University of New South Wales

School of Mathematics and Statistics

Assignment

Functional Analysis

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Declaration

I declare that this assessment item is my own work, except where acknowledged, and has not been submitted for academic credit elsewhere.
I certify that I have read and understood the University Rules in respect of Student Academic Misconduct.

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Question 1

See that

$$\begin{aligned}\langle Tx, x\rangle &= \overline{\langle x, Tx\rangle} \\ &\geq 0\end{aligned}$$

which means

$$\begin{aligned}\overline{\langle x, Tx\rangle} &= \langle x, Tx\rangle \\ &= \langle T^* x, x\rangle\end{aligned}$$

so

$$\langle (T - T^*)x, x\rangle = 0$$

for all $x \in \mathcal{H}$.
We just want to show that $(T - T^*) = 0$. Say $\langle Jx, x\rangle = 0$ for all $x \in \mathcal{H}$ then for $x, y \in \mathcal{H}$ see that

$$\begin{aligned}\langle J(x+y), x+y\rangle &= \langle Jx, y\rangle + \langle Jy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jy, y\rangle}_{=0} \\ &= \langle Jx, y\rangle + \langle Jy, x\rangle \qquad (1)\end{aligned}$$

and see that

$$\begin{aligned}\langle J(x+iy), x+iy\rangle &= \langle Jx, iy\rangle + \langle Jiy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jiy, iy\rangle}_{=0} \\ &= -i\langle Jx, y\rangle + i\langle Jy, x\rangle \qquad (2)\end{aligned}$$

Now multiplying (1) by i and subtracting it from (2) yields

$$\langle Jx, y\rangle = 0$$

for all $x, y \in \mathcal{H}$ which implies $J \equiv 0$. Letting $J = (T - T^*)$, yields $(T - T^*) = 0$ as required.
So $T \equiv T^*$. $\square$

Question 2

Part 1

From a toeplitz operator $T = \{a_{i-j}\}_{i,j\in\mathbb{Z}}$ define the diagonal operator D_j as the matrix-type operator taken as then j-th diagonal of T. Ie.

$$D_j = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \\ a_j & \ddots & \ddots & \ddots \\ 0 & a_j & \ddots & \ddots \\ 0 & 0 & a_j & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

If $\mathbf{x} = (x_1, x_2, \ldots)$ then it is clear that $D_j\mathbf{x} = (0, \ldots, 0, a_jx_1, a_jx_2, \ldots)$. if $j > 0$, and that $D_j\mathbf{x} = (a_jx_{j+1}, a_jx_{j+2}, a_jx_{j+3}, \ldots)$ when $j < 0$ In the case $j = 0$ then $D_0\mathbf{x} = (a_0x_1, a_0x_2, a_0x_3, \ldots)$. In all three cases we have bonded linear operators. We can regard $D_j = a_j(S^*)^j$ when $j < 0$, and $D_j = a_jS^j$ when $j > 0$ and the a_0 times the identity operator when $j = 0$. Importantly in all three cases the norm of the operator is just 1.
Also it is clear that

$$T = \sum_{j\in\mathbb{Z}} D_j.$$

Since we have that

$$\sum_{j\in\mathbb{Z}} |a_j| = M < \infty$$

we can write

$$\begin{aligned} ||T\mathbf{x}|| &= ||\sum_{j\in\mathbb{Z}} D_j\mathbf{x}|| \\ &\leq \sum_{j\in\mathbb{Z}} ||D_j\mathbf{x}|| \\ &\leq \sum_{j\in\mathbb{Z}} |a_j|||x|| \\ &= ||x||\sum_{j\in\mathbb{Z}} |a_j| \\ &= ||x||M \end{aligned}$$

and so T is a bounded on ℓ^2.

Part 3

Since Hilbert-Schmidt operators are independent of the basis we can just use the standard ℓ^2 basis for this argument. Using the idea that $T = \sum_{j\in Z} D_j$ from part 1 we consider

$$\begin{aligned} \sum_{n=1}^{\infty} ||Te_n||^2 &= \sum_{n=1}^{\infty} ||\sum_{j\in\mathbb{Z}} D_j\mathbf{e}_n||^2 \\ &= \sum_{n=1}^{\infty} ||(a_{n-1}, a_{n-2}, \ldots)||^2 \\ &= \sum_{n=1}^{\infty} ||\sum_{j=0}^{\infty} a_{n-j}^2|| \end{aligned}$$

Now suppose that $a_k \neq 0$ for some k then there exits an N such that $n > N$ implies that a_k^2 is in the sum $\sum_{j=0}^{\infty} a_{n-j}^2$ and consequently $\sum_{n=1}^{\infty} ||\sum_{j=0}^{\infty} a_{n-j}^2|| = \infty$.
So $a_k = 0$ for all k and Hilbert-Schmidt Toeplitz operators are just the 0 operator.

Question 3

Part 1

$x \in (\operatorname{Im} T)^{\perp}$ iff

$$\begin{aligned} &\langle Ty, x\rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow \langle y, T^{*}x\rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow T^{*}x = 0 \\ &\Leftrightarrow x \in \ker T* \end{aligned}$$

So $(\operatorname{Im} T)^{\perp} = \ker T^{*}$. $\square$

Part 2

Setting $T = T^{*}$ in the result of *part a* yields

$$(\operatorname{Im} T^{*})^{\perp} = \ker T^{**} \qquad \Leftrightarrow (\operatorname{Im} T^{*})^{\perp} = \ker T$$

and taking orthogonal complements of both sides gets

$$\begin{aligned} &(\operatorname{Im} T^{*})^{\perp\perp} = (\ker T)^{\perp} \\ &\Leftrightarrow \overline{\operatorname{Im} T^{*}} = (\ker T)^{\perp}. \end{aligned}$$

So $\overline{\operatorname{Im} T^{*}} = (\ker T)^{\perp}$. $\square$

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Thank you to Roberto Riedig for suggesting a method which I latter developed for the solution to question 2.