University of New South Wales

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Lecture Notes

Functional Analysis

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Hilbert Spaces

Definition 1. *A Hilbert space $\mathcal{H}$ is a real or complex inner product space that is also a complete metric space with respect to the metric induced by the inner product.* [1]

We say that the inner product is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ (or $\mathbb{R}$). which satisfies the following properties:

- $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for $x, y \in \mathcal{H}$.
- $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$ for $x_1, x_2, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.
- $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Moreover $\langle x, x \rangle = 0$ if and only if $x = 0$.

Note that from this inner product we can easily define a norm by writing

$$||x|| = \sqrt{\langle x, x \rangle}.$$

We will now look at a couple of examples of Hilbert spaces.

$\mathbb{C}$ and $\mathbb{R}$

Without being tedious checking that all the requirements are fulfilled we will simply state that both $\mathbb{C}$ and $\mathbb{R}$ are Hilbert spaces.

ℓ^2

We will show that many of the properties of a Hilbert space are fulfilled by ℓ^2 and we will claim that ℓ^2 is infect a Hilbert space. To do this will will first define ℓ^2 and then check that the inner product axioms are satisfied.

Definition 2 (ℓ^2).

$$\ell^2 = \left\{ (x_1, x_2, x_3, \dots) : x_k \in \mathbb{C}, \sum_{k=1}^{\infty} |x_k|^2 \right\}$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{\infty} x_k \overline{y}_k$$

Firstly we show that all of the inner product axioms are satisfied for the inner product here.
For $\mathbf{x}, \mathbf{y} \in \ell^2$ see that

$$\begin{aligned} \langle \mathbf{y}, \mathbf{x} \rangle &= \sum_{k=1}^{\infty} y_k \overline{x}_k \\ &= \overline{\sum_{k=1}^{\infty} \overline{y}_k x_k} \\ &= \overline{\langle \mathbf{x}, \mathbf{y} \rangle}. \end{aligned}$$

[1] This definition is more or less taken straight from Wikipedia (3 June 2014).

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \ell^2$ and $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned}\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}\rangle &= \sum_{k=1}^{\infty}(\alpha x_k + \beta y_k)\overline{z}_k \\ &= \alpha\sum_{k=1}^{\infty} x_k\overline{z}_k + \beta\sum_{k=1}^{\infty} y_k\overline{z}_k \\ &= \alpha\langle \mathbf{x}, \mathbf{z}\rangle + \beta\langle \mathbf{y}, \mathbf{z}\rangle.\end{aligned}$$

For $\mathbf{x} \in \ell^2$ see that

$$\begin{aligned}\langle \mathbf{x}, \mathbf{x}\rangle &= \sum_{k=1}^{\infty} x_k\overline{x}_k \\ &= \sum_{k=1}^{\infty} |x_k|^2 \\ &\geq 0\end{aligned}$$

and in particular it is clear to see that $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ if and only if $\mathbf{x} = 0$.
We now wish to show that for two elements $\mathbf{x}, \mathbf{y} \in \ell^2$ that the inner product does in fact map to a complex number (and not just map to infinity).
For $\mathbf{x}, \mathbf{y} \in \ell^2$ see that

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \sum_{k=1}^{\infty} |x_k| \cdot |y_k|$$

and by geometric - arithmetic mean inequality

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k| \cdot |y_k| &\leq \frac{1}{2} \underbrace{\sum_{k=1}^{\infty} |x_k|^2}_{<\infty \text{ because } \mathbf{x}\in\ell^2} + \frac{1}{2} \underbrace{\sum_{k=1}^{\infty} |y_k|^2}_{<\infty \text{ because } \mathbf{x}\in\ell^2} \\ &< \infty.\end{aligned}$$

We now show that ℓ^2 is in fact closed under addition, and hence a linear space.
For $\mathbf{x}, \mathbf{y} \in \ell^2$ we see that by finite dimension Cauchy-Schwartz

$$\sum_{k=1}^{\infty} |x_k + y_k| \leq \underbrace{\sum_{k=1}^{N} |x_k|^2}_{<\infty} + \underbrace{\sum_{k=1}^{N} |y_k|}_{<\infty}$$

and by letting $N \longrightarrow \infty$ we see that $||\mathbf{x} + \mathbf{y}|| < \infty$ and hence $\mathbf{x} + \mathbf{y} \in \ell^2$.
While we have not shown that ℓ^2 is complete we do claim that ℓ^2 is complete and hence a Hilbert space.

