Question 1

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(a) Let $X_1, X_2, \dots, X_n \sim Exp(b)$ (iid)

Then let h_n be the law of $Y_n = \sum_{k=1}^{n} X_n$

Then $h_2(z) = \int_0^z b e^{-b(z-y)} b e^{-by} \, dy$

$$= b^2 e^{-bz} \int_0^z dy$$

$$= z b^2 e^{-bz}$$

Now assume $h_n(z) = \frac{b^n z^{n-1}}{(n-1)!} e^{-bz}$

then $h_{n+1}(z) = \int_0^z \frac{b^n}{(n-1)!} (z-y)^{n-1} e^{-b(z-y)} b e^{-by} \, dy$

$$= \frac{b^{n+1}}{(n-1)!} e^{-bz} \int_0^z (z-y)^{n-1} \, dy$$

$$= \frac{b^{n+1}}{(n-1)!} e^{-bz} \left[-\frac{(z-y)^n}{n} \right]_{y=0}^{y=z}$$

$$= \frac{b^{n+1}}{(n-1)!} e^{-bz} \frac{1}{n} z^n$$

$$= \frac{b^{n+1}}{n!} e^{-bz} z^n$$

which is the distribution of $\Gamma(n+1, b)$

so by induction the distribution of $Y_n = \sum_{k=1}^{n} X_k$

is $h_n(z) = \frac{b^n}{(n-1)!} e^{-bz} z^{(n-1)}$. ∎

(b) We have $\phi(k) = \left(1 - \frac{ik}{b}\right)^{-a}$.

Then $\sqrt[n]{\phi(k)} = \left(1 - \frac{ik}{b}\right)^{-a/n}$

and by substituting variables this would be the characteristic function of

$\Gamma\left(\frac{a}{n}, b\right)$. ∎

Question 2

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(a) Suppose X and Y are infinitely divisible random vectors with laws μ and ν respectively. $X+Y$ has CF $\hat{\mu}\hat{\nu}$.
We need to verify $(\hat{\mu}\hat{\nu})^{1/n}$ corresponds to a random vector.
Well $\hat{\mu}^{1/n}$ corresponds to a random vector as does $\hat{\nu}^{1/n}$, say $X^{(n)}$ and $Y^{(n)}$ respectively. Then by Thm 6.8 and 7.2 $\hat{\mu}^{1/n}\hat{\nu}^{1/n}$ corresponds to $X^{(n)}+Y^{(n)}$ which is a random vector and so we are done. ■

(b) Suppose μ_k is a sequence of laws such that $\mu_k \to \mu$ weakly and that each μ_k is infinitely divisible. We wish to show μ is also infinitely divisible.
For a contradiction suppose μ is not infinitely divisible. Then there exists an n such that $\hat{\mu}^{1/n}$ does not correspond to the law of a random variable.
Now as μ_k is infinitely divisible $\hat{\mu}_k^{1/n}$ corresponds to the CF of a random variable. Further as $\mu_k \to \mu$ weakly then $\hat{\mu}_k^{1/n}(\omega) \to \hat{\mu}^{1/n}(\omega)$ pointwise because $\hat{\mu}_k(\omega) \to \hat{\mu}(\omega)$ pointwise. Further as $\hat{\mu}(\omega)$ is continuous and equal to 1 at $\omega=0$, then $\hat{\mu}^{1/n}(\omega)$ is also continuous in some nhd of $\omega=0$ at equal to 1 at $\omega=0$, but then by Lévy continuity $\mu_k^{1/n} \to \mu^{1/n}$ weakly, and $\mu^{1/n}$ is the law of some random variable, this is a contradiction of our assumption. Therefore μ must also be infinitely divisible! ■

Homework Qn 3.

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(a) By the result of assignment 2 qn 3.b $\frac{\partial}{\partial k}\hat{\mu}(k)\Big|_{k=0} = i\mathbb{E}[X]$ and $\frac{\partial^2}{\partial k^2}\hat{\mu}(k)\Big|_{k=0} = -\mathbb{E}[X^2]$.

