University of New South Wales

School of Mathematics and Statistics

Assignment 3

Ergodic Theory

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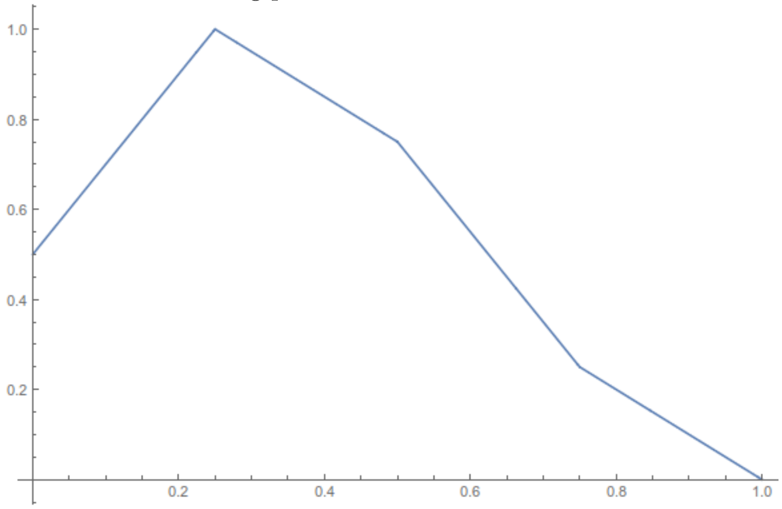
We consider the map

$$T(x) = \begin{cases} 2x + \frac{1}{2} & 0 \leq x < \frac{1}{4} \\ -x + \frac{5}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ -2x + \frac{7}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ -x + 1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{1}$$

This has the following plot:



1.1

We have that in general

$$\mathcal{P}_T f(x) = \sum_{z \in T^{-1}x} \frac{f(z)}{|T'(z)|}. \tag{2}$$

Then in general we have

$$T'(x) = \begin{cases} 2 & 0 \leq x < \frac{1}{4} \\ -1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -2 & \frac{1}{2} \leq x < \frac{3}{4} \\ -1 & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{3}$$

Also we can also see that

$$T^{-1}\{x\} = \begin{cases} \{1 - x\} & 0 \leq x < \frac{1}{4} \\ \{\frac{7}{8} - \frac{x}{2}\} & \frac{1}{4} \leq x < \frac{1}{2} \\ \{\frac{7}{8} - \frac{x}{2}\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{1}{2} \leq x < \frac{3}{4} \\ \{\frac{5}{4} - x\} \cup \{\frac{x}{2} - \frac{1}{4}\} & \frac{3}{4} \leq x < 1 \end{cases} \tag{4}$$

thus for $f_i(x) = \mathbb{1}_{I_i}(x)$ we have that

$$\mathcal{P}_T f_1(x) = \frac{1}{2}\mathbb{1}_{I_3 \cup I_4}(x) = \frac{1}{2}\mathbb{1}_{I_3}(x) + \frac{1}{2}\mathbb{1}_{I_4}(x) \tag{5}$$

$$\mathcal{P}_T f_2(x) = \mathbb{1}_{I_4}(x) \tag{6}$$

$$\mathcal{P}_T f_3(x) = \frac{1}{2}\mathbb{1}_{I_2 \cup I_3}(x) = \frac{1}{2}\mathbb{1}_{I_2}(x) + \frac{1}{2}\mathbb{1}_{I_3}(x) \tag{7}$$

$$\mathcal{P}_T f_4(x) = \mathbb{1}_{I_1}(x) \tag{8}$$

1.2

Let $\mathcal{S} = \operatorname{span} \bigcup_{i=1}^{4} \mathbb{1}_{I_i}$.
Let $f \in \mathcal{S}$ then $f = \sum_{i=1}^{4} \lambda_4 \mathbb{1}_{I_i}$ and so by linearity of the Perron-Frobenius operator, we have that $\mathcal{P}_T(f) = \sum_{i=1}^{4} \lambda_4 \mathcal{P}_T(\mathbb{1}_{I_i})$ but from the result above we see that $\mathcal{P}_T(\mathbb{1}_{I_i}) \in \mathcal{S}$ for each i and so $\mathcal{P}$ preserves $\mathcal{S}$

