



UNSW
A U S T R A L I A



UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment

Functional Analysis

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Declaration

I declare that this assessment item is my own work, except where acknowledged, and has not been submitted for academic credit elsewhere.

I certify that I have read and understood the University Rules in respect of Student Academic Misconduct.

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Question 1

See that

$$\begin{aligned}\langle Tx, x \rangle &= \overline{\langle x, Tx \rangle} \\ &\geq 0\end{aligned}$$

which means

$$\begin{aligned}\overline{\langle x, Tx \rangle} &= \langle x, Tx \rangle \\ &= \langle T^*x, x \rangle\end{aligned}$$

so

$$\langle (T - T^*)x, x \rangle = 0$$

for all $x \in \mathcal{H}$.

We just want to show that $(T - T^*) = 0$. Say $\langle Jx, x \rangle = 0$ for all $x \in \mathcal{H}$ then for $x, y \in \mathcal{H}$ see that

$$\begin{aligned}\langle J(x + y), x + y \rangle &= \langle Jx, y \rangle + \langle Jy, x \rangle + \underbrace{\langle Jx, x \rangle}_{=0} + \underbrace{\langle Jy, y \rangle}_{=0} \\ &= \langle Jx, y \rangle + \langle Jy, x \rangle\end{aligned}\tag{1}$$

and see that

$$\begin{aligned}\langle J(x + iy), x + iy \rangle &= \langle Jx, iy \rangle + \langle Jiy, x \rangle + \underbrace{\langle Jx, x \rangle}_{=0} + \underbrace{\langle Jiy, iy \rangle}_{=0} \\ &= -i\langle Jx, y \rangle + i\langle Jy, x \rangle\end{aligned}\tag{2}$$

Now multiplying (1) by i and subtracting it from (2) yields

$$\langle Jx, y \rangle = 0$$

for all $x, y \in \mathcal{H}$ which implies $J \equiv 0$. Letting $J = (T - T^*)$, yields $(T - T^*) = 0$ as required.

So $T \equiv T^*$. □

Question 2

Part 1

From a toeplitz operator $T = \{a_i, j\}_{i,j \in \mathbb{Z}}$ define the diagonal operator D_j as the matrix-type operator taken as then j -th diagonal of T . Ie.

$$D_j = \begin{pmatrix} 0 & \cdots & a_j & 0 & 0 & \cdots \\ 0 & \cdots & 0 & a_j & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \ddots & \cdots \end{pmatrix}$$

If $\mathbf{x} = (x_1, x_2, \dots)$ then it is clear that $D_j \mathbf{x} = (0, \dots, 0, a_j x_j, 0, \dots)$. Also it is clear that

$$T = \sum_{j \in \mathbb{Z}} D_j.$$

Since we have that $\sum_{j \in \mathbb{Z}} |a_j| = M < \infty$ we can write

$$\begin{aligned} \|T\mathbf{x}\| &= \left\| \sum_{j \in \mathbb{Z}} D_j \mathbf{x} \right\| \\ &\leq \sum_{j \in \mathbb{Z}} \|D_j \mathbf{x}\| \\ &\leq \sum_{j \in \mathbb{Z}} |a_j| \|\mathbf{x}\| \\ &= \|\mathbf{x}\| \sum_{j \in \mathbb{Z}} |a_j| \\ &= \|\mathbf{x}\| M \end{aligned}$$

and so T is a bounded on ℓ^2 .

Part 3

Using the idea that $T = \sum_{j \in \mathbb{Z}} D_j$ from part 1 we consider

$$\begin{aligned} \sum_{n=1}^{\infty} \|Te_n\|^2 &= \sum_{n=1}^{\infty} \left\| \sum_{j \in \mathbb{Z}} D_j e_n \right\|^2 \\ &= \sum_{n=1}^{\infty} \|(a_{n-1}, a_{n-2}, \dots)\|^2 \\ &= \sum_{n=1}^{\infty} \left\| \sum_{j=0}^{\infty} a_{n-j}^2 \right\| \end{aligned}$$

Now suppose that $a_k \neq 0$ for some k then there exists an N such that $n > N$ implies that a_k^2 is in the sum $\sum_{j=0}^{\infty} a_{n-j}^2$ and consequently $\sum_{n=1}^{\infty} \left\| \sum_{j=0}^{\infty} a_{n-j}^2 \right\| = \infty$.

So $a_k = 0$ for all k and Hilbert-Schmidt Toeplitz operators are 0.

Question 3

Part 1

$x \in (\operatorname{Im} T)^\perp$ iff

$$\begin{aligned} \langle Ty, x \rangle &= 0 & \forall y \in \mathcal{H} \\ \Leftrightarrow \langle y, T^*x \rangle &= 0 & \forall y \in \mathcal{H} \\ \Leftrightarrow T^*x &= 0 \\ \Leftrightarrow x &\in \ker T^* \end{aligned}$$

So $(\operatorname{Im} T)^\perp = \ker T^*$. □

Part 2

Setting $T = T^*$ in the result of *part a* yields

$$(\operatorname{Im} T^*)^\perp = \ker T^{**} \quad \Leftrightarrow (\operatorname{Im} T^*)^\perp = \ker T$$

and taking orthogonal complements of both sides gets

$$\begin{aligned} (\operatorname{Im} T^*)^{\perp\perp} &= (\ker T)^\perp \\ \Leftrightarrow \overline{\operatorname{Im} T^*} &= (\ker T)^\perp. \end{aligned}$$

So $\overline{\operatorname{Im} T^*} = (\ker T)^\perp$. □