University of New South Wales

School of Mathematics and Statistics

Assignment

Functional Analysis

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Declaration

I declare that this assessment item is my own work, except where acknowledged, and has not been submitted for academic credit elsewhere.
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Question 1

Define a map $T : X/X_0 \longrightarrow (X_0^\perp)*$ by having $T(x + X_0)(f) = f(x)$.
T Well Defined:
If $x + X_0 = y + X_0$ then $f(x) = f(y)$ because $x - y \in X_0$ and $f \in X_0^\perp$, so $f(x) - f(y) = f(x - y) = 0$.
T Linear:

$$T(\alpha x + \beta y + X_0) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha T(x) + \beta T(y)$$

T Isometric:
Firstly prove $||T(x + X_0)||_{(X_0^\perp)^*} \leq ||x + X_0||_{X/X_0}$:

$$||T(x + X_0)|| = \sup_{\substack{||f||\leq 1 \\ f\in X_0^\perp}} |f(x)| \leq \inf_{x\in X_0} ||x - x'||$$

because for all $x' \in X_0$, $|f(x - x')| = |f(x)| = ||x - x'||$.
Now prove $||T(x + X_0)||_{(X_0^\perp)^*} \geq ||x + X_0||_{X/X_0}$. Fix $x^* \in X$ and let
$V = \text{span}\{x^*\}$. Define $\omega \in V^*$ by $\omega(\lambda x) = \lambda||x^* + X_0||_{X/X_0}$. This functional is clearly linear on V.
$|\omega(x^*)| = ||x^* + X_0||_{X/X_0} \leq ||x^*||_X$ so

$$||\omega||_{V^*} = \sup_{\substack{x\in X \\ x\neq 0}} \frac{|\omega(x)|}{||x||_X} \leq 1.$$

By Hahn-Banach $\exists\ \ \overline{\omega} \in X^*$ with $\overline{\omega}(x) = \omega(x)\ \ \forall\ \ x \in V$ and $|\overline{\omega}(x)| \leq |\omega(x)| = ||x + X_0||$.
Then for $z \in X_0$, $\overline{\omega}(z) = 0$ because $||z + X_0|| = 0$ and so $\overline{\omega} \in X_0^\perp$
$|\overline{\omega}(z)| \leq ||z||$.
Now $\overline{\omega}(x^*) = ||x^* + X_0||_{X/X_0}$ therefore taking the sup over all $f \in X_0^\perp$ we have

$$\sup_{\substack{f\in X_0^\perp \\ ||f||_{X^*}\leq 1}} |f(x^*)| \geq ||x^* + X_0||_{X/X_0}$$

and so $||T(x^* + X_0)||_{(X_0^\perp)^*} \geq ||x^* + X_0||_{X/X_0}$ for all $x^* \in X$. $\square$

Question 2

Definition of T $\Leftrightarrow$ T continuous $\Leftrightarrow$ T bounded. (You probably didn't want that proof...)
Another method
T bounded $\Rightarrow T^{-1}(U)$ *open:*
For some open $U \in Y$ such that $T^{-1}(U) \neq \emptyset$, pick an $x_1 \in T^{-1}(U)$. Because U is open $\exists\ \ \varepsilon > 0$ such that $B_1 = \{y \in Y : ||Tx_1 - y|| < \varepsilon\} \subset U$. Let $\delta = \frac{\varepsilon}{||T||}$. Then any $x_2 \in B_2 = \{x \in X : ||x_1 - x_2|| < \delta\}$ is such that

$||Tx_1 - Tx_2|| \leq ||T|| \cdot ||x_1 - x_2|| < \varepsilon$ because T is bounded. This means $Tx_2 \in U$ and importantly $x_2 \in T^{-1}(U)$. It follows that $B_2 \subset T^{-1}(U)$ because this holds for any $x_2 in B_2$.
$T^{-1}(U)$ open $\Rightarrow$ T bounded:
Let $U = \{y \in Y : ||y|| < 1\}$ then we have that $T^{-1}(U)$ is open, which specifically means that there exists some ε such that $B = \{X \in X : ||x|| < \varepsilon\} \subset T^{-1}(U)$. The fact that this ball contains 0 is guarenteed by the linearity of T.
Then for $x \in B$, $T(x) \in U$ so

$$\sup_{x \in B} ||Tx|| \leq 1$$

and

$$\sup_{||x|| \leq 1} ||Tx|| = \sup_{x \in B} ||T(\frac{x}{\varepsilon})|| \leq \frac{1}{\varepsilon}.$$

