University of New South Wales

School of Mathematics and Statistics

Assignment 3

Ergodic Theory

Author:
Adam J. Gray

Student Number:
3329798

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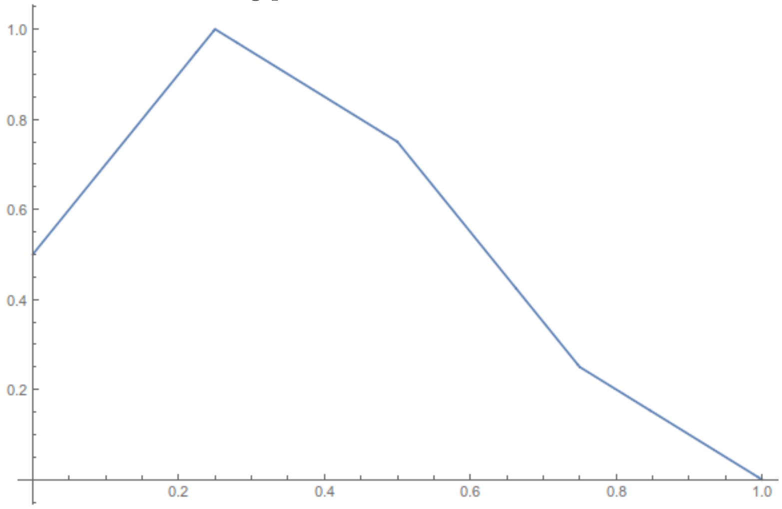
We consider the map

$$T(x) = \begin{cases} 2x + \frac{1}{2} & 0 \leq x < \frac{1}{4} \\ -x + \frac{5}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ -2x + \frac{7}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ -x + 1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{1}$$

This has the following plot:



1.1

We have that in general

$$\mathcal{P}_T f(x) = \sum_{z \in T^{-1}x} \frac{f(z)}{|T'(z)|}. \tag{2}$$

Then in general we have

$$T'(x) = \begin{cases} 2 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -2 & \frac{1}{2} \leq x < \frac{3}{4} \\ -1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{3}$$

Also we can also see that

$$T^{-1}\{x\} = \begin{cases} \{1 - x\} & 0 \leq x < \frac{1}{4} \\ \{\frac{7}{8} - \frac{x}{2}\} & \frac{1}{4} \leq x < \frac{1}{2} \\ \{\frac{7}{8} - \frac{x}{2}\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{1}{2} \leq x < \frac{3}{4} \\ \{\frac{5}{4} - x\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{3}{4} \leq x < 1 \end{cases} \tag{4}$$

thus for $f_i(x) = \mathbb{1}_{I_i}(x)$ we have that

$$\mathcal{P}_T f_1(x) = \frac{1}{2}\mathbb{1}_{I_3 \cup I_4}(x) = \frac{1}{2}\mathbb{1}_{I_3}(x) + \frac{1}{2}\mathbb{1}_{I_4}(x) \tag{5}$$

$$\mathcal{P}_T f_2(x) = \mathbb{1}_{I_4}(x) \tag{6}$$

$$\mathcal{P}_T f_3(x) = \frac{1}{2}\mathbb{1}_{I_2 \cup I_3}(x) = \frac{1}{2}\mathbb{1}_{I_2}(x) + \frac{1}{2}\mathbb{1}_{I_3}(x) \tag{7}$$

$$\mathcal{P}_T f_4(x) = \mathbb{1}_{I_1}(x) \tag{8}$$

1.2

Let $\mathcal{S} = \text{span} \bigcup_{i=1}^4 \mathbb{1}_{I_i}$.
Let $f \in \mathcal{S}$ then $f = \sum_{i=1}^4 \lambda_4 \mathbb{1}_{I_i}$ and so by linearity of the Perron-Frobenius operator, we have that $\mathcal{P}_T(f) = \sum_{i=1}^4 \lambda_4 \mathcal{P}_T(\mathbb{1}_{I_i})$ but from the result above we see that $\mathcal{P}_T(\mathbb{1}_{I_i}) \in \mathcal{S}$ for each i and so $\mathcal{P}$ preserves $\mathcal{S}$

