownian motion on the probability space $(\Omega, \mathbf{F}, \mathbf{Q})$. It is thus possible to construct, by defining in a standard way the class of admissible trading strategies, an arbitrage-free market in which a risk-free bond and a dividend-paying stock are primary securities. Assuming that this is done, the valuation of stock-dependent contingent claims is now standard. In particular, we have the following result.

Proposition 3.2.2 *The arbitrage price at time $t \leq T$ of a call option on a stock which pays dividends at a constant rate κ during the option's lifetime is given by the risk-neutral formula*

$$C_t^\kappa = B_t \, \mathbf{E}_{\mathbf{Q}} \big(B_T^{-1} (S_T - K)^+ \, \big| \, \mathcal{F}_t \big), \quad \forall \, t \in [0, T], \tag{3.75}$$

or explicitly

$$C_t^\kappa = \bar{S}_t N\big(d_1(\bar{S}_t, T - t)\big) - K e^{-r(T-t)} N\big(d_2(\bar{S}_t, T - t)\big), \tag{3.76}$$

where $\bar{S}_t = S_t e^{-\kappa(T-t)}$, and d_1, d_2 are given by (3.43)–(3.44). *Equivalently,*

$$C_t^\kappa = e^{-\kappa(T-t)} \Big(S_t N\big(\hat{d}_1(S_t, T-t)\big) - K e^{-(r-\kappa)(T-t)} N\big(\hat{d}_2(S_t, T-t)\big) \Big),$$

where

$$\hat{d}_{1,2}(s,t) = \frac{\ln(s/K) + (r - \kappa \pm \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \, . \tag{3.77}$$

Proof. The first equality is obvious. For the second, note first that we may rewrite (3.75) as follows

$$C_t^\kappa = e^{-r(T-t)} \, \mathbf{E}_{\mathbf{Q}} \big((S_T - K)^+ \, \big| \, \mathcal{F}_t \big) = e^{-\kappa T} e^{-r(T-t)} \, \mathbf{E}_{\mathbf{Q}} \big((\tilde{S}_T - e^{\kappa T} K)^+ \, \big| \, \mathcal{F}_t \big).$$

Using (3.74), and proceeding along the same lines as in the proof of Theorem 3.2.1, we find that

$$C_t^\kappa = e^{-\kappa T} \, c\,(\tilde{S}_t, T - t, e^{\kappa T} K),$$

where c is the standard Black-Scholes call option valuation function. Put another way, $C_t^\kappa = c^\kappa(S_t, T-t)$, where

$$c^\kappa(s,t) = se^{-\kappa t} N\big(\hat{d}_1(s,t)\big) - Ke^{-rt} N\big(\hat{d}_2(s,t)\big)$$

and $\hat{d}_1$, $\hat{d}_2$ are given by (3.77). □

Alternatively, to derive the valuation formula for a call option (or for any European claim of the form $X = g(S_T)$), we may first show that its arbitrage price equals $v(t, S_t)$, where v solves the following backward PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + (r-\kappa)s\frac{\partial v}{\partial s} - rv = 0 \tag{3.78}$$

on $(0,\infty)\times(0,T)$, subject to the standard terminal condition $v(s,T) = g(s)$. Under the assumptions of Proposition 3.2.2, one can show also that the value at time t of the forward contract with expiry date T and delivery price K is given by the equality

$$V_t(K) = e^{-\kappa(T-t)} S_t - e^{-r(T-t)} K.$$

Consequently, the forward price at time $t \le T$ of the stock S, for settlement at date T, equals

$$F_S^\kappa(t,T) = e^{(r-\kappa)(T-t)} S_t. \tag{3.79}$$

It is not difficult to check that the following version of the put-call parity relationship is valid

$$c^\kappa(S_t, T-t) - p^\kappa(S_t, T-t) = e^{-\kappa(T-t)} S_t - e^{-r(T-t)} K, \tag{3.80}$$

where $p^\kappa(S_t, T-t)$ stands for the arbitrage price at time t of the European put option with maturity date T and strike price K. In particular, if the exercise price equals the forward price of the underlying stock, then

$$c^\kappa(S_t, T-t) - p^\kappa(S_t, T-t) = 0.$$

As already mentioned, formula (3.76) is commonly used by market practitioners when valuing stock index options. For this purpose, one needs to assume that the stock index follows a lognormal process – actually, a geometric Brownian motion.[8] The dividend yield κ, which can be estimated from the historical data, slowly varies on a monthly (or quarterly) basis. Therefore, for options with a relatively short maturity, it is reasonable to assume that the dividend yield is constant.

3.2.8 Historical Volatility

All potential practical applications of the Black-Scholes formula hinge on knowledge of the volatility parameter of the return of stock prices. Indeed, of the five variables necessary to specify the model, all are directly observable except for the stock price volatility. The most natural approach uses an estimate of the standard deviation based upon an ex-post series of returns from the underlying stock. In the first empirical tests of the Black-Scholes model, performed by Black and Scholes (1972), the authors used over-the-counter data covering the 1966-1969 period. The stock volatilities were estimated from daily data over the year preceding each option price observation. They concluded that the model overpriced (underpriced, respectively) options on stocks with high (low, respectively) historical volatilities. More generally, they suggested that the usefulness of the model depends to a great extent upon investors' abilities to make good forecasts of the volatility. In subsequent years, the model was tested by several authors on exchange-traded options (see, e.g., Galai (1977)) confirming the bias in theoretical option prices observed originally by Black and Scholes. Although the estimation of stock price volatility from historical data is a fairly straightforward procedure, some important points should be mentioned. Firstly, to reduce the estimation risk arising from the sampling error, it seems natural to increase the sample size, e.g., by using a longer series of historical

[8] This does not follow from the assumption that each underlying stock follows a lognormal process, unless the stock index is calculated on the basis of the geometric average, as opposed to the more commonly used arithmetic average.

observations or by increasing the frequency of observations. Unfortunately, there is evidence to suggest that the variance is non-stationary, so that extending the observation period may make matters even worse. Furthermore, in many cases only daily data are available, so that there is a limit on the number of observations available within a given period. Finally, since the option pricing formula is non-linear in the standard deviation, an unbiased estimate of the standard deviation does not produce an unbiased estimate of the option price. To summarize, since the volatility is usually unstable through time, historical precedent is a poor guide for estimating future volatility. Moreover, estimates of option prices based on historical volatilities are systematically biased.

3.2.9 Implied Volatility

Alternatively, one can infer the investment community's consensus outlook as to the volatility of a given asset by examining the prices at which options on that asset trade. It was observed in the research of Black and Scholes (1972) that the actual standard deviation that would result over the life of an option would be a better input into the model if it were known in advance. Since an option price appears to be an increasing function of the underlying stock volatility, and all other factors determining the option price are known with certainty, one can infer the volatility that is implied in the observed market price of an option. More specifically, the *implied volatility,* σ_{imp} say, is derived from the non-linear equation[9]

$$C_t = sN\big(d_1(s, T-t)\big) - Ke^{-r(T-t)}N\big(d_2(s, T-t)\big),$$

where the only unknown parameter is σ, since for C_t we take the current market price of the call option. In other words, the implied volatility is the value of the standard deviation of stock returns that, when put in the Black-Scholes formula, results in a model price equal to the current market price. The actual value of the implied volatility σ_{imp} determined in this way depends, in general, on an option's contractual features – that is, on the value K of the strike price, as well as on the time $T-t$ to maturity. A properly weighted average of these implied standard deviations is used as a measure of the market forecasts of return variability. The implied volatility, considered as a function of the option's strike price, sometimes exhibits a specific U-shape. One of the long-standing problems has been how to reconcile this peculiar feature of empirical option prices, referred to as the *smile effect,* with the Black-Scholes model. A typical solution to this problem relies on a judicious choice of a discrete- or continuous-time model for stock price returns (see, e.g., Rubinstein (1994) or Derman et al. (1996)). Let us mention in this context the growing interest in financial modelling based on *stable* and *hyperbolic* distributions.

Brenner and Subrahmanyam (1988) and Corrado and Miller (1996) provide explicit approximate formulas for the implied volatility in the Black-Scholes setting. To derive the Corrado-Miller formulas, we start with the expansion of the Gaussian probability distribution function (see Stuart and Ord (1987), p.184)

$$N(x) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}}\Big(x - \frac{x^3}{6} + \frac{x^5}{40} + \dots\Big).$$

By substituting this expansion into the Black-Scholes call price, we obtain the following approximate formula for the call price (note that cubic and higher order terms are ignored)

$$C_0 \approx S_0\Big(\frac{1}{2} + \frac{d_1(S_0, T)}{2\sqrt{\pi}}\Big) - Ke^{-rT}\Big(\frac{1}{2} + \frac{d_2(S_0, T)}{2\sqrt{\pi}}\Big).$$

After standard manipulations, we arrive at the following quadratic equation in the quantity $\tilde{\sigma} = \sigma\sqrt{T}$

$$\tilde{\sigma}^2(S_0 + \tilde{K}) - \tilde{\sigma}\sqrt{8\pi}\big(C_0 - \tfrac{1}{2}(S_0 - \tilde{K})\big) + 2(S_0 - \tilde{K})\ln(S_0/\tilde{K}) = 0,$$

[9]To solve this equation explicitly, one needs to make use of numerical methods, such as, e.g., the Newton-Raphson method. See Manaster and Koehler (1982) for the reasonable choice of a starting value for the first iteration.

where we write $\tilde{K} = Ke^{-rT}$. The largest root of the last equation is

$$\tilde{s} = \sqrt{2\pi}\left(\frac{C_0 - \frac{1}{2}(S_0 - \tilde{K})}{S_0 + \tilde{K}}\right) + \sqrt{2\pi\left(\frac{C_0 - \frac{1}{2}(S_0 - \tilde{K})}{S_0 + \tilde{K}}\right)^2 - \frac{2(S_0 - \tilde{K})\ln(S_0/\tilde{K})}{S_0 + \tilde{K}}}.$$

In particular, when the current stock price S_0 equals the discounted exercise price, this reduces to the original Brenner-Subrahmanyam formula. A further gain in accuracy can be obtained using

$$\ln(S_0/\tilde{K}) \approx 2(S_0 - \tilde{K})/(S_0 + K),$$

and by substituting the number 4 with the parameter α. In this way, we arrive at the following approximate formula (which still reduces to the Brenner-Subrahmanyam formula if $S_0 = Ke^{-rT}$)

$$\tilde{s} \approx \sqrt{2\pi}\left(\frac{C_0 - \frac{1}{2}(S_0 - \tilde{K})}{S_0 + \tilde{K}}\right) + \sqrt{2\pi\left(\frac{C_0 - \frac{1}{2}(S_0 - \tilde{K})}{S_0 + \tilde{K}}\right)^2 - \alpha\left(\frac{S_0 - \tilde{K}}{S_0 + \tilde{K}}\right)^2}.$$

The clever choice of α allows us to improve the accuracy of (without affecting at-the-money accuracy). It appears (see Corrado and Miller (1996) for the details) that $\alpha = 2$ is a reasonable choice; we thus have

$$\tilde{s} \approx \frac{\sqrt{2\pi}}{S_0 + \tilde{K}}\left(C_0 - \tfrac{1}{2}(S_0 - \tilde{K}) + \sqrt{\left(C_0 - \tfrac{1}{2}(S_0 - \tilde{K})\right)^2 - \tfrac{1}{\pi}(S_0 - \tilde{K})^2}\right).$$

For the discussion of the accuracy of the last formula, we refer to the original paper by Corrado and Miller (1996).

3.2.10 Numerical Methods

Since a closed-form expression for the arbitrage price of a claim is not always available, an important issue is the study of numerical methods which give approximations of arbitrage prices and hedging strategies. In the Black-Scholes framework, these methods include: the use of multinomial lattices (trees) to approximate continuous-time models of security prices; procedures based on the Monte Carlo simulation of random variables; and finite-difference methods of solving the associated partial differential equations. Let us survey briefly the relevant literature.

Binomial (or, more generally, multinomial) models were studied by, among others, Cox et al. (1979) and Boyle (1986), who proposed an approximation of the stock price by means of the trinomial tree. The CRR binomial approximation of the exponential Brownian motion was examined in Sect. 2.1.2. It should be pointed out that if a continuous-time framework is taken as a benchmark, an increase in accuracy is gained by assuming that the first two moments of an approximating binomial process coincide with the corresponding moments of the exponential Brownian motion used to model the stock price. The corresponding modification of the CRR model of Chap. 2 runs as follows. For a fixed $T > 0$ and arbitrary $n \in \mathbf{N}$, we denote $\Delta_n = T/n$. Notice that for every $j = 0, \dots, n-1$, we have

$$S_{(j+1)\Delta_n} = S_{j\Delta_n} \exp\Big(\sigma\big(W^*_{(j+1)\Delta_n} - W^*_{j\Delta_n}\big) + \big(r - \tfrac{1}{2}\sigma^2\big)\Delta_n\Big)$$

under the martingale measure $\mathbf{P}^*$. Therefore, the expected value of the ratio $S_{(j+1)\Delta_n}/S_{j\Delta_n}$ equals

$$m_1(n) = \mathbf{E}_{\mathbf{P}^*}(S_{(j+1)\Delta_n}/S_{j\Delta_n}) = \exp(r\Delta_n)$$

for every j, and the second moment of this ratio is

$$m_2(n) = \mathbf{E}_{\mathbf{P}^*}(S_{(j+1)\Delta_n}/S_{j\Delta_n})^2 = \exp(2r\Delta_n + \sigma^2\Delta_n).$$

The corresponding values of the parameters u_n, d_n and p_n can be found by solving the following equations

$$\begin{cases} p_n u_n + (1-p_n) d_n &= m_1(n), \\ p_n u_n^2 + (1-p_n) d_n^2 &= m_2(n), \end{cases}$$

with $d_n = u_n^{-1}$. It is worthwhile to point out that the convergence result of Sect. 2.1.2 remains valid for the above modification of the CRR binomial model. More accurate approximation results were obtained by considering a trinomial tree, as proposed in Boyle (1986). In a trinomial tree, there are three possible future states for every node. For a fixed j, let us denote by $p_{jk}(i)$ the transition probability from the state $s_j(i)$ at time $j\Delta_n$ to the state $s_{j+1}(k)$ at time $(j+1)\Delta_n$. The transition probabilities $p_{jk}(i)$ and the state values $s_j(i)$ must be chosen in such a way that the lattice accurately approximates the behavior of the stock price S under the martingale probability. Once the lattice is already constructed, the valuation of European (and American) contingent claims is done by the standard backward induction method. In order to evaluate the prices of options which depend on two underlying assets, Boyle (1988) extends this technique to the case of a bivariate multinomial model. For further developments of the binomial approach, which take into account the presence of the volatility smile effect, we refer to Rubinstein (1994) and Derman et al. (1996). The efficient valuation of derivative securities using the Monte Carlo simulation was studied by, among others, Boyle (1977), Johnson and Shanno (1987), and Duffie and Glynn (1996). Hull and White (1988) apply the control variate technique in order to improve the efficiency of the finite-difference method when valuing American options on a dividend-paying stock.

Finite-difference methods of solving PDEs are examined in the context of option valuation in Schwartz (1977), Brennan and Schwartz (1978), and Courtadon (1982). Since a presentation of these methods is beyond the scope of this book, let us mention only that one can use both explicit and implicit finite-difference schemes, as well as the Crank-Nicolson method. For a detailed analysis of finite-difference methods in option valuation, we refer the reader to the monographs by Wilmott et al. (1993, 1995), in which the authors successfully apply the PDE approach to all kinds of standard and exotic options.

3.3 Futures Market

Let us denote by $f_S(t,T^*)$, $t \in [0,T^*]$, the futures price of a certain stock S for the date T^*. Our aim is to study the arbitrage pricing of a European contingent claim which settles at time T, with $T \le T^*$. The dynamics of futures prices $f_t = f_S(t,T^*)$ are given by the familiar expression

$$df_t = \mu_f f_t \, dt + \sigma_f f_t \, dW_t, \quad f_0 > 0, \tag{3.81}$$

where μ_f and $\sigma_f > 0$ are real numbers, and W_t, $t \in [0,T^*]$, stands for a one-dimensional standard Brownian motion, defined on a probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where $\mathbf{F} = \mathbf{F}^W$. The unique solution of SDE (3.81) is given by the formula (cf. (3.28))

$$f_t = f_0 \exp\big(\sigma_f W_t + (\mu_f - \tfrac{1}{2}\sigma_f^2)\, t\big), \quad \forall\, t \in [0,T]. \tag{3.82}$$

The price of the second security, a risk-free bond, is given as before by (3.30). In the Black-Scholes setting, the futures price dynamics of a stock S can be found by combining (3.27) with the following chain of equalities

$$f_t = f_S(t,T^*) = F_S(t,T^*) = S_t e^{r(T^*-t)}, \quad \forall\, t \in [0,T^*], \tag{3.83}$$

where, as usual, we write $F_S(t,T^*)$ to denote the forward price of the stock for the settlement date T^*. The last equality in (3.83) can be easily derived from the absence of arbitrage in the spot/forward market; the second is a consequence of the assumption that the interest rate is deterministic. If the dynamics of the stock price S are given by the SDE (3.27), then Itô's formula yields

$$df_t = (\mu - r) f_t \, dt + \sigma f_t \, dW_t,$$

with $f_0 = S_0 e^{rT^*}$, so that f satisfies (3.81) with $\mu_f = \mu - r$ and $\sigma_f = \sigma$. Since futures contracts are not necessarily associated with a physical underlying security, such as a stock or a bond, we prefer to study the case of futures options in an abstract way. This means that we consider (3.81) as the exogenously given dynamics of the futures price f. However, for the sake of notational simplicity, we shall write $\mu = \mu_f$ and $\sigma = \sigma_f$ in what follows. It follows from (3.83) that

$$f_S(t,T^*) = F_S(t,T^*) = \mathbf{E}_{\mathbf{P}^*}(S_{T^*} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T^*], \tag{3.84}$$

but also

$$f_S(t,T^*) = F_S(t,T^*) = S_t / B(t,T^*), \quad \forall\, t \in [0,T^*], \tag{3.85}$$

where $B(t,T^*)$ stands for the price at time t of the zero-coupon bond that matures at T^*. It appears that under uncertainty of interest rates, the right-hand sides of (3.84) and (3.85) characterize the futures and the forward price of S, respectively (see Chap. 10).

3.3.1 Self-financing Strategies

Let us fix a time horizon $T \le T^*$. We consider a European contingent claim X which settles at time T. By a *futures strategy* we mean a pair $\phi_t = (\phi^1_t, \phi^2_t)$ of real-valued adapted stochastic processes, defined on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Since it costs nothing to take a long or short position in a futures contract, the wealth process $V^f(\phi)$ of a futures strategy ϕ equals

$$V^f_t(\phi) = \phi^2_t B_t, \quad \forall\, t \in [0,T]. \tag{3.86}$$

We say that a futures strategy $\phi = (\phi^1, \phi^2)$ is *self-financing* if its wealth process $V^f(\phi)$ satisfies for every $t \in [0,T]$

$$V^f_t(\phi) = V^f_0(\phi) + \int_0^t \phi^1_u \, df_u + \int_0^t \phi^2_u \, dB_u. \tag{3.87}$$

We write Φ^f to denote the class of all self-financing futures strategies.

3.3.2 Martingale Measure for the Futures Market

A probability measure $\tilde{\mathbf{P}}$ equivalent to $\mathbf{P}$ is called the *futures martingale measure* if the discounted wealth $\tilde{V}^f(\phi)$ of any strategy $\phi \in \Phi^f$, which equals $\tilde{V}^f_t(\phi) = V^f_t(\phi)/B_t$, follows a local martingale under $\tilde{\mathbf{P}}$.

Lemma 3.3.1 *Let $\tilde{\mathbf{P}}$ be a probability measure on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbf{P}$. Then $\tilde{\mathbf{P}}$ is a futures martingale measure if and only if the futures price f follows a local martingale under $\tilde{\mathbf{P}}$.*

Proof. The discounted wealth $\tilde{V}^f$ for any trading strategy $\phi \in \Phi^f$ satisfies

$$d\tilde{V}^f_t(\phi) = B_t^{-1}(\phi^1_t \, df_t + \phi^2_t \, dB_t) - rB_t^{-1} V^f_t(\phi)\, dt = \phi^1_t B_t^{-1} \, df_t,$$

as (3.86) yields the equality

$$B_t^{-1}\big(\phi^2_t \, dB_t - rV^f_t(\phi)\, dt\big) = B_t^{-1}\big(B_t^{-1} V^f_t(\phi)\, dB_t - rV^f_t(\phi)\, dt\big) = 0.$$

The statement of the lemma now easily follows. □

The next result is an immediate consequence of Girsanov's theorem.

Proposition 3.3.1 *The unique martingale measure $\tilde{\mathbf{P}}$ for the process f is given by the Radon-Nikodým derivative*

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \exp\Big(-\frac{\mu}{\sigma} W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2} T\Big), \quad \mathbf{P}\text{-a.s.} \tag{3.88}$$

The dynamics of the futures price f under $\tilde{\mathbf{P}}$ are

$$df_t = \sigma f_t \, d\tilde{W}_t, \tag{3.89}$$

and the process $\tilde{W}_t = W_t + \mu t/\sigma$ follows a standard Brownian motion on the probability space $(\Omega, \mathbf{F}, \tilde{\mathbf{P}})$.

It is clear from (3.89) that

$$f_t = f_0 \exp\bigl(\sigma \tilde{W}_t - \tfrac{1}{2}\sigma^2 t\bigr), \quad \forall\, t \in [0, T^*], \tag{3.90}$$

so that f follows a strictly positive martingale under $\tilde{\mathbf{P}}$. As expected, we say that a futures strategy $\phi \in \Phi^f$ is $\tilde{\mathbf{P}}$-*admissible* if the discounted wealth $\tilde{V}^f(\phi)$ follows a martingale under $\tilde{\mathbf{P}}$. We shall study an arbitrage-free futures market $\mathcal{M}^f = (f, B, \Phi^f(\tilde{\mathbf{P}}))$, where $\Phi^f(\tilde{\mathbf{P}})$ is the class of all $\tilde{\mathbf{P}}$-admissible futures trading strategies. The futures market model $\mathcal{M}^f$ is referred to as the *Black futures market* in what follows. The notion of an arbitrage price is defined in a similar way to the case of the Black-Scholes market.

3.3.3 The Black Futures Option Formula

We shall now derive the valuation formula for futures options, due to Black (1976). Let the function $c^f : \mathbf{R}_+ \times [0, T] \to \mathbf{R}$ be given by *Black's futures formula*

$$c^f(f, t) = e^{-rt}\Bigl(f\, N\bigl(\tilde{d}_1(f, t)\bigr) - K\, N\bigl(\tilde{d}_2(f, t)\bigr)\Bigr), \tag{3.91}$$

where

$$\tilde{d}_{1,2}(f, t) = \frac{\ln(f/K) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \tag{3.92}$$

and N denotes the standard Gaussian cumulative distribution function.

Before we formulate the main result of this section, let us consider once again the futures contract written on a stock whose dynamics are given by (3.27). If $T = T^*$, the futures option valuation result (3.91)–(3.92) can be found directly from the Black-Scholes formula by setting $S_t = f_t e^{-r(T-t)}$ (this applies also to the replicating strategy). Intuitively, this follows from the simple observation that in this case we have $f_T = S_T$ at the option's expiry, and thus the payoffs from both options coincide. In practice, the expiry date of a futures option usually precedes the settlement date of the underlying futures contract – that is, $T < T^*$. In such a case we have

$$C_T^f = \bigl(f_S(T, T^*) - K\bigr)^+ = e^{r(T^*-T)}\,\bigl(S_T - Ke^{-r(T^*-T)}\,\bigr)^+,$$

and we may still value the futures option as if it were the spot option. Such considerations rely on the equality $f_S(t, T^*) = F_S(t, T^*)$, which in turn hinges on the assumption that the interest rate is a deterministic function. They thus cannot be easily extended to the case of stochastic interest rates. For this reason, we prefer to give below a straightforward derivation of the Black futures formula.

Theorem 3.3.1 *The arbitrage price C^f in the arbitrage-free futures market $\mathcal{M}^f$ of a European futures call option, with expiry date T and strike price K, is given by the equality $C_t^f = c^f(f_t, T-t)$. The futures strategy $\phi \in \Phi^f(\tilde{\mathbf{P}})$ that replicates a European futures call option is given by*

$$\phi_t^1 = \frac{\partial c^f}{\partial f}(f_t, T-t), \quad \phi_t^2 = e^{-rt} c^f(f_t, T-t), \tag{3.93}$$

for every $t \in [0, T]$.

Proof. We shall follow rather closely the proof of Theorem 3.2.1. Therefore, we shall focus mainly on the derivation of (3.91)–(3.93). Some technical details, such as integrability of random variables or admissibility of trading portfolios, are left aside.

First method. Assume that the price process C^f_t is of the form $C^f_t = v(f_t,t)$ for some function $v : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$, and consider a futures strategy $\phi \in \Phi^f$ of the form $\phi_t = (g(f_t,t), h(f_t,t))$ for some functions $g,h : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$. Since the replicating portfolio ϕ is assumed to be self-financing, the wealth process $V^f(\phi)$, which equals

$$V^f_t(\phi) = h(S_t,t)B_t = v(f_t,t), \tag{3.94}$$

satisfies

$$dV^f_t(\phi) = g(f_t,t)\,df_t + h(f_t,t)\,dB_t,$$

or more explicitly

$$dV^f_t(\phi) = f_t\mu g(f_t,t)\,dt + f_t\sigma g(f_t,t)\,dW_t + rv(f_t,t)dt. \tag{3.95}$$

On the other hand, assuming that the function v is sufficiently smooth, we find that

$$dv(f_t,t) = \big(v_t(f_t,t) + \mu f_t v_f(f_t,t) + \tfrac{1}{2}\sigma^2 f_t^2\, v_{ff}(f_t,t)\big)\,dt + \sigma f_t\, v_f(f_t,t)\,dW_t.$$

Combining the last equality with (3.95), we get the following expression for the Itô differential of the process $Y_t = v(f_t,t) - V^f_t(\phi)$

$$\begin{aligned} dY_t \;\; = \;\; & \big(v_t(f_t,t) + \mu f_t v_f(f_t,t) + \tfrac{1}{2}\sigma^2 f_t^2\, v_{ff}(f_t,t)\big)\,dt + \sigma f_t\, v_f(f_t,t)\,dW_t \\ & - \mu f_t g(f_t,t)\,dt - \sigma f_t g(f_t,t)\,dW_t - rv(f_t,t)\,dt = 0. \end{aligned}$$

Arguing along the same lines as in the proof of Theorem 3.2.1, we infer that

$$g(f,t) = v_f(f,t), \quad \forall\,(f,t) \in \mathbf{R}_+ \times [0,T], \tag{3.96}$$

and thus also

$$Y_t = \int_0^t \Big\{ v_t(f_u,u) + \tfrac{1}{2}\sigma^2 f_u^2 v_{ff}(f_u,u) - rv(f_u,u) \Big\}\,du = 0, \tag{3.97}$$

where the last equality follows from the definition of Y. To guarantee the last equality we assume that v satisfies the following partial differential equation (referred to as the *Black PDE*)

$$v_t + \tfrac{1}{2}\sigma^2 f^2 v_{ff} - rv = 0 \tag{3.98}$$

on $(0,\infty) \times (0,T)$, with the terminal condition $v(f,T) = (f-K)^+$. Since the function $v(f,t) = c^f(f,T-t)$, where c^f is given by (3.91)–(3.93), is easily seen to solve this problem, to complete the proof it is sufficient to note that, by virtue of (3.96) and (3.94), the unique $\tilde{\mathbf{P}}$-admissible strategy ϕ that replicates the option satisfies

$$\phi^1_t = g(f_t,t) = v_f(f_t,t), \quad \phi^2_t = h(f_t,t) = B_t^{-1} v_f(f_t,t).$$

Details are left to the reader.

Second method. Since the random variable $X^* = B_T^{-1}(S_T - K)^+$ is integrable with respect to the martingale measure $\tilde{\mathbf{P}}$, it is enough to evaluate the conditional expectation

$$C^f_t = B_t\,\mathbf{E}_{\tilde{\mathbf{P}}}\big((f_T-K)^+ B_T^{-1}\,\big|\,\mathcal{F}^f_t\big) = B_t\,\mathbf{E}_{\tilde{\mathbf{P}}}\big((f_T-K)^+ B_T^{-1}\,\big|\,f_t\big).$$

This means that, in particular for $t=0$, we need to find the expectation

$$\mathbf{E}_{\tilde{\mathbf{P}}}\big((f_T-K)^+ B_T^{-1}\big) = \mathbf{E}_{\tilde{\mathbf{P}}}(f_T B_T^{-1}\,\mathrm{I}_D) - \mathbf{E}_{\tilde{\mathbf{P}}}(K B_T^{-1}\,\mathrm{I}_D) = I_1 - I_2,$$

where D denotes the set $\{f_T > K\}$. For I_2, we have

$$I_2 = e^{-rT} K\, \tilde{\mathbf{P}}\{f_T > K\} = e^{-rT} K\, \tilde{\mathbf{P}}\Big\{\, f_0 \exp\big(\sigma \tilde{W}_T - \tfrac{1}{2}\sigma^2 T\big) > K\Big\},$$

and thus

$$\begin{aligned} I_2 &= e^{-rT} K\, \tilde{\mathbf{P}}\Big\{ -\sigma \tilde{W}_T < \ln(f_0/K) - \tfrac{1}{2}\sigma^2 T\Big\} \\ &= e^{-rT} K\, \tilde{\mathbf{P}}\Big\{ \xi < \frac{\ln(f_0/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}}\Big\} \\ &= e^{-rT} K N\big(\tilde{d}_2(f_0, T)\big), \end{aligned}$$

since the random variable $\xi = -\tilde{W}_T/\sqrt{T}$ has under $\tilde{\mathbf{P}}$ the standard Gaussian law. To evaluate I_1, we define an auxiliary probability measure $\hat{\mathbf{P}}$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\hat{\mathbf{P}}}{d\tilde{\mathbf{P}}} = \exp\big(\sigma \tilde{W}_T - \tfrac{1}{2}\sigma^2 T\big), \quad \tilde{\mathbf{P}}\text{-a.s.},$$

and thus (cf. (3.90))

$$I_1 = \tilde{\mathbf{P}}\{f_T B_T^{-1}\, \mathrm{I}_D\} = e^{-rT} f_0\, \hat{\mathbf{P}}\{f_T > K\}.$$

Moreover, the process $\hat{W}_t = \tilde{W}_t - \sigma t$ follows a standard Brownian motion on the filtered probability space $(\Omega, \mathbf{F}, \hat{\mathbf{P}})$, and

$$f_T = f_0 \exp\big(\sigma \hat{W}_T + \tfrac{1}{2}\sigma^2 T\big).$$

Consequently,

$$\begin{aligned} I_1 &= e^{-rT} f_0\, \hat{\mathbf{P}}\,\{f_T > K\} \\ &= e^{-rT} f_0\, \hat{\mathbf{P}}\Big\{ f_0 \exp\big(\sigma \hat{W}_T + \tfrac{1}{2}\sigma^2 T\big) > K\Big\} \\ &= e^{-rT} f_0\, \hat{\mathbf{P}}\Big\{ -\sigma \hat{W}_T < \ln(f_0/K) + \tfrac{1}{2}\sigma^2 T\Big\} \\ &= e^{-rT} f_0\, N\big(\tilde{d}_1(f_0, T)\big). \end{aligned}$$

The general valuation result for any date t is a consequence of the Markov property of f. □

The method of arbitrage pricing in the Black futures market can be easily extended to any path-independent claim contingent on the futures price. In fact, the following corollary follows easily from the first proof of Theorem 3.3.1. As already mentioned, in financial literature, the partial differential equation (3.99) is commonly referred to as the *Black PDE.*

Corollary 3.3.1 *The arbitrage price in $\mathcal{M}^f$ of any attainable contingent claim $X = g(f_T)$ which settles at time T is given by $\pi_t^f(X) = v(f_t, t)$, where the function $v : \mathbf{R}_+ \times [0, T] \to \mathbf{R}$ is a solution of the following partial differential equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 v}{\partial f^2} - rv = 0, \quad \forall\, (f, t) \in (0, \infty) \times (0, T), \tag{3.99}$$

subject to the terminal condition $v(f, T) = g(f)$.

Let us denote by $P_t^f = p^f(f_t, T - t)$ the price of a put futures option with strike price K and $T - t$ to its expiry date, provided that the current futures price is f_t. To find the price of a futures put option, we can use the following result, whose easy proof is left to the reader.

Corollary 3.3.2 *The following relationship, known as the put-call parity for futures options, holds for every $t \in [0,T]$*

$$C_t^f - P_t^f = c^f(f_t, T-t) - p^f(f_t, T-t) = e^{-r(T-t)}(f_t - K). \tag{3.100}$$

Consequently,

$$p^f(f_t, T-t) = e^{-r(T-t)} \Big(KN\big(-\tilde{d}_2(f_t, T-t)\big) - f_t N\big(-\tilde{d}_1(f_t, T-t)\big)\Big),$$

where $\tilde{d}_1(f,t)$ and $\tilde{d}_2(f,t)$ are given by (3.92).

Example 3.3.1 Suppose that the call option considered in Example 3.2.1 is a futures option. This means, in particular, that the price is now interpreted as the futures price. Using (3.91), one finds that the arbitrage price of a futures call option equals (approximately) $C_0^f = 1.22$. Moreover, the portfolio that replicates the option is composed at time 0 of ϕ_0^1 futures contracts and ϕ_0^2 invested in risk-free bonds, where $\phi_0^1 = 0.75$ and $\phi_0^2 = 1.22$. Since the number ϕ_0^1 is positive, it is clear that an investor who assumes a short option position needs to enter ϕ_0^1 (long) futures contracts. Such a position, commonly referred to as the *long hedge,* is also a generally accepted practical strategy for a party who expects to purchase a given asset at some future date. To find the arbitrage price of the corresponding put futures option, we make use of the put-call parity relationship (3.100). We find that $P_0^f = 0.23$; moreover, for the replicating portfolio of the put option we have $\phi_0^1 = -0.25$ and $\phi_0^2 = 0.23$. Since now $\phi_0^1 < 0$, we deal here with the *short hedge* – a strategy typical for an investor who expects to sell a given asset at some future date.

3.3.4 Options on Forward Contracts

We adopt the classic Black-Scholes framework of Sect. 3.2.1. We will consider a forward contract with delivery date $T^* > 0$ written on a non-dividend-paying stock S. Recall that the forward price at time t of a stock S for the settlement date T^* equals

$$F_S(t, T^*) = S_t e^{r(T^*-t)}, \quad \forall\, t \in [0, T^*].$$

This means that the forward contract, established at time t, in which the delivery price is set to be equal to $F_S(t,T^*)$ is worthless at time t. It should be stressed that the value of such a contract at time $u \in [t, T^*]$ is no longer zero, in general. It is intuitively clear that the value $V^F(t,u,T^*)$ of such a contract at time u equals the discounted value of the difference between the current forward price of S at time u and its value at time t, that is

$$V^F(t,u,T^*) = e^{-r(T^*-u)}\big(S_u e^{r(T^*-u)} - S_t e^{r(T^*-t)}\big) = S_u - S_t e^{r(u-t)}$$

for every $u \in [t, T^*]$. The last equality can also be derived by applying directly the risk-neutral valuation formula to the claim $X = S_{T^*} - F_S(t,T^*)$, which settles at time T^*. Indeed, we have

$$\begin{aligned} V^F(t,u,T^*) &= B_u\, \mathbf{E}_{\mathbf{P}^*}\big(B_{T^*}^{-1} S_{T^*} - B_{T^*}^{-1} S_t e^{r(T^*-t)} \,|\, \mathcal{F}_u\big) \\ &= B_u\, \mathbf{E}_{\mathbf{P}^*}(S_{T^*}^* \,|\, \mathcal{F}_u) - S_t e^{r(T^*-t)} e^{-r(T^*-u)} \\ &= S_u - S_t e^{r(u-t)} = S_u - F_S(t,u), \end{aligned}$$

since the random variable S_t is $\mathcal{F}_u$-measurable. It is worthwhile to observe that $V^F(t,u,T^*)$ is in fact independent of the settlement date T^*, therefore we may and do write $V^F(t,u,T^*) = V^F(t,u)$ in what follows. By definition,[10] a call option written at time t on a forward contract with the expiry

[10] Since options on forward contracts are not traded on exchanges, the definition of an option written on a forward contract is largely a matter of convention.

date $t < T < T^*$ is simply a call option, with zero strike price, which is written on the value of the underlying forward contract. The terminal option's payoff thus equals

$$C_T^F = \big(V^F(t,T)\big)^+ = \big(S_T - S_t e^{r(T-t)}\big)^+.$$

It is clear that the call option on the forward contract purchased at time t gives the right to enter at time T into the forward contract on the stock S with delivery date T^* and delivery price $F_S(t,T^*)$. If the forward price at time T is less than it was at time t, the option is abandoned. In the opposite case, the holder exercises the option, and either enters, at no additional cost, into a forward contract under more favorable conditions than those prevailing at time T, or simply takes the payoff of the option. Assume now that the option was written at time 0, so that

$$C_T^F = (V^F(0,T))^+ = (S_T - S_0 e^{rT})^+.$$

To value such an option at time $t \le T$, we can make use of the Black-Scholes formula with the (fixed) strike price $K = S_0 e^{rT}$. After simple manipulations, we find that the option's value at time t is

$$C_t^F = S_t N\big(d_1(S_t,t)\big) - S_0 e^{rt} N\big(d_2(S_t,t)\big), \tag{3.101}$$

where

$$d_{1,2}(S_t,t) = \frac{\ln S_t - \ln(S_0 e^{rt}) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Alternatively, we can make use of Black's futures formula. Since the futures price $f_S(T,T)$ coincides with S_T, we have

$$C_T^F = (f_S(T,T) - S_0 e^{rT})^+.$$

An application of Black's formula yields

$$C_t^F = e^{-r(T-t)}\Big(f_t\, N\big(\tilde d_1(f_t,t)\big) - S_0 e^{rT}\, N\big(\tilde d_2(f_t,t)\big)\Big), \tag{3.102}$$

where $f_t = f_S(t,T)$, and

$$\tilde d_{1,2}(f_t,t) = \frac{\ln f_t - \ln(S_0 e^{rT}) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Since in the Black-Scholes setting the relationship $f_S(t,T) = S_t e^{r(T-t)}$ is satisfied, it is apparent that expressions (3.101) and (3.102) are equivalent.

Chapter 4

Foreign Market Derivatives

In this chapter, an arbitrage-free model of the domestic security market is extended by assuming that trading in foreign assets, such as foreign risk-free bonds and foreign stocks (and their derivatives), is allowed. We shall work within the classic Black-Scholes framework. More specifically, both domestic and foreign risk-free interest rates are assumed throughout to be non-negative constants, and the foreign stock price and the exchange rate are modelled by means of geometric Brownian motions. This implies that the foreign stock price, as well as the price in domestic currency of one unit of foreign currency (i.e., the exchange rate) will have lognormal probability distributions at future times. Notice, however, that in order to avoid perfect correlation between these two processes, the underlying noise process should be modelled by means of a multidimensional, rather than a one-dimensional, Brownian motion. Our main goal is to establish explicit valuation formulae for various kinds of currency and foreign equity options. Also, we will provide some indications concerning the form of the corresponding hedging strategies. It is clear that foreign market contracts of certain kinds should be hedged both against exchange rate movements and against the fluctuations of relevant foreign equities.

4.1 Cross-currency Market Model

All processes considered in what follows are defined on a common filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where the filtration $\mathbf{F}$ is assumed to be the $\mathbf{P}$-augmentation of the natural filtration generated by a d-dimensional Brownian motion $W = (W^1, \ldots, W^d)$. The domestic and foreign interest rates, r_d and r_f, are assumed to be given real numbers. Consequently, the domestic and foreign savings accounts satisfy

$$B_t^d = \exp(r_d t), \quad B_t^f = \exp(r_f t), \quad \forall\, t \in [0, T^*], \tag{4.1}$$

where B_t^d and B_t^f are denominated in units of domestic and foreign currency, respectively. The exchange rate process Q, which is used to convert foreign payoffs into domestic currency, is modelled by the following stochastic differential equation

$$dQ_t = Q_t \big(\mu_Q \, dt + \sigma_Q \cdot dW_t\big), \quad Q_0 > 0, \tag{4.2}$$

where $\mu_Q \in \mathbf{R}$ is a constant drift coefficient and $\sigma_Q \in \mathbf{R}^d$ denotes a constant volatility vector. As usual, the dot "$\cdot$" stands for the Euclidean inner product in $\mathbf{R}^d$, for instance

$$\sigma_Q \cdot dW_t = \sum_{i=1}^{d} \sigma_Q^i \, dW_t^i.$$

Also, we write $|\cdot|$ to denote the Euclidean norm in $\mathbf{R}^d$. Using this notation, we can make a clear distinction between models which are based on a one-dimensional Brownian motion, and those models in which the multidimensional character of the underlying noise process is essential. Let us make clear that we adopt here the convention that the exchange rate process Q is denominated in units of domestic currency per unit of foreign currency; that is, Q_t represents the domestic price at time t of one unit of the foreign currency. It should be stressed, however, that the exchange rate process Q cannot be treated on an equal basis with the price processes of domestic assets; put another way, the foreign currency cannot be seen as just an additional traded security in the domestic market model, unless the impact of the foreign interest rate is taken into account. The process Q plays an important role as a tool which allows the conversion of foreign market cash flows into units of domestic currency. Moreover, it can also play the role of an option's underlying "asset".

4.1.1 Domestic Martingale Measure

In view of (4.2), the exchange rate at time t equals

$$Q_t = Q_0 \exp\bigl(\sigma_Q \cdot W_t + (\mu_Q - \tfrac{1}{2}|\sigma_Q|^2)t\bigr). \tag{4.3}$$

Let us introduce an auxiliary process Q^*, given by the equality

$$Q_t^* \stackrel{\text{def}}{=} B_t^f Q_t / B_t^d = e^{(r_f - r_d)t}\, Q_t, \quad \forall\, t \in [0, T^*].$$

It is clear that Q_t^* represents the value at time t of the foreign savings account, when converted into the domestic currency, and discounted by the current value of the domestic savings account. Moreover, it is useful to observe that Q^* satisfies

$$Q_t^* = Q_0 \exp\bigl(\sigma_Q \cdot W_t + (\mu_Q + r_f - r_d - \tfrac{1}{2}|\sigma_Q|^2)t\bigr),$$

or equivalently, that the dynamics of Q^* are

$$dQ_t^* = Q_t^* \bigl((\mu_Q + r_f - r_d)\, dt + \sigma_Q \cdot dW_t\bigr). \tag{4.4}$$

It is clear that the process Q^* follows a martingale under the original probability measure $\mathbf{P}$ if and only if the drift coefficient μ_Q satisfies $\mu_Q = r_d - r_f$. We shall frequently make use of the process $\tilde{B}_t^f$, which equals

$$\tilde{B}_t^f \stackrel{\text{def}}{=} B_t^f Q_t = e^{r_f t} Q_t, \quad \forall\, t \in [0, T^*]. \tag{4.5}$$

Note that $\tilde{B}_t^f$ represents the value at time t of a unit investment in a foreign savings account, expressed in units of the domestic currency. In order to exclude arbitrage between investments in domestic and foreign bonds, we have to assume that the drift coefficient of the exchange rate process equals $r_d - r_f$ under an equivalent probability measure $\mathbf{P}^*$, hereafter referred to as the *martingale measure of the domestic market,* or briefly, the *domestic martingale measure.* It is worthwhile to observe that a martingale measure $\mathbf{P}^*$ is not unique, in general. Indeed, in our framework, the martingale measure $\mathbf{P}^*$ is associated with a solution $\hat{\eta} \in \mathbf{R}^d$ of the following equation

$$\mu_Q + r_f - r_d + \sigma_Q \cdot \hat{\eta} = 0,$$

for which the uniqueness of a solution need not hold, in general. Still, for any solution $\hat{\eta}$ of this equation, the probability measure $\mathbf{P}^*$ given by the usual exponential formula

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\bigl(\hat{\eta} \cdot W_{T^*} - \tfrac{1}{2}|\hat{\eta}|^2 T^*\bigr), \quad \mathbf{P}\text{-a.s.}, \tag{4.6}$$

can play the role of a martingale measure associated with the domestic market. In addition, the process W^*, which equals

$$W_t^* = W_t - \hat{\eta} t, \quad \forall\, t \in [0, T^*], \tag{4.7}$$

follows a d-dimensional Brownian motion under $\mathbf{P}^*$ (with respect to the underlying filtration). The uniqueness of the martingale measure can be gained by introducing the possibility of trading in additional foreign or domestic assets, foreign or domestic *stocks,* say. In other words, the uniqueness of a martingale measure holds if the number of (non-redundant) traded assets, including the domestic savings account, equals $d+1$, where d stands for the dimensionality of the underlying Brownian motion. For instance, if no domestic stocks are traded and only one foreign stock is considered, to guarantee the uniqueness of a martingale measure $\mathbf{P}^*$, and thus the completeness of the market, it is enough to assume that W, and thus also W^*, is a two-dimensional Brownian motion. In such a case, the market model involves three primary securities – the domestic and foreign savings accounts (or equivalently, domestic and foreign bonds) and a foreign stock. In any case, the dynamics of the exchange rate Q under the domestic martingale measure $\mathbf{P}^*$ are easily seen to be

$$dQ_t = Q_t \big((r_d - r_f)\, dt + \sigma_Q \cdot dW_t^*\big), \quad Q_0 > 0, \tag{4.8}$$

where W^* follows a d-dimensional Brownian motion under $\mathbf{P}^*$. Intuitively, the domestic martingale measure $\mathbf{P}^*$ is a risk-neutral probability as seen from the perspective of a domestic investor – that is, an investor who constantly denominates the prices of all assets in units of domestic currency. It is clear that the arbitrage price $\pi_t(X)$, in units of domestic currency, of any contingent claim X, which settles at time T and is also denominated in the domestic currency, equals

$$\pi_t(X) = e^{-r_d(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\big(X \mid \mathcal{F}_t\big). \tag{4.9}$$

If a time T claim Y is denominated in units of foreign currency, its arbitrage price at time t, expressed in units of domestic currency, is given by the formula

$$\pi_t(Y) = e^{-r_d(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\big(Q_T Y \mid \mathcal{F}_t\big). \tag{4.10}$$

Notice that the arbitrage price of such a claim can be alternatively evaluated using the martingale measure associated with the foreign market, and ultimately converted into domestic currency using the current exchange rate Q_t. For this purpose, we need to introduce an arbitrage-free probability measure associated with the foreign market, referred to as the *foreign martingale measure.*

4.1.2 Foreign Martingale Measure

We shall now take the perspective of a foreign-based investor – that is, an investor who consistently denominates her profits and losses in units of foreign currency. Since Q_t is the price at time t of one unit of foreign currency in the domestic currency, it is evident that the price at time t of one unit of the domestic currency, expressed in units of foreign currency, equals $R_t = 1/Q_t$. From Itô's formula, we have

$$dQ_t^{-1} = Q_t^{-2}\, dQ_t + Q_t^{-3}\, d\langle Q, Q\rangle_t,$$

or more explicitly

$$dR_t = -R_t \big((r_d - r_f)\, dt + \sigma_Q \cdot dW_t^*\big) + R_t |\sigma_Q|^2\, dt.$$

Therefore, the dynamics of R under $\mathbf{P}^*$ are given by the expression

$$dR_t = R_t \big((r_f - r_d)\, dt - \sigma_Q \cdot (dW_t^* - \sigma_Q\, dt)\big). \tag{4.11}$$

Equivalently,

$$dR_t^* = -R_t^*\, \sigma_Q \cdot (dW_t^* - \sigma_Q\, dt), \tag{4.12}$$

where we denote by R^* the following process

$$R_t^* \stackrel{\text{def}}{=} R_t e^{(r_d - r_f)t} = e^{-r_f t} R_t B_t^d, \quad \forall\, t \in [0, T^*].$$

Observe that R^* represents the price process of the domestic savings account, expressed in units of foreign currency, and discounted using the foreign risk-free interest rate. By virtue of (4.12), it is easily seen that the process R^* follows a martingale under a probability measure $\tilde{\mathbf{P}}$, equivalent to $\mathbf{P}^*$, which satisfies

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}^*} = \eta_{T^*}, \quad \mathbf{P}^*\text{-a.s.} \tag{4.13}$$

on $(\Omega, \mathcal{F}_{T^*})$, where η equals

$$\eta_t = \exp\big(\sigma_Q \cdot W_t^* - \tfrac{1}{2}|\sigma_Q|^2 t\big), \quad \forall\, t \in [0, T^*]. \tag{4.14}$$

Any probability measure $\tilde{\mathbf{P}}$ defined in this way is referred to as the *martingale measure of the foreign market.* If the uniqueness of a domestic martingale measure $\mathbf{P}^*$ is not valid, the uniqueness of a foreign market martingale measure $\tilde{\mathbf{P}}$ does not hold either. However, under any foreign martingale measure $\tilde{\mathbf{P}}$, we have

$$dR_t^* = -R_t^* \, \sigma_Q \cdot d\tilde{W}_t, \tag{4.15}$$

where the process $\tilde{W}_t = W_t^* - \sigma_Q t$ follows a d-dimensional Brownian motion under $\tilde{\mathbf{P}}$. It is useful to observe that the dynamics of R under the martingale measure $\tilde{\mathbf{P}}$ are given by the following counterpart of (4.8)

$$dR_t = R_t \left((r_f - r_d)\, dt - \sigma_Q \cdot d\tilde{W}_t\right). \tag{4.16}$$

In financial interpretation, a foreign market martingale measure $\tilde{\mathbf{P}}$ is any probability measure on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbf{P}$ that excludes arbitrage opportunities between risk-free and risky investments in both economies, as seen from the perspective of a foreign-based investor. For any attainable contingent claim X, which settles at time T and is denominated in units of domestic currency, the arbitrage price at time t in units of foreign currency is given by the equality

$$\tilde{\pi}_t(X) = e^{-r_f(T-t)}\, \mathbf{E}_{\tilde{\mathbf{P}}}\big(R_T X \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, T]. \tag{4.17}$$

We are now in a position to establish a relationship which links a conditional expectation evaluated under the foreign market martingale measure $\tilde{\mathbf{P}}$ to its counterpart evaluated under the domestic martingale measure $\mathbf{P}^*$. We assume here that $\tilde{\mathbf{P}}$ is associated with $\mathbf{P}^*$ through (4.13).

Proposition 4.1.1 *The following formula is valid for any $\mathcal{F}_T$-measurable random variable X (provided that the conditional expectation is well-defined)*

$$\mathbf{E}_{\tilde{\mathbf{P}}}(X \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}^*}\Big(X \exp\big(\sigma_Q \cdot (W_T^* - W_t^*) - \tfrac{1}{2}|\sigma_Q|^2 (T-t)\big)\Big|\, \mathcal{F}_t\Big). \tag{4.18}$$

Proof. By virtue of the Bayes formula we get

$$\mathbf{E}_{\tilde{\mathbf{P}}}(X \,|\, \mathcal{F}_t) = \frac{\mathbf{E}_{\mathbf{P}^*}(\eta_T X \,|\, \mathcal{F}_t)}{\mathbf{E}_{\mathbf{P}^*}(\eta_T \,|\, \mathcal{F}_t)},$$

and thus (observe that η follows a martingale under $\mathbf{P}^*$)

$$\mathbf{E}_{\tilde{\mathbf{P}}}(X \,|\, \mathcal{F}_t) = \eta_t^{-1}\, \mathbf{E}_{\mathbf{P}^*}(\eta_T X \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}^*}(\eta_T \eta_t^{-1} X \,|\, \mathcal{F}_t),$$

as expected. □

Let S_t^f be the foreign currency price at time t of a foreign traded stock which pays no dividends. In order to exclude arbitrage, we assume that the dynamics of the price process S^f under the foreign martingale measure $\tilde{\mathbf{P}}$ are

$$dS_t^f = S_t^f \left(r_f\, dt + \sigma_{S^f} \cdot d\tilde{W}_t\right), \quad S_0^f > 0, \tag{4.19}$$

with a constant volatility coefficient $\sigma_{S^f} \in \mathbf{R}^d$. This means that the stock price process S^f follows

$$dS_t^f = S_t^f \left((r_f - \sigma_Q \cdot \sigma_{S^f})\, dt + \sigma_{S^f} \cdot dW_t^*\right) \tag{4.20}$$

under the domestic martingale measure $\mathbf{P}^*$ associated with $\tilde{\mathbf{P}}$. For the purpose of pricing foreign equity options, we will sometimes find it useful to convert the price of the underlying foreign stock into the domestic currency. We write $\tilde{S}^f_t = Q_t S^f_t$ to denote the price of a foreign stock S^f expressed in units of domestic currency. Using Itô's formula, and the dynamics under $\mathbf{P}^*$ of the exchange rate Q, which are

$$dQ_t = Q_t\left((r_d - r_f)\,dt + \sigma_Q \cdot dW^*_t\right), \tag{4.21}$$

one finds that under the domestic martingale measure $\mathbf{P}^*$, the process $\tilde{S}^f$ satisfies

$$d\tilde{S}^f_t = \tilde{S}^f_t\left(r_d\,dt + (\sigma_{S^f} + \sigma_Q)\cdot dW^*_t\right). \tag{4.22}$$

The last equality shows that the price process $\tilde{S}^f$ behaves as the price process of a domestic stock in the classic Black-Scholes framework; however, the corresponding volatility coefficient is equal to the superposition $\sigma_{S^f} + \sigma_Q$ of two volatilities – the foreign stock price volatility and the exchange rate volatility. By defining in the usual way the class of admissible trading strategies, one may now easily construct a market model in which there is no arbitrage between investments in foreign and domestic bonds and stocks. Since this can be easily done, we leave the details to the reader.