1.3

We do this by just pluging the basis of $\mathcal{S}$ into the $\mathcal{P}_T$ and putting the results in each row of the matrix. Seeing as we already have first part we can just write down the matrix M as

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

1.4

Using Mathematica we can calculate the left eigenvector corresponding to eigenvalue 1 which is

$$\mathbf{v} = (1, \frac{1}{2}, 1, 1) \tag{10}$$

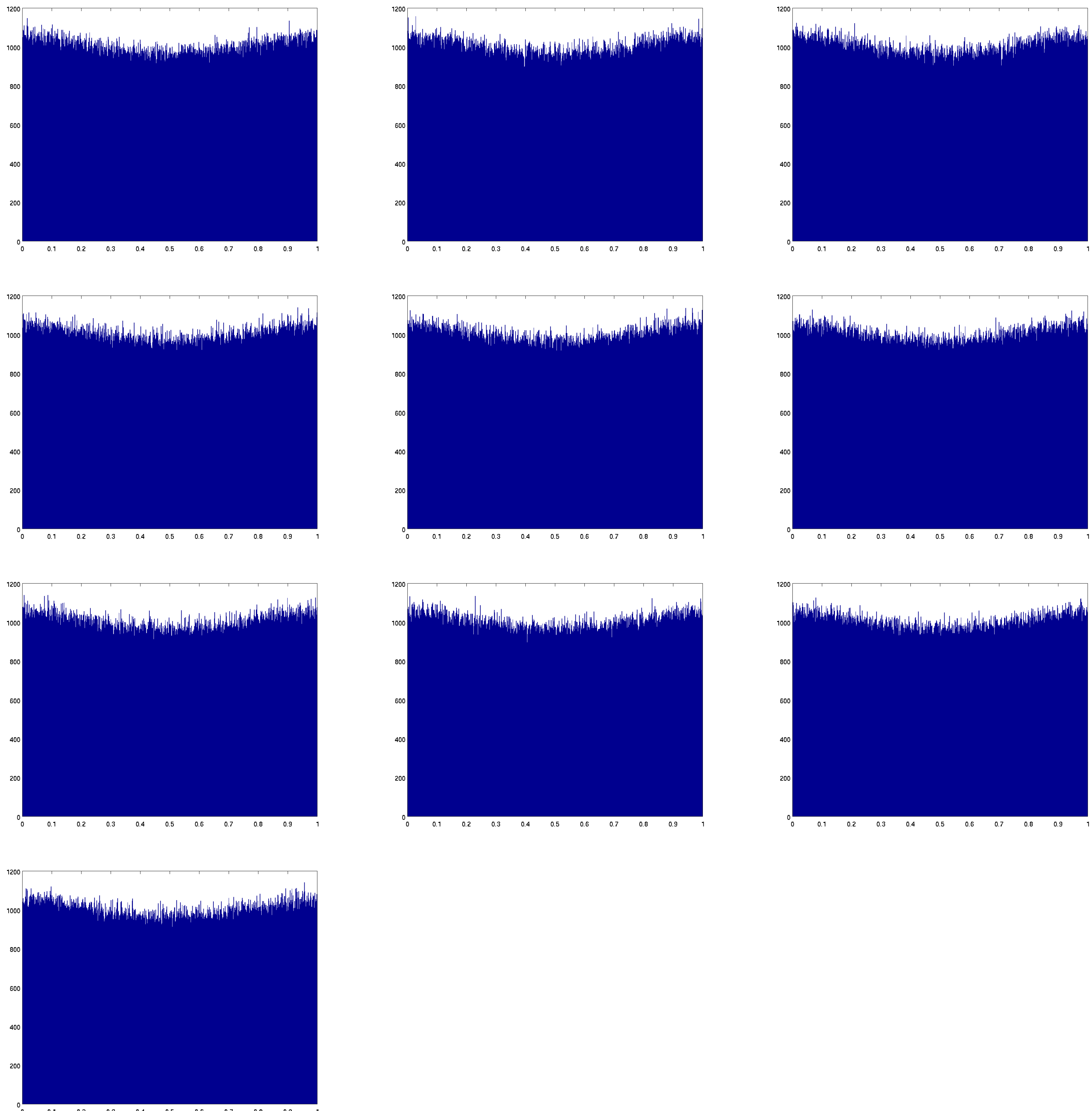
which corresponds to the ACIM (after rescaling)

