



UNSW  
A U S T R A L I A



UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

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## Assignment 3

Ergodic Theory

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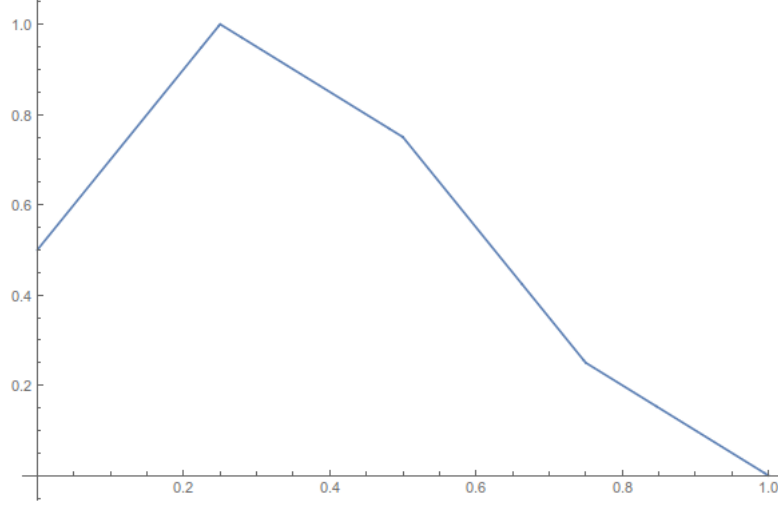
*Student Number:*  
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# 1

We consider the map

$$T(x) = \begin{cases} 2x + \frac{1}{2} & 0 \leq x < \frac{1}{4} \\ -x + \frac{5}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ -2x + \frac{7}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ -x + 1 & \frac{3}{4} \leq x \leq 1. \end{cases} \quad (1)$$

This has the following plot:



## 1.1

We have that in general

$$\mathcal{P}_T f(x) = \sum_{z \in T^{-1}x} \frac{f(z)}{|T'(z)|}. \quad (2)$$

Then in general we have

$$T'(x) = \begin{cases} 2 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -2 & \frac{1}{2} \leq x < \frac{3}{4} \\ -1 & \frac{3}{4} \leq x \leq 1. \end{cases} \quad (3)$$

Also we can also see that

$$T^{-1}\{x\} = \begin{cases} \{1-x\} & 0 \leq x < \frac{1}{4} \\ \{\frac{7}{8} - \frac{x}{2}\} & \frac{1}{4} \leq x < \frac{1}{2} \\ \{\frac{7}{8} - \frac{x}{2}\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{1}{2} \leq x < \frac{3}{4} \\ \{\frac{5}{4} - x\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{3}{4} \leq x < 1 \end{cases} \quad (4)$$

thus for  $f_i(x) = \mathbb{1}_{I_i}(x)$  we have that

$$\mathcal{P}_T f_1(x) = \frac{1}{2} \mathbb{1}_{I_3 \cup I_4}(x) = \frac{1}{2} \mathbb{1}_{I_3}(x) + \frac{1}{2} \mathbb{1}_{I_4}(x) \quad (5)$$

$$\mathcal{P}_T f_2(x) = \mathbb{1}_{I_4}(x) \quad (6)$$

$$\mathcal{P}_T f_3(x) = \frac{1}{2} \mathbb{1}_{I_2 \cup I_3}(x) = \frac{1}{2} \mathbb{1}_{I_2}(x) + \frac{1}{2} \mathbb{1}_{I_3}(x) \quad (7)$$

$$\mathcal{P}_T f_4(x) = \mathbb{1}_{I_1}(x) \quad (8)$$

## 1.2

Let  $\mathcal{S} = \text{span} \bigcup_{i=1}^4 \mathbb{1}_{I_i}$ .