$C[-1,1]$

The aim with $C[-1,1]$ is to show that with the usual inner product, $C[-1,1]$ is not a Hilbert space, because it is not complete.
For clarity we define the inner product on $C[-1,1]$, for $f,g \in C[-1,1]$ by

$$\langle f,g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt.$$

Do show that $C[-1,1]$ is not complete define the sequence of function f_n by

$$f_n(t) = \begin{cases} 0 & \text{if } t \notin \left[\frac{-1}{n}, \frac{1}{n}\right] \\ \sqrt{n} & \text{if } x \in \left[\frac{-1}{2n}, \frac{1}{2n}\right] \\ \text{linear} & \text{elsewhere} \end{cases}$$

Now see that for each n, $f_n \in C[-1,1]$. It is also easy to see that for all n, $||f_n|| \leq \sqrt{2}$.
Now consider the function F defined by

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}} f_n(x).$$

Observe that

$$||F|| \leq \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}} < \infty$$

but see that

$$F(0) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}} \longrightarrow \infty.$$

This means that $F \notin C[-1,1]$ and hence $C[-1,1]$ is not complete and therefore not a Hilbert space.

Completion Procedure for Hilbert Spaces

Let $(\mathcal{H}_0, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space (i.e. not complete). We have the usual norm induced by the inner product, that is $||x|| = \sqrt{\langle x,x\rangle}$.
To "complete" this as forma Hilbert space define the set of Cauchy Sequences

$$H = \{X = (x_k)_{k=1}^{\infty} : x_k \in \mathcal{H}_0\}.$$

We then claim that

$$\mathcal{H} = \{\overline{X} : X \in H\}$$

is a Hilbert space where we define

$$\overline{X} = \{Y \in H : Y \sim X\}.$$

We say that

$$X \sim Y$$

when $\lim_{k\longrightarrow\infty} ||x_k - y_k|| = 0$.
We define the inner product on this new Hilbert space $\mathcal{H}$ by

$$\langle \overline{X}, \overline{Y} \rangle = \lim_{k\longrightarrow\infty} \langle x_k, y_k \rangle.$$

Simple verification shows that is is infact a "valid" inner product.
We also clarify addition an scalar multiplcation in this new Hilbert space by saying that

$$\overline{X} + \overline{Y} = \overline{(x_k + y_k)}$$
$$\alpha X = (\alpha x_k).$$

Since we have defined al lthe basic properties of this new Hilbert space, $\mathcal{H}$, we check the "correctness" of this definition.
Suppose we have

$$X' = (x'_k) \text{ and } X = (x_k)$$
$$Y' = (y'_k) \text{ and } Y = (y_k)$$

such that

$$X' \sim X \text{ and } Y' \sim Y$$

then we wish to verify that

$$X + Y \sim X' + Y'$$

and to do this see that

$$\lim_{k\longrightarrow\infty} x_k + y_k = \lim_{k\longrightarrow\infty} x'_k + y'_k.$$

Also observe that

$$\alpha X \sim \alpha X'$$

by the same reasoning.
To verify correctness we also need to show that the limit $\lim_{k\longrightarrow\infty}\langle x_k, y_k\rangle$ exists and that

$$\lim_{k\longrightarrow\infty} \langle x_k y_k \rangle = \lim_{k\longrightarrow\infty} \langle x'_k, y'_k \rangle.$$

Firstly see that

$$\lim_{k\longrightarrow\infty} \langle x_k, y_k \rangle \leq \lim_{k\longrightarrow\infty} ||x_k|| \cdot ||y_k||$$

by the Cuachy-Schwartz inequality and so the limit exists.
See that

$$\lim_{k\longrightarrow\infty} \langle x_k - x'_k, y_k - y'_k \rangle = \langle 0, 0 \rangle = 0$$

and so the desired equality holds.
To make this whole exercise purposful we need to check that the space we have defined is infact complete, and therefore a Hilbert space.
Let $\overline{X}_n \in \mathcal{H}$ be a sequence of elements of $\mathcal{H}$ such that

$$\sum_{k\longrightarrow\infty} ||\overline{X}_k|| < \infty.$$

Let $\overline{X} = \sum_{k\longrightarrow\infty} \overline{X}_k$ and note that by the definition of $\mathcal{H}$ we must that $\overline{X} \in \mathcal{H}$ and thus $\mathcal{H}$ is complete.

