University of New South Wales

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Assignment 1

Ergodic Theory

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1

Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_1, T_2)$ be two measure theoretically isomorphic p.p.ts ($\pi : X_1 \longrightarrow X_2$). Suppose $(X_1, \mathcal{B}_1, \mu_1, T_1)$ is mixing.
Then for $E, F \in \mathcal{B}_2$ see that

$$\begin{aligned}\mu_2(E \cap T_2^{-n} F) &= \mu_1(\pi^{-1}(E \cap T_2^{-n} F)) \\ &= \mu_1(\pi^{-1}(E) \cap T_1^{-n} \pi^{-1}(F))\end{aligned}$$

and since μ_1 is mixing then

$$\begin{aligned}\lim_{n \longrightarrow \infty} \mu_1(\pi^{-1}(E) \cap T_1^{-n} \pi^{-1}(F)) &= \mu_1(\pi^{-1}(E))\mu_1(\pi^{-1}(F)) \\ &= \mu_2(E)\mu_2(F)\end{aligned}$$

and so

$$\lim_{n \longrightarrow \infty} \mu_2(E \cap T_2^{-n} F) = \mu_2(E)\mu_2(F)$$

2

Consider $g_t : S^1 \longrightarrow S^1$ where $g_t(x) \mapsto x + t(\mod 1)$. The measure space is $(S^1, \mu_{\mathcal{L}})$ where $\mu_{\mathcal{L}}$ is the Lebesgue measure. Clearly the flow is ergodic because the only sets which are invariant for all t are S and $\emptyset$. However if we consider g_τ with $\tau = \frac{1}{2}$ then there are lots of invariant sets which are not of full or zero measure. For example $[0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$ is invariant but $\mu_{\mathcal{L}}([0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]) = \frac{1}{2}$ and so $g_{\frac{1}{2}}$ with the Lebesgue measure is not ergodic.

3

(We show the contrapositive)
Suppose $(X, \mathcal{B}, \mu, T)$ is a p.p.t which is not ergodic. Then there exists another T invaraint measure ν. Since $(X, \mathcal{B}, \mu, T)$ is not ergodic then there exists $E \in \mathcal{B}$ such that $T^{-1}E = E$ but $0 < \mu(E) < 1$. Fix E and say $\mu(E) = \alpha$. Then we can define a new measure

$$\nu(F) = (1 - \alpha)^{-1} \mu(F \setminus E).$$

ν is clearly $\mathcal{B}$ additive so it's a measure. We just have to verify that ν is T invariant.

$$\begin{aligned}\nu(T^{-1}F) &= (1 - \alpha)^{-1} \mu((T^{-1}F) \setminus E) \\ &= (1 - \alpha)^{-1} \mu(\{x : T(x) \in F, x \in E^c\}) \\ &= (1 - \alpha)^{-1} \mu(\{x : T(x) \in (F \cap E^c)\}) \qquad \text{because } E^c \text{ is } T \text{ invariant} \\ &= (1 - \alpha)^{-1} \mu(T^{-1}(F \setminus E)) \\ &= (1 - \alpha)^{-1} \mu(F \setminus E) \\ &= \nu(F)\end{aligned}$$

and so the contrapositive is proved.

4

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A is aperiodic and because $A^n_{i,j}$ represents the number of paths of length n from i to j then if $A^n_{1,1} \neq 0$ then $A^n_{1,2} = 0$. In other words there will never be a n such that all the entries of the matrix A^n are non-zero.
Another way of saying this is that state 2 can only be reached from state 1 in an odd number of steps whereas state 1 can only get back to state 1 in an even number of steps.

5

5.1

(By induction)
In the case $n = 1$ the result holds trivially.
Suppose $A^n_{i,j} = \#\{\text{paths from } i \text{ to } j \text{ of length } n\}$ (For all $j, k \leq N$.) Then

$$\begin{aligned} A^{n+1}_{i,j} &= (A^n A)_{i,j} \\ &= \sum_{k=1}^{N} A^n_{i,k} A_{k,j} \\ &= \sum_{k=1}^{N} \#\{\text{paths from } i \text{ to } k \text{ of length } n\} A_{k,j} \\ &= \sum_{k \in \{1,\ldots,N\}, A_{k,j}=1} \#\{\text{paths from } i \text{ to } k \text{ of length } n\} \\ &= \#\{\text{paths from } i \text{ to } j \text{ of length } n\} \end{aligned}$$

5.2

(By induction)
The case $n = 1$ holds by definition of P.
Suppose $P^n_{i,j} = \mathcal{P}[\text{ going from } i \text{ to } j \text{ in exactly } n \text{ steps }]$.

Then

$$
\begin{aligned}
P_{i,j}^{n+1} &= (P^n P)_{i,j} \\
&= \sum_{k=1}^{N} P_{i,k}^{n} P_{k,j} \\
&= \sum_{k=1}^{N} P_{k,j} \mathcal{P}[\text{ going from } i \text{ to } k \text{ in exactly } n \text{ steps }] \\
&= \sum_{k=1}^{N} \mathcal{P}[\text{ going from } k \text{ to } j \text{ in exactly 1 step }] \mathcal{P}[\text{ going from } i \text{ to } k \text{ in exactly } n \text{ steps }] \\
&= \sum_{k=1}^{N} \mathcal{P}[\text{ going from } i \text{ to } k \text{ in exactly } n \text{ steps then directly to } j] \\
&= \mathcal{P}[\text{ going from } i \text{ to } j \text{ in exactly } n+1 \text{ steps }].
\end{aligned}
$$

5.3

5.3.1

$$
\begin{aligned}
q_j &= \sum_{k=1}^{N} P_{k,j} q_j \\
&= P_{j,j} q_j
\end{aligned}
$$

which implies either $P_{j,j} = 1$ or $q_j = 0$. Suppose $P_{j,j} = 1$, then $P_{j,k} = 0$ for all $k \neq j$. This implies

$$
\begin{aligned}
P_{j,k}^2 &= \sum_{i=1}^{N} P_{j,i} P_{i,k} \\
&= P_{j,j} P_{j,k} \\
&= \delta_{j,k}
\end{aligned}
$$

and by induction implies $(P^n)_{j,k} = 0$ for all $j \neq k$, which is not allowed, so $q_j = 0$.

5.3.2

Group into communicating classes $[i]$, $[j]$ respectively. (There may be other communicating classes but we can neglect them) Then there is a stochastic matrix

$$
R = \begin{bmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{bmatrix}
$$

If $\mathbf{w}$ is the stationary vector for R (defined above) then it is easy to see $w_1 = 0$. The result for q_i follows as $i \in [i]$.