Let  $f \in \mathcal{S}$  then  $f = \sum_{i=1}^4 \lambda_i \mathbb{1}_{I_i}$  and so by linearity of the Perron-Frobenius operator, we have that  $\mathcal{P}_T(f) = \sum_{i=1}^4 \lambda_i \mathcal{P}_T(\mathbb{1}_{I_i})$  but from the result above we see that  $\mathcal{P}_T(\mathbb{1}_{I_i}) \in \mathcal{S}$  for each  $i$  and so  $\mathcal{P}$  preserves  $\mathcal{S}$

## 1.3

We do this by just plugging the basis of  $\mathcal{S}$  into the  $\mathcal{P}_T$  and putting the results in each row of the matrix. Seeing as we already have first part we can just write down the matrix  $M$  as

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

## 1.4

Using Mathematica we can calculate the left eigenvector corresponding to eigenvalue 1 which is

$$\mathbf{v} = (1, \frac{1}{2}, 1, 1) \quad (10)$$

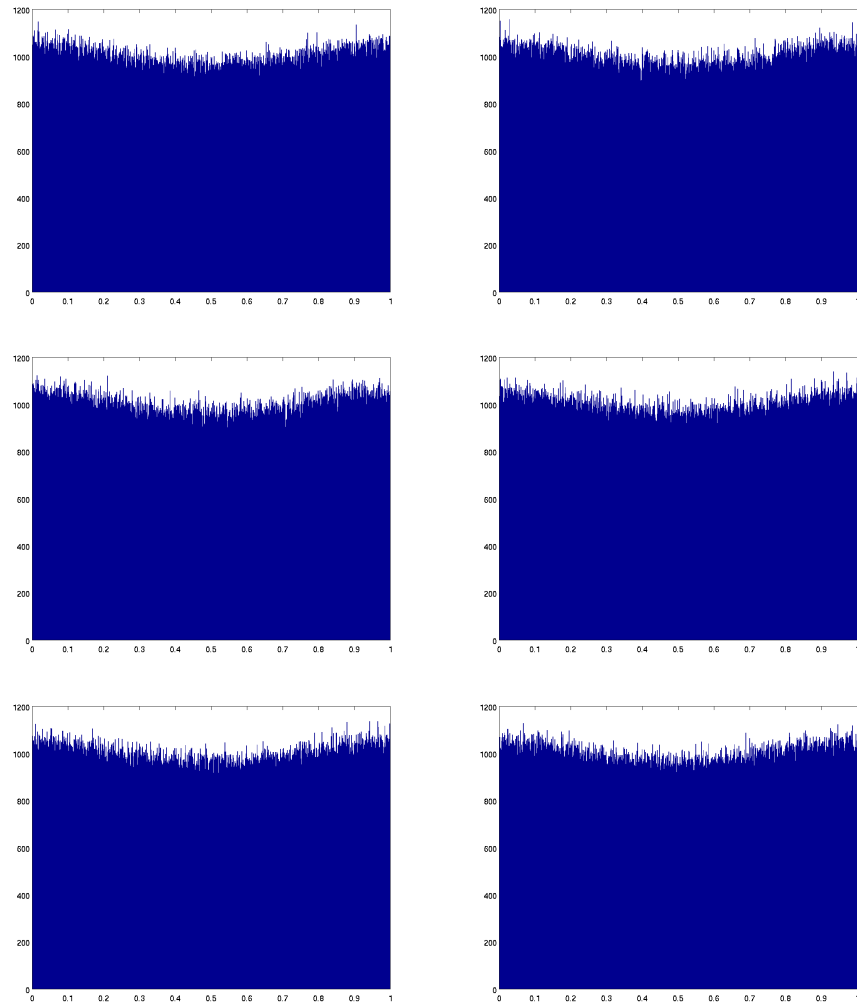
which corresponds to the ACIM (after rescaling)

