University of New South Wales

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Assignment 2

Measure Theory

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1.1

Let μ and ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We wish to show that the convolution

$$\mu * \nu(B) = \int \nu(B - x)\mu(dx) \tag{1}$$

of these two measures is well defined in the sense that $\nu(B - x)$ is measurable in x and the integral exists. Firstly see that

$$\begin{align} \nu(B - x) &= \int \chi_{B-x}(y)\nu(dy) \tag{2} \\ &= \int \chi_B(x + y)\nu(dy) \tag{3} \end{align}$$

and by seeing that $\phi(x, y) = \chi_B(x+y)$ is $\mathcal{B}(R)\otimes\mathcal{B}(R)$ measurable and so applying Tonelli's theorem $\nu(B-x)$ is measurable.
We now need to show that the integral exists. This is clear because

$$0 \leq \nu(B - x) \leq 1 \tag{4}$$

and thus

$$\begin{align} \int \nu(B - x)\mu(dx) &\leq \int 1\mu(dx) \tag{5} \\ &= 1 \tag{6} \end{align}$$

because μ is a probability measure.

1.2

Suppose there exists a bounded set $F \in \mathcal{B}(\mathbb{R})$ such that

$$\mu * \nu(F) = 1 \tag{7}$$

we wish to show that there exists bounded sets $G, H \in \mathcal{B}(\mathbb{R})$ such that

$$\mu(G) = 1 \quad \text{and} \quad \nu(H) = 1. \tag{8}$$

See that if

$$\begin{align} 1 &= \mu * \nu(F) \tag{9} \\ &= \int \nu(F - x)\mu(dx) \tag{10} \\ &= \int\int \chi_F(x + y)\nu(dy)\mu(dx) \tag{11} \end{align}$$

Now as F is bounded there there exist intervals $G = [a, b]$ and $H = [c, d]$ such that

$$\int\int \chi_F(x + y)\chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \tag{12}$$

and further

$$1 = \int \int \chi_F(x+y)\chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \le \int \int \chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \tag{13}$$

$$= \int \chi_H(y)\nu(dy) \int \chi_G(x)\mu(dx) \tag{14}$$

[1] which implies that

$$\int \chi_G(x)\mu(dx) = 1 \quad \text{and} \quad \int \chi_H(y)\nu(dy) = 1 \tag{15}$$

that is $\mu(G) = 1$ and $\nu(H) = 1$.
Suppose that F is now countable but such that $\mu * \nu(F) = 1$. Then

$$1 = \int \nu(F - x)\mu(dx) \tag{16}$$

$$= \int \int \chi_F(x+y)\nu(dy)\mu(dx) \tag{17}$$

now since $F = \{(x_k, y_k)\}_{k\in\mathbb{N}}$ is countable then there must exist countable sets $G = \{x_k\}_{k\in\mathbb{N}}$, $H = \{y_k\}_{k\in\mathbb{N}}$ such that

$$\int \int \chi_F(x+y)\nu(dy)\mu(dx) = \int \int \chi_F(x+y)\chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \tag{18}$$

and further

$$1 = \int \int \chi_F(x+y)\chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \le \int \int \chi_G(x)\chi_H(y)\nu(dy)\mu(dx) \tag{19}$$

$$= \int \chi_H(y)\nu(dy) \int \chi_G(x)\mu(dx) \tag{20}$$

which as before implies $\mu(G) = 1$ and $\nu(H) = 1$.
Then argument above also holds for finite F with exactly the same construction and so if F is finite and such that $\mu * \nu(F) = 1$ then there must exist finite sets G and H such that $\mu(G) = 1$ and $\nu(H) = 1$.

2

2.1

Suppose u and ν are σ-finite *positive* measures on $(\Omega, \mathcal{F})$. Then suppose that $\mu << \nu$ and $\nu << \mu$. Then for $A \in \mathcal{F}$ we have that $\mu(a) = 0 \Rightarrow \nu(A) = 0$ and $\nu(A) = 0 \Rightarrow \mu(A)$. That is to say that $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$. That is to say that ν and μ have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.
We now wish to show that there is an $\mathcal{F}$-measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$.

