



# UNSW

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UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

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## Assignment 2

Measure Theory

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# 1

## 2

### 2.1

Suppose  $\mu$  and  $\nu$  are  $\sigma$ -finite *positive* measures on  $(\Omega, \mathcal{F})$ . Then suppose that  $\mu \ll \nu$  and  $\nu \ll \mu$ . Then for  $A \in \mathcal{F}$  we have that  $\mu(A) = 0 \Rightarrow \nu(A) = 0$  and  $\nu(A) = 0 \Rightarrow \mu(A) = 0$ . That is to say that  $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$ . That is to say that  $\nu$  and  $\mu$  have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.

We now wish to show that there is an  $\mathcal{F}$ -measurable function  $g$  that satisfies  $0 < g(\omega) < +\infty$  at each  $\omega \in \Omega$  and is such that  $\nu(A) = \int_A g d\mu$  for all  $A \in \mathcal{F}$ .

### 2.2

Let  $\{B_n\}_{n \in \mathbb{N}}$  be a covering of  $\Omega$  by disjoint sets with  $0 < \mu(B_n) < \infty$  for all  $n$ . Such a covering exists because  $\mu$  is  $\sigma$ -finite. Now select a sequence of constants  $\{\alpha_n\}_{n \in \mathbb{N}}$  such that  $\alpha_n > 0$  for all  $n$  and

$$\sum_{n=1}^{\infty} \alpha_n = 1. \quad (1)$$

For example we could select  $\alpha_n = 2^{-n}$ .

We claim that the function  $\nu : \mathcal{F} \rightarrow \mathbb{R}$  defined by

$$\nu(A) = \sum_{n=1}^{\infty} \alpha_n \frac{\mu(A \cap B_n)}{\mu(B_n)} \quad (2)$$

is a probability measure with the same null sets as  $\mu$ . Firstly we show that it is a measure, that is we show  $\sigma$ -additivity.

For a collection of disjoint sets  $\{A_n\}_{n \in \mathbb{N}}$  we have that

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{k=1}^{\infty} \alpha_k \frac{\mu(\bigcup_{n=1}^{\infty} A_n \cap B_k)}{\mu(B_k)} \quad (3)$$

and by  $\sigma$ -additivity of  $\mu$  we get

$$\sum_{k=1}^{\infty} \alpha_k \frac{\mu(\bigcup_{n=1}^{\infty} A_n \cap B_k)}{\mu(B_k)} = \sum_{k=1}^{\infty} \alpha_k \frac{\sum_{n=1}^{\infty} \mu(A_n \cap B_k)}{\mu(B_k)} \quad (4)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k \frac{\mu(A_n \cap B_k)}{\mu(B_k)} \quad (5)$$

$$= \sum_{n=1}^{\infty} \nu(A_n). \quad (6)$$

The interchange of the order of summation can be justified by the fact that  $\mu(A_n \cap B_k) \geq 0$  and

$$\sum_{k=1}^{\infty} \alpha_k \frac{\sum_{n=1}^{\infty} \mu(A_n \cap B_k)}{\mu(B_k)} < \infty \quad (7)$$

which we prove now (by proving  $\nu$  is a probability measure).

See that

$$\nu(\Omega) = \sum_{k=1}^{\infty} \alpha_k \frac{\mu(\Omega \cap B_k)}{\mu(B_k)} \quad (8)$$

$$= \sum_{k=1}^{\infty} \alpha_k \frac{\mu(B_k)}{\mu(B_k)} \quad (9)$$

$$= \sum_{k=1}^{\infty} \alpha_k \quad (10)$$

$$= 1. \quad (11)$$

We now just have to show that  $\mu$  and  $\nu$  share the same null sets.  
Suppose  $\mu(A) = 0$  then

$$\mu(A) = \sum_{k=1}^{\infty} \alpha_k \frac{\mu(A \cap B_k)}{\mu(B_k)} \quad (12)$$

