



UNSW
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UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 3

Ergodic Theory

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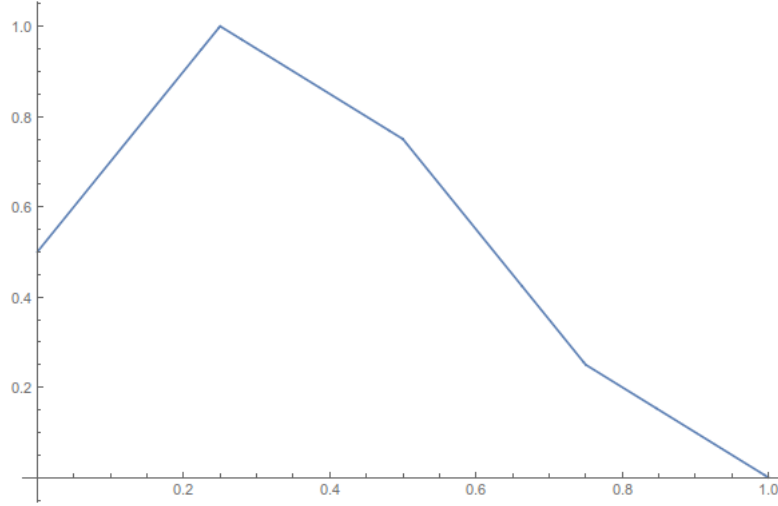
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We consider the map

$$T(x) = \begin{cases} 2x + \frac{1}{2} & 0 \leq x < \frac{1}{4} \\ -x + \frac{5}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ -2x + \frac{7}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ -x + 1 & \frac{3}{4} \leq x \leq 1. \end{cases} \quad (1)$$

This has the following plot:



1.1

We have that in general

$$\mathcal{P}_T f(x) = \sum_{z \in T^{-1}x} \frac{f(z)}{|T'(z)|}. \quad (2)$$

Then in general we have

$$T'(x) = \begin{cases} 2 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -2 & \frac{1}{2} \leq x < \frac{3}{4} \\ -1 & \frac{3}{4} \leq x \leq 1. \end{cases} \quad (3)$$

Also we can also see that

$$T^{-1}\{x\} = \begin{cases} \{1-x\} & 0 \leq x < \frac{1}{4} \\ \{\frac{7}{8} - \frac{x}{2}\} & \frac{1}{4} \leq x < \frac{1}{2} \\ \{\frac{7}{8} - \frac{x}{2}\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{1}{2} \leq x < \frac{3}{4} \\ \{\frac{5}{4} - x\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{3}{4} \leq x < 1 \end{cases} \quad (4)$$

thus for $f_i(x) = \mathbb{1}_{I_i}(x)$ we have that

$$\mathcal{P}_T f_1(x) = \frac{1}{2} \mathbb{1}_{I_3 \cup I_4}(x) = \frac{1}{2} \mathbb{1}_{I_3}(x) + \frac{1}{2} \mathbb{1}_{I_4}(x) \quad (5)$$

$$\mathcal{P}_T f_2(x) = \mathbb{1}_{I_4}(x) \quad (6)$$

$$\mathcal{P}_T f_3(x) = \frac{1}{2} \mathbb{1}_{I_2 \cup I_3}(x) = \frac{1}{2} \mathbb{1}_{I_2}(x) + \frac{1}{2} \mathbb{1}_{I_3}(x) \quad (7)$$

$$\mathcal{P}_T f_4(x) = \mathbb{1}_{I_1}(x) \quad (8)$$

1.2

Let $\mathcal{S} = \text{span} \bigcup_{i=1}^4 \mathbb{1}_{I_i}$.

Let $f \in \mathcal{S}$ then $f = \sum_{i=1}^4 \lambda_i \mathbb{1}_{I_i}$ and so by linearity of the Perron-Frobenius operator, we have that $\mathcal{P}_T(f) = \sum_{i=1}^4 \lambda_i \mathcal{P}_T(\mathbb{1}_{I_i})$ but from the result above we see that $\mathcal{P}_T(\mathbb{1}_{I_i}) \in \mathcal{S}$ for each i and so \mathcal{P} preserves \mathcal{S}

1.3

We do this by just plugging the basis of \mathcal{S} into the \mathcal{P}_T and putting the results in each row of the matrix. Seeing as we already have first part we can just write down the matrix M as

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

1.4

Using Mathematica we can calculate the left eigenvector corresponding to eigenvalue 1 which is

$$\mathbf{v} = (1, \frac{1}{2}, 1, 1) \quad (10)$$

which corresponds to the ACIM

$$\mathbb{1}_{I_1} + \frac{1}{2} \mathbb{1}_{I_2} + \mathbb{1}_{I_3} + \mathbb{1}_{I_4} \quad (11)$$

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