2.2

Let $\{B_n\}_{n\in\mathbb{N}}$ be a covering of Ω by disjoint sets with $0 < \mu(B_n) < \infty$ for all n. Such a covering exists because μ is σ-finite. Now select a sequence of constants $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n > 0$ for all n and

$$\sum_{n=1}^{\infty} \alpha_n = 1. \tag{21}$$

[1]Note that the reason we could split up the integrals to a product in both cases is because $\int \chi_H(y)\nu(y)$ is a constant with respect to x.

For example we could select $\alpha_n = 2^{-n}$.
We claim that the function $\nu : \mathcal{F} \longrightarrow \mathbb{R}$ defined by

$$\nu(A) = \sum_{n=1}^{\infty} \alpha_n \frac{\mu(A \cap B_n)}{\mu(B_n)} \tag{22}$$

is a probability measure with the same null sets as μ. Firstly we show that it is a measure, that is we show σ-additivity.
For a collection of disjoint sets $\{A_n\}_{n\in\mathbb{N}}$ we have that

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{k=1}^{\infty} \alpha_k \frac{\mu\left(\bigcup_{n=1}^{\infty} A_n \cap B_k\right)}{\mu(B_k)} \tag{23}$$

and by σ-additivity of μ we get

$$\begin{aligned}\sum_{k=1}^{\infty} \alpha_k \frac{\mu\left(\bigcup_{n=1}^{\infty} A_n \cap B_k\right)}{\mu(B_k)} &= \sum_{k=1}^{\infty} \alpha_k \frac{\sum_{n=1}^{\infty} \mu\left(A_n \cap B_k\right)}{\mu(B_k)} && (24)\\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k \frac{\mu\left(A_n \cap B_k\right)}{\mu(B_k)} && (25)\\ &= \sum_{n=1}^{\infty} \nu(A_n). && (26)\end{aligned}$$

The interchange of the order of summation can be justified by the fact that $\mu(A_n \cap B_k) \geq 0$ and

$$\sum_{k=1}^{\infty} \alpha_k \frac{\sum_{n=1}^{\infty} \mu\left(A_n \cap B_k\right)}{\mu(B_k)} < \infty \tag{27}$$

which we prove now (by proving ν is a probability measure).
See that

$$\begin{aligned}\nu(\Omega) &= \sum_{k=1}^{\infty} \alpha_k \frac{\mu(\Omega \cap B_k)}{\mu(B_k)} && (28)\\ &= \sum_{k=1}^{\infty} \alpha_k \frac{\mu(B_k)}{\mu(B_k)} && (29)\\ &= \sum_{k=1}^{\infty} \alpha_k && (30)\\ &= 1. && (31)\end{aligned}$$

We now just have to show that μ and ν share the same null sets.
Suppose $\mu(A) = 0$ then

$$\begin{aligned}\mu(A) &= \sum_{k=1}^{\infty} \alpha_k \frac{\mu(A \cap B_k)}{\mu(B_k)} && (32)\\ &\leq \sum_{k=1}^{\infty} \alpha_k \frac{\mu(A)}{\mu(B_k)} && (33)\\ &= 0 && (34)\end{aligned}$$

and thus $\nu \ll \mu$. Now suppose $\nu(A) = 0$ then

$$\sum_{k=1}^{\infty} \alpha_k \frac{\mu(A \cap B_k)}{\mu(B_k)} = 0 \tag{35}$$

and as $\mu(B_k) < \infty$ for all k this implies that $\mu(A \cap B_k) = 0$ for all k. Now

$$\begin{aligned} 0 &= \sum_{k=1}^{\infty} \mu(A \cap B_k) &(36)\\ &= \mu\left(\bigcup_{k=1}^{\infty}(A \cap B_k)\right) &(37)\\ &= \mu\left(A \cap \bigcup_{k=1}^{\infty} B_k\right) &(38)\\ &= \mu(A \cap \Omega) &(39)\\ &= \mu(A). &(40) \end{aligned}$$

Thus $\mu \ll \nu$.
So ν is a finite measure on $(\Omega, \mathcal{F})$ which is equivalent to μ.