4.2 Currency Forward Contracts and Options

In this section, we consider derivative securities whose value depends exclusively on the fluctuations of exchange rate Q, as opposed to those securities which depend also on some foreign equities. Currency options, forward contracts and futures contracts provide an important financial instrument through which to control the risk exposure induced by the uncertain future exchange rate. The deliverable instrument in a classic foreign exchange option is a fixed amount of underlying foreign currency. The valuation formula that provides the arbitrage price of foreign exchange European-style options was established independently in Biger and Hull (1983) and Garman and Kohlhagen (1983). They have shown that if the domestic and foreign risk-free rates are constant, and the dynamics of the exchange rate are given by (4.2), then a foreign currency option may be valued by means of a suitable variant of the Black-Scholes option valuation formula. More precisely, one may apply formula (3.76), which gives the arbitrage price of a European option written on a stock which pays a constant dividend yield.

4.2.1 Forward Exchange Rate

Let us first consider a foreign exchange forward contract, written at time t, which settles at the future date T. The asset to be delivered by the party assuming a short position in the contract is a prespecified amount of foreign currency, say 1 unit. The party who assumes a long position in a currency forward contract is obliged to pay a certain number of units of a domestic currency, the *delivery price*. As usual, the delivery price that makes the forward contract worthless at time $t \le T$ is called the *forward price* at time t of one unit of the foreign currency to be delivered at the settlement date T. In the present context, it is natural to refer to this forward price as the *forward exchange rate*. We will write $F_Q(t,T)$ to denote the forward exchange rate.

Proposition 4.2.1 *The forward exchange rate $F_Q(t,T)$ at time t for the settlement date T is given by the following formula*

$$F_Q(t,T) = e^{(r_d - r_f)\,(T-t)}\,Q_t, \quad \forall\, t \in [0,T]. \tag{4.23}$$

Proof. It is easily seen that if (4.23) does not hold, risk-free profitable opportunities arise between the domestic and the foreign market. □

Relationship (4.23), commonly known as the *interest rate parity,* asserts that the forward exchange premium must equal, in the market equilibrium, the interest rate differential $r_d - r_f$. A relatively simple version of the interest rate parity still holds even when the domestic and foreign interest rates are no longer deterministic constants, but follow stochastic processes. Under uncertain interest rates, we need to to introduce the price processes $B_d(t,T)$ and $B_f(t,T)$ of the domestic and foreign zero-coupon bonds with maturity T. A zero-coupon bond with a given maturity T is a financial security which pays one unit of the corresponding currency at the future date T. Suppose that zero-coupon bonds with maturity T are traded in both domestic and foreign markets. Then equality (4.23) may be extended to cover the case of stochastic interest rates. Indeed, it is not hard to show, by means of no-arbitrage arguments, that

$$F_Q(t,T) = \frac{B_f(t,T)}{B_d(t,T)}\, Q_t, \quad \forall\, t \in [0,T], \tag{4.24}$$

where $B_d(t,T)$ and $B_f(t,T)$ stand for the respective time t prices of the domestic and foreign zero-coupon bonds with maturity T. Notice that in (4.24), both $B_d(t,T)$ and $B_f(t,T)$ should be seen as the domestic and foreign discount factors rather than the prices. Indeed, prices should be expressed in units of the corresponding currencies, while discount factors are merely the corresponding real numbers. Finally, it follows immediately from (4.21) that for any fixed settlement date T, the forward price dynamics under the martingale measure (of the domestic economy) $\mathbf{P}^*$ are

$$dF_Q(t,T) = F_Q(t,T)\, \sigma_Q \cdot dW_t^*, \tag{4.25}$$

and $F_Q(T,T) = Q_T$.

4.2.2 Currency Option Valuation Formula

As a first example of a currency option, we consider a standard European call option, whose payoff at the expiry date T equals

$$C_T^Q \stackrel{\text{def}}{=} N(Q_T - K)^+,$$

where Q_T is the spot price of the deliverable currency (i.e., the spot exchange rate at the option's expiry date), K is the strike price in units of domestic currency per foreign unit, and $N > 0$ is the nominal value of the option, expressed in units of the underlying foreign currency. It is clear that payoff from the option is expressed in the domestic currency; also, there is no loss of generality if we assume that $N = 1$. Summarizing, we consider an option to buy one unit of a foreign currency at a prespecified price K, which may be exercised at the date T only.

Proposition 4.2.2 *The arbitrage price, in units of domestic currency, of a currency European call option is given by the risk-neutral valuation formula*

$$C_t^Q = e^{-r_d(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\big((Q_T - K)^+ \,\big|\, \mathcal{F}_t\big), \quad \forall\, t \in [0,T]. \tag{4.26}$$

Moreover, the price C_t^Q is given by the following expression

$$C_t^Q = Q_t e^{-r_f(T-t)}\, N\big(h_1(Q_t, T-t)\big) - K e^{-r_d(T-t)} N\big(h_2(Q_t, T-t)\big),$$

where N is the standard Gaussian cumulative distribution function, and

$$h_{1,2}(q,t) = \frac{\ln(q/K) + (r_d - r_f \pm \frac{1}{2}\sigma_Q^2)t}{\sigma_Q \sqrt{t}}\,.$$

Proof. Let us first examine a trading strategy in risk-free domestic and foreign bonds, which we call a *currency trading strategy* in what follows. Formally, by a currency trading strategy we mean an adapted stochastic process $\phi = (\phi^1, \phi^2)$. In financial interpretation, $\phi^1 \tilde{B}_t^f$ and $\phi^2 B_t^d$ represent the amounts of money invested at time t in foreign and domestic bonds. It is important to note that both amounts are expressed in units of domestic currency (see, in particular, (4.4)). A currency trading strategy ϕ is said to be self-financing if its wealth process $V(\phi)$, which equals

$$V_t(\phi) = \phi_t^1 \tilde{B}_t^f + \phi_t^2 B_t^d, \quad \forall\, t \in [0,T],$$

where $\tilde{B}_t^f = B_t^f Q_t,\ B_t^d = e^{r_d t}$, satisfies the following relationship

$$dV_t(\phi) = \phi_t^1\, d\tilde{B}_t^f + \phi_t^2\, dB_t^d.$$

For the discounted wealth process $V_t^*(\phi) = e^{-r_d t}\, V_t(\phi)$ of a self-financing currency trading strategy, we easily get

$$dV_t^*(\phi) = \phi_t^1\, d(e^{-r_d t}\tilde{B}_t^f) = \phi_t^1\, dQ_t^*.$$

On the other hand, by virtue of (4.21), the dynamics of the process Q^*, under the domestic martingale measure $\mathbf{P}^*$, are given by the expression

$$dQ_t^* = \sigma_Q\, Q_t^*\, dW_t^*.$$

Therefore, the discounted wealth $V^*(\phi)$ of any self-financing currency trading strategy ϕ follows a martingale under $\mathbf{P}^*$. This justifies the risk-neutral valuation formula (4.26). Taking into account the equality $Q_T = \tilde{B}_T^f e^{-r_f T}$, one gets also

$$\begin{aligned} C_t^Q &= e^{-r_d(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\big((Q_T - K)^+ \,|\, \mathcal{F}_t\big) \\ &= e^{-r_f T} e^{-r_d(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\big((\tilde{B}_T^f - K e^{r_f T})^+ \,|\, \mathcal{F}_t\big) \\ &= e^{-r_f T} C(\tilde{B}_t^f, T-t, K e^{r_f T}, r_d, \sigma_Q), \end{aligned}$$

where C stands for the standard Black-Scholes call option price. More explicitly, we have

$$\begin{aligned} C_t^Q &= e^{-r_f T}\Big(\tilde{B}_t^f\, N\big(d_1(\tilde{B}_t^f, T-t)\big) - K e^{r_f T} e^{-r_d(T-t)}\, N\big(d_2(\tilde{B}_t^f, T-t)\big)\Big) \\ &= Q_t e^{-r_f(T-t)} N\big(d_1(\tilde{B}_t^f, T-t)\big) - K e^{-r_d(T-t)}\, N\big(d_2(\tilde{B}_t^f, T-t)\big). \end{aligned}$$

This proves the formula we wish to show, since

$$d_i(\tilde{B}_t^f, T-t, K e^{r_f T}, r_d, \sigma_Q) = h_i(Q_t, T-t)$$

for $i = 1, 2$. Finally, one finds immediately that the first component of the self-financing currency trading strategy that replicates the option equals

$$\phi_t^1 = e^{-r_f T}\, N\big(d_1(\tilde{B}_t^f, T-t)\big) = e^{-r_f T}\, N\big(h_1(Q_t, T-t)\big).$$

Therefore, to hedge a short position, the writer of the currency call should invest at time $t \le T$ the amount (expressed in units of foreign currency)

$$\phi_t^1 B_t^f = e^{-r_f(T-t)}\, N\big(h_1(Q_t, T-t)\big)$$

in foreign market risk-free bonds (or equivalently, in the foreign savings account). On the other hand, she should also invest the amount (denominated in domestic currency)

$$C_t^Q - Q_t e^{-r_f(T-t)}\, N\big(h_1(Q_t, T-t)\big)$$

in the domestic savings account. □

Remarks. (a) As mentioned earlier, a comparison of the currency option valuation formula established in Proposition 4.2.2 with expression (3.76) shows that the exchange rate Q can be formally seen as the price of a fictitious domestic "stock". Under such a convention, the foreign interest rate r_f can be interpreted as a dividend yield that is continuously paid by this fictitious stock.

(b) It is easy to derive the put-call relationship for currency options. Indeed, the payoff in domestic currency of a portfolio composed of one long call option and one short put option is

$$C_T^Q - P_T^Q = (Q_T - K)^+ - (K - Q_T)^+ = Q_T - K,$$

where we assume, as before, that the options are written on one unit of foreign currency. Consequently, for any $t \in [0, T]$, we have

$$C_t^Q - P_t^Q = e^{-r_f(T-t)} Q_t - e^{-r_d(T-t)} K. \tag{4.27}$$

(c) We may also rewrite the currency option valuation formula of Proposition 4.2.2 in the following way

$$C_t^Q = e^{-r_d(T-t)} \Big(F_t \, N\big(\tilde{d}_1(F_t, T-t)\big) - K N\big(\tilde{d}_2(F_t, T-t)\big)\Big), \tag{4.28}$$

where $F_t = F_Q(t, T)$ and

$$\tilde{d}_{1,2}(F, t) = \frac{\ln(F/K) \pm \frac{1}{2}\sigma_Q^2 \, t}{\sigma_Q \sqrt{t}}, \quad \forall \, (F, t) \in \mathbf{R}_+ \times (0, T].$$

This shows that the currency option valuation formula can be seen as a variant of the Black futures formula (3.91) of Sect. 3.3. Furthermore, it is possible to re-express the replicating strategy of the option in terms of domestic bonds and currency forward contracts. Let us mention that under the present assumptions of deterministic domestic and foreign interest rates, the distinction between the currency futures price and forward exchange rate is not essential. In market practice, currency options are frequently hedged by taking positions in forward and futures contracts, rather than by investing in foreign risk-free bonds.

4.3 Foreign Equity Forward Contracts

In a global equity market, an investor may link his foreign stock and currency exposures in a large variety of ways. More specifically, he may choose to combine his investments in foreign equities with differing degrees of protection against adverse moves in exchange rates and stock prices, using forward and futures contracts as well as a variety of options.

4.3.1 Forward Price of a Foreign Stock

Let us first consider an ordinary forward contract with a foreign stock being the underlying asset to be delivered – that is, an agreement to buy a stock on a certain date at a certain delivery price in a specified currency. We shall distinguish between the two following cases: (a) when the delivery price K^f is denominated in the foreign currency, and (b) when it is expressed in the domestic currency; in the latter case the delivery price will be denoted by K^d. It should be emphasized that in both situations, the value of the forward contract at the settlement date T is equal to the spread between the stock price at time T and the delivery price expressed in foreign currency. The terminal payoff is then converted into units of domestic currency at the exchange rate that prevails at the settlement date T. Summarizing, in units of domestic currency, the terminal payoffs from the long positions are

$$V_T^d(K^f) = Q_T(S_T^f - K^f)$$

in the first case, and

$$V_T^d(K^d) = Q_T(S_T^f - R_T^{-1} K^d) = (Q_T S_T^f - K^d) = (\tilde{S}_T^f - K^d)$$

if the second case is considered.

Case (a). Observe that the foreign-currency payoff at settlement of the forward contract equals $X_T = S^f_T - K^f$. Therefore its value at time t, denominated in the foreign currency, is

$$V^f_t(K^f) = e^{-r_f(T-t)}\,\mathbf{E}_{\tilde{\mathbf{P}}}\big(S^f_T - K^f \,|\, \mathcal{F}_t\big) = S^f_t - e^{-r_f(T-t)}K^f.$$

Consequently, when expressed in the domestic currency, the value of the contract at time t equals

$$V^d_t(K^f) = Q_t(S^f_t - e^{-r_f(T-t)}K^f).$$

We conclude that the forward price of the stock S^f, expressed in units of foreign currency, equals

$$F^f_{S^f}(t,T) = e^{r_f\,(T-t)}S^f_t, \quad \forall\, t \in [0,T]. \tag{4.29}$$

Case (b). In this case, equality (4.9) yields immediately

$$V^d_t(K^d) = e^{-r_d(T-t)}\,\mathbf{E}_{\mathbf{P}^*}\big(\tilde{S}^f_T - K^d \,|\, \mathcal{F}_t\big);$$

hence, by virtue of (4.22), the domestic-currency value of the forward contract with the delivery price K^d denominated in domestic currency equals

$$V^d_t(K^d) = Q_tS^f_t - e^{-r_d\,(T-t)}K^d.$$

This implies that the forward price of a foreign stock in domestic currency equals

$$F^d_{S^f}(t,T) = e^{r_d\,(T-t)}\tilde{S}^f_t, \quad \forall\, t \in [0,T], \tag{4.30}$$

so that, somewhat surprisingly, it is independent of the foreign risk-free interest rate r_f.

4.3.2 Quanto Forward Contracts

In this section, we shall examine a *quanto forward contract* on a foreign stock.[1] Such a contract is also known as a *guaranteed exchange rate forward contract* (a *GER forward contract* for short). To describe the intuition that underpins the concept of a quanto forward contract, let us consider an investor who expects a certain foreign stock to appreciate significantly over the next period, and who wishes to capture this appreciation in his portfolio. Buying the stock, or taking a long position in it through a forward contract or call option, leaves the investor exposed to exchange rate risk. To avoid having his return depend on the performance of the foreign currency, he needs a guarantee that he can close his foreign stock position at an exchange rate close to the one that prevails at present. This can be done by entering a quanto forward or option contract in a foreign stock. In this subsection, we shall study the case of quanto forward contracts, leaving the analysis of quanto options to the next section. We start by defining precisely what is meant by a quanto forward contract in a foreign stock S^f. As before, the payoff of a guaranteed exchange rate forward contract on a foreign stock at settlement date T is the difference between the stock price at time T and the delivery price denominated in the foreign currency, say K^f. However, this payoff is converted into domestic currency at a predetermined exchange rate, denoted by $\bar{Q}$ in what follows. More formally, denoting by $V^d_t(K^f,\bar{Q})$ the time t value in domestic currency of the quanto forward contract, we have

$$V^d_T(K^f,\bar{Q}) = \bar{Q}(S^f_T - K^f).$$

We wish to determine the right value of such a contract at time t before the settlement. Notice that the terminal payoff of a quanto forward contract is independent of the future exchange rate fluctuations during the life of a contract. Still, as we shall see in what follows, its value $V^d_t(K^f,\bar{Q})$ depends on the volatility coefficient σ_Q of the exchange rate process Q – more precisely, on the scalar

[1] Generally speaking, a financial asset is termed to be a *quanto* product if it is denominated in a currency other than that in which it is usually traded.

product $\sigma_Q \cdot \sigma_{S^f}$ that determines the instantaneous covariance between the logarithmic returns of the stock price and the exchange rate. By virtue of the risk-neutral valuation formula, the value at time t of the quanto forward contract equals (in domestic currency)

$$V_t^d(K^f, \bar{Q}) = \bar{Q} e^{-r_d(T-t)} \left(\mathbf{E}_{\mathbf{P}^*}(S_T^f \,|\, \mathcal{F}_t) - K^f \right).$$

To find the conditional expectation $\mathbf{E}_{\mathbf{P}^*}(S_T^f \,|\, \mathcal{F}_t)$, observe that by virtue of (4.20), the process $\hat{S}_t = e^{-\delta t} S_t^f$ follows a martingale under $\mathbf{P}^*$, provided that we take $\delta = r_f - \sigma_Q \cdot \sigma_{S^f}$. Consequently, we find easily that

$$\mathbf{E}_{\mathbf{P}^*}(S_T^f \,|\, \mathcal{F}_t) = e^{\delta T} \mathbf{E}_{\mathbf{P}^*}(\hat{S}_T \,|\, \mathcal{F}_t) = e^{\delta T} \hat{S}_t = e^{\delta(T-t)} S_t^f,$$

and thus

$$V_t^d(K^f, \bar{Q}) = \bar{Q} e^{-r_d(T-t)} \big(e^{(r_f - \sigma_Q \cdot \sigma_{S^f})(T-t)} S_t^f - K^f \big). \tag{4.31}$$

This in turn implies that the forward price at time t associated with the quanto forward contract that settles at time T equals (in units of foreign currency)

$$\hat{F}_{S^f}^f(t,T) = e^{(r_f - \sigma_Q \cdot \sigma_{S^f})(T-t)} S_t^f = \mathbf{E}_{\mathbf{P}^*}(S_T^f \,|\, \mathcal{F}_t). \tag{4.32}$$

It is interesting to note that $\hat{F}_{S^f}^f(t,T)$ is simply the conditional expectation of the stock price at the settlement date T, as seen at time t from the perspective of a domestic-based investor. Furthermore, at least when $\kappa = \sigma_Q \cdot \sigma_{S^f} \geq 0$, it can also be interpreted as the forward price of a fictitious dividend-paying stock, with $\kappa = \sigma_Q \cdot \sigma_{S^f}$ playing the role of the dividend yield (cf. formula (3.79) in Sect. 3.2.7).

4.4 Foreign Equity Options

In this section, we shall study examples of *foreign equity options* – that is, options whose terminal payoff (in units of domestic currency) depends not only on the future behavior of the exchange rate, but also on the price fluctuations of a certain foreign stock.

4.4.1 Options Struck in a Foreign Currency

Assume first that an investor wants to participate in gains in foreign equity, desires protection against losses in that equity, but is unconcerned about the translation risk arising from the potential drop in the exchange rate. We denote by T the expiry date and by K^f the exercise price of an option. It is essential to note that K^f is expressed in units of foreign currency. The terminal payoff from a *foreign equity call struck in foreign currency* equals

$$C_T^1 \stackrel{\text{def}}{=} Q_T (S_T^f - K^f)^+.$$

This means, in particular, that the terminal payoff is assumed to be converted into domestic currency at the spot exchange rate that prevails at the expiry date. By reasoning in much the same way as in the previous section, one can check that the arbitrage price of a European call option at time t equals

$$C_t^1 = e^{-r_d(T-t)} \, \mathbf{E}_{\mathbf{P}^*}\big(Q_T (S_T^f - K^f)^+ \big| \mathcal{F}_t \big).$$

Using (4.21), we find that

$$C_t^1 = e^{-r_d(T-t)} Q_t \, \mathbf{E}_{\mathbf{P}^*}\Big\{ (S_T^f - K^f)^+ \, \exp\big(\sigma_Q \cdot (W_T^* - W_t^*) + \lambda(T-t) \big) \,\Big|\, \mathcal{F}_t \Big\},$$

where $\lambda = r_d - r_f - \frac{1}{2}|\sigma_Q|^2$. Equivalently, using (4.18), we get

$$C_t^1 = e^{-r_f(T-t)} Q_t \, \mathbf{E}_{\tilde{\mathbf{P}}}\big((S_T^f - K^f)^+ \big| \mathcal{F}_t \big).$$

Since $\tilde{\mathbf{P}}$ is the arbitrage-free measure of the foreign economy, it is not hard to establish the following expression

$$C_t^1 = Q_t \Big(S_t^f \, N\big(g_1(S_t^f, T-t)\big) - K^f e^{-r_f(T-t)} \, N\big(g_2(S_t^f, T-t)\big)\Big),$$

where

$$g_{1,2}(s,t) = \frac{\ln(s/K^f) + (r_f \pm \frac{1}{2}\,|\sigma_{S^f}|^2)t}{|\sigma_{S^f}|\sqrt{t}}\,.$$

An inspection of the valuation formula above makes clear that a hedging portfolio involves at any instant t the number $N(g_1(S_t^f, T-t))$ shares of the underlying stock; this stock investment demands the additional borrowing of $\beta_t^d = Q_t K^f e^{-r_f(T-t)} \, N\big(g_2(S_t^f, T-t)\big)$ units of the domestic currency, or equivalently, the borrowing of $\beta_t^f = K^f e^{-r_f(T-t)} \, N\big(g_2(S_t^f, T-t)\big)$ units of the foreign currency.

Remarks. The valuation result established above is in fact quite natural. Indeed, seen from the foreign market perspective, the foreign equity option struck in foreign currency can be priced directly by means of the standard Black-Scholes formula. As mentioned, when dealing with foreign equity options, one can either do the calculations with reference to the domestic economy, or equivalently, one may work within the framework of the foreign economy and then convert the final result into units of domestic currency. For example, in the case considered above, to complete the calculations in the domestic economy, on may use the following elementary lemma, whose proof is left to the reader.

Lemma 4.4.1 *Let (ξ, η) be a zero-mean, jointly Gaussian (non-degenerate), two-dimensional random variable on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then for arbitrary positive real numbers a and b, we have*

$$\mathbf{E}_{\mathbf{P}} \Big(a e^{\xi - \frac{1}{2}\operatorname{Var}\xi} - b e^{\eta - \frac{1}{2}\operatorname{Var}\eta}\Big)^+ = aN(h) - bN(h-k), \tag{4.33}$$

where $h = \frac{1}{k}\ln(a/b) + \frac{1}{2}k$ and $k = \sqrt{\operatorname{Var}(\xi - \eta)}$.

4.4.2 Options Struck in Domestic Currency

Assume now that an investor wishes to receive any positive returns from the foreign market, but wants to be certain that those returns are meaningful when translated back into his own currency. In this case he might be interested in a *foreign equity call struck in domestic currency,* with payoff at expiry

$$C_T^2 \stackrel{\text{def}}{=} (S_T^f Q_T - K^d)^+ = (\tilde{S}_T^f - K^d)^+,$$

where the strike price K^d is expressed in domestic currency. Due to the particular form of the option's payoff, it is clear that it is now convenient to study the option from the domestic perspective. To find the arbitrage price of the option at time t, it is sufficient to calculate the following conditional expectation

$$C_t^2 = e^{-r_d(T-t)} \, \mathbf{E}_{\mathbf{P}^*} \big((\tilde{S}_T^f - K^d)^+ \,\big|\, \mathcal{F}_t \big),$$

and by virtue of (4.21)–(4.20), the stock price expressed in units of domestic currency $\tilde{S}_t^f$ has the following dynamics under $\mathbf{P}^*$

$$d\tilde{S}_t^f = \tilde{S}_t^f \,\big(r_d \, dt + (\sigma_{S^f} + \sigma_Q) \cdot dW_t^* \big).$$

Therefore, arguing as in the proof of the classic Black-Scholes formula, one finds easily that the option's price, expressed in units of domestic currency, is given by the formula

$$C_t^2 = \tilde{S}_t^f \, N\big(l_1(\tilde{S}_t^f, T-t)\big) - e^{-r_d(T-t)} K^d \, N\big(l_2(\tilde{S}_t^f, T-t)\big),$$

where

$$l_{1,2}(s,t) = \frac{\ln(s/K^d) + (r_d \pm \frac{1}{2}\,|\sigma_{S^f} + \sigma_Q|^2)t}{|\sigma_{S^f} + \sigma_Q|\sqrt{t}}\,.$$

4.4.3 Quanto Options

Assume, as before, that an investor wishes to capture positive returns on his foreign equity investment, but now desires to eliminate all exchange risk by fixing an advance rate at which the option's payoff will be converted into domestic currency. At the intuitive level, such a contract can be seen as a combination of a foreign equity option with a currency forward contract. By definition, the payoff of a *quanto call* (i.e., a *guaranteed exchange rate foreign equity call option*) at expiry is set to be

$$C_T^3 \stackrel{\text{def}}{=} \bar{Q}(S_T^f - K^f)^+,$$

where $\bar{Q}$ is the prespecified exchange rate at which the conversion of the option's payoff is made. Notice that the quantity $\bar{Q}$ is denominated in the domestic currency per unit of foreign currency, and the strike price K^f is expressed in units of foreign currency. Since the payoff from the quanto option is expressed in units of domestic currency, its arbitrage price equals

$$C_t^3 = \bar{Q}e^{-r_d(T-t)}\,\mathbf{E}_{\mathbf{P}^*}\big((S_T^f - K^f)^+ \,\big|\, \mathcal{F}_t\big). \tag{4.34}$$

The proofs of the next two propositions are left to the reader as exercises.

Proposition 4.4.1 *The arbitrage price at time t of a European quanto call option with expiry date T and strike price K^f equals (in units of domestic currency)*

$$C_t^3 = \bar{Q}e^{-r_d(T-t)}\Big(S_t^f e^{\delta(T-t)}N\big(c_1(S_t^f, T-t)\big) - K^f N\big(c_2(S_t^f, T-t)\big)\Big),$$

where $\delta = r_f - \sigma_Q \cdot \sigma_{S^f}$ and

$$c_{1,2}(s,t) = \frac{\ln(s/K^f) + (\delta \pm \frac{1}{2}\,|\sigma_{S^f}|^2)t}{|\sigma_{S^f}|\sqrt{t}}\,.$$

4.4.4 Equity-linked Foreign Exchange Options

Finally, assume that an investor desires to hold foreign equity regardless of whether the stock price rises or falls (that is, he is indifferent to the foreign equity exposure), however, wishes to place a floor on the exchange rate risk of his foreign investment. An *equity-linked foreign exchange call* (an *Elf-X call,* for short) with payoff at expiry (in units of domestic currency)

$$C_T^4 \stackrel{\text{def}}{=} (Q_T - K)^+ S_T^f,$$

where K is a *strike exchange rate* expressed in domestic currency per unit of foreign currency, is thus a combination of a currency option with an equity forward. The arbitrage price (in units of domestic currency) of a European Elf-X call with expiry date T equals

$$C_t^4 = e^{-r_d(T-t)}\,\mathbf{E}_{\mathbf{P}^*}\big((Q_T - K)^+ S_T^f \,\big|\, \mathcal{F}_t\big).$$

We shall first value an equity-linked foreign exchange call option using the domestic martingale measure.

Proposition 4.4.2 *The arbitrage price, expressed in domestic currency, of a European equity-linked foreign exchange call option, with strike exchange rate K and expiry date T, is given by the following formula*

$$C_t^4 = S_t^f\,\Big(Q_t N\big(w_1(Q_t, T-t)\big) - K\,e^{-\gamma(T-t)}\,N\big(w_2(Q_t, T-t)\big)\Big), \tag{4.35}$$

where $\gamma = r_d - r_f + \sigma_Q \cdot \sigma_{S^f}$ and

$$w_{1,2}(q,t) = \frac{\ln(q/K) + (\gamma \pm \frac{1}{2}\,|\sigma_Q|^2)t}{|\sigma_Q|\sqrt{t}}\,.$$

Chapter 5

American Options

In contrast to the holder of a European option, the holder of an *American option* is allowed to exercise his right to buy (or sell) the underlying asset at any time before or at the expiry date. This special feature makes the arbitrage pricing of American options much more involved than the valuation of standard European claims. We know already that arbitrage valuation of American claims (for instance, within the framework of the binomial CRR model studied in Chap. 2) is closely related to specific optimal stopping problems. Intuitively, one might expect that the holder of an American option will choose her exercise policy in such a way that the expected payoff from the option will be maximized. Maximization of the expected discounted payoff under subjective probability would lead, of course, to non-uniqueness of the price. It appears, however, that for the purpose of arbitrage valuation, the maximization of the expected discounted payoff should be done under the martingale measure (that is, under risk-neutral probability). Therefore, the uniqueness of the arbitrage price of an American claim holds. One of the earliest works to examine the relationship between the early exercise feature of American options and optimal stopping problems was the paper in McKean (1965). It should be made clear that the arbitrage valuation of derivative securities was not yet discovered at this time, however. For this reason, the optimal stopping problem associated with the optimal exercise of American put was studied in McKean (1965) under an actual probability $\mathbf{P}$, rather than under the martingale measure $\mathbf{P}^*$, as is done nowadays. Basic features of American options, within the framework of arbitrage valuation theory, were already examined in van Moerbeke (1976). However, mathematically rigorous valuation results for American claims were first established by means of arbitrage arguments in Bensoussan (1984) and Karatzas (1988, 1989). For an exhaustive survey of results and techniques related to the arbitrage pricing of American options, we refer the reader to Myneni (1992).

5.1 Valuation of American Claims

We place ourselves within the classic Black-Scholes setup. Hence, the prices of primary securities – that is, the stock price, S, and the savings account, B – are modelled by means of the following differential equations

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad S_0 > 0, \tag{5.1}$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$ are real numbers, and

$$dB_t = rB_t \, dt, \quad B_0 = 1, \tag{5.2}$$

with $r \in \mathbf{R}$, respectively. As usual, we denote by W the standard Brownian motion defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where $\mathbf{F} = \mathbf{F}^W$. For the sake of notational convenience, we assume here that the underlying Brownian motion W is one-dimensional.

In the context of arbitrage valuation of American contingent claims, it is convenient to assume that an individual may withdraw funds to finance his consumption needs. For any fixed t, we denote by A_t the cumulative amount of funds that are withdrawn and consumed by an investor up to time t. The term "consumed" refers to the fact that the wealth is dynamically diminished according to the process A. The process A is assumed to be progressively measurable with non-decreasing and RCLL sample paths; also, by convention, $A_0 = A_{0-} = 0$. We say that A represents the *consumption strategy,* as opposed to the *trading strategy* ϕ. It is thus natural to call a pair (ϕ, A) a *trading and consumption strategy* in (S, B). In the present context, the formal definition of a self-financing strategy reads as follows.

Definition 5.1.1 A trading and consumption strategy (ϕ, A) in (S, B) is *self-financing* on $[0, T]$ if its wealth process $V(\phi, A)$, which equals

$$V_t(\phi, A) = \phi_t^1 S_t + \phi_t^2 B_t, \quad \forall\, t \in [0, T], \tag{5.3}$$

satisfies for every $t \in [0, T]$

$$V_t(\phi, A) = V_0(\phi, A) + \int_0^t \phi_u^1 \, dS_u + \int_0^t \phi_u^2 \, dB_u - A_t. \tag{5.4}$$

In view of (5.4), it is clear that A models the flow of funds that are not reinvested in primary securities, but rather are put aside forever. By convention, we say that the amount of funds represented by A_t is *consumed* by the holder of the dynamic portfolio (ϕ, A) up to time t. Eliminating the component ϕ^2 yields the following equivalent form of (5.4)

$$dV_t = rV_t \, dt + \phi_t^1 S_t \big((\mu - r) \, dt + \sigma \, dW_t \big) - dA_t,$$

where we write V to denote the wealth process $V(\phi, A)$. Equivalently,

$$dV_t = rV_t \, dt + \zeta_t (\mu - r) \, dt + \sigma \zeta_t \, dW_t - dA_t, \tag{5.5}$$

where $\zeta_t = \phi_t^1 S_t$ represents the amount of cash invested in shares at time t. The unique solution of the linear SDE (5.5) is given by an explicit formula

$$V_t = B_t \Big(V_0 + \int_0^t (\mu - r) \zeta_u B_u^{-1} \, du - \int_0^t B_u^{-1} \, dA_u + \int_0^t \sigma \zeta_u B_u^{-1} \, dW_u \Big),$$

which holds for every $t \in [0, T]$. We conclude that the wealth process of any self-financing trading and consumption strategy is uniquely determined by the following quantities: the initial endowment V_0, the consumption process A, and the process ζ representing the amount of cash invested in shares. In other words, given an initial endowment V_0, there is one-to-one correspondence between self-financing trading and consumption strategies (ϕ, A) and two-dimensional processes (ζ, A). We will sometimes find it convenient to identify a self-financing trading and consumption strategy (ϕ, A) with the corresponding pair (ζ, A), where $\zeta = \phi_t^1 S_t$. Recall that the unique martingale measure $\mathbf{P}^*$ for the Black-Scholes spot market satisfies

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big(\frac{r - \mu}{\sigma} \, W_{T^*} - \frac{1}{2} \, \frac{(r - \mu)^2}{\sigma^2} \, T^* \Big), \quad \mathbf{P}\text{-a.s.}$$

It is easily seen that the dynamics of the wealth process V under the martingale measure $\mathbf{P}^*$ are given by the following expression

$$dV_t = rV_t \, dt + \sigma \zeta_t \, dW_t^* - dA_t,$$

where W^* follows the standard Brownian motion under $\mathbf{P}^*$. This yields immediately

$$V_t = B_t \Big(V_0 - \int_0^t B_u^{-1} \, dA_u + \int_0^t \sigma \zeta_u B_u^{-1} \, dW_u^* \Big).$$

Therefore, an auxiliary process Z, which is given by the formula

$$Z_t \stackrel{\text{def}}{=} V_t^* + \int_0^t B_u^{-1}\,dA_u = V_0 + \int_0^t \sigma\zeta_u B_u^{-1}\,dW_u^*,$$

where $V_t^* = V_t/B_t$, follows a local martingale under $\mathbf{P}^*$. We say that a self-financing trading and consumption strategy (ϕ, A) is *admissible* if the condition

$$\mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T \zeta_u^2\,du\Big) = \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T (\phi_u^1 S_u)^2\,du\Big) < \infty$$

is satisfied so that Z is a $\mathbf{P}^*$-martingale. Similarly to Sect. 3.2.1, this assumption is imposed in order to exclude pathological examples of arbitrage opportunities from the market model. We are now in a position to formally introduce the concept of a contingent claim of American style. To this end, we take an arbitrary continuous *reward function* $g : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ satisfying the linear growth condition. An *American claim* with the reward function g and expiry date T is a financial security which pays to its holder the amount $g(S_t, t)$ when exercised at time t.

The writer of an American claim with the reward function g accepts the obligation to pay the amount $g(S_t, t)$ at any time t. It should be emphasized that the choice of the exercise time is at discretion of the holder of an American claim (that is, of a party assuming a long position). In order to formalize the concept of an American claim, we need to introduce first a suitable class of admissible exercise times. Since we exclude clairvoyance, the admissible exercise time τ is assumed to be a stopping time of filtration $\mathbf{F}$. Let us recall that a random variable $\tau : (\Omega, \mathcal{F}_T, \mathbf{P}) \to [0,T]$ is a *stopping time of filtration* $\mathbf{F}$ if, for every $t \in [0,T]$, the event $\{\tau \le t\}$ belongs to the σ-field $\mathcal{F}_t$. Since in the Black-Scholes model we have $\mathbf{F} = \mathbf{F}^W = \mathbf{F}^{W^*} = \mathbf{F}^S$, any stopping time of the filtration $\mathbf{F}$ is also a stopping time of the filtration $\mathbf{F}^S$ generated by the stock price process S. In intuitive terms, it is assumed throughout that the decision to exercise an American claim at time t is based on the observations of stock price fluctuations up to time t, but not after this date. This interpretation is consistent with our general assumption that the σ-field $\mathcal{F}_t$ represents the information available to all investors at time t. Let us denote by $\mathcal{T}_{[t,T]}$ the set of all stopping times of the filtration $\mathbf{F}$ which satisfy $t \le \tau \le T$ (with probability 1).

Definition 5.1.2 An American contingent claim X^a with the reward function $g : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ is a financial instrument consisting of: (a) an expiry date T; (b) the selection of a stopping time $\tau \in \mathcal{T}_{[0,T]}$; and (c) a payoff $X_\tau^a = g(S_\tau, \tau)$ on exercise.

Typical examples of American claims are American options with constant strike price K and expiry date T. The payoffs of American call and put options, when exercised at the random time τ, are equal to $X_\tau = (S_\tau - K)^+$ and $Y_\tau = (K - S_\tau)^+$ respectively. Our aim is to derive the "rational" price and to determine the "rational" exercise time of an American contingent claim by means of purely arbitrage arguments. To this end, we shall first introduce a specific class of trading strategies. For expositional simplicity, we shall search for the price of an American claim X^a at time 0; the general case can be treated along the same lines, but is more cumbersome from the notational viewpoint. It will be sufficient to consider a very special class of trading strategies associated with the American contingent claim X^a, namely the *buy-and-hold* strategies. By a *buy-and-hold* strategy associated with an American claim X^a, we mean a pair (c, τ), where $\tau \in \mathcal{T}_{[0,T]}$ and c is a real number. In financial interpretation, a buy-and-hold strategy (c, τ) assumes that $c > 0$ units of the American security X^a are acquired (or shorted, if $c < 0$) at time 0, and then held in the portfolio up to the exercise time τ. Observe that such a strategy excludes trading in the American claim after the initial date. In other words, dynamic trading in the American claim is not considered at this stage.

Let us assume that there exists a "market" price, say U_0, at which the American claim X^a trades in the market at time 0. Our first task is to find the right value of U_0 by means of no-arbitrage arguments (as mentioned above, the arguments which lead to the arbitrage valuation of the claim

X^a at time $t > 0$ are much the same as in the case of $t = 0$, therefore the general case is left to the reader).

Definition 5.1.3 By a *self-financing trading strategy in* (S, B, X^a), we mean a collection (ϕ, A, c, τ), where (ϕ, A) is a trading and consumption strategy in (S, B) and (c, τ) is a buy-and-hold strategy associated with X^a. In addition, we assume that on the random interval $(\tau, T]$ we have

$$\phi_t^1 = 0, \quad \phi_t^2 = \phi_\tau^1 S_\tau B_\tau^{-1} + \phi_\tau^2 + cg(S_\tau, \tau) B_\tau^{-1}. \tag{5.6}$$

It will soon become apparent that it is enough to consider the cases of $c = 1$ and $c = -1$; that is, the long and short positions in the American claim X^a. An analysis of condition (5.6) shows that the definition of a self-financing strategy (ϕ, A, c, τ) implicitly assumes that the American claim is exercised at a random time τ, existing positions in shares are closed at time τ, and all the proceeds are invested in risk-free bonds. For brevity, we shall sometimes write $\tilde{\psi}$ to denote the dynamic portfolio (ϕ, A, c, τ) in what follows. Note that the wealth process $V(\tilde{\psi})$ of any self-financing strategy in (S, B, X^a) satisfies the following initial and terminal conditions

$$V_0(\tilde{\psi}) = \phi_0^1 S_0 + \phi_0^2 + cU_0 \tag{5.7}$$

and

$$V_T(\tilde{\psi}) = e^{r(T-\tau)}(\phi_\tau^1 S_\tau + cg(S_\tau, \tau)) + e^{rT} \phi_\tau^2. \tag{5.8}$$

In what follows, we shall restrict our attention to the class of *admissible* trading strategies $\tilde{\psi} = (\phi, A, c, \tau)$ in (S, B, X^a), which are defined in the following way.

Definition 5.1.4 A self-financing trading strategy (ϕ, A, c, τ) in (S, B, X^a) is said to be *admissible* if a trading and consumption strategy (ϕ, A) is admissible and $A_T = A_\tau$. The class of all admissible strategies (ϕ, A, c, τ) is denoted by $\tilde{\Psi}$.

Let us introduce the class $\tilde{\Psi}^0$ of those admissible trading strategies $\tilde{\psi}$ for which the initial wealth satisfies $V_0(\tilde{\psi}) < 0$, and the terminal wealth has the non-negative value; that is[1] $V_T(\tilde{\psi}) = \phi_T^2 B_t \geq 0$. In order to precisely define an arbitrage opportunity, we have to take into account the early exercise feature of American claims. It is intuitively clear that it is enough to consider two cases – a long and a short position in one unit of an American claim. This is due to the fact that we need to exclude the existence of arbitrage opportunities for both the seller and the buyer of an American claim. Indeed, the position of both parties involved in a contract of American style is no longer symmetric, as it was in the case of European claims. The holder of an American claim can actively choose his exercise policy. The seller of an American claim, on the contrary, should be prepared to meet his obligations at any (random) time. We therefore set down the following definition of arbitrage and an arbitrage-free market model.

Definition 5.1.5 There is *arbitrage* in the market model with trading in the American claim X^a with initial price U_0 if either (a) there is *long arbitrage,* i.e., there exists a stopping time τ such that for some trading and consumption strategy (ϕ, A) the strategy $(\phi, A, 1, \tau)$ belongs to the class $\tilde{\Psi}^0$, or (b) there is *short arbitrage,* i.e., there exists a trading and consumption strategy (ϕ, A) such that for any stopping time τ the strategy $(\phi, A, -1, \tau)$ belongs to the class $\tilde{\Psi}^0$. In the absence of arbitrage in the market model, we say that the model is *arbitrage-free.*

Definition 5.1.5 can be reformulated in the following way: there is *absence of arbitrage* in the market if the following conditions are satisfied: (a) for any stopping time τ and any trading and consumption strategy (ϕ, A), the strategy $(\phi, A, 1, \tau)$ is not in $\tilde{\Psi}^0$; and (b) for any trading and consumption strategy (ϕ, A), there exists a stopping time τ such that the strategy $(\phi, A, -1, \tau)$ is

[1] Since the existence of a strictly positive savings account is assumed, one can alternatively define the class Ψ^0 as the set of those strategies $\tilde{\psi}$ from $\tilde{\Psi}$ for which $V_0(\tilde{\psi}) = 0$, $V_T(\tilde{\psi}) = \phi_T^2 B_T \geq 0$, and the latter inequality is strict with positive probability.

not in $\tilde{\Psi}^0$. Intuitively, under the absence of arbitrage in the market, the holder of an American claim is unable to find an exercise policy τ and a trading and consumption strategy (ϕ, A) that would yield a risk-free profit. Also, under the absence of arbitrage, it is not possible to make risk-free profit by selling the American claim at time 0, provided that the buyer makes a clever choice of the exercise date. More precisely, there exists an exercise policy for the long party which prevents the short party from locking in a risk-free profit.

By definition, the *arbitrage price* at time 0 of the American claim X^a, denoted by $\pi_0(X^a)$, is that level of the price U_0 which makes the model arbitrage-free. Our aim is now to show that the assumed absence of arbitrage in the sense of Definition 5.1.5 leads to a unique value for the arbitrage price $\pi_0(X^a)$ of X^a (as already mentioned, it is not hard to extend this reasoning in order to determine the arbitrage price $\pi_t(X^a)$ of the American claim X^a at any date $t \in [0,T]$). Also, we shall find the *rational* exercise policy of the holder – that is, the stopping time that excludes the possibility of short arbitrage.

The following auxiliary result relates the value process associated with the specific optimal stopping problem to the wealth process of a certain admissible trading strategy. For any reward function g, we define an adapted process V by setting

$$V_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(e^{-r(\tau - t)} g(S_\tau, \tau) \,|\, \mathcal{F}_t \big) \tag{5.9}$$

for every $t \in [0,T]$, provided that the right-hand side in (5.9) is well-defined.

Proposition 5.1.1 *Let V be an adapted process defined by formula* (5.9) *for some reward function g. Then there exists an admissible trading and consumption strategy (ϕ, A) such that $V_t = V_t(\phi, A)$ for every $t \in [0,T]$.*

Proof. We shall give the outline of the proof (for technical details, we refer to Karatzas (1988) and Myneni (1992)). Let us introduce the Snell envelope J of the discounted reward process $Z^*_t = e^{-rt} g(S_t, t)$. By definition, the process J is the smallest supermartingale majorant to the process Z^*. From the general theory of optimal stopping, we know that

$$J_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(e^{-r\tau} g(S_\tau, \tau) \,|\, \mathcal{F}_t \big) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(Z^*_\tau \,|\, \mathcal{F}_t \big)$$

for every $t \in [0,T]$, so that $V_t = e^{rt} J_t$. Since J is a RCLL *regular* supermartingale of class DL,[2] it follows from general results that J admits the unique Doob-Meyer decomposition $J = M - H$, where M is a (square-integrable) martingale and H is a continuous non-decreasing process with $H_0 = 0$. Consequently,

$$d\big(e^{rt} J_t\big) = r e^{rt} J_t \, dt + e^{rt} \, dM_t - e^{rt} \, dH_t.$$

By virtue of the predictable representation property (see Theorem 3.1.2) we have

$$M_t = M_0 + \int_0^t \xi_u \, dW^*_u, \quad \forall \, t \in [0,T],$$

for some progressively measurable process ξ with $\mathbf{E}_{\mathbf{P}^*} \big(\int_0^T \xi_u^2 \, du \big) < \infty$. Hence, upon setting

$$\phi^1_t = e^{rt} \xi_t \sigma^{-1} S_t^{-1}, \quad \phi^2_t = J_t - \xi_t \sigma^{-1}, \quad A_t = \int_0^t e^{ru} \, dH_u, \tag{5.10}$$

we conclude that the process V represents the wealth process of some (admissible) trading and consumption strategy. □

[2]Basically, one needs to check that the family $\{J_\tau \,|\, \tau \in \mathcal{T}_{[0,T]}\}$ of random variables is *uniformly integrable* under $\mathbf{P}^*$. We refer the reader to Sect. 1.4 in Karatzas and Shreve (1998) for the definition of a *regular* process and for the concept of the *Doob-Meyer decomposition* of a semimartingale.

By the general theory of optimal stopping, we know also that the random time τ_t that maximizes the expected discounted reward after the date t is the first instant at which the process J drops to the level of the discounted reward, that is

$$\tau_t = \inf\{u \in [t,T] \mid J_u = Z^*_u\}, \quad \forall\, t \in [0,T]. \tag{5.11}$$

In other words, the optimal (under $\mathbf{P}^*$) exercise policy of the American claim with reward function g is given by the equality

$$\tau_0 = \inf\{u \in [0,T] \mid J_u = e^{-ru} g(S_u, u)\}. \tag{5.12}$$

Observe that the stopping time τ_0 is well-defined (i.e., the set on the right-hand side is non-empty with probability 1), and necessarily

$$V_{\tau_0} = g(S_{\tau_0}, \tau_0). \tag{5.13}$$

In addition, the stopped process $J_{t\wedge\tau_0}$ is a martingale, so that the process H is constant on the interval $[0,\tau_0]$. This means also that $A_t = 0$ on the random interval $[0,\tau_0]$, so that no consumption is present before time τ_0. We find it convenient to introduce the following definition.

Definition 5.1.6 An admissible trading and consumption strategy (ϕ, A) is said to be a *perfect hedging* against the American contingent claim X^a with reward function g if, with probability 1,

$$V_t(\phi) \geq g(S_t, t), \quad \forall\, t \in [0,T]. \tag{5.14}$$

We write $\Phi(X^a)$ to denote the class of all perfect hedging strategies against the American contingent claim X^a.

From the majorizing property of the Snell envelope, we infer that the trading and consumption strategy (ϕ, A) introduced in the proof of Proposition 5.1.1 is a perfect hedging against the American claim with reward function g. Moreover, this strategy has the special property of minimal initial endowment amongst all admissible perfect hedging strategies against the American claim. We shall now explicitly determine $\pi_0(X^a)$ by assuming that trading in the American claim X^a would not destroy the arbitrage-free features of the Black-Scholes model.

Theorem 5.1.1 *There is absence of arbitrage (in the sense of Definition 5.1.5) in the market model with trading in an American claim if and only if the price $\pi_0(X^a)$ is given by the formula*

$$\pi_0(X^a) = \sup\nolimits_{\tau\in\mathcal{T}_{[0,T]}} \mathbf{E}_{\mathbf{P}^*}\big(e^{-r\tau} g(S_\tau, \tau)\big). \tag{5.15}$$

More generally, the arbitrage price at time t of an American claim with reward function g equals

$$\pi_t(X^a) = \operatorname{ess\,sup}\nolimits_{\tau\in\mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*}\big(e^{-r(\tau-t)} g(S_\tau, \tau)\,|\mathcal{F}_t\big).$$

Proof. We shall follow Myneni (1992). Let us assume that the "market" price of the option is $U_0 > V_0$. We shall show that in this case, the American claim is overpriced – that is, a short arbitrage is possible. Let (ϕ, A) be the trading and consumption strategy considered in the proof of Proposition 5.1.1 (see formula (5.10)). Suppose that the option's buyer selects an arbitrary stopping time $\tau \in \mathcal{T}_{[0,T]}$ as his exercise policy. Let us consider the following strategy $(\hat\phi, \hat A, -1, \tau)$ (observe that in implementing this strategy, we do not need to assume that the exercise time τ is known in advance)

$$\begin{aligned}
\hat\phi^1_t &= \phi^1_t\, \mathrm{I}_{[0,\tau]}(t),\\
\hat\phi^2_t &= \phi^2_t\, \mathrm{I}_{[0,\tau]}(t) + \big(\,\phi^2_\tau + \phi^1_\tau S_\tau B^{-1}_\tau - g(S_\tau,\tau) B^{-1}_\tau\,\big)\, \mathrm{I}_{(\tau,T]}(t),
\end{aligned}$$

and $\hat A_t = A_{t\wedge\tau}$. Since (ϕ, A) is assumed to be a perfect hedging, we have $\hat\phi^1_\tau S_\tau + \hat\phi^2_\tau B_\tau \geq g(S_\tau, \tau)$, so that $\hat\phi^2_T B_T \geq 0$, $\mathbf{P}^*$-a.s. On the other hand, by construction, the initial wealth of $(\hat\phi, \hat A, -1, \tau)$

satisfies $\hat{\phi}_0^1 S_0 + \hat{\phi}_0^2 - U_0 = V_0 - U_0 < 0$. We conclude that the strategy $(\hat{\phi}, \hat{A}, -1, \tau)$ is a (short) arbitrage opportunity – that is, a risk-free profitable strategy for the seller of the American claim X^a.

Suppose now that $U_0 < V_0$, so that the American claim is underpriced. We shall now construct an arbitrage opportunity for the buyer of this claim. In this case, we may and do assume that he chooses the stopping time τ_0 as an exercise time. In addition, we assume that he holds a dynamic portfolio $(-\hat{\phi}, -\hat{A})$. Notice that the process $\hat{A}$ vanishes identically, since $\tau = \tau_0$. This means that no consumption is involved in the strategy chosen by the buyer. Furthermore, the initial wealth of his portfolio satisfies

$$-\hat{\phi}_0^1 S_0 - \hat{\phi}_0^2 + U_0 = U_0 - V_0 < 0$$

and the terminal wealth is zero, since in view of (5.13) the wealth of the portfolio at the exercise time τ_0 vanishes. This shows that by making a clever choice of exercise policy, the buyer of the American claim is able to lock in a risk-free profit. We conclude that the arbitrage price $\pi_0(X^a)$ necessarily coincides with V_0, since otherwise arbitrage opportunities would exist in the market model. □

5.2 American Call and Put Options

We shall now focus our attention on the case of American call and put options with constant strike price K. The reward functions we shall study in what follows are $g^c(s,t) = (s-K)^+$ and $g^p(s,t) = (K-s)^+$, where the rewards g^c and g^p correspond to the call and the put options, respectively. It will be convenient to introduce the discounted rewards $X_t^* = (S_t - K)^+/B_t$ and $Y_t^* = (K - S_t)^+/B_t$. For a continuous semimartingale Z, and a fixed $a \in \mathbf{R}$, we denote by $L_t^a(Z)$ the (right) semimartingale *local time* of Z, given explicitly by the formula

$$L_t^a(Z) \stackrel{\text{def}}{=} |Z_t - a| - |Z_0 - a| - \int_0^t \operatorname{sgn}(Z_u - a)\, dZ_u$$

for every $t \in [0,T]$ (by convention we set $\operatorname{sgn}(0) = -1$). It is well known that the local time $L^a(Z)$ of a continuous semimartingale Z is an adapted process whose sample paths are almost all continuous, non-decreasing functions. Moreover, for an arbitrary convex function $f : \mathbf{R} \to \mathbf{R}$, the following decomposition, referred to as the Itô-Tanaka-Meyer formula, is valid

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_u)\, dZ_u + \frac{1}{2}\int_{\mathbf{R}} L_t^a(Z)\, \mu(da),$$

where f' is the left-hand side derivative of f, and μ denotes the second derivative of f, in the sense of distributions. An application of the Itô-Tanaka-Meyer formula yields[3]

$$X_t^* = X_0^* + \int_0^t \mathrm{I}_{\{S_u > K\}}\, B_u^{-1}\big(\sigma S_u\, dW_u^* + rK\, du\big) + \frac{1}{2}\int_0^t B_u^{-1}\, dL_u^K(S)$$

and

$$Y_t^* = Y_0^* - \int_0^t \mathrm{I}_{\{S_u < K\}}\, B_u^{-1}\big(\sigma S_u\, dW_u^* + rK\, du\big) + \frac{1}{2}\int_0^t B_u^{-1}\, dL_u^K(S).$$

Since the local time is known to be an increasing process, it is evident that if the strike price is a positive constant and the interest rate is non-negative, then the discounted reward process X^* follows a submartingale under the martingale measure $\mathbf{P}^*$, that is

$$\mathbf{E}_{\mathbf{P}^*}(X_t^* \,|\, \mathcal{F}_u) \geq X_u^*, \quad \forall\, u \leq t \leq T.$$

On the other hand, the discounted reward Y^* of the American put option with constant exercise price is a submartingale under $\mathbf{P}^*$, provided that $r \leq 0$. Summarizing, we have the following useful result.

[3]Obviously we have $L_t^0(S-K) = L_t^K(S)$ and $L_t^0(S-K) = L^{-K}(-S)$. It is also possible to show that $L^{-K}(-S) = L^K(S)$.