1.3

We do this by just pluging the basis of $\mathcal{S}$ into the $\mathcal{P}_T$ and putting the results in each row of the matrix. Seeing as we already have first part we can just write down the matrix M as

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

1.4

Using Mathematica we can calculate the left eigenvector corresponding to eigenvalue 1 which is

$$\mathbf{v} = (1, \frac{1}{2}, 1, 1) \tag{10}$$

which corresponds to the ACIM (after rescaling)

$$\frac{8}{7}\left(\mathbb{1}_{I_1} + \frac{1}{2}\mathbb{1}_{I_2} + \mathbb{1}_{I_3} + \mathbb{1}_{I_4}\right) \tag{11}$$

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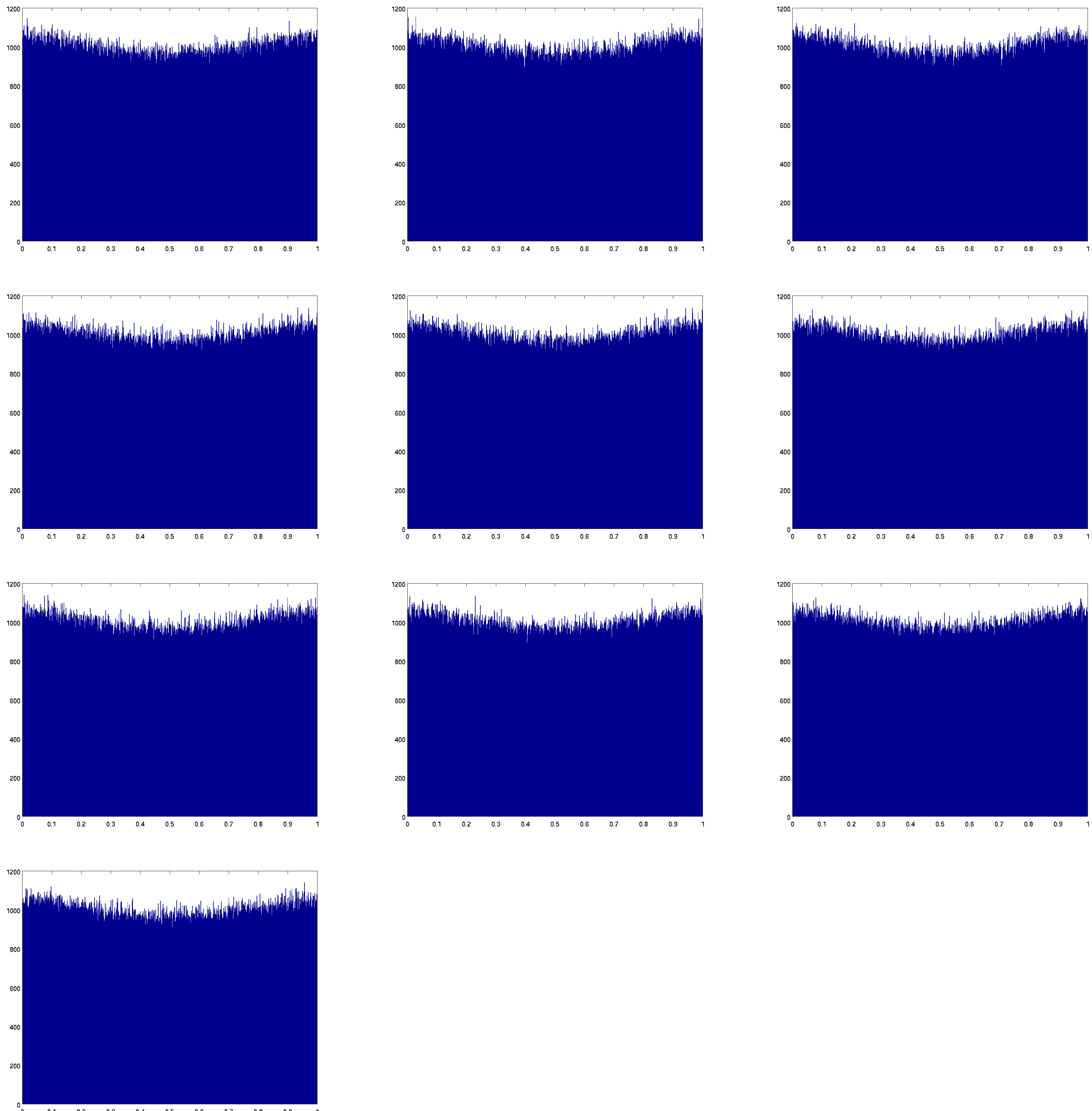
2.1

The following plots we produced by code in *Question_2_a.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.