Important Identities and Inequalities

Theorem 1 (Cauchy-Schwartz)**.** *If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$ then*

$$|\langle x, y\rangle| \leq ||x|| \cdot ||y||.$$

Proof. Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$f(t) = ||x + ty||^2$$

and note that $f(t) \geq 0$ for all $t \in \mathbb{R}$. See that

$$f(t) = ||x||^2 + t^2||y||^2 + 2t\Re\langle x, y\rangle \geq 0$$

and since $f(t)$ is a quadratic in t we can say that

$$|\Re\langle x, y\rangle| \leq ||x|| \cdot ||y||$$

by considering the discriminant.
We wish to "get rid of" the $\Re$ in this result.
Now if $\alpha = \langle x, y\rangle$ with $\alpha \in \mathbb{C}$ then we can define $\text{sgn}(\alpha) = \frac{\overline{\alpha}}{|\alpha|}$.
Let $\tilde{x} = \text{sgn}(\alpha) \cdot x$ and from the already established result we can say that

$$|\Re\langle \tilde{x}, y\rangle| \leq ||\tilde{x}|| \cdot ||y||$$

and

$$\begin{aligned}\Re\langle \tilde{x}, y\rangle &= \Re(\text{sgn}(\alpha) \cdot \langle x, y\rangle) \\ &= |\langle x, y\rangle|.\end{aligned}$$

Also

$$||\tilde{x}|| = |\,\alpha\,| \cdot ||x|| = ||x||$$

and so we can remove the $\Re$ part of the result by saying

$$|\langle x, y\rangle| = |\Re\langle \tilde{x}, y\rangle| \leq ||\tilde{x}|| \cdot ||y|| = ||x|| \cdot ||y||.$$

□

Theorem 2 (Triangle Inequality)**.** *If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$ then*

$$||x+y|| \leq ||x|| + ||y||.$$

Proof. See that

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y \rangle \\ &= ||x||^2 + ||y||^2 + 2 \times \underbrace{\Re\langle x, y \rangle}_{\leq ||x|| \cdot ||y|| \text{ by Cauchy-Schwartz}} \\ &\leq ||x||^2 + ||y||^2 + 2||x|| \cdot ||y|| \\ &= (||x|| + ||y||)^2 \end{aligned}$$

and so

$$||x+y|| \leq ||x|| + ||y||.$$

□

Theorem 3 (Parallelogram Law)**.** *If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$ then*

$$2||x||^2 + 2||y||^2 = ||x+y||^2 + ||x-y||^2.$$

Proof. See that

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= |x||^2 + ||y||^2 + ||x||^2 + ||y||^2 \\ &\quad + \langle x, y \rangle + \langle y, x \rangle - \langle x, y \rangle - \langle y, x \rangle \\ &= 2||x||^2 + 2||y||^2. \end{aligned}$$

□

Theorem 4 (Polarization Identity)**.** *If $\mathcal{H}$ is a complex Hilbert space and $x, y \in \mathcal{H}$ then*

$$\langle x, y \rangle = \frac{1}{2} \left(||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2 \right).$$

If $\mathcal{H}$ is a real Hilbert space then this can be reduced to

$$\langle x, y \rangle = \frac{1}{4} \left(||x+y||^2 - ||x-y||^2 \right).$$

The proof of these facts follows in much that same way as for the parallelogram law.

Orthonormal Basis in Hilbert Spaces

An orthonormal basis is a family of elements of $\mathcal{H}$ which satisfy the following conditions

- Mutual orthogonality

- Normality

- Completeness.