3

3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u\rangle)] \tag{41}$$

and so for the random vector cX with $c \in \mathbb{R}$ we have that

$$\begin{aligned} \hat{\mu}_{cX}(u) &= \mathbb{E}[\exp(i\langle cX, u\rangle)] &(42)\\ &= \mathbb{E}[\exp(i\langle X, cu\rangle)] &(43)\\ &= \hat{\mu}_X(cu) &(44) \end{aligned}$$

3.2

By definition we have that

$$\hat{\mu}(\mathbf{u}) = \int \exp(i\langle \mathbf{x}, \mathbf{u}\rangle) d\mathbb{P}_X(\mathbf{x}). \tag{45}$$

We can say that

$$\frac{d^\alpha \hat{\mu}(\mathbf{u})}{d\mathbf{u}^\alpha} = \frac{d^\alpha}{d\mathbf{u}^\alpha} \int \exp(i\langle \mathbf{x}, \mathbf{u}\rangle) d\mathbb{P}_X(\mathbf{x}). \tag{46}$$

We wish to justify taking this derivative through the integral sign. To do this we use an extension of corollary 2.28 (2) from the notes.

Claim
If the partial derivative

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \tag{47}$$

exists for all $(\mathbf{x}, \mathbf{u}) \in X \times [a, b]^d$ and if there is a function $g \in L^1(\mu)$ such that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \right| \leq g \tag{48}$$

for every $\mathbf{x} \in X$ and $\mathbf{u} \in (a, b)^d$ then

$$\frac{d^\alpha}{d\mathbf{u}^\alpha} \int f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) = \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) \qquad \text{for } \mathbf{u} \in (a, b)^d. \tag{49}$$

The proof follows from induction on the dimension of $\mathbf{x}$ and $\mathbf{u}$ and the order of the derivative. The base case is specifically the statement of corollary 2.28 (2) from the notes.

We have that

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) = i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \tag{50}$$

and that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \right| \leq \underbrace{\prod_{k=1}^d |x_k|^{\alpha_k}}_{\circledast}. \tag{51}$$

Now because

$$\mathbf{E}\left(\prod_{k=1}^d |X_k|^{\alpha_k} \right) = \int \prod_{k=1}^d |x_k|^{\alpha_k} \, d\mathbb{P}_X(\mathbf{x}) < \infty \tag{52}$$

then $\circledast \in L^1(\mathbb{P}_X)$ and so we can apply our claim (the DCT) to get

$$\frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha} = \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \tag{53}$$

$$= \int i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \tag{54}$$

$$= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \tag{55}$$

and so

$$\left.\frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha}\right|_{\mathbf{u}=\mathbf{0}} = i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{0} \rangle) d\mathbb{P}_X(\mathbf{x}) \tag{56}$$

$$= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} d\mathbb{P}_X(\mathbf{x}) \tag{57}$$

$$= i^{|\alpha|} \mathbb{E}(X^\alpha) \tag{58}$$

3.3

Let $d = 1$ and let μ have the Lebesgue density,

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \qquad x \in \mathbb{R}. \tag{59}$$

We wish to show that $E[X]$ is not defined but $\hat{\mu}(u)$ is differentiable at 0. Firstly we show that $E[X]$ is not defined. If $\mathbb{E}(X)$ were defined then

$$\mathbb{E}(X) = \int \frac{xC}{(1+x^2)\log(e+x^2)}dx \tag{60}$$

$$= \underbrace{\int_{-\infty}^{0} \frac{xC}{(1+x^2)\log(e+x^2)}dx}_{\mathbb{E}(X^-)} + \underbrace{\int_{0}^{\infty} \frac{xC}{(1+x^2)\log(e+x^2)}dx}_{\mathbb{E}(X^+)} \tag{61}$$

but

$$\int_0^\infty \frac{xC}{(1+x^2)\log(e+x^2)}dx \geq \int_1^\infty \frac{xC}{2x^2\log(e+x^2)}dx \tag{62}$$