What these historgrams show is the average time the trajectory spends in each bin. The Birkhoff ergodic theorem implies that we can approximate the invariant measure for T using this histogram. Formally this is done by using indicator functions across each bin and by weighting by the proportion of time that the process spends in each bin.

2.2

This code is available in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.3

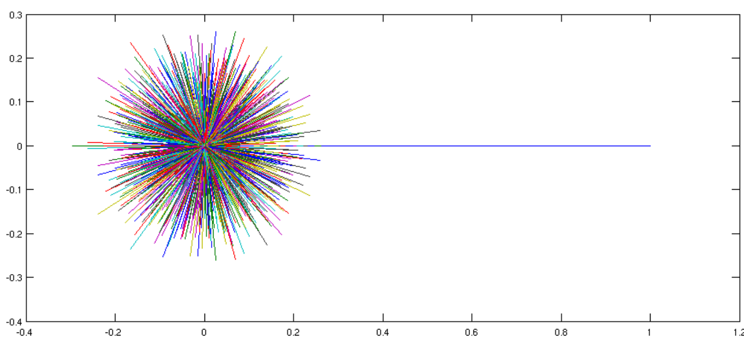
Its obvious this is a stochastic matrix...
Let r_j represent the j-th row sum. The

$$r_j = \sum_{i=1}^{1000} Q_{ji} = \sum_{i=1} \frac{\#\{p \in S_j | T(p) \in B_i\}}{1000} = 1 \tag{12}$$

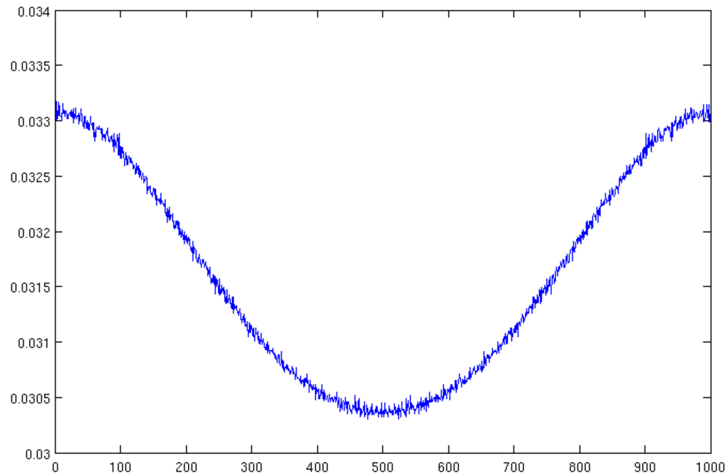
because $|B_i| = 1000$.
Also we can see that each entry must be positive (from the definition $Q_{ji} = \#\{p \in S_j | T(p) \in B_j\}$) so this is a stochastic matrix.
Note that numerical verification is done in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.4

The following is a plot of all the left eigenvalues of Q in the complex plane.



It is clear to see that there is only one eigenvalue equal to 1. We have also checked this in code in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.
The following is a plot of the left eigenvector corresponding to the eigenvalue 1.



Appart from the change of scale of the axis this plot aggrees with the histograms from before.

3

3.1

Firstly we see that because h is a diffeomorphism,

$$\mathcal{P}_h f(x) = \frac{f(h^{-1}(x))}{|h'(h^{-1}(x))|} \tag{13}$$

and by the inverse function theorem

$$\frac{f(h^{-1}(x))}{|h'(h^{-1}(x))|} = f(h^{-1}(x)) \cdot |(h^{-1}(x)'| \tag{14}$$

which by definition is g.
Now note that $S = h \circ T \circ h^{-1}$ or equivalently $T = h^{-1} \circ S \circ h$.
As h and T are non-singular, so is S and thus by the composition of Perron-Frobenius operators we can write

$$\mathcal{P}_T f = (\mathcal{P}_{h^{-1}} \circ \mathcal{P}_S \circ \mathcal{P}_h)\, f \tag{15}$$

or equivalently

$$(\mathcal{P}_h \circ \mathcal{P}_T)\, f = (\mathcal{P}_S \circ \mathcal{P}_h)\, f.. \tag{16}$$

Now as $\mathcal{P}_h f = g$ and by assumption $\mathcal{P}_T f = f$ then

$$\mathcal{P}_h f = \mathcal{P}_s g \tag{17}$$

but we have also shown that $\mathcal{P}_h f = g$ and so the result follows.