Spectral Theory

Spectral theory is analogous to eigen-value theory for matrices. In this case we look at spectral theory on bounded linear operators on a Hilbert space $\mathcal{H}$.
We define the following
For $T \in B(\mathcal{H})$ we define the following

Definition 3 (Resolvent Set)**.** *Define* $\rho(T) := \{\lambda \in \mathbb{C} : \exists (T - \lambda)^{-1} \in B(\mathcal{H})\}$.

Definition 4 (Resolvent)**.** *Define* $R_\lambda(T) := (T - \lambda)^{-1}$

Definition 5 (Spectrum)**.** *Define* $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

On could heuristically think of λ as being an eigen-value and we formalize this idea as follows.
If $\mathcal{H} = \mathbb{C}^n$ and $T = T_A$ where $A = (a_{j,k})_{j,k=1}^n$ (a matrix) then if $(T - \lambda I)^{-1}$ does not exist, i.e. $\ker(T - \lambda I) \neq \mathbf{0}$ then λ is an eigen-value.
We would like to formalize and prove the following three ideas;

- $\sigma(T)$ is bounded,
- $\sigma(T)$ is closed, and
- $\sigma(T)$ is non-empty.

Lemma 1 ($\sigma(T)$ is bounded)**.** *If* $|\lambda| > ||T||$ *then* $\lambda \in \rho(T)$.

Proof. If we have that $|\lambda| < ||T||$ then

$$\sum_{k=0}^{\infty} \frac{||T||^k}{|\lambda|^k} < \infty$$

and so it makes since to define an operator

$$S_\lambda := \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}$$

where $S_\lambda \in B(\mathcal{H})$.
Now clearly we can write

$$(T-\lambda I)S_\lambda = (T + \frac{T^2}{\lambda} + \frac{T^3}{\lambda^2} + \cdots) - (\lambda I + T + \frac{T^2}{\lambda} + \frac{T^3}{\lambda^2} + \cdots)$$

and therefore $(T-\lambda I)S_\lambda = -\lambda I$. Similarly $S_\lambda(T-\lambda I) = -\lambda I$. That means that $\lambda \in \rho(T)$, because we have found an inverse of $(T-\lambda I)$, namely

$$R_\lambda(T) = \frac{-S_\lambda}{\lambda}.$$

□

Due to the way that $\sigma(T)$ is defined it would be appropriate to prove that $\rho(T)$ is open in order to show that $\sigma(T)$ is closed.

Lemma 2 ($\sigma(T)$ is closed)**.** *If $\lambda_0 \in \rho(T)$ and $a = ||R_{\lambda_0}(T)||^{-1} > 0$ then $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < a\} \subseteq \rho(T)$.*

What this essentially says is that we can always cook up an open a-ball around any $\lambda_0 \in \rho(T)$.

Proof. Let $\lambda \in \mathbb{C}$ be such that $|\lambda - \lambda_0| < a$. We can see that since

$$|\lambda - \lambda_0| \cdot ||R_{\lambda_0}(T)|| < 1$$

then

$$\sum_{k=0}^{\infty} |\lambda - \lambda_0|^k \cdot ||R_{\lambda_0}(T)||^k < \infty$$

and so it makes since to define an operator

$$S_\lambda := \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(T)^k$$

where $S_\lambda \in B(\mathcal{H})$. Now by writing $\mu = \lambda - \lambda_0$ we have that

$$\begin{aligned}(T-\lambda)S_\lambda &= (T - \lambda_0 I - \mu I)S_\lambda \\ &= \big((T-\lambda_0 I) + \mu I + \mu^2 R_{\lambda_0}(T) + \mu^3 R_{\lambda_0}(T) + \cdots\big) \\ &\quad - \big(\mu I + \mu^2 R_{\lambda_0}(T) + \mu^3 R_{\lambda_0}(T)^2 + \cdots\big) \\ &= (T - \lambda_0 I).\end{aligned}$$

In exactly the same manner we see that $S_\lambda(T-\lambda I) = (T-\lambda_0 I)$.
Now since $(T-\lambda_0 I)$ is invertible, with inverse R_{λ_0} it follows that

$$R_{\lambda_0} S\lambda (T - \lambda I) = I$$

and so $\lambda \in \rho(T)$ and in particular $R_\lambda = R_{\lambda_0} S_\lambda$. This shows that $\rho(T)$ is open and hence $\sigma(T)$ is closed. □