$$= \int_1^\infty \frac{C}{2x\log(e+x^2)}dx \tag{63}$$

$$\geq \int_4^\infty \frac{C}{2x\log(4x^2)}dx \tag{64}$$

$$\geq \int_6^\infty \frac{C}{10x\log(x)}dx. \tag{65}$$

Let $u = \log(x)$ so that $du = \frac{1}{x}dx$ and so

$$\int_6^\infty \frac{C}{10x\log(x)}dx = \underbrace{\int_{\log(6)}^\infty \frac{C}{10u}du}_{\circledast} \tag{66}$$

and $\circledast$ diverges. So $\mathbb{E}(X^+)$ does not exist and in a similar manner we can see that $\mathbb{E}(X^-)$ does not exist and thus $E(X)$ is not defined.
However we can calculate $\hat{\mu}(u)$ as

$$\hat{\mu}(u) = \int \frac{e^{ixu}C}{(1+x^2)\log(e+x^2)}dx \tag{67}$$

and note that $\hat{\mu}(u)$ is differentiable at 0 if the following limit exists

$$\lim_{h \longrightarrow 0} \int \frac{e^{ixh}C - C}{h(1+x^2)\log(e+x^2)}dx \tag{68}$$

4

Let μ be the binomial distribution with n trials and probability of success p, that is $\mu = \text{Bin}(n,p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

4.1

We wish to verify that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$. Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that $(1-p+pe^{iu}$ is the characteristic function for $Bernoulli(p)$.
If ν is the Bernoulli measure and X has law ν then

$$\begin{aligned}
\hat{\nu}(u) &= \mathbb{E}[\exp(iuX)] &(69)\\
&= \sum_{k\in\{0,1\}} e^{iuk}\nu_X(k) &(70)\\
&= pe^{iu} + (1-p). &(71)
\end{aligned}$$

Then by repeated application of the convolution theorem we get that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$.

4.2

We wish to verify that $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$. The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!}e^{-\lambda} \tag{72}$$

and thus

$$\begin{aligned}
\mathbb{E}[\exp(iuX)] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda}e^{iuk} &(73)\\
&= e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(e^{iu})^k &(74)\\
&= e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} &(75)\\
&= e^{-\lambda}e^{\lambda e^{iu}} &(76)\\
&= e^{\lambda(e^{iu}-1)} &(77)
\end{aligned}$$

4.3

We wish to show that if p_n is a sequence in $[0, 1]$ such that $p_n \downarrow 0$ and $np_n \longrightarrow \lambda$ then $\mu_n \longrightarrow \nu$ in the weak sense where $\mu_n = \text{Bin}(n, p_n)$. Let $f \in C_b$ then

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \sum_{k=0}^{n} f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k} &= \lim_{n\longrightarrow\infty} \sum_{k=0}^{\infty} \chi_{k\le n} \cdot f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k} &(78)\\
&= \sum_{k=0}^{\infty} \lim_{n\longrightarrow\infty} \chi_{k\le n} \cdot f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k}. &(79)
\end{aligned}$$

The interchange of the order of the limit and the sum is justified by the uniform convergence of the sum. To see this let $M = \sup_{k\in\mathbb{N}^0} f(k)$ (which exists because $f \in C_b$) and then note that

$$\begin{aligned}
\sum_{k=0}^{\infty} \chi_{k\le n} \cdot f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k} &\le \sum_{k=0}^{\infty} \chi_{k\le n} \cdot M\binom{n}{k}p_n^k(1-p_n)^{n-k} &(80)\\
&= M < \infty &(81)
\end{aligned}$$

and so by the Weierstrass M test the series converges uniformly.
Now as $np_n \longrightarrow \lambda$ or $p_n \longrightarrow \frac{\lambda}{n}$ we get

$$\sum_{k=0}^{\infty} \lim_{n\longrightarrow\infty} \chi_{k\leq n} \cdot f(k)\binom{n}{k} p_n^k (1-p_n)^{n-k} = \sum_{k=0}^{\infty} \lim_{n\longrightarrow\infty} \chi_{k\leq n} \cdot f(k)\frac{n!}{k!(n-k)!} p_n^k (1-p_n)^n (1-p_n)^{-k} \tag{82}$$