$$\frac{8}{7}\left(\mathbb{1}_{I_1} + \frac{1}{2}\mathbb{1}_{I_2} + \mathbb{1}_{I_3} + \mathbb{1}_{I_4}\right) \tag{11}$$

2

2.1

The following plots we produced by code in *Question_2_a.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.



What these historgrams show is that TODO

2.2

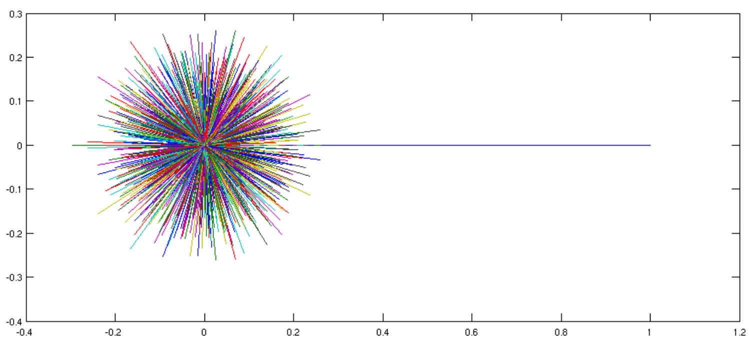
This code is available in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.3

Show TODO
Note that numerical verification is done in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.

2.4

The following is a plot of all the left eigenvalues of Q in the complex plane.



It is clear to see that there is only one eigenvalue equal to 1. We have also checked this in code in *Question_2_b_c_d.m* available at `https://github.com/adamjoshuagray/Honours_Ergodic_Theory/tree/master/Assignment_3`.
The following is a plot of the left eigenvector corresponding to the eigenvalue 1.

3

3.1

Firstly we see that because h is a diffeomorphism,

$$\mathcal{P}_h f(x) = \frac{f(h^{-1}(x))}{|h'(h^{-1}(x))|} \tag{12}$$

and by the inverse function theorem

$$\frac{f(h^{-1}(x))}{|h'(h^{-1}(x))|} = f(h^{-1}(x)) \cdot |(h^{-1}(x)'| \tag{13}$$

which by definition is g.
Now note that $S = h \circ T \circ h^{-1}$ or equivalently $T = h^{-1} \circ S \circ h$.
As h and T are non-singular, so is S and thus by the composition of Perron-Frobenius operators we can write

$$\mathcal{P}_T f = (\mathcal{P}_{h^{-1}} \circ \mathcal{P}_S \circ \mathcal{P}_h) f \tag{14}$$

or equivalently

$$(\mathcal{P}_h \circ \mathcal{P}_T)\, f = (\mathcal{P}_S \circ \mathcal{P}_h)\, f.. \tag{15}$$

Now as $\mathcal{P}_h f = g$ and by assumption $\mathcal{P}_T f = f$ then

$$\mathcal{P}_h f = \mathcal{P}_s g \tag{16}$$

but we have also shown that $\mathcal{P}_h f = g$ and so the result follows.

3.2

From the lectures we know that $\mathcal{P}_S g = g$ implies that g is an S invariant density, so the result is immediate. A formal statement of that theorem along with it's proof is as follows

Theorem 1.