Question 3

Firstly show that $X_0^{\perp} = \text{span}\{x\}$ where $x = (1, 1, 1, \ldots)$.
Note that $X^{\perp} = \ell^{\infty}$ from lectures.
Let $y = (\xi_k)_{k \in \mathbb{N}} \in \ell^1$ with $\xi_1 = 1$ and $\xi_n = 1$ but $\xi_k = 0$ otherwise and let $z = (\eta_k)_{k \in \mathbb{N}} \in X_0^{\perp}$. We require that for all $y \in X_0$

$$\sum_{k \in \mathbb{N}} \overline{\eta_k} \xi_k = 0 \qquad \circledast$$

This therefore requires that $\eta_n = -\eta_1$, but as n was abitrary and $\circledast$ must hold for all $y \in X_0$ so it follows that $X_0^{\perp} \subseteq \text{span}\{x\}$ where $x = (1, 1, 1, \ldots)$.
It is clear that if $z = (\eta, \eta, \eta, \ldots) \in \text{span}\{x\}$ then

$$\sum_{k \in \mathbb{N}} \overline{\eta} \xi_k = \overline{\eta} \sum_{k \in \mathbb{N}} \xi_k = 0$$

so $z \in X_0^{\perp}$ and $X_0^{\perp} = \text{span}\{x\}$.
Secondly show that there exists an isometrical isomorphism $T : X/X_0 \longrightarrow \mathbb{C}$.
Define a mapping $T : X/X_0 \longrightarrow \mathbb{C}$ by $T(x + X_0) = \sum_{k \in \mathbb{N}} \xi_k$ where $x = (\xi_k)_{k \in \mathbb{N}}$
T Well Defined:
Suppose $y = (\eta_k)_{k \in \mathbb{N}} \in X_0$ and $x = (\xi_k)_{k \in \mathbb{N}} \in X$ then

$$T(x + y) = \sum_{k \in \mathbb{N}} (\xi_k + \eta_k) = \sum_{k \in \mathbb{N}} \xi_k + \underbrace{\sum_{k \in \mathbb{N}} \eta_k}_{=0} = \sum_{k \in \mathbb{N}} \xi_k = T(x)$$

T Linear:
This follows directly from the linearity of the sum.
T Surjective:
This is clear, because for any $a \in \mathbb{C}$ we just notice that $x = (a, 0, 0, \ldots)$ is such that $T(x + X_0) = a$.

T Injective:
Suppose $T(x + X_0) = T(y + X_0)$ then $T(x - y + X_0) = 0$ by linearity and by the definition of T and X_0 it follows that $x - y \in X_0$ so $x + X_0 = y + X_0$.
T injective and T surjective implies X/X_0 is ismorphic to $\mathbb{C}$ and importantly 1 dimensional.
T Isometric:
Because we have shown that X/X_0 then we have that any $y + X_0 \in X/X_0$ is such that $y = \lambda x + X_0$. We choose $x = (\xi_k)_{k\in\mathbb{N}}$ with $\xi_k = \left(\frac{1}{2}\right)^k$ [1] so that $T(x + X_0) = 1$ and $\xi_k \geq 0$ for all $k \in \mathbb{N}$. Now $|T(y + X_0)| = |\lambda T(x + X_0)| = |\lambda|$ and $||y+X_0|| = \inf_{x\in X_0} ||y-x|| = ||y||$ by the definition of X_0. So $|T(y+X_0)| = ||y + X_0||$ and T is an isometry.

[1] $0 \notin \mathbb{N}$