3.2

From the lectures we know that $\mathcal{P}_S g = g$ implies that g is an S invariant density, so the result is immediate.
A formal statement of that theorem along with it's proof is as follows

Theorem 1.

Proof. Suppose $\mathcal{P}_S g = g$. Then we can say

$$\int_A \mathcal{P}_S g d\mu = \int_A g d\mu. \tag{18}$$

Further we can say from the definition of the Perron-Frobenius operator that

$$\int_A \mathcal{P}_S f d\mu = \int_{S^{-1}A} g d\mu \tag{19}$$

and so

$$\int_A f d\mu = \int_{S^{-1}A} f d\mu \tag{20}$$

that is g is the density of a S invariant measure defined by

$$\nu(A) = \int_A g d\mu. \tag{21}$$

□

3.3

We wish to show that

$$\int_I \log|T'(x)| d\mu_f(x) = \int_I \log|S'(x)| d\mu_g(x). \tag{22}$$

Starting from the definition of S we can write

$$\begin{aligned}\int_I \log|S'(x)| d\mu_g(x) &= \int_I \log|\left(h \circ T \circ h^{-1}(x)\right)'(x)| d\mu_g(x) &(23)\\ &= \int_I \log|\left(h \circ T \circ h^{-1}(x)\right)'(x)| g(x) d\mu(x) &(24)\end{aligned}$$

but then using the definition of g we can further write

$$\begin{aligned}\int_I \log|\left(h \circ T \circ h^{-1}(x)\right)'(x)| g(x) d\mu(x) &= \int_I \log|\left(h \circ T \circ h^{-1}(x)\right)'(x)| (f \circ h^{-1})(x) |h^{-1}(x)'| d\mu(x) &(25)\\ &= \int_I \log|h'(T(h^{-1}(x))) \cdot T'(h^{-1}(x)) \cdot h^{-1}(x)'| (f \circ h^{-1})(x) |h^{-1}(x)'| d\mu(x) &(26)\\ &= \int_I \left(\log|h'(T(h^{-1}(x)))| + \log|T'(h^{-1}(x))| + \log|h^{-1}(x)'|\right) f(h^{-1}(x)) |h^{-1}(x)'| d\mu(x &(27)\end{aligned}$$

Now let $y = h(x)$ so we have $d\mu(y) = h'(x)d\mu(x)$, that is $d\mu(x) = \frac{1}{h'(x)}d\mu(y)$ and by the inverse function theorem $d\mu(x) = h^{-1}(y)'d\mu(y)$ so we have

$$\int_{h(I)} \big(-\log|h'(h^{-1}(S(y))| + \log|S'| + \log|h'(h^{-1}(y))|\big) \cdot \frac{h^{-1}(x)'}{|h^{-1}(x)'|} g(y)d\mu(y) \tag{28}$$

$$= \int_{h(I)} \big(-\log|h'(h^{-1}(S(y))| + \log|S'| + \log|h'(h^{-1}(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')g(y)d\mu(y) \tag{29}$$

$$= \underbrace{\int_{h(I)} \big(-\log|h'(h^{-1}(S(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y)}_{(1)} + \int_{h(I)} (\log|S'|) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y) \tag{30}$$

$$+ \underbrace{\int_{h(I)} \big(\log|h'(h^{-1}(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y)}_{(2)} \tag{31}$$

$$\tag{32}$$

by recognising the definition of g. Now from the previous result we have that S is μ_g invariant and so it follows that (1) and (2) just have opposite signs and hence cancel out. This leaves us with

$$\int_{h(I)} (\log|S'|) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y) \tag{33}$$

Now if h (and h^{-1}) has positive gradient then $h(I) = I$ and so we are left with just

$$\int_I \log|S(x)|d\mu_g(x). \tag{34}$$

On the other hand if h has negative gradient then $h(I) = -I$ in the sense that the direction of integration has changed. This means again we are left with

$$\int_I \log|S(x)|d\mu_g(x). \tag{35}$$

and so we have the result.