Proof. Suppose $\mathcal{P}_S g = g$. Then we can say

$$\int_A \mathcal{P}_S g d\mu = \int_A g d\mu. \tag{17}$$

Further we can say from the definition of the Perron-Frobenius operator that

$$\int_A \mathcal{P}_S f d\mu = \int_{S^{-1}A} g d\mu \tag{18}$$

and so

$$\int_A f d\mu = \int_{S^{-1}A} f d\mu \tag{19}$$

that is g is the density of a S invariant measure defined by

$$\nu(A) = \int_A g d\mu. \tag{20}$$

□

3.3

We wish to show that

$$\int_I \log|T'(x)| d\mu_f(x) = \int_I \log|S'(x)| d\mu_g(x). \tag{21}$$

Starting from the definition of S we can write

$$\int_I \log|S'(x)| d\mu_g(x) = \int_I \log\left| \left(h \circ T \circ h^{-1}(x)\right)'(x) \right| d\mu_g(x) \tag{22}$$

$$= \int_I \log\left| \left(h \circ T \circ h^{-1}(x)\right)'(x) \right| g(x) d\mu(x) \tag{23}$$

but then using the definition of g we can further write

$$\int_I \log\left| \left(h \circ T \circ h^{-1}(x)\right)'(x) \right| g(x) d\mu(x) = \int_I \log\left| \left(h \circ T \circ h^{-1}(x)\right)'(x) \right| (f \circ h^{-1})(x) |h^{-1}(x)'| d\mu(x) \tag{24}$$

$$= \int_I \log|h'(T(h^{-1}(x))) \cdot T'(h^{-1}(x)) \cdot h^{-1}(x)'| (f \circ h^{-1})(x) |h^{-1}(x)'| d\mu(x) \tag{25}$$

$$= \int_I \left(\log|h'(T(h^{-1}(x)))| + \log|T'(h^{-1}(x))| + \log|h^{-1}(x)'|\right) f(h^{-1}(x)) |h^{-1}(x)'| d\mu(x) \tag{26}$$

Now let $y = h(x)$ so we have $d\mu(y) = h'(x)d\mu(x)$, that is $d\mu(x) = \frac{1}{h'(x)}d\mu(y)$ and by the inverse function theorem $d\mu(x) = h^{-1}(y)'d\mu(y)$ so we have

$$\int_{h(I)} \big(-\log|h'(h^{-1}(S(y))| + \log|S'| + \log|h'(h^{-1}(y))|\big) \cdot \frac{h^{-1}(x)'}{|h^{-1}(x)'|} g(y)d\mu(y) \tag{27}$$

$$= \int_{h(I)} \big(-\log|h'(h^{-1}(S(y))| + \log|S'| + \log|h'(h^{-1}(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')g(y)d\mu(y) \tag{28}$$

$$= \underbrace{\int_{h(I)} \big(-\log|h'(h^{-1}(S(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y)}_{(1)} + \int_{h(I)} (\log|S'|) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y) \tag{29}$$

$$+ \underbrace{\int_{h(I)} \big(\log|h'(h^{-1}(y))|\big) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y)}_{(2)} \tag{30}$$

$$\tag{31}$$

by recognising the definition of g. Now from the previous result we have that S is μ_g invariant and so it follows that (1) and (2) just have opposite signs and hence cancel out. This leaves us with

$$\int_{h(I)} (\log|S'|) \cdot \operatorname{sgn}(h^{-1}(x)')d\mu_g(y) \tag{32}$$

Now if h (and h^{-1}) has positive gradient then $h(I) = I$ and so we are left with just

$$\int_I \log|S(x)|d\mu_g(x). \tag{33}$$

On the other hand if h has negative gradient then $h(I) = -I$ in the sense that the direction of integration has changed. This means again we are left with

$$\int_I \log|S(x)|d\mu_g(x). \tag{34}$$

and so we have the result.