Lemma 3 ($\sigma(T)$ is non-empty)**.**

Proof. We assume $\sigma(T) \neq \emptyset$ and arrive at a contradiction.
Define a function

$$f_T(z) := R_z(T).$$

By lemma 2 for any $\lambda_0 \in \mathbb{C}$ we can write

$$\sum_{k=0}^{\infty}(z-\lambda_0)^k R_{\lambda_0}(T)^{k+1}.$$

We can also further say that this series converges for all $z \in \mathbb{C}$. This means f is an entire function. By using the ideas of lemma 1 we can also write

$$f_T(z) = R_z(T) = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{T^k}{z^k}$$

and so $f(z)$ is bounded. Now by Liouville's theorem $f \equiv C$ where C is a constant, but this is clearly a contradiction. □

Spectral Radius

In this section we define the spectral radius of an operator and show a result that relates the norm of the operator to the spectral radius. We then go on to show that if T is self adjoint then this relationship can be further simplified.

Definition 6 (Spectral Radius)**.** *For any $T \in B(\mathcal{H})$ define*

$$r(t) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

.

Theorem 5. *For any $T \in B(\mathcal{H})$ we have that*

$$r(T) = \lim_{n \longrightarrow \infty} ||T^n||^{\frac{1}{n}}$$

We use several lemmas to prove this result. Define $\alpha(T) = \lim_{n\longrightarrow\infty} ||T^n||^{\frac{1}{n}}$.

Lemma 4. *$\alpha(T)$ is well defined.*

Proof. Let $\alpha_n = ||T^n||$ and observe that by Cauchy-Schwartz

$$||T^{n+k}|| \leq ||T^n|| \cdot ||T^k||. \qquad \circledast$$

Let

$$\alpha = \inf_{n \geq 1} \alpha_n^{\frac{1}{n}}$$

and fix any $\epsilon > 0$.

Pick an $m \in \mathbb{N}$ such that $\alpha \leq \alpha_m^{\frac{1}{m}} < \alpha + \epsilon$.
For any $n \in \mathbb{N}$ we can pick $k, b \in \mathbb{N}$ such that $n = km + b$ and $b < m$.
Then by using the observation from $\circledast$ we can write

$$\begin{aligned} \alpha &\leq \alpha_n^{\frac{1}{n}} \\ &\leq \alpha_{km}^{\frac{1}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &\leq \alpha_m^{\frac{k}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &= \left(\alpha_m^{\frac{1}{m}}\right)^{\frac{km}{n}} \cdot \alpha_b^{\frac{1}{n}} \\ &\leq (\alpha + \epsilon)^{1-\frac{b}{n}} \cdot \left(\max_{1 \leq b \leq m} a_b\right)^{\frac{1}{n}}. \end{aligned}$$

By letting $n \longrightarrow \infty$ then we can pinch $\alpha_n^{\frac{1}{n}}$ between α and $\alpha + \epsilon$, i.e.

$$\alpha \leq \liminf_{n \longrightarrow \infty} \alpha_n^{\frac{1}{n}} \leq \limsup_{n \longrightarrow \infty} \alpha_n^{\frac{1}{n}} \leq \alpha + \epsilon.$$

So we can say that $\alpha(T) = \alpha$. □

Lemma 5. *If* $|\lambda| > \alpha(T)$ *then* $\lambda \in \rho(T)$

Proof. Note that if $|\lambda| > \alpha(T)$ then there exists some $N \in \mathbb{N}$ such that for any $n \geq N$ we have that

$$||T^n|| \frac{1}{n} < \alpha(T).$$

If we let $q = \frac{\alpha(T)}{|\lambda|}$ then we can re-write this as

$$||T^n|| < q^n |\lambda|^n$$

and so $\frac{||T^n||}{|\lambda|^n} < q^n$. By noting that $0 < q < 1$ we can say that the operator defined by

$$S_\lambda = -\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}$$

converges absolutely.
Now since $(T - \lambda I)S_\lambda = S_\lambda(T - \lambda I) = I$ (by a result similar to that in lemma 1) we can say that $\lambda \in \rho(T)$. □

Lemma 6. *If* $|\lambda| > \alpha(T)$ *then* $|\lambda| \geq r(T)$.

