



UNSW

A U S T R A L I A



UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment 2

Measure Theory

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2.1

Suppose u and ν are σ -finite *positive* measures on (Ω, \mathcal{F}) . Then suppose that $\mu \ll \nu$ and $\nu \ll \mu$. Then for $A \in \mathcal{F}$ we have that $\mu(A) = 0 \Rightarrow \nu(A) = 0$ and $\nu(A) = 0 \Rightarrow \mu(A) = 0$. That is to say that $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$. That is to say that ν and μ have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.

We now wish to show that there is an \mathcal{F} -measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$.

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3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u \rangle)] \quad (1)$$

and so for the random vector cX with $c \in \mathbb{R}$ we have that

$$\hat{\mu}_{cX}(u) = \mathbb{E}[\exp(i\langle cX, u \rangle)] \quad (2)$$

$$= \mathbb{E}[\exp(i\langle X, cu \rangle)] \quad (3)$$

$$= \hat{\mu}_X(cu) \quad (4)$$

3.2

3.3

Let $d = 1$ and let μ have the Lebesgue density,

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \quad x \in \mathbb{R}. \quad (5)$$

We wish to show that $E[X]$ is not defined but $\hat{\mu}(u)$ is differentiable at 0. Firstly we show that $E[X]$ is not defined.

$$E[X] = \int_{-\infty}^{\infty} \frac{Cx}{(1+x^2)\log(e+x^2)} dx \quad (6)$$

$$= \int_{-\infty}^{\infty} \frac{C}{(x^{-1}+x)\log(e+x^2)} dx \quad (7)$$

note that

$$\frac{C}{(x^{-1}+x)\log(e+x^2)} \sim \frac{C}{2x\log(x)} \quad (8)$$

and

$$\int_a^{\infty} \frac{C}{2x\log(x)} dx, \text{ and } \int_{-\infty}^b \frac{C}{2x\log(x)} dx \quad (9)$$

do not converge so $\mathbb{E}[X]$ does not exist.

We now just have to show that $\hat{m}u(u)$ is differentiable at $u = 0$.

$$\hat{\mu}(u) = \int_{-\infty}^{\infty} \frac{e^{iux}C}{(1+x^2)\log(e+x^2)} dx \quad (10)$$

and

$$\left. \frac{d}{du} \hat{\mu}(u) \right|_{u=0} = \int_{-\infty}^{\infty} \frac{ixe^{iux}C}{(1+x^2)\log(e+x^2)} dx \quad (11)$$

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Let μ be the binomial distribution with n trials and probability of success p , that is $\mu = \text{Bin}(n, p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

4.1

We wish to verify that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$. Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that $(1 - p + pe^{iu})$ is the characteristic function for *Bernoulli*(p).

If ν is the Bernoulli measure and X has law ν then

$$\hat{\nu}(u) = \mathbb{E}[\exp(iuX)] \quad (12)$$

$$= \sum_{k \in \{0,1\}} e^{iuk} \nu_X(k) \quad (13)$$

$$= pe^{iu} + (1 - p). \quad (14)$$

Then by repeated application of the convolution theorem we get that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$.

4.2

We wish to verify that $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$. The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!} e^{-\lambda} \quad (15)$$

and thus

$$\mathbb{E}[\exp(iuX)] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} \quad (16)$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (e^{iu})^k \quad (17)$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} \quad (18)$$

$$= e^{-\lambda} e^{\lambda e^{iu}} \quad (19)$$

$$= e^{\lambda(e^{iu} - 1)} \quad (20)$$

4.3

We wish to show that if p_n is a sequence in $[0, 1]$ such that $p_n \downarrow 0$ and $np_n \rightarrow \lambda$ then $\mu_n \rightarrow \nu$ in the weak sense where $\mu_n = \text{Bin}(n, p_n)$.

Let $f \in C_b$ then

$$E[f(X_n)] = \sum_{k=0}^n f(k) \mu_n(k) \quad (21)$$

$$= \sum_{k=0}^n f(k) \mu_n(k) \quad (22)$$

$$= \sum_{k=0}^n f(k) \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad (23)$$

then

$$\lim_{n \rightarrow \infty} E[f(X_n)] = \lim_{n \rightarrow \infty} \sum_{k=0}^n f(k) \binom{n}{k} p_n^k (1 - p_n)^{n-k} \quad (24)$$

and

$$\mathbb{E}[f(Y)] = f(x) \nu(x) \quad (25)$$

$$= \sum_{k=0}^{\infty} f(k) \frac{\lambda^k}{k!} e^{-\lambda} \quad (26)$$

$$(27)$$

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5.1

We wish to show that $\mathbb{P}(B_n) = 1/2$ for every $n \geq 1$. Note that

$$B_n = \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \quad (28)$$

and thus

$$\mathbb{P}(B_n) = \mathbb{P} \left(\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \right) \quad (29)$$

$$= \sum_{k=0}^{2^{n-1}-1} \mathbb{P} \left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n} \right) \right) \quad (30)$$

$$= \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^n} \quad (31)$$

$$= 2^{n-1} \frac{1}{2^n} \quad (32)$$

$$= \frac{1}{2}. \quad (33)$$

5.2

We now wish to show that the sequence of events B_n form an infinite sequence of independent events. Take a finite subset $J \subset \mathbb{N}$ with $|J| = m$ and $\max J = r$ then

$$\mathbb{P}\left(\bigcap_{n \in J} B_n\right) = \mathbb{P}\left(\bigcap_{n \in J} \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) \quad (34)$$

$$= \mathbb{P}\left(\bigcup_{k=0}^{2^{r-m}-1} \left[\frac{2k}{2^r}, \frac{2k+1}{2^r}\right)\right) \quad (35)$$

$$= \sum_{k=0}^{2^{r-m}-1} \mathbb{P}\left(\left[\frac{2k}{2^r}, \frac{2k+1}{2^r}\right)\right) \quad (36)$$

$$= 2^{r-m} \frac{1}{2^r} \quad (37)$$

$$= \frac{1}{2^m} \quad (38)$$

$$= \prod_{n \in J} \mathbb{P}(B_n) \quad (39)$$

and so the sequence of events B_n form an infinite sequence of independent events.

5.3

We wish to show / argue that the probability that a randomly sampled number ω will have the sequence 5825 occur infinitely often in its decimal expansion is 1.

We use the Borel-Cantelli lemma. Ignoring possible overlaps (on the 5s) we can see that we can break any decimal expansion of ω up into blocks of 4 digits.

Then by we can define E_i as the probability of obtaining 5285 in the i -th block position. By the same argument as above these events are independent.

The for any i we have that $\mathbb{P}(E_i) = \frac{1}{10000}$ (the same argument as above applied to a decimal expansion). Then clearly

$$\mathbb{P}(E_i) = \infty. \quad (40)$$

By the Borel-Cantelli lemma this implies

$$\mathbb{P}(\limsup_n (E_n)) = 1. \quad (41)$$

Now

$$\limsup_n (E_n) = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j \quad (42)$$

can be intuitively read as E_j happens infinitely often. Which is to say that 5285 occurs *blockwise* in the expansion of ω infinitely often. Clearly as allowing for overlaps allows for more configurations then the probability is 1 (it can be no more).