University of New South Wales

School of Mathematics and Statistics

Assignment

Functional Analysis

Author:
Adam J. Gray

Student Number:
3329798

Question 1

$$\langle B_k, e_n \rangle = \int_0^1 B_k(x) e^{-2\pi inx} dx.$$

Integrate by parts with:

$$u = B_k(x) \qquad du = B'{}_k(x) \qquad v = \frac{-e^{2\pi inx}}{2\pi in}$$

So

$$\begin{aligned}\langle B_k, e_n \rangle &= \underbrace{\left[\frac{-B_k(x)e^{-2\pi inx}}{2\pi in}\right]_0^1}_{=0} + \int_0^1 \frac{e^{-2\pi inx}}{2\pi in} B'{}_k(x) dx \\ &= \int_0^1 \frac{e^{-2\pi inx}}{2\pi in} B'{}_k(x) dx \\ &= \frac{k}{2\pi in} \int_0^1 e^{-2\pi inx} B_{k-1}(x) dx.\end{aligned}$$

Integrating by parts again with:

$$u = B_{k-1}(x) \qquad du = B'{}_{k-1}(x) \qquad v = \frac{e^{-2\pi inx}}{2\pi in}$$

So

$$\begin{aligned}\langle B_k, e_n \rangle &= \frac{k}{2\pi in}\left(\underbrace{\left[\frac{B_{k-1}(x)e^{-2\pi inx}}{2\pi in}\right]_0^1}_{=0} + \frac{1}{2\pi in} \int_0^1 e^{-2\pi inx} B'_{k-1}(x) dx \right) \\ &= \frac{k(k-1)}{(2\pi in)^2} \int_0^1 e^{-2\pi inx} B_{k-2}(x) dx.\end{aligned}$$

Continuing thus

$$\begin{aligned}\langle B_k, e_n \rangle &= \frac{k!}{(2\pi in)^{k-1}} \int_0^1 e^{-2\pi inx} (x - \frac{1}{2}) dx \\ &= \frac{-k!}{(2\pi in)^k}.\end{aligned}$$

Thus

$$B_k(x) = \sum_{n \in \mathbb{Z}} \frac{-k!}{(2\pi in)^k} e^{2\pi inx}.$$

Question 2

Consider the set $S_n := \text{span}\{x^k\}_{k=0}^n$ and note that

$$S_n = S_{n-1} \oplus S_{n-1}^{\perp}.$$

Let ℓ_n be the nth Legendre polynomial. Now $S_{n-1} \cap S_n$ has dimension 1 in so if we can show

$$\ell_n \in S_{n-1}^{\perp} \cap S_n$$

then $S_{n-1}^{\perp} = \text{span}\{\ell_n\}$.
To see this just see that for $k < n$

$$\langle x^k, \ell_n(x) \rangle = R(n) \int_{-1}^{1} t^k \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right] dx$$

where

$$R(n) = \sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!}.$$

Applying the lemma discussed in class (by applying repeated integration by parts) we have that

$$\begin{aligned}
\langle x^k, \ell_n(x) \rangle &= R(n)(-1)^k k! \int_{-1}^{1} \frac{d^{n-k}}{dx^{n-k}} \left[(x^2 - 1)^n \right] dx \\
&= R(n)(-1)^k k! \left[\frac{d^{n-k-1}}{dx^{n-k-1}} \left[(x^2 - 1)^n \right] \right]_{-1}^{1} \\
&= R(n)(-1)^k k! \left[S(x)(x^2 - 1) \right]_{-1}^{1} \\
&= 0
\end{aligned}$$

where S(x) is a polynomial function of x.
So for $n > k$ we have that $\ell_n \in S_{n-1}^{\perp}$ and since ℓ_n is an nth degree polynomial $\ell_n \in S_{n-1}^{\perp} \cap S_n$
Now consider $fn = x^n - \text{proj}_{S_{n-1}} x^n$ and note that this must be the nth Legendre polynomial (unnormalized) and note that because by design $f_n \in S_{n-1}^{\perp} \cap S_n$. We have that $S_{n-1}^{\perp} \cap S_n = \text{span}\{\ell_n\}$ and so $\ell_n = \alpha f_n$, where alpha is a normalization constant, and hence the Legendre polynomials are the polynomials which arise from the Gram-Schmidt process applied to $\{x^n\}_{n=0}^{\infty}$.

Question 3

If we write $P_n(x) = \sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$ then we can define $Q_n(x) = \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$ and consider $\langle x^k, Q_n(x) \rangle$.

$$\langle x^k, Q_n(x) \rangle = \int_{-1}^{1} x^k \frac{d^n}{dx^n} \left[(x^2-1)^n\right] dx$$

By applying repeated integration by parts and the lemma shown in class we have that if $k < n$ then $\langle x^k, Q_n(x) \rangle = 0$ otherwise

$$\int_{-1}^{1} x^k \frac{d^n}{dx^n} \left[(x^2-1)^n\right] dx = (-1)^{k-n} \frac{k!}{n!} \int_{-1}^{1} x^{k-n} (x^2-1)^n dx$$

Now as $(x^2-1)^n$ is always even, then when $k-n$ is odd the integral is 0. When $k-n$ is even the integrand is even and so

$$(-1)^{k-n} \frac{k!}{n!} \int_{-1}^{1} x^{k-n} (x^2-1)^n dx = 2\frac{k!}{n!} \int_{0}^{1} x^{k-n} (x^2-1)^n dx.$$

Performing the substitution $u = x^2$ yields

$$\begin{aligned} 2\frac{k!}{n!} \int_0^1 x^{k-n}(x^2-1)^n dx &= (-1)^n \frac{k!}{n!} \int_0^1 u^{\frac{k-n-1}{2}} (1-u)^n du \\ &= (-1)^n \frac{k!}{n!} B\left(\frac{k-n}{2} + \frac{1}{2}, n+1\right) \end{aligned}$$

Thus

$$\langle x^k, P_n(x) \rangle = \begin{cases} \sqrt{\frac{2n+1}{2}} \frac{1}{2^n} (-1)^n \frac{k!}{(n!)^2} B\left(\frac{k-n}{2} + \frac{1}{2}, n+1\right) & \text{when } k > n \text{ and } k-n \text{ is even.} \\ 0 & \text{otherwise.} \end{cases}$$

This can also be expressed as

$$\langle x^k, P_n(x) \rangle = \begin{cases} \sqrt{\frac{2n+1}{2}} (-1)^n \frac{k!}{n!} \frac{(k-n)! 2^{n+2} \left(\frac{k+n+2}{2}\right)!}{\left(\frac{k-n}{2}\right)! (k+n+2)!} & \text{when } k > n \text{ and } k-n \text{ is even.} \\ 0 & \text{otherwise.} \end{cases}$$

but this is silly.

Acknowledgements

Thank you to Peter Nguyen for his simplified solution method for question 2. It certainly made the solution fit on one page.