Corollary 5.2.1 *The discounted reward X^* (Y^*, respectively) of the American call option (put option, respectively) with constant strike price follows a submartingale under $\mathbf{P}^*$ if $r \geq 0$ (if $r \leq 0$, respectively).*

Let us now examine the rational exercise policy of the holder of an option of American style. We shall use throughout the superscripts c and p to denote the quantities associated with the reward functions g^c and g^p, respectively. In particular,

$$\begin{aligned} V_t^c &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(e^{-r(\tau - t)} (S_\tau - K)^+ \,|\mathcal{F}_t \big) \\ &= e^{rt} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(X_\tau^* \,|\mathcal{F}_t \big), \end{aligned}$$

and

$$\begin{aligned} V_t^p &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(e^{-r(\tau - t)} (K - S_\tau)^+ \,|\mathcal{F}_t \big) \\ &= e^{rt} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(Y_\tau^* \,|\mathcal{F}_t \big). \end{aligned}$$

We know already that in the case of a constant strike price and under a non-negative interest rate r, the discounted reward X^* of a call option follows a $\mathbf{P}^*$-submartingale. Consequently, the Snell envelope J^c of X^* equals

$$J_t^c = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \big(X_\tau^* \,|\mathcal{F}_t \big) = \mathbf{E}_{\mathbf{P}^*} (X_T^* \,|\, \mathcal{F}_t). \tag{5.16}$$

This in turn implies that for every date t, the rational exercise time after time t of the American call option with a constant strike price K is the option's expiry date T (the same property holds for the American put option, provided that $r \leq 0$). In other words, under a non-negative interest rate, an American call option with constant strike price should never be exercised before its expiry date, and thus its arbitrage price coincides with the Black-Scholes price of a European call. In other words, in usual circumstances, the American call option written on a non-dividend-paying stock is always worth more alive than dead, hence it is formally equivalent to the European call option with the same contractual features. This shows that in the case of constant strike price and non-negative interest rate r, only an American put option written on a non-dividend-paying stock requires further examination.

5.3 Early Exercise Representation of an American Put

The following result is a consequence of a much more general theorem established in El Karoui and Karatzas (1991). For the sake of concreteness, we shall focus on the case of the put option. We refer the reader to Myneni (1992) for the proof.

Proposition 5.3.1 *The Snell envelope J^p admits the following decomposition*

$$J_t^p = \mathbf{E}_{\mathbf{P}^*} \Big(e^{-rT} (K - S_T)^+ \,\Big|\, \mathcal{F}_t \Big) + \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T e^{-ru} \, \mathrm{I}_{\{\tau_u = u\}} \, rK \, du \,\Big|\, \mathcal{F}_t \Big).$$

Recall that $V_t^p = e^{rt} J_t^p$ for every t. Consequently, the price P_t^a of an American put option satisfies

$$\begin{aligned} P_t^a = V_t^p &= \mathbf{E}_{\mathbf{P}^*} \Big(e^{-r(T-t)} (K - S_T)^+ \,\Big|\, \mathcal{F}_t \Big) \\ &\quad + \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T e^{-r(u-t)} \, \mathrm{I}_{\{\tau_u = u\}} \, rK \, du \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

Furthermore, in view of the Markov property of the stock price process S, it is clear that for any stopping time $\tau \in \mathcal{T}_{[t,T]}$, we have

$$\mathbf{E}_{\mathbf{P}^*} \big(e^{-r(\tau - t)} (K - S_\tau)^+ \,|\, \mathcal{F}_t \big) = \mathbf{E}_{\mathbf{P}^*} \big(e^{-r(\tau - t)} (K - S_\tau)^+ \,|\, S_t \big).$$

We conclude that the price of an American put equals $P_t^a = P^a(S_t, T-t)$ for a certain function $P^a : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$. To be a bit more explicit, we define the function $P^a(s,t)$ by setting

$$P^a(s, T-t) = \sup{}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*} \Big(e^{-r(\tau - t)} (K - S_\tau)^+ \,\Big|\, S_t = s \Big), \tag{5.17}$$

where the expectation on the right-hand side is conditional on the event $\{S_t = s\}$. It is possible to show that the function $P^a(s,u)$ is decreasing and convex in s, and increasing in u. Let us denote by $\mathcal{C}$ and $\mathcal{D}$ the *continuation region* and *stopping region,* respectively. The stopping region $\mathcal{D}$ is defined as that subset of $\mathbf{R}_+ \times [0,T]$ for which the stopping time τ_t satisfies

$$\tau_t = \inf \{ u \in [t,T] \,|\, (S_u, u) \in \mathcal{D} \}$$

for every $t \in [0,T]$. The continuation region $\mathcal{C}$ is, of course, the complement of $\mathcal{D}$ in $\mathbf{R}_+ \times [0,T]$. Note that in terms of the function P^a, we have

$$\mathcal{D} = \{ (s,t) \in \mathbf{R}_+ \times [0,T] \,|\, P^a(s, T-t) = (K-s)^+ \}$$

and

$$\mathcal{C} = \{ (s,t) \in \mathbf{R}_+ \times [0,T] \,|\, P^a(s, T-t) > (K-s)^+ \}.$$

Let us define the function $b^* : [0,T] \to \mathbf{R}_+$ by setting

$$b^*(T-t) = \sup \{ s \in \mathbf{R}_+ \,|\, P^a(s, T-t) = (K-s)^+ \}.$$

It can be shown that the graph of b^* is contained in the stopping region $\mathcal{D}$. This means that it is only rational to exercise the put option at time t if the current stock price S_t is at or below the level $b^*(T-t)$. For this reason, the value $b^*(T-t)$ is commonly referred to as the *critical stock price* at time t. It is sometimes convenient to consider the function $c^* : [0,T] \to \mathbf{R}_+$ which is given by the equality $c^*(t) = b^*(T-t)$. For any $t \in [0,T]$, the optimal exercise time τ_t after time t satisfies

$$\tau_t = \inf \{ u \in [t,T] \,|\, (K - S_u)^+ = P^a(S_u, T-u) \},$$

or equivalently

$$\tau_t = \inf \{ u \in [t,T] \,|\, S_u \le b^*(T-u) \} = \inf \{ u \in [t,T] \,|\, S_u \le c^*(u) \}.$$

We quote the following lemma from van Moerbeke (1976).

Lemma 5.3.1 *If the strike price is constant, then c^* is non-decreasing, infinitely smooth over $(0,T)$ and $\lim_{t\uparrow T} c^*(t) = \lim_{T-t\downarrow 0} b^*(T-t) = K$.*

By virtue of the next result, the price of an American put option may be represented as the sum of the arbitrage price of the corresponding European call option and the so-called early exercise premium.

Corollary 5.3.1 *The following decomposition of the price of an American put option is valid*

$$P_t^a = P_t + \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T e^{-r(u-t)} \, \mathrm{I}_{\{S_u < b^*(T-u)\}} rK \, du \,\Big|\, \mathcal{F}_t \Big),$$

where $P_t^a = P^a(S_t, T-t)$ is the price of the American put option, and $P_t = P(S_t, T-t)$ is the Black-Scholes price of a European put with strike K.

Decomposition of the price provided by Corollary 5.3.1 is commonly referred to as the *early exercise premium representation* of an American put. It was derived independently, by different means and at various levels of strictness, by Kim (1990), Jacka (1991), and Jamshidian (1992) (see also Carr et al. (1992) for other representations of the price of an American put). For $t = 0$ we have

$$P_0^a = P_0 + \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T e^{-ru}\, \mathrm{I}_{\{S_u < b^*(T-u)\}}\, rK\, du \Big), \tag{5.18}$$

where P_0 is the price at time 0 of the European put. Observe that if $r = 0$, then the early exercise premium vanishes, which means that the American put is equivalent to the European put and thus should not be exercised before the expiry date. Taking into account the dynamics of the stock price under $\mathbf{P}^*$, we can make representation (5.18) more explicit, namely

$$P^a(S_0, T) = P(S_0, T) + rK \int_0^T e^{-ru}\, N\left(\frac{\ln(b^*(T-u)/S_0) - \rho u}{\sigma\sqrt{u}} \right) du,$$

where $\rho = r - \sigma^2/2$, and N denotes the standard Gaussian cumulative distribution function. A similar decomposition is valid for any instant $t \in [0, T]$, provided that the current stock price S_t belongs to the continuation region $\mathcal{C}$; that is, the option should not be exercised immediately. We have

$$\begin{aligned} P^a(S_t, T-t) &= P(S_t, T-t) \\ &\quad + rK \int_t^T e^{-r(u-t)}\, N\left(\frac{\ln(b^*(T-u)/S_t) - \rho(u-t)}{\sigma\sqrt{u-t}} \right) du, \end{aligned}$$

where $P(S_t, T-t)$ stands for the price of a European put option of maturity $T - t$, and S_t is the current level of the stock price. A change of variables leads to the following equivalent expression

$$P^a(s, t) = P(s, t) + rK \int_0^t e^{-ru}\, N\left(\frac{\ln(b^*(t-u)/s) - \rho u}{\sigma\sqrt{u}} \right) du,$$

which is valid for every $(s, t) \in \mathcal{C}$. If $S_t = b^*(T-t)$, then we have necessarily $P^a(S_t, T-t) = K - b^*(T-t)$, so that clearly $P^a(b^*(t), t) = K - b^*(t)$ for every $t \in [0, T]$. This simple observation leads to the following integral equation, which is satisfied by the optimal boundary function b^*

$$K - b^*(t) = P(b^*(t), t) + rK \int_0^t e^{-ru}\, N\left(\frac{\ln(b^*(t-u)/b^*(t)) - \rho u}{\sigma\sqrt{u}} \right) du.$$

Unfortunately, a solution to this integral equation is not explicitly known, and thus it needs to be solved numerically. On the other hand, the following bounds for the price $P^a(s, t)$ are easy to derive

$$P^a(s, t) - P(s, t) \le rK \int_0^t e^{-ru}\, N\left(\frac{\ln(K/s) - \rho u}{\sigma\sqrt{u}} \right) du$$

and

$$P^a(s, t) - P(s, t) \ge rK \int_0^t e^{-ru}\, N\left(\frac{\ln(b^*_\infty/s) - \rho u}{\sigma\sqrt{u}} \right) du,$$

where b^*_∞ stands for the optimal exercise boundary of a *perpetual put* – that is, an American put option with expiry date $T = \infty$.

To this end, it is enough to show that for any maturity T, the values of the optimal stopping boundary b^* lie between the strike price K and the level b^*_∞, i.e.,

$$K \le b^*(t) \le b^*_\infty, \quad \forall\, t \in [0, T].$$

It should be stressed that the value of b^*_∞ is known to be (see McKean (1965), van Moerbeke (1976) or the next section)

$$b^*_\infty = \frac{2rK}{2r + \sigma^2}. \tag{5.19}$$

5.4 Free Boundary Problem

The *free boundary problem* related to the optimal stopping problem for an American put option was first examined by McKean (1965) and van Moerbeke (1976). For a fixed expiry date T and constant strike price K, we denote by $P^a(S_t, T-t)$ the price of an American put at time $t \in [0,T]$; in particular, $P^a(S_T, 0) = (K - S_T)^+$. It will be convenient to denote by $\mathcal{L}$ the following differential operator

$$\mathcal{L}v = \frac{1}{2}\,\sigma^2 s^2\,\frac{\partial^2 v}{\partial s^2} + rs\,\frac{\partial v}{\partial s} - rv = \mathcal{A}v - rv, \tag{5.20}$$

where $\mathcal{A}$ stands for the infinitesimal generator of the one-dimensional diffusion process S (considered under the martingale measure $\mathbf{P}^*$). Also, let $\mathcal{L}_t$ stand for the following differential operator

$$\mathcal{L}_t v = \frac{\partial v}{\partial t} + \mathcal{L}v = v_t + \mathcal{A}v - rv.$$

The proof of the following proposition, which focuses on the properties of the price function $P^a(s,t)$ of an American put, can be found in van Moerbeke (1976) (see also McKean (1965) and Jacka (1991)). Recall that we denote $c^*(t) = b^*(T-t)$.

Proposition 5.4.1 *The American put value function $P^a(s,t)$ is smooth on the continuation region $\mathcal{C}$ with $-1 \le P^a_s(s,t) \le 0$ for all $(s,t) \in \mathcal{C}$. The optimal stopping boundary b^* is a continuous and non-increasing function on $(0,T]$, hence c^* is a non-decreasing function on $[0,T)$, and*

$$\lim_{T-t\,\downarrow\,0} b^*(T-t) = \lim_{t\,\uparrow\,T} c^*(t) = K.$$

On $\mathcal{C}$, the function P^a satisfies $P^a_t(s,t) = \mathcal{L}P^a(s,t)$; that is,

$$P^a_t(s,t) = \tfrac{1}{2}\sigma^2 s^2\, P^a_{ss}(s,t) + rs\, P^a_s(s,t) - rP^a(s,t), \quad \forall\,(s,t) \in \mathcal{C}.$$

Furthermore, we have

$$\begin{aligned}
\lim_{s\,\downarrow\,b^*(t)} P^a(s,t) &= K - b^*(t), && \forall\, t \in (0,T],\\
\lim_{t\to 0} P^a(s,t) &= (K-s)^+, && \forall\, s \in \mathbf{R}_+,\\
\lim_{s\to\infty} P^a(s,t) &= 0, && \forall\, t \in (0,T],\\
P^a(s,t) &\ge (K-s)^+, && \forall\,(s,t) \in \mathbf{R}_+ \times (0,T].
\end{aligned}$$

In order to determine the optimal stopping boundary, one needs to impose an additional condition, known as the *smooth fit principle*, which reads as follows (for the proof of the next result, we refer the reader to van Moerbeke (1976)).

Proposition 5.4.2 *The partial derivative $P^a_s(s,t)$ is continuous a.e. across the stopping boundary c^*; that is*

$$\lim_{s\downarrow c^*(t)} P^a_s(s,t) = -1 \tag{5.21}$$

for almost every $t \in [0,T]$.

We are now in a position to state (without proof) the result, which characterizes the price of an American put option as the solution to the free boundary problem (see van Moerbeke (1976), Jacka (1991)).

Theorem 5.4.1 *Let $\mathcal{G}$ be an open domain in $\mathbf{R}_+ \times [0,T)$ with continuously differentiable boundary c. Assume that $v : \mathbf{R}_+ \times [0,T]$ is a continuous function such that $u \in C^{3,1}(\mathcal{G})$, the function $g(s,t) = v(e^s,t)$ has Tychonov growth (see Myneni (1992)) and v satisfies $\mathcal{L}_t v = 0$ on $\mathcal{G}$; that is*

$$v_t(s,t) + \tfrac{1}{2}\sigma^2 s^2\, v_{ss}(s,t) + rs\, v_s(s,t) - rv(s,t) = 0 \tag{5.22}$$

for every $(s,t) \in \mathcal{G}$, *and*

$$\begin{aligned} v(s,t) &> (K-s)^+, && \forall\,(s,t) \in \mathcal{G}, \\ v(s,t) &= (K-s)^+, && \forall\,(s,t) \in \mathcal{G}^c, \\ v(s,T) &= (K-s)^+, && \forall\, s \in \mathbf{R}_+, \\ \lim_{s \downarrow c(t)} v_s(s,t) &= -1, && \forall\, t \in [0,T). \end{aligned}$$

Then the function $P^a(s,t) = v(s, T-t)$ *for every* $(s,t) \in \mathbf{R}_+ \times [0,T]$ *is the value function of the American put option with strike price* K *and maturity* T*. Moreover, the set* $\mathcal{C} = \mathcal{G}$ *is the option's continuation region, and the function* $b^*(t) = c(T-t)$, $t \in [0,T]$, *represents the critical stock price.*

We shall now apply the above theorem to a *perpetual put* – that is, an American put option which has no expiry date (i.e., with maturity $T = \infty$). Since the time to expiry of a perpetual put is always infinite, the critical stock price becomes a real number $b^*_\infty \le K$, and the PDE (5.22) becomes the following ordinary differential equation (ODE) for the function $v_\infty(s) = v(s,\infty)$

$$\frac{1}{2}\,\sigma^2 s^2 \,\frac{d^2 v_\infty}{ds^2}(s) + rs\,\frac{dv_\infty}{ds}(s) - r v_\infty(s) = 0, \quad \forall\, s \in (b^*_\infty, \infty), \tag{5.23}$$

with $v_\infty(s) = K - s$ for every $s \in [0, b^*_\infty]$. Our aim is to show that equality (5.19) is valid. For this purpose, observe that ODE (5.23) admits a general solution of the form

$$v_\infty(s) = c_1 s^{d_1} + c_2 s^{d_2}, \quad \forall\, s \in (b^*_\infty, \infty), \tag{5.24}$$

where c_1, c_2, d_1 and d_2 are constants. Using the boundary condition, which reads $v_\infty(b_\infty) = K - b_\infty$, and the smooth fit condition

$$\lim_{s \downarrow b_\infty} \frac{dv_\infty}{ds}(s) = -1,$$

we find that

$$v_\infty(s) = (K - b_\infty)\left(\frac{b_\infty}{s}\right)^{2r\sigma^{-2}}, \quad \forall\, s \in (b^*_\infty, \infty), \tag{5.25}$$

and thus, in particular, equality (5.19) is valid.

5.5 Approximations of the American Put Price

Since no closed-form expression for the value of an American put is available, in order to value American options one needs to use a numerical procedure. It appears that the use of the CRR binomial tree, although remarkably simple, is far from being the most efficient way of pricing American options. Various approximations of the American put price on a non-dividend-paying stock were examined in Brennan and Schwartz (1977), Johnson (1983), MacMillan (1986), and Broadie and Detemple (1996).

Let us comment briefly on the approximate valuation method proposed by Geske and Johnson (1984). Basically, the *Geske-Johnson approximation* relies on the discretization of the time parameter and the application of backward induction, as in any other standard discrete-time approach. However, in contrast to the space-time discretization used in the multinomial trees approach or in the finite difference methods, the approach of Geske and Johnson makes use of the exact distribution of the vector of stock prices $(S_{t_1}, \ldots, S_{t_n})$, where $t_1 < \ldots < t_n = T$ are the only admissible (deterministic) exercise times. [4] In other words, the decision to exercise an option can be made at any of the dates $t_1, \ldots, t_n$ only. Let us start by considering the special case when $n = 2$ and $t_1 = T/2$, $t_2 = T$. Note that if $n = 1$, the option can be exercised at $t_1 = T$ only, so that it is

[4] An option which may be exercised early, but only on predetermined dates, is commonly referred to as *Bermudan option.*

equivalent to a European put. To find an approximate value for an American put, we shall argue by backward induction. Suppose that the option was not exercised at time t_1. Then the value of the option at time t_1 is equal to the value of a European put option with maturity $t_2 - t_1 = T/2$, given the initial stock price $S_{t_1} = S_{T/2}$. The price of a European put is given, of course, by the standard Black-Scholes formula, denoted by $P(S_{T/2}, T/2)$. The critical stock price b_1^* at time $t_1 = T/2$ solves the equation $K - S_{T/2} = P(S_{T/2}, T/2)$, hence it can be found by numerical methods. Moreover, it is clear that the value $V_{T/2}$ of the option at time $T/2$ satisfies

$$V_{T/2} = \begin{cases} P(S_{T/2}, T/2) & \text{if } S_{T/2} > b_1^*, \\ K - S_{T/2} & \text{if } S_{T/2} \le b_1^*. \end{cases}$$

Note that it is optimal to exercise the option at time t_1 if and only if $S_{T/2} \le b_1^*$. To find the value of the option at time 0, we need first to evaluate the expectation $V_0(S_0) = \mathbf{E}_{\mathbf{P}^*}(e^{-rT/2}\, V_{T/2})$, or equivalently

$$V_0(S_0) = e^{-rT/2}\,\mathbf{E}_{\mathbf{P}^*}\Big(e^{-rT/2}(S_T - K)^+ \mathrm{I}_{\{S_{T/2} > b_1^*\}} + (K - S_{T/2})\mathrm{I}_{\{S_{T/2} \le b_1^*\}}\Big).$$

Notice that the latter expectation can be expressed in terms of the probability law of the two-dimensional random variable $(S_{T/2}, S_T)$; equivalently, one exploit the joint law of $W_{T/2}$ and W_T. A specific, quasi-explicit representation of $V_0(S_0)$ in terms of two-dimensional Gaussian cumulative distribution function is in fact a matter of convenience. The approximate value of an American put with two admissible exercise times, $T/2$ and T, equals $P_2^a(S_0, T) = V_0(S_0)$. The same iterative procedure may be applied to an arbitrary finite sequence of times $t_1 < \ldots < t_n = T$. In this case, the Geske-Johnson approximation formula involves integration with respect to a n-dimensional Gaussian probability density function. It appears that for three admissible exercise times, $T/3$, $2T/3$ and T, the approximate quasi-analytical valuation formula provided by the Geske and Johnson method is roughly as accurate as the binomial tree with 150 time steps. For any natural n, let us denote by $P_n^a(S_0, T)$ the Geske-Johnson option's approximate value associated with admissible dates $t_i = Ti/n$, $i = 1, \ldots, n$. It is possible to show that the sequence $P_n^a(S_0, T)$ converges to the option's exact price $P^a(S_0, T)$ when the number of steps tends to infinity, so that the step length tends to zero. To estimate the limit $P^a(S_0, T)$, one can make use of any extrapolation technique, for instance Richardson's approximation scheme. Let us briefly describe the latter technique. Suppose that the function F satisfies

$$F(h) = F(0) + c_1 h + c_2 h^2 + o(h^2)$$

in the neighborhood of zero, so that

$$F(kh) = F(0) + c_1 kh + c_2 k^2 h^2 + o(h^2)$$

and

$$F(lh) = F(0) + c_1 lh + c_2 l^2 h^2 + o(h^2)$$

for arbitrary $l > k > 1$. Ignoring the term $o(h^2)$, and solving the above system of equations for $F(0)$, we obtain ("$\approx$" denotes approximate equality)

$$F(0) \approx F(h) + \frac{a}{c}\big(F(h) - F(kh)\big) + \frac{b}{c}\big(F(kh) - F(lh)\big), \tag{5.26}$$

where $a = l(l-1) - k(k-1)$, $b = k(k-1)$ and $c = l^2(k-1) - l(k^2-1) + k(k-1)$. Let us write P_n^a to denote $P_n^a(S_0, T)$ for $n = 1, 2, 3$ (in particular, $P_1^a(S_0, T)$ is the European put price $P(S_0, T)$). For $n = 3$ upon setting $k = 3/2$, $l = 3$ and $P_1^a = F(lh)$, $P_2^a = F(kh)$, $P_3^a = F(h)$, we get the following approximate formula

$$P^a(S_0, T) \approx P_3^a + \tfrac{7}{2}(P_3^a - P_2^a) - \tfrac{1}{2}(P_2^a - P_1^a).$$

Bunch and Johnson (1992) argue that the Geske-Johnson method can be further improved if the exercise times are chosen iteratively in such a way that the option's approximate value is maximized.

5.6 Option on a Dividend-paying Stock

Since most traded options on stocks are *unprotected* American call options written on dividend-paying stocks, it is worthwhile to comment briefly on the valuation of these contracts. A call option is said to be *unprotected* if it has no contracted "protection" against the stock price decline that occurs when a dividend is paid. It is intuitively clear that an unprotected American call written on a dividend-paying stock is not equivalent to the corresponding option of European style, in general. Suppose that a known dividend, D, will be paid to each shareholder with certainty at a prespecified date T_D during the option's lifetime. Furthermore, assume that the ex-dividend stock price decline equals δD for a given constant $\delta \in [0,1]$. Let us denote by S_{T_D} and $P_{T_D} = S_{T_D} - \delta D$ respectively the cum-dividend and ex-dividend stock prices at time T_D. It is clear that the option should eventually be exercised just before the dividend is paid – that is, an instant before T_D. Consequently, as first noted by Black (1975), the lower bound for the price of such an option is the price of the European call option with expiry date T_D and strike price K. This lower bound is a good estimate of the exact value of the price of the American option whenever the probability of early exercise is large – that is, when the probability $\mathbf{P}\{C_{T_D} < S_{T_D} - K\}$ is large, where $C_{T_D} = C(P_{T_D}, T - T_D, K)$ is the Black-Scholes price of the European call option with maturity $T - T_D$ and exercise price K. Hence, early exercise of the American call is more likely the larger the dividend, the higher the stock price S_{T_D} relative to the strike price K, and the shorter the time-period $T - T_D$ between expiry and dividend payment dates. An analytic valuation formula for unprotected American call options on stocks with known dividends was established by Roll (1977). However, it seems to us that Roll's original reasoning, which refers to options that expire an instant before the ex-dividend date, assumes implicitly that the holder of an option may exercise it before the ex-dividend date, but apparently is not allowed to sell it before the ex-dividend date. To avoid this discrepancy, we prefer instead to consider European options which expire on the ex-dividend date – i.e., after the ex-dividend stock price decline.

Before formulating the next result, we need to introduce some notation. Let us denote by b^* the cum-dividend stock price level above which the original American option will be exercised at time T_D, so that

$$C(b^* - \delta D, T - T_D, K) = b^* - K. \tag{5.27}$$

It is worthwhile to observe that $C(s - \delta D, T - T_D, K) < s - K$ when $s \in (b^*, \infty)$, and $C(s - \delta D, T - T_D, K) > s - K$ for every $s \in (0, b^*)$. Note that the first two terms on the right-hand side of equality (5.28) below represent the values of European options, written on a stock S, which expire at time T and on the ex-dividend date T_D, respectively. The last term, $\mathbf{CO}_t(T_D, b^* - K)$, represents the price of a so-called *compound option* (see Sect. 6.4). To be more specific, we deal here with a European call option with strike price $b^* - K$ which expires on the ex-dividend date T_D, and whose underlying asset is the European call option, written on S, with maturity T and strike price K. The compound option will be exercised by its holder at the ex-dividend date T_D if and only if he is prepared to pay $b^* - K$ for the underlying European option. Since the value of the underlying option after the ex-dividend stock price decline equals $C(P_{T_D}, T - T_D, K)$, the compound option is exercised whenever

$$C(P_{T_D}, T - T_D, K) = C(S_{T_D} - \delta D, T - T_D, K) > b^* - K,$$

that is, when the cum-dividend stock price exceeds b^* (this follows from the fact that the price of a standard European call option is an increasing function of the stock price, combined with equality (5.27)).

Proposition 5.6.1 *The arbitrage price $\tilde{C}^a_t(T, K)$ of an unprotected American call option with expiry date $T > T_D$ and strike price K, written on a stock which pays a known dividend D at time T_D, equals*

$$\tilde{C}^a_t(T, K) = \tilde{C}_t(T, K) + C_t(T_D, b^*) - \mathbf{CO}_t(T_D, b^* - K) \tag{5.28}$$

for $t \in [0, T_D]$, where b^* is the solution to (5.27).

Chapter 6

Exotic Options

In the preceding chapters, we have focused on the two standard classes of options – that is, call and put options of European and American style. The aim of this chapter is to study examples of more sophisticated option contracts. For convenience, we give the generic name *exotic option* to any option contract which is not a standard European or American option. It should be made clear that we shall restrict our attention to the case of exotic spot options. We find it convenient to classify the large family of exotic options as follows:
(a) *packages* – options that are equivalent to a portfolio of standard European options, cash and the underlying asset (stock, say);
(b) *forward-start options* – options that are paid for in the present but received by holders at a prespecified future date;
(c) *chooser options* – option contracts that are chosen by their holders to be call or put at a prescribed future date;
(d) *compound options* – option contracts with other options playing the role of the underlying assets;
(e) *binary options* – contracts whose payoff is defined by means of some binary function;
(f) *barrier options* – options whose payoff depends on whether the underlying asset price reaches some barrier during the option's lifetime;
(g) *Asian options* – options whose payoff depends on the average price of the underlying asset during a prespecified period;
(h) *basket options* – options with a payoff depending on the average of prices of several assets;
(i) *lookback options* – options whose payoff depends, in particular, on the minimum or maximum price of the underlying asset during options' lifetimes;

6.1 Packages

An arbitrary financial contract whose terminal payoff is a piecewise linear function of the terminal price of the underlying asset may be seen as a *package option* – that is, a combination of standard options, cash and the underlying asset. Unless explicitly stated otherwise, we shall place ourselves within the classic Black-Scholes framework.

6.1.1 Collars

Let $K_2 > K_1 > 0$ be fixed real numbers. The payoff at expiry date T from the long position in a *collar option* equals

$$\mathbf{CL}_T \stackrel{\text{def}}{=} \min\big\{ \max\{S_T, K_1\}, K_2\big\}.$$

It is easily seen that the payoff $\mathbf{CL}_T$ can be represented as follows

$$\mathbf{CL}_T = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+,$$

so that a collar option can be seen as a portfolio of cash and two standard call options. This implies that the arbitrage price of a collar option at any date t before expiry equals

$$\mathbf{CL}_t = K_1 e^{-r(T-t)} + C(S_t, T-t, K_1) - C(S_t, T-t, K_2),$$

where $C(s, T-t, K) = C(s, T-t, K, r, \sigma)$ stands for the Black-Scholes call option price at time t, where the current level of the stock price is s, and the exercise price of the option equals K (see formula (3.71)).

6.1.2 Break Forwards

By a *break forward* we mean a modification of a typical forward contract, in which the potential loss from the long position is limited by some prespecified number. More explicitly, the payoff from the long *break forward* is defined by the equality

$$\mathbf{BF}_T \stackrel{\text{def}}{=} \max\{S_T, F\} - K,$$

where $F = F_S(0,T) = S_0 e^{rT}$ is the forward price of a stock for settlement at time T, and $K > F$ is some constant. The delivery price K is set in such a manner that the break forward contract is worthless when it is entered into. Since

$$\mathbf{BF}_T = (S_T - F)^+ + F - K,$$

it is clear that for every $t \in [0,T]$,

$$\mathbf{BF}_t = C(S_t, T-t, F) + (F-K)e^{-r(T-t)}.$$

In particular, the right level of K, K_0 say, is given by the expression

$$K_0 = e^{rT}\big(S_0 + C(S_0, T, S_0 e^{rT})\big).$$

Using the Black-Scholes valuation formula, we end up with the following equality

$$K_0 = e^{rT} S_0 \Big(1 + N\big(d_1(S_0,T)\big) - N\big(d_2(S_0,T)\big)\Big),$$

where d_1 and d_2 are given by (3.43)–(3.44).

6.1.3 Range Forwards

A *range forward* may be seen as a special case of a collar – one with zero initial cost. Its payoff at expiry is

$$\mathbf{RF}_T \stackrel{\text{def}}{=} \max\big\{\min\{S_T, K_2\}, K_1\big\} - F = \max\big\{\min\{S_T - F, K_2 - F\}, K_1 - F\big\},$$

where $K_1 < F < K_2$, and as before $F = F_S(0,T) = S_0 e^{rT}$. It appears convenient to decompose the payoff of a range forward in the following way

$$\mathbf{RF}_T = S_T - F + (K_1 - S_T)^+ - (S_T - K_2)^+.$$

Indeed, the above representation of the payoff implies directly that a range forward may be seen as a portfolio composed of a long forward contract, a long put option with strike price K_1, and finally a short call option with strike price K_2. Furthermore, its price at t equals

$$\mathbf{RF}_t = S_t - S_0 e^{rt} + P(S_t, T-t, K_1) - C(S_t, T-t, K_2).$$

As mentioned earlier, the levels K_1 and K_2 should be chosen in such a way that the initial value of a range forward equals 0.

6.2 Forward-start Options

Let us consider two dates, say T_0 and T, with $T_0 < T$. A *forward-start option* is a contract in which the holder receives, at time T_0 (at no additional cost), an option with expiry date T and exercise price K equal to S_{T_0}. On the other hand, the holder must pay at time 0 an up-front fee, the price of a forward-start option. Let us consider the case of a forward-start call option, with terminal payoff

$$\mathbf{FS}_T \stackrel{\text{def}}{=} (S_T - S_{T_0})^+.$$

To find the price at time $t \in [0, T_0]$ of such an option, it suffices to consider its value at the delivery date T_0, that is

$$\mathbf{FS}_T = C(S_{T_0}, T - T_0, S_{T_0}).$$

Since we restrict our attention to the classic Black-Scholes model, it is easily seen that

$$C(S_{T_0}, T - T_0, S_{T_0}) = S_{T_0} C(1, T - T_0, 1),$$

and thus the option's value at time 0 equals

$$\mathbf{FS}_0 = S_0 C(1, T - T_0, 1) = C(S_0, T - T_0, S_0).$$

If a stock continuously pays dividends at a constant rate κ, the above equality should be modified as follows

$$\mathbf{FS}_0^\kappa = e^{-\kappa T_0} C^\kappa (S_0, T - T_0, S_0),$$

where C^κ stands for the call option price derived in Proposition 3.2.2. Similar formulae can be derived for the case of a forward-start put option.

6.3 Chooser Options

A *chooser option* is an agreement in which one party has the right to choose at some future date T_0 whether the option is to be a call or put option with a common exercise price K and remaining time to expiry $T - T_0$. Therefore, the payoff at T_0 of a chooser option is

$$\mathbf{CH}_{T_0} \stackrel{\text{def}}{=} \max\big\{C(S_{T_0}, T - T_0, K), P(S_{T_0}, T - T_0, K)\big\},$$

while its terminal payoff is given by the expression

$$\mathbf{CH}_{T_0} = (S_T - K)^+ \, \mathrm{I}_A + (K - S_T)^+ \, \mathrm{I}_{A^c},$$

where A stands for the following event, which belongs to the σ-field $\mathcal{F}_{T_0}$

$$A = \{\, \omega \in \Omega \,|\, C(S_{T_0}, T - T_0, K) > P(S_{T_0}, T - T_0, K) \,\}$$

and A^c is the complement of A in Ω. The call-put parity implies that

$$P(S_{T_0}, T - T_0, K) = C(S_{T_0}, T - T_0, K) - S_{T_0} + K e^{-r(T - T_0)},$$

and thus

$$\mathbf{CH}_{T_0} = \max\big\{C(S_{T_0}, T - T_0, K), C(S_{T_0}, T - T_0, K) - S_{T_0} + K e^{-r(T - T_0)}\big\},$$

or finally

$$\mathbf{CH}_{T_0} = C(S_{T_0}, T - T_0, K) + (K e^{-r(T - T_0)} - S_{T_0})^+.$$

The last equality implies immediately that the standard chooser option is equivalent to the portfolio composed of a long call option and a long put option (with different exercise prices and different expiry dates), so that its arbitrage price equals

$$\mathbf{CH}_t = C(S_t, T - t, K) + P(S_t, T_0 - t, K e^{-r(T - T_0)})$$

for every $t \in [0, T_0]$. In particular, using the Black-Scholes formula, we get for $t = 0$

$$\mathbf{CH}_0 = S_0 \big(N(d_1) - N(-\bar{d}_1)\big) + Ke^{-rT} \big(N(-\bar{d}_2) - N(d_2)\big),$$

where

$$d_{1,2} = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$\bar{d}_{1,2} = \frac{\ln(S_0/K) + rT \pm \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}.$$

6.4 Compound Options

A *compound option* (see Geske (1977, 1979a)) is a standard option with another standard option being the underlying asset. One can distinguish four basic types of compound options: call on a call, put on a call, call on a put, and, finally, put on a put. Let us consider, for instance, the case of a *call on a call* compound option. For two future dates T_0 and T, with $T_0 < T$, and two exercise prices K_0 and K, consider a call option with exercise price K_0 and expiry date T_0 on a call option with strike price K and maturity T. It is clear that the payoff of the compound option at time T_0 is

$$\mathbf{CO}_{T_0} \stackrel{\text{def}}{=} \big(C(S_{T_0}, \tau, K) - K_0\big)^+,$$

where $C(S_{T_0}, \tau, K)$ stands for the value at time T_0 of a standard call option with strike price K and expiry date $T = T_0 + \tau$. In the Black-Scholes framework, we obtain the following equality

$$C(s, \tau, K) = sN\big(d_1(s, \tau, K)\big) - Ke^{-r\tau}N\big(d_2(s, \tau, K)\big).$$

Moreover, since under $\mathbf{P}^*$ we have

$$S_{T_0} = S_0 \exp\big(\sigma\sqrt{T_0}\,\xi + (r - \tfrac{1}{2}\sigma^2)\,T_0\big),$$

where ξ has a standard Gaussian probability law under $\mathbf{P}^*$, the price of the compound option at time 0 equals

$$\mathbf{CO}_0 = e^{-rT_0} \int_{x_0}^{\infty} \big(g(x)N(\hat{d}_1) - Ke^{-r\tau}N(\hat{d}_2) - K_0\big)\, n(x)\, dx,$$

where $\hat{d}_i = d_i(g(x), \tau, K)$ for $i = 1, 2$, the function $g : \mathbf{R} \to \mathbf{R}$ equals

$$g(x) = S_0 \exp\big(\sigma\sqrt{T_0}\,x + (r - \tfrac{1}{2}\sigma^2)\,T_0\big)$$

and, finally, the constant $x_0 = \inf\{x \in \mathbf{R} \,|\, C(g(x), \tau, K) \geq K_0\}$. Straightforward calculations yield

$$d_1(g(x), \tau, K) = \frac{\ln(S_0/K) + \sigma\sqrt{T_0}\,x + rT - \sigma^2 T_0 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T - T_0}}$$

and

$$d_2(g(x), \tau, K) = \frac{\ln(S_0/K) + \sigma\sqrt{T_0}\,x + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T - T_0}}.$$

6.5 Digital Options

By a *digital* (or *binary*) option we mean a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset. The simplest examples of binary options are *cash-or-nothing* options and *asset-or-nothing* options. The payoffs at expiry of a cash-or-nothing call and put options are

$$\mathbf{BCC}_T \stackrel{\text{def}}{=} X\,\mathrm{I}_{\{S_T > K\}}, \quad \mathbf{BCP}_T \stackrel{\text{def}}{=} X\,\mathrm{I}_{\{S_T < K\}},$$

where in both cases X stands for a prespecified amount of cash. Similarly, for the asset-or-nothing option we have

$$\mathbf{BAC}_T \stackrel{\text{def}}{=} S_T \, \mathrm{I}_{\{S_T > K\}}, \quad \mathbf{BAP}_T \stackrel{\text{def}}{=} S_T \, \mathrm{I}_{\{S_T < K\}}$$

for a call and put respectively. All options introduced above may be easily priced by means of the risk-neutral valuation formula. Somewhat more complex binary options are the so-called *gap options,* whose payoff at expiry equals

$$\mathbf{GC}_T \stackrel{\text{def}}{=} (S_T - X) \, \mathrm{I}_{\{S_T > K\}} = \mathbf{BAC}_T - \mathbf{BCC}_T$$

for the call option, and

$$\mathbf{GP}_T \stackrel{\text{def}}{=} (X - S_T) \, \mathrm{I}_{\{S_T < K\}} = \mathbf{BCP}_T - \mathbf{BAP}_T$$

for the corresponding put option. Once again, pricing these options involves no difficulties. As a last example of a binary option, let us mention a *supershare,* whose payoff is

$$\mathbf{SS}_T \stackrel{\text{def}}{=} \frac{S_T}{K_1} \, \mathrm{I}_{\{K_1 < S_T < K_2\}}$$

for some positive constants $K_1 < K_2$. The price of such an option at time 0 is easily seen to equal

$$\mathbf{SS}_0 = \frac{S_0}{K_1} \Big(N\big(h_1(S_0, T)\big) - N\big(h_2(S_0, T)\big) \Big),$$

where

$$h_i(s,t) = \frac{\ln(S_0/K_i) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \, .$$

6.6 Barrier Options

The generic term *barrier options* refers to the class of options whose payoff depends on whether or not the underlying prices hit a prespecified barrier during the options' lifetimes. For closed-form expressions for prices of various barrier options – such as *knock-in* and *knock-out* options – we refer to Rubinstein and Reiner (1991), Kunitomo and Ikeda (1992), and Carr (1995) (see also Cheuk and Vorst (1996) for a numerical approach). Let us also mention the paper by Heynen and Kat (1994), which is devoted to so-called *partial barrier options* – that is, barrier options in which the underlying price is monitored for barrier hits only during a prespecified period during an option's lifetime. To give the flavor of the mathematical techniques used when dealing with barrier options, we shall examine here a specific kind of currency barrier option, namely the *down-and-out call option.* The payoff at expiry of a down-and-out call option equals (in units of domestic currency)

$$C_T^1 \stackrel{\text{def}}{=} (Q_T - K)^+ \, \mathrm{I}_{\{\min_{0 \le t \le T} Q_t \, \ge H\}},$$

where K and H are constants. It follows from the formula above that the down-and-out option becomes worthless (or is knocked out) if, at any time t prior to the expiry date T, the current exchange rate Q_t falls below a predetermined level H. It is thus evident that a down-and-out option is less valuable than a standard currency option. Our aim is to find an explicit formula for the so-called *knock-out discount.*

Out-of-the-money knock-out option. Suppose first that the inequalities $H < K$ and $H < Q_0$ are satisfied. From the general features of a down-and-out call, it is clear that the option is knocked out when it is out-of-the-money. Recall that under the domestic martingale measure $\mathbf{P}^*$ we have (cf. (4.8))

$$Q_t = Q_0 \, e^{\sigma_Q W_t^* + \lambda t} = Q_0 \, e^{X_t},$$

where $X_t = \sigma_Q W_t^* + \lambda t$ for $t \in [0,T]$, and $\lambda = r_d - r_f - \frac{1}{2}\sigma_Q^2$. Therefore,

$$\{\omega \in \Omega \,|\, \min_{0\le t\le T} Q_t \ge H\} = \{\omega \in \Omega \,|\, m_T \ge \ln(H/Q_0)\},$$

where $m_T = \min_{0\le t\le T} X_t$, and thus

$$C_T^1 = (Q_T - K)\,\mathrm{I}_{\{Q_T \ge K,\, \min_{0\le t\le T} Q_t \ge H\}} = Q_0\, e^{X_T}\,\mathrm{I}_D - K\mathrm{I}_D,$$

where D stands for the set

$$D = \{\omega \in \Omega \,|\, X_T \ \ge \ \ln(K/Q_0),\ m_T \ \ge \ \ln(H/Q_0)\}.$$

We conclude that the price at time 0 of a down-and-out call option admits the following representation

$$C_0^1 = e^{-r_d T} Q_0\, \mathbf{E}_{\mathbf{P}^*}\big(e^{X_T}\mathrm{I}_D\big) - e^{-r_d T} K \mathbf{P}^*\{D\},$$

where $\mathbf{P}^*$ is the martingale measure of the domestic market. In order to directly calculate C_0^1 by means of integration, we need to find first the joint probability distribution of random variables X_T and m_T. One can show that for all x, y such that $y \le 0$ and $y \le x$, we have

$$\mathbf{P}^*\{\, X_T \ge x,\ m_T \ge y\} = N\left(\frac{-x + \lambda T}{\sigma\sqrt{T}}\right) - e^{2\lambda y \sigma^{-2}}\, N\left(\frac{-x + 2y + \lambda T}{\sigma\sqrt{T}}\right),$$

where, for the sake of notational convenience, we write σ in place of σ_Q. Consequently, the probability density function of (X_T, m_T) equals

$$f(x,y) = \frac{-2(2y-x)}{\sigma^3 T^{3/2}}\, e^{2\lambda y\sigma^{-2}}\, n\left(\frac{-x+2y+\lambda T}{\sigma\sqrt{T}}\right)$$

for $y \le 0,\ y \le x,$ where n stands for the standard Gaussian density function. From the above it follows, in particular, that

$$\mathbf{P}^*\{D\} = N\left(\frac{\ln(Q_0/K) + \lambda T}{\sigma\sqrt{T}}\right) - \big(H/Q_0\big)^{2\lambda\sigma^{-2}}\, N\left(\frac{\ln(H^2/Q_0K) + \lambda T}{\sigma\sqrt{T}}\right).$$

To find the expectation

$$I_1 \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}^*}\big(e^{X_T}\mathrm{I}_D\big) = \mathbf{E}_{\mathbf{P}^*}\Big(e^{X_T}\,\mathrm{I}_{\{X_T \ge \ln(K/Q_0),\, m_T \ge \ln(H/Q_0)\}}\Big),$$

we need to evaluate the double integral

$$\int\!\!\int_A e^x f(x,y)\,dx dy,$$

where $A = \{(x,y);\ x \ \ge\ \ln(K/Q_0),\ y \ \ge\ \ln(H/Q_0),\ y \ \le\ 0,\ y \ \le\ x\}$. Straightforward (but rather cumbersome) integration leads to the following result

$$I_1 = e^{(r_d - r_f)T}\,\Big(N\big(h_1(Q_0,T)\big) - (H/Q_0)^{2\lambda\sigma^{-2}+2}\, N\big(c_1(Q_0,T)\big)\Big),$$

where

$$h_{1,2}(q,t) = \frac{\ln(q/K) + (r_d - r_f \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

and

$$c_{1,2}(q,t) = \frac{\ln(H^2/qK) + (r_d - r_f \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}.$$

By collecting and rearranging the formulae above, we conclude that the price at initiation of the knock-out option admits the following representation (recall that we write $\sigma = \sigma_Q$)

$$C_0^1 = C_0^Q - J_0 = \textit{ Standard Call Price } - \textit{ Knockout Discount,} \tag{6.1}$$

where (cf. Proposition 4.2.2)

$$C_0^Q = Q_0 e^{-r_f T} N(h_1) - K e^{-r_d T} N(h_2)$$

and

$$J_0 = Q_0 e^{-r_f T} \, (H/Q_0)^{2\lambda\sigma^{-2}+2} \, N(c_1) - K e^{-r_d T} \, (H/Q_0)^{2\lambda\sigma^{-2}} \, N(c_2),$$

where $h_{1,2} = h_{1,2}(Q_0, T)$ and $c_{1,2} = c_{1,2}(Q_0, T)$. Notice that the proof of this formula can be substantially simplified by an application of Girsanov's theorem. We define an auxiliary probability measure $\bar{\mathbf{P}}$ by setting

$$\frac{d\bar{\mathbf{P}}}{d\mathbf{P}^*} = \exp\bigl(\sigma W_T^* - \tfrac{1}{2}\sigma^2 T\bigr) = \eta_T, \quad \mathbf{P}^*\text{-a.s.}$$

It follows from the Girsanov theorem that the process $\bar{W}_t = W_t^* - \sigma t$ follows a standard Brownian motion under the probability measure $\bar{\mathbf{P}}$. Moreover, taking into account the definition of X, we find that

$$\mathbf{E}_{\mathbf{P}^*}\bigl(e^{X_T}\, \mathrm{I}_D\bigr) = e^{(r_d - r_f)T}\, \mathbf{E}_{\mathbf{P}^*}\bigl(\eta_T\, \mathrm{I}_D\bigr)$$

and thus

$$I_1 = e^{(r_d - r_f)T}\, \bar{\mathbf{P}}\{D\} = e^{(r_d - r_f)T}\, \bar{\mathbf{P}}\bigl\{X_T \ge \ln(K/Q_0),\, m_T \ge \ln(H/Q_0)\bigr\}.$$

Finally, the semimartingale decomposition of the process X under $\bar{\mathbf{P}}$ is

$$X_t = \sigma \bar{W}_t + (r_d - r_f + \tfrac{1}{2}\sigma^2)t, \quad \forall\, t \in [0, T],$$

hence for every $y \le 0,\ y \le x,$ we have

$$\bar{\mathbf{P}}\{D\} = N\bigl(h_1(Q_0, T)\bigr) - (H/Q_0)^{2\lambda\sigma^{-2}+2}\, N\bigl(c_1(Q_0, T)\bigr).$$

Representation (6.1) of the option's price now follows easily.

6.7 Asian Options

An *Asian option* (or an *average option*) is a generic name for the class of options (of European or American style) whose terminal payoff is based on average asset values during some period within the options' lifetimes. Due to their averaging feature, Asian options are particularly suitable for thinly traded assets (or commodities). Actually, in contrast to standard options, Asian options are more robust with respect to manipulations near their expiry dates. Typically, they are also less expensive than standard options. Let T be the exercise date, and let $0 \le T_0 < T$ stand for the beginning date of the averaging period. Then the payoff at expiry of an Asian call option equals

$$C_T^A \stackrel{\text{def}}{=} \bigl(A_S(T_0, T) - K\bigr)^+, \tag{6.2}$$

where

$$A_S(T_0, T) = \frac{1}{T - T_0} \int_{T_0}^{T} S_u \, du \tag{6.3}$$

is the arithmetic average of the asset price over the time interval $[T_0, T],\ K$ is the fixed strike price, and the price S of the stock is assumed to follow a geometric Brownian motion. The main difficulty in pricing and hedging Asian options is due to the fact that the random variable $A_S(T_0, T)$ does not have a lognormal distribution. This feature makes the task of finding an explicit formula for the price of an Asian option surprisingly involved. For this reason, early studies of Asian options were based either on approximations or on the direct application of the Monte Carlo method. The numerical approach to the valuation of Asian options, proposed independently in Ruttiens (1990) and Vorst (1992), is based on the approximation of the arithmetic average using the geometric average. Note first that it is natural to substitute the continuous-time average $A_S(T_0, T)$ with its discrete-time counterpart

$$A_S^n(T_0, T) = \frac{1}{n} \sum_{i=0}^{n-1} S_{T_i}, \tag{6.4}$$

where $T_i = T_0 + i(T - T_0)/n$. Furthermore, the arithmetic average $A_S^n(T_0, T)$ can be replaced with the geometric average, denoted by $G_S^n(T_0, T)$ in what follows. Recall that the random variables S_{T_i}, $i = 1, \dots, n$, are given explicitly by the expression

$$S_{T_i} = S_{T_0} \exp\big(\sigma(W_{T_i}^* - W_{T_0}^*) + (r - \tfrac{1}{2}\sigma^2)(T_i - T_0)\big),$$

where W^* is a standard Brownian motion under the martingale measure $\mathbf{P}^*$. Therefore, the geometric average admits the following representation

$$G_S^n(T_0, T) = \Big(\prod_{i=0}^{n-1} S_{T_i}\Big)^{1/n} = c S_{T_0} \exp\Big(\frac{\sigma}{n} \sum_{i=0}^{n-1} (n - i - 1)(W_{T_{i+1}}^* - W_{T_i}^*)\Big)$$

for a strictly positive constant c. In view of the independence of increments of the Brownian motion, the last formula makes clear that the geometric average $G_S^n(T_0, T)$ has a lognormal distribution under $\mathbf{P}^*$. The approximate Black-Scholes-like formula for the price of an Asian call option can thus be easily found by the direct evaluation of the conditional expectation

$$\tilde{C}_{T_0}^n = e^{-r(T-T_0)}\, \mathbf{E}_{\mathbf{P}^*}\big((G_S^n(T_0, T) - K)^+ \,\big|\, \mathcal{F}_{T_0}\big).$$

It appears, however, that such an approach significantly underprices Asian call options. To overcome this deficiency, one may directly approximate the true distribution of the arithmetic average using an approximate distribution, typically a lognormal law with the appropriate parameters[1] (see Levy (1992), Turnbull and Wakeman (1991), and Bouaziz et al. (1994)). Another approach, initiated by Carverhill and Clewlow (1990), relies on the use of the fast Fourier transform to calculate the density of the sum of random variables as the convolution of individual densities. The second step in this method involves numerical integration of the option's payoff function with respect to this density function. Kemna and Vorst (1990) apply the Monte Carlo simulation with variance reduction to price Asian options. They replace $A_S(T_0, T)$ with the arithmetic average (6.4), so that the approximate value of an Asian call option is given by the formula

$$\bar{C}_{T_0}^n = e^{-r(T-T_0)}\, \mathbf{E}_{\mathbf{P}^*}\Big\{\Big(\frac{1}{n}\sum_{i=0}^{n-1} S_{T_i} - K\Big)^+ \,\Big|\, \mathcal{F}_{T_0}\Big\}.$$

Since the random variables S_{T_i}, $i = 1, \dots, n$, are given by an explicit formula, they can easily be generated using any standard procedure.

Let us first consider the special case of an Asian option which is already known to be in-the-money, i.e., assume that $t < T$ belongs to the averaging period, and the past values of stock price are such that

$$A_S(T_0, T) = \frac{1}{T - T_0} \int_{T_0}^{T} S_u\, du > \frac{1}{T - T_0} \int_{T_0}^{t} S_u\, du \geq K. \tag{6.5}$$

In this case, the value at time t of the Asian option equals

$$C_t^A = \frac{S_t(1 - e^{-r(T-t)})}{r(T - T_0)} - e^{-r(T-t)}\Big(K - \frac{1}{T - T_0} \int_{T_0}^{t} S_u\, du\Big). \tag{6.6}$$

[1] It should be noted that explicit formulae for all moments of the arithmetic average are available (see Geman and Yor (1993)).

Indeed, under (6.5), the price of the option satisfies

$$C_t^A = e^{-r(T-t)}\, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{1}{T-T_0} \int_{T_0}^T S_u\, du - K \,\Big|\, \mathcal{F}_t \Big),$$

or equivalently

$$C_t^A = \frac{e^{-r(T-t)}}{T-T_0}\, \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T S_u\, du \,\Big|\, \mathcal{F}_t \Big) + e^{-r(T-t)} \Big(\frac{1}{T-T_0} \int_{T_0}^t S_u\, du - K \Big).$$

Furthermore, we have

$$\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T S_u\, du \,\Big|\, \mathcal{F}_t \Big) = r^{-1} S_t (e^{r(T-t)} - 1),$$

since (recall that $S_t^* = e^{-rt} S_t$)

$$S_t(e^{r(T-t)} - 1) = e^{rT} S_T^* - e^{rt} S_t^* = \int_t^T d(e^{ru} S_u^*) = \int_t^T r S_u\, du + \int_t^T e^{ru}\, dS_u^*.$$

For an Asian option which is not known at time t to be in-the-money at time T, an explicit valuation formula is not available (see, however, Geman and Yor (1992, 1993) for quasi-explicit pricing formulas).