$$\leq \sum_{k=1}^{\infty} \alpha_k \frac{\mu(A)}{\mu(B_k)} \quad (13)$$

$$= 0 \quad (14)$$

and thus  $\nu \ll \mu$ . Now suppose  $\nu(A) = 0$  then

$$\sum_{k=1}^{\infty} \alpha_k \frac{\mu(A \cap B_k)}{\mu(B_k)} = 0 \quad (15)$$

and as  $\mu(B_k) < \infty$  for all  $k$  this implies that  $\mu(A \cap B_k) = 0$  for all  $k$ . Now

$$0 = \sum_{k=1}^{\infty} \mu(A \cap B_k) \quad (16)$$

$$= \mu \left( \bigcup_{k=1}^{\infty} (A \cap B_k) \right) \quad (17)$$

$$= \mu \left( A \cap \bigcup_{k=1}^{\infty} B_k \right) \quad (18)$$

$$= \mu(A \cap \Omega) \quad (19)$$

$$= \mu(A). \quad (20)$$

Thus  $\mu \ll \nu$ .

So  $\nu$  is a finite measure on  $(\Omega, \mathcal{F})$  which is equivalent to  $\mu$ .

## 3

### 3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u \rangle)] \quad (21)$$

and so for the random vector  $cX$  with  $c \in \mathbb{R}$  we have that

$$\hat{\mu}_{cX}(u) = \mathbb{E}[\exp(i\langle cX, u \rangle)] \quad (22)$$

$$= \mathbb{E}[\exp(i\langle X, cu \rangle)] \quad (23)$$

$$= \hat{\mu}_X(cu) \quad (24)$$

## 3.2

By definition we have that

$$\hat{\mu}(\mathbf{u}) = \int \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}). \quad (25)$$

We can say that

$$\frac{d^\alpha \hat{\mu}(\mathbf{u})}{d\mathbf{u}^\alpha} = \frac{d^\alpha}{d\mathbf{u}^\alpha} \int \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}). \quad (26)$$

We wish to justify taking this derivative through the integral sign. To do this we use an extension of corollary 2.28 (2) from the notes.

**Claim**

If the partial derivative

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \quad (27)$$

exists for all  $(\mathbf{x}, \mathbf{u}) \in X \times [a, b]^d$  and if there is a function  $g \in L^1(\mu)$  such that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \right| \leq g \quad (28)$$

for every  $\mathbf{x} \in X$  and  $\mathbf{u} \in (a, b)^d$  then

$$\frac{d^\alpha}{d\mathbf{u}^\alpha} \int f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) = \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) \quad \text{for } \mathbf{u} \in (a, b)^d. \quad (29)$$

The proof follows from induction on the dimension of  $\mathbf{x}$  and  $\mathbf{u}$  and the order of the derivative. The base case is specifically the statement of corollary 2.28 (2) from the notes.

We have that

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) = i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \quad (30)$$

and that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \right| \leq \underbrace{\prod_{k=1}^d |x_k|^{\alpha_k}}_{\circledast} \quad (31)$$

Now because

$$\mathbf{E} \left( \prod_{k=1}^d |X_k|^{\alpha_k} \right) = \int \prod_{k=1}^d |x_k|^{\alpha_k} d\mathbb{P}_X(\mathbf{x}) < \infty \quad (32)$$

then  $\circledast \in L^1(\mathbb{P}_X)$  and so we can apply our claim (the DCT) to get

$$\frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha} = \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \quad (33)$$

$$= \int i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \quad (34)$$

$$= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) \quad (35)$$

and so

$$\left. \frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha} \right|_{\mathbf{u}=\mathbf{0}} = i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{0} \rangle) d\mathbb{P}_X(\mathbf{x}) \quad (36)$$

$$= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} d\mathbb{P}_X(\mathbf{x}) \quad (37)$$

$$= i^{|\alpha|} \mathbb{E}(X^\alpha) \quad (38)$$

### 3.3

Let  $d = 1$  and let  $\mu$  have the Lebesgue density,

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \quad x \in \mathbb{R}. \quad (39)$$

