UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 3

Measure Theory

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1.1

1.1.1 $f + g$

$$f(x) + g(x) < t \Leftrightarrow f(x) < r \text{ and } g(x) < t + r$$

for some $r \in \mathbb{Q}$.
Thus

$$\{x : f(x) + g(x) < t\} = \bigcup_{r \in \mathbb{Q}} \left[f^{-1}([-\infty, r)) \cap g^{-1}([-\infty, t - r))\right].$$

This is the countable union of measurable sets and is thus measurable itself. So $f + g$ is measurable.

1.1.2 αf

$$\{x : \alpha f(x) < t\} = \bigcup_{q \in \mathbb{Q}, q < \alpha} \left[f^{-1}([-\infty, \frac{t}{\alpha}))\right].$$

This is the countable union of measurable sets and is thus measurable itself. So αf is measurable.

1.1.3 fg

$$fg = \frac{(f + g)^2 - f^2 - g^2}{2}$$

and

$$\{x : f^2(x) < t\} = \{x : f(x) < \sqrt{t}\} = \bigcup_{q \in \mathbb{Q}, q < \sqrt{t}} f^{-1}([-\infty, q))$$

This is the countable union of measurable sets so it is measurable and so f^2 is measurable. The measurability of fg follows from this and the last two results.

1.1.4 f/g

$$\{x : \frac{1}{g(x)} < t\} = \begin{cases} \{x : \frac{1}{t} < g(x) < 0\} & t < 0 \\ \{x : -\infty < g(x) < 0\} & t = 0 \\ \{x : -\infty < g(x) < 0\} \cup \{x : \frac{1}{t} < g(x) < 0\} & b > 0 \end{cases}$$

Which is clearly measurable so $\frac{1}{g}$ is measurable and thus $\frac{f}{g}$ is measurable.

1.1.5 $f \wedge g$

$$f \wedge g = \frac{f+g+|f-g|}{2}$$

and

$$\{x : |f(x)| < t\} = \bigcup_{q \in \mathbb{Q}, q<t} f^{-1}([-\infty, q)) \cap f^{-1}((-q, \infty])$$

which is the countable union of measurable sets so $|f|$ is measurable. It follows that $f \wedge g$ is measurable by previous results.

1.1.6 $f \vee g$

$$f \vee g = \frac{f+g-|f-g|}{2}.$$

The result then follows from previous results.

1.2

1.2.1 $\{x : f(x) < g(x)\}$

$\exists r : f(x) < r < g(x)$ for each x.

$$\{x : f(x) < g(x)\} = \bigcup f^{-1}([-\infty, r)) \cap g^{-1}((r, \infty])$$

which is the countable union of measurable sets so $\{x : f(x) < g(x)\}$ is measurable.

1.2.2 $\{x : f(x) \leq g(x)\}$

$$\{x : f(x) \leq g(x)\} = \{x : g(x) < f(x)\}^c$$

and the result follows from above because the complement of a measurable set is measurable.

1.2.3 $\{x : f(x) = g(x)\}$

$\{x : f(x) = g(x)\} = \{x : f(x) < g(x)\} \cap \{x : f(x) \leq g(x)\}$ and the result follows from finite intersections.

1.3

1.3.1 $\sup_n f_n$

$$\{\sup_n f_n < t\} = \bigcap_{n \in \mathbb{N}} \{x : f_n(x) < t\}$$

which is the countable intersection of measurable sets and therefore $\sup_n f_n$ is measurable.

1.3.2 $\inf_n f_n$

Likewise

$$\{\inf_n f_n < t\} = \bigcup_{n\in\mathbb{N}} \{x : f_n(x) < t\}$$

and so by the same argument with unions the result follows.

1.3.3 $\limsup_n f_n$

$$\limsup_n f_n = \inf_k \sup_{n\geq k} f_n$$

and the result follows immediatly by putting $\sup_{n\geq k} f_n$ in the previous result and noting that $\sup_{n\geq k} f_n$ is measurable by the result for sup .

1.3.4 $\liminf_n f_n$

$$\liminf_n f_n = \sup_k \inf_{n\geq k} f_n$$

and using an essentially identical argument as in the previous result.

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2.1

$$F(t) = \mu\left(\{x : f(x) > t\}\right)$$
$$F(t+\varepsilon) = \mu\left(\{x : f(x) > t+\varepsilon\}\right)$$

and as

$$\{x : f(x) > t+\varepsilon\} \subseteq \{x : f(x) > t\}$$

then

$$\mu\left(\{x : f(x) > t+\varepsilon\}\right) \leq \mu\left(\{x : f(x) > t\}\right)$$

and so $F(t+\varepsilon) \leq F(t)$.
Also as f is a bounded function there must exist an N such that $f(x) \leq N$ $\forall$ x, which means $\{x : f(x) > N\} = \emptyset$ and so $F(t) = 0$ $\forall$ $t > N$.
Note that $\lim_{t\longrightarrow 0} F(t)$ is finite if f is of finite support.

$$\lim_{n\longrightarrow\infty} \int_0^n F(t)dt = \int_0^N F(t)dt + \underbrace{\lim_{n\longrightarrow\infty} \int_N^n F(t)dt}_{=0} \qquad \leq NF(0).$$

Now as N and $F(0)$ are finite then $\lim_{n\longrightarrow\infty} \int_0^n F(t)dt < \infty$. Now as $F(t)$ is non-negative then $G(n) = \int_0^n F(t)dt$ is an increasing function, bounded by $NF(0)$ and thus, $\lim_{n\longrightarrow\infty} G(n)$ exists.

2.2

$$L_n = \sum_{k=1}^{N2^n} \frac{1}{2^n} F(\frac{k}{2^n})$$

which is clearly the Lebesgue integral of

$$S_n(x) = \sum_{k=1}^{N2^n} \frac{1}{2^n} \chi_{\{t:f(t)>\frac{k}{2^n}\}}(x).$$

2.3

It is also clear that

$$S_n(x) = \frac{1}{2^n} \lfloor 2^n f(x) \rfloor$$

and so $\lim_{n\longrightarrow\infty} S_n(x) = f(x)$. Also by noting that

$$\frac{1}{2} \lfloor 2^{n+1} f(x) \rfloor \geq \lfloor 2^n f(x) \rfloor$$

we have that $S_{n+1}(x) \geq S_n(x)$. Thus

$$\lim_{n\longrightarrow\infty} \int S_n d\mu = \int f d\mu.$$

by the monotone convergence theorem. Now as L_n was also the lower Riemann integral of F and as $n \longrightarrow \infty$, $L_n \longrightarrow \int_0^N F(x)dx = \int_0^\infty F(x)dx$ then

$$\int f d\mu = \int_0^\infty F(x)dx.$$

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Let

$$f_n(x) = \chi_{A_n}(x) f(x) + n\chi_{A_n^c}(x)$$

then

$$\int f d\mu = \int_{A_n} f d\mu + \int_{A_n^c} f d\mu > \int f_n d\mu + n\mu(A_n^c).$$

Then as clearly $f_n \leq f$ and $\lim_{n\longrightarrow\infty} f_n(x) = f(x)$ and $f \in \mathcal{L}^1(\mu)$,by the dominated convergence theorem

$$\lim_{n\longrightarrow\infty} \int f_n d\mu = \int f d\mu$$

which implies that $\lim_{n\longrightarrow\infty} n\mu(A_n^c) = 0$