6.8 Basket Options

A *basket option,* as suggested by its name, is a kind of option contract which serves to hedge against the risk exposure of a basket of assets – that is, a prespecified portfolio of assets. Generally speaking, a basket option is more cost-effective than a portfolio of single options, as the latter over-hedges the exposure, and costs more than a basket option. An intuitive explanation for this feature is that the basket option takes into account the correlation between different risk factors. For instance, in the case of a strong negative correlation between two or more underlying assets, the total risk exposure may almost vanish, and this nice feature is not reflected in payoffs and prices of single options. Let us observe that from the analytical viewpoint, there is a close analogy between basket options and Asian options. Let us denote by S^i, $i = 1, \dots, k$ the price processes of k underlying assets, which will be referred to as stocks in what follows. In this case, it seems natural to refer to such a basket option as the *stock index option* (in market practice, options on a basket of currencies are also quite common). The payoff at expiry of a basket call option is defined in the following way

$$C_T^B \stackrel{\text{def}}{=} \Big(\sum_{i=1}^k w_i S_T^i - K \Big)^+ = (A_T - K)^+, \tag{6.7}$$

where $w_i \geq 0$ is the weight of the i^{th} asset, so that $\sum_{i=1}^k w_i = 1$. Note that by A_T we denote here the weighted arithmetic average

$$A_T = \sum_{i=1}^k w_i S_T^i.$$

We assume that each stock price S^i follows a geometric Brownian motion. More explicitly, under the martingale measure $\mathbf{P}^*$ we have

$$dS_t^i = S_t^i \big(r\, dt + \hat{\sigma}_i \cdot dW_t^* \big) \tag{6.8}$$

for some non-zero vectors $\hat{\sigma}_i \in \mathbf{R}^k$, where $W^* = (W^{1*}, \dots, W^{k*})$ stands for a k-dimensional Brownian motion under $\mathbf{P}^*$, and r is the risk-free interest rate. Observe that for any fixed i, we can find a standard one-dimensional Brownian motion $\tilde{W}^i$ such that

$$dS_t^i = S_t^i \big(r\, dt + \sigma_i\, d\tilde{W}_t^i \big) \tag{6.9}$$

and $\sigma_i = |\hat{\sigma}_i|$, where $|\hat{\sigma}_i|$ is the Euclidean norm of $\hat{\sigma}_i$. Let us denote by $\rho_{i,j}$ the instantaneous correlation coefficient

$$\rho_{i,j} = \frac{\hat{\sigma}_i \cdot \hat{\sigma}_j}{\sigma_i \sigma_j} = \frac{\hat{\sigma}_i \cdot \hat{\sigma}_j}{|\hat{\sigma}_i||\hat{\sigma}_j|}\,.$$

We may thus alternatively assume that the dynamics of price processes S^i are given by (6.9), where $\tilde{W}^i$, $i = 1, \ldots, k$ are one-dimensional Brownian motions, whose cross-variations satisfy

$$\langle \tilde{W}^i, \tilde{W}^j \rangle_t = \rho_{i,j}\, t, \quad \forall\, i, j = 1, \ldots, k.$$

Let us return to the problem of valuation of basket options. For similar reasons as those applying to Asian options, basket options are rather intractable analytically. Rubinstein (1991) developed a simple technique of pricing basket options on a bivariate binomial lattice, thus generalizing the standard Cox-Ross-Rubinstein methodology. Unfortunately, this numerical method is very time-consuming, especially where there are several underlying assets. To overcome this, Gentle (1993) proposed valuation of a basket using an approximation of the weighted arithmetic mean in the form of its geometric counterpart (this follows the approach of Ruttiens (1990) and Vorst (1992) to Asian options). For a fixed $t \le T$, let us denote by $\hat{w}_i$ the modified weights

$$\hat{w}_i = \frac{w_i S^i_t}{\sum_{j=1}^k w_j S^j_t} = \frac{w_i F_{S^i}(t,T)}{\sum_{j=1}^k w_j F_{S^j}(t,T)}, \tag{6.10}$$

where $F_{S^i}(t,T)$ is the forward price at time t of the i^{th} asset for the settlement date T. We may rewrite (6.7) as follows[2]

$$C^B_T = \Big(\sum_{j=1}^k w_j F_{S^j}(t,T)\Big)\Big(\sum_{i=1}^k \hat{w}_i \tilde{S}^i_T - \tilde{K}\Big)^+ = \Big(\sum_{j=1}^k w_j F_{S^j}(t,T)\Big)\big(\tilde{A}_T - \tilde{K}\big)^+,$$

where $\tilde{S}^i_T = S^i_T / F_{S^i}(t,T)$, $\tilde{A}_T = \sum_{i=1}^k \hat{w}_i \tilde{S}^i_T$, and

$$\tilde{K} = \frac{K}{\sum_{j=1}^k w_j F_{S^j}(t,T)} = \frac{e^{-r(T-t)}\, K}{\sum_{j=1}^k w_j S^j_t}\,. \tag{6.11}$$

The arbitrage price at time t of a basket call option thus equals

$$C^B_t = e^{-r(T-t)}\Big(\sum_{j=1}^k w_j F_{S^j}(t,T)\Big)\,\mathbf{E}_{\mathbf{P}^*}\Big(\big(\tilde{A}_T - \tilde{K}\big)^+ \,\Big|\, \mathcal{F}_t\Big),$$

or equivalently

$$C^B_t = \Big(\sum_{j=1}^k w_j S^j_t\Big)\,\mathbf{E}_{\mathbf{P}^*}\Big(\big(\tilde{A}_T - \tilde{K}\big)^+ \,\Big|\, \mathcal{F}_t\Big). \tag{6.12}$$

The next step relies on an approximation of the weighted arithmetic mean $\sum_{i=1}^k \hat{w}_i \tilde{S}^i_T$ using a similarly weighted geometric mean. More specifically, we approximate the price C^B_0 of the basket option using $\hat{C}^B_0$, which is given by the formula (for the sake of notational simplicity, we put $t = 0$ in what follows)

$$\hat{C}^B_0 = \Big(\sum_{j=1}^k w_j S^j_0\Big)\,\mathbf{E}_{\mathbf{P}^*}\big(\tilde{G}_T - \hat{K}\big)^+, \tag{6.13}$$

where $\tilde{G}_T = \prod_{i=1}^k (\tilde{S}^i_T)^{\hat{w}_i}$, and

$$\hat{K} = \tilde{K} + \mathbf{E}_{\mathbf{P}^*}(\tilde{G}_T - \tilde{A}_T). \tag{6.14}$$

[2] This representation is introduced because it appears to give a better approximation of the price of a basket option than formula (6.7).

In view of (6.8), we have (recall that $F_{S^i}(0,T) = e^{rT} S_0^i$)

$$\tilde{S}_T^i = S_T^i / F_{S^i}(0,T) = e^{\hat{\sigma}_i \cdot W_T^* - \sigma_i^2 T/2},$$

and thus the weighted geometric average $\tilde{G}_T$ equals

$$\tilde{G}_T = e^{c_1 \cdot W_T^* - c_2 T/2} = e^{\eta_T - c_2 T/2}, \tag{6.15}$$

with $\eta_T = c_1 \cdot W_T^*$, where $c_1 = \sum_{i=1}^k \hat{w}_i \hat{\sigma}_i$ and $c_2 = \sum_{i=1}^k \hat{w}_i \sigma_i^2$. We conclude that the random variable $\tilde{G}_T$ is lognormally distributed under $\mathbf{P}^*$. More precisely, the random variable η_T in (6.15) has Gaussian law with zero mean and the variance[3]

$$\begin{aligned} \mathrm{Var}\,(\eta_T) &= \mathbf{E}_{\mathbf{P}^*}\Big\{\Big(\sum_{i,l=1}^k \hat{w}_i \hat{\sigma}_{il} W_T^{l*}\Big)\Big(\sum_{j,m=1}^k \hat{w}_j \hat{\sigma}_{jm} W_T^{m*}\Big)\Big\} \\ &= \sum_{i,j,l,m=1}^k \hat{w}_i \hat{w}_j \, \hat{\sigma}_{il} \hat{\sigma}_{jm} \, \mathbf{E}_{\mathbf{P}^*}\big(W_T^{l*} W_T^{m*}\big) \\ &= \sum_{i,j=1}^k \hat{w}_i \hat{w}_j \, \hat{\sigma}_i \cdot \hat{\sigma}_j \, T = v^2 T, \end{aligned}$$

where $v^2 = \sum_{i,j=1}^k \rho_{i,j} \, \hat{w}_i \hat{w}_j \, \sigma_i \sigma_j$. Notice also that the last term on the right-hand side of (6.14) equals 1, since

$$\mathbf{E}_{\mathbf{P}^*}(\tilde{A}_T) = \sum_{j=1}^k \hat{w}_j S_0^j \, \mathbf{E}_{\mathbf{P}^*}\big(e^{-rT} S_T^i\big) = \sum_{j=1}^k \hat{w}_j = 1,$$

and the expected value $\mathbf{E}_{\mathbf{P}^*}(\tilde{G}_T)$ equals

$$\mathbf{E}_{\mathbf{P}^*}(\tilde{G}_T) = e^{(v^2 - c_2)T/2} \, \mathbf{E}_{\mathbf{P}^*}\big(e^{\eta_T - \frac{1}{2}\mathrm{Var}\,(\eta_T)}\big) = e^{(v^2 - c_2)T/2} \stackrel{\mathrm{def}}{=} c.$$

We conclude that $\hat{K} = \tilde{K} + c - 1$. The expectation in (6.13) can now be evaluated explicitly, using the following simple lemma (cf. Lemma 4.4.1).

Lemma 6.8.1 *Let ξ be a Gaussian random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ with zero mean and the variance $\sigma^2 > 0$. For any strictly positive real numbers a and b, we have*

$$\mathbf{E}_{\mathbf{P}}\big(a e^{\xi - \frac{1}{2}\sigma^2} - b\big)^+ = aN(h) - bN(h - \sigma), \tag{6.16}$$

where $h = \sigma^{-1} \ln(a/b) + \frac{1}{2}\sigma$.

We have

$$\mathbf{E}_{\mathbf{P}^*}\big(\tilde{G}_T - \hat{K}\big)^+ = \mathbf{E}_{\mathbf{P}^*}\Big(c e^{\eta_T - \frac{1}{2}\mathrm{Var}\,(\eta_T)} - (\tilde{K} + c - 1)\Big)^+,$$

so that $\xi = \eta_T$, $a = c$ and $b = \tilde{K} + c - 1$. In view of Lemma 6.8.1, the following result is straightforward.

Proposition 6.8.1 *The approximate value $\hat{C}_t^B$ of the price C_t^B of a basket call option with strike price K and expiry date T equals*

$$\hat{C}_t^B = \Big(\sum_{j=1}^k w_j S_t^j\Big)\Big(cN\big(l_1(T-t)\big) - (\tilde{K} + c - 1)N\big(l_2(T-t)\big)\Big), \tag{6.17}$$

[3] We use here, in particular, the equality $\mathbf{E}_{\mathbf{P}^*}\big(W_T^{l*} W_T^{m*}\big) = \delta_{lm} T$, where δ_{lm} stands for Kronecker's delta – that is, δ_{lm} equals 1 if $l = m$, and zero otherwise.

where

$$c = \exp\Big\{\Big(\frac{1}{2}\sum_{i,j=1}^{k}\rho_{i,j}\hat{w}_i\hat{w}_j\sigma_i\sigma_j - \sum_{j=1}^{k}\hat{w}_j\sigma_j^2\Big)(T-t)\Big\},$$

and where the modified weights $\hat{w}_i$ *are given by* (6.10)*,* $\tilde{K}$ *is given by* (6.11)*, and*

$$l_{1,2}(t) = \frac{\ln c - \ln(\tilde{K}+c-1) \pm \frac{1}{2}v^2 t}{v\sqrt{t}}.$$

6.9 Lookback Options

Lookback options are another example of path-dependent options – i.e., option contracts whose payoff at expiry depends not only on the terminal prices of the underlying assets, but also on asset price fluctuations during the options' lifetimes. We shall first examine the two following cases: that of a *standard lookback call option,* with payoff at expiry

$$\mathbf{LC}_T \stackrel{\text{def}}{=} (S_T - m_T^S)^+ = S_T - m_T^S, \tag{6.18}$$

where $m_T^S = \min_{t\in[0,T]} S_t$; and that of a *standard lookback put option,* whose terminal payoff equals

$$\mathbf{LP}_T \stackrel{\text{def}}{=} (M_T^S - S_T)^+ = M_T^S - S_T, \tag{6.19}$$

where $M_T^S = \max_{t\in[0,T]} S_t$. Note that a *lookback option* is not a genuine option contract since the (European) lookback option is always exercised by its holder at its expiry date. It is clear that the arbitrage prices of a lookback option are

$$\mathbf{LC}_t = e^{-r(T-t)}\,\mathbf{E}_{\mathbf{P}^*}(S_T\,|\,\mathcal{F}_t) - e^{-r(T-t)}\,\mathbf{E}_{\mathbf{P}^*}(m_T^S\,|\,\mathcal{F}_t) = I_1 - I_2$$

and

$$\mathbf{LP}_t = e^{-r(T-t)}\,\mathbf{E}_{\mathbf{P}^*}(M_T^S\,|\,\mathcal{F}_t) - e^{-r(T-t)}\,\mathbf{E}_{\mathbf{P}^*}(S_T\,|\,\mathcal{F}_t) = J_1 - J_2$$

for the lookback call and put, respectively. Standard lookback options were first studied by Goldman et al. (1979). So-called *limited risk* and *partial* lookback options (both of European and American style) were examined in Conze and Viswanathan (1991).

Proposition 6.9.1 *The price at time* $t\in[0,T]$ *of a European lookback call option equals*

$$\begin{aligned}\mathbf{LC}_t = {} & sN\left(\frac{\ln(s/m)+r_1\tau}{\sigma\sqrt{\tau}}\right) - me^{-r\tau}N\left(\frac{\ln(s/m)+r_2\tau}{\sigma\sqrt{\tau}}\right)\\ & - \frac{s\sigma^2}{2r}N\left(\frac{\ln(m/s)-r_1\tau}{\sigma\sqrt{\tau}}\right) + e^{-r\tau}\frac{s\sigma^2}{2r}\Big(\frac{m}{s}\Big)^{2r\sigma^{-2}}N\left(\frac{\ln(m/s)+r_2\tau}{\sigma\sqrt{\tau}}\right),\end{aligned}$$

where $s=S_t$*,* $m=m_t^S$*,* $\tau=T-t$*, and* $r_{1,2}=r\pm\frac{1}{2}\sigma^2$*. Equivalently,*

$$\begin{aligned}\mathbf{LC}_t &= sN(\tilde{d}) - me^{-r\tau}N(\tilde{d}-\sigma\sqrt{\tau}) - \frac{s\sigma^2}{2r}N(-\tilde{d})\\ &\quad + e^{-r\tau}\frac{s\sigma^2}{2r}\Big(\frac{m}{s}\Big)^{2r\sigma^{-2}}N(-\tilde{d}+2r\sigma^{-1}\sqrt{\tau}),\end{aligned}$$

where

$$\tilde{d} = \frac{\ln(s/m)+r_1\tau}{\sigma\sqrt{\tau}}.$$

In particular, if $s=S_t=m=m_t^S$*, then by setting* $d=r_1\sqrt{\tau}/\sigma$*, we get*

$$\mathbf{LC}_t = s\Big(N(d) - e^{-r\tau}N(d-\sigma\sqrt{\tau}) - \frac{\sigma^2}{2r}N(-d) + e^{-r\tau}\frac{\sigma^2}{2r}N(d-\sigma\sqrt{\tau})\Big).$$

The next result, which is also stated without proof, deals with the lookback put option (see Goldman et al. (1979)).

Proposition 6.9.2 *The price of a European lookback put option at time t equals*

$$\begin{aligned}\mathbf{LP}_t = -sN\left(-\frac{\ln(s/M)+r_1\tau}{\sigma\sqrt{\tau}}\right) + Me^{-r\tau}N\left(-\frac{\ln(s/M)+r_2\tau}{\sigma\sqrt{\tau}}\right)\\ + \frac{s\sigma^2}{2r}N\left(\frac{\ln(s/M)+r_1\tau}{\sigma\sqrt{\tau}}\right) - e^{-r\tau}\frac{s\sigma^2}{2r}\Big(\frac{M}{s}\Big)^{2r\sigma^{-2}} N\left(\frac{\ln(s/M)-r_2\tau}{\sigma\sqrt{\tau}}\right),\end{aligned}$$

where $s = S_t$, $M = M_t^S$, $\tau = T - t$, and $r_{1,2} = r \pm \frac{1}{2}\sigma^2$. Equivalently,

$$\begin{aligned}\mathbf{LP}_t &= -sN(-\hat{d}) + Me^{-r\tau}N(-\hat{d}+\sigma\sqrt{\tau}) + s\,\frac{\sigma^2}{2r}\,N(\hat{d})\\ &\quad - e^{-r\tau}\,\frac{s\sigma^2}{2r}\Big(\frac{M}{s}\Big)^{2r\sigma^{-2}}\,N(\hat{d}-2r\sigma^{-1}\sqrt{\tau}),\end{aligned}$$

where

$$\hat{d} = \frac{\ln(s/M) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

In particular, if $s = S_t = M = M_t^S$, then, denoting $d = r_1\sqrt{\tau}/\sigma$, we obtain

$$\mathbf{LP}_t = s\Big(-N(-d) + e^{-r\tau}N(-d+\sigma\sqrt{\tau}) + \frac{\sigma^2}{2r}\,N(d) - e^{-r\tau}\frac{\sigma^2}{2r}\,N(-d+\sigma\sqrt{\tau})\Big).$$

Notice that an American lookback call option is equivalent to its European counterpart. Indeed, the process $Z_t = e^{-rt}(S_t - m_t^S) = S_t^* - A_t$ is a submartingale, since the process A has non-increasing sample paths with probability 1. American and European lookback put options are not equivalent, however. The following bounds for the price $\mathbf{LP}_t^a$ of an American lookback put option can be established

$$\mathbf{LP}_t \le \mathbf{LP}_t^a \le e^{r\tau}\mathbf{LP}_t + S_t(e^{r\tau}-1).$$

Part II

Fixed-income Markets

Chapter 7

Interest Rates and Related Contracts

By a *fixed-income market* we mean that sector of the global financial market on which various interest rate-sensitive instruments, such as *bonds, swaps, swaptions, caps,* etc. are traded. In real-world practice, several fixed-income markets operate; as a result, many concepts of interest rates have been developed. There is no doubt that management of interest rate risk, by which we mean the control of changes in value of a stream of future cash flows resulting from changes in interest rates, or more specifically the pricing and hedging of interest rate products, is an important and complex issue. It creates a demand for mathematical models capable of covering all sorts of interest rate risks. Due to the somewhat peculiar way in which fixed-income securities and their derivatives are quoted in existing markets, theoretical term structure models are often easier to formulate and analyse in terms of interest rates which are different from the conventional market rates. In this chapter, we give an overview of various concepts of interest rates. We also describe the most important financial contracts related to interest rates, and markets at which they are traded. A more detailed description of real-world bond and swap markets can be found in Ray (1993) and Das (1994), respectively. Grabbe (1995) focuses on contracts related to international financial markets.

7.1 Zero-coupon Bonds

Let $T^* > 0$ be a fixed horizon date for all market activities. By a *zero-coupon bond* (a *discount bond*) of maturity T we mean a financial security paying to its holder one unit of cash at a prespecified date T in the future. This means that, by convention, the bond's *principal* (known also as *face value* or *nominal value*) is one dollar. We assume throughout that bonds are *default-free*; that is, the possibility of default by the bond's issuer is excluded. The price of a zero-coupon bond of maturity T at any instant $t \leq T$ will be denoted by $B(t,T)$; it is thus obvious that $B(T,T) = 1$ for any maturity date $T \leq T^*$. Since there are no other payments to the holder, in practice a discount bond sells for less than the principal before maturity – that is, at a discount (hence its name). This is because one could carry cash at virtually no cost, and thus would have no incentive to invest in a discount bond costing more than its face value. We shall usually assume that, for any fixed maturity $T \leq T^*$, the bond price $B(\cdot,T)$ follows a strictly positive and adapted process on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$.

7.1.1 Term Structure of Interest Rates

Let us consider a zero-coupon bond with a fixed maturity date $T \leq T^*$. The simple rate of return from holding the bond over the time interval $[t, T]$ equals

$$\frac{1 - B(t,T)}{B(t,T)} = \frac{1}{B(t,T)} - 1.$$

The equivalent rate of return, with continuous compounding, is commonly referred to as a (continuously compounded) *yield-to-maturity* on a bond. Formally, we have the following definition.

Definition 7.1.1 An adapted process $Y(t,T)$ defined by the formula

$$Y(t,T) = -\frac{1}{T-t} \ln B(t,T), \quad \forall\, t \in [0,T), \tag{7.1}$$

is called the *yield-to-maturity* on a zero-coupon bond maturing at time T.

The *term structure of interest rates,* known also as the *yield curve*, is the function that relates the yield $Y(t,T)$ to maturity T. It is obvious that, for arbitrary fixed maturity date T, there is a one-to-one correspondence between the bond price process $B(t,T)$ and its yield-to-maturity process $Y(t,T)$. Given the yield-to-maturity process $Y(t,T)$, the corresponding bond price process $B(t,T)$ is uniquely determined by the formula[1]

$$B(t,T) = e^{-Y(t,T)(T-t)}, \quad \forall\, t \in [0,T]. \tag{7.2}$$

The *discount function* relates the discount bond price $B(t,T)$ to maturity T. At the theoretical level, the initial term structure of interest rates may be represented either by the family of current bond prices $B(0,T)$, or by the initial yield curve $Y(0,T)$, as

$$B(0,T) = e^{-Y(0,T)T}, \quad \forall\, T \in [0,T^*]. \tag{7.3}$$

In practice, the term structure of interest rates is derived from the prices of several actively traded interest rate instruments, such as Treasury bills, Treasury bonds, swaps and futures. Note that the yield curve at any given day is determined exclusively by market prices quoted on that day. The shape of an historically observed yield curve varies over time; the observed yield curve may be upward sloping, flat, descending, or humped. There is also strong empirical evidence that the movements of yields of different maturities are not perfectly correlated. Also, the short-term interest rates fluctuate more than long-term rates; this may be partially explained by the typical shape of the term structure of yield volatilities, which is downward sloping. These features mean that the construction of a reliable model for stochastic behavior of the term structure of interest rates is a task of considerable complexity.

7.1.2 Forward Interest Rates

Let $f(t,T)$ be the forward interest rate at date $t \leq T$ for instantaneous risk-free borrowing or lending at date T. Intuitively, $f(t,T)$ should be interpreted as the interest rate over the infinitesimal time interval $[T, T+dT]$ as seen from time t. As such, $f(t,T)$ will be referred to as the *instantaneous, continuously compounded forward rate,* or shortly, *instantaneous forward rate.* In contrast to bond prices, the concept of an instantaneous forward rate is a mathematical idealization rather than a quantity observable in practice. As we shall see in what follows, however, one of the popular approaches to the bond price modelling is based on the exogenous specification of a family $f(t,T)$, $t \leq T \leq T^*$, of instantaneous forward interest rates. Given such a family $f(t,T)$, the bond prices are then defined by setting

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\, du\Big), \quad \forall\, t \in [0,T]. \tag{7.4}$$

[1] We assume that the limit of $Y(t,T)$, as t tends to T, exists.

On the other hand, if the family of bond prices $B(t,T)$ is sufficiently smooth with respect to maturity T, the implied instantaneous forward interest rate $f(t,T)$ is given by the formula

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} \tag{7.5}$$

which, indeed, can be seen as the formal definition of the instantaneous forward rate $f(t,T)$.

Alternatively, the instantaneous forward rate can be seen as a limit case of a *forward rate* $f(t,T,U)$ which prevails at time t for riskless borrowing or lending over the future time interval $[T,U]$. The rate $f(t,T,U)$ is in turn tied to the zero-coupon bond price by means of the formula

$$\frac{B(t,U)}{B(t,T)} = e^{-f(t,T,U)(U-T)}, \quad \forall\, t \le T \le U, \tag{7.6}$$

or equivalently

$$f(t,T,U) = \frac{\ln B(t,T) - \ln B(t,U)}{U-T}. \tag{7.7}$$

Observe that $Y(t,T) = f(t,t,T)$, as expected – indeed, investing at time t in T-maturity bonds is clearly equivalent to lending money over the time interval $[t,T]$. On the other hand, under suitable technical assumptions, the convergence $f(t,T) = \lim_{U\downarrow T} f(t,T,U)$ holds for every $t \le T$. For convenience, we focus on interest rates that are subject to continuous compounding. In practice, interest rates are quoted on an *actuarial basis,* rather than as continuously compounded rates. For instance, the *actuarial rate* (or *effective rate*) $a(t,T)$ at time t for maturity T (i.e., over the time interval $[t,T]$) is given by the following relationship

$$(1 + a(t,T))^{T-t} = e^{f(t,t,T)(T-t)} = e^{Y(t,T)(T-t)}, \quad \forall\, t \le T.$$

This means, of course, that the bond price $B(t,T)$ equals

$$B(t,T) = \frac{1}{(1 + a(t,T))^{T-t}}, \quad \forall\, t \le T.$$

Similarly, the *forward actuarial rate* $a(t,T,U)$ prevailing at time t over the future time period $[T,U]$ is set to satisfy

$$(1 + a(t,T,U))^{U-T} = \exp\big(f(t,T,U)(U-T)\big) = B(t,T)/B(t,U).$$

7.1.3 Short-term Interest Rate

Most traditional stochastic interest rate models are based on the exogenous specification of a short-term rate of interest. We write r_t to denote the *instantaneous interest rate* (also referred to as a *short-term interest rate,* or *spot interest rate*) for risk-free borrowing or lending prevailing at time t over the infinitesimal time interval $[t, t+dt]$. In a stochastic setup, the short-term interest rate is modelled as an adapted process, say r, defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$ for some $T^* > 0$. We assume throughout that r is a stochastic process with almost all sample paths integrable on $[0,T^*]$ with respect to the Lebesgue measure. We may then introduce an adapted process B of finite variation and with continuous sample paths, given by the formula

$$B_t = \exp\Big(\int_0^t r_u\, du\Big), \quad \forall\, t \in [0,T^*]. \tag{7.8}$$

Equivalently, for almost all $\omega \in \Omega$, the function $B_t = B_t(\omega)$ solves the differential equation $dB_t = r_t B_t dt$, with the conventional initial condition $B_0 = 1$. In financial interpretation, B represents the price process of a risk-free security which continuously compounds in value at the rate r. The process B is referred to as an *accumulation factor* or a *savings account* in what follows. Intuitively, B_t represents the amount of cash accumulated up to time t by starting with one unit of cash at time 0, and continually rolling over a bond with infinitesimal time to maturity.

7.2 Coupon-bearing Bonds

A *coupon-bearing bond* is a financial security which pays to its holder the amounts $c_1, \dots, c_m$ at the dates $T_1, \dots, T_m$. Unless explicitly stated otherwise, we assume that the time variable is expressed in years. Obviously the bond price, say $B_c(t)$, at time t can be expressed as a sum of the cash flows $c_1, \dots, c_m$ discounted to time t, namely

$$B_c(t) = \sum_{j=1}^{m} c_j B(t, T_j). \tag{7.9}$$

A real bond typically pays a fixed coupon c and repays the principal N. Therefore, we have $c_j = c$ for $j = 1, \dots, m-1$ and $c_m = c+N$. The main difficulty in dealing with bond portfolios is due to the fact that most bonds involved are coupon-bearing bonds, rather than zero-coupon bonds. Although the coupon payments and the relevant dates are preassigned in a bond contract, the future cash flows from holding a bond are reinvested at rates that are not known in advance. Therefore, the total return on a coupon-bearing bond which is held to maturity (or for a lesser period of time) appears to be uncertain. As a result, bonds with different coupons and cash flow dates may not be easy to compare. The standard way to circumvent this difficulty is to extend the notion of a yield-to-maturity to coupon-bearing bonds.

Consider a bond which pays m identical yearly coupons c at the dates $1, \dots, m$, and the principal N after m years. Its yield-to-maturity at time 0, denoted by $\tilde{Y}_c(0)$, may be found from the following relationship

$$B_c(0) = \sum_{j=1}^{m} \frac{c}{(1+\tilde{Y}_c(0))^j} + \frac{N}{(1+\tilde{Y}_c(0))^m} \, . \tag{7.10}$$

Since the coupon payments are usually determined by a preassigned interest rate $r_c > 0$ (known as a *coupon rate*), this may be rewritten as follows

$$B_c(0) = \sum_{j=1}^{m} \frac{r_c N}{(1+\tilde{Y}_c(0))^j} + \frac{N}{(1+\tilde{Y}_c(0))^m} \, . \tag{7.11}$$

It is clear that in this case the yield does not depend on the face value of the bond. Notice that the price $B_c(0)$ equals the bond's face value N whenever $r_c = \tilde{Y}_c(0)$; in this case a bond is said to be priced at *par*. Similarly, we say that a bond is priced below par (i.e., *at a discount*) when its current price is lower than its face value: $B_c(0) < N$, or equivalently, when its yield-to-maturity exceeds the coupon rate: $\tilde{Y}_c(0) > r_c$. Finally, a bond is priced above par (i.e., *at a premium*) when $B_c(0) > N$, that is, when $\tilde{Y}_c(0) < r_c$. In the case of continuous compounding, the corresponding yield-to-maturity $Y_c(0)$ satisfies

$$B_c(0) = \sum_{j=1}^{m} c e^{-jY_c(0)} + N e^{-mY_c(0)}, \tag{7.12}$$

where $B_c(0)$ stands for the current market price of the bond. Let us now focus on zero-coupon bonds (i.e., $c = 0$ and $N = 1$). The initial price $B(0, m)$ of a zero-coupon bond can easily be found provided its yield-to-maturity $\tilde{Y}(0, m)$ is known. Indeed, we have

$$B(0, m) = \frac{1}{(1+\tilde{Y}(0, m))^m}.$$

Similarly, in a continuous-time framework, we have $B(0, T) = e^{-Y(0,T)T}$, where $B(0, T)$ is the initial price of a unit zero-coupon bond of maturity T, and $Y(0, T)$ stands for its yield-to-maturity.

Definition 7.2.1 The discretely compounded *yield-to-maturity* $\tilde{Y}_c(i)$ at time i on a coupon-bearing bond which pays the positive deterministic cash flows $c_1, \ldots, c_m$ at the dates $1 < \ldots < m \leq T^*$ is given implicitly by means of the formula

$$B_c(i) = \sum_{j=i+1}^{m} \frac{c_j}{(1 + \tilde{Y}_c(i))^{j-i}}, \tag{7.13}$$

where $B_c(i)$ stands for the price of a bond at the date $i < m$.

In a continuous-time framework, we have the following definition of the bond's yield-to-maturity.

Definition 7.2.2 If a bond pays the positive cash flows $c_1, \ldots, c_m$ at the dates $T_1 < \ldots < T_m \leq T^*$, then its continuously compounded *yield-to-maturity* $Y_c(t) = Y_c(t; c_1, \ldots, c_m, T_1, \ldots, T_m)$ is uniquely determined by the following relationship

$$B_c(t) = \sum_{T_j > t} c_j e^{-Y_c(t)(T_j - t)}, \tag{7.14}$$

where $B_c(t)$ denotes the bond price at time $t < T_m$.

Note that on the right-hand side of (7.13) ((7.14), respectively), the coupon payment at time i (at time t, respectively) is not taken into account. Consequently, the price $B_c(i)$ ($B_c(t)$, respectively) is the price of a bond after the coupon at time i (at time t, respectively) has been paid. We focus mainly on the continuously compounded yield-to-maturity $Y_c(t)$. It is common to interpret the yield-to-maturity $Y_c(t)$ as a proxy for the uncertain return on a bond; this means that it is implicitly assumed that all coupon payments occurring after the date t are reinvested at the rate $Y_c(t)$. Since this cannot, of course, be guaranteed, the yield-to-maturity should be seen as a very rough approximation of the uncertain return on a coupon-bearing bond. On the other hand, the return on a discount bond is certain, therefore the yield-to-maturity determines exactly the return on a discount bond. It is worthwhile to note that for every t, an $\mathcal{F}_t$-adapted random variable $Y_c(t)$ is uniquely determined for any given collection of positive cash flows $c_1, \ldots, c_m$ and dates $T_1, \ldots, T_m$, provided that the bond price at time t is known.

Let us conclude this section by observing that the bond price moves inversely to the bond's yield-to-maturity. Moreover, it can also be checked that the moves are asymmetric, so that a decrease in yields raises bond prices more than the same increase lowers bond prices. This specific feature of the bond price is referred to as *convexity*. Finally, it should be stressed that the uncertain return on a bond comes from both the interest paid and from the potential capital gains (or losses) caused by the future fluctuations of the bond price. Therefore, the term *fixed-income security* should not be taken literally, unless we consider a bond which is held to its maturity.

7.3 Interest Rate Futures

Until the introduction of financial futures, the futures market consisted only of contracts for delivery of commodities. In 1975, the Chicago Board of Trade (CBT) created the first financial futures contract, a futures contract for so-called *mortgage-backed securities.* Mortgage-backed securities are bonds collaterized with a pool of government-guaranteed home mortgages. Since these securities are issued by the Government National Mortgage Association (GNMA), the corresponding futures contract is commonly referred to as *Ginnie Mae futures.*

7.3.1 Treasury Bond Futures

Treasury bond futures contracts were introduced on the Chicago Board of Trade in 1977. Nominally, the underlying instrument of a T-bond futures contract is a 15-year T-bond with an 8% coupon.

T-bond futures contracts and T-bond futures options trade with up to one year to maturity. As usual, the futures contract specifies precisely the time and other relevant conditions of delivery. Delivery is made on any business day of the delivery month, two days after the *delivery notice* (i.e., the declaration of intention to make delivery) is passed to the exchange. The *invoice price* received by the party with a short position equals the bond futures settlement price multiplied by the *delivery factor* for the bond to be delivered, plus the accrued interest. The delivery factor, calculated for each deliverable bond issue, is based on the coupon rate and the time to the bond's expiry date. Basically, it equals the price of a unit bond with the same coupon rate and maturity, assuming that the yield-to-maturity of the bond equals 8%. For instance, for a bond with m years to maturity and coupon rate r_c, the *conversion factor* δ equals

$$\delta = \sum_{j=1}^{2m} \frac{r_c/2}{(1+0.04)^j} + \frac{1}{(1+0.04)^{2m}}\,, \tag{7.15}$$

so that $\delta > 1$ ($\delta < 1$, respectively) whenever $r_c > 0.08$ ($r_c < 0.08$, respectively). Note that the adjustment factor δ makes the yields of each deliverable bond roughly equal for a party paying the invoice price. In particular, if the settlement futures price[2] is close to 100, this yield is approximately 8%. At any given time, there are about 30 bonds that can be delivered in the T-bond futures contract (basically, any bond with at least 15 years to maturity). The *cheapest-to-deliver* bond is that deliverable issue for which the difference

$$\text{Quoted bond price} \;-\; \text{Settlement futures price} \;\times\; \text{Conversion factor}$$

is least. Put another way, the cheapest-to-deliver bond is the one for which the *basis* $b_t^i = B_c^i(t) - f_t\delta^i$ is minimal, where $B_c^i(t)$ is the current price of the i^{th} deliverable bond, f_t is the bond futures settlement price, and δ^i is the conversion factor of the i^{th} bond. Usually, the market is able to forecast the cheapest-to-deliver bond for a given delivery month. A change in the shape of the yield curve or a change in the level of yields often means a switch in cheapest-to-deliver bond, however. This is because, as yields change, a security with a slightly different coupon or maturity may become cheaper for market makers to deliver. Before the delivery month, the calculation of the cheapest-to-deliver issue also involves the *cost of carry* (net financing cost) of a given bond; the top delivery choice is the issue with the lowest after-carry basis. Due to the change of yield level (or new bond issue) as time passes, the top delivery choice also changes. The possibility of such an event, which may be seen as an additional source of risk, makes the valuation of futures contracts and their use for hedging purposes more involved.

7.3.2 Bond Options

Currently traded bond-related options split into two categories: OTC bond options and T-bond futures options. The market for the first class of bond options is made by primary dealers and some active trading firms. The long (i.e., 30-year) bond is the most popular underlying instrument of OTC bond options; however, options on shorter-term issues are also available to customers. Since a large number of different types of OTC bond options exists in the market, the market is rather illiquid. Most options are written with one month or less to expiry. They usually trade at-the-money. This convention simplifies quotation of bond option prices. Options with exercise prices that are up to two points out-of-the-money are also common. Bond options are used by traders to immunize their positions from the direction of future price changes. For instance, if a dealer buys call options from a client, he usually sells cash bonds in the open market at the same time.

Like all typical exchange-traded options, T-bond futures options have fixed strike prices and expiry dates. Strike prices come in two-point increments. The options are written on the first four delivery months of a futures contract (note that the delivery of the T-bond futures contract occurs only every three months). In addition, a 1-month option is traded (unless the next month is the

[2] It is customary to quote both the bond price and the bond futures price for a $100 face value bond.

delivery month of the futures contract). The options stop trading a few days before the corresponding delivery month of the underlying futures contract. The T-bond futures option market is highly liquid. An open interest in one option contract may amount to \$5 billion in face value (this corresponds to 50,000 option contracts). For a detailed description of the T-bond market, we refer to Ray (1993).

7.3.3 Treasury Bill Futures

The Treasury bill (T-bill, for short) is a bill of exchange issued by the U.S. Treasury to raise money for temporary needs. It pays no coupons, and the investor receives the face value at maturity. T-bills are issued on a regular schedule in 3-month, 6-month and 1-year maturities. In the T-bill futures contract, the underlying asset is a 90-day T-bill. The common market practice is to quote a discount bond, such as the T-bill, not in terms of the yield-to-maturity, but rather in terms of so-called *discount rates.* The discount rate represents the size of the price reduction for a 360-day period (for instance, a bill of face value 100 which matures in 90 days and is sold at a discount rate 10% is priced at 97.5). Formally, a *discount rate* $R_b(t,T)$ (known also as *bankers' discount yield*) of a security which pays a deterministic cash flow X_T at the future date $T,$ and has the price X_t at time $t<T,$ equals

$$R_b(t,T)=\frac{X_T-X_t}{X_T}\,\frac{360}{T-t}, \tag{7.16}$$

where $T-t$ is now expressed in days. In particular, for a discount bond this gives

$$R_b(t,T)=\frac{100-P(t,T)}{100}\,\frac{360}{T-t}=(1-B(t,T))\,\frac{360}{T-t},$$

where $P(t,T)$ ($B(t,T)$, respectively) stands for the cash price of a bill with face value 100 (with the unit face value, respectively) and $T-t$ days to maturity. Conversely, given a discount rate $R_b(t,T)$ of a bill, we find its cash price from the following formula

$$P(t,T)=100\Big(1-R_b(t,T)\frac{T-t}{360}\Big).$$

For a just-issued 90-day T-bill, the above formulae can be further simplified. Indeed, we have

$$R_b(0,90)=4(1-P(0,90)/100),\quad P(0,90)=100\big(1-\tfrac{1}{4}R_b(0,90)\big).$$

The *bill yield* on a T-bill equals

$$Y_b(t,T)=\frac{100-P(t,T)}{P(t,T)}\,\frac{360}{T-t}=\frac{1-B(t,T)}{B(t,T)}\,\frac{360}{T-t}=\frac{R_b(t,T)}{B(t,T)},$$

so that it represents the annualized (with no compounding) interest rate earned by the bill owner. In terms of a bill yield $Y_b(t,T),$ its price $B(t,T)$ equals

$$B(t,T)=\frac{1}{1+Y_b(t,T)(T-t)/360}\,.$$

Let us now examine market conventions associated with T-bill futures contracts. In the T-bill futures contract, the underlying asset is the 90-day Treasury bill. In contrast to T-bills, which are quoted in terms of the discount rate, T-bill futures are quoted in terms of the price for a 100 face value bill. In particular, the T-bill futures quoted price at maturity equals 100 minus the T-bill quote (in percentage terms). The marking to market procedure is based, however, on the corresponding cash price of a given futures contract, which is calculated from the formula

$$P_t^b=100-\tfrac{1}{4}(100-\tilde f_t^b)=100-\tfrac{1}{4}\tilde R_t^f,$$

where $\tilde f_t^b$ is the market quotation of T-bill futures at time $t,$ and $\tilde R_t^f=\tilde R_t^f(T,T+90)$ represents the futures discount rate over the future time interval $[T,T+90]$ implied by T-bill futures contract (note that $\tilde{R}^f_t$ is expressed here in percentage terms). For instance, if the quotation for T-bill futures is $\tilde{f}^b_t = 95.02$, the implied futures discount rate equals $\tilde{R}^f_t = 4.98\%$ and thus the corresponding cash futures price, which is used in marking to the market, equals

$$P^b_t = 100 - \tfrac{1}{4}(100 - 95.02) = 100 - \tfrac{1}{4}4.98 = 98.755$$

per \$100 face value bill, or equivalently, \$987,550 per futures contract (the nominal size of one T-bill futures contract which trades on the CME is \$1 million).

7.3.4 Eurodollar Futures

Since conventions associated with market quotations of the LIBOR rate and Eurodollar futures contracts are close to those examined above, we shall describe them in a rather succinct way. Eurodollar futures and futures options have traded on the CME since 1981 and 1985, respectively.[3] Eurodollar futures and related Eurodollar futures options trade with up to five years to maturity. A Eurodollar futures option is of American style; one option covers one futures contract and it expires at the settlement date of the underlying Eurodollar futures contract. Formally, the underlying instrument of a Eurodollar futures contract is the 3-month LIBOR rate. At the settlement date of a Eurodollar futures contract, the CME surveys 12 randomly selected London banks, which are asked to give their perception of the rate at which 3-month Eurodollar deposit funds are currently offered by the market to prime banks. A suitably rounded average of these quotes serves to calculate the Eurodollar futures price at settlement (cf. Amin and Morton (1994)). Let us stress that the LIBOR is defined as the *add-on yield*; that is, the actual interest payment on a 3-month Eurodollar time deposit equals "LIBOR $\times$ numbers of days for investment/360" per unit investment. In our framework, the (spot) LIBOR at time t on a Eurodollar deposit with a maturity of τ days is formally defined as

$$l(t, t+\tau) = \frac{360}{\tau}\Big(\frac{1}{B(t, t+\tau)} - 1\Big),$$

or equivalently

$$B(t, t+\tau) = \frac{1}{1 + l(t, t+\tau)\tau/360}.$$

In particular, a 3-month LIBOR equals

$$l(t, t+90) = 4\,(B^{-1}(t, t+90) - 1), \quad \tilde{l}(t, t+90) = 400\,(B^{-1}(t, t+90) - 1),$$

where $\tilde{l}(t, t+90)$ represents a 3-month LIBOR expressed in percentage terms. Eurodollar futures contract is always settled in cash (no physical delivery is possible). A Eurodollar futures price f^l_T on the settlement day T is tied to the current level of a 3-month LIBOR at time T (and thus to the price of a zero-coupon bond) through the conventional formula

$$f^l_T = \big(1 - l(T, T+90)\big) = \Big(1 - 4\,\big(B^{-1}(T, T+90) - 1\big)\Big).$$

More specifically, the quoted price $\tilde{f}^l_T$ is given for 100\$ of a nominal value, so that at maturity date T we have

$$\tilde{f}^l_T = 100\big(1 - l(T, T+90)\big) = 100 - \tilde{l}(T, T+90) = 100\,\Big(1 - 4\,\big(B^{-1}(T, T+90) - 1\big)\Big).$$

Therefore, a 3-month futures LIBOR $\tilde{l}^f_t = 100 - \tilde{f}^l_t$ implicit in a Eurodollar futures contract converges to a 3-month LIBOR (expressed in percentage terms) as the time argument t tends to the settlement date T. Let us now focus on the valuation of a Eurodollar futures contract. A market quotation of Eurodollar futures contracts is based on the same rule as the quotation of T-bill futures. Explicitly,

[3] Eurodollar futures trade also on the LIFFE (since 1982) and SIMEX (since 1984).

if $\tilde{f}_t^l$ stands for the market quotation of Eurodollar futures, then the value of a contract (which is used in marking to the market) equals

$$P_t^l = 100 - \tfrac{1}{4}\big(100 - \tilde{f}_t^l\big) = 100 - \tfrac{1}{4}\tilde{l}_t^f$$

per \$100 of nominal value. The nominal value of one Eurodollar futures contract is \$1 million; one basis point is thus worth \$25 when the contract is marked to the market daily. For instance, the quoted Eurodollar futures price $\tilde{f}_t^l = 94.47$ corresponds to a 3-month LIBOR futures rate of $\tilde{l}_t^f = 5.53\%$, and to the price \$986,175 of one Eurodollar futures contract. If the next day the quoted price rises to 94.48 (i.e., the 3-month LIBOR futures rate declines to 5.52%), the value of one contract appreciates by \$25 to \$986,200. We end this section by a short description of market conventions related to Eurodollar futures options. The owner of a Eurodollar futures call option obtains a long position in the futures contract with a futures price equal to the option's exercise price; the call writer obtains a short futures position. On marking to market, the call owner receives the cash difference between the marked-to-market futures price and the exercise price.

7.4 Interest Rate Swaps

Generally speaking, a *swap* is an agreement between two parties to exchange cash flows at some future dates according to a prearranged formula. In a classic swap contract, the value of the swap at the time it is entered into, as well as at the end of its life, is zero. In a *plain vanilla interest rate swap,* one party, say A, agrees to pay to the other party, say B, amounts determined by a fixed interest rate on a *notional principal* at each of the payment dates. At the same time, the party B agrees to pay to the party A interest at a floating reference rate on the same notional principal for the same period of time. Thus an interest rate swap can be used to transform a floating-rate loan into a fixed-rate loan or vice versa. In essence, a swap is a long position in a fixed-rate coupon bond combined with short positions in floating-rate notes (alternatively, it can be seen as a portfolio of specific forward contracts). In a *payer swap,* the fixed rate is paid at the end (or, depending on contractual features of the swap, at the beginning) of each period, and the floating rate is received (therefore, it may also be termed a *fixed-for-floating swap*). Similarly, a *receiver swap* (or a *floating-for-fixed swap*) is one in which an investor pays a floating rate and receives a fixed rate on the same notional principal. In a payer swap settled *in arrears,* the floating rate paid at the end of a period is set at the beginning of this period. We say that a swap is settled *in advance* if payments are made at the beginning of each period. Notice that payments of a swap which settles in advance are the payments, discounted to the beginning of each period, of the corresponding swap settled in arrears. However, the discounting conventions vary from country to country. In some cases, both sides of a swap are discounted using the same floating rate; in others, the floating is discounted using the floating and the fixed using the fixed. Let us consider an arbitrary collection $T_0 = T < T_1 < \ldots < T_n$ of future dates. Formally, a *forward start swap* (or briefly, a *forward swap*) is a swap agreement entered at the trade date $t \le T_0$ with payment dates $T_1 < \ldots < T_n$ (if a swap is settled in arrears) or $T_0 < \ldots < T_{n-1}$ (if a swap is settled in advance). For most swaps, a fee (the up-front cost) is negotiated between the counter-parties at the trade date t. The *forward swap rate* is that value of the fixed rate which makes the value of the forward swap zero. The market gives quotes on *swap rates,* i.e., the fixed rates at which financial institutions offer to their clients interest rate swap contracts of differing maturities, with fixed quarterly, semi-annual or annual payment schedules. The most typical option contract associated with swaps is a *swaption* – that is, an option on the value of the underlying swap or, equivalently, on the *swap rate.*

Let us comment briefly on a more conventional class of contracts, widely used by companies to hedge the interest rate risk. Consider a company which forecasts that it will need to borrow cash at a future date, say T, for the period $[T, U]$. The company will be, of course, unhappy if the interest rate rises by the time the loan is required. A commonly used contract, which serves to reduce interest rate risk exposure by locking into a rate of interest, is a *forward rate agreement.* A forward rate agreement (an FRA) is a contract in which two parties (a seller of a contract and a buyer) agree to exchange, at some future date, interest payments on the notional principal of a contract. It will be convenient to assume that this payment is made at the end of the period, say at time U. The cash flow is determined by the length of the time-period, say $[T, U]$, and by two relevant interest rates: the prespecified rate of interest and the risk-free rate of interest prevailing at time T. The buyer of an FRA thus pays interest at a preassigned rate and receives interest at a floating reference rate which prevails at time T. Note that an FRA may be seen as an example of a forward contract, the contract's underlier being an uncertain future cash flow (interests to be paid at time U). A typical use of a forward rate agreement is a long position in an FRA combined with a loan taken at time T over the period $[T, U]$. A synthetic version of such a strategy is a *forward-forward loan* – that is, a combination of a longer-term loan and a shorter-term deposit (a company just takes a loan over $[0, U]$ and makes a deposit over $[0, T]$). Assuming a frictionless market, the rate of interest a company manages to lock into on its loan using the above strategy will coincide with the prespecified rate of interest in forward rate agreements proposed to customers by financial institutions at no additional charge. Indeed, instead of manufacturing a forward-forward loan, a company may alternatively buy (at no charge) a forward rate agreement and take at time T a loan on the spot market (both contracts should refer to the same notional principal).

We shall examine first a forward rate agreement written at time 0 with the reference period $[T, U]$. We may and do assume, without loss of generality, that the notional principal of the contract is 1. Let us first introduce some notation. We denote by $r(0, T)$ the continuously compounded interest rate for risk-free borrowing and lending over the time-period $[0, T]$. It is clear that, barring arbitrage opportunities between bank deposits and the zero-coupon bond market, the T-maturity spot rate $r(0, T)$ should satisfy $e^{r(0,T)T} = B^{-1}(0, T)$. In other words, the interest rate $r(0, T)$ coincides with the continuously compounded yield on a T-maturity discount bond – that is, $r(0, T) = Y(0, T)$ for every T. As mentioned earlier, the buyer of an FRA receives at time U a cash flow corresponding to an interest rate set at time T, and pays interest according to a rate preassigned at time 0. The level of the prespecified rate, loosely termed a *forward interest rate,* is chosen in such a way that the contract is worthless at the date it is entered into.

Let us denote by $f(0, T, U)$ this level of interest rate, corresponding to an FRA written at time 0 and referring to the period $[T, U]$. The forward rate $f(0, T, U)$ may alternatively be seen as a continuously compounded interest rate, prevailing at time 0, for risk-free borrowing or lending over the time period $[T, U]$. It is not difficult to determine the "right" level of the forward rate $f(0, T, U)$ by standard no-arbitrage arguments. By considering two alternative trading strategies, it is easy to establish the following relationship

$$e^{Ur(0,U)} = e^{Tr(0,T)}\, e^{f(0,T,U)(U-T)}.$$

More generally, the forward rate $f(t, T, U)$ satisfies

$$f(t, T, U) = \frac{Ur(t, U) - Tr(t, T)}{U - T}$$

for every $t \leq T \leq U$, where $r(t, T)$ is the future spot rate, as from time t, for risk-free borrowing or lending over the time period $[t, T]$. Note that $f(0, T, T) = f(0, T)$, i.e., the rate $f(0, T, T)$ (if well-defined) coincides with the instantaneous forward interest rate $f(0, T)$. For similar reasons, the equality $f(t, T, T) = f(t, T)$ is valid. If $r(t, T) = Y(t, T)$, then in terms of bond prices we have (cf. (7.7))

$$f(t, T, U) = \frac{\ln B(t, T) - \ln B(t, U)}{U - T}.$$

Chapter 8

Models of the Short-term Rate

The aim of this chapter is to survey the most popular models of the short-term interest rate. For convenience, we shall work throughout within a continuous-time framework; a detailed presentation of a discrete-time approach to term structure modelling is done in Jarrow (1996). We start this chapter by addressing the existence and uniqueness of an arbitrage-free family of bond prices related to a given short-term rate process. To obtain more explicit results, we then assume that the short-term interest rate is modelled either as an Itô process or, even more specifically, as a one-dimensional diffusion process.

8.1 Arbitrage-free Family of Bond Prices

Recall that, by convention, a zero-coupon bond pays one unit of cash at a prescribed date T in the future. The price at any instant $t \le T$ of a zero-coupon bond of maturity T is denoted by $B(t,T)$; it is thus clear that, necessarily, $B(T,T) = 1$ for any maturity date $T \le T^*$. Furthermore, since there are no intervening interest payments, in market practice the bond sells for less than the principal before the maturity date. Essentially, this follows from the fact that it is possible to invest money in a risk-free savings account yielding a non-negative interest rate (or at least to carry cash at virtually no cost). We assume throughout that for any maturity $T \le T^*$, the price process $B(t,T)$, $t \in [0,T]$, follows a strictly positive and adapted process on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$, where the filtration $\mathbf{F}$ is generated by the underlying d-dimensional standard Brownian motion W. Suppose that an adapted process r, given on $(\Omega, \mathbf{F}, \mathbf{P})$, models the short-term interest rate, meaning that the savings account process B satisfies (7.8).

Definition 8.1.1 A family $B(t,T)$, $t \le T \le T^*$, of adapted processes is called an *arbitrage-free family of bond prices relative to* r if: (a) $B(T,T) = 1$ for every $T \in [0,T^*]$; and (b) there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that for any $T \in [0,T^*]$, the relative bond price

$$Z^*(t,T) = B(t,T)/B_t, \quad \forall\, t \in [0,T], \tag{8.1}$$

follows a martingale under $\mathbf{P}^*$.

Any probability measure $\mathbf{P}^*$ of Definition 8.1.1 is called a *martingale measure for the family* $B(t,T)$ *relative to* r, or briefly, a *martingale measure for the family* $B(t,T)$ if no confusion may arise.[1] The reader might wonder why it is assumed that the relative price Z^* follows a martingale, and not merely a local martingale, under $\mathbf{P}^*$. The main reason is that under such an assumption we have trivially $Z^*(t,T) = \mathbf{E}_{\mathbf{P}^*}(Z^*(T,T) \,|\, \mathcal{F}_t)$ for $t \le T$, so that the bond price satisfies

$$B(t,T) = B_t \mathbf{E}_{\mathbf{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{8.2}$$

[1] In what follows, we shall distinguish between *spot* and *forward* martingale measures. In this context, the martingale measure of Definition 8.1.1 should be seen as a spot martingale measure for the family $B(t,T)$.

