University of New South Wales

School of Mathematics and Statistics

Assignment

Functional Analysis

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Declaration

I declare that this assessment item is my own work, except where acknowledged, and has not been submitted for academic credit elsewhere.
I certify that I have read and understood the University Rules in respect of Student Academic Misconduct.

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Date

Question 1

See that

$$\begin{aligned}\langle Tx, x\rangle &= \overline{\langle x, Tx\rangle} \\ &\geq 0\end{aligned}$$

which means

$$\begin{aligned}\overline{\langle x, Tx\rangle} &= \langle x, Tx\rangle \\ &= \langle T^* x, x\rangle\end{aligned}$$

so

$$\langle (T - T^*)x, x\rangle = 0$$

for all $x \in \mathcal{H}$.
We just want to show that $(T - T^*) = 0$. Say $\langle Jx, x\rangle = 0$ for all $x \in \mathcal{H}$ then for $x, y \in \mathcal{H}$ see that

$$\begin{aligned}\langle J(x+y), x+y\rangle &= \langle Jx, y\rangle + \langle Jy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jy, y\rangle}_{=0} \\ &= \langle Jx, y\rangle + \langle Jy, x\rangle \qquad (1)\end{aligned}$$

and see that

$$\begin{aligned}\langle J(x+iy), x+iy\rangle &= \langle Jx, iy\rangle + \langle Jiy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jiy, iy\rangle}_{=0} \\ &= -i\langle Jx, y\rangle + i\langle Jy, x\rangle \qquad (2)\end{aligned}$$

Now multiplying (1) by i and subtracting it from (2) yields

$$\langle Jx, y\rangle = 0$$

for all $x, y \in \mathcal{H}$ which implies $J \equiv 0$. Letting $J = (T - T^*)$, yields $(T - T^*) = 0$ as required.
So $T \equiv T^*$. □

Question 2

Part 1

From a toeplitz operator $T = \{a_i, j\}_{i,j\in\mathbb{Z}}$ define the diagonal operator D_j as the matrix-type operator taken as then j-th diagonal of T. Ie.

$$D_j = \begin{pmatrix} 0 & \cdots & a_j & 0 & 0 & \cdots \\ 0 & \cdots & 0 & a_j & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \ddots & \cdots \end{pmatrix}$$

If $\mathbf{x} = (x_1, x_2, \ldots)$ then it is clear that $D_j \mathbf{x} = (0, \ldots, 0, a_j x_j, 0, \ldots)$. Also it is clear that

$$T = \sum_{j \in \mathbb{Z}} D_j.$$

Since we have that $\sum_{j \in \mathbb{Z}} |a_j| = M < \infty$ we can write

$$\begin{aligned} ||T\mathbf{x}|| &= ||\sum_{j \in \mathbb{Z}} D_j \mathbf{x}|| \\ &\leq \sum_{j \in \mathbb{Z}} ||D_j \mathbf{x}|| \\ &\leq \sum_{j \in \mathbb{Z}} |a_j| ||\mathbf{x}|| \\ &= ||x|| \sum_{j \in \mathbb{Z}} |a_j| \\ &= ||x|| M \end{aligned}$$

and so T is a bounded on ℓ^2.

Question 3

Part 1

$x \in (\operatorname{Im} T)^{\perp}$ iff

$$\begin{aligned} &\langle Ty, x \rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow \langle y, T^* x \rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow T^* x = 0 \\ &\Leftrightarrow x \in \ker T* \end{aligned}$$

So $(\operatorname{Im} T)^{\perp} = \ker T^*$. □

Part 2

Setting $T = T^*$ in the result of *part a* yields

$$(\operatorname{Im} T^*)^{\perp} = \ker T^{**} \qquad \Leftrightarrow (\operatorname{Im} T^*)^{\perp} = \ker T$$

and taking orthogonal complements of both sides gets

$$\begin{aligned} &(\operatorname{Im} T^*)^{\perp\perp} = (\ker T)^{\perp} \\ &\Leftrightarrow \overline{\operatorname{Im} T^*} = (\ker T)^{\perp}. \end{aligned}$$

So $\overline{\operatorname{Im} T^*} = (\ker T)^{\perp}$. □