University of New South Wales

School of Mathematics and Statistics

Assignment 3

Ergodic Theory

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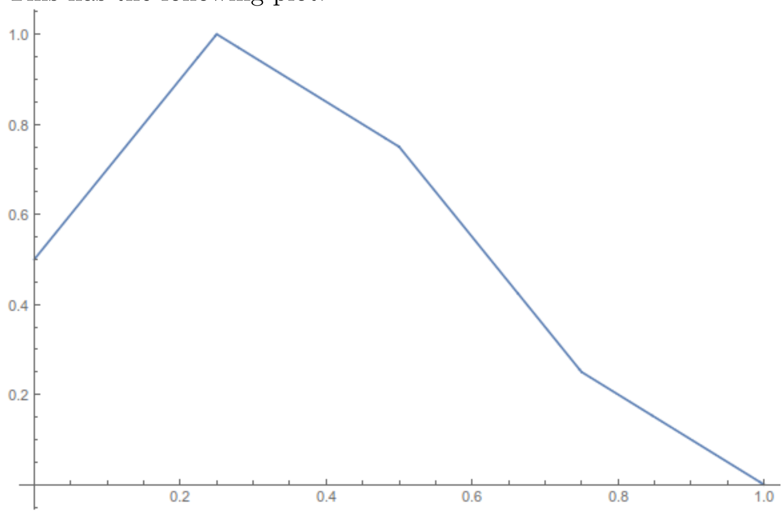
1

We consider the map

$$T(x) = \begin{cases} 2x + \frac{1}{2} & 0 \leq x < \frac{1}{4} \\ -x + \frac{5}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ -2x + \frac{7}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ -x + 1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{1}$$

This has the following plot:



1.1

We have that in general

$$\mathcal{P}_T f(x) = \sum_{z \in T^{-1}x} \frac{f(z)}{|T'(z)|}. \tag{2}$$

Then in general we have

$$T'(x) = \begin{cases} 2 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -2 & \frac{1}{2} \leq x < \frac{3}{4} \\ -1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{3}$$

Also we can also see that

$$T^{-1}\{x\} = \begin{cases} \{1 - x\} & 0 \leq x < \frac{1}{4} \\ \{\frac{7}{8} - \frac{x}{2}\} & \frac{1}{4} \leq x < \frac{1}{2} \\ \{\frac{7}{8} - \frac{x}{2}\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{1}{2} \leq x < \frac{3}{4} \\ \{\frac{5}{4} - x\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{3}{4} \leq x < 1 \end{cases} \tag{4}$$

thus for $f_i(x) = \mathbb{1}_{I_i}(x)$ we have that

$$\mathcal{P}_T f_1(x) = \frac{1}{2}\mathbb{1}_{I_3 \cup I_4}(x) = \frac{1}{2}\mathbb{1}_{I_3}(x) + \frac{1}{2}\mathbb{1}_{I_4}(x) \tag{5}$$

$$\mathcal{P}_T f_2(x) = \mathbb{1}_{I_4}(x) \tag{6}$$

$$\mathcal{P}_T f_3(x) = \frac{1}{2}\mathbb{1}_{I_2 \cup I_3}(x) = \frac{1}{2}\mathbb{1}_{I_2}(x) + \frac{1}{2}\mathbb{1}_{I_3}(x) \tag{7}$$

$$\mathcal{P}_T f_4(x) = \mathbb{1}_{I_1}(x) \tag{8}$$

1.2

Let $\mathcal{S} = \operatorname{span} \bigcup_{i=1}^4 \mathbb{1}_{I_i}$.
Let $f \in \mathcal{S}$ then $f = \sum_{i=1}^4 \lambda_4 \mathbb{1}_{I_i}$ and so by linearity of the Perron-Frobenius operator, we have that $\mathcal{P}_T(f) = \sum_{i=1}^4 \lambda_4 \mathcal{P}_T(\mathbb{1}_{I_i})$ but from the result above we see that $\mathcal{P}_T(\mathbb{1}_{I_i}) \in \mathcal{S}$ for each i and so $\mathcal{P}$ preserves $\mathcal{S}$

1.3

We do this by just pluging the basis of $\mathcal{S}$ into the $\mathcal{P}_T$ and putting the results in each row of the matrix. Seeing as we already have first part we can just write down the matrix M as

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

1.4

Using Mathematica we can calculate the left eigenvector corresponding to eigenvalue 1 which is

$$\mathbf{v} = (1, \frac{1}{2}, 1, 1) \tag{10}$$

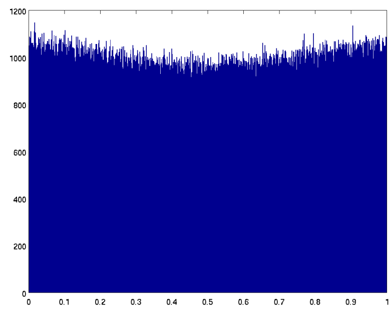
which corresponds to the ACIM

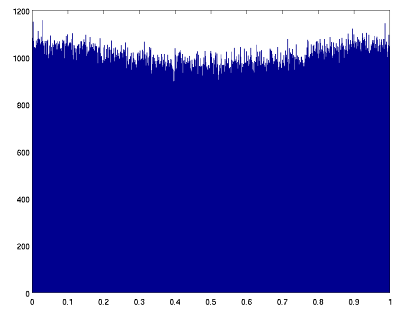
$$\mathbb{1}_{I_1} + \frac{1}{2}\mathbb{1}_{I_2} + \mathbb{1}_{I_3} + \mathbb{1}_{I_4 git} \tag{11}$$