In other words, for any martingale measure $\mathbf{P}^*$ of an arbitrage-free family of bond prices, we have

$$B(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big(e^{-\int_t^T r_u\,du}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T]. \tag{8.3}$$

Conversely, given any non-negative short-term interest rate r defined on a probability space $(\Omega, \mathbf{F}, \mathbf{P})$, and any probability measure $\mathbf{P}^*$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbf{P}$, the family $B(t,T)$ given by (8.3) is easily seen to be an arbitrage-free family of bond prices relative to r. Let us observe that if a family $B(t,T)$ satisfies Definition 8.1.1, then necessarily the bond price process $B(\cdot,T)$ is a $\mathbf{P}^*$-semimartingale, as a product of a martingale and a process of finite variation (that is, a product of two $\mathbf{P}^*$-semimartingales). Therefore, it is also a $\mathbf{P}$-semimartingale, since the probability measures $\mathbf{P}$ and $\mathbf{P}^*$ are assumed to be mutually equivalent (see Theorem III.20 in Protter (1990)).

8.1.1 Expectations Hypotheses

Suppose that equality (8.3) is satisfied under the actual probability measure $\mathbf{P}$, that is

$$B(t,T) = \mathbf{E}_{\mathbf{P}}\Big(e^{-\int_t^T r_u\,du}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T]. \tag{8.4}$$

Equality (8.4) is traditionally referred to as the *local expectations hypothesis* ((L-EH) for short), or a *risk-neutral expectations hypothesis.* The term "local expectations" refers to the fact that under (8.4), the current bond price equals the expected value, under the actual probability, of the bond price in the next (infinitesimal) period, discounted at the current short-term rate. This property can be made more explicit in a discrete-time setting (see Ingersoll (1987) or Jarrow (1996)). In our framework, given an arbitrage-free family of bond prices relative to a short-term rate r, it is evident that (8.3) holds, by definition, under any martingale measure $\mathbf{P}^*$. This does not mean, however, that the local expectations hypothesis, or any other traditional form of expectations hypothesis, is satisfied under the actual probability $\mathbf{P}$. The *return-to-maturity expectations hypothesis* (RTM-EH) assumes that the return from holding any discount bond to maturity is equal to the return expected from rolling over a series of a single-period bonds. Its continuous-time counterpart reads as follows

$$\frac{1}{B(t,T)} = \mathbf{E}_{\mathbf{P}}\Big(e^{\int_t^T r_u\,du}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T],$$

for every $T \le T^*$. Finally, the *yield-to-maturity expectations hypothesis* (YTM-EH) asserts that the yield from holding any bond is equal to the yield expected from rolling over a series of a single-period bonds. In a continuous-time framework, this means that for any maturity date $T \le T^*$, we have

$$B(t,T) = \exp\Big\{-\mathbf{E}_{\mathbf{P}}\Big(\int_t^T r_u\,du\,\Big|\,\mathcal{F}_t\Big)\Big\}, \quad \forall\, t \in [0,T].$$

The last formula may also be given the following equivalent form

$$Y(t,T) = \frac{1}{T-t}\,\mathbf{E}_{\mathbf{P}}\Big(\int_t^T r_u\,du\,\Big|\,\mathcal{F}_t\Big),$$

or finally

$$f(t,T) = \mathbf{E}_{\mathbf{P}}(r_T\,|\,\mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{8.5}$$

In view of (8.5), under the yield-to-maturity expectations hypothesis, the forward interest rate $f(t,T)$ is an unbiased estimate, under the actual probability $\mathbf{P}$, of the future short-term interest rate r_T. For this reason, the YTM-EH is also frequently referred to as the *unbiased expectations hypothesis.* We shall see in what follows that condition (8.5) is always satisfied – not under the actual probability, however, but under the so-called *forward martingale measure* for the given date T. Note that if the short-term rate r is a deterministic function, then all expectations hypotheses coincide, and follow easily from the absence of arbitrage.

8.2 Case of Itô Processes

In a continuous-time framework, it is customary to model the short-term rate of interest by means of an Itô process, or more specifically, as a one-dimensional diffusion[2] process. We shall first examine the general case of a short-term interest rate which follows an Itô process. We thus assume that the dynamics of r are given in a differential form

$$dr_t = \mu_t \, dt + \sigma_t \cdot dW_t, \quad r_0 > 0, \tag{8.6}$$

where μ and σ are adapted stochastic processes with values in $\mathbf{R}$ and $\mathbf{R}^d$, respectively. Recall that (8.6) is a shorthand form of the following integral representation

$$r_t = r_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \cdot dW_u, \quad \forall \, t \in [0, T].$$

It is thus implicitly assumed that μ and σ satisfy the suitable integrability conditions, so that the process r is well-defined. In financial interpretation, the underlying probability measure $\mathbf{P}$ is believed to reflect a subjective assessment of the "market" upon the future behavior of the short-term interest rate. In other words, the underlying probability $\mathbf{P}$ is the actual probability, as opposed to a martingale probability measure for the bond market, which we are now going to construct. Let us recall, for the reader's convenience, a few basic facts concerning the notion of equivalence of probability measures on a filtration of a Brownian motion. Firstly, it is well known that any probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ is given by the Radon-Nikodým derivative

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} \lambda_u \cdot dW_u \Big) \stackrel{\text{def}}{=} \eta_{T^*}, \quad \mathbf{P}\text{-a.s.} \tag{8.7}$$

for some predictable $\mathbf{R}^d$-valued process λ. The member on the right-hand side of (8.7) is the Doléans exponential, which is given by the following expression

$$\eta_t \stackrel{\text{def}}{=} \mathcal{E}_t\Big(\int_0^{\cdot} \lambda_u \cdot dW_u \Big) = \exp\Big(\int_0^t \lambda_u \cdot dW_u - \frac{1}{2} \int_0^t |\lambda_u|^2 \, du \Big).$$

Given an adapted process λ, we write $\mathbf{P}^\lambda$ to denote the probability measure whose Radon-Nikodým derivative with respect to $\mathbf{P}$ is given by the right-hand side of (8.7). In view of Girsanov's theorem, the process

$$W_t^\lambda = W_t - \int_0^t \lambda_u \, du, \quad \forall \, t \in [0, T^*], \tag{8.8}$$

follows a d-dimensional standard Brownian motion under $\mathbf{P}^\lambda$. It should be stressed that the natural filtrations of the Brownian motions W and W^λ do not coincide, in general. The following result deals with the behavior of the short-term interest rate r and the bond price $B(t,T)$ under a probability measure $\mathbf{P}^\lambda$ equivalent to $\mathbf{P}$ – more specifically, under a probability measure $\mathbf{P}^\lambda$ which is a martingale measure in the sense of Definition 8.1.1 (see Artzner and Delbaen (1989) for related results).

Proposition 8.2.1 *Assume the short-term interest rate r follows an Itô process under the actual probability $\mathbf{P}$, as specified by* (8.6). *Let $B(t,T)$ be an arbitrage-free family of bond prices relative to r. For any martingale measure $\mathbf{P}^* = \mathbf{P}^\lambda$ of Definition* 8.1.1, *the following holds.* (i) *The process r satisfies under $\mathbf{P}^\lambda$*

$$dr_t = (\mu_t + \sigma_t \cdot \lambda_t) \, dt + \sigma_t \cdot dW_t^\lambda. \tag{8.9}$$

(ii) *There exists an adapted $\mathbf{R}^d$-valued process $b^\lambda(t,T)$ such that*

$$dB(t,T) = B(t,T)\big(r_t \, dt + b^\lambda(t,T) \cdot dW_t^\lambda \big). \tag{8.10}$$

[2]Generally speaking, a *diffusion process* is an arbitrary strong Markov process with continuous sample paths. In our framework, a diffusion process is given as a strong solution of a stochastic differential equation (SDE) driven by the underlying Brownian motion W.

Consequently, for any fixed maturity $T \in (0, T^*]$, we have

$$B(t,T) = B(0,T)\, B_t \,\exp\Big(\int_0^t b^\lambda(u,T)\cdot dW_u^\lambda - \frac{1}{2}\int_0^t |b^\lambda(u,T)|^2\, du\Big).$$

Proof. To show (i), it is enough to combine (8.6) with (8.8). For (ii), it is sufficient to observe that the process $M = Z^*\eta$ follows a (local) martingale under $\mathbf{P}$. In view of Theorem 3.1.2, we have

$$M_t = Z^*(t,T)\eta_t = Z^*(0,T) + \int_0^t \gamma_u \cdot dW_u, \quad \forall\, t \in [0,T],$$

for some $\mathbf{F}$-adapted process γ. Applying Itô's formula, we obtain

$$dZ^*(t,T) = d(M_t\eta_t^{-1}) = \eta_t^{-1}\big(\gamma_t - M_t\lambda_t\big)\cdot dW_u^\lambda, \tag{8.11}$$

where we have used

$$d\eta_t^{-1} = -\eta_t^{-1}\lambda_t \cdot dW_t^\lambda = -\eta_t^{-1}\lambda_t\cdot(dW_t - \lambda_t\, dt).$$

Equality (8.10) now follows easily from (8.11), once again by Itô's formula. The last asserted formula is also evident. □

Let us comment on the consequences of the results above. Suppose that the short-term interest rate r satisfies (8.6) under a probability measure $\mathbf{P}$. Let $\mathbf{P}^* = \mathbf{P}^\lambda$ be an arbitrary probability measure equivalent to $\mathbf{P}$. Then we may define a bond price $B(t,T)$ by setting

$$B(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big(e^{-\int_t^T r_u\, du}\,\Big|\,\mathcal{F}_t^{W^\lambda}\Big), \quad \forall\, t\in[0,T]. \tag{8.12}$$

It follows from (8.10) that the bond price $B(t,T)$ satisfies, under the actual probability $\mathbf{P}$,

$$dB(t,T) = B(t,T)\Big(\big(r_t - \lambda_t\cdot b^\lambda(t,T)\big)\,dt + b^\lambda(t,T)\cdot dW_t\Big). \tag{8.13}$$

This means that the instantaneous returns from holding the bond differ, in general, from the short-term interest rate r. In financial literature, the additional term is commonly referred to as the *risk premium* or the *market price for risk*. It is usually argued that due to the riskiness of a zero-coupon bond, it is reasonable to expect that the instantaneous return from holding the bond will exceed that of a risk-free security (i.e, of a savings account) in a market equilibrium.Unfortunately, since our arguments refer only to the absence of arbitrage between primary securities and derivatives (that is, we place ourselves in a partial equilibrium framework), we are unable to identify the risk premium. Summarizing, we have a certain degree of freedom: if the short-term rate r is given by (8.6), then any probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ can formally be used to construct an arbitrage-free family of bond prices through formula (8.12). Notice, however, that if the actual probability measure $\mathbf{P}$ is used to define the bond price through (8.12), the market prices for risk vanish.

We end this section by examining the problem of matching the initial yield curve. Given a short-term interest rate process r and a probability measure $\mathbf{P}^*$, the initial term structure $B(0,T)$ is uniquely determined by the formula

$$B(0,T) = \mathbf{E}_{\mathbf{P}^*}\Big(e^{-\int_0^T r_u\, du}\Big), \quad \forall\, T \in [0,T^*]. \tag{8.14}$$

This feature of bond price models based on a specified short-term interest rate process makes the problem of matching the current yield curve much more cumbersome than in the case of models which incorporate the initial term structure as an input of the model.

8.3 Single-factor Models

In this section, we survey the most widely accepted single-factor models of the short-term rate. It is assumed throughout that the dynamics of r are specified under the martingale probability measure $\mathbf{P}^*$ (i.e., the risk premium vanishes identically). The underlying Brownian motion W^* is assumed to be one-dimensional. In this sense, the models are based on a single source of uncertainty, i.e., they belong to the class of single-factor models.

Vasicek's model. The model analysed in Vasicek (1977) is one of the earliest models of term structure (see also Richard (1978) and Dothan (1978)). The diffusion process proposed by Vasicek is a mean-reverting version of the Ornstein-Uhlenbeck process. The short-term interest rate r is defined as the unique strong solution of the SDE

$$dr_t = \big(a - br_t\big)\,dt + \sigma\,dW_t^*, \tag{8.15}$$

where a, b and σ are strictly positive constants. It is well known that the solution of (8.15) is a Markov process with continuous sample paths and Gaussian increments.[3] It is evident that Vasicek's model, as any Gaussian model, allows for negative values of (nominal) interest rates. This property is manifestly incompatible with no-arbitrage in the presence of cash in the economy. Let us consider any security whose payoff depends on the short-term rate r as the only state variable. More specifically, we assume that this security is of European style, pays dividends continuously at a rate $h(r_t, t)$, and yields a terminal payoff $G_T = g(r_T)$ at time T. Using the well-known relationship between diffusion processes and the PDEs, one can show that the price process G_t of such a security admits the representation $G_t = v(r_t, t)$, where the function $v : \mathbf{R} \times [0, T^*] \to \mathbf{R}$ solves the following valuation PDE

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\,\sigma^2\,\frac{\partial^2 v}{\partial r^2}(r,t) + (a - br)\,\frac{\partial v}{\partial r}(r,t) - rv(r,t) + h(r,t) = 0,$$

subject to the terminal condition $v(r,T) = g(r)$. Solving this equation with $h = 0$ and $g(r) = 1$, Vasicek showed that the price of a zero-coupon bond is

$$B(t,T) = v(r_t, t, T) = e^{m(t,T) - n(t,T) r_t}, \tag{8.16}$$

where

$$n(t,T) = \frac{1}{b}\Big(1 - e^{-b(T-t)}\Big) \tag{8.17}$$

and

$$m(t,T) = \frac{\sigma^2}{2}\int_t^T n^2(u,T)\,du - a\int_t^T n(u,T)\,du. \tag{8.18}$$

To establish this result, it is enough to assume that the bond price is given by (8.16), with the functions m and n satisfying $m(T,T) = n(T,T) = 0$, and to make use of the fundamental PDE. By separating terms which do not depend on r, and those that are linear in r, we arrive at the following system of differential equations

$$m_t(t,T) = an(t,T) - \tfrac{1}{2}\sigma^2 n^2(t,T), \quad n_t(t,T) = bn(t,T) - 1, \tag{8.19}$$

with $m(T,T) = n(T,T) = 0$, which in turn yields easily the expressions above. One may check that we have

$$dB(t,T) = B(t,T)\big(r_t\,dt + \sigma n(t,T)\,dW_t^*\big), \tag{8.20}$$

so that the bond price volatility equals $b(t,T) = \sigma n(t,T)$, with $n(t,T)$ given by (8.17). If the bond price admits representation (8.16), then obviously

$$Y(t,T) = \frac{n(t,T) r_t - m(t,T)}{T - t},$$

[3] The solution to (8.15) is known to admit a stationary distribution, namely, a Gaussian distribution with the mean a/b and the variance $\sigma^2/2b$.

and thus the bond's yield, $Y(t,T)$, is an affine function of the short-term rate r_t. For this reason, models of the short-term rate in which the bond price satisfies (8.16) for some functions m and n are termed *affine models of the term structure.* Jamshidian (1989a) obtained closed-form solutions for the prices of a European option written on a zero-coupon and on a coupon-bearing bond for Vasicek's model. He showed that the arbitrage price at time t of a call option on a U-maturity zero-coupon bond, with strike price K and expiry $T \le U$, equals (let us mention that Jamshidian implicitly used the *forward measure* technique, which is presented in Sect. 9.2)

$$C_t = B(t,T)\mathbf{E}_{\mathbf{Q}}\big((\xi\eta - K)^+ \big| \mathcal{F}_t\big),$$

where $\eta = B(t,U)/B(t,T)$, and $\mathbf{Q}$ stands for some probability measure equivalent to $\mathbf{P}^*$. The random variable ξ is independent of the σ-field $\mathcal{F}_t$ under $\mathbf{Q}$, and has under $\mathbf{Q}$ a lognormal law such that the variance $\mathrm{Var}_{\mathbf{Q}}(\ln\xi)$ equals $v_U(t,T)$, where

$$v_U^2(t,T) = \int_t^T |b(t,T) - b(t,U)|^2\,du = \frac{\sigma^2}{2b^3}\big(1 - e^{-2b(T-t)}\big)\big(1 - e^{-b(U-t)}\big)^2.$$

The bond option valuation formula established in Jamshidian (1989a) reads as follows

$$C_t = B(t,U)N\big(h_1(t,T)\big) - KB(t,T)N\big(h_2(t,T)\big), \tag{8.21}$$

where for every $t \le T \le U$

$$h_{1,2}(t,T) = \frac{\ln\big(B(t,U)/B(t,T)\big) - \ln K \pm \frac{1}{2}\,v_U^2(t,T)}{v_U(t,T)}\,. \tag{8.22}$$

It is important to observe that the coefficient a does not enter the bond option valuation formula. This suggests that the actual value of the risk premium has no impact whatsoever on the bond option price (at least if it is deterministic); the only relevant quantities are in fact the bond price volatilities $b(t,T)$ and $b(t,U)$. To account for the risk premium, it is enough to make an equivalent change of the probability measure in (8.20). Since the volatility of the bond price is invariant with respect to such a transformation of the underlying probability measure, the bond option price is independent of the risk premium, provided that the bond price volatility is deterministic.

Cox-Ingersoll-Ross model. The general equilibrium approach to term structure modelling developed by Cox et al. (1985b) (CIR, for short) leads to the following modification of the mean-reverting diffusion of Vasicek, known as the *square-root process*

$$dr_t = \big(a - br_t\big)\,dt + \sigma\sqrt{r_t}\,dW_t^*, \tag{8.23}$$

where a, b and σ are strictly positive constants. Due to the presence of the square-root in the diffusion coefficient, the CIR diffusion takes only positive values; it can reach zero, but it never becomes negative. In a way similar to the previous case, the price process $G_t = v(r_t,t)$ of any standard European interest rate derivative, which settles at time T, can be found, in principle, by solving the valuation PDE

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\,\sigma^2 r\,\frac{\partial^2 v}{\partial r^2}(r,t) + (a - br)\,\frac{\partial v}{\partial r}(r,t) - rv(r,t) + h(r,t) = 0,$$

subject to the terminal condition $v(r,T) = g(r)$. Cox et al. (1985b) found closed-form solutions for the price of a zero-coupon bond. If we assume that the bond price $B(t,T)$ satisfies (8.16), then using the valuation PDE above, we find that the function n solves, for each fixed maturity date T, the Riccati equation

$$n_t(t,T) - \tfrac{1}{2}\sigma^2 n^2(t,T) - bn(t,T) + 1 = 0, \quad n(T,T) = 0, \tag{8.24}$$

and m satisfies

$$m_t(t,T) = an(t,T), \quad m(T,T) = 0. \tag{8.25}$$

Solving this, we obtain

$$m(t,T)=\frac{2a}{\sigma^2}\,\ln\left\{\frac{\gamma e^{b\tau/2}}{\gamma\cosh\gamma\tau+\frac{1}{2}b\sinh\gamma\tau}\right\},\quad n(t,T)=\frac{\sinh\gamma\tau}{\gamma\cosh\gamma\tau+\frac{1}{2}b\sinh\gamma\tau},$$

where $\tau=T-t$ and $2\gamma=(b^2+2\sigma^2)^{1/2}$. Closed-form expressions for the price of an option on a zero-coupon bond and an option on a coupon-bearing bond in the CIR framework were derived in Cox et al. (1985b) and in Longstaff (1993), respectively. Since they are rather involved, and will not be used in what follows, we refer the interested reader to the original papers for details. Let us only mention that they involve the cumulative non-central chi-square distribution function, and depend on the deterministic risk premium (it is easily seen that the bond price volatility is now stochastic).

Longstaff (1990) has shown how to value European call and put *options on yields* in the CIR model. For a fixed time to maturity, the yield on a zero-coupon bond in the CIR framework is, of course, a linear function of the short-term rate, since

$$Y(t,t+\tau)=\tilde{Y}(r_t,\tau)=\tilde{m}(\tau)+\tilde{n}(\tau)r_t,$$

where $\tau=T-t$ is fixed. The number $\tilde{Y}(r,\tau)$ represents the yield at time t for zero-coupon bonds with a constant maturity τ, provided that the current level of the short-term rate is $r_t=r$. According to the contractual features, for a fixed τ, a European *yield call option* entitles its owner to receive the payoff C_T^Y, which is expressed in monetary units and equals $C_T^Y=(\tilde{Y}(r_T,\tau)-K)^+$, where K is the fixed level of the yield.

Longstaff's model. Longstaff (1989) modified the CIR model by postulating the following dynamics for the short-term rate

$$dr_t=a\big(b-c\sqrt{r_t}\big)\,dt+\sigma\sqrt{r_t}\,dW_t^*, \tag{8.26}$$

referred to as the *double square-root* (DSR) process. Longstaff derived a closed-form expression for the price of a zero-coupon bond

$$B(t,T)=v(r_t,t,T)=e^{m(t,T)-n(t,T)r_t-p(t,T)\sqrt{r_t}}$$

for some explicitly known functions m,n and p, which are not reproduced here. The bond's yield is thus a non-linear function of the short-term rate. Also, the bond price is not a monotone (decreasing) function of the current level of the short-term rate. This feature makes the valuation of a bond option less straightforward than usual. Indeed, typically, it is possible to represent the exercise set of a bond option in terms of r as the interval $[r^*,\infty)$ or $(-\infty,r^*]$ for some constant r^*, depending on whether an option is a put or a call (see Sect. 8.3.1).

Hull-White model. Note that both Vasicek's and the CIR models are special cases of the following mean-reverting diffusion process

$$dr_t=a\big(b-cr_t\big)dt+\sigma r_t^\beta\,dW_t^*,$$

where $0\le\beta\le 1$ is a constant. These models of the short-term rate are thus built upon a certain diffusion process with constant (i.e., time-independent) coefficients. In practical applications, it is more reasonable to expect that in some situations, the market's expectations about future interest rates involve time-dependent coefficients. Also, it would be a plausible feature if a model fitted not merely the initial value of the short-term rate, but rather the whole initial yield curve. This desirable property of a bond price model motivated Hull and White (1990a) to propose an essential modification of the models above. In its most general form, the Hull-White methodology assumes that

$$dr_t=\big(a(t)-b(t)r_t\big)\,dt+\sigma(t)r_t^\beta\,dW_t^* \tag{8.27}$$

for some constant $\beta \geq 0$, where W^* is a one-dimensional Brownian motion, and a, b, $\sigma : \mathbf{R}_+ \to \mathbf{R}$ are locally bounded functions. By setting $\beta = 0$ in (8.27), we obtain the *generalized Vasicek model,* in which the dynamics of r are[4]

$$dr_t = \big(a(t) - b(t)r_t\big)\,dt + \sigma(t)\,dW_t^*. \tag{8.28}$$

To explicitly solve this equation, let us denote $l(t) = \int_0^t b(u)\,du$. Then we have

$$d\big(e^{l(t)}r_t\big) = e^{l(t)}\big(a(t)\,dt + \sigma(t)\,dW_t^*\big),$$

so that

$$r_t = e^{-l(t)}\Big(r_0 + \int_0^t e^{l(u)}a(u)\,du + \int_0^t e^{l(u)}\sigma(u)\,dW_u^*\Big).$$

It is thus not surprising that closed-form solutions for bond and bond option prices are not hard to derive in this setting. On the other hand, if we put $\beta = 1/2$, then we obtain the *generalized CIR model*

$$dr_t = \big(a(t) - b(t)r_t\big)\,dt + \sigma(t)\sqrt{r_t}\,dW_t^*.$$

In this case, however, the closed-form expressions for the bond and option prices are not easily available (this would require solving (8.24)–(8.25) with time-dependent coefficients $a(t), b(t)$ and $\sigma(t)$). The most important feature of the Hull-White approach is the possibility of the exact fit of the initial term structure and, in some circumstances, also of the term structure of forward rate volatilities. This can be done, for instance, by means of a judicious choice of the functions a, b and σ. Since the details of the fitting procedure depend on the particular model (i.e., on the choice of β), let us illustrate this point by restricting our attention to the generalized Vasicek model. We start by assuming that the bond price $B(t,T)$ can be represented in the following way

$$B(t,T) = B(r_t,t,T) = e^{m(t,T)-n(t,T)r_t} \tag{8.29}$$

for some functions m and n, with $m(T,T) = 0$ and $n(T,T) = 0$. Plugging (8.29) into the fundamental PDE for the zero-coupon bond, which is

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\,\sigma^2(t)\,\frac{\partial^2 v}{\partial r^2}(r,t) + \big(a(t) - b(t)r\big)\,\frac{\partial v}{\partial r}(r,t) - rv(r,t) = 0,$$

we obtain

$$m_t(t,T) - a(t)n(t,T) + \tfrac{1}{2}\sigma^2(t)n^2(t,T) - \big(1 + n_t(t,T) - b(t)n(t,T)\big)r = 0.$$

Since the last equation holds for every t, T and r, we deduce that m and n satisfy the following system of differential equations (cf. (8.19))

$$m_t(t,T) = a(t)n(t,T) - \tfrac{1}{2}\sigma^2(t)n^2(t,T), \quad n_t(t,T) = b(t)n(t,T) - 1 \tag{8.30}$$

with $M(T,T) = n(T,T) = 0$. Suppose that an initial term structure $P(0,T)$ is exogenously given. We adopt the convention to denote by $P(0,T)$ the initial term structure, which is given (it can, for instance, be inferred from the market data), as opposed to the initial term structure $B(0,T)$, which is implied by a particular stochastic model of the term structure. Assume also that the forward rate volatility is not prespecified. In this case, we may and do assume that $b(t) = b$ and $\sigma(t) = \sigma$ are given constants. We shall thus search for the function a only. Since b and σ are constants, n is given by (8.17). Furthermore, in view of (8.30), m equals

$$m(t,T) = \frac{1}{2}\int_t^T \sigma^2 n^2(u,T)\,du - \int_t^T a(u)n(u,T)\,du. \tag{8.31}$$

[4] A special case of such a model, with $b = 0$, was considered in Merton (1973).

Since the forward rates implied by the model equal (cf. (7.5))

$$f(0,T) = -\frac{\partial \ln B(0,T)}{\partial T} = n_T(0,T)r_0 - m_T(0,T), \tag{8.32}$$

easy calculations involving (8.17) and (8.31) show that

$$\hat{f}(0,T) \stackrel{\text{def}}{=} -\frac{\partial \ln P(0,T)}{\partial T} = e^{-bT} r_0 + \int_0^T e^{-b(T-u)} a(u)\, du - \frac{\sigma^2}{2b^2}\big(1 - e^{-bT}\big)^2.$$

Put another way, $\hat{f}(0,T) = g(T) - h(T)$, where $g'(T) = -bg(T) + a(T)$, with $g(0) = r_0$, and $h(T) = \sigma^2(1 - e^{-bT})^2/(2b^2)$. Consequently, we obtain

$$a(T) = g'(T) + bg(T) = \hat{f}_T(0,T) + h'(T) + b(\hat{f}(0,T) + h(T)),$$

and thus the function a is indeed uniquely determined. This terminates the fitting procedure. Though, at least theoretically, this procedure can be extended to fit the volatility structure, is should be stressed that the possibility of an exact match with the historical data is only one of several desirable properties of a model of the term structure.

Let us now summarize the most important features of term structure models which assume the diffusion-type dynamics of the short-term rate. Suppose that the dynamics of r under the actual probability $\mathbf{P}$ satisfy

$$dr_t = \mu(r_t,t)\, dt + \sigma(r_t,t)\, dW_t \tag{8.33}$$

for some sufficiently regular functions μ and σ. Assume, in addition, that the risk premium process equals $\lambda_t = \lambda(r_t,t)$ for some function $\lambda = \lambda(r,t)$. In financial interpretation, the last condition means that the excess rate of return of a given zero-coupon bond depends only on the current short-term rate and the price volatility of this bond. Using (8.33) and Girsanov's theorem,[5] we conclude that under the martingale measure $\mathbf{P}^* = \mathbf{P}^\lambda$, the process r satisfies

$$dr_t = \mu^\lambda(r_t,t) dt + \sigma(r_t,t)\, dW_t^*, \tag{8.34}$$

where

$$\mu^\lambda(r,t) = \mu(r,t) + \lambda(r,t)\sigma(r,t).$$

Let us stress once again that it is essential to assume that the functions μ, σ and λ are sufficiently regular (for instance, locally Lipschitz with respect to the first variable, and satisfying the linear growth condition), so that the SDE (8.34), with initial condition $r_0 > 0$, admits a unique global strong solution. Under such assumptions, the process r is known to follow, under the martingale measure $\mathbf{P}^*$, a strong Markov process with continuous sample paths. The arbitrage price $\pi_t(X)$ of any attainable contingent claim X, which is of the form $X = g(r_T)$ for some function $g : \mathbf{R} \to \mathbf{R}$, is given by the risk-neutral valuation formula

$$\pi_t(X) = \mathbf{E}_{\mathbf{P}^*}\Big(g(r_T)\, e^{-\int_t^T r_u\, du} \,\Big|\, \mathcal{F}_t\Big) = v(r_t,t),$$

where $v : \mathbf{R} \times [0,T^*] \to \mathbf{R}$. It follows from the general theory of diffusion processes, more precisely from the result known as the *Feynman-Kac formula* (see Theorem 5.7.6 in Karatzas and Shreve (1988)), that under mild technical assumptions, if a security pays continuously at a rate $h(r_t,t)$ and yields a terminal payoff $G_T = g(r_T)$ at time T, then the valuation function v solves the following *fundamental PDE*

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\,\sigma^2(r,t)\,\frac{\partial^2 v}{\partial r^2}(r,t) + \mu^\lambda(r,t)\,\frac{\partial v}{\partial r}(r,t) - rv(r,t) + h(r,t) = 0,$$

[5] We need, of course, to show that an application of Girsanov's theorem is justified. Essentially, this means that we need to impose certain conditions which would guarantee that $\mathbf{P}^\lambda$ is indeed a probability measure equivalent to $\mathbf{P}$.

subject to the terminal condition $v(r,T) = g(r)$. Existence of a closed-form solution of this equation for the most typical derivative securities (in particular, for a zero-coupon bond and an option on such a bond) is, of course, a desirable property of a term structure model of diffusion type. Otherwise, the efficiency of numerical procedures used to solve the fundamental PDE becomes an important practical issue.

8.3.1 American Bond Options

Let us denote by $P^a(r_t,t,T)$ the price at time t of an American put option with strike price K and expiry date T, written on a zero-coupon bond of maturity $U \geq T$. Arguing along similar lines as in Chap. 5, it is possible to show that

$$P^a(r_t,t,T) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbf{E}_{\mathbf{P}^*}\Big(e^{-\int_t^\tau r_v\,dv}\big(K - B(r_\tau,\tau,U)\big)^+ \,\Big|\, r_t\Big),$$

where $\mathcal{T}_{[t,T]}$ is the class of all stopping times with values in the interval $[t,T]$. For a detailed justification of the application of standard valuation procedures to American contingent claims under uncertain interest rates, we refer the reader to Amin and Jarrow (1992). For any $t \in [0,T]$, the optimal exercise time τ_t equals

$$\tau_t = \inf\{u \in [t,T] \,|\, (K - B(r_u,u,U))^+ = P^a(r_u,u,T)\}.$$

Assume that the bond price is a decreasing function of the rate r (this holds in most, but not all, single-factor models). Then

$$\tau_t = \inf\{u \in [t,T] \,|\, r_u \geq r_u^*\}$$

for a certain process r^*, which represents the critical level of the short-term interest rate. Using this, one can derive the following early exercise premium representation of the price of an American put option on a zero-coupon bond

$$P^a(r_t,t,T) = P(r_t,t,T) + \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T e^{-\int_t^u r_v\,dv}\, \mathrm{I}_{\{r_u \geq r_u^*\}}\, r_u K\,du \,\Big|\, \mathcal{F}_t\Big),$$

where $P(r_t,t,T)$ stands for the price of the corresponding European-style put option. The quasi-analytical forms of this representation for Vasicek's model and the CIR model were found in Jamshidian (1992). More recently, Chesney et al. (1993) have studied bond and yield options of American style for the CIR model, using the properties of the Bessel bridges.

8.3.2 Options on Coupon-bearing Bonds

A coupon-bearing bond is formally equivalent to a portfolio of discount bonds with different maturities. To value European options on coupon-bearing bonds, we take into account the fact that the zero-coupon bond price is typically a decreasing function of the short-term rate r. This implies that an option on a portfolio of zero-coupon bonds is equivalent to a portfolio of options on zero-coupon bonds with appropriate strike prices. Let us consider, for instance, a European call option with exercise price K and expiry date T on a coupon-bearing bond which pays coupons $c_1,\dots,c_m$ at dates $T_1 < \ldots < T_m \leq T^*$. The payoff of the option at expiry equals

$$C_T = \Big(\sum_{j=1}^m c_j\, B(r_T,T,T_j) - K\Big)^+.$$

Therefore the option will be exercised if and only if $r_t < r^*$, where the critical interest rate r^* solves the equation $\sum_{j=1}^m c_j\, B(r^*,T,T_j) = K$. The option's payoff can be represented in the following way

$$C_T = \sum_{j=1}^m c_j \big(B(r_T,T,T_j) - K_j\big)^+,$$

where $K_j = B(r^*,T,T_j)$. The valuation of a call option on a coupon-bearing bond thus reduces to the pricing of options on zero-coupon bonds.

Chapter 9

Models of Forward Rates

The Heath, Jarrow and Morton approach to term structure modelling is based on an exogenous specification of the dynamics of instantaneous, continuously compounded forward rates $f(t,T)$. For any fixed maturity $T \le T^*$, the dynamics of the forward rate $f(t,T)$ are (cf. Heath et al. (1990, 1992a))

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW_t, \tag{9.1}$$

where α and σ are adapted stochastic processes with values in $\mathbf{R}$ and $\mathbf{R}^d$, respectively, and W is a d-dimensional standard Brownian motion with respect to the underlying probability measure $\mathbf{P}$ (to be interpreted as the actual probability). For any fixed maturity date T, the initial condition $f(0,T)$ is determined by the current value of the continuously compounded forward rate for the future date T which prevails at time 0. The price $B(t,T)$ of a zero-coupon bond which matures at the date $T \le T^*$ can be recovered from the formula (cf. (7.4))

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big), \quad \forall\, t \in [0,T], \tag{9.2}$$

provided that the integral on the right-hand side of (9.2) exists (for almost all ω's). Leaving the technical assumptions aside, the first question that should be addressed is the absence of arbitrage in a financial market model which involves all bonds, with differing maturities, as primary traded securities. As expected, the answer to this question can be formulated in terms of the existence of a suitably defined martingale measure. It appears that in an arbitrage-free setting – that is, under the martingale probability – the drift coefficient α in the dynamics (9.1) of the forward rate is uniquely determined by the volatility coefficient σ, and a stochastic process which can be interpreted as the *risk premium.* More importantly, if σ follows a deterministic function, then the valuation results for interest rate-sensitive derivatives appear to be independent of the choice of the risk premium. In this sense, the choice of a particular model from the broad class of Heath-Jarrow-Morton (HJM) models hinges uniquely on the specification of the volatility coefficient σ. It should be stressed that for this specific feature of continuous-time forward rate modelling to hold, we need to restrict our attention to the class of HJM models with deterministic coefficient σ; that is, to *Gaussian HJM models.*

9.1 HJM Model

Let W be a d-dimensional standard Brownian motion given on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. As usual, the filtration $\mathbf{F} = \mathbf{F}^W$ is assumed to be the right-continuous and $\mathbf{P}$-completed version of the natural filtration of W. We consider a continuous-time trading economy with a trading interval $[0, T^*]$ for the fixed horizon date T^*. We are in a position to formulate the basic postulates of the HJM approach.

(HJM.1) For every fixed $T \le T^*$, the dynamics of the instantaneous forward rate $f(t,T)$ are given by the integrated version of (9.1)

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\cdot dW_u, \quad \forall\, t \in [0,T], \tag{9.3}$$

for a Borel-measurable function $f(0,\cdot) : [0,T^*] \to \mathbf{R}$, and some applications $\alpha : C \times \Omega \to \mathbf{R}$, $\sigma : C \times \Omega \to \mathbf{R}^d$, where $C = \{\,(u,t)\,|\, 0 \le u \le t \le T^*\,\}$.

(HJM.2) For any maturity T, $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ follow adapted processes, such that

$$\int_0^T |\alpha(u,T)|\,du + \int_0^T |\sigma(u,T)|^2\,du < \infty, \quad \mathbf{P}\text{-a.s.}$$

Though unnecessary, one may find it useful to introduce also a savings account as an additional primary security. For this purpose, assume that there exists a measurable version of the process $f(t,t)$, $t \in [0,T^*]$. It is then natural to postulate that the short-term interest rate satisfies $r_t = f(t,t)$ for every t. Consequently, the savings account equals (cf. (7.8))

$$B_t = \exp\Big(\int_0^t f(u,u)\,du\Big), \quad \forall\, t \in [0,T^*]. \tag{9.4}$$

The following auxiliary lemma deals with the dynamics of the bond price process $B(t,T)$ under the actual probability $\mathbf{P}$. It is interesting to observe that the drift and volatility coefficients in the dynamics of $B(t,T)$ can be expressed in terms of the coefficients α and σ of forward rate dynamics, and the short-term interest rate $r_t = f(t,t)$.

Lemma 9.1.1 *The dynamics of the bond price $B(t,T)$ are determined by the expression*

$$dB(t,T) = B(t,T)\big(a(t,T)\,dt + b(t,T)\cdot dW_t\big), \tag{9.5}$$

where a and b are given by the following formulae

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|\sigma^*(t,T)|^2, \quad b(t,T) = -\sigma^*(t,T), \tag{9.6}$$

and for any $t \in [0,T]$ we have

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)\,du, \quad \sigma^*(t,T) = \int_t^T \sigma(t,u)\,du. \tag{9.7}$$

Proof. Let us denote $I_t = \ln B(t,T)$. It follows directly from (9.2)–(9.3) that

$$I_t = -\int_t^T f(0,u)\,du - \int_t^T\!\int_0^t \alpha(v,u)\,dv\,du - \int_t^T\!\int_0^t \sigma(v,u)\cdot dW_v\,du.$$

Applying Fubini's standard and stochastic theorems (for the latter, see Theorem IV.45 in Prottter (1990)), we find that

$$I_t = -\int_t^T f(0,u)\,du - \int_0^t\!\int_t^T \alpha(v,u)\,du\,dv - \int_0^t\!\int_t^T \sigma(v,u)\,du\cdot dW_v,$$

or equivalently

$$\begin{aligned} I_t \;=\; & -\int_0^T f(0,u)\,du - \int_0^t\!\int_v^T \alpha(v,u)\,du\,dv - \int_0^t\!\int_v^T \sigma(v,u)\,du\cdot dW_v \\ & + \int_0^t f(0,u)\,du + \int_0^t\!\int_v^t \alpha(v,u)\,du\,dv + \int_0^t\!\int_v^t \sigma(v,u)\,du\cdot dW_v. \end{aligned}$$

Consequently,

$$I_t = I_0 + \int_0^t r_u \, du - \int_0^t \int_u^T \alpha(u,v) \, dv \, du - \int_0^t \int_u^T \sigma(u,v) \, dv \cdot dW_u,$$

where we have used the representation

$$r_u = f(u,u) = f(0,u) + \int_0^u \alpha(v,u) \, dv + \int_0^u \sigma(v,u) \cdot dW_v. \tag{9.8}$$

Taking into account equations (9.7), we obtain

$$I_t = I_0 + \int_0^t r_u \, du - \int_0^t \alpha^*(u,T) \, du - \int_0^t \sigma^*(u,T) \cdot dW_u.$$

To check that (9.5) holds, it suffices to apply Itô's formula. □

9.1.1 Absence of Arbitrage

In the present setting, a continuum of bonds with different maturities is available for trade. We shall assume, however, that any particular portfolio involves investments in an arbitrary, but finite, number of bonds. An alternative approach, in which infinite portfolios are also allowed, can be found in Björk et al. (1997a, 1997b). For any collection of maturities $0 < T_1 < T_2 < \ldots < T_k = T^*$, we write $\mathcal{T}$ to denote the vector $(T_1, \ldots, T_k)$; similarly, $B(\cdot, \mathcal{T})$ stands for the $\mathbf{R}^k$-valued process $(B(t,T_1), \ldots, B(t,T_k))$. We find it convenient to extend the $\mathbf{R}^k$-valued process $B(\cdot,\mathcal{T})$ over the time interval $[0,T^*]$ by setting $B(t,T) = 0$ for any $t \in (T, T^*]$ and any maturity $0 < T < T^*$. By a *bond trading strategy* we mean a pair $(\phi, \mathcal{T})$, where ϕ is a predictable $\mathbf{R}^k$-valued stochastic process which satisfies $\phi_t^i = 0$ for every $t \in (T_i, T^*]$ and any $i = 1, \ldots, k$. A bond trading strategy $(\phi, \mathcal{T})$ is said to be *self-financing* if the wealth process $V(\phi)$, which equals

$$V_t(\phi) \stackrel{\text{def}}{=} \phi_t \cdot B(t,\mathcal{T}) = \sum_{i=1}^k \phi_t^i B(t,T_i),$$

satisfies

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \cdot dB(u,\mathcal{T}) = V_0(\phi) + \sum_{i=1}^k \int_0^t \phi_u^i \, dB(u,T_i)$$

for every $t \in [0,T^*]$. To ensure the arbitrage-free properties of the bond market model, we need to examine the existence of a martingale measure for a suitable choice of a numeraire; in the present setup, we can take either the bond price $B(t,T^*)$ or the savings account B.

Assume, for simplicity, that the coefficient σ in (9.3) is bounded. We are looking for a condition ensuring the absence of arbitrage opportunities across all bonds of different maturities. Let us introduce an auxiliary process F_B by setting

$$F_B(t,T,T^*) \stackrel{\text{def}}{=} \frac{B(t,T)}{B(t,T^*)}, \quad \forall\, t \in [0,T].$$

In view of (9.5), the dynamics of the process $F_B(\cdot,T,T^*)$ are given by

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\Big(\tilde{a}(t,T)\,dt + \big(b(t,T) - b(t,T^*)\big) \cdot dW_t\Big),$$

where for every $t \in [0,T]$

$$\tilde{a}(t,T) = a(t,T) - a(t,T^*) - b(t,T^*) \cdot \big(b(t,T) - b(t,T^*)\big).$$

We know that any probability measure equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ satisfies

$$\frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big), \qquad \mathbf{P}\text{-a.s.} \tag{9.9}$$

for some predictable $\mathbf{R}^d$-valued process h. Let us fix a maturity date T. It is easily seen from Girsanov's theorem and the dynamics of $F_B(t,T,T^*)$ that $F_B(t,T,T^*)$ follows a martingale[1] under $\hat{\mathbf{P}}$, provided that for every $t \in [0,T]$

$$a(t,T) - a(t,T^*) = \big(b(t,T^*) - h_t\big) \cdot \big(b(t,T) - b(t,T^*)\big). \tag{9.10}$$

In order to exclude arbitrage opportunities between all bonds with different maturities, it suffices to assume that a martingale measure $\hat{\mathbf{P}}$ can be chosen simultaneously for all maturities. The following condition is thus sufficient for the absence of arbitrage between all bonds. As usual, we restrict our attention to the class of admissible trading strategies.

Condition (M.1) There exists an adapted $\mathbf{R}^d$-valued process h such that

$$\mathbf{E}_{\mathbf{P}}\Big\{\mathcal{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big)\Big\} = 1$$

and, for every $T \le T^*$, equality (9.10) is satisfied, or equivalently

$$\int_T^{T^*} \alpha(t,u)\, du + \frac{1}{2}\Big|\int_T^{T^*} \sigma(t,u)\, du\Big|^2 + h_t \cdot \int_T^{T^*} \sigma(t,u)\, du = 0.$$

By taking the partial derivative with respect to T, we obtain

$$\alpha(t,T) + \sigma(t,T) \cdot \Big(h_t + \int_T^{T^*} \sigma(t,u)\, du\Big) = 0, \tag{9.11}$$

for every $0 \le t \le T \le T^*$. For any process h of condition (M.1), the probability measure $\hat{\mathbf{P}}$ given by (9.9) will later be interpreted as the *forward martingale measure* for the date T^* (see Sect. 9.2). Assume now, in addition, that one may invest also in the savings account given by (9.4). In view of (9.5), the relative bond price $Z^*(t,T) = B(t,T)/B_t$ satisfies under $\mathbf{P}$

$$dZ^*(t,T) = -Z^*(t,T)\Big(\big(\alpha^*(t,T) - \tfrac{1}{2}|\sigma^*(t,T)|^2\big)\, dt + \sigma^*(t,T) \cdot dW_t\Big).$$

The following no-arbitrage condition excludes arbitrage not only across all bonds, but also between bonds and the savings account.

Condition (M.2) There exists an adapted $\mathbf{R}^d$-valued process λ such that

$$\mathbf{E}_{\mathbf{P}}\Big\{\mathcal{E}_{T^*}\Big(\int_0^{\cdot} \lambda_u \cdot dW_u\Big)\Big\} = 1$$

and, for any maturity $T \le T^*$, we have

$$\alpha^*(t,T) = \tfrac{1}{2}|\sigma^*(t,T)|^2 - \sigma^*(t,T) \cdot \lambda_t.$$

Differentiation of the last equality with respect to T gives

$$\alpha(t,T) = \sigma(t,T)(\sigma^*(t,T) - \lambda_t), \quad \forall t \in [0,T], \tag{9.12}$$

[1] The martingale property of $F_B(t,T,T^*)$, as opposed to the local martingale property, follows from the assumed boundedness of σ.

which holds for any $T \le T^*$. A probability measure $\mathbf{P}^*$, which satisfies

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} \lambda_u \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.}$$

for some process λ satisfying (M.2), can be seen as a *spot martingale measure* for the HJM model; in this context, the process λ is associated with the *risk premium.* Define a $\mathbf{P}^*$-Brownian motion W^* by setting

$$W_t^* = W_t - \int_0^t \lambda_u \, du, \quad \forall\, t \in [0, T].$$

The next result, whose proof is straightforward, deals with the dynamics of bond prices and interest rates under the spot martingale measure $\mathbf{P}^*$.

Corollary 9.1.1 *For any fixed maturity $T \le T^*$, the dynamics of the bond price $B(t,T)$ under the spot martingale measure $\mathbf{P}^*$ are*

$$dB(t,T) = B(t,T)\big(r_t \, dt - \sigma^*(t,T) \cdot dW_t^*\big), \tag{9.13}$$

and the forward rate $f(t,T)$ satisfies

$$df(t,T) = \sigma(t,T) \cdot \sigma^*(t,T) \, dt + \sigma(t,T) \cdot dW_t^*. \tag{9.14}$$

Finally, the short-term interest rate $r_t = f(t,t)$ is given by the expression

$$r_t = f(0,t) + \int_0^t \sigma(u,t) \cdot \sigma^*(u,t) \, du + \int_0^t \sigma(u,t) \cdot dW_u^*. \tag{9.15}$$

It follows from (9.15) that the expectation of the future short-term rate under the spot martingale measure $\mathbf{P}^*$ does not equal the current value $f(0,T)$ of the instantaneous forward rate; that is, $f(0,T) \neq \mathbf{E}_{\mathbf{P}^*}(r_T)$, in general. We shall see soon that $f(0,T)$ equals the expectation of r_T under the *forward martingale measure* for the date T (see Corollary 10.1.1). In view of (9.13), the relative bond price $Z^*(t,T) = B(t,T)/B_t$ satisfies

$$dZ^*(t,T) = -Z^*(t,T)\sigma^*(t,T) \cdot dW_t^*, \tag{9.16}$$

and thus

$$Z^*(t,T) = B(0,T) \, \exp\Big(-\int_0^t \sigma^*(u,T) \cdot dW_u^* - \frac{1}{2}\int_0^t |\sigma^*(u,T)|^2 \, du\Big),$$

or equivalently

$$\ln B(t,T) = \ln B(0,T) + \int_0^t \Big(r_u - \tfrac{1}{2}|\sigma^*(u,T)|^2\Big) \, du - \int_0^t \sigma^*(u,T) \cdot dW_u^*.$$

It is not hard to check that, under mild technical assumptions, the no-arbitrage conditions (M.1) and (M.2) are equivalent.
We assume from now on that the following assumption is satisfied.

(HJM.3) No-arbitrage condition (M.1) (or equivalently, (M.2)) is satisfied.

It is not essential to assume that the martingale measure for the bond market is unique, so long as we are not concerned with the completeness of the model. Recall that if a market model is arbitrage-free, any attainable claim admits a unique arbitrage price anyway (it is uniquely determined by the replicating strategy), whether a market model is complete or not.

9.2 Forward Measure Approach

The aim of this section is to describe the specific features that distinguish the arbitrage valuation of contingent claims within the classic Black-Scholes framework from the pricing of options on stocks and bonds under stochastic interest rates. We assume throughout that the price $B(t,T)$ of a zero-coupon bond of maturity $T \le T^*$ ($T^* > 0$ is a fixed horizon date) follows an Itô process under the martingale measure $\mathbf{P}^*$[2]

$$dB(t,T) = B(t,T)\left(r_t\,dt + b(t,T)\cdot dW_t^*\right), \tag{9.17}$$

with $B(T,T) = 1$, where W^* denotes a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbf{F}, \mathbf{P}^*)$, and r_t stands for the instantaneous, continuously compounded rate of interest. In other words, we take for granted the existence of an arbitrage-free family $B(t,T)$ of bond prices associated with a certain process r which models the short-term interest rate. Moreover, it is implicitly assumed that we have already constructed an arbitrage-free model of a market in which all bonds of different maturities, as well as a certain number of other assets (called *stocks* in what follows), are primary traded securities. It should be stressed that the way in which such a construction is achieved is not relevant for the results presented in what follows. In particular, the concept of the instantaneous forward interest rate, which is known to play an essential role in the HJM methodology, is not employed. As already mentioned, in addition to zero-coupon bonds, we shall also consider other primary assets, referred to as *stocks* in what follows. The dynamics of a stock price S^i, $i = 1,\dots,M$, under the martingale measure $\mathbf{P}^*$ are given by the following expression

$$dS_t^i = S_t^i\left(r_t\,dt + \sigma_t^i\cdot dW_t^*\right), \quad S_0^i > 0, \tag{9.18}$$

where σ^i represents the volatility of the stock price S^i. Unless explicitly stated otherwise, for every T and i, the bond price volatility $b(t,T)$ and the stock price volatility σ_t^i are assumed to be $\mathbf{R}^d$-valued, bounded, adapted processes. Generally speaking, we assume that the prices of all primary securities follow strictly positive processes with continuous sample paths. It should be observed, however, that certain results presented in this section are independent of the particular form of bond and stock prices introduced above. We denote by $\pi_t(X)$ the arbitrage price at time t of an attainable contingent claim X which settles at time T. Therefore

$$\pi_t(X) = B_t\,\mathbf{E}_{\mathbf{P}^*}(XB_T^{-1}\,|\,\mathcal{F}_t), \quad \forall\, t\in[0,T], \tag{9.19}$$

by virtue of the standard risk-neutral valuation formula. In (9.19), B represents the savings account given by (7.8). Recall that the price $B(t,T)$ of a zero-coupon bond which matures at time T admits the following representation (cf. (8.2))

$$B(t,T) = B_t\,\mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t), \quad \forall\, t\in[0,T], \tag{9.20}$$

for any maturity $0 \le T \le T^*$. Suppose now that we wish to price a European call option, with expiry date T, which is written on a zero-coupon bond of maturity $U > T$. The option's payoff at expiry equals

$$C_T = (B(T,U) - K)^+,$$

so that the option price C_t at any date $t \le T$ is

$$C_t = B_t\,\mathbf{E}_{\mathbf{P}^*}\left(B_T^{-1}(B(T,U)-K)^+\,\middle|\,\mathcal{F}_t\right).$$

To find the option's price using the last equality, we need to know the joint (conditional) probability law of $\mathcal{F}_T$-measurable random variables B_T and $B(T,U)$. The technique which was developed to circumvent this step is based on an equivalent change of probability measure. It appears that it is possible to find a probability measure $\mathbf{P}_T$ such that the following holds

$$C_t = B(t,T)\mathbf{E}_{\mathbf{P}_T}\left((B(T,U)-K)^+\,\middle|\,\mathcal{F}_t\right).$$

[2]The reader may find it convenient to assume that the probability measure $\mathbf{P}^*$ is the unique martingale measure for the family $B(t,T)$, $T \le T^*$; this is not essential, however.

Consequently,

$$C_t = B(t,T)\mathbf{E}_{\mathbf{P}_T}\big((F_B(T,U,T)-K)^+ \,\big|\, \mathcal{F}_t\big),$$

where $F_B(t,U,T)$ is the forward price at time t, for settlement at the date T, of the U-maturity zero-coupon bond (see formula (9.22)). If $b(t,U)-b(t,T)$ is a deterministic function, then the forward price $F_B(t,U,T)$ can be shown to follow a lognormal martingale under $\mathbf{P}_T$; therefore, a Black-Scholes-like expression for the option's price is available.

9.2.1 Forward Price

Recall that a *forward contract* is an agreement, established at the date $t<T$, to pay or receive on settlement date T a preassigned payoff, say X, at an agreed forward price. It should be emphasized that there is no cash flow at the contract's initiation and the contract is not marked to market. We may and do assume, without loss of generality, that a forward contract is settled by cash on date T. Therefore, a forward contract written at time t with the underlying contingent claim X and prescribed settlement date $T>t$ may be summarized by the following two basic rules: (a) a cash amount X will be received at time T, and a preassigned amount $F_X(t,T)$ of cash will be paid at time T; (b) the amount $F_X(t,T)$ should be predetermined at time t (according to the information available at this time) in such a way that the arbitrage price of the forward contract at time t is zero. In fact, since nothing is paid up front, it is natural to admit that a forward contract is worthless at its initiation. We adopt the following formal definition of a forward contract.

Definition 9.2.1 Let us fix $0\le t\le T\le T^*$. A *forward contract* written at time t on a time T contingent claim X is represented by the time T contingent claim $G_T = X - F_X(t,T)$ that satisfies the following conditions: (a) $F_X(t,T)$ is a $\mathcal{F}_t$-measurable random variable; (b) the arbitrage price at time t of a contingent claim G_T equals zero, i.e., $\pi_t(G_T)=0$.