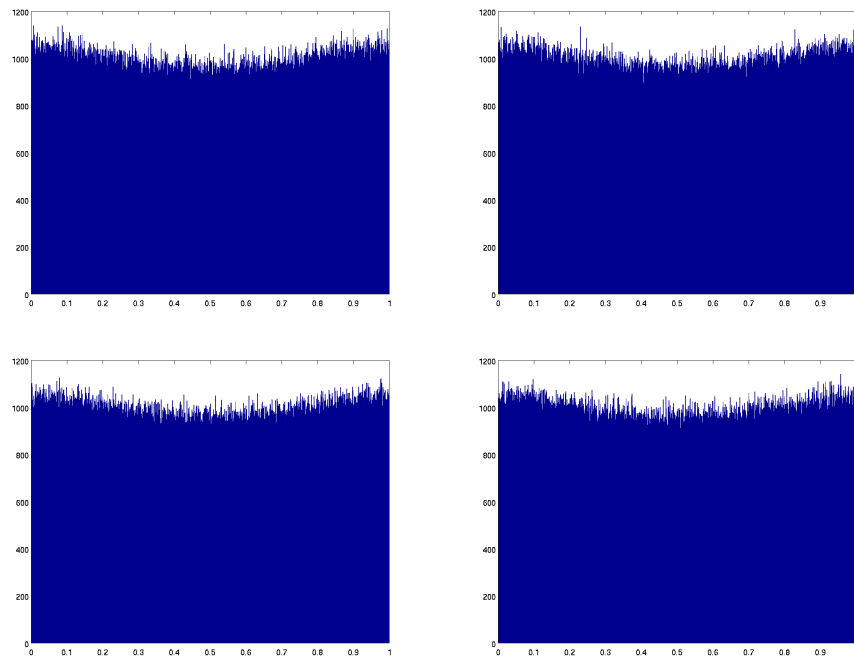
$$\frac{8}{7} \left( \mathbb{1}_{I_1} + \frac{1}{2} \mathbb{1}_{I_2} + \mathbb{1}_{I_3} + \mathbb{1}_{I_4} \right) \quad (11)$$

## 2

### 2.1

The following plots we produced by code in *Question\_2.a.m* available at [https://github.com/adamjoshuagray/Honours\\_Ergodic\\_Theory/tree/master/Assignment\\_3](https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3).





What these histograms show is that TODO

## 2.2

This code is available in *Question\_2\_b\_c\_d.m* available at [https://github.com/adamjoshuagray/Honours\\_Ergodic\\_Theory/tree/master/Assignment\\_3](https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3).

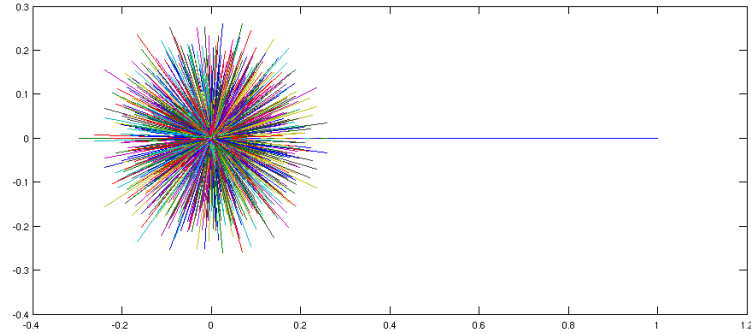
## 2.3

Show TODO

Note that numerical verification is done in *Question\_2\_b\_c\_d.m* available at [https://github.com/adamjoshuagray/Honours\\_Ergodic\\_Theory/tree/master/Assignment\\_3](https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3).

## 2.4

The following is a plot of all the left eigenvalues of  $Q$  in the complex plane.



It is clear to see that there is only one eigenvalue equal to 1. We have also checked this in code in *Question\_2\_b.c.d.m* available at [https://github.com/adamjoshuagray/Honours\\_Ergodic\\_Theory/tree/master/Assignment\\_3](https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3). The following is a plot of the left eigenvector corresponding to the eigenvalue 1.

### 3

#### 3.1

Now note that  $S = h \circ T \circ h^{-1}$  or equivalently  $T = h^{-1} \circ S \circ h$ .

As  $h$  and  $T$  are non-singular, so is  $S$  and thus by the composition of Perron-Frobenius operators we can write

$$\mathcal{P}_T f = (\mathcal{P}_{h^{-1}} \circ \mathcal{P}_S \circ \mathcal{P}_h) f \quad (12)$$

or equivalently

$$(\mathcal{P}_h \circ \mathcal{P}_T) f = (\mathcal{P}_S \circ \mathcal{P}_h) f. \quad (13)$$

Now as  $\mathcal{P}_h f = g$  and by assumption  $\mathcal{P}_T f = f$  then

$$\mathcal{P}_h f = \mathcal{P}_s g \quad (14)$$

but we have also shown that  $\mathcal{P}_h f = g$  and so the result follows.