Proof. If $|\lambda| > \alpha(T)$ and $|\lambda| < r(T)$ then because of the way we define r there must exist a $\lambda_1 \in \rho(T)$ such that $\alpha(T) < |\lambda| < |\lambda_1| < r(T)$. This is a contradiction because lemma 5 shows $\lambda_0 \in \rho(T)$. □

What 6 says is that $\alpha(T) \geq r(T)$. We can now prove a final result which will amount to a proof of theorem 5 *of theorem 5.* Assume, for a contradiction that $\alpha(T) > r(T)$. Then we can pick some $\lambda \in \mathbb{C}$ such that

$$\alpha(T) < |\lambda| < r(T).$$

Fix some $x, y \in \mathcal{H}$ and define a function

$$f_T(z) := R_z(T) = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{\langle T^n x, y\rangle}{z^n}.$$

This function is defined if $|z| > \alpha(T)$ (by applying the same idea as in lemma 5).

Since $\sigma(T) \subseteq \{|z| \leq r(T)\}$ and so f is holomorphic for all $|z| > r(T)$ so it admits a power series expansion on the whole disc. It follows that the expansion given above for the wider disc

$$\{|z| > r(T)\}$$

This means that the series

$$\sum_{n=0}^{\infty}\frac{\langle T^n x, y\rangle}{\lambda^n}$$

converges.

This means that we must have the tail terms of this series going to zero, i.e.

$$\lim_{n\longrightarrow\infty}\frac{|\langle T^n x, y\rangle|}{|\lambda|^n} = 0$$

and so by the uniform boundedness principle we must have a $c > 0$ such that

$$||T^n|| \leq c|\lambda|^n$$

for all n and so

$$||T^n||^{\frac{1}{n}} \leq c^{\frac{1}{n}}|\lambda|$$

which leads to a contradiction because it implies that $\alpha(t) = \lim_{n\longrightarrow\infty}||T^n||^{\frac{1}{n}} \leq |\lambda|$. $\square$

We now consider the case of a self-adjoint bounded linear operator. In this case a formula formula for $r(T)$ is rather much more simple.

Lemma 7. *If $T \in B(\mathcal{H})$ and $T = T^*$ then $r(T) = ||T||$.*

Proof. From the C^* identity we have that $||T||^2 = ||T^*T|| = ||T^2||$.

We can generalize this to say $||T||^{2n} = ||T^{2n}||$ for all $n \in \mathbb{N}$. By noting that we can write

$$r(T) = \lim_{n\longrightarrow\infty}||T^n||\frac{1}{n} = \lim_{n\longrightarrow\infty}||T^{2^n}||^{2^{-n}} = ||T||$$

we have the result. $\square$

We now prove a collection of results about the spectrum of various types of operators.

Theorem 6. *If $T \in B(\mathcal{H})$ and T is self adjoint then $\sigma(T) \subseteq \mathbb{R}$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $\lambda = \alpha + i\beta$ with $\beta \neq 0$. Then

$$\begin{aligned}
||(T - \lambda I)x||^2 &= ||(T - \alpha I - i\beta I)x||^2 \\
&= \langle (T - \alpha I - i\beta I)x, (T - \alpha I - i\beta I)x \rangle \\
&= \langle (T - \alpha I + i\beta I)(T - \alpha I - i\beta I)x, x \rangle \\
&= \langle \left((T - \alpha I)^2 + \beta I\right) x, x \rangle \\
&= \langle (T - \alpha I)^2 x + \beta I x, x \rangle \\
&= \langle (T - \alpha I)^2 x, x \rangle + \beta^2 \langle x, x \rangle \\
&= ||(T - \alpha I)x||^2 + \beta^2 ||x||^2.
\end{aligned}$$