The random variable $F_X(t,T)$ is referred to as the *forward price* of a contingent claim X at time t for the settlement date T. The contingent claim X may be defined in particular as a preassigned amount of the underlying financial asset to be delivered at the settlement date. For instance, if the underlying asset of a forward contract is one share of a stock S, then clearly $X=S_T$. Similarly, if the asset to be delivered at time T is a zero-coupon bond of maturity $U\ge T$, we have $X=B(T,U)$. Note that both S_T and $B(T,U)$ are attainable contingent claims in our market model. The following well-known result expresses the forward price of a claim X in terms of its arbitrage price $\pi_t(X)$ and the price $B(t,T)$ of a zero-coupon bond which matures at time T.

Lemma 9.2.1 *The forward price $F_X(t,T)$ at time $t\le T$, for the settlement date T, of an attainable contingent claim X equals*

$$F_X(t,T) = \frac{\mathbf{E}_{\mathbf{P}^*}(XB_T^{-1}\,|\,\mathcal{F}_t)}{\mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t)} = \frac{\pi_t(X)}{B(t,T)}. \tag{9.21}$$

Proof. It is sufficient to observe that

$$\begin{aligned}
\pi_t(G_T) &= B_t\,\mathbf{E}_{\mathbf{P}^*}(G_T B_T^{-1}\,|\,\mathcal{F}_t)\\
&= B_t\Big(\mathbf{E}_{\mathbf{P}^*}(XB_T^{-1}\,|\,\mathcal{F}_t) - F_X(t,T)\mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t)\Big) = 0,
\end{aligned}$$

where the last equality follows by condition (b) of Definition 9.2.1. This proves the first equality; the second follows immediately from (9.18)–(9.20). □

Let us examine the two typical cases of forward contracts mentioned above. If the underlying asset for delivery at time T is a zero-coupon bond of maturity $U\ge T$, then (9.21) becomes

$$F_{B(T,U)}(t,T) = \frac{B(t,U)}{B(t,T)},\quad \forall\, t\in[0,T]. \tag{9.22}$$

On the other hand, the forward price of a stock S (S stands hereafter for S^i for some i) equals

$$F_{S_T}(t,T) = \frac{S_t}{B(t,T)}, \quad \forall\, t \in [0,T]. \tag{9.23}$$

For the sake of brevity, we shall write $F_B(t,U,T)$ and $F_S(t,T)$ instead of $F_{B(T,U)}(t,T)$ and $F_{S_T}(t,T)$, respectively. More generally, for any tradable asset Z, we write $F_Z(t,T)$ to denote the forward price of the asset – that is, $F_Z(t,T) = Z_t/B(t,T)$ for $t \in [0,T]$.

9.2.2 Forward Martingale Measure

To the best of our knowledge, within the framework of arbitrage valuation of interest rate derivatives, the method of a forward risk adjustment was pioneered under the name of a *forward risk-adjusted process* in Jamshidian (1987) (the corresponding equivalent change of probability measure was then used by Jamshidian (1989a) in the Gaussian framework). The formal definition of a forward probability measure was explicitly introduced in Geman (1989) under the name of *forward neutral probability.* In particular, Geman observed that the forward price of any financial asset follows a (local) martingale under the forward neutral probability associated with the settlement date of a forward contract. Most results in this section do not rely on specific assumptions imposed on the dynamics of bond and stock prices. We assume that we are given an arbitrage-free family $B(t,T)$ of bond prices and the related savings account B. Note that by assumption, $0 < B(0,T) = \mathbf{E}_{\mathbf{P}^*}(B_T^{-1}) < \infty$.

Definition 9.2.2 A probability measure $\mathbf{P}_T$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbf{P}^*$, with the Radon-Nikodým derivative given by the formula

$$\frac{d\mathbf{P}_T}{d\mathbf{P}^*} = \frac{B_T^{-1}}{\mathbf{E}_{\mathbf{P}^*}(B_T^{-1})} = \frac{1}{B_T B(0,T)}, \quad \mathbf{P}^*\text{-a.s.}, \tag{9.24}$$

is called the *forward martingale measure* (or briefly, the *forward measure*) for the settlement date T.

Notice that the above Radon-Nikodým derivative, when restricted to the σ-field $\mathcal{F}_t$, satisfies for every $t \in [0,T]$

$$\eta_t \stackrel{\text{def}}{=} \frac{d\mathbf{P}_T}{d\mathbf{P}^*}_{\big| \mathcal{F}_t} = \mathbf{E}_{\mathbf{P}^*}\Big(\frac{1}{B_T B(0,T)} \Big| \mathcal{F}_t \Big) = \frac{B(t,T)}{B_t B(0,T)}.$$

When the bond price is governed by (9.17), an explicit representation for the density process η_t is available. Namely, we have

$$\eta_t = \exp\Big(\int_0^t b(u,T)\cdot dW_u^* - \frac{1}{2}\int_0^t |b(u,T)|^2\, du \Big). \tag{9.25}$$

In other words, $\eta_t = \mathcal{E}_t(U^T)$, where $U_t^T = \int_0^t b(u,T)\cdot dW_u^*$. Furthermore, the process W^T given by the formula

$$W_t^T = W_t^* - \int_0^t b(u,T)\, du, \quad \forall\, t \in [0,T], \tag{9.26}$$

follows a standard Brownian motion under the forward measure $\mathbf{P}_T$. We shall sometimes refer to W^T as the *forward Brownian motion* for the date T. The next result shows that the forward price of a European contingent claim X which settles at time T can be easily expressed in terms of the conditional expectation under the forward measure $\mathbf{P}_T$.

Lemma 9.2.2 *The forward price at t for the date T of an attainable contingent claim X which settles at time T equals*

$$F_X(t,T) = \mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T], \tag{9.27}$$

provided that X is $\mathbf{P}_T$-integrable. In particular, the forward price process $F_X(t,T)$, $t \in [0,T]$, follows a martingale under the forward measure $\mathbf{P}_T$.

Proof. The Bayes rule yields

$$\mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t) = \frac{\mathbf{E}_{\mathbf{P}^*}(\eta_T X \,|\, \mathcal{F}_t)}{\mathbf{E}_{\mathbf{P}^*}(\eta_T \,|\, \mathcal{F}_t)} = \mathbf{E}_{\mathbf{P}^*}(\eta_T \eta_t^{-1} X \,|\, \mathcal{F}_t), \tag{9.28}$$

where

$$\eta_T = \frac{d\mathbf{P}_T}{d\mathbf{P}^*} = \frac{1}{B_T B(0,T)}$$

and $\eta_t = \mathbf{E}_{\mathbf{P}^*}(\eta_T \,|\, \mathcal{F}_t)$. Combining (9.21) with (9.28), we obtain the desired result. □

Under (9.17), (9.28) can be given a more explicit form, namely

$$\mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}^*}\Big\{ X \exp\Big(\int_t^T b(u,T) \cdot dW_u^* - \frac{1}{2} \int_t^T |b(u,T)|^2 \, du \Big) \,\Big|\, \mathcal{F}_t \Big\}.$$

The following equalities:

$$F_B(t,T,U) = \mathbf{E}_{\mathbf{P}_T}(B(T,U) \,|\, \mathcal{F}_t), \quad \forall\, 0 < t \le T \le U \le T^*,$$

and

$$F_S(t,T) = \mathbf{E}_{\mathbf{P}_T}(S_T \,|\, \mathcal{F}_t) \quad \forall t \in [0,T],$$

are immediate consequences of the last lemma. More generally, the relative price of any traded security (which pays no coupons or dividends) follows a local martingale under the forward probability measure $\mathbf{P}_T$, provided that the price of a bond which matures at time T is taken as a numeraire. The next lemma establishes a version of the risk-neutral valuation formula that is tailored to the stochastic interest rate framework.

Lemma 9.2.3 *The arbitrage price of an attainable contingent claim X which settles at time T is given by the formula*

$$\pi_t(X) = B(t,T)\, \mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{9.29}$$

Proof. Equality (9.29) is an immediate consequence of (9.21) combined with (9.27). For a more direct proof, note that the price $\pi_t(X)$ can be re-expressed as follows

$$\pi_t(X) = B_t\, \mathbf{E}_{\mathbf{P}^*}(X B_T^{-1} \,|\, \mathcal{F}_t) = B_t B(0,T)\, \mathbf{E}_{\mathbf{P}^*}(\eta_T X \,|\, \mathcal{F}_t).$$

An application of the Bayes rule yields

$$\begin{aligned} \pi_t(X) &= B_t B(0,T)\, \mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t)\, \mathbf{E}_{\mathbf{P}^*}(\eta_T \,|\, \mathcal{F}_t) \\ &= B_t B(0,T)\, \mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t)\, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{1}{B_T B(0,T)} \,\Big|\, \mathcal{F}_t \Big) \\ &= B(t,T)\, \mathbf{E}_{\mathbf{P}_T}(X \,|\, \mathcal{F}_t), \end{aligned}$$

as expected. □

The following corollary deals with a contingent claim which settles at time $U \neq T$. Our aim is to express the value of this claim in terms of the forward measure for the date T.

Corollary 9.2.1 *Let X be an arbitrary attainable contingent claim which settles at time U.* (i) *If $U \le T$, then the price of X at time $t \le U$ equals*

$$\pi_t(X) = B(t,T)\, \mathbf{E}_{\mathbf{P}_T}(X B^{-1}(U,T) \,|\, \mathcal{F}_t). \tag{9.30}$$

(ii) *If $U \ge T$ and X is $\mathcal{F}_T$-measurable, then for any $t \le U$ we have*

$$\pi_t(X) = B(t,T)\, \mathbf{E}_{\mathbf{P}_T}(X B(T,U) \,|\, \mathcal{F}_t). \tag{9.31}$$

Proof. Both equalities are intuitively clear. In case (i), we invest at time U a $\mathcal{F}_U$-measurable payoff X in zero-coupon bonds which mature at time T. For the second case, observe that in order to replicate a $\mathcal{F}_T$-measurable claim X at time U, it is enough to purchase, at time T, X units of a zero-coupon bond maturing at time U. Both strategies are manifestly self-financing, and thus the result follows.

An alternative way of deriving (9.30) is to observe that since X is $\mathcal{F}_U$-measurable, we have for every $t \in [0, U]$

$$\begin{aligned} B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{X}{B_T B(U,T)} \,\Big|\, \mathcal{F}_t \Big) &= B_t \, \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{X}{B_U B(U,T)} \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_U}{B_T} \,\Big|\, \mathcal{F}_U \Big) \,\Big|\, \mathcal{F}_t \Big\} \\ &= B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{X}{B_U} \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

This means that the claim X that settles at time U has, at any date $t \in [0, U]$, an identical arbitrage price to the claim $Y = XB^{-1}(U,T)$ that settles at time T. Formula (9.30) now follows from relation (9.29) applied to the claim Y. Similarly, to prove the second formula, we observe that since X is $\mathcal{F}_T$-measurable, we have for $t \in [0, T]$

$$\begin{aligned} B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{X}{B_U} \,\Big|\, \mathcal{F}_t \Big) &= B_t \, \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{X}{B_T} \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_T}{B_U} \,\Big|\, \mathcal{F}_T \Big) \,\Big|\, \mathcal{F}_t \Big\} \\ &= B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{XB(T,U)}{B_T} \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

We conclude once again that a $\mathcal{F}_T$-measurable claim X which settles at time $U \geq T$ is essentially equivalent to a claim $Y = XB(T,U)$ which settles at time T. □

9.3 Gaussian HJM Model

In this section, we assume that the volatility σ of the forward rate is deterministic; such a case will be referred to as the Gaussian HJM model. This terminology refers to the fact that the forward rate $f(t,T)$ and the spot rate r_t have Gaussian probability laws under the martingale measure $\mathbf{P}^*$ (cf. (9.14)–(9.15)). Our aim is to show that the arbitrage price of any attainable interest rate-sensitive claim can be evaluated by each of the following procedures.

(I) We start with arbitrary dynamics of the forward rate such that condition (M.1) (or (M.2)) is satisfied. We then find a martingale measure $\mathbf{P}^*$, and apply the risk-neutral valuation formula.

(II) We assume instead that the underlying probability measure $\mathbf{P}$ is actually the spot (forward, respectively) martingale measure. In other words, we assume that condition (M.2) (condition (M.1), respectively) is satisfied, with the process λ (h, respectively) equal to zero.

Since both procedures give the same valuation results, we conclude that the specification of the risk premium is not relevant in the context of arbitrage valuation of interest rate-sensitive derivatives in the Gaussian HJM framework. Put another way, when the coefficient σ is deterministic, we can assume, without loss of generality, that $\alpha(t,T) = \sigma(t,T)\sigma^*(t,T)$. Observe that by combining the last equality with (9.6), we find immediately that $a(t,T) = f(t,t) = r_t$. To formulate a result which justifies the considerations above, we need to introduce some additional notation. Let a function α_0 be given by (9.12), with $\lambda = 0$, i.e.,

$$\alpha_0(t,T) = \sigma(t,T) \cdot \sigma^*(t,T), \quad \forall\, t \in [0,T], \tag{9.32}$$

so that

$$\alpha_0^*(t,T) = \int_t^T \alpha_0(u,T)\, du = \frac{1}{2}\, |\, \sigma^*(t,T)\, |^2.$$

Finally, we denote by $B_0(t,T)$ the bond price specified by the equality

$$B_0(t,T) = \exp\Big(-\int_t^T f_0(t,u)\,du\Big),$$

where the dynamics under $\mathbf{P}$ of the instantaneous forward rate $f_0(t,T)$ are

$$f_0(t,T) = f(0,T) + \int_0^t \alpha_0(u,T)\,du + \int_0^t \sigma(u,T)\cdot dW_u.$$

Let us put

$$Z^*(t,\mathcal{T}) = B(t,\mathcal{T})/B_t, \quad Z_0^*(t,\mathcal{T}) = B_0(t,\mathcal{T})/B_t,$$

where $\mathcal{T} = (T_1,\dots,T_k)$ is any finite collection of maturity dates.

Proposition 9.3.1 *Suppose that the coefficient σ is deterministic. Then for any choice $\mathcal{T}$ of maturity dates and of a spot martingale measure $\mathbf{P}^*$, the probability law of the process $Z^*(t,\mathcal{T})$, $t\in[0,T^*]$, under the martingale measure $\mathbf{P}^*$ coincides with the probability law of the process $Z_0^*(t,\mathcal{T})$, $t\in[0,T^*]$, under $\mathbf{P}$.*

Proof. The assertion easily follows by Girsanov's theorem. Indeed, for any fixed $0<T\le T^*$, the dynamics of $Z^*(t,T)$ under a spot martingale measure $\mathbf{P}^* = \mathbf{P}^\lambda$ are

$$dZ^*(t,T) = -Z^*(t,T)\,\sigma^*(t,T)\cdot dW_t^\lambda, \tag{9.33}$$

where W^λ follows a standard Brownian motion under $\mathbf{P}^\lambda$. On the other hand, under $\mathbf{P}$ we have

$$dZ_0^*(t,T) = -Z_0^*(t,T)\,\sigma^*(t,T)\cdot dW_t. \tag{9.34}$$

Moreover, for every $0<T\le T^*$

$$Z^*(0,T) = B(0,T) = e^{-\int_0^T f(0,u)\,du} = B_0(0,T) = Z_0^*(0,T).$$

Since σ is deterministic, the assertion follows easily from (9.33)–(9.34). □

Example 9.3.1 Let us assume that the volatility of each forward rate is constant, i.e., independent of the maturity date and the level of the forward interest rate. Taking $d=1$, we thus have $\sigma(t,T)=\sigma$ for a strictly positive constant $\sigma>0$. By virtue of (9.14), the dynamics of the forward rate process $f(t,T)$ under the martingale measure are given by the expression

$$df(t,T) = \sigma^2(T-t)\,dt + \sigma\,dW_t^*, \tag{9.35}$$

so that the dynamics of the bond price $B(t,T)$ are

$$dB(t,T) = B(t,T)\big(r_t\,dt - \sigma(T-t)\,dW_t^*\big),$$

where the short-term rate of interest r satisfies

$$r_t = f(0,t) + \tfrac{1}{2}\sigma^2 t^2 + \sigma\,W_t^*.$$

It follows from the last formula that

$$dr_t = \big(f_T(0,t) + \sigma^2 t\big)\,dt + \sigma\,dW_t^*.$$

Since this agrees with the general form of the continuous-time Ho-Lee model, we conclude that in the HJM framework, the Ho-Lee model corresponds to the constant volatility of forward rates. Dynamics (9.35) make apparent that the only possible movements of the yield curve in the Ho-Lee model are parallel shifts; that is, all rates along the yield curve fluctuate in the same way. The price

$B(t,T)$ of a bond maturing at T equals (it follows from (9.36) that bond prices of all maturities are perfectly correlated)

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\Big(-\tfrac{1}{2}(T-t)Tt\sigma^2 - (T-t)\sigma W_t^*\Big). \tag{9.36}$$

It can also be expressed in terms of r, namely

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\Big((T-t)f(0,t) - \tfrac{1}{2}t(T-t)^2\sigma^2 - (T-t)r_t\Big).$$

Example 9.3.2 It is a conventional wisdom that forward rates of longer maturity fluctuate less than rates of shorter maturity. To account for this feature in the HJM framework, we assume now that the volatility of a forward rate is a decreasing function of the time to its effective date. For instance, we may assume that the volatility structure is exponentially dampened:

$$\sigma(t,T) = \sigma e^{-\gamma(T-t)}, \quad \forall\, t \in [0,T],$$

for strictly positive real numbers σ, $\gamma > 0$. Then $\sigma^*(t,T)$ equals

$$\sigma^*(t,T) = \int_t^T \sigma e^{-\gamma(u-t)}\, du = -\sigma\gamma^{-1}\left(e^{-\gamma(T-t)} - 1\right), \tag{9.37}$$

and consequently

$$df(t,T) = -\sigma^2\gamma^{-1}\, e^{-\gamma(T-t)}\big(e^{-\gamma(T-t)} - 1\big)\, dt + \sigma e^{-\gamma(T-t)}\, dW_t^*. \tag{9.38}$$

It is thus clear that for any maturity T, the bond price $B(t,T)$ satisfies

$$dB(t,T) = B(t,T)\Big(r_t dt + \sigma\gamma^{-1}\big(e^{-\gamma(T-t)} - 1\big)\, dW_t^*\Big). \tag{9.39}$$

Substituting (9.37) into (9.15), we obtain

$$r_t = f(0,t) - \int_0^t \sigma^2\gamma^{-1}\, e^{-\gamma(t-u)}\big(e^{-\gamma(t-u)} - 1\big)\, du + \int_0^t \sigma e^{-\gamma(t-u)}\, dW_u^*,$$

so that, as in the previous example, the negative values of the short-term interest rate are not excluded. Denoting

$$m(t) = f(0,t) + \frac{\sigma^2}{2\gamma^2}\left(1 - e^{-\gamma t}\right)^2,$$

we arrive at the following formula

$$r_t = m(t) + \int_0^t \sigma e^{-\gamma(t-u)}\, dW_u^*,$$

so that finally

$$dr_t = (a(t) - \gamma r_t)dt + \sigma dW_t^*, \tag{9.40}$$

where $a(t) = \gamma m(t) + m'(t)$. This means that (9.37) leads to a generalized version of Vasicek's model (cf. (8.15)). Notice that in the present framework, the perfect fit of the initial term structure is trivially achieved.

9.4 Model of LIBOR Rates

The Heath-Jarrow-Morton methodology of term structure modelling presented in the previous section is based on the arbitrage-free dynamics of instantaneous, continuously compounded forward rates. The assumption that instantaneous rates exist is not always convenient, since it requires a certain degree of smoothness with respect to the *tenor* (i.e., maturity) of bond prices and their volatilities. An alternative construction of an arbitrage-free family of bond prices, making no reference to the instantaneous, continuously compounded rates, is in some circumstances more suitable.

By definition, the forward δ-LIBOR rate[3] $L(t,T)$ for the future date $T \le T^* - \delta$ prevailing at time t is given by the conventional market formula

$$1 + \delta L(t,T) = F_B(t,T,T+\delta), \quad \forall\, t \in [0,T]. \tag{9.41}$$

The forward LIBOR rate $L(t,T)$ represents the add-on rate prevailing at time t over the future time interval $[T, T+\delta]$. We can also re-express $L(t,T)$ directly in terms of bond prices, as for any $T \in [0, T^* - \delta]$, we have

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}, \quad \forall\, t \in [0,T]. \tag{9.42}$$

In particular, the initial term structure of forward LIBOR rates satisfies

$$L(0,T) = f_s(0,T,T+\delta) = \delta^{-1}\Big(\frac{B(0,T)}{B(0,T+\delta)} - 1\Big). \tag{9.43}$$

Under the forward measure $\mathbf{P}_{T+\delta}$, we have

$$dL(t,T) = \delta^{-1} F_B(t,T,T+\delta)\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta},$$

where $W_t^{T+\delta}$ and $\mathbf{P}_{T+\delta}$ are yet unspecified. This means that $L(\cdot,T)$ solves the equation

$$dL(t,T) = \delta^{-1}(1 + \delta L(t,T))\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta}, \tag{9.44}$$

subject to the initial condition (9.43). Suppose that forward LIBOR rates $L(t,T)$ are strictly positive. Then formula (9.44) can be rewritten as follows

$$dL(t,T) = L(t,T)\, \lambda(t,T) \cdot dW_t^{T+\delta}, \tag{9.45}$$

where for any $t \in [0,T]$

$$\lambda(t,T) = \frac{1 + \delta L(t,T)}{\delta L(t,T)}\, \gamma(t,T,T+\delta). \tag{9.46}$$

The construction of a model of forward LIBOR rates relies on the following assumptions.

(LR.1) For any maturity $T \le T^* - \delta$, we are given a $\mathbf{R}^d$-valued, bounded, deterministic function[4] $\lambda(\cdot,T)$, which represents the volatility of the forward LIBOR rate process $L(\cdot,T)$.

(LR.2) We assume a strictly decreasing and strictly positive initial term structure $P(0,T), T \in [0,T^*]$, and thus an initial term structure $L(0,T)$ of forward LIBOR rates

$$L(0,T) = \delta^{-1}\Big(\frac{P(0,T)}{P(0,T+\delta)} - 1\Big), \quad \forall\, T \in [0, T^* - \delta]. \tag{9.47}$$

[3]In practice, several types of LIBOR rates occur, e.g., 3-month LIBOR and 6-month LIBOR. For ease of exposition, we consider a fixed maturity δ.

[4]Volatility λ could follow a stochastic process; we deliberately focus here on a *lognormal model* of forward LIBOR rates in which λ is deterministic.

9.4.1 Discrete-tenor Case

We start by studying a *discrete-tenor* version of a lognormal model of forward LIBOR rates. It should be stressed that a so-called discrete-tenor model still possesses certain continuous-time features; for instance, forward LIBOR rates follow continuous-time processes. For ease of notation, we shall assume that the horizon date T^* is a multiple of δ, say $T^* = M\delta$ for a natural M. We shall focus on a finite number of dates, $T^*_m = T^* - m\delta$ for $m = 1, \dots, M-1$. The construction is based on backward induction, therefore we start by defining the forward LIBOR rate with the longest maturity, $L(t, T^*_1)$. We postulate that the rate $L(t, T^*_1)$ is governed under the probability measure $\mathbf{P}$ by the following SDE

$$dL(t, T^*_1) = L(t, T^*_1)\, \lambda(t, T^*_1) \cdot dW_t, \tag{9.48}$$

with the initial condition

$$L(0, T^*_1) = \delta^{-1}\Big(\frac{P(0, T^*_1)}{P(0, T^*)} - 1\Big). \tag{9.49}$$

Put another way, we postulate that for every $t \in [0, T^*_1]$

$$L(t, T^*_1) = \delta^{-1}\Big(\frac{P(0, T^*_1)}{P(0, T^*)} - 1\Big)\, \mathcal{E}_t \left(\int_0^{\cdot} \lambda(u, T^*_1) \cdot dW_u\right). \tag{9.50}$$

Since $P(0, T^*_1) > P(0, T^*)$, it is clear that $L(t, T^*_1)$ follows a strictly positive continuous martingale under $\mathbf{P}$. Also, for any fixed $t \le T^*_1$, the random variable $L(t, T^*_1)$ has a lognormal probability law under $\mathbf{P}$. The next step is to define the forward LIBOR rate for the date T^*_2,

$$\gamma(t, T^*_1, T^*) = \frac{\delta L(t, T^*_1)}{1 + \delta L(t, T^*_1)}\, \lambda(t, T^*_1), \quad \forall\, t \in [0, T^*_1]. \tag{9.51}$$

Given that the volatility $\gamma(t, T^*_1, T^*)$ is determined by (9.51), the forward process $F_B(t, T^*_1, T^*)$ is known to solve, under $\mathbf{P}$

$$dF_B(t, T^*_1, T^*) = F_B(t, T^*_1, T^*)\, \gamma(t, T^*_1, T^*) \cdot dW_t \tag{9.52}$$

and the initial condition is $F_B(0, T^*_1, T^*) = P(0, T^*_1)/P(0, T^*)$. The forward process $F_B(t, T^*_1, T^*)$ is a continuous martingale under $\mathbf{P}$, since the volatility $\gamma(t, T^*_1, T^*)$ follows a bounded process. We introduce a d-dimensional process $W^{T^*_1}$, which corresponds to the date T^*_1, by setting

$$W^{T^*_1}_t = W_t - \int_0^t \gamma(u, T^*_1, T^*)\, du, \quad \forall\, t \in [0, T^*_1]. \tag{9.53}$$

Due to the boundedness of the process $\gamma(t, T^*_1, T^*)$, the existence of the process $W^{T^*_1}$ and of the associated probability measure $\mathbf{P}_{T^*_1}$, equivalent to $\mathbf{P}$, under which the process $W^{T^*_1}$ follows a Browinian motion, and which is given by the formula

$$\frac{d\mathbf{P}_{T^*_1}}{d\mathbf{P}} = \mathcal{E}_{T^*_1} \left(\int_0^{\cdot} \gamma(u, T^*_1, T^*) \cdot dW_u\right), \quad \mathbf{P}\text{-a.s.}, \tag{9.54}$$

is trivial. The process $W^{T^*_1}$ may be interpreted as the forward Brownian motion for the date T^*_1. We are in a position to specify the dynamics of the forward LIBOR rate for the date T^*_2 under the forward probability measure $\mathbf{P}_{T^*_1}$. Analogously to (9.48), we set

$$dL(t, T^*_2) = L(t, T^*_2)\, \lambda(t, T^*_2) \cdot dW^{T^*_1}_t, \tag{9.55}$$

with the initial condition

$$L(0, T^*_2) = \delta^{-1}\Big(\frac{P(0, T^*_2)}{P(0, T^*_1)} - 1\Big). \tag{9.56}$$

Solving equation (9.55) and comparing with (9.46) for $T = T_2^*$, we obtain

$$\gamma(t, T_2^*, T_1^*) = \frac{\delta L(t, T_2^*)}{1 + \delta L(t, T_2^*)}\, \lambda(t, T_2^*), \quad \forall\, t \in [0, T_2^*]. \tag{9.57}$$

To find $\gamma(t, T_2^*, T^*)$, we make use of the relationship

$$\gamma(t, T_2^*, T_1^*) = \gamma(t, T_2^*, T^*) - \gamma(t, T_1^*, T^*), \quad \forall\, t \in [0, T_2^*]. \tag{9.58}$$

Given the process $\gamma(t, T_2^*, T_1^*)$, we can define the pair $(W^{T_2^*}, \mathbf{P}_{T_2^*})$ corresponding to the date T_2^* and so forth. By working backwards to the first relevant date $T_{M-1}^* = \delta$, we construct a family of forward LIBOR rates $L(t, T_m^*), m = 1, \ldots, M-1$. Notice that the lognormal probability law of every process $L(t, T_m^*)$ under the corresponding forward probability measure $\mathbf{P}_{T_{m-1}^*}$ is ensured. Indeed, for any $m = 1, \ldots, M-1$, we have

$$dL(t, T_m^*) = L(t, T_m^*)\, \lambda(t, T_m^*) \cdot dW_t^{T_{m-1}^*}, \tag{9.59}$$

where $W^{T_{m-1}^*}$ is a standard Brownian motion under $\mathbf{P}_{T_{m-1}^*}$. This completes the derivation of the *lognormal model* of forward LIBOR rates in a discrete-tenor framework. Note that in fact we have simultaneously constructed a family of forward LIBOR rates and a family of associated forward processes. Let us now examine the existence and uniqueness of the implied savings account, in a discrete-time setting. The implied savings account is thus seen as a discrete-time process, B_t^*, $t = 0, \delta, \ldots, T^* = M\delta$. Intuitively, the value B_t^* of a savings account at time t can be interpreted as the cash amount accumulated up to time t by rolling over a series of zero-coupon bonds with the shortest maturities available. To find the process B^* in a discrete-tenor framework, we do not have to specify explicitly all bond prices; the knowledge of forward bond prices is sufficient. Indeed,

$$F_B(t, T_j, T_{j+1}) = \frac{F_B(t, T_j, T^*)}{F_B(t, T_{j+1}, T^*)} = \frac{B(t, T_j)}{B(t, T_{j+1})},$$

where we write $T_j = j\delta$. This in turn yields, upon setting $t = T_j$

$$F_B(T_j, T_j, T_{j+1}) = 1/B(T_j, T_{j+1}), \tag{9.60}$$

so that the price $B(T_j, T_{j+1})$ of a one-period bond is uniquely specified for every j. Though the bond that matures at time T_j does not physically exist after this date, it seems justifiable to consider $F_B(T_j, T_j, T_{j+1})$ as its forward value at time T_j for the next future date T_{j+1}. In other words, the spot value at time T_{j+1} of one cash unit received at time T_j equals $B^{-1}(T_j, T_{j+1})$. The discrete-time savings account B^* thus equals

$$B_{T_k}^* = \prod_{j=1}^{k} F_B\big(T_{j-1}, T_{j-1}, T_j\big) = \Big(\prod_{j=1}^{k} B\big(T_{j-1}, T_j\big)\Big)^{-1}$$

for $k = 0, \ldots, M-1$, since by convention $B_0^* = 1$. Note that

$$F_B\big(T_j, T_j, T_{j+1}\big) = 1 + \delta L(T_j, T_{j+1}) > 1$$

for $j = 1, \ldots, M-1$, and since

$$B_{T_{j+1}}^* = F_B(T_j, T_j, T_{j+1})\, B_{T_j}^*,$$

we find that $B_{T_{j+1}}^* > B_{T_j}^*$ for every $j = 0, \ldots, M-1$. We conclude that the implied savings account B^* follows a strictly increasing discrete-time process. We define the probability measure $\mathbf{P}^* \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ by the formula

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = B_{T^*}^* P(0, T^*), \quad \mathbf{P}\text{-a.s.} \tag{9.61}$$

The probability measure $\mathbf{P}^*$ appears to be a plausible candidate for a spot martingale measure. Indeed, if we set

$$B(T_l, T_k) = \mathbf{E}_{\mathbf{P}^*}(B_{T_l}^*/B_{T_k}^* \,|\, \mathcal{F}_{T_l}) \tag{9.62}$$

for every $l \le k \le M$, then in the case of $l = k-1$, equality (9.62) coincides with (9.60).

9.4.2 Continuous-tenor Case

By a *continuous-tenor model* we mean a model in which all forward LIBOR rates $L(t,T)$ with $T \in [0,T^*]$ are specified. Given the discrete-tenor skeleton constructed in the previous section, it is sufficient to fill the gaps between the discrete dates to produce a continuous-tenor model. To construct a model in which each forward LIBOR rate $L(t,T)$ follows a lognormal process under the forward measure for the date $T+\delta$, we shall proceed by backward induction.

First step. We construct a discrete-tenor model using the previously described method.

Second step. We focus on the forward rates and forward measures for maturities $T \in (T_1^*, T^*)$. In this case we do not have to take into account the forward LIBOR rates $L(t,T)$ (such rates do not exist in the present model after the date T_1^*). From the previous step, we are given the values $B^*_{T_1^*}$ and $B^*_{T^*}$ of a savings account. It is important to observe that $B^*_{T_1^*}$ and $B^*_{T^*}$ are $\mathcal{F}_{T_1^*}$-measurable random variables. We start by defining a spot martingale measure $\mathbf{P}^*$ associated with the discrete-tenor model, using formula (9.61). Since the model needs to match a given initial term structure, we search for an increasing function $\alpha : [T_1^*, T^*] \to [0,1]$ such that $\alpha(T_1^*) = 0$, $\alpha(T^*) = 1$, and the process

$$\ln B_t^* = (1-\alpha(t)) \ln B^*_{T_1^*} + \alpha(t) \ln B^*_{T^*}, \quad \forall\, t \in [T_1^*, T^*],$$

satisfies $P(0,t) = \mathbf{E}_{\mathbf{P}^*}(1/B_t^*)$ for every $t \in [T_1^*, T^*]$. Since $0 < B^*_{T_1^*} < B^*_{T^*}$, and $P(0,t)$, $t \in [T_1^*, T^*]$, is assumed to be a strictly decreasing function, a function α with desired properties exists and is unique.

Third step. In the previous step, we have constructed the savings account B_T^* for every $T \in [T_1^*, T^*]$. Hence the forward measure for any date $T \in (T_1^*, T^*)$ can be defined by setting

$$\frac{d\mathbf{P}_T}{d\mathbf{P}^*} = \frac{1}{B_T^* P(0,T)}, \quad \mathbf{P}^*\text{-a.s.} \tag{9.63}$$

Combining (9.63) with (9.61), we obtain

$$\frac{d\mathbf{P}_T}{d\mathbf{P}} = \frac{d\mathbf{P}_T}{d\mathbf{P}^*}\frac{d\mathbf{P}^*}{d\mathbf{P}} = \frac{B^*_{T^*} P(0,T^*)}{B_T^* P(0,T)}, \quad \mathbf{P}\text{-a.s.},$$

for every $T \in [T_1^*, T^*]$, so that

$$\frac{d\mathbf{P}_T}{d\mathbf{P}}_{\mid \mathcal{F}_t} = \mathbf{E}_{\mathbf{P}}\Big(\frac{B^*_{T^*} P(0,T^*)}{B_T^* P(0,T)} \,\Big|\, \mathcal{F}_t \Big), \quad \forall\, t \in [0,T].$$

Exponential representation of the above martingale – that is, the formula

$$\frac{d\mathbf{P}_T}{d\mathbf{P}}_{\mid \mathcal{F}_t} = \frac{P(0,T^*)}{P(0,T)}\, \mathcal{E}_t \Big(\int_0^{\cdot} \gamma(u,T,T^*) \cdot dW_u^* \Big), \quad \forall\, t \in [0,T],$$

yields the forward volatility $\gamma(t,T,T^*)$ for any maturity $T \in (T_1^*, T^*)$. This in turn allows us to define also the associated $\mathbf{P}_T$-Brownian motion W^T. Given the forward measure $\mathbf{P}_T$ and the associated Brownian motion W^T, we define the forward LIBOR rate process $L(t, T-\delta)$ for arbitrary $T \in (T_1^*, T^*)$ by setting (cf. (9.48)–(9.49))

$$dL(t,T_\delta) = L(t,T_\delta)\, \lambda(t,T_\delta) \cdot dW_t^T,$$

where $T_\delta = T - \delta$, with initial condition

$$L(0,T_\delta) = \delta^{-1}\Big(\frac{P(0,T_\delta)}{P(0,T)} - 1 \Big).$$

Finally, we set (cf. (9.51))

$$\gamma(t,T_1^*,T^*) = \frac{\delta L(t,T_1^*)}{1+\delta L(t,T_1^*)}\, \lambda(t,T_1^*), \quad \forall\, t \in [0,T_1^*],$$

hence we are in a position to introduce also the forward measure $\mathbf{P}_T$ for the date $T = T_1^*$. To define the forward measure $\mathbf{P}_U$ and the corresponding Brownian motion W^U for any maturity $U \in (T_2^*, T_1^*)$, we put (cf. (9.46))

$$\gamma(t, U, T) = \gamma(t, T_\delta, T) = \frac{\delta L(t, T_\delta)}{1 + \delta L(t, T_\delta)} \lambda(t, T_\delta), \quad \forall\, t \in [0, T_\delta],$$

where $U = T_\delta$ so that $T = U + \delta$ belongs to (T_1^*, T^*). The coefficient $\gamma(t, U, T^*)$ is found from the relationship

$$\gamma(t, U, T^*) = \gamma(t, U, T) - \gamma(t, T, T^*), \quad \forall\, t \in [0, U].$$

Proceeding by backward induction, we are able to specify a fully continuous-time family $L(t, T)$ of forward LIBOR rates with desired properties. mild regularity assumptions, this system can be solved recursively.

9.5 Model of Forward Swap Rates

Let us now describe briefly the model of forward swap rates developed by Jamshidian (1997). Generally speaking, the goal is to construct a lognormal model of the term structure of forward swap rates with a fixed end date. Let us observe that the lognormal model of forward LIBOR rates of Section 9.4 and the lognormal model of forward swap rates introduced below are inconsistent with each other.

Formally, for a given collection of dates $T_j = j\delta,\ j = 1, \dots, M,$ we consider a forward start fixed-for-floating interest rate swap (see Sect. 11.1) which starts at time T_j and has $M - j$ accrual periods. The *forward swap rate* $\kappa(t, T_j, M - j)$ – that is, that value of a fixed rate κ for which such a swap is worthless at time t – is known to be given by the expression (cf. (11.4))

$$\kappa(t, T_j, M - j) = (B(t, T_j) - B(t, T_M)) \Big(\delta \sum_{l=j+1}^{M} B(t, T_l) \Big)^{-1}$$

for every $t \in [0, T_j]$ and every $j = 1, \dots, M - 1$. We consider a family of forward swap rates

$$\tilde{\kappa}(t, T_j) = \kappa(t, T_j, M - j), \quad \forall\, t \in [0, T_j],$$

for $j = 1, \dots, M - 1$; that is, the underlying swaps differ in length, but have a common expiry date, $T^* = T_M$. Let us denote $T_k^* = T^* - k\delta$, in particular $T_0^* = T^*$. The forward swap rate for the date T_m^* equals

$$\tilde{\kappa}(t, T_m^*) = \frac{B(t, T_m^*) - B(t, T^*)}{\delta\big(B(t, T_{m-1}^*) + \ldots + B(t, T^*)\big)}, \quad \forall\, t \in [0, T_m^*]. \tag{9.64}$$

Suppose that bond prices $B(t, T_m^*)$, $m = 0, \dots, M - 1$, are given on a probability space $(\Omega, \mathbf{F}, \mathbf{P})$ equipped with a Brownian motion W. We find it convenient to assume that $\mathbf{P} = \mathbf{P}_{T^*}$ is the forward measure for the date T^*, and $W = W^{T^*}$ is the corresponding forward Brownian motion. For any $m = 1, \dots, M - 1,$ we introduce the *coupon process* $G(m)$ by setting

$$G_t(m) = \sum_{k=0}^{m-1} B(t, T_k^*), \quad \forall\, t \in [0, T_{m-1}^*]. \tag{9.65}$$

By definition, *the forward swap measure* $\tilde{\mathbf{P}}_{T_{m-1}^*}$ for the date T_{m-1}^* is that probability measure equivalent to $\mathbf{P}$, which corresponds to the choice of the process $G(m)$ as a numeraire. In other words, for a fixed m and any $k = 0, \dots, M - 1,$ the relative bond prices

$$Z_m(t, T_k^*) \overset{\text{def}}{=} \frac{B(t, T_k^*)}{G_t(m)} = \frac{B(t, T_k^*)}{B(t, T_{m-1}^*) + \ldots + B(t, T^*)}$$

for $t \in [0, T_k^* \wedge T_m^*]$ are bound to follow a local martingale under the forward swap measure $\tilde{\mathbf{P}}_{T_{m-1}^*}$. Since obviously $G_t(1) = B(t, T^*)$, it is clear that

$$Z_1(t, T_k^*) = F_B(t, T_k^*, T^*), \quad \forall\, t \in [0, T_k^*],$$

and thus the probability measure $\tilde{\mathbf{P}}_{T^*}$ can be chosen to coincide with the forward martingale measure $\mathbf{P}_{T^*}$. More noticeably, it follows from (9.64)–(9.65) that the forward swap rate $\tilde{\kappa}(t, T_m^*)$ is also a local martingale under $\tilde{\mathbf{P}}_{T_{m-1}^*}$, since it equals

$$\tilde{\kappa}(t, T_m^*) = \delta^{-1}\big(Z_m(t, T_m^*) - Z_m(t, T^*)\big), \quad \forall\, t \in [0, T_m^*].$$

As already mentioned, our aim is to directly construct a model of forward swap rates; the underlying bond price processes will not be explicitly specified. For the sake of concreteness, we shall focus on the lognormal version of the model of forward swap rates; this restriction is not essential, however. We assume that we are given a family of bounded (deterministic) functions $\nu(\cdot, T_m^*) : [0, T_m^*] \to \mathbf{R}$, $m = 1, \dots, M-1$, which represent the volatilities of the forward swap rates. In addition, an initial term structure, represented by a family $P(0, T_m^*)$, $m = 0, \dots, M-1$, of bond prices, is known. Our goal is to construct a model of forward swap rates in such a way that

$$d\tilde{\kappa}(t, T_{m+1}^*) = \tilde{\kappa}(t, T_{m+1}^*)\nu(t, T_{m+1}^*) \cdot d\tilde{W}_t^{T_m^*} \tag{9.66}$$

for every $m = 0, \dots, M-2$, where $\tilde{W}^{T_m^*}$ is a Brownian motion under the corresponding forward swap measure $\tilde{\mathbf{P}}_{T_m^*}$. The model should be consistent with the initial term structure, meaning that

$$\tilde{\kappa}(0, T_{m+1}^*) = \frac{P(0, T_{m+1}^*) - P(0, T^*)}{\delta\big(P(0, T_m^*) + \ldots + P(0, T^*)\big)}. \tag{9.67}$$

We proceed by backward induction. The first step is to introduce the forward swap rate $\tilde{\kappa}(t, T_1^*)$ by setting (note that $\tilde{W}^{T^*} = W^{T^*} = W$)

$$d\tilde{\kappa}(t, T_1^*) = \tilde{\kappa}(t, T_1^*)\nu(t, T_1^*) \cdot d\tilde{W}_t^{T^*} \tag{9.68}$$

with the initial condition

$$\tilde{\kappa}(0, T_1^*) = \frac{P(0, T_1^*) - P(0, T^*)}{\delta P(0, T^*)}.$$

To specify the process $\tilde{\kappa}(\cdot, T_2^*)$, we need first to introduce a forward swap measure $\tilde{\mathbf{P}}_{T_1^*}$ and an associated Brownian motion $\tilde{W}^{T_1^*}$. The following auxiliary lemma is a straightforward consequence of Itô's formula.

Lemma 9.5.1 *Let G and H be real-valued adapted processes, such that $dG_t = G_t g_t \cdot dW_t$ and $dH_t = H_t h_t \cdot dW_t$. Assume that $H > -1$. Then the process $Y_t = G_t/(1 + H_t)$ satisfies*

$$dY_t = Y_t\Big(g_t - \frac{H_t h_t}{1 + H_t}\Big) \cdot \Big(dW_t - \frac{H_t h_t}{1 + H_t}\, dt\Big).$$

In the next step, our aim is to define the process $\tilde{\kappa}(\cdot, T_2^*)$. Notice that each process $Z_1(\cdot, T_k^*) = F_B(\cdot, T_k^*, T^*)$ follows a strictly positive local martingale under $\tilde{\mathbf{P}}_{T^*} = \mathbf{P}_{T^*}$; more precisely, we have

$$dZ_1(t, T_k^*) = Z_1(t, T_k^*)\gamma_1(t, T_k^*) \cdot dW_t^{T^*} \tag{9.69}$$

for some process $\gamma_1(\cdot, T_k^*)$. According to the definition of a forward swap measure, we postulate that for every k, the process

$$Z_2(t, T_k^*) = \frac{B(t, T_k^*)}{B(t, T_1^*) + B(t, T^*)} = \frac{Z_1(t, T_k^*)}{1 + Z_1(t, T_1^*)}$$

follows a local martingale under $\tilde{\mathbf{P}}_{T_1^*}$. Applying Lemma 9.5.1 to processes $G = Z_1(\cdot, T_k^*)$ and $H = Z_1(\cdot, T_1^*)$, we see that for this property to hold, it is enough to assume that the process $W^{T_1^*}$, which equals

$$\tilde{W}_t^{T_1^*} = \tilde{W}_t^{T^*} - \int_0^t \frac{Z_1(u, T_1^*)\gamma_1(u, T_1^*)}{1 + Z_1(u, T_1^*)}\, du, \quad \forall\, t \in [0, T_1^*],$$

follows a Brownian motion under $\tilde{\mathbf{P}}_{T_1^*}$ (probability measure $\tilde{\mathbf{P}}_{T_1^*}$ is yet unspecified, but can be found easily using Girsanov's theorem). Note that

$$Z_1(t, T_1^*) = \frac{B(t, T_1^*)}{B(t, T^*)} = \delta\tilde{\kappa}(t, T_1^*) + Z_1(t, T^*) = \delta\tilde{\kappa}(t, T_1^*) + 1.$$

Differentiating both sides of the last equality, we get (cf. (9.68)–(9.69))

$$Z_1(t, T_1^*)\gamma_1(t, T_1^*) = \delta\tilde{\kappa}(t, T_1^*)\nu(t, T_1^*).$$

Consequently, $\tilde{W}^{T_1^*}$ is explicitly given by the formula

$$\tilde{W}_t^{T_1^*} = \tilde{W}_t^{T^*} - \int_0^t \frac{\delta\tilde{\kappa}(u, T_1^*)}{\delta\tilde{\kappa}(u, T_1^*) + 2}\, \nu(u, T_1^*)\, du, \quad \forall\, t \in [0, T_1^*].$$

We may now define, using Girsanov's theorem, the associated forward swap measure $\tilde{\mathbf{P}}_{T_1^*}$. We are thus in a position to define the process $\tilde{\kappa}(\cdot, T_2^*)$, which solves the SDE

$$d\tilde{\kappa}(t, T_2^*) = \tilde{\kappa}(t, T_2^*)\nu(t, T_2^*) \cdot d\tilde{W}_t^{T_1^*} \tag{9.70}$$

with the initial condition

$$\tilde{\kappa}(0, T_2^*) = \frac{P(0, T_2^*) - P(0, T^*)}{\delta\big(P(0, T_1^*) + P(0, T^*)\big)}.$$

For the reader's convenience, let us consider one more inductive step, in which we are looking for $\tilde{\kappa}(\cdot, T_3^*)$. We now consider processes

$$Z_3(t, T_k^*) = \frac{B(t, T_k^*)}{B(t, T_2^*) + B(t, T_1^*) + B(t, T^*)} = \frac{Z_2(t, T_k^*)}{1 + Z_2(t, T_2^*)},$$

so that

$$\tilde{W}_t^{T_2^*} = \tilde{W}_t^{T_1^*} - \int_0^t \frac{Z_2(u, T_2^*)}{1 + Z_2(u, T_2^*)}\, \gamma_2(u, T_2^*)\, du$$

for every $t \in [0, T_2^*]$. It is crucial to note that

$$Z_2(t, T_2^*) = \frac{B(t, T_2^*)}{B(t, T_1^*) + B(t, T^*)} = \delta\tilde{\kappa}(t, T_2^*) + Z_2(t, T^*),$$

where in turn

$$Z_2(t, T^*) = \frac{Z_1(t, T^*)}{\delta\tilde{\kappa}(t, T_1^*) + Z_1(t, T^*) + 1}$$

and the process $Z_1(\cdot, T^*)$ is already known from the previous step.

Let us now turn to the general case. We assume that we have found forward swap rates $\tilde{\kappa}(\cdot, T_1^*), \ldots, \tilde{\kappa}(\cdot, T_m^*)$, the swap forward measure $\tilde{\mathbf{P}}_{T_{m-1}^*}$, and the associated Brownian motion $\tilde{W}^{T_{m-1}^*}$. Our aim is to determine the forward swap measure $\tilde{\mathbf{P}}_{T_m^*}$, the associated Brownian motion $\tilde{W}^{T_m^*}$, and, of course, the forward swap rate $\tilde{\kappa}(\cdot, T_{m+1}^*)$. We postulate that processes

$$Z_{m+1}(t, T_k^*) = \frac{B(t, T_k^*)}{B(t, T_m^*) + \ldots + B(t, T^*)} = \frac{Z_m(t, T_k^*)}{1 + Z_m(t, T_m^*)}$$

follow local martingales under $\tilde{\mathbf{P}}_{T_m^*}$. In view of Lemma 9.5.1, applied to processes $G = Z_m(\cdot, T_k^*)$ and $H = Z_m(\cdot, T_m^*)$, it is clear that we may set

$$\tilde{W}_t^{T_m^*} = \tilde{W}_t^{T^*} - \int_0^t \frac{Z_m(u, T_m^*)}{1 + Z_m(u, T_m^*)} \, \gamma_m(u, T_m^*) \, du \tag{9.71}$$

for $t \in [0, T_m^*]$. Therefore, it is sufficient to analyze the process

$$Z_m(t, T_m^*) = \frac{B(t, T_m^*)}{B(t, T_{m-1}^*) + \ldots + B(t, T^*)} = \delta\tilde{\kappa}(t, T_m^*) + Z_m(t, T^*).$$

Observe that

$$Z_m(t, T^*) = \frac{Z_{m-1}(t, T^*)}{\delta\tilde{\kappa}(t, T_{m-1}^*) + Z_{m-1}(t, T^*) + 1}$$

and, from the preceding step, the process $Z_{m-1}(\cdot, T^*)$ is a (rational) function of forward swap rates $\tilde{\kappa}(\cdot, T_1^*), \ldots, \tilde{\kappa}(\cdot, T_{m-1}^*)$. Consequently, the process under the integral sign on the right-hand side of (9.71) can be expressed using the terms $\tilde{\kappa}(\cdot, T_1^*), \ldots, \tilde{\kappa}(\cdot, T_{m-1}^*)$ and their volatilities (since the explicit formula is rather involved, we do not report it here). Having found the process $\tilde{W}^{T_m^*}$ and probability measure $\tilde{\mathbf{P}}_{T_m^*}$, we introduce the forward swap rate $\tilde{\kappa}(\cdot, T_{m+1}^*)$ through (9.66)–(9.67), and so forth.

Remarks. It is worthwhile to notice that lognormal models of forward LIBOR and swap rates can be easily generalized to the case of accrual periods with variable length. In market practice, the length of accrual periods of caps or swaps is known to vary slightly, even in the case when it is formally defined as a fixed time interval (for instance, a quarter of the year may amount equally well to 89, 90 or 91 days).

Chapter 10

Option Valuation in Gaussian Models

In this chapter, the forward measure methodology is employed in arbitrage pricing of interest rate derivative securities in a Gaussian framework. By a Gaussian framework we mean any model of the term structure, either based on the short-term rate or on forward rates, in which all bond price volatilities (as well as the volatility of any other underlying asset) follow deterministic functions. This assumption is made for expositional simplicity; it is not a necessary condition in order to obtain a closed-form solution for the price of a particular option, however. For instance, when a European option on a specific asset is examined in order to obtain an explicit expression for its arbitrage price, it is in fact enough to assume that the volatility of the forward price of the underlying asset for the settlement date coinciding with the option's maturity date is deterministic.

This chapter is organized as follows. In the first section, we examine typical questions related to the valuation of European options on stocks, zero-coupon bonds and coupon-bearing bonds. As already indicated, we postulate that the bond price volatilities, as well as the volatility of the option's underlying asset, follow deterministic functions. The next section is devoted to the study of futures prices and to arbitrage valuation of futures options.

10.1 European Spot Options

The first step towards explicit valuation of European options is to observe that Lemma 9.2.3 provides a simple formula which expresses the price of a European call option written on a tradable asset, Z say, in terms of the forward price process $F_Z(t,T)$ and the forward probability measure $\mathbf{P}_T$. Indeed, we have for every $t \in [0,T]$

$$\pi_t\big((Z_T - K)^+\big) = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\big((F_Z(T,T) - K)^+ \,|\, \mathcal{F}_t\big), \tag{10.1}$$

since manifestly $Z_T = F_Z(T,T)$. To evaluate the conditional expectation on the right-hand side of (10.1), we need to find first the dynamics, under the forward probability measure $\mathbf{P}_T$, of the forward price $F_Z(t,T)$. The following auxiliary result is an easy consequence of (9.22)–(9.23) and (9.25).

Lemma 10.1.1 *For any fixed $T > 0$, the process W^T given by the formula*

$$W_t^T = W_t^* - \int_0^t b(u,T)\,du, \quad \forall\, t \in [0,T], \tag{10.2}$$

follows a standard d-dimensional Brownian motion under the forward measure $\mathbf{P}_T$. The forward price process for the settlement date T of a zero-coupon bond which matures at time U satisfies

$$dF_B(t,U,T) = F_B(t,U,T)\,\big(b(t,U) - b(t,T)\big)\cdot dW_t^T, \tag{10.3}$$

subject to the terminal condition $F_B(T,U,T) = B(T,U)$. *The forward price of a stock* S *satisfies* $F_S(T,T) = S_T$, *and*

$$dF_S(t,T) = F_S(t,T)\,\big(\sigma_t - b(t,T)\big) \cdot dW_t^T. \tag{10.4}$$

The next result, which uses the HJM framework, shows that the yield-to-maturity expectations hypothesis (cf. Sect. 8.1) is satisfied for any fixed maturity T under the corresponding forward probability measure $\mathbf{P}_T$. This feature is merely a distant reminder of the classic hypothesis that for every maturity T, the instantaneous forward rate $f(0,T)$ is an unbiased estimate, under the actual probability $\mathbf{P}$, of the future short-term rate r_T.

Corollary 10.1.1 *For any fixed* $T \in [0,T^*]$, *the forward rate* $f(0,T)$ *is equal to the expected value of the spot rate* r_T *under the forward probability measure* $\mathbf{P}_T$.