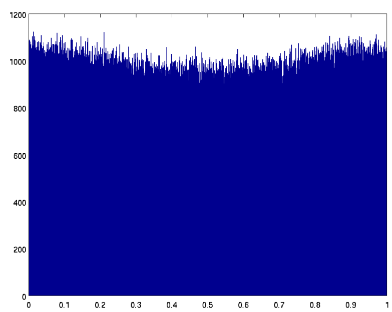
2

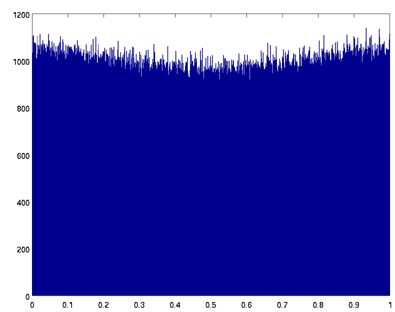
2.1

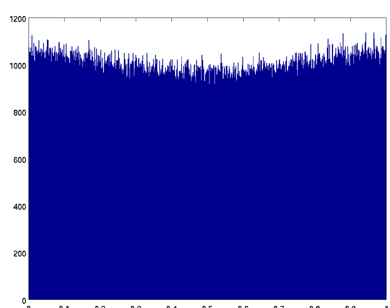
The following plots we produced by code in *Question_2_a.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

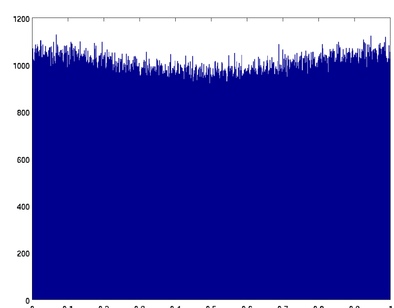


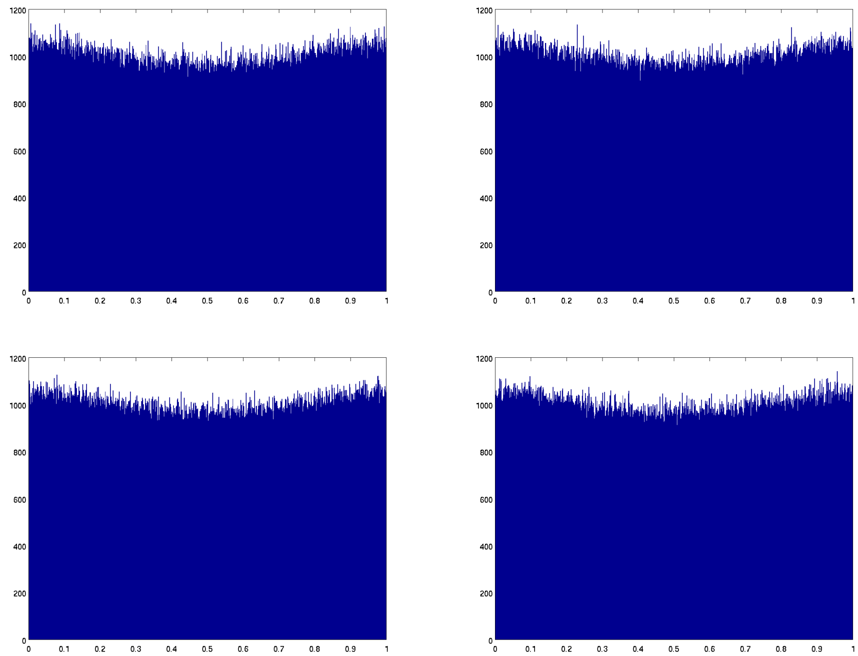












What these historgrams show is that TODO

2.2

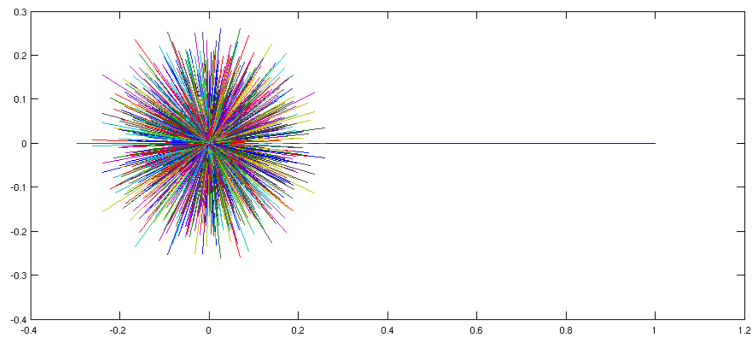
This code is available in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.3

Show TODO
Note that numerical verification is done in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.4

The following is a plot of all the left eigenvalues of Q in the complex plane.



It is clear to see that there is only one eigenvalue equal to 1. We have also checked this in code in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.
The following is a plot of the left eigenvector corresponding to the eigenvalue 1.

3

3.1

Now note that $S = h \circ T \circ h^{-1}$ or equivelently $T = h^{-1} \circ S \circ h$.
As h and T are non-singular, so is S and thus by the composition of Perron-Frobenius operators we can write

$$\mathcal{P}_T f = (\mathcal{P}_{h^{-1}} \circ \mathcal{P}_S \circ \mathcal{P}_h) f \tag{12}$$

or equivelently

$$(\mathcal{P}_h \circ \mathcal{P}_T) f = (\mathcal{P}_S \circ \mathcal{P}_h) f.. \tag{13}$$

Now as $\mathcal{P}_h f = g$ and by assumption $\mathcal{P}_T f = f$ then

$$\mathcal{P}_h f = \mathcal{P}_s g \tag{14}$$

but we have also shown that $\mathcal{P}_h f = g$ and so the result follows.

3.2

From the lectures we know that $\mathcal{P}_S g = g$ implies that g is an S invariant density, so the result is immediate. A formal statement of that theorem along with it's proof is as follows

Theorem 1.

Proof. □