Then by Taylor expansion
$$\hat{\mu}(k) = \hat{\mu}(0) + i\mathbb{E}[X]k - \frac{\mathbb{E}[X^2]k^2}{2} + O(k^3)$$
$$= 1 - \frac{\sigma^2 k^2}{2} + O(k^3)$$

(b) By Thm 6.8 and 7.2 the characteristic function of $X_1 + X_2 + \cdots + X_n$ is
$$\hat{\nu}_n(k) = \left(\hat{\mu}(k)\right)^n$$
$$= \left(1 - \frac{\sigma^2 k^2}{2} + O(k^3)\right)^n$$
$$= 1 - \frac{n\sigma^2 k^2}{2} + O(nk^3)$$
$$= 1 - \frac{(\sqrt{n}\,k)^2\sigma^2}{2} + O(nk^3) = \exp\left(-(\sqrt{n}\,k)^2\sigma^2\right) + O(nk^3)$$

Then by the result of assignment 2 qn 3.a the characteristic function of
$$Z = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \quad \text{is} \quad \hat{\nu}_n\left(\frac{k}{\sqrt{n}}\right) = \exp\left(-k^2\sigma^2\right) + O\left(\frac{k^3}{\sqrt{n}}\right)$$

Now as $n \to \infty$ $\hat{\nu}_n\left(\frac{k}{\sqrt{n}}\right) \xrightarrow[\text{pointwise}]{} \hat{\nu}(k) =: \exp(-k^2\sigma^2)$. Notice that $\hat{\nu}(k)$ is continuous in some nbd of $k=0$ and so by the Levy continuity theorem

$\nu_n \to \nu$ weakly. Notice that $\hat{\nu}(k)$ is the characteristic function of the law with distribution $N(0,\sigma^2)$, so
$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \to \sigma Z \quad \text{where } Z \sim N(0,1).$$

(c) Suppose $X_1, \ldots, X_n$ are random <u>vectors</u> in $\mathbb{R}^d$ then assignment 2 qn 3.b implies that X_m has characteristic function $\hat{\mu}(k)$ with
$$\frac{\partial}{\partial k_i}\hat{\mu}(k)\Big|_{k=0} = i\,\mathbb{E}[X_{m,i}] \text{ and } \frac{\partial^2}{\partial k_i \partial k_j}\hat{\mu}(k)\Big|_{k=0} = -1\,\mathbb{E}[X_{m,i}X_{m,j}]$$

and so
$$\hat{\mu}(k) = 1 + i\underbrace{\sum_{i=1}^{d}\mathbb{E}[X_{m,i}]k}_{=0} - \sum_{\substack{j=1\\ i=1}}^{d}\mathbb{E}[X_{m,i}X_{m,j}]k^2 + O(k^3)$$

Tutorial Qns 3 continued...

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$$= \left(1 + \sum_{\substack{i=1 \\ j=1}}^{d} \sigma^2 k^2 + O(k^3)\right)^n$$

Thus the characteristic function $\hat{\nu}_n(k)$ of $X_1 + \cdots + X_n$ is

$$\hat{\nu}_n(k) = \left(1 - \frac{1}{2}\sum_{\substack{j=1 \\ i=1}}^{d} \mathbb{E}[X_j X_i] k^2 + O(k^3)\right)^n$$

$$= 1 - \frac{1}{2}\sum_{\substack{i=1 \\ j=1}}^{d} n\sigma^2 k^2 + O(nk^3)$$

$$= \exp\left(-\left(\sqrt{nd}\ k\right)^2 \sigma^2\right) + O(nk^3)$$

Then the characteristic function of $z = \frac{X_1 + \cdots X_n}{\sqrt{n}}$ is

$$\hat{\nu}_n(k) = \exp\left(-k^2 d\sigma^2\right) + O\left(\frac{k^3}{\sqrt{n}}\right)$$

And as $n \to \infty$ $\hat{\nu}_n(k) \to \exp\left(-k^2 d\sigma^2\right)$ pointwise.

Thus by Levy continuity $\nu_n \to \nu$ weakly where ν is the law of $\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{pmatrix}$ with $X_1, \ldots, X_d \sim N(0, \sigma^2)$ i.i.d. $\blacksquare$