Proof. Observe that in view of (9.15), we have

$$r_T = f(0,T) + \int_0^T \sigma(t,T) \cdot \big(\sigma^*(t,T)\,dt + dW_t^*\big) = f(0,T) + \int_0^T \sigma(t,T) \cdot dW_t^T,$$

since $\sigma^*(t,T) = -b(t,T)$. Therefore, $\mathbf{E}_{\mathbf{P}_T}(r_T) = f(0,T)$, as expected. □

10.1.1 Bond Options

For the reader's convenience, we shall examine separately options written on zero-coupon bonds and on stocks (a general valuation result is given in Proposition 10.1.3). At expiry date T, the payoff of a European call option written on a zero-coupon bond which matures at time $U \geq T$ equals

$$C_T = \big(B(T,U) - K\big)^+. \tag{10.5}$$

Since $B(T,U) = F_B(T,U,T)$, the payoff C_T can alternatively be re-expressed in the following way

$$C_T = (F_B(T,U,T) - K)^+ = F_B(T,U,T)\,\mathrm{I}_D - K\,\mathrm{I}_D,$$

where

$$D = \{B(T,U) > K\} = \{F_B(T,U,T) > K\}$$

is the exercise set. The next proposition provides a closed-form expression for the arbitrage price of a European bond option. Valuation results of this form were derived previously by several authors, including El Karoui and Rochet (1989), Amin and Jarrow (1992), Brace and Musiela (1994). For the sake of expositional simplicity, we assume that the volatilities are bounded; however, this assumption can be weakened.

Proposition 10.1.1 *Assume that the bond price volatilities* $b(\cdot,T)$ *and* $b(\cdot,U)$ *are bounded deterministic functions. The arbitrage price at time* $t \in [0,T]$ *of a European call option with expiry date* T *and strike price* K, *written on a zero-coupon bond which matures at time* $U \geq T$, *equals*

$$C_t = B(t,U)N\big(h_1(B(t,U),t,T)\big) - KB(t,T)N\big(h_2(B(t,U),t,T)\big), \tag{10.6}$$

where

$$h_{1,2}(b,t,T) = \frac{\ln(b/K) - \ln B(t,T) \pm \frac{1}{2}\,v_U^2(t,T)}{v_U(t,T)} \tag{10.7}$$

for $(b,t) \in \mathbf{R}_+ \times [0,T]$, *and*

$$v_U^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\,du, \quad \forall\, t \in [0,T]. \tag{10.8}$$

The arbitrage price of the corresponding European put option written on a zero-coupon bond equals

$$P_t = KB(t,T)N\big(-h_2(B(t,U),t,T)\big) - B(t,U)N\big(-h_1(B(t,U),t,T)\big).$$

Proof. In view of the general valuation formula (10.1), it is clear that we have to evaluate the conditional expectations

$$C_t = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\big(F_B(T,U,T)\,\mathrm{I}_D \,\big|\, \mathcal{F}_t\big) - KB(t,T)\,\mathbf{P}_T\{D \,|\, \mathcal{F}_t\} = I_1 - I_2.$$

We known from Lemma 10.1.1 that the dynamics of $F_B(t,U,T)$ under $\mathbf{P}_T$ are given by formula (10.3), so that

$$F_B(T,U,T) = F_B(t,U,T)\exp\Big(\int_t^T \gamma(u,U,T)\cdot dW_u^T - \frac{1}{2}\int_t^T |\gamma(u,U,T)|^2\,du\Big),$$

where $\gamma(u,U,T) = b(u,U) - b(u,T)$. This can be rewritten as follows

$$F_B(T,U,T) = F_B(t,U,T)\exp\big(\zeta(t,T) - \tfrac{1}{2}v_U^2(t,T)\big),$$

where $F_B(t,U,T)$ is $\mathcal{F}_t$-measurable, and $\zeta(t,T) = \int_t^T \gamma(u,U,T)\cdot dW_u^T$ is, under $\mathbf{P}_T$, a real-valued Gaussian random variable, independent of the σ-field $\mathcal{F}_t$, with zero expected value and the variance $\mathrm{Var}_{\mathbf{P}_T}(\zeta(t,T)) = v_U^2(t,T)$. Using the properties of conditional expectation, we obtain

$$\mathbf{P}_T\{D \,|\, \mathcal{F}_t\} = \mathbf{P}_T\Big\{\zeta(t,T) < \ln\big(F/K\big) - \tfrac{1}{2}\,v_U^2(t,T)\Big\},$$

where $F = F_B(t,U,T)$, so that

$$I_2 = KB(t,T)\,N\left(\frac{\ln\big(F_B(t,U,T)/K\big) - \frac{1}{2}v_U^2(t,T)}{v_U(t,T)}\right).$$

To evaluate I_1, we introduce an auxiliary probability measure $\tilde{\mathbf{P}}_T \sim \mathbf{P}_T$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\tilde{\mathbf{P}}_T}{d\mathbf{P}_T} = \exp\Big(\int_0^T \gamma(u,U,T)\cdot dW_u^T - \frac{1}{2}\int_0^T |\gamma(u,U,T)|^2\,du\Big) \stackrel{\mathrm{def}}{=} \tilde{\eta}_T.$$

By Girsanov's theorem, it is clear that the process $\tilde{W}^T$, which equals

$$\tilde{W}_t^T = W_t^T - \int_0^t \gamma(u,U,T)\,du, \quad \forall\, t \in [0,T],$$

follows a standard Brownian motion under $\tilde{\mathbf{P}}_T$. Note also that the forward price $F_B(T,U,T)$ admits the following representation under $\tilde{\mathbf{P}}_T$

$$F_B(T,U,T) = F_B(t,U,T)\exp\Big(\int_t^T \gamma(u,U,T)\cdot d\tilde{W}_u^T + \frac{1}{2}\int_t^T |\gamma(u,U,T)|^2\,du\Big),$$

so that

$$F_B(T,U,T) = F_B(t,U,T)\,\exp\big(\tilde{\zeta}(t,T) + \tfrac{1}{2}v_U^2(t,T)\big), \tag{10.9}$$

where we denote $\tilde{\zeta}(t,T) = \int_t^T \gamma(u,U,T)\cdot d\tilde{W}_u^T$. The random variable $\tilde{\zeta}(t,T)$ has under $\tilde{\mathbf{P}}_T$ a Gaussian law with zero mean value and variance $v_U^2(t,T)$, and it is also independent of the σ-field $\mathcal{F}_t$. Furthermore, once again using Lemma 10.1.1, we obtain

$$I_1 = B(t,U)\mathbf{E}_{\mathbf{P}_T}\Big\{\mathrm{I}_D\exp\Big(\int_t^T \gamma(u,U,T)\cdot dW_u^T - \frac{1}{2}\int_t^T |\gamma(u,U,T)|^2\,du\Big)\,\Big|\,\mathcal{F}_t\Big\},$$

that is

$$I_1 = B(t,U)\,\mathbf{E}_{\mathbf{P}_T}\big(\tilde{\eta}_T\,\tilde{\eta}_t^{-1}\,\mathrm{I}_D \,|\, \mathcal{F}_t\big).$$

By virtue of the Bayes rule (see (9.28)), we find that $I_1 = B(t,U)\,\tilde{\mathbf{P}}_T\{D \,|\, \mathcal{F}_t\}$. Taking into account (10.9), we conclude that

$$\tilde{\mathbf{P}}_T\{D \,|\, \mathcal{F}_t\} = \tilde{\mathbf{P}}_T\Big\{\tilde{\zeta}(t,T) \le \ln\big(F_B(t,U,T)/K\big) + \tfrac{1}{2}v_U^2(t,T)\Big\},$$

and thus

$$I_1 = B(t,U)\,N\left(\frac{\ln\big(F_B(t,U,T)/K\big) + \frac{1}{2}v_U^2(t,T)}{v_U(t,T)}\right).$$

This completes the proof of the valuation formula (10.6). The formula that gives the price of the put option can be established along the same lines. Alternatively, to find the price of a European put option written on a zero-coupon bond, one may combine equality (10.6) with the put-call parity relationship (10.22). □

Formula (10.6) can be re-expressed as follows

$$C_t = B(t,T)\Big(F_t N\big(\tilde{d}_1(F_t,t,T)\big) - KN\big(\tilde{d}_2(F_t,t,T)\big)\Big), \tag{10.10}$$

where we write briefly F_t to denote the forward price $F_B(t,U,T)$, and

$$\tilde{d}_{1,2}(F,t,T) = \frac{\ln(F/K) \pm \frac{1}{2}\,v_U^2(t,T)}{v_U(t,T)} \tag{10.11}$$

for $(F,t) \in \mathbf{R}_+ \times [0,T]$, where $v_U(t,T)$ is given by (10.8). Note also that we have

$$\frac{d\tilde{\mathbf{P}}_T}{d\mathbf{P}^*} = \frac{d\tilde{\mathbf{P}}_T}{d\mathbf{P}_T}\,\frac{d\mathbf{P}_T}{d\mathbf{P}^*} = \exp\Big(\int_0^T b(u,U)\cdot dW_u - \frac{1}{2}\int_0^T |b(u,U)|^2\,du\Big).$$

It is thus apparent that the auxiliary probability measure $\tilde{\mathbf{P}}_T$ is in fact the restriction of the forward measure $\mathbf{P}_U$ to the σ-field $\mathcal{F}_T$. Since the exercise set D belongs to the σ-field $\mathcal{F}_T$, we have $\tilde{\mathbf{P}}_T\{D \,|\, \mathcal{F}_t\} = \mathbf{P}_U\{D \,|\, \mathcal{F}_t\}$. Therefore, formula (10.6) admits the following alternative representation

$$C_t = B(t,U)\mathbf{P}_U\{D \,|\, \mathcal{F}_t\} - KB(t,T)\mathbf{P}_T\{D \,|\, \mathcal{F}_t\}. \tag{10.12}$$

10.1.2 Stock Options

The payoff at expiry of a European call option written on a stock S equals $C_T = \big(S_T - K\big)^+$, where T is the expiry date and K denotes the strike price. The next result, which is a straightforward generalization of the Black-Scholes formula, provides an explicit solution for the arbitrage price of a stock call option. We assume that the dynamics of S under the martingale measure $\mathbf{P}^*$ are

$$dS_t = S_t\,\big(r_t\,dt + \sigma_t \cdot dW_t^*\big),$$

where $\sigma : [0,T^*] \to \mathbf{R}$ is a deterministic function.

Proposition 10.1.2 *Assume that the bond price volatility $b(\cdot,T)$ and the stock price volatility σ are bounded deterministic functions. Then the arbitrage price of a European call option with expiry date T and exercise price K, written on a stock S, equals*

$$C_t = S_t N\big(h_1(S_t,t,T)\big) - KB(t,T)N\big(h_2(S_t,t,T)\big), \tag{10.13}$$

where

$$h_{1,2}(s,t,T) = \frac{\ln(s/K) - \ln B(t,T) \pm \frac{1}{2}v^2(t,T)}{v(t,T)} \tag{10.14}$$

for $(s,t) \in \mathbf{R}_+ \times [0,T]$, and

$$v^2(t,T) = \int_t^T |\sigma_u - b(u,T)|^2\,du, \quad \forall\, t \in [0,T]. \tag{10.15}$$

Proof. The proof goes along the same lines as the proof of Proposition 10.1.1. □

Example 10.1.1 Let us examine a very special case of the pricing formula established in Proposition 10.1.2. Let $W^* = (W^{1*}, W^{2*})$ be a two-dimensional standard Brownian motion given on a probability space $(\Omega, \mathbf{F}, \mathbf{P}^*)$. We assume that the bond price $B(t,T)$ satisfies, under $\mathbf{P}^*$

$$dB(t,T) = B(t,T)\big(r_t\, dt + \hat{b}(t,T)(\rho, \sqrt{1-\rho^2}) \cdot dW_t^*\big),$$

where $\hat{b}(\cdot, T) : [0,T] \to \mathbf{R}$ is a real-valued, bounded deterministic function, and the dynamics of the stock price S are

$$dS_t = S_t\big(r_t\, dt + (\hat{\sigma}(t), 0) \cdot dW_t^*\big)$$

for some function $\hat{\sigma} : [0, T^*] \to \mathbf{R}$. Let us introduce the real-valued stochastic processes $\hat{W}^1$ and $\hat{W}^2$ by setting $\hat{W}_t^1 = W_t^{1*}$ and $\hat{W}_t^2 = \rho W_t^{1*} + \sqrt{1-\rho^2}\, W_t^{2*}$. It is not hard to check that $\hat{W}^1$ and $\hat{W}^2$ follow standard one-dimensional Brownian motions under the martingale measure $\mathbf{P}^*$, and their cross-variation equals $\langle \hat{W}^1, \hat{W}^2 \rangle_t = \rho t$ for $t \in [0, T^*]$. It is evident that

$$dB(t,T) = B(t,T)\,\big(r_t\, dt + \hat{b}(t,T)\, d\hat{W}_t^2\big) \tag{10.16}$$

and

$$dS_t = S_t\,\big(r_t\, dt + \hat{\sigma}(t)\, d\hat{W}_t^1\big). \tag{10.17}$$

An application of Proposition 10.1.2 yields the following result, first established in Merton (1973).

Corollary 10.1.2 *Assume that the dynamics of a bond and a stock price are given by* (10.16) *and* (10.17), *respectively. If the volatility coefficients $\hat{b}$ and $\hat{\sigma}$ are deterministic functions, then the arbitrage price of a European call option written on a stock S is given by* (10.13)–(10.14), *with*

$$v^2(t,T) = \int_t^T \big(\hat{\sigma}^2(u) - 2\rho\hat{\sigma}(u)\hat{b}(u,T) + \hat{b}^2(u,T)\big)\, du. \tag{10.18}$$

We are in a position to formulate a result which encompasses both cases studied above. The dynamics of the spot price Z of a tradable asset are assumed to be given by the expression

$$dZ_t = Z_t\,\big(r_t\, dt + \xi_t \cdot dW_t^*\big). \tag{10.19}$$

It is essential to assume that the volatility $\xi_t - b(t,T)$ of the forward price of Z for the settlement date T is deterministic.

Proposition 10.1.3 *The arbitrage price of a European call option with expiry date T and exercise price K, written on an asset Z, is given by the expression*

$$C_t = B(t,T)\Big(F_Z(t,T) N\big(\tilde{d}_1(F_Z(t,T), t, T)\big) - K N\big(\tilde{d}_2(F_Z(t,T), t, T)\big)\Big),$$

where

$$\tilde{d}_{1,2}(F,t,T) = \frac{\ln(F/K) \pm \frac{1}{2}\, v^2(t,T)}{v(t,T)} \tag{10.20}$$

for $(F,t) \in \mathbf{R}_+ \times [0,T]$, and

$$v^2(t,T) = \int_t^T |\xi_u - b(u,T)|^2\, du, \quad \forall t \in [0,T]. \tag{10.21}$$

Let P_t stand for the price at time $t \le T$ of a European put option written on an asset Z, with expiry date T and strike price K. Then the following useful result is valid. The reader may find it instructive to derive (10.22) by constructing particular trading portfolios.

Corollary 10.1.3 *The following put-call parity relationship is valid*

$$C_t - P_t = Z_t - B(t,T)K, \quad \forall\, t \in [0,T]. \tag{10.22}$$

Proof. We make use of the forward measure method. We have

$$C_t - P_t = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\big(F_Z(T,U,T) - K \,\big|\, \mathcal{F}_t\big),$$

and thus

$$C_t - P_t = B(t,T)F_Z(t,U,T) - B(t,T)K = Z_t - B(t,T)K$$

for every $t \in [0,T]$. □

10.1.3 Option on a Coupon-bearing Bond

Our aim is to value a European option written on a coupon-bearing bond. For a given selection of dates $T_1 < \ldots < T_m \leq T^*$, we consider a coupon-bearing bond whose value Z_t at time $t \leq T_1$ is

$$Z_t = \sum_{j=1}^{m} c_j\, B(t,T_j), \quad \forall\, t \in [0,T_1], \tag{10.23}$$

where c_j are real numbers. We shall study a European call option with expiry date $T \leq T_1$, whose payoff at expiry has the following form

$$C_T = (Z_T - K)^+ = \Big(\sum_{j=1}^{m} c_j\, B(T,T_j) - K\Big)^+. \tag{10.24}$$

Proposition 10.1.4 *The arbitrage price of a European call option on a coupon-bearing bond is given by the formula*

$$C_t = \sum_{j=1}^{m} c_j\, B(t,T_j) J_1^j - KB(t,T)J_2, \tag{10.25}$$

where

$$J_1^j = \mathbf{Q}\Big\{\sum_{l=1}^{m} c_l B(t,T_l)\, e^{\zeta_l + v_{lj} - v_{ll}/2} > KB(t,T)\Big\} \tag{10.26}$$

for $j = 1, \ldots, m$,

$$J_2 = \mathbf{Q}\Big\{\sum_{l=1}^{m} c_l B(t,T_l)\, e^{\zeta_l - v_{ll}/2} > KB(t,T)\Big\}, \tag{10.27}$$

and $(\zeta_1, \ldots, \zeta_m)$ is a random variable whose law under $\mathbf{Q}$ is Gaussian, with zero expected value, and which has the following variance-covariance matrix

$$\mathrm{Cov}_{\mathbf{Q}}(\zeta_k, \zeta_l) = v_{kl} = \int_t^T \gamma(u,T_k,T) \cdot \gamma(u,T_l,T)\, du \tag{10.28}$$

for $k,l = 1, \ldots, m$, where $\gamma(t,T_k,T) = b(t,T_k) - b(t,T)$.

Proof. We need to evaluate the conditional expectation

$$C_t = B(t,T) \sum_{j=1}^{m} c_j \mathbf{E}_{\mathbf{P}_T}(F_B(T,T_j,T)\mathrm{I}_D|\mathcal{F}_t) - KB(t,T)\mathbf{P}_T\{D \,|\, \mathcal{F}_t\} = I_1 - I_2,$$

where D stands for the exercise set

$$D = \{\sum_{j=1}^{m} c_j\, B(T,T_j) > K\} = \{\sum_{j=1}^{m} c_j F_B(T,T_j,T) > K\}.$$

Let us first examine the conditional probability $\mathbf{P}_T\{D \,|\, \mathcal{F}_t\}$. By virtue of Lemma 10.1.1, the process $F_B(t) = F_B(t, T_l, T)$ satisfies

$$F_B(T) = F_B(t) \exp\Big(\int_t^T \gamma(u, T_l, T) \cdot dW_u^T - \frac{1}{2} \int_t^T |\gamma(u, T_l, T)|^2 \, du \Big),$$

where $\gamma(u, T_l, T) = b(u, T_l) - b(u, T)$. In other words,

$$F_B(T, T_l, T) = F_B(t, T_l, T) \, e^{\xi_l^T - v_{ll}/2},$$

where ξ_l^T is a random variable independent of the σ-field $\mathcal{F}_t$, and such that the probability law of ξ_l^T under $\mathbf{P}_T$ is the Gaussian law $N(0, v_{ll})$. Therefore

$$\mathbf{P}_T\{D \,|\, \mathcal{F}_t\} = \mathbf{P}_T\Big\{ \sum_{l=1}^m c_l B(t, T_l) \, e^{\xi_l^T - v_{ll}^2/2} > KB(t, T) \Big\}.$$

This proves that $I_2 = KB(t, T)J_2$. Let us show that $I_1 = \sum_{j=1}^m c_j \, B(t, T_j) J_1^j$. To this end, it is sufficient to check that for any fixed j we have

$$B(t, T) \mathbf{E}_{\mathbf{P}_T} \big(F_B(T, T_j, T) \, \mathrm{I}_D \,\big|\, \mathcal{F}_t \big) = B(t, T_j) J_1^j. \tag{10.29}$$

This can be done by proceeding in much the same way as in the proof of Proposition 10.1.1. Let us fix j and introduce an auxiliary probability measure $\tilde{\mathbf{P}}_{T_j}$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\tilde{\mathbf{P}}_{T_j}}{d\mathbf{P}_T} = \exp\Big(\int_0^T \gamma(u, T_j, T) \cdot dW_u^T - \frac{1}{2} \int_0^T |\gamma(u, T_j, T)|^2 \, du \Big).$$

Then the process

$$\tilde{W}_t^j = W_t^T - \int_0^t \gamma(u, T_j, T) \, du$$

follows a standard Brownian motion under $\tilde{\mathbf{P}}_{T_j}$. Recall that $\tilde{\mathbf{P}}_{T_j} = \mathbf{P}_{T_j}$ on $\mathcal{F}_T$, hence we shall write simply $\mathbf{P}_{T_j}$ in place of $\tilde{\mathbf{P}}_{T_j}$ in what follows. For any l, the forward price $F_B(t, T_l, T)$ has the following representation under $\mathbf{P}_{T_j}$

$$dF_B(t, T_l, T) = F_B(t, T_l, T) \gamma(t, T_l, T) \cdot \big(\tilde{W}_t^j + \gamma(t, T_j, T) \, dt \big). \tag{10.30}$$

For a fixed j, we define the random variable $(\xi_1, \ldots, \xi_m)$ by the formula

$$\xi_l = \int_t^T \gamma(u, T_l, T) \cdot d\tilde{W}_u^j.$$

It is clear that the random variable $(\xi_1, \ldots, \xi_m)$ is independent of $\mathcal{F}_t$, with Gaussian law under $\mathbf{P}_{T_j}$. More precisely, the expected value of each random variable ξ_i is zero, and for every $k, l = 1, \ldots, m$, we have

$$v_{kl} = \mathrm{Cov}_{\mathbf{P}_{T_j}} \, (\xi_k, \xi_l) = \int_t^T \gamma(u, T_k, T) \cdot \gamma(u, T_l, T) \, du.$$

On the other hand, using (10.30) we find that

$$F_B(T, T_l, T) = F_B(t, T_l, T) \, \exp\big(\xi_l - \tfrac{1}{2} v_{ll} + v_{lj} \big)$$

for every $l = 1, \ldots, m$. The Bayes rule yields

$$B(t, T) \mathbf{E}_{\mathbf{P}_T} \big(F_B(T, T_j, T) \, \mathrm{I}_D \,\big|\, \mathcal{F}_t \big) = B(t, T_j) \mathbf{P}_{T_j}\{D \,|\, \mathcal{F}_t\}.$$

Furthermore,

$$\mathbf{P}_{T_j}\{D \,|\, \mathcal{F}_t\} = \mathbf{P}_{T_j}\Big\{\sum_{l=1}^{m} c_l B(t,T_l) \exp\big(\xi_l - \tfrac{1}{2}v_{ll} + v_{lj}\big) > KB(t,T)\Big\}.$$

By combining the last two equalities, we arrive at (10.29). □

The next result suggests an alternative way to prove Proposition 10.1.4.

Lemma 10.1.2 *Let us denote $D = \{Z_T > K\}$. Then the arbitrage price of a European call option written on a coupon bond satisfies*

$$C_t = \sum_{j=1}^{m} c_j\, B(t,T_j)\mathbf{P}_{T_j}\{D \,|\, \mathcal{F}_t\} - KB(t,T)\mathbf{P}_T\{D \,|\, \mathcal{F}_t\}. \tag{10.31}$$

Proof. We have $Z_T = \sum_{j=1}^m c_j B_T\, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_j}^{-1} \,|\, \mathcal{F}_T\big)$, and thus

$$\begin{aligned} C_t &= B_t\, \mathbf{E}_{\mathbf{P}^*}\Big\{\, \mathrm{I}_D B_T^{-1}\Big(\sum_{j=1}^{m} c_j B_T\, \mathbf{E}_{\mathbf{P}^*}(B_{T_j}^{-1} \,|\, \mathcal{F}_T) - K\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &= \sum_{j=1}^{m} c_j B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_j}^{-1}\, \mathrm{I}_D \,\big|\, \mathcal{F}_t\big) - KB_t \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1}\, \mathrm{I}_D \,\big|\, \mathcal{F}_t\big), \end{aligned}$$

since $D \in \mathcal{F}_T$. Using (9.28), we get for every j

$$\begin{aligned} B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_j}^{-1}\, \mathrm{I}_D \,\big|\, \mathcal{F}_t\big) &= B_t\, B(0,T_j)\, \mathbf{E}_{\mathbf{P}^*}\big(\eta_{T_j}\, \mathrm{I}_D \,\big|\, \mathcal{F}_t\big) \\ &= B_t\, \mathbf{E}_{\mathbf{P}_{T_j}}(\mathrm{I}_D \,|\, \mathcal{F}_t)\, \mathbf{E}_{\mathbf{P}^*}(B_{T_j}^{-1} \,|\, \mathcal{F}_t) \\ &= B(t,T_j)\, \mathbf{P}_{T_j}\{D \,|\, \mathcal{F}_t\}. \end{aligned}$$

Since a similar relation holds for the last term, this ends the proof. □

10.1.4 Pricing of General Contingent Claims

Let us consider a European contingent claim X, which settles at time T, of the form $X = g(Z_T^1, \dots, Z_T^n)$, where $g : \mathbf{R}^n \to \mathbf{R}$ is a bounded Borel-measurable function. Assume that the price process Z^i of the i^{th} asset satisfies, under $\mathbf{P}^*$

$$dZ_t^i = Z_t^i\, \big(r_t\, dt + \xi_t^i \cdot dW_t^*\big). \tag{10.32}$$

Then

$$F_{Z^i}(T,T) = F_{Z^i}(t,T) \exp\Big(\int_t^T \gamma_i(u,T) \cdot dW_u^T - \frac{1}{2}\int_t^T |\gamma_i(u,T)|^2\, du\Big),$$

where $\gamma_i(u,T) = \xi_u^i - b(u,T)$, or in short

$$F_{Z^i}(T,T) = F_{Z^i}(t,T) \exp\big(\zeta_i(t,T) - \tfrac{1}{2}\gamma_{ii}\big),$$

where $\zeta_i(t,T) = \int_t^T \gamma_i(u,T) \cdot dW_u^T$ and $\gamma_{ii} = \int_t^T |\gamma_i(u,T)|^2\, du$. The forward price $F_{Z^i}(t,T)$ is a random variable measurable with respect to the σ-field $\mathcal{F}_t$, while the random variable $\zeta_i(t,T)$ is independent of this σ-field. Moreover, it is clear that the probability distribution under the forward measure $\mathbf{P}_T$ of the vector-valued random variable

$$(\zeta_1(t,T), \dots, \zeta_n(t,T)) = \Big(\int_t^T \gamma_1(u,T) \cdot dW_u^T, \dots, \int_t^T \gamma_n(u,T) \cdot dW_u^T\Big)$$

is Gaussian $N(0,\Gamma)$, where the entries of the $n\times n$ matrix Γ are

$$\gamma_{ij}=\int_t^T \gamma_i(u,T)\cdot\gamma_j(u,T)\,du.$$

Let us introduce a $k\times n$ matrix $\Theta=[\theta_1,\dots,\theta_n]$ such that $\Gamma=\Theta^t\Theta$.

Proposition 10.1.5 *Assume that γ_i is a deterministic function for $i=1,\dots,n$. Then the arbitrage price at time $t\in[0,T]$ of a European contingent claim $X=g(Z_T^1,\dots,Z_T^n)$ which settles at time T equals*

$$\pi_t(X)=B(t,T)\int_{\mathbf{R}^k} g\left(\frac{Z_t^1 n_k(x+\theta_1)}{B(t,T)n_k(x)},\dots,\frac{Z_t^n n_k(x+\theta_n)}{B(t,T)n_k(x)}\right)n_k(x)\,dx,$$

where n_k is the standard k-dimensional Gaussian density

$$n_k(x)=(2\pi)^{-k/2}e^{-|x|^2/2},\quad \forall\, x\in\mathbf{R}^k,$$

and the vectors $\theta_1,\dots,\theta_n\in\mathbf{R}^k$ are such that for every $i,j=1,\dots,n,$ we have

$$\theta_i\cdot\theta_j=\int_t^T \gamma_i(u,T)\cdot\gamma_j(u,T)\,du.$$

Proof. We have

$$\begin{aligned}\pi_t(X) &= B_t\,\mathbf{E}_{\mathbf{P}^*}\Big(B_T^{-1}\,g(Z_T^1,\dots,Z_T^n)\Big|\mathcal{F}_t\Big)\\ &= B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\Big(g(F_{Z^1}(T,T),\dots,F_{Z^n}(T,T))\Big|\mathcal{F}_t\Big)=B(t,T)J.\end{aligned}$$

In view of the definition of the matrix $\Theta,$ it is clear that

$$\begin{aligned}J &= \int_{\mathbf{R}^k} g\Big(F_{Z^1}(t,T)\,e^{\theta_1\cdot x-|\theta_1|^2/2},\dots,F_{Z^n}(t,T)\,e^{\theta_n\cdot x-|\theta_n|^2/2}\Big)\,n_k(x)\,dx\\ &= \int_{\mathbf{R}^k} g\left(\frac{Z_t^1 n_k(x+\theta_1)}{B(t,T)n_k(x)},\dots,\frac{Z_t^n n_k(x+\theta_n)}{B(t,T)n_k(x)}\right)n_k(x)dx.\end{aligned}$$

This ends the proof of the proposition. □

10.1.5 Replication of Options

In preceding sections, we have valued options using a risk-neutral valuation approach, assuming implicitly that options correspond to attainable claims. In this section, we focus on the construction of a replicating portfolio. Consider a contingent claim X which settles at time $T,$ and is represented by a $\mathbf{P}_T$-integrable, strictly positive random variable $X.$ The forward price of X for the settlement date T satisfies

$$F_X(t,T)=\mathbf{E}_{\mathbf{P}_T}(X\,|\,\mathcal{F}_t)=F_X(0,T)+\int_0^t F_X(u,T)\gamma_u\cdot dW_u^T \tag{10.33}$$

for some predictable process $\gamma.$ Assume, in addition, that γ is a deterministic function. Let us denote $F_t=F_X(t,T).$ Our aim is to show, by means of a replicating strategy, that the arbitrage price of a European call option written on a claim $X,$ with expiry date T and strike price $K,$ equals

$$C_t=B(t,T)\Big(F_tN\big(\tilde d_1(F_t,t,T)\big)-KN\big(\tilde d_2(F_t,t,T)\big)\Big), \tag{10.34}$$

where $\tilde d_1$ and $\tilde d_2$ are given by (10.20) with $v^2(t,T)=v_X^2(t,T)=\int_t^T|\gamma_u|^2\,du.$ Equality (10.34) yields the following expression for the forward price of the option

$$F_C(t,T)=F_tN\big(\tilde d_1(F_t,t,T)\big)-KN\big(\tilde d_2(F_t,t,T)\big). \tag{10.35}$$

Note that by applying Itô's formula to (10.35), we obtain

$$dF_C(t,T) = N(\tilde{d}_1(F_t,t,T))\,dF_t. \tag{10.36}$$

Forward asset/bond market. Let us consider a *T-forward market,* i.e., a financial market in which the forward contracts for settlement at time T play the role of primary securities. Consider a *forward strategy* $\psi = (\psi^1, \psi^2)$, where ψ^1 and ψ^2 stand for the number of forward contracts on the underlying claim X and on the zero-coupon bond with maturity T, respectively. Observe that the T-forward market differs essentially from a futures market. The *forward wealth* process $\tilde{V}$ of a T-forward market portfolio ψ equals

$$\tilde{V}_t(\psi) = \psi^1_t F_X(t,T) + \psi^2_t F_B(t,T,T). \tag{10.37}$$

Since clearly $F_B(t,T,T) = 1$ for any $t \in [0,T]$, a portfolio ψ is self-financing in the T-forward market if its forward wealth satisfies

$$d\tilde{V}_t(\psi) = \psi^1_t dF_X(t,T) = \psi^1_t F_X(t,T)\gamma_t \cdot dW^T_t,$$

where the last equality follows from (10.33). Our aim is to find the forward portfolio ψ that replicates the forward contract written on the option, and to subsequently rederive pricing formulae (10.34)–(10.35). To replicate the forward contract written on the option, it is enough to take positions in forward contracts on a claim X and in forward contracts on T-maturity bonds. Suppose that the option's forward price equals $F_C(t,T) = u(F_X(t,T),t)$ for some function u. Arguing along similar lines as in the first proof of Theorem 3.2.1, with constant interest rate $r = 0$ and time-variable deterministic volatility γ_t, one may derive the following PDE

$$u_t(x,t) + \tfrac{1}{2}\,\gamma_t x^2\, u_{xx}(x,t) = 0,$$

with $u(x,T) = (x-K)^+$ for $x \in \mathbf{R}_+$. The solution u to this problem is given by the formula

$$u(x,t) = xN\big(\tilde{d}_1(x,t,T)\big) - KN\big(\tilde{d}_2(x,t,T)\big).$$

The corresponding strategy $\psi = (\psi^1, \psi^2)$ in the T-forward market is

$$\psi^1_t = u_x\big(F_X(t,T),t\big) = N\big(\tilde{d}_1(F_X(t,T),t,T)\big) \tag{10.38}$$

and $\psi^2_t = u(F_X(t,T),t) - \psi^1_t F_X(t,T)$. It can be checked, using Itô's formula, that the strategy ψ is self-financing in the T-forward market; moreover, $\tilde{V}_T(\psi) = V_T(\psi) = (X-K)^+$. The forward price of the option is thus given by (10.35), and consequently its spot price at time t equals

$$C_t = B(t,T)\tilde{V}_t(\psi) = B(t,T)u(F_X(t,T),t). \tag{10.39}$$

The last formula coincides with (10.34).

Forward/spot asset/bond market. It may be convenient to replicate the terminal payoff of an option by means of a combined spot/forward trading strategy. Let the date t be fixed, but arbitrary. Consider an investor who purchases at time t the number $F_C(t,T)$ of T-maturity bonds and holds them to maturity. In addition, at any date $s \ge t$ she takes ψ^1_s positions in T-maturity forward contracts on the underlying claim, where ψ^1_s is given by (10.38). The terminal wealth of such a strategy at the date T equals

$$F_C(t,T) + \int_t^T \psi^1_s\, dF_X(s,T) = F_C(t,T) + \tilde{V}_T(\psi) - \tilde{V}_t(\psi) = (X-K)^+,$$

since $\tilde{V}_t(\psi) = F_C(t,T)$ and $\tilde{V}_T(\psi) = (X-K)^+$.

Spot asset/bond market. To replicate an option in a spot market, we need to assume that it is written on an asset which is tradable in the spot market. As the second asset, we use a T-maturity bond, with the spot price $B(t,T)$. Assume that a claim X corresponds to the value Z_T of a tradable asset, whose spot price at time t equals Z_t. To replicate an option in the spot market, we consider the spot trading strategy $\phi = \psi$, where Z and a T-maturity bond are primary securities. We deduce easily from (10.39) that the wealth $V(\phi)$ equals

$$V_t(\phi) = \phi_t^1 Z_t + \phi_t^2 B(t,T) = B(t,T)V_t(\psi) = C_t,$$

so that the strategy ϕ replicates the option value at any date $t \leq T$. It remains to check that ϕ is self-financing. The following property is a general feature of self-financing strategies in the T-forward market: a T-forward trading strategy ψ is self-financing if and only if the spot market strategy $\phi = \psi$ is self-financing. Replication of a European call option with terminal payoff $(Z_T - K)^+$ can thus be done using the spot trading strategy $\phi = (\phi^1, \phi^2)$, where $\phi_t^1 = N\big(\tilde{d}_1\big(F_Z(t,T), t, T\big)\big)$ and

$$\phi_t^2 = (C_t - \phi_t^1 Z_t)/B(t,T) = -KN\big(\tilde{d}_2(F_t, t, T)\big).$$

Here, ϕ_t^1 and ϕ_t^2 represent the number of units of the underlying asset and of T-maturity bonds held at time t, respectively.

Spot asset/cash market. Let us show that since a savings account follows a process of finite variation, replication of an option written on Z in the spot asset/cash market is not always possible. Suppose that $\hat{\phi} = (\hat{\phi}^1, \hat{\phi}^2)$ is an asset/cash self-financing trading strategy which replicates an option. In particular, we have

$$\hat{\phi}_t^1 dZ_t + \hat{\phi}_t^2 dB_t = dC_t. \tag{10.40}$$

On the other hand, from the preceding paragraph, we know that

$$\phi_t^1 dZ_t + \phi_t^2 dB(t,T) = dC_t = C_t\big(r_t\, dt + \xi_t^C \cdot dW_t^*\big), \tag{10.41}$$

where

$$\xi_t^C = (\phi_t^1 Z_t \xi_t + \phi_t^2 Z_t b(t,T))/C_t.$$

A comparison of martingale parts in (10.40) and (10.41) yields

$$\phi_t^1 Z_t \xi_t \cdot dW_t^* + \phi_t^2 B(t,T) b(t,T) \cdot dW_t^* = \hat{\phi}_t^1 Z_t \xi_t \cdot dW_t^*.$$

When the underlying Brownian motion is multidimensional, we cannot solve the last equality for $\hat{\phi}_t^1$, in general. If, however, W^* is one-dimensional and processes Z and ξ are strictly positive, then we have

$$\hat{\phi}_t^1 = \phi_t^1 + \phi_t^2 b(t,T) B(t,T)/(\xi_t Z_t).$$

We put, in addition, $\hat{\phi}_t^2 = B_t^{-1}(C_t - \hat{\phi}_t^1 Z_t)$. It is clear that such a strategy replicates the option. Moreover, it is self-financing, since simple calculations show that

$$\hat{\phi}_t^1 dZ_t + \hat{\phi}_t^2 dB_t = r_t C_t\, dt + \xi_t^C C_t \cdot dW_t^* = dC_t = dV_t(\hat{\phi}).$$

For instance, the stock/cash trading strategy that involves at time t

$$\hat{\phi}_t^1 = N\big(\tilde{d}_1(F_t, t, T)\big) - K\, \frac{b(t,T)B(t,T)}{\xi_t Z_t}\, N\big(\tilde{d}_2(F_t, t, T)\big)$$

shares of stock, and the amount $C_t - \hat{\phi}_t^1 Z_t$ held in a savings account, is a self-financing strategy replicating a European call option written on Z.

10.2 Futures Prices

Our next goal is to establish the relationship between forward and futures prices. We consider an arbitrary tradable asset, whose spot price Z has the dynamics given by the expression

$$dZ_t = Z_t \left(r_t \, dt + \xi_t \cdot dW_t^* \right).$$

The forward price of Z for settlement at the date T is already known to satisfy

$$F_Z(T,T) = F_Z(t,T) \, \exp\Big(\int_t^T \gamma_Z(u,T) \cdot dW_u^T - \frac{1}{2} \int_t^T |\gamma_Z(u,T)|^2 \, du \Big),$$

where $\gamma_Z(u,T) = \xi_u - b(u,T)$, and $W_t^T = W_t^* - \int_0^t b(u,T) \, du$ is a Brownian motion under the forward measure $\mathbf{P}_T$. Since the martingale measure $\mathbf{P}^*$ for the spot market is assumed to be unique, it is natural to introduce the futures price by means of the following definition.

Definition 10.2.1 The *futures price* $f_Z(t,T)$ of an asset Z, in the futures contract that expires at time T, is given by the formula

$$f_Z(t,T) = \mathbf{E}_{\mathbf{P}^*}(Z_T \,|\, \mathcal{F}_t), \quad \forall \, t \in [0,T]. \tag{10.42}$$

Equality (10.42) defines the futures price of a contingent claim Z_T which settles at time T; hence it applies to any contingent claim which settles at time T. We are in a position to establish the relationship between the forward and futures prices of an arbitrary asset.

Proposition 10.2.1 *Assume that the volatility $\gamma_Z(\cdot,T) = \xi - b(\cdot,T)$ of the forward price process $F_Z(t,T)$ follows a deterministic function. Then the futures price $f_Z(t,T)$ equals*

$$f_Z(t,T) = F_Z(t,T) \exp\Big(\int_t^T \big(b(u,T) - \xi_u \big) \cdot b(u,T) \, du \Big). \tag{10.43}$$

Proof. It is clear that

$$F_Z(T,T) = F_Z(t,T) \, \zeta_t \, \exp\Big(\int_t^T \big(b(u,T) - \xi_u \big) \cdot b(u,T) \, du \Big),$$

where ζ_t stands for the following random variable

$$\zeta_t = \exp\Big(\int_t^T \big(\xi_u - b(u,T) \big) \cdot dW_u^* - \frac{1}{2} \int_t^T |\xi_u - b(u,T)|^2 \, du \Big).$$

The random variable ζ_t is independent of the σ-field $\mathcal{F}_t$, and its expectation under $\mathbf{P}^*$ is equal to 1 – that is, $\mathbf{E}_{\mathbf{P}^*}(\zeta_t) = 1$. Since by definition

$$f_Z(t,T) = \mathbf{E}_{\mathbf{P}^*}(Z_T \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}^*}(F_Z(T,T) \,|\, \mathcal{F}_t),$$

using the well-known properties of conditional expectation, we obtain

$$f_Z(t,T) = F_Z(t,T) \, \exp\Big(\int_t^T \big(b(u,T) - \xi_u \big) \cdot b(u,T) \, du \Big) \, \mathbf{E}_{\mathbf{P}^*}(\zeta_t),$$

which is the desired result. □

Observe that the dynamics of the futures price process $f_Z(t,T)$, $t \in [0,T]$, under the martingale measure $\mathbf{P}^*$ are

$$df_Z(t,T) = f_Z(t,T) \big(\xi_t - b(t,T) \big) \cdot dW_t^*. \tag{10.44}$$

It is interesting to note that the dynamics of the forward price $F_Z(t,T)$ under the forward measure $\mathbf{P}_T$ are given by the analogous expression

$$dF_Z(t,T) = F_Z(t,T) \big(\xi_t - b(t,T) \big) \cdot dW_t^T. \tag{10.45}$$

10.2.1 Futures Options

We shall now focus on an explicit solution for the arbitrage price of a European call option written on a futures contract on a zero-coupon bond. Let us denote by $f_B(t,U,T)$ the futures price for settlement at the date T of a U-maturity zero-coupon bond. From (10.44), we have

$$df_B(t,U,T) = f_B(t,U,T)\big(b(t,U) - b(t,T)\big) \cdot dW_t^*, \tag{10.46}$$

subject to the terminal condition $f_B(T,U,T) = B(T,U)$. The wealth process $V^f(\psi)$ of any futures trading strategy $\psi = (\psi^1, \psi^2)$ equals

$$V_t^f(\psi) = \psi_t^2 B(t,T), \quad \forall\, t \in [0,T]. \tag{10.47}$$

A futures trading strategy $\psi = (\psi^1, \psi^2)$ is said to be *self-financing* if its wealth process $V^f = V^f(\psi)$ satisfies the standard relationship

$$V_t^f(\psi) = V_0^f(\psi) + \int_0^t \psi_u^1\, df_u + \int_0^t \psi_u^2\, dB(u,T). \tag{10.48}$$

We fix U and T, and we write briefly f_t instead of $f_B(t,U,T)$ in what follows. Let us consider the relative wealth process $\tilde{V}_t^f = V_t^f(\psi)B^{-1}(t,T)$. As one might expect, the relative wealth of a self-financing futures trading strategy follows a local martingale under the forward measure $\mathbf{P}_T$. Indeed, using Itô's formula we get

$$d\tilde{V}_t^f = B^{-1}(t,T)\, dV_t^f + V_t^f\, dB^{-1}(t,T) + d\,\langle V^f, B^{-1}(\cdot,T)\rangle_t,$$

so that

$$\begin{aligned} d\tilde{V}_t^f &= B^{-1}(t,T)\psi_t^1\, df_t + B^{-1}(t,T)\psi_t^2\, dB(t,T) + \psi_t^2 B(t,T)\, dB^{-1}(t,T) \\ &\quad + \psi_t^1\, d\,\langle f, B^{-1}(\cdot,T)\rangle_t + \psi_t^2\, d\,\langle B(\cdot,T), B^{-1}(\cdot,T)\rangle_t \\ &= B^{-1}(t,T)\psi_t^1\, df_t + \psi_t^1\, d\,\langle f, B^{-1}(\cdot,T)\rangle_t. \end{aligned}$$

On the other hand, we have

$$d\langle f, B^{-1}(,T)\rangle_t = -f_t B^{-1}(t,T)\big(b(t,U) - b(t,T)\big) \cdot b(t,T)\, dt.$$

Combining these formulae , we arrive at the expression

$$d\tilde{V}_t^f(\psi) = \psi_t^1 f_t B^{-1}(t,T)\big(b(t,U) - b(t,T)\big) \cdot (dW_t^* - b(t,T)\, dt),$$

which is valid under $\mathbf{P}^*$, or equivalently, at the formula

$$d\tilde{V}_t^f(\psi) = \psi_t^1 f_t B^{-1}(t,T)\big(b(t,U) - b(t,T)\big) \cdot dW_t^T, \tag{10.49}$$

which in turn is satisfied under the forward probability measure $\mathbf{P}_T$. We conclude that the relative wealth of any self-financing futures strategy follows a local martingale under the forward measure for the date T. Therefore, to find the arbitrage price $\pi_t^f(X)$ at time $t \in [0,T]$ of any $\mathbf{P}_T$-integrable contingent claim[1] X of the form $X = g(f_T, T)$, we can make use of the equality

$$\pi_t^f(X) = B(t,T)\, \mathbf{E}_{\mathbf{P}_T}\big(X \,\big|\, \mathcal{F}_t\big), \quad \forall\, t \in [0,T]. \tag{10.50}$$

To check this, note that if ψ is a futures trading strategy replicating X, then the process $\tilde{V}^f(\psi)$ is a $\mathbf{P}_T$-martingale, and thus

$$\mathbf{E}_{\mathbf{P}_T}\big(X \,\big|\, \mathcal{F}_t\big) = \mathbf{E}_{\mathbf{P}_T}\big(\tilde{V}_T^f(\psi) \,\big|\, \mathcal{F}_t\big) = B^{-1}(t,T)\, V_t^f(\psi) = B^{-1}(t,T)\, \pi_t^f(X).$$

[1] As usual, it is implicitly assumed that a claim X is also attainable.

Proposition 10.2.2 *Assume that $U \geq T$. The arbitrage price at time $t \in [0,T]$ of a European call option with expiry date T and exercise price K, written on the futures contract for a U-maturity zero-coupon bond with delivery date T, equals*

$$C_t^f = B(t,T)\Big(f_t h_U(t,T) N\big(g_1(f_t,t,T)\big) - KN\big(g_2(f_t,t,T)\big)\Big), \tag{10.51}$$

where

$$g_1(f,t,T) = \frac{\ln(f/K) + \frac{1}{2}\int_t^T (|b(u,U)|^2 - |b(u,T)|^2)\,du}{v_U(t,T)}, \tag{10.52}$$

$g_2(f,t,T) = g_1(f,t,T) - v_U(t,T)$, *the function $v_U(t,T)$ is given by* (10.8), *and*

$$h_U(t,T) = \exp\Big(\int_t^T \big(b(u,U) - b(u,T)\big)\cdot b(u,T)\,du\Big). \tag{10.53}$$

Proof. We need to evaluate

$$C_t^f = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\big((f_B(T,U,T) - K)^+ \,\big|\, \mathcal{F}_t\big) = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\big((F_B(T,U,T) - K)^+ \,\big|\, \mathcal{F}_t\big).$$

Proceeding as in the proof of Proposition 10.1.1, we find that (cf. (10.10)–(10.11))

$$C_t^f = B(t,T)\Big(F_t N\big(\tilde{d}_1(F_t,t,T)\big) - KN\big(\tilde{d}_2(F_t,t,T)\big)\Big), \tag{10.54}$$

where $F_t = F_B(t,U,T)$,

$$\tilde{d}_1(F,t,T) = \frac{\ln(F/K) + \frac{1}{2}v_U^2(t,T)}{v_U(t,T)}, \tag{10.55}$$

$\tilde{d}_2(F,t,T) = \tilde{d}_1(F,t,T) - v_U(t,T)$, and $v_U(t,T)$ is given by (10.8). On the other hand, (10.43) yields

$$F_B(t,U,T) = f_B(t,U,T)\exp\Big(-\int_t^T \big(b(u,T) - b(u,U)\big)\cdot b(u,T)\,du\Big) + f_B(t,U,T)h_U(t,T).$$

Substituting the last formula into (10.54), we find the desired formula. □

For ease of notation, we write f_t to denote the futures price $f_Z(t,T)$ of a tradable security Z. We assume that the volatilities $b(\cdot,T)$ and $\gamma_Z(\cdot,T)$ are deterministic functions. The proof of the next result is similar to that of Proposition 10.2.2, and is thus omitted.

Proposition 10.2.3 *The arbitrage price of a European call option with expiry date T and strike price K, written on a futures contract which settles at time T for delivery of one unit of security Z, is given by the formula*

$$C_t^f = B(t,T)\Big(f_t h(t,T) N\big(g_1(f_t,t,T)\big) - KN\big(g_2(f_t,t,T)\big)\Big), \tag{10.56}$$

where

$$g_{1,2}(f,t,T) = \frac{\ln(f/K) + \ln h(t,T) \pm \frac{1}{2}v^2(t,T)}{v(t,T)} \tag{10.57}$$

for $(f,t) \in \mathbf{R}_+ \times [0,T]$, the function $v(t,T)$ is given by (10.21), *and*

$$h(t,T) = \exp\Big(\int_t^T \gamma_Z(u,T)\cdot b(u,T)\,du\Big). \tag{10.58}$$

Let us examine the put-call parity for futures options. We write P_t^f to denote the price of a European put futures option with expiry date T and strike price K. Arguing as in the proof of Proposition 10.1.3, one may establish the following result.

Proposition 10.2.4 *Under the assumptions of Proposition* 10.2.3, *the following put-call parity relationship is valid*

$$C_t^f - P_t^f = B(t,T)\Big(f_Z(t,T)\exp\Big(\int_t^T \gamma_Z(u,T)\cdot b(u,T)\,du\Big) - K\Big).$$

10.3 PDE Approach to Interest Rate Derivatives

This section presents the PDE approach to the hedging and valuation of contingent claims in the Gaussian HJM setting. As discussed in Chap. 8, PDEs play an important role in pricing of term-structure derivatives in the framework of diffusion models of short-term interest rates. In such a case, one works with the PDE satisfied by the price process of an interest rate-sensitive security, considered as a function of the time parameter t and the current value of a short-term rate r_t. In the present setting, however, it is not assumed that the short-term rate follows a diffusion process. The PDEs examined in this section are directly related to the price dynamics of bonds and underlying assets. To be more specific, the arbitrage price of a derivative security is expressed in terms of the time parameter t, the current price of an underlying asset and the price of a certain zero-coupon bond. For ease of exposition, we focus on the case of spot and futures European call options (for the proofs, see Rutkowski (1996)).

10.3.1 PDEs for Spot Derivatives

We start by examining the case of a European call option with expiry date T written on a tradable asset Z. We assume throughout that the dynamics of the (spot) price process of Z are governed under a probability measure $\mathbf{P}$ by the expression[2]

$$dZ_t = Z_t \big(\mu_t \, dt + \xi_t \cdot dW_t\big), \tag{10.59}$$

where μ is a stochastic process. For a fixed date $D \geq T,$ the price of a bond which matures at time D is assumed to follow, under $\mathbf{P}$

$$dB(t,D) = B(t,D) \big(\kappa_t \, dt + b(t,D) \cdot dW_t\big), \tag{10.60}$$

where κ is a stochastic process. Volatilities ξ and $b(\cdot, D)$ can also follow stochastic processes; we shall assume, however, that the volatility $\xi_t - b(t,D)$ of the forward price of Z is deterministic.

We consider a European option, with expiry date $T,$ written on the forward price of Z for the date $D,$ where $D \geq T.$ More precisely, by definition the option's payoff at expiry equals

$$C_T = B(T,D)(F_Z(T,D) - K)^+ = (Z_T - KB(T,D))^+.$$

When $D = T,$ we deal with a standard option written on $Z.$ For $D > T,$ the option can be interpreted either as an option written on the forward price of $Z,$ with deferred payoff at time $D,$ or simply as an option to exchange one unit of an asset Z for K units of D-maturity bonds.

Proposition 10.3.1 *Assume that the price processes Z and $B(t,D)$ follow* (10.59) *and* (10.60), *respectively, and the volatility $\xi_t - b(t,D)$ of the forward price is deterministic. Consider a European contingent claim X, of the form $X = B(T,D)g(Z_T/B(T,D))$, which settles at time T. The arbitrage price of X equals*

$$\pi_t(X) = v\big(Z_t, B(t,D), t\big) = B(t,D) H\big(Z_t B^{-1}(t,D), t\big)$$

for every $t \in [0,T],$ where the function $H : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ solves the following PDE

$$\frac{\partial H}{\partial t}(z,t) + \frac{1}{2}\, |\xi_t - b(t,D)|^2 \, z^2 \frac{\partial^2 H}{\partial z^2}(z,t) = 0,$$

with the terminal condition $H(z,T) = g(z)$ for every $z \in \mathbf{R}_+$.

Remarks. A savings account can be used in the replication of European claims which settle at time T and have the form $X = B_T g(Z_T/B_T)$ for some function g. Let us consider, for instance,

[2] We assume implicitly that Z follows a strictly positive process. It should be stressed that $\mathbf{P}$ is not necessarily a martingale measure.

a European option with expiry date T and terminal payoff $(Z_T - KB_T)^+$. If the volatility of the underlying asset is deterministic, Proposition 10.3.1 is in force, and thus replication of such an option involves $N\big(k_1(Z_t, B_t, t, T)\big)$ units of the underlying asset, combined with the amount $-KB_tN\big(k_2(Z_t, B_t, t, T)\big)$ held in a savings account. In general, a standard European option cannot be replicated using a savings account.

10.3.2 PDEs for Futures Derivatives

Let us fix three dates T, D and R, such that $T \le \min\{D, R\}$. The futures price of an asset Z in a contract which settles at time R satisfies

$$df_Z(t,R) = f_Z(t,R)\big(\xi_t - b(t,R)\big) \cdot dW_t. \tag{10.61}$$

We have assumed that the drift coefficient in the dynamics of $f_Z(t,R)$ vanishes. This is not essential, however. Indeed, suppose that a non-zero drift in the dynamics of the futures price is present. Then we may either modify all foregoing considerations in a suitable way, or, more conveniently, we may first make, using Girsanov's theorem, an equivalent change of an underlying probability measure in such a way that the drift of the futures price will disappear. The drift of bond price will thus change – however, it is arbitrary, and it does not enter the final result anyway. It is convenient to assume that the volatility ζ of the futures price and the bond price volatility $b(\cdot, R)$ are deterministic functions.

