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UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Lecture Notes

Functional Analysis

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Spectral Thoery

Spectral theory is analogous to eigen-value theory for matrices. In this case we look at spectral theory on bounded linear operators on a Hilbert space \mathcal{H} .

We define the following

For $T \in B(\mathcal{H})$ we define the following

Definition 1 (Resolvent Set). *Define $\rho(T) := \{\lambda \in \mathbb{C} : \exists (T - \lambda)^{-1} \in B(\mathcal{H})\}$.*

Definition 2 (Resolvent). *Define $R_\lambda(T) := (T - \lambda)^{-1}$*

Definition 3 (Spectrum). *Define $\sigma(T) := \mathbb{C} \setminus \rho(T)$.*

One could heuristically think of λ as being an eigen-value and we formalize this idea as follows.

If $\mathcal{H} = \mathbb{C}^n$ and $T = T_A$ where $A = (a_{j,k})_{j,k=1}^n$ (a matrix) then if $(T - \lambda I)^{-1}$ does not exist, i.e. $\ker(T - \lambda I) \neq \mathbf{0}$ then λ is an eigen-value.

We would like to formalize and prove the following three ideas;

- $\sigma(T)$ is bounded,
- $\sigma(T)$ is closed, and
- $\sigma(T)$ is non-empty.

Lemma 1 ($\sigma(T)$ is bounded). *If $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$.*

Proof. If we have that $|\lambda| < \|T\|$ then

$$\sum_{k=0}^{\infty} \frac{\|T\|^k}{|\lambda|^k} < \infty$$

and so it makes sense to define an operator

$$S_\lambda := \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}$$

where $S_\lambda \in B(\mathcal{H})$.

Now clearly we can write

$$(T - \lambda I)S_\lambda = (T + \frac{T^2}{\lambda} + \frac{T^3}{\lambda^2} + \cdots) - (\lambda I + T + \frac{T^2}{\lambda} + \frac{T^3}{\lambda^2} + \cdots)$$

and therefore $(T - \lambda I)S_\lambda = -\lambda I$. Similarly $S_\lambda(T - \lambda I) = -\lambda I$. That means that $\lambda \in \rho(T)$, because we have found an inverse of $(T - \lambda I)$, namely

$$R_\lambda(T) = \frac{-S_\lambda}{\lambda}.$$

□

Due to the way that $\sigma(T)$ is defined it would be appropriate to prove that $\rho(T)$ is open in order to show that $\sigma(T)$ is closed.

Lemma 2 ($\sigma(T)$ is closed). *If $\lambda_0 \in \rho(T)$ and $a = \|R_{\lambda_0}(T)\|^{-1} > 0$ then $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < a\} \subseteq \rho(T)$.*

What this essentially says is that we can always cook up an open a -ball around any $\lambda_0 \in \rho(T)$.

Proof. Let $\lambda \in \mathbb{C}$ be such that $|\lambda - \lambda_0| < a$. We can see that since

$$|\lambda - \lambda_0| \cdot \|R_{\lambda_0}(T)\| < 1$$

then

$$\sum_{k=0}^{\infty} |\lambda - \lambda_0|^k \cdot \|R_{\lambda_0}(T)\|^k < \infty$$

and so it makes sense to define an operator

$$S_{\lambda} := \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(T)^k$$

where $S_{\lambda} \in B(\mathcal{H})$. Now by writing $\mu = \lambda - \lambda_0$ we have that

$$\begin{aligned} (T - \lambda)S_{\lambda} &= (T - \lambda_0 I - \mu I)S_{\lambda} \\ &= ((T - \lambda_0 I) + \mu I + \mu^2 R_{\lambda_0}(T) + \mu^3 R_{\lambda_0}(T) + \cdots) \\ &\quad - (\mu I + \mu^2 R_{\lambda_0}(T) + \mu^3 R_{\lambda_0}(T)^2 + \cdots) \\ &= (T - \lambda_0 I). \end{aligned}$$

In exactly the same manner we see that $S_{\lambda}(T - \lambda I) = (T - \lambda_0 I)$. Now since $(T - \lambda_0 I)$ is invertible, with inverse R_{λ_0} it follows that

$$R_{\lambda_0} S_{\lambda} (T - \lambda I) = I$$

and so $\lambda \in \rho(T)$ and in particular $R_{\lambda} = R_{\lambda_0} S_{\lambda}$. This shows that $\rho(T)$ is open and hence $\sigma(T)$ is closed. \square

Lemma 3 ($\sigma(T)$ is non-empty).

Proof. We assume $\sigma(T) \neq \emptyset$ and arrive at a contradiction. Define a function

$$f_T(z) := R_z(T).$$

By lemma 2 for any $\lambda_0 \in \mathbb{C}$ we can write

$$\sum_{k=0}^{\infty} (z - \lambda_0)^k R_{\lambda_0}(T)^{k+1}.$$

We can also further say that this series converges for all $z \in \mathbb{C}$. This means f is an entire function. By using the ideas of lemma 1 we can also write

$$f_T(z) = R_z(T) = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{T^k}{z^k}$$

and so $f(z)$ is bounded. Now by Liouville's theorem $f \equiv C$ where C is a constant, but this is clearly a contradiction. \square

Spectral Radius

In this section we define the spectral radius of an operator and show a result that relates the norm of the operator to the spectral radius. We then go on to show that if T is self adjoint then this relationship can be further simplified.