We can therefore write that $||(T - \lambda I)x|| \geq \Im(\lambda)||x||$ which means that $(T - \lambda I)$ is injective.
We can also see that T is subjective because

$$(\mathrm{Im}(T - \lambda I))^{\perp} = \ker(T - \overline{\lambda}) = \{0\}.$$

This means that

$$||(T - \lambda I)^{-1}|| \leq \Im(\lambda)^{-1}$$

which is to say that $\lambda \in \rho(T)$.
This means that $\lambda \notin \sigma(T)$ which means that any $z \in \sigma(T)$ must be such that $\Im(z) = 0$. □

Rational Calculus

Here we generalist ideas about functions on the real numbers to functions of bounded linear operators.
Firstly we consider a function

$$f(z) = \frac{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)}{(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)}$$

and call this function a rational function. This is because it can be considered the quotient of two polynomials. (The complete factorization here being permitted by the fundamental theorem of calculus).
From this definition of f is clear to see that the α_j constitute the roots of f and the λ_j constitute the poles of f.
Now for any $T \in B(\mathcal{H})$ and $\lambda_1, \ldots, \lambda_m \notin \sigma(T)$ we define a rational calculus by

$$f(T) := (T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I) R_{\lambda_1}(T) R_{\lambda_2}(T) \cdots R_{\lambda_m}(T)$$

We now have a simple theorem we wish to consider which relates the spectrum of $f(T)$ to the spectrum of T.

Theorem 7. *For $T \in B(\mathcal{H})$ and with f defined as above (i.e. with $\lambda_j \notin \sigma(T)$) then*

$$\sigma(f(T)) = f(\sigma(T)).^2$$

Proof. This proof proceeds in two parts:

- $f(\sigma(T)) \subseteq \sigma(f(T))$

  Fix a $\lambda \in \sigma(T)$ and let $\mu = f(\lambda)$. Define a function g such that

  $$g(z) = f(z) - \mu.$$

  Now since $g(\lambda) = 0$, g admits a factorization $g(z) = (z - \lambda)\frac{p(z)}{q(z)}$ where p, q are polynomials.

  Now consider

  $$g(T) = f(T) - \mu I = \underbrace{(T - \lambda I)}_{\circledast} p(T)q(T)^{-1}$$

  and note that $\circledast$ is not invertible because $\lambda \in \sigma(T)$ and hence

  $$f(T) - \mu$$

  is not invertible and so $\mu \in \sigma(f(T))$.

- $\sigma(f(T)) \subseteq f(\sigma(T))$

  Let $\mu \in \sigma(f(T))$ and define a function

  $$g(z) = f(z) - \mu = \frac{(z - \alpha_1)((z - \alpha_2) \cdots (z - \alpha_n)}{(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)}$$

  so we can write

  $$f(T) - \mu I = \underbrace{(T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I)}_{\circledast\circledast} \underbrace{R_{\lambda_1}(T) R_{\lambda_2}(T) \cdots R_{\lambda_m}(T)}_{\circledast\circledast\circledast}.$$

  Note $f(T) - \mu I$ is not invertible because $\mu \in \sigma(f(T))$. By definition everything in $\circledast\circledast\circledast$ must be invertible so something in $\circledast\circledast$ must not be invertible. This means that one of $\alpha_1, \alpha_2 \ldots \alpha_n$ is in $\sigma(T)$.

  Now for all $k \in \{1, \ldots, n\}$ we have that

  $$g(\alpha_k) = 0$$

  which would force $f(\alpha_k) = \mu$ and so $\mu \in f(\sigma(T))$.

$\square$

[2] We define $f(\sigma(T)) := \{f(z) : z \in \sigma(T)\}$. That is to say $f(\sigma(T))$ returns a set with f applied to each element of the set which was supplied as the argument.

We have an immediate corollary in the case where T is self adjoint.

Corollary 1. *If* $T = T^*$ *then*

$$||f(T)|| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$$

Proof. By the C^* identity we have that

$$\begin{aligned} ||f(T)||^2 &= ||f(T)^* f(T)|| \\ &= \overline{f}(T^*) f(T)||. \end{aligned}$$

Letting $g(z) = \overline{f}(z) f(z)$ we have that since T, and hence $g(T)$, is self adjoint that

$$||g(T)|| = r(g(T)) = \sup_{z \in \sigma(T)} |g(z)| = \sup_{z \in \sigma(T)} |f(z)|^2.$$

That is to say

$$||f(T)|| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|.$$

□