$$= \sum_{k=0}^{\infty} \lim_{n\longrightarrow\infty} \chi_{k\leq n} \cdot f(k)\frac{n^k + O(n^{k-1})}{k!} p_n^k (1-p_n)^n (1-p_n)^{-k} \tag{83}$$

$$= \sum_{k=0}^{\infty} f(k) \lim_{n\longrightarrow\infty} \frac{n^k + O(n^{k-1})}{k!} p_n^k (1-p_n)^n \underbrace{(1-p_n)^{-k}}_{\longrightarrow 0} \tag{84}$$

$$= \sum_{k=0}^{\infty} f(k) \lim_{n\longrightarrow\infty} \underbrace{\frac{n^k + O(n^{k-1})}{k!} p_n^k}_{\longrightarrow \frac{\lambda^k}{k!}} (1-p_n)^n \tag{85}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f(k) \lim_{n\longrightarrow\infty} \underbrace{(1-\frac{\lambda}{n})^n}_{\longrightarrow e^{-\lambda}} \tag{86}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} f(k). \tag{87}$$

This proves the weak convergence.

4.4

This argument holds whether one takes the integral (sum) or not. So $\mu_n(\{k\}) \longrightarrow \nu(\{k\})$ for all $k \in \mathbb{N}^0$.

5

5.1

We wish to show that $\mathbb{P}(B_n) = 1/2$ for every $n \geq 1$. Note that

$$B_n = \bigcup_{k=0}^{2^{n-1}-1} \Big[\frac{2k}{2^n}, \frac{2k+1}{2^n}\Big) \tag{88}$$

and thus

$$\mathbb{P}(B_n) = \mathbb{P}\left(\bigcup_{k=0}^{2^{n-1}-1} \Big[\frac{2k}{2^n}, \frac{2k+1}{2^n}\Big)\right) \tag{89}$$

$$= \sum_{k=0}^{2^{n-1}-1} \mathbb{P}\left(\Big[\frac{2k}{2^n}, \frac{2k+1}{2^n}\Big)\right) \tag{90}$$

$$= \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^n} \tag{91}$$

$$= 2^{n-1}\frac{1}{2^n} \tag{92}$$

$$= \frac{1}{2}. \tag{93}$$

5.2

We now wish to show that the sequence of events B_n form an infinite sequence of independent events. Take a finite subset $J \subset \mathbb{N}$ with $|J| = m$ and $\max J = r$ then

$$\mathbb{P}\left(\bigcap_{n\in J} B_n\right) = \mathbb{P}\left(\bigcap_{n\in J} \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) \tag{94}$$

$$= \mathbb{P}\left(\bigcup_{k=0}^{2^{r-m}-1} \left[\frac{2k}{2^r}, \frac{2k+1}{2^r}\right)\right) \tag{95}$$

$$= \sum_{k=0}^{2^{r-m}-1} \mathbb{P}\left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) \tag{96}$$

$$= 2^{r-m}\frac{1}{2^r} \tag{97}$$

$$= \frac{1}{2^m} \tag{98}$$

$$= \prod_{n\in J} \mathbb{P}(B_n) \tag{99}$$

and so the sequence of events B_n form an infinite sequence of independent events.

5.3

We wish to show / argue that the probability that a randomly sampled number ω will have the sequence 5825 occur infinitely often in its decimal expansion is 1.
We use the Borel-Cantelli lemma. Ignoring possible overlaps (on the 5s) we can see that we can break any decimal expansion of ω up into blocks of 4 digits.
Then by we can define E_i as the probability of obtaining 5285 in the i-th block possition. By the same argument as above these events are independent.
The for any i we have that $\mathbb{P}(E_i) = \frac{1}{10000}$ (the same argument as above applied to a decimal expansion). Then clearly

$$\mathbb{P}(E_i) = \infty. \tag{100}$$

By the Borel-Cantelli lemma this implies

$$\mathbb{P}(\limsup_n(E_n) = 1. \tag{101}$$

Now

$$\limsup_n(E_n) = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j \tag{102}$$

can be intuatively read as E_j happens infinitely often. Which is to say that 5285 occurs *blockwise* in the expansion of ω infinitely often. Clearly as allowing for overlaps allows for more configurations then the probability is 1 (it can be no more).