University of New South Wales

School of Mathematics and Statistics

Assignment

Functional Analysis

Author:
Adam J. Gray

Student Number:
3329798

Declaration

I declare that this assessment item is my own work, except where acknowledged, and has not been submitted for academic credit elsewhere.
I certify that I have read and understood the University Rules in respect of Student Academic Misconduct.

Adam J. Gray
3329798

Date

Question 1

See that

$$\begin{aligned}\langle Tx, x\rangle &= \overline{\langle x, Tx\rangle} \\ &\geq 0\end{aligned}$$

which means

$$\begin{aligned}\overline{\langle x, Tx\rangle} &= \langle x, Tx\rangle \\ &= \langle T^*x, x\rangle\end{aligned}$$

so

$$\langle (T - T^*)x, x\rangle = 0$$

for all $x \in \mathcal{H}$.
We just want to show that $(T - T^*) = 0$. Say $\langle Jx, x\rangle = 0$ for all $x \in \mathcal{H}$ then for $x, y \in \mathcal{H}$ see that

$$\begin{aligned}\langle J(x+y), x+y\rangle &= \langle Jx, y\rangle + \langle Jy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jy, y\rangle}_{=0} \\ &= \langle Jx, y\rangle + \langle Jy, x\rangle \qquad (1)\end{aligned}$$

and see that

$$\begin{aligned}\langle J(x+iy), x+iy\rangle &= \langle Jx, iy\rangle + \langle Jiy, x\rangle + \underbrace{\langle Jx, x\rangle}_{=0} + \underbrace{\langle Jiy, iy\rangle}_{=0} \\ &= -i\langle Jx, y\rangle + i\langle Jy, x\rangle \qquad (2)\end{aligned}$$

Now multiplying (1) by i and subtracting it from (2) yields

$$\langle Jx, y\rangle = 0$$

for all $x, y \in \mathcal{H}$ which implies $J \equiv 0$. Letting $J = (T - T^*)$, yields $(T - T^*) = 0$ as required.
So $T \equiv T^*$. □

Question 3

Part 1

$x \in (\operatorname{Im} T)^\perp$ iff

$$\begin{aligned}&\langle Ty, x\rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow \langle y, T^*x\rangle = 0 && \forall y \in \mathcal{H} \\ &\Leftrightarrow T^*x = 0 \\ &\Leftrightarrow x \in \ker T*\end{aligned}$$

So $(\operatorname{Im} T)^\perp = \ker T^*$. □

Part 2

Setting $T = T^*$ in the result of *part a* yeilds

$$(\operatorname{Im} T^*)^\perp = \ker T^{**} \qquad \Leftrightarrow (\operatorname{Im} T^*)^\perp = \ker T$$

and taking orthogonal complements of both sides gets

$$\begin{aligned}(\operatorname{Im} T^*)^{\perp\perp} &= (\ker T)^\perp \\ \Leftrightarrow \overline{\operatorname{Im} T^*} &= (\ker T)^\perp.\end{aligned}$$

So $\overline{\operatorname{Im} T^*} = (\ker T)^\perp$. $\square$