#### 3.2

From the lectures we know that  $\mathcal{P}_S g = g$  implies that  $g$  is an  $S$  invariant density, so the result is immediate. A formal statement of that theorem along with it's proof is as follows

**Theorem 1.**

*Proof.*

□

### 3.3

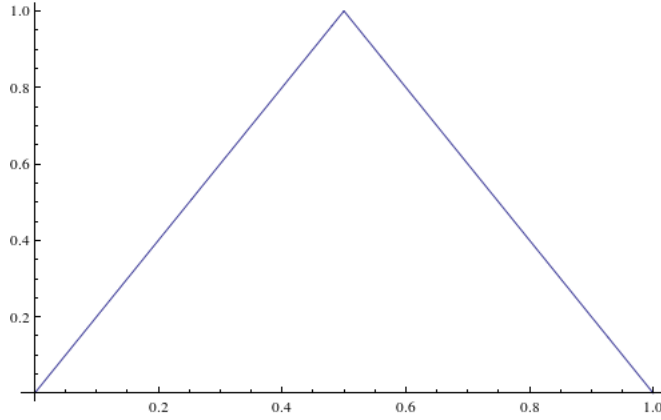
## 4

### 4.1

For  $T : I \rightarrow I$  defined as

$$T(x) := \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \leq 1 \end{cases} \quad (15)$$

we have the following plot:



The  $T$  invariant density may be found in much the same way as for question 1. If we let  $I_1 = [0, \frac{1}{2}]$  and  $I_2 = [\frac{1}{2}, 1]$  and choose the basis  $\{\mathbb{1}_{I_1}, \mathbb{1}_{I_2}\}$  then the associated matrix representation of the Perron-Frobenius operator is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (16)$$

which has eigenvector  $(1, 1)$  associated with eigenvalue 1 and so the  $T$  invariant density is  $\mathbb{1}_{[0,1]}$ , in other words the density is the uniform density (1).

### 4.2

We have to do is show that

$$g = (f \circ h^{-1}) |(h^{-1})'| \quad (17)$$

where  $f$  is the density of the ACIM associated with  $T$ .

Now since  $f$  is the uniform density all we have to show is that

$$|(h^{-1})'| = \frac{1}{\pi \sqrt{x(1-x)}}. \quad (18)$$

Now since  $h(x) = \sin^2(\frac{\pi x}{2})$  we have that  $h^{-1}(x) = \frac{2}{\pi} \sin^{-1}(\sqrt{x})$  and by differentiating we see that

$$(h^{-1}(x))' = \frac{1}{\pi \sqrt{x(1-x)}} \quad (19)$$

and note that for  $x \in [0, 1]$  this is positive so

$$|(h^{-1}(x))'| = \frac{1}{\pi \sqrt{x(1-x)}} \quad (20)$$

The last thing we have to show is that  $S(x) = (h \circ T \circ h^{-1})(x)$  or equivalently  $T(x) = (h^{-1} \circ S \circ h)(x)$ .

See that

$$T(x) = \frac{2}{\pi} \sin^{-1} \left( \sqrt{4 \sin^2 \left( \frac{\pi x}{2} \right) \left( 1 - \sin^2 \left( \frac{\pi x}{2} \right) \right)} \right) \quad (21)$$

$$= \frac{2}{\pi} \sin^{-1} \left( \sqrt{4 \sin^2 \left( \frac{\pi x}{2} \right) \cos^2 \left( \frac{\pi x}{2} \right)} \right) \quad (22)$$

$$= \frac{2}{\pi} \sin^{-1} \left( 2 \sin \left( \frac{\pi x}{2} \right) \cos \left( \frac{\pi x}{2} \right) \right) \quad (23)$$

$$= \frac{2}{\pi} \sin^{-1} (\sin(\pi x)) \quad (24)$$

$$= \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \leq 1 \end{cases} \quad (25)$$

and so this resolves the problem in that by using the result of Section 3. That is, the density  $g$  is  $S$ -invariant.

### 4.3

By using the result of question 3.3 we can calculate the Lyapunov exponent of  $(I, \mathcal{B}, \mu_g, S)$  by just calculating the Lyapunov exponent of  $(I, \mathcal{B}, \mu_f, T)$ . Since we have that  $f = 1$  then we can just write

$$\int_I \log |T'(x)| d\mu_f(x) = \int_I \log |T'(x)| d\mu \quad (26)$$

$$= \int_0^1 \log 2 d\mu \quad (27)$$

$$= \log 2 \quad (28)$$