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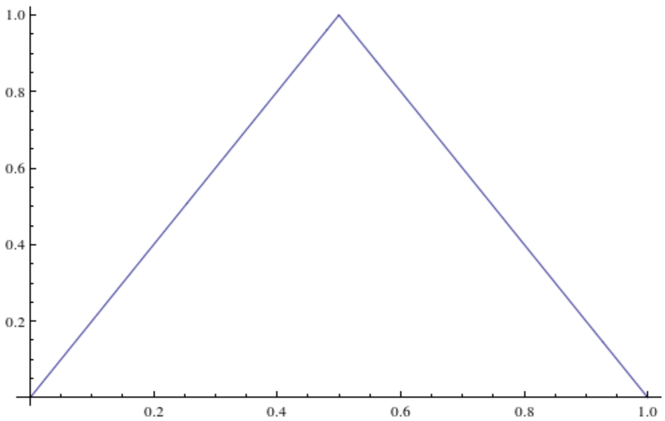
4.1

For $T : I \longrightarrow I$ defined as

$$T(x) := \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \le 1 \end{cases} \tag{36}$$

we have the following plot:



The T invariant density may be found in much the same way as for question 1. If we let $I_1 = \left[0, \frac{1}{2}\right]$ and $I_2 = \left[\frac{1}{2}, 1\right]$ and choose the basis $\{\mathbb{1}_{I_1}, \mathbb{1}_{I_2}\}$ then the associated matrix representation of the Perron-Frobenius operator is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{37}$$

which has eigenvector $(1, 1)$ associated with eigenvalue 1 and so the T invariant density is $\mathbb{1}_{[0,1]}$, in other words the density is the uniform density (1).

4.2

We have to do is show that

$$g = (f \circ h^{-1})|\left(h^{-1}\right)'| \tag{38}$$

where f is the density of the ACIM associated with T.
Now since f is the uniform density all we have to show is that

$$|\left(h^{-1}\right)'| = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{39}$$

Now since $h(x) = \sin^2(\frac{\pi x}{2})$ we have that $h^{-1}(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$ and by differentiating we see that

$$\left(h^{-1}(x)\right)' = \frac{1}{\pi\sqrt{x(1-x)}} \tag{40}$$

and note that for $x \in [0, 1]$ this is positive so

$$|(h^{-1}(x))'| = \frac{1}{\pi\sqrt{x(1-x)}} \tag{41}$$

The last thing we have to show is that $S(x) = \left(h \circ T \circ h^{-1}\right)(x)$ or equivelently $T(x) = \left(h^{-1} \circ S \circ h\right)(x)$.

See that

$$T(x) = \frac{2}{\pi}\sin^{-1}\left(\sqrt{4\sin^2\left(\frac{\pi x}{2}\right)\left(1-\sin^2\left(\frac{\pi x}{2}\right)\right)}\right) \tag{42}$$

$$= \frac{2}{\pi}\sin^{-1}\left(\sqrt{4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right)}\right) \tag{43}$$

$$= \frac{2}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right) \tag{44}$$

$$= \frac{2}{\pi}\sin^{-1}\left(\sin\left(\pi x\right)\right) \tag{45}$$

$$= \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \leq 1 \end{cases} \tag{46}$$

and so this resolves the problem in that by using the result of Section 3. That is, the density g is S-invariant.

4.3

By using the result of question 3.3 we can calculate the Lyapunov exponent of $(I, \mathcal{B}, \mu_g, S)$ by just calculating the Lyapunov exponent of $(I, \mathcal{B}, \mu_f, T)$. Since we have that $f = 1$ then we can just write

$$\int_I \log|T'(x)|d\mu_f(x) = \int_I \log|T'(x)|d\mu \tag{47}$$

$$= \int_0^1 \log 2 d\mu \tag{48}$$

$$= \log 2 \tag{49}$$