We wish to show that  $E[X]$  is not defined but  $\hat{\mu}(u)$  is differentiable at 0. Firstly we show that  $E[X]$  is not defined. If  $\mathbb{E}(X)$  were defined then

$$\mathbb{E}(X) = \int \frac{xC}{(1+x^2)\log(e+x^2)} dx \quad (40)$$

$$= \underbrace{\int_{-\infty}^0 \frac{xC}{(1+x^2)\log(e+x^2)} dx}_{\mathbb{E}(X^-)} + \underbrace{\int_0^{\infty} \frac{xC}{(1+x^2)\log(e+x^2)} dx}_{\mathbb{E}(X^+)} \quad (41)$$

but

$$\int_0^{\infty} \frac{xC}{(1+x^2)\log(e+x^2)} dx \geq \int_1^{\infty} \frac{xC}{2x^2\log(e+x^2)} dx \quad (42)$$

$$= \int_1^{\infty} \frac{C}{2x\log(e+x^2)} dx \quad (43)$$

$$\geq \int_4^{\infty} \frac{C}{2x\log(4x^2)} dx \quad (44)$$

$$\geq \int_6^{\infty} \frac{C}{10x\log(x)} dx. \quad (45)$$

Let  $u = \log(x)$  so that  $du = \frac{1}{x}dx$  and so

$$\int_6^{\infty} \frac{C}{10x\log(x)} dx = \underbrace{\int_{\log(6)}^{\infty} \frac{C}{10u} du}_{\otimes} \quad (46)$$

and  $\otimes$  diverges. So  $\mathbb{E}(X^+)$  does not exist and in a similar manner we can see that  $\mathbb{E}(X^-)$  does not exist and thus  $E(X)$  is not defined.

However we can calculate  $\hat{\mu}(u)$  as

$$\hat{\mu}(u) = \int \frac{e^{ixu}C}{(1+x^2)\log(e+x^2)} dx \quad (47)$$

and note that  $\hat{\mu}(u)$  is differentiable at 0 if the following limit exists

$$\lim_{h \rightarrow 0} \int \frac{e^{ixh}C - C}{h(1+x^2)\log(e+x^2)} dx \quad (48)$$

## 4

Let  $\mu$  be the binomial distribution with  $n$  trials and probability of success  $p$ , that is  $\mu = \text{Bin}(n, p)$ , and let  $\nu$  be the Poisson distribution with mean  $\lambda > 0$ .

## 4.1

We wish to verify that  $\hat{\mu}(u) = (1 - p + pe^{iu})^n$ . Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that  $(1 - p + pe^{iu})$  is the characteristic function for *Bernoulli*( $p$ ).

If  $\nu$  is the Bernoulli measure and  $X$  has law  $\nu$  then

$$\hat{\nu}(u) = \mathbb{E}[\exp(iuX)] \quad (49)$$

$$= \sum_{k \in \{0,1\}} e^{iuk} \nu_X(k) \quad (50)$$

$$= pe^{iu} + (1 - p). \quad (51)$$

Then by repeated application of the convolution theorem we get that  $\hat{\mu}(u) = (1 - p + pe^{iu})^n$ .

## 4.2

We wish to verify that  $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$ . The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!} e^{-\lambda} \quad (52)$$

and thus

$$\mathbb{E}[\exp(iuX)] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} \quad (53)$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (e^{iu})^k \quad (54)$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} \quad (55)$$

$$= e^{-\lambda} e^{\lambda e^{iu}} \quad (56)$$

$$= e^{\lambda(e^{iu} - 1)} \quad (57)$$

## 4.3

We wish to show that if  $p_n$  is a sequence in  $[0, 1]$  such that  $p_n \downarrow 0$  and  $np_n \rightarrow \lambda$  then  $\mu_n \rightarrow \nu$  in the weak sense where  $\mu_n = \text{Bin}(n, p_n)$ .