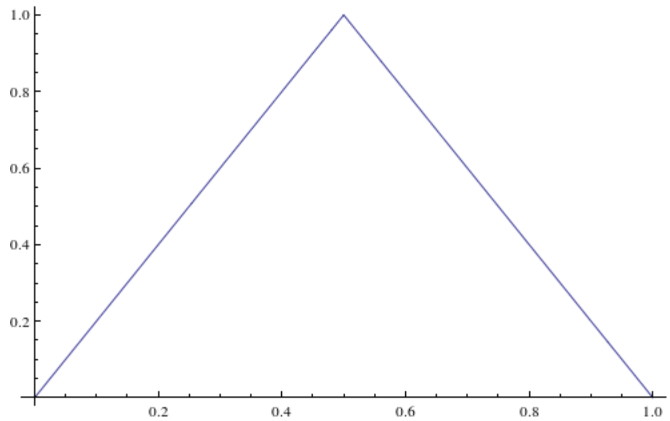
4

4.1

For $T : I \longrightarrow I$ defined as

$$T(x) := \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \le 1 \end{cases} \tag{35}$$

we have the following plot:



The T invariant density may be found in much the same way as for question 1. If we let $I_1 = \left[0, \frac{1}{2}\right]$ and $I_2 = \left[\frac{1}{2}, 1\right]$ and choose the basis $\{\mathbb{1}_{I_1}, \mathbb{1}_{I_2}\}$ then the associated matrix representation of the Perron-Frobenius operator is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{36}$$

which has eigenvector $(1, 1)$ associated with eigenvalue 1 and so the T invariant density is $\mathbb{1}_{[0,1]}$, in other words the density is the uniform density (1).

4.2

We have to do is show that

$$g = (f \circ h^{-1}) | \left(h^{-1}\right)' | \tag{37}$$

where f is the density of the ACIM associated with T.
Now since f is the uniform density all we have to show is that

$$| \left(h^{-1}\right)' | = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{38}$$

Now since $h(x) = \sin^2(\frac{\pi x}{2})$ we have that $h^{-1}(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$ and by differentiating we see that

$$\left(h^{-1}(x)\right)' = \frac{1}{\pi\sqrt{x(1-x)}} \tag{39}$$

and note that for $x \in [0, 1]$ this is positive so

$$|(h^{-1}(x))'| = \frac{1}{\pi\sqrt{x(1-x)}} \tag{40}$$

The last thing we have to show is that $S(x) = \left(h \circ T \circ h^{-1}\right)(x)$ or equivelently $T(x) = \left(h^{-1} \circ S \circ h\right)(x)$.

See that

$$
\begin{aligned}
T(x) &= \frac{2}{\pi}\sin^{-1}\left(\sqrt{4\sin^2\left(\frac{\pi x}{2}\right)\left(1-\sin^2\left(\frac{\pi x}{2}\right)\right)}\right) && (41)\\
&= \frac{2}{\pi}\sin^{-1}\left(\sqrt{4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right)}\right) && (42)\\
&= \frac{2}{\pi}\sin^{-1}\left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right) && (43)\\
&= \frac{2}{\pi}\sin^{-1}\left(\sin\left(\pi x\right)\right) && (44)\\
&= \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \le 1 \end{cases} && (45)
\end{aligned}
$$

and so this resolves the problem in that by using the result of Section 3. That is, the density g is S-invariant.

4.3

By using the result of question 3.3 we can calculate the Lyapunov exponent of $(I, \mathcal{B}, \mu_g, S)$ by just calculating the Lyapunov exponent of $(I, \mathcal{B}, \mu_f, T)$. Since we have that $f = 1$ then we can just write

$$
\begin{aligned}
\int_I \log|T'(x)|d\mu_f(x) &= \int_I \log|T'(x)|d\mu && (46)\\
&= \int_0^1 \log 2 d\mu && (47)\\
&= \log 2 && (48)
\end{aligned}
$$