Definition 4 (Spectral Radius). *For any $T \in B(\mathcal{H})$ define*

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

Theorem 1. *For any $T \in B(\mathcal{H})$ we have that*

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

We use several lemmas to prove this result. Define $\alpha(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$.

Lemma 4. $\alpha(T)$ is well defined.

Proof. Let $\alpha_n = \|T^n\|$ and observe that by Cauchy-Schwartz

$$\|T^{n+k}\| \leq \|T^n\| \cdot \|T^k\|. \quad \circledast$$

Let

$$\alpha = \inf_{n \geq 1} \alpha_n^{\frac{1}{n}}$$

and fix any $\epsilon > 0$.

Pick an $m \in \mathbb{N}$ such that $\alpha \leq \alpha_m^{\frac{1}{m}} < \alpha + \epsilon$.

For any $n \in \mathbb{N}$ we can pick $k, b \in \mathbb{N}$ such that $n = km + b$ and $b < m$.

Then by using the observation from \circledast we can write

$$\begin{aligned} \alpha &\leq \alpha_n^{\frac{1}{n}} \\ &\leq \alpha_{km}^{\frac{1}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &\leq \alpha_m^{\frac{k}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &= \left(\alpha_m^{\frac{1}{m}}\right)^{\frac{km}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &\leq (\alpha + \epsilon)^{1 - \frac{b}{n}} \cdot \left(\max_{1 \leq b \leq m} \alpha_b\right)^{\frac{1}{n}}. \end{aligned}$$

By letting $n \rightarrow \infty$ then we can pinch $\alpha_n^{\frac{1}{n}}$ between α and $\alpha + \epsilon$, i.e.

$$\alpha \leq \liminf_{n \rightarrow \infty} \alpha_n^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \alpha_n^{\frac{1}{n}} \leq \alpha + \epsilon.$$

So we can say that $\alpha(T) = \alpha$. □

Lemma 5. *If $|\lambda| > \alpha(T)$ then $\lambda \in \rho(T)$*

Proof. Note that if $|\lambda| > \alpha(T)$ then there exists some $N \in \mathbb{N}$ such that for any $n \geq N$ we have that

$$\|T^n\| \frac{1}{n} < \alpha(T).$$

If we let $q = \frac{\alpha(T)}{|\lambda|}$ then we can re-write this as

$$\|T^n\| < q^n |\lambda|^n$$

and so $\frac{\|T^n\|}{|\lambda|^n} < q^n$. By noting that $0 < q < 1$ we can say that the operator defined by

$$S_\lambda = - \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}$$

converges absolutely.

Now since $(T - \lambda I)S_\lambda = S_\lambda(T - \lambda I) = I$ (by a result similar to that in lemma 1) we can say that $\lambda \in \rho(T)$. □

Lemma 6. *If $|\lambda| > \alpha(T)$ then $|\lambda| \geq r(T)$.*

Proof. If $|\lambda| > \alpha(T)$ and $|\lambda| < r(T)$ then because of the way we define r there must exist a $\lambda_1 \in \rho(T)$ such that $\alpha(T) < |\lambda| < |\lambda_1| < r(T)$. This is a contradiction because lemma 5 shows $\lambda_0 \in \rho(T)$. □

What 6 says is that $\alpha(T) \geq r(T)$. We can now prove a final result which will amount to a proof of theorem 1

of theorem 1. Assume, for a contradiction that $\alpha(T) > r(T)$. Then we can pick some $\lambda \in \mathbb{C}$ such that

$$\alpha(T) < |\lambda| < r(T).$$

Fix some $x, y \in \mathcal{H}$ and define a function

$$f_T(z) := R_z(T) = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{\langle T^k x, y \rangle}{z^k}.$$

This function is defined if $|z| > \alpha(T)$ (by applying the same idea as in lemma 5).

Since $\sigma(T) \subseteq \{|z| \leq r(T)\}$ and so f is holomorphic for all $|z| > r(T)$ so it admits a power series expansion on the whole disc. It follows that the expansion given above for the wider disc

$$\{|z| > r(T)\}$$

This means that the series

$$\sum_{n=0}^{\infty} \frac{\langle T^n x, y \rangle}{\lambda^n}$$

converges.

This means that we must have the tail terms of this series going to zero, i.e.

$$\lim_{n \rightarrow \infty} \frac{|\langle T^n x, y \rangle|}{|\lambda|^n} = 0$$

and so by the uniform boundedness principle we must have a $c > 0$ such that

$$\|T^n\| \leq c|\lambda|^n$$

for all n and so

$$\|T^n\|^{\frac{1}{n}} \leq c^{\frac{1}{n}}|\lambda|$$

which leads to a contradiction because it implies that $\alpha(t) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq |\lambda|$. \square

We now consider the case of a self-adjoint bounded linear operator. In this case a formula for $r(T)$ is rather much more simple.

Lemma 7. *If $T \in B(\mathcal{H})$ and $T = T^*$ then $r(T) = \|T\|$.*

Proof. From the C^* identity we have that $\|T\|^2 = \|T^*T\| = \|T^2\|$.

We can generalise this to say $\|T\|^{2n} = \|T^{2n}\|$ for all $n \in \mathbb{N}$. By noting that we can write

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T^{2^n}\|^{2^{-n}} = \|T\|$$

we have the result. \square

Rational Calculus