Let  $f \in C_b$  then

$$E[f(X_n)] = \sum_{k=0}^n f(k) \mu_n(k) \quad (58)$$

$$= \sum_{k=0}^n f(k) \mu_n(k) \quad (59)$$

$$= \sum_{k=0}^n f(k) \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad (60)$$

then

$$\lim_{n \rightarrow \infty} E[f(X_n)] = \lim_{n \rightarrow \infty} \sum_{k=0}^n f(k) \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad (61)$$

and

$$\mathbb{E}[f(Y)] = f(x)\nu(x) \quad (62)$$

$$= \sum_{k=0}^{\infty} f(k) \frac{\lambda^k}{k!} e^{-\lambda} \quad (63)$$

$$(64)$$

## 5

### 5.1

We wish to show that  $\mathbb{P}(B_n) = 1/2$  for every  $n \geq 1$ . Note that

$$B_n = \bigcup_{k=0}^{2^{n-1}-1} \left[ \frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \quad (65)$$

and thus

$$\mathbb{P}(B_n) = \mathbb{P} \left( \bigcup_{k=0}^{2^{n-1}-1} \left[ \frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \right) \quad (66)$$

$$= \sum_{k=0}^{2^{n-1}-1} \mathbb{P} \left( \left[ \frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \right) \quad (67)$$

$$= \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^n} \quad (68)$$

$$= 2^{n-1} \frac{1}{2^n} \quad (69)$$

$$= \frac{1}{2}. \quad (70)$$

### 5.2

We now wish to show that the sequence of events  $B_n$  form an infinite sequence of independent events.

Take a finite subset  $J \subset \mathbb{N}$  with  $|J| = m$  and  $\max J = r$  then

$$\mathbb{P} \left( \bigcap_{n \in J} B_n \right) = \mathbb{P} \left( \bigcap_{n \in J} \bigcup_{k=0}^{2^{n-1}-1} \left[ \frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \right) \quad (71)$$

$$= \mathbb{P} \left( \bigcup_{k=0}^{2^{r-m}-1} \left[ \frac{2k}{2^r}, \frac{2k+1}{2^r} \right) \right) \quad (72)$$

$$= \sum_{k=0}^{2^{r-m}-1} \mathbb{P} \left( \left[ \frac{2k}{2^r}, \frac{2k+1}{2^r} \right) \right) \quad (73)$$

$$= 2^{r-m} \frac{1}{2^r} \quad (74)$$

$$= \frac{1}{2^m} \quad (75)$$

$$= \prod_{n \in J} \mathbb{P}(B_n) \quad (76)$$



and so the sequence of events  $B_n$  form an infinite sequence of independent events.

### 5.3

We wish to show / argue that the probability that a randomly sampled number  $\omega$  will have the sequence 5825 occur infinitely often in its decimal expansion is 1.

We use the Borel-Cantelli lemma. Ignoring possible overlaps (on the 5s) we can see that we can break any decimal expansion of  $\omega$  up into blocks of 4 digits.

Then by we can define  $E_i$  as the probability of obtaining 5285 in the  $i$ -th block position. By the same argument as above these events are independent.

The for any  $i$  we have that  $\mathbb{P}(E_i) = \frac{1}{10000}$  (the same argument as above applied to a decimal expansion). Then clearly

$$\mathbb{P}(E_i) = \infty. \quad (77)$$

By the Borel-Cantelli lemma this implies

$$\mathbb{P}(\limsup_n(E_n)) = 1. \quad (78)$$

Now

$$\limsup_n(E_n) = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j \quad (79)$$

can be intuitively read as  $E_j$  happens infinitely often. Which is to say that 5285 occurs *blockwise* in the expansion of  $\omega$  infinitely often. Clearly as allowing for overlaps allows for more configurations then the probability is 1 (it can be no more).