For convenience, we shall write f_t instead of $f_Z(t,R)$. Consider a European option with the terminal payoff $C_T^f = B(T,D)(f_T - K)^+$ at time T, or equivalently $(f_T - K)^+$ at time D (hence it is a standard futures option with deferred payoff).

We are in a position to formulate a counterpart of Proposition 10.3.1.

Proposition 10.3.2 *Suppose that the futures price $f_t = F_Z(t,R)$ of an asset Z satisfies* (10.61), *where the volatility $\zeta_t = \xi_t - b(t,R)$ is such that $\zeta_t \cdot b(t,D)$ is a deterministic function. Let X be a European contingent claim which settles at time T and has the form $X = B(T,D)g(f_T)$ for some function $g : \mathbf{R}_+ \to \mathbf{R}$. The arbitrage price of X equals*

$$\pi_t^f(X) = v\big(f_t, B(t,D), t\big) = B(t,D)L\big(f_t e^{\eta(t,T)}, t\big)$$

for every $t \in [0,T]$, where

$$\eta(t,T) = \int_t^T (\xi_u - b(u,R)) \cdot b(u,D)\, du, \quad \forall\, t \in [0,T],$$

and the function $L = L(z,t)$ solves the PDE

$$\frac{\partial L}{\partial t}(z,t) + \frac{1}{2}\,|\zeta_t|^2\, z^2\, \frac{\partial^2 L}{\partial z^2}(z,t) = 0,$$

with the terminal condition $L(z,T) = g(z)$ for every $z \in \mathbf{R}_+$.

Example 10.3.1 Assume that a futures contract has a zero-coupon bond which matures at time $U \ge R$ as the underlying asset. Then $\xi_u = b(u,U)$ and thus

$$v_f^2(t,T) = \int_t^T |b(u,U) - b(u,R)|^2\, du.$$

Moreover, in this case we have (writing f in place of x)

$$l_1(f,t,T) = \frac{\ln(f/K) + \int_t^T \gamma(u,U,R) \cdot b(u,D)\, du + \frac{1}{2}\int_t^T |\gamma(u,U,R)|^2\, du}{v(t,T)},$$

where $\gamma(u,U,R) = b(u,U) - b(u,R)$. In particular, if $D = R = T$ we obtain $l_1(f,t,T) = g_1(f,t,T)$ and $\eta(t,T) = h(t,T)$, where g_1 and h are given by (10.52) and (10.53) respectively.

Chapter 11

Valuation of Swap Derivatives

The aim of this chapter is twofold. First, we give a general description of basic swap derivatives. We shall examine the most typical examples of interest rate derivatives such as *interest rate swaps, caps, floors* and *swaptions.* Second, we provide the explicit valuation solutions for some of these instruments within the Gaussian HJM framework and in the case of lognormal models of LIBOR. In the last section, an alternative model of the term structure, put forward by Jamshidian (1997), is presented.

11.1 Interest Rate Swaps

Let us consider a *forward start payer swap* settled *in arrears,* with notional principal N. We consider a finite collection of dates $T_j,\ j=0,\dots,n,$ where, for simplicity, $T_j - T_{j-1} = \delta$ for every $j=1,\dots,n.$ The floating rate $L(T_j)$ received at time T_{j+1} is set at time T_j by reference to the price of a zero-coupon bond over that period – namely, $L(T_j)$ satisfies

$$B(T_j,T_{j+1})^{-1} = 1 + (T_{j+1}-T_j)L(T_j) = 1 + \delta L(T_j). \tag{11.1}$$

Formula (11.1) agrees with market quotations of LIBOR; indeed, $L(T_j)$ is the spot LIBOR rate prevailing at time T_j for the period of length $\delta = T_{j+1} - T_j$. More generally, the forward LIBOR rate $L(t,T_j)$ for the future time period $[T_j, T_{j+1}]$ of length δ satisfies

$$1 + \delta L(t,T_j) = \frac{B(t,T_j)}{B(t,T_{j+1})} = F_B(t,T_j,T_{j+1}), \tag{11.2}$$

so that $L(T_j)$ coincides with $L(T_j,T_j)$. At any date $T_j,\ j=1,\dots n,$ the cash flows of a payer swap are $L(T_{j-1})\delta N$ and $-\kappa\delta N$, where κ is a preassigned fixed rate of interest (the cash flows of a receiver swap have the same size, but opposite signs). The number $n,$ which coincides with the number of payments, is referred to as the *length* of a swap, the dates $T_0,\dots,T_{n-1}$ are known as *reset dates,* and the dates $T_1,\dots,T_n$ as *settlement dates.* We shall refer to the first reset date T_0 as the *start date* of a swap. Finally, the time interval $[T_{j-1},T_j]$ is referred to as the j^{th} *accrual period.* We may and do assume, without loss of generality, that the notional principal $N=1$.

The value at time t of a forward start payer swap, which is denoted by $\mathbf{FS}_t$ or $\mathbf{FS}_t(\kappa)$, equals

$$\begin{aligned} \mathbf{FS}_t(\kappa) &= \mathbf{E}_{\mathbf{P}^*}\Big\{ \sum_{j=1}^n \frac{B_t}{B_{T_j}} (L(T_{j-1})-\kappa)\delta \,\Big|\, \mathcal{F}_t \Big\} \\ &= \sum_{j=1}^n \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{B_t}{B_{T_j}} \Big(B(T_{j-1},T_j)^{-1} - \tilde{\delta} \Big) \,\Big|\, \mathcal{F}_t \Big\}, \end{aligned}$$

where we write $\tilde{\delta} = 1 + \kappa\delta$. Consequently,

$$\begin{aligned} \mathbf{FS}_t(\kappa) &= \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*}\Big\{ B(T_{j-1},T_j)^{-1}\, \frac{B_t}{B_{T_{j-1}}}\, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_{T_{j-1}}}{B_{T_j}} \,\Big|\, \mathcal{F}_{T_{j-1}}\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &\quad - \sum_{j=1}^{n} \tilde{\delta}\, \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t}{B_{T_j}} \,\Big|\, \mathcal{F}_t\Big) = \sum_{j=1}^{n}\Big(B(t,T_{j-1}) - \tilde{\delta} B(t,T_j)\Big), \end{aligned}$$

which, after rearranging, yields

$$\mathbf{FS}_t(\kappa) = B(t,T_0) - \sum_{j=1}^{n} c_j B(t,T_j) \tag{11.3}$$

for every $t \in [0,T]$, where $c_j = \kappa\delta$ for $j = 1,\ldots,n-1$, and $c_n = \tilde{\delta} = 1 + \kappa\delta$. The last equality makes clear that a forward swap settled in arrears is, essentially, a contract to deliver a specific coupon-bearing bond and to receive in the same time a zero-coupon bond. This relationship, which in fact may be inferred by a straightforward comparison of the future cash flows from these bonds, provides a simple method for the replication of a swap contract. As mentioned, a swap agreement is worthless at initiation. This important feature of a swap leads to the following definition, which refers in fact to the more general concept of a forward swap. Basically, a forward swap rate is that fixed rate of interest which makes a forward swap worthless.

Definition 11.1.1 The *forward swap rate* $\kappa(t,T,n)$ at time t for the date T is that value of the fixed rate κ which makes the value of the forward swap zero, i.e., that value of κ for which $\mathbf{FS}_t(\kappa) = 0$. Using (11.3), we obtain

$$\kappa(t,T,n) = (B(t,T) - B(t,T_n))\Big(\delta \sum_{j=1}^{n} B(t,T_j)\Big)^{-1}. \tag{11.4}$$

A *swap* (*swap rate,* respectively) is the forward swap (forward swap rate, respectively) with $t = T$. The swap rate, $\kappa(T,T,n)$, equals

$$\kappa(T,T,n) = (1 - B(T,T_n))\Big(\delta \sum_{j=1}^{n} B(T,T_j)\Big)^{-1}. \tag{11.5}$$

Note that the definition of a forward swap rate implicitly refers to a swap contract of length n which starts at time T. It would thus be more correct to refer to $\kappa(t,T,n)$ as the *n-period forward swap rate* prevailing at time t, for the future date T. A forward swap rate is a rather theoretical concept, as opposed to swap rates, which are quoted daily (subject to an appropriate bid-ask spread) by financial institutions who offer interest rate swap contracts to their institutional clients. In practice, swap agreements of various lengths are offered. Also, typically, the length of the reference period varies over time; for instance, a 5-year swap may be settled quarterly during the first three years, and semi-annually during the last two. For the sake of simplicity, we assume throughout that all reference periods $T_j - T_{j-1}$ of a swap are of the same length, which is denoted by δ. Swap rates also play an important role as a basis for several derivative instruments. For instance, an appropriate swap rate is commonly used as a strike level for an option written on the value of a swap; that is, a *swaption.*

Remarks. Let us examine one leg of a swap – that is, an interest rate swap agreement with only one payment date. For $n = 1$, (11.4) gives

$$\kappa(t,T,U) = \frac{B(t,T) - B(t,U)}{B(t,U)(U-T)},$$

where we write U and $\kappa(t,T,U)$ instead of T_1 and $\kappa(t,T,1)$, respectively. Using (7.3), we find that

$$\kappa(0,T,U) \approx \frac{UY(0,U)-TY(0,T)}{U-T} = f(0,T,U),$$

where "$\approx$" denotes approximate equality. This shows that the swap rate does not coincide with the forward interest rate $f(t,T,U)$ determined by a forward rate agreement.

11.2 Gaussian Model

The basic swap derivatives we shall now examine are: *caps, floors, captions, swaptions, options on a swap rate spread,* and *yield curve swaps.* Further examples of swap derivatives will be discussed in the next chapter.

11.2.1 Forward Caps and Floors

An *interest rate cap* (known also as a *ceiling rate agreement,* or briefly *CRA*) is a contractual arrangement where the grantor (seller) has an obligation to pay cash to the holder (buyer) if a particular interest rate exceeds a mutually agreed level at some future date or dates. Similarly, in an *interest rate floor,* the grantor has an obligation to pay cash to the holder if the interest rate is below a preassigned level. When cash is paid to the holder, the holder's net position is equivalent to borrowing (or depositing) at a rate fixed at that agreed level. This assumes that the holder of a cap (or floor) agreement also holds an underlying asset (such as a deposit) or an underlying liability (such as a loan). Finally, the holder is not affected by the agreement if the interest rate is ultimately more favorable to him than the agreed level. This feature of a cap (or floor) agreement makes it similar to an option. Specifically, a *forward start cap* (or a *forward start floor*) is a strip of caplets (floorlets), each of which is a call (put) option on a forward rate, respectively. Let us denote by κ and by δ the cap strike rate and the length of a caplet, respectively. We shall check that an interest rate caplet (i.e., one leg of a cap) may also be seen as a put option with strike price 1 (per dollar of notional principal) which expires at the caplet start day on a discount bond with face value $1+\kappa\delta$ which matures at the caplet end date. This property makes the valuation of a cap relatively simple; essentially, it can be reduced to the problem of option pricing on zero-coupon bonds.

Similarly to swap agreements, interest rate caps and floors may be settled either *in arrears* or *in advance.* In a forward cap or floor with the notional principal N settled in arrears at dates T_j, $j=1,\dots,n,$ where $T_j - T_{j-1} = \delta$ and $T_0 = T,$ the cash flows at times T_j are $N(L(T_{j-1})-\kappa)^+\delta$ and $N(\kappa - L(T_{j-1}))^+\delta$, respectively. As usual, the rate $L(T_{j-1})$ is determined at the reset date T_{j-1}, and it satisfies

$$B(T_{j-1},T_j)^{-1} = 1 + L(T_{j-1})(T_j - T_{j-1}). \tag{11.6}$$

The arbitrage price at time $t \le T_0$ of a *forward cap,* denoted by $\mathbf{FC}_t,$ is (we assume that $N=1$)

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t}{B_{T_j}}(L(T_{j-1})-\kappa)^+\delta \,\Big|\, \mathcal{F}_t\Big). \tag{11.7}$$

Let us consider a *caplet* (i.e., one leg of a cap) with reset date T and settlement date $T+\delta$. The value at time t of a caplet equals

$$\begin{aligned}
\mathbf{Cpl}_t &= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_{T+\delta}}\Big((B(T,T+\delta)^{-1}-1)\delta^{-1}-\kappa\Big)^+\delta \,\Big|\, \mathcal{F}_t\Big\} \\
&= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_{T+\delta}}\Big(\frac{1}{B(T,T+\delta)}-\tilde{\delta}\Big)^+ \,\Big|\, \mathcal{F}_t\Big\} \\
&= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(\frac{1}{B(T,T+\delta)}-\tilde{\delta}\Big)^+ \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t}{B_{T+\delta}}\,\Big|\,\mathcal{F}_T\Big) \,\Big|\, \mathcal{F}_t\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(1-\tilde\delta B(T,T+\delta)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}\\
&= B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\Big\{\big(1-\tilde\delta B(T,T+\delta)\big)^+\,\Big|\,\mathcal{F}_t\Big\},
\end{aligned}
$$

where the last equality was deduced from Lemma 9.2.3. It is apparent that a caplet is a put option on a zero-coupon bond; it is also an option on a one-period swap.

Remarks. Since the cash flow of the j^{th} caplet at time T_j is manifestly a $\mathcal{F}_{T_{j-1}}$-measurable random variable, we may use Corollary 9.2.1 to express the value of the cap in terms of expectations under forward measures. Indeed, from (9.31) we have

$$
\mathbf{FC}_t=\sum_{j=1}^{n} B(t,T_{j-1})\,\mathbf{E}_{\mathbf{P}_{T_{j-1}}}\Big(B(T_{j-1},T_j)(L(T_{j-1})-\kappa)^+\delta\,\Big|\,\mathcal{F}_t\Big). \tag{11.8}
$$

Consequently, using (11.6) we get equality

$$
\mathbf{FC}_t=\sum_{j=1}^{n} B(t,T_{j-1})\,\mathbf{E}_{\mathbf{P}_{T_{j-1}}}\Big(\big(1-\tilde\delta B(T_{j-1},T_j)\big)^+\,\Big|\,\mathcal{F}_t\Big), \tag{11.9}
$$

which is valid for every $t\in[0,T]$. The equivalence of a cap and a put option on a zero-coupon bond can be explained in an intuitive way. For this purpose, it is enough to examine two basic features of both contracts: the exercise set and the payoff value. Let us consider the j^{th} caplet. A caplet is exercised at time T_{j-1} if and only if $L(T_{j-1})-\kappa>0$, or equivalently, if

$$
B(T_{j-1},T_j)^{-1}=1+L(T_{j-1})(T_j-T_{j-1})>1+\kappa\delta=\tilde\delta.
$$

The last inequality holds whenever $\tilde\delta B(T_{j-1},T_j)<1$. This shows that both of the considered options are exercised in the same circumstances. If exercised, the caplet pays $\delta(L(T_{j-1})-\kappa)$ at time T_j, or equivalently

$$
\delta B(T_{j-1},T_j)(L(T_{j-1})-\kappa)=1-\tilde\delta B(T_{j-1},T_j)=\tilde\delta\left(\tilde\delta^{-1}-B(T_{j-1},T_j)\right)
$$

at time T_{j-1}. This shows once again that the caplet with strike level κ and nominal value 1 is essentially equivalent to a put option with strike price $(1+\kappa\delta)^{-1}$ and nominal value $(1+\kappa\delta)$ written on the corresponding zero-coupon bond with maturity T_j.

We assume that the bond price volatility is a deterministic function – as before, such an assumption is referred to as the Gaussian model. The following lemma is an immediate consequence of Proposition 10.1.1. Recall that for any dates $U,T\in[0,T^*]$, we denote $F_B(t,U,T)=B(t,U)/B(t,T)$ and $\gamma(t,U,T)=b(t,U)-b(t,T)$.

Lemma 11.2.1 *For any $T\le T^*-\delta$, the arbitrage price at time $t\in[0,U]$ of a caplet with expiry date T, settlement date $T+\delta$, and strike level κ equals*

$$
\mathbf{Cpl}_t=B(t,T)\Big(N\big(e_1(t,T)\big)-\tilde\delta F_B(t,T+\delta,T)N\big(e_2(t,T)\big)\Big), \tag{11.10}
$$

where

$$
e_{1,2}(t,T)=\frac{\ln F_B(t,T,T+\delta)-\ln\tilde\delta\pm\frac12 v^2(t,T)}{v(t,T)}
$$

and

$$
v^2(t,T)=\int_t^T|\gamma(u,T,T+\delta)|^2\,du.
$$

The next result provides a general pricing formula for a forward cap in the Gaussian case.

Proposition 11.2.1 *The price at time $t \le T_0$ of an interest rate cap with strike level κ, settled in arrears at times T_j, $j = 1, \ldots, n$, equals*

$$\mathbf{FC}_t = \sum_{j=1}^{n} B(t, T_{j-1}) \Big(N\big(e_1^j(t)\big) - \tilde{\delta} F_B(t, T_j, T_{j-1}) N\big(e_2^j(t)\big)\Big), \tag{11.11}$$

where

$$e_{1,2}^j(t) = \frac{\ln F_B(t, T_{j-1}, T_j) - \ln \tilde{\delta} \pm \frac{1}{2} v_j^2(t,)}{v_j(t)} \tag{11.12}$$

and

$$v_j^2(t) = \int_t^{T_{j-1}} |\gamma(u, T_{j-1}, T_j)|^2 \, du. \tag{11.13}$$

Proof. We represent the price of a forward cap in the following way

$$\begin{aligned}
\mathbf{FC}_t &= \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)^+ \delta \,\Big|\, \mathcal{F}_t \Big\} \\
&= \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_{T_j}} \Big((B(T_{j-1}, T_j)^{-1} - 1)\delta^{-1} - \kappa \Big)^+ \delta \,\Big|\, \mathcal{F}_t \Big\} \\
&= \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_{T_{j-1}}} \Big(1 - \tilde{\delta} B(T_{j-1}, T_j) \Big)^+ \,\Big|\, \mathcal{F}_t \Big\} = \sum_{j=1}^{n} \mathbf{Cpl}_t^j,
\end{aligned}$$

where $\mathbf{Cpl}_t^j$ stands for the price at time t of the j^{th} caplet. The assertion now follows from Lemma 11.2.1. □

The price of a *forward floor* at time $t \in [0, T]$ equals

$$\mathbf{FF}_t = \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*} \Big(\frac{B_t}{B_{T_j}} (\kappa - L(T_{j-1}))^+ \delta \,\Big|\, \mathcal{F}_t \Big). \tag{11.14}$$

Using a trivial equality

$$(\kappa - L(T_{j-1}))^+ \delta = (L(T_{j-1}) - \kappa)^+ \delta - (L(T_{j-1}) - \kappa)\delta,$$

we find that the following cap-floor parity relationship is satisfied at any time $t \in [0, T]$ (the three contracts are assumed to have the same payment dates)

$$\textit{Forward Cap}\,(t) \;-\; \textit{Forward Floor}\,(t) \;=\; \textit{Forward Swap}\,(t).$$

This relationship can also be verified by a straightforward comparison of the corresponding cash flows of both portfolios. By combining the valuation formulae for caps and swaps, we find easily that

$$\mathbf{FF}_t = \mathbf{FC}_t - \mathbf{FS}_t = \sum_{j=1}^{n} \Big(\tilde{\delta} B(t, T_j) N\big(-e_2^j(t)\big) - B(t, T_{j-1}) N\big(-e_1^j(t)\big)\Big).$$

Let us mention that by a *cap* (*floor*, respectively), we mean a forward cap (forward floor, respectively) with $t = T$.

11.2.2 Captions

Since a caplet is essentially a put option on a zero-coupon bond, a European call option on a caplet is an example of a compound option. More exactly, it is a call option on a put option with a zero-coupon bond as the underlying asset of the underlying put option. Therefore, the valuation of a call option on a caplet can be done along the same lines as in Chap. 6 (provided, of course, that the model of a zero-coupon bond price has sufficiently good properties). A call option on a cap, or a *caption,* is thus a call on a portfolio of put options. To price a caption observe that its payoff at expiry is

$$\mathbf{CC}_T = \Big(\sum_{j=1}^{n} \mathbf{Cpl}_T^j - K \Big)^+,$$

where as usual $\mathbf{Cpl}_T^j$ stands for the price at time T of the j^{th} caplet of the cap, T is the call option's expiry date and K is its strike price. Suppose that we place ourselves within the framework of the spot rate models of Chap. 8 – for instance, the Hull-White model. Typically, the caplet price is an increasing function of the current value of the spot rate r_t. Let r^* be the critical level of interest rate, which is implicitly determined by the equality $\sum_{j=1}^{n} \mathbf{Cpl}_T^j(r^*) = K$. It is clear that the option is exercised when the rate r_T is greater than r^*. Let us introduce numbers K_j by setting $K_j = \mathbf{Cpl}_T^j(r^*)$ for $j = 1, \ldots, n$. It is easily seen that the caption's payoff is equal to the sum of the payoffs of n call options on particular caplets, with K_j being the corresponding strike prices. Consequently, the caption's price $\mathbf{CC}_t$ at time $t \le T$ is given by the formula

$$\mathbf{CC}_t = \sum_{j=1}^{n} C_t(\mathbf{Cpl}^j, T, K_j),$$

where $C_t(\mathbf{Cpl}^j, T, K_j)$ is the price at time t of a call option with expiry date T and strike level K_j written on the j^{th} caplet (see Hull and White (1994)). An option on a cap (or floor) can also be studied within the Gaussian HJM framework (see Brace and Musiela (1997)). However, results concerning caption valuation within this framework are less explicit than in the case of the Hull-White model.

11.2.3 Swaptions

The owner of a *payer* (*receiver,* respectively) *swaption* with strike rate κ, maturing at time $T = T_0$, has the right to enter at time T the underlying forward payer (receiver, respectively) swap settled in arrears. Because $\mathbf{FS}_T(\kappa)$ is the value at time T of the payer swap with the fixed interest rate κ, it is clear that the price of the payer swaption at time t equals

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{B_t}{B_T} \Big(\mathbf{FS}_T(\kappa) \Big)^+ \Big| \mathcal{F}_t \Big\}.$$

More explicitly, we have

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{B_t}{B_T} \Big(\mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (L(T_{j-1}) - \kappa)\delta \Big| \mathcal{F}_T \Big) \Big)^+ \Big| \mathcal{F}_t \Big\}. \tag{11.15}$$

For the receiver swaption, we have

$$\mathbf{RS}_t = \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{B_t}{B_T} \Big(-\mathbf{FS}_T(\kappa) \Big)^+ \Big| \mathcal{F}_t \Big\},$$

that is

$$\mathbf{RS}_t = \mathbf{E}_{\mathbf{P}^*}\Big\{ \frac{B_t}{B_T} \Big(\mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (\kappa - L(T_{j-1}))\delta \Big| \mathcal{F}_T \Big) \Big)^+ \Big| \mathcal{F}_t \Big\}, \tag{11.16}$$

where we write $\mathbf{RS}_t$ to denote the price at time t of a receiver swaption. We will now focus on a payer swaption. In view of (11.15), it is apparent that a payer swaption is exercised at time T if and only if the value of the underlying swap is positive at this date. It should be made clear that a swaption may be exercised by its owner only at its maturity date T. If exercised, a swaption gives rise to a sequence of cash flows at prescribed future dates. By considering the future cash flows from a swaption and from the corresponding *market swap*[1] available at time T, it is easily seen that the owner of a swaption is protected against the adverse movements of the swap rate that may occur before time T. Suppose, for instance, that the swap rate at time T is greater than κ. Then by combining the swaption with a market swap, the owner of a swaption with strike rate κ is entitled to enter at time T, at no additional cost, a swap contract in which the fixed rate is κ. If, on the contrary, the swap rate at time T is less than κ, the swaption is worthless, but its owner is, of course, able to enter a market swap contract based on the current swap rate $\kappa(T,T,n) \leq \kappa$. Concluding, the fixed rate paid by the owner of a swaption who intends to initiate a swap contract at time T will never be above the preassigned level κ. Since we may rewrite (11.15) as follows

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(1-\sum_{j=1}^n c_j B(T,T_j)\Big)^+ \,\Big|\, \mathcal{F}_t\Big\}, \tag{11.17}$$

the payer swaption may also be seen as a put option on a coupon-bearing bond with the coupon rate κ. Similar remarks are valid for the receiver swaption. It follows easily from (11.15)–(11.16) that $\mathbf{PS}_t - \mathbf{RS}_t = \mathbf{FS}_t$, i.e.,

$$\textit{Payer Swaption}\,(t) \;-\; \textit{Receiver Swaption}\,(t) \;=\; \textit{Forward Swap}\,(t)$$

provided that both swaptions expire at the same date T (and have the same contractual features). We shall now show that a payer (receiver, respectively) swaption can also be viewed as a sequence of call (put, respectively) options on a swap rate which are not allowed to be exercised separately. At time T the long party receives the value of a sequence of cash flows, discounted from time $T_j,\ j=1,\dots,n$, to the date T, defined by

$$d_j^p = \delta\,(\kappa(T,T,n)-\kappa)^+, \quad d_j^r = \delta\,(\kappa-\kappa(T,T,n))^+,$$

for the payer option and the receiver option, respectively, where

$$\kappa(T,T,n) = (1-B(T,T_n))\Big(\delta\sum_{j=1}^n B(T,T_j)\Big)^{-1}$$

is the corresponding swap rate at the option's expiry. Indeed, the price at time t of the call (payer) option on a swap rate is

$$\begin{aligned} C_t &= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\mathbf{E}_{\mathbf{P}^*}\Big(\sum_{j=1}^n \frac{B_T}{B_{T_j}}(\kappa(T,T,n)-\kappa)^+\delta \,\Big|\, \mathcal{F}_T\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(1-\sum_{j=1}^n c_j B(T,T_j)\Big)^+ \,\Big|\, \mathcal{F}_t\Big\} \\ &= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbf{E}_{\mathbf{P}^*}\Big(\sum_{j=1}^n \frac{B_T}{B_{T_j}}(L(T_{j-1})-\kappa)\delta \,\Big|\, \mathcal{F}_T\Big)\Big)^+ \,\Big|\, \mathcal{F}_t\Big\}, \end{aligned}$$

which is the payer swaption price $\mathbf{PS}_t$. Equality $C_t = \mathbf{PS}_t$ may also be derived by directly verifying that the future cash flows from the following portfolios established at time T are identical: portfolio A – a swaption and a market swap; and portfolio B – an option on a swap rate and a market swap. Indeed, both portfolios correspond to a payer swap with the fixed rate equal to κ. Similarly, for every $t \leq T$, the price of the put (receiver) option on a swap rate is (as before, $c_j = \kappa\delta$, $j=1,\dots,n-1$,

[1]At any time t, a *market swap* is that swap whose current value equals zero. Put more explicitly, it is the swap in which the fixed rate κ equals the current swap rate.

and $c_n = 1 + \kappa\delta$)

$$
\begin{aligned}
P_t &= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\mathbf{E}_{\mathbf{P}^*}\Big(\sum_{j=1}^n \frac{B_T}{B_{T_j}}(\kappa-\kappa(T,T,n))^+\delta\,\Big|\,\mathcal{F}_T\Big)\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(\sum_{j=1}^n c_j B(T,T_j)-1\Big)^+\,\Big|\,\mathcal{F}_t\Big\}\\
&= \mathbf{E}_{\mathbf{P}^*}\Big\{\frac{B_t}{B_T}\Big(\mathbf{E}_{\mathbf{P}^*}\Big(\sum_{j=1}^n \frac{B_T}{B_{T_j}}(\kappa-L(T_{j-1}))\delta\,\Big|\,\mathcal{F}_T\Big)\Big)^+\,\Big|\,\mathcal{F}_t\Big\},
\end{aligned}
$$

which equals the price $\mathbf{RS}_t$ of the receiver swaption. As mentioned earlier, a payer (receiver, respectively) swaption may be seen as a put (call, respectively) option on a coupon bond with strike price 1 and coupon rate equal to the strike rate κ of the underlying forward swap. Therefore, the arbitrage price of payer and receiver swaptions can be evaluated by applying the general valuation formula to the functions

$$
g^p(x_1,\dots,x_n) = \Big(1-\sum_{j=1}^n c_j x_j\Big)^+, \quad g^r(x_1,\dots,x_n) = \Big(\sum_{j=1}^n c_j x_j - 1\Big)^+
$$

for a payer and a receiver swaption, respectively. Let us rederive the valuation formula for the payer swaption in a more intuitive way. Recall that a payer swaption is essentially a sequence of fixed payments $d_j^p = \delta(\kappa(T,T,n)-\kappa)^+$ which are received at settlement dates $T_1,\dots,T_n$, but whose value is known already at the expiry date T. Therefore, the random variable d_j^p is $\mathcal{F}_T$-measurable, and thus we may directly apply Corollary 9.2.1, which gives

$$
\mathbf{PS}_t = B(t,T)\sum_{j=1}^n \mathbf{E}_{\mathbf{P}_T}\Big(\delta B(T,T_j)(\kappa(T,T,n)-\kappa)^+\,\Big|\,\mathcal{F}_t\Big)
$$

for every $t \in [0,T]$. After simple manipulations, this yields, as expected

$$
\mathbf{PS}_t = B(t,T)\,\mathbf{E}_{\mathbf{P}_T}\Big\{\Big(1-\sum_{j=1}^n c_j B(T,T_j)\Big)^+\,\Big|\,\mathcal{F}_t\Big\}. \tag{11.18}
$$

Let us now consider a *forward swaption.* In this case, we assume that the expiry date $\hat{T}$ of the swaption precedes the initiation date T of the underlying payer swap – that is, $\hat{T} \le T$. Notice that if κ is a fixed strike level, then we have always

$$
\mathbf{FS}_t(\kappa) = \mathbf{FS}_t(\kappa) - \mathbf{FS}_t(\kappa(t,T,n)),
$$

as by the definition of the forward swap rate we have $\mathbf{FS}_t(\kappa(t,T,n)) = 0$. A direct application of valuation result (11.3) to both members on the right-hand side of the last equality yields

$$
\mathbf{FS}_t(\kappa) = \sum_{j=1}^n \big(\kappa(t,T,n)-\kappa\big)B(t,T_j)
$$

for $t \in [0,T]$. It is thus clear that the payoff $\mathbf{PS}_{\hat{T}}$ at expiry $\hat{T}$ of the forward swaption (with strike 0) is either 0, if $\kappa \ge \kappa(\hat{T},T,n)$, or

$$
\mathbf{PS}_{\hat{T}} = \sum_{j=1}^n \big(\kappa(\hat{T},T,n)-\kappa\big)B(\hat{T},T_j)
$$

if, on the contrary, inequality $\kappa(\hat{T}, T, n) > \kappa$ holds. We conclude that the payoff $\mathbf{PS}_{\hat{T}}$ of the forward swaption can be represented in the following way

$$\mathbf{PS}_{\hat{T}} = \sum_{j=1}^{n} \big(\kappa(\hat{T}, T, n) - \kappa\big)^{+} B(\hat{T}, T_j). \tag{11.19}$$

This means that, if exercised, the forward swaption gives rise to a sequence of equal payments $\kappa(\hat{T}, T, n) - \kappa$ at each settlement date $T_1, \ldots, T_n$. By substituting $\hat{T} = T$ we recover, in a more intuitive way and in a more general setting, the previously observed dual nature of the swaption: it may be seen either as an option on the value of a particular (forward) swap or, equivalently, as an option on the corresponding (forward) swap rate. It is also clear that the owner of a forward swaption is able to enter at time $\hat{T}$ (at no additional cost) into a forward payer swap with preassigned fixed interest rate κ.

The following result provides a quasi-explicit formula for the arbitrage price of a payer swaption in the Gaussian framework (the price of a receiver swaption is given by an analogous expression). Formula (11.20) can be easily generalized to the case of a forward swaption. To this end, it is enough to consider the following claim which settles at time $\hat{T}$ (cf. (11.4))

$$\mathbf{PS}_{\hat{T}} = \sum_{j=1}^{n} B(\hat{T}, T_j) \left(\frac{B(\hat{T}, T) - B(\hat{T}, T_n)}{\delta \sum_{i=1}^{n} B(\hat{T}, T_i)} - \kappa \right)^{+}.$$

To value such a claim within the Gaussian framework, it is enough to apply Proposition 10.1.5. As usual, we write n_k to denote the standard k-dimensional Gaussian density function.

Proposition 11.2.2 *Assume the Gaussian model of the term structure of interest rates. For $t \in [0, T]$, the arbitrage price of a payer swaption equals*

$$\mathbf{PS}_t = \int_{\mathbf{R}^k} \Big(B(t, T) n_k(x) - \sum_{i=1}^{n} c_i B(t, T_i) n_k(x + \theta_i) \Big)^{+} dx, \tag{11.20}$$

where n_k is the standard k-dimensional Gaussian probability density function, and vectors $\theta_1, \ldots, \theta_n \in \mathbf{R}^k$ satisfy for every $i, j = 1, \ldots, n$

$$\theta_i \cdot \theta_j = \int_t^T \gamma(u, T_i, T) \cdot \gamma(u, T_j, T)\, du. \tag{11.21}$$

Remarks. Traded caps and swaptions are of American rather than European style. More exactly, they typically have semi-American features, since exercising is allowed on a finite number of dates (for instance, on reset dates). As a simple example of such a contract, let us consider a *Bermudan swaption.* Consider a fixed collection of reset dates $T_0, \ldots, T_{n-1}$ and an associated family of exercise dates $\tilde{T}_1, \ldots, \tilde{T}_k$ with $\tilde{T}_i \in [T_{j_i}, T_{j_i+1})$. It should be stressed that the exercise dates are known in advance; that is, they cannot be chosen freely by the long party. A Bermudan swaption gives its holder the right to enter at time $\tilde{T}_m$ a forward swap which starts at T_{j_i+1} and ends at time T_n, provided that this right has not already been exercised at a previous time $\tilde{T}_p$ for some $p < m$. Let us observe that Bermudan swaptions arise as embedded options in cancellable swaps.

11.3 Model of Forward LIBOR Rates

The main motivation for the introduction of a lognormal model of LIBOR rates was the market practice of pricing caps and swaptions by means of Black-Scholes-like expressions. For this reason, we shall first describe how market practitioners value caps and swaptions. The formulae commonly used by practitioners assume that the underlying instrument follows a geometric Brownian motion under some probability measure, $\mathbf{Q}$ say. Since the formal definition of this probability measure is not available, we shall informally refer to $\mathbf{Q}$ as the *market probability.*

11.3.1 Caps

Let us consider an interest rate cap with expiry date T and fixed strike level κ. Market practice is to price the option assuming that the underlying forward interest rate process is lognormally distributed with zero drift. Let us first consider a caplet – that is, one leg of a cap. Assume that the forward LIBOR rate $L(t,T)$, $t \in [0,T]$, for the period of length δ follows a geometric Brownian motion under the "market probability", $\mathbf{Q}$ say. More specifically

$$dL(t,T) = L(t,T)\sigma \, dW_t, \tag{11.22}$$

where W follows a one-dimensional standard Brownian motion under $\mathbf{Q}$, and σ is a strictly positive constant. The unique solution of (11.22) is

$$L(t,T) = L(0,T) \exp\big(\sigma W_t - \tfrac{1}{2}\sigma^2 t^2\big), \quad \forall\, t \in [0,T], \tag{11.23}$$

where the initial condition is derived from the yield curve $Y(0,T)$, namely

$$1 + \delta L(0,T) = \frac{B(0,T)}{B(0,T+\delta)} = \exp\Big((T+\delta)Y(0,T+\delta) - TY(0,T)\Big).$$

The "market price" at time t of a caplet with expiry date T and strike level κ is calculated by means of the formula

$$\mathbf{Cpl}_t = \delta B(t,T+\delta)\, \mathbf{E_Q}\Big((L(T,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \Big).$$

More explicitly, for any $t \in [0,T]$ we have

$$\mathbf{Cpl}_t = \delta B(t,T+\delta)\Big(L(t,T) N\big(\hat{e}_1(t,T)\big) - \kappa N\big(\hat{e}_1(t,T)\big)\Big), \tag{11.24}$$

where

$$\hat{e}_{1,2}(t,T) = \frac{\ln(L(t,T)/\kappa) \pm \frac{1}{2}\,\hat{v}_0^2(t,T)}{\hat{v}_0(t,T)}$$

and $\hat{v}_0^2(t,T) = \sigma^2(T-t)$. This means that market practitioners price caplets using Black's formula, with discount from the settlement date $T+\delta$. A cap settled in arrears at times T_j, $j = 1,\ldots,n$, where $T_j - T_{j-1} = \delta$, $T_0 = T$, is priced by the formula

$$\mathbf{FC}_t = \delta \sum_{j=1}^{n} B(t,T_j)\Big(L(t,T_{j-1}) N\big(\hat{e}_1^j(t)\big) - \kappa N\big(\hat{e}_2^j(t)\big)\Big), \tag{11.25}$$

where for every $j = 0,\ldots,n-1$

$$\hat{e}_{1,2}^j(t) = \frac{\ln(L(t,T_{j-1})/\kappa) \pm \frac{1}{2}\,\hat{v}_j^2(t)}{\hat{v}_j(t)} \tag{11.26}$$

and $\hat{v}_j^2(t) = \sigma_j^2(T_{j-1} - t)$ for some constants σ_j, $j = 1,\ldots,n$. Apparently, the market assumes that for any maturity T_j, the corresponding forward LIBOR rate has a lognormal probability law under the "market probability". As we shall see in what follows, the valuation result obtained for caps in the lognormal case agrees with market practice.

Recall that in a general framework of stochastic interest rates, the price of a *forward cap* equals (see formula (11.7))

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t}{B_{T_j}} \big(L(T_{j-1}) - \kappa\big)^+ \delta \,\Big|\, \mathcal{F}_t \Big) = \sum_{j=1}^{n} \mathbf{Cpl}_t^j,$$

where

$$\mathbf{Cpl}_t^j = B(t,T_j)\, \mathbf{E}_{\mathbf{P}_{T_j}}\Big(\big(L(T_{j-1},T_{j-1}) - \kappa\big)^+ \delta \,\Big|\, \mathcal{F}_t \Big)$$

for every $j = 1, \dots, n$. We shall now examine the valuation of caps within the lognormal model of forward LIBOR rates. The dynamics of the forward LIBOR rate $L(t, T_{j-1})$ under the forward probability measure $\mathbf{P}_{T_j}$ are

$$dL(t, T_{j-1}) = L(t, T_{j-1})\, \lambda(t, T_{j-1}) \cdot dW_t^{T_j}, \tag{11.27}$$

where W^{T_j} follows a d-dimensional Brownian motion under the forward measure $\mathbf{P}_{T_j}$, and $\lambda(\cdot, T_{j-1}) : [0, T_{j-1}] \to \mathbf{R}^d$ is a deterministic function. Consequently, under $\mathbf{P}_{T_j}$ we have

$$L(t, T_{j-1}) = L(0, T_{j-1})\mathcal{E}_t\left(\int_0^{\cdot} \lambda(u, T_{j-1}) \cdot dW_u^{T_j}\right)$$

for $t \in [0, T_{j-1}]$. Let us first consider a particular caplet, with expiry date T and strike rate κ. Since the proof of the next result is standard, it is left to the reader.

Lemma 11.3.1 *The price at time $t \in [0, T]$ of a caplet with strike rate κ, maturing at T, equals*

$$\mathbf{Cpl}_t = \delta B(t, T+\delta)\Big(L(t, T)N\big(\tilde{e}_1(t, T)\big) - \kappa\, N\big(\tilde{e}_2(t, T)\big)\Big), \tag{11.28}$$

where

$$\tilde{e}_{1,2}(t, T) = \frac{\ln(L(t, T)/\kappa) \pm \frac{1}{2}\, \tilde{v}_0^2(t, T)}{\tilde{v}_0(t, T)}$$

and $\tilde{v}_0^2(t, T) = \int_t^T |\lambda(u, T)|^2\, du$.

To the best of our knowledge, the cap pricing formula (11.29) was first established in a formal way in Sandmann et al. (1995), who focused on the dynamics of the forward LIBOR rate for a given date. Equality (11.29) was subsequently rederived in Brace et al. (1997), where a method of continuous-time modelling of all forward LIBOR rates was presented. The reader may find it instructive to compare valuation formula (11.28) with formula (11.10), which holds for a Gaussian case. The following proposition is an immediate consequence of Lemma 11.3.1 and formula (11.27).

Proposition 11.3.1 *Consider an interest rate cap with strike level κ, settled in arrears at times T_j, $j = 1, \dots, n$. Assuming the lognormal model of LIBOR rates, the price of a cap at time $t \in [0, T]$ equals*

$$\mathbf{FC}_t = \delta \sum_{j=1}^{n} B(t, T_j)\Big(L(t, T_{j-1})N\big(\tilde{e}_1^j(t)\big) - \kappa\, N\big(\tilde{e}_2^j(t)\big)\Big), \tag{11.29}$$

where

$$\tilde{e}_{1,2}^j(t) = \frac{\ln(L(t, T_{j-1})/\kappa) \pm \frac{1}{2}\, \tilde{v}_j^2(t)}{\tilde{v}_j(t)}$$

and $\tilde{v}_j^2(t) = \int_t^{T_{j-1}} |\lambda(u, T_{j-1})|^2\, du$.

11.3.2 Swaptions

The commonly used formula for pricing swaptions, based on the assumption that the underlying swap rate follows a geometric Brownian motion under the "market probability" $\mathbf{Q}$, is given by the *Black swaption formula*

$$\mathbf{PS}_t = \delta \sum_{j=1}^{n} B(t, T_j)\Big(\kappa(t, T, n)N\big(h_1(t, T)\big) - \kappa N\big(h_2(t, T)\big)\Big), \tag{11.30}$$

where

$$h_{1,2}(t, T) = \frac{\ln(\kappa(t, T, n)/\kappa) \pm \frac{1}{2}\, \sigma^2(T - t)}{\sigma\sqrt{T - t}}$$

for some constant $\sigma > 0$. To examine equality (11.30) in an intuitive way, let us assume, for simplicity, that $t = 0$. In this case, using general valuation results, we obtain the following equality

$$\mathbf{PS}_0 = \delta \sum_{j=1}^{n} B(0, T_j)\, \mathbf{E}_{\mathbf{P}_{T_j}} \Big((\kappa(T, T, n) - \kappa)^+ \Big).$$

Apparently, market practitioners assume lognormal probability law for the forward swap rate $\kappa(T, T, n)$ under $\mathbf{P}_{T_j}$. The swaption valuation formula obtained in the framework of the lognormal model of LIBOR rates appears to be more involved. It reduces to the "market formula" (11.30) only in very special circumstances. On the other hand, the swaption price derived within the lognormal model of forward swap rates agrees with the "market formula". To be more precise, this holds for a specific family of swaptions. This is by no means surprising, as the model was exactly tailored to handle a particular family of swaptions, or rather, to analyze certain path-dependent swaptions (such as *Bermudan swaptions*). The price of a cap in the lognormal model of swap rates is not given by a closed-form expression, however.

Recall that within the general framework, the price at time $t \in [0, T]$ of a payer swaption with expiry date $T = T_0$ and strike level κ equals

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_T} \Big(\mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (L(T_{j-1}) - \kappa)\delta \,\Big|\, \mathcal{F}_T \Big) \Big)^+ \,\Big|\, \mathcal{F}_t \Big\},$$

or equivalently

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_T} \mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (\kappa(T, T, n) - \kappa)^+ \delta \,\Big|\, \mathcal{F}_T \Big) \,\Big|\, \mathcal{F}_t \Big\}.$$

Let $D \in \mathcal{F}_T$ be the exercise set of a swaption; that is

$$D = \{\omega \in \Omega \,|\, (\kappa(T, T, n) - \kappa)^+ > 0\} = \{\omega \in \Omega \,|\, \sum_{j=1}^{n} c_j B(T, T_j) < 1\}.$$

Lemma 11.3.2 *The following equality holds for every* $t \in [0, T]$

$$\mathbf{PS}_t = \delta \sum_{j=1}^{n} B(t, T_j)\, \mathbf{E}_{\mathbf{P}_{T_j}} \Big((L(T, T_{j-1}) - \kappa)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_t \Big). \tag{11.31}$$

Proof. Since

$$\mathbf{PS}_t = \mathbf{E}_{\mathbf{P}^*} \Big\{ \frac{B_t}{B_T}\, \mathrm{I}_D\, \mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}} (L(T_{j-1}) - \kappa)\delta \,\Big|\, \mathcal{F}_T \Big) \,\Big|\, \mathcal{F}_t \Big\},$$

we have

$$\begin{aligned} \mathbf{PS}_t &= \delta\, \mathbf{E}_{\mathbf{P}^*} \Big\{ \mathbf{E}_{\mathbf{P}^*} \Big(\sum_{j=1}^{n} \frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_T \Big) \,\Big|\, \mathcal{F}_t \Big\} \\ &= \delta \sum_{j=1}^{n} B(t, T_j)\, \mathbf{E}_{\mathbf{P}_{T_j}} \Big((L(T_{j-1}) - \kappa)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_t \Big), \end{aligned}$$

where $L(T_{j-1}) = L(T_{j-1}, T_{j-1})$. For any $j = 1, \dots, n$, we have

$$\mathbf{E}_{\mathbf{P}_{T_j}} \Big((L(T_{j-1}) - \kappa)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_t \Big) = \mathbf{E}_{\mathbf{P}_{T_j}} \Big(\mathbf{E}_{\mathbf{P}_{T_j}} \big(L(T_{j-1}) - \kappa \,|\, \mathcal{F}_T \big)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_t \Big) = \mathbf{E}_{\mathbf{P}_{T_j}} \Big((L(T, T_{j-1}) - \kappa)\, \mathrm{I}_D \,\Big|\, \mathcal{F}_t \Big),$$

since $\mathcal{F}_t \subset \mathcal{F}_T$ and the process $L(t, T_{j-1})$ is a $\mathbf{P}_{T_j}$-martingale. □

For any $k = 1, \dots, n,$ we define the random variable $\zeta_k(t)$ by setting

$$\zeta_k(t) = \int_t^T \lambda(u, T_{k-1}) \cdot dW_u^{T_k}, \quad \forall\, t \in [0, T]. \tag{11.32}$$

We write also

$$\lambda_k^2(t) = \int_t^T |\lambda(u, T_{k-1})|^2 \, du, \quad \forall\, t \in [0, T]. \tag{11.33}$$

Note that for every $k = 1, \dots, n$ and $t \in [0, T],$ we have

$$L(T, T_{k-1}) = L(t, T_{k-1}) \, e^{\zeta_k(t) - \lambda_k^2(t)/2}.$$

Recall also that the processes W^{T_k} satisfy the following relationship

$$W_t^{T_{k+1}} = W_t^{T_k} + \int_0^t \frac{\delta L(u, T_k)}{1 + \delta L(u, T_k)} \, \lambda(u, T_k) \, du, \quad \forall\, t \in [0, T_k]. \tag{11.34}$$

For ease of notation, we formulate the next result for $t = 0$ only; a general case can be treated along the same lines. For any fixed $j,$ we denote by G_j the joint probability distribution function of the n-dimensional random variable $(\zeta_1(0), \dots, \zeta_n(0))$ under the forward measure $\mathbf{P}_{T_j}$.

Proposition 11.3.2 *Assume the lognormal model of LIBOR rates. The price at time* 0 *of a payer swaption with expiry date $T = T_0$ and strike level κ equals*

$$\mathbf{PS}_0 = \delta \sum_{j=1}^n B(0, T_j) \int_{\mathbf{R}^n} \Big(L(0, T_{j-1}) e^{y_j - \lambda_j^2(0)/2} - \kappa \Big) \, \mathrm{I}_{\tilde{D}} \, dG_j(y_1, \dots, y_n),$$

where $\mathrm{I}_{\tilde{D}} = \mathrm{I}_{\tilde{D}}(y_1, \dots, y_n),$ *and* $\tilde{D}$ *stands for the set*

$$\tilde{D} = \Big\{ (y_1, \dots, y_n) \in \mathbf{R}^n \, \Big| \, \sum_{j=1}^n c_j \prod_{k=1}^j \Big(1 + \delta \, L(0, T_{k-1}) \, e^{y_k - \lambda_k^2(0)/2} \Big)^{-1} < 1 \Big\}.$$

Proof. Let us start by considering arbitrary $t \in [0, T].$ Notice that

$$\frac{B(t, T_j)}{B(t, T)} = \prod_{k=1}^j \frac{B(t, T_k)}{B(t, T_{k-1})} = \prod_{k=1}^j (F_B(t, T_{k-1}, T_k))^{-1}$$

and thus, in view of (9.42), we have

$$B(T, T_j) = \prod_{k=1}^j \Big(1 + \delta L(T, T_{k-1}) \Big)^{-1}.$$

Consequently, the exercise set D can be re-expressed in terms of forward LIBOR rates. Indeed, we have

$$D = \Big\{ \omega \in \Omega \, \Big| \, \sum_{j=1}^n c_j \prod_{k=1}^j \Big(1 + \delta L(T, T_{k-1}) \Big)^{-1} < 1 \Big\},$$

or more explicitly

$$D = \Big\{ \omega \in \Omega \, \Big| \, \sum_{j=1}^n c_j \prod_{k=1}^j \Big(1 + \delta \, L(t, T_{k-1}) \, e^{\zeta_k(t) - \lambda_k^2(t)/2} \Big)^{-1} \, < \, 1 \Big\}.$$

Let us put $t = 0$. In view of Lemma 11.3.2, to find the arbitrage price of a swaption at time 0, it is sufficient to determine the joint law under the forward measure $\mathbf{P}_{T_j}$ of the random variable $(\zeta_1(0), \ldots, \zeta_n(0))$, where $\zeta_1(0), \ldots, \zeta_n(0)$ are given by (11.32). Note also that

$$D = \Big\{\omega \in \Omega \,\Big|\, \sum_{j=1}^{n} c_j \prod_{k=1}^{j} \Big(1 + \delta\, L(0, T_{k-1})\, e^{\zeta_k(0) - \lambda_k^2(0)/2}\Big)^{-1} < 1\Big\}.$$

This shows the validity of the swaption valuation formula for $t = 0$. It is clear that this result admits a rather straightforward generalization to arbitrary $0 < t \le T$. When $t > 0$, one needs to examine the conditional probability law of $(\zeta_1(t), \ldots, \zeta_n(t))$ with respect to the σ-field $\mathcal{F}_t$. □

11.4 Model of Forward Swap Rates

For any fixed, but otherwise arbitrary, date T_j, $j = 1, \ldots, M-1$, we consider a swaption with expiry date T_j, written on a payer swap settled in arrears, with fixed rate κ, which starts at date T_j and has $M - j$ accrual periods. The j^{th} swaption may be seen as a contract which pays to its owner the amount $\delta(\kappa(T_j, T_j, M-j) - \kappa)^+$ at each settlement date $T_{j+1}, \ldots, T_M$. Equivalently, we may assume that it pays an amount

$$Y = \sum_{k=j+1}^{M} B(T_j, T_k)\delta\big(\kappa(T_j, T_j, M-j) - \kappa\big)^+$$

at time T_j. Note that Y admits the following representation (cf. (9.65))

$$Y = \delta G_{T_j}(M-j)\big(\kappa(T_j, T_j, M-j) - \kappa\big)^+.$$

Recall that the lognormal model of forward swap rates specifies the dynamics of $\kappa(t, T_j, M-j)$ by means of the following SDE (see Sect. 9.5)

$$d\kappa(t, T_j, M-j) = \kappa(t, T_j, M-j)\nu(t, T_j) \cdot d\tilde{W}_t^{T_{j+1}}, \tag{11.35}$$

where $\nu(\cdot, T_j) : [0, T_j] \to \mathbf{R}^d$ is a bounded deterministic function, and $\tilde{W}^{T_{j+1}}$ follows a standard d-dimensional Brownian motion under $\tilde{\mathbf{P}}_{T_{j+1}}$. Furthermore, by the definition of the forward swap measure $\tilde{\mathbf{P}}_{T_{j+1}}$, any process of the form $B(t, T_k)/G_t(M-j)$ is a local martingale under $\tilde{\mathbf{P}}_{T_{j+1}}$. From the general considerations concerning the choice of a numeraire, it follows that for any attainable claim $X = g(B(T_j, T_{j+1}), \ldots, B(T_j, T_M))$, which settles at time T_j, the arbitrage price $\pi_t(X)$ satisfies

$$\pi_t(X) = G_t(M-j)\, \mathbf{E}_{\tilde{\mathbf{P}}_{T_{j+1}}}\big(X G_{T_j}^{-1}(M-j) \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0, T_j].$$

Applying this equality to the swaption's payoff Y, we obtain

$$\mathbf{PS}_t^j = \pi_t(Y) = G_t(M-j)\, \mathbf{E}_{\tilde{\mathbf{P}}_{T_{j+1}}}\big((\kappa(T_j, T_j, M-j) - \kappa)^+ \,|\, \mathcal{F}_t\big),$$

where we write $\mathbf{PS}_t^j$ to denote the price at time t of the j^{th} swaption.

Proposition 11.4.1 *Assume the lognormal model of forward swap rates. For any $j = 1, \ldots, M-1$, the arbitrage price $\mathbf{PS}_t^j$ of the j^{th} swaption at time $t \le T_j$ equals*

$$\mathbf{PS}_t^j = \delta \sum_{k=j+1}^{M} B(t, T_k)\Big(\kappa(t, T_j, M-j) N\big(h_1(t, T_j)\big) - \kappa N\big(h_2(t, T_j)\big)\Big),$$

where

$$h_{1,2}(t, T_j) = \frac{\ln(\kappa(t, T_j, M-j)/\kappa) \pm \frac{1}{2} v_S^2(t, T_j)}{v_S(t, T_j)}$$

and $v_S^2(t, T_j) = \int_t^{T_j} |\nu(u, T_j)|^2\, du.$

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