University of New South Wales

School of Mathematics and Statistics

Assignment 2

Measure Theory

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2.1

Suppose u and ν are σ-finite *positive* measures on $(\Omega, \mathcal{F})$. Then suppose that $\mu << \nu$ and $\nu << \mu$. Then for $A \in \mathcal{F}$ we have that $\mu(a) = 0 \Rightarrow \nu(A) = 0$ and $\nu(A) = 0 \Rightarrow \mu(A)$. That is to say that $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$. That is to say that ν and μ have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.
We now wish to show that there is an $\mathcal{F}$-measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$.

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3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u\rangle)] \tag{1}$$

and so for the random vector cX with $c \in \mathbb{R}$ we have that

$$\begin{aligned} \hat{\mu}_{cX}(u) &= \mathbb{E}[\exp(i\langle cX, u\rangle)] &(2)\\ &= \mathbb{E}[\exp(i\langle X, cu\rangle)] &(3)\\ &= \hat{\mu}_X(cu) &(4) \end{aligned}$$

3.2

3.3

Let $d = 1$ and let μ have the Lebesgue density,

$$f(x) = \frac{C}{(1 + x^2)\log(e + x^2)}, \qquad x \in \mathbb{R}. \tag{5}$$

We wish to show that $E[X]$ is not defined but $\hat{\mu}(u)$ is differentiable at 0. Firstly we show that $E[X]$ is not defined.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} \frac{Cx}{(1 + x^2)\log(e + x^2)} dx &(6)\\ &= \int_{-\infty}^{\infty} \frac{C}{(x^{-1} + x)\log(e + x^2)} &(7) \end{aligned}$$

note that

$$\frac{C}{(x^{-1} + x)\log(e + x^2)} \sim \frac{C}{2x\log(x)} \tag{8}$$

and

$$\int_a^{\infty} \frac{C}{2x\log(x)} dx, \text{ and } \int_{-\infty}^{b} \frac{C}{2x\log(x)} dx \tag{9}$$

do not converge so $\mathbb{E}[X]$ does not exist.
We now just have to show that $\hat{mu}(u)$ is differentiable at $u = 0$.

$$\hat{\mu}(u) = \int_{-\infty}^{\infty} \frac{e^{iux}C}{(1+x^2)\log(e+x^2)}dx \tag{10}$$

and

$$\frac{d}{du}\hat{\mu}(u)\Big|_{u=0} = \int_{-\infty}^{\infty} \frac{ixe^{iux}C}{(1+x^2)\log(e+x^2)}dx \tag{11}$$

4

Let μ be the binomial distribution with n trials and probability of success p, that is $\mu = \mathrm{Bin}(n, p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

4.1

We wish to verify that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$. Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that $(1-p+pe^{iu}$ is the characteristic function for $Bernoulli(p)$.
If ν is the Bernoulli measure and X has law ν then

$$\begin{aligned}
\hat{\nu}(u) &= \mathbb{E}[\exp(iuX)] && (12)\\
&= \sum_{k\in\{0,1\}} e^{iuk}\nu_X(k) && (13)\\
&= pe^{iu} + (1-p). && (14)
\end{aligned}$$

Then by repeated application of the convolution theorem we get that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$.

4.2

We wish to verify that $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$. The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!}e^{-\lambda} \tag{15}$$

and thus

$$\begin{aligned}
\mathbb{E}[\exp(iuX)] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda}e^{iuk} && (16)\\
&= e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(e^{iu})^k && (17)\\
&= e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} && (18)\\
&= e^{-\lambda}e^{\lambda e^{iu}} && (19)\\
&= e^{\lambda(e^{iu}-1)} && (20)
\end{aligned}$$

4.3

We wish to show that if p_n is a sequence in $[0,1]$ such that $p_n \downarrow 0$ and $np_n \longrightarrow \lambda$ then $\mu_n \longrightarrow \nu$ in the weak sense where $\mu_n = \text{Bin}(n, p_n)$.
Let $f \in C_b$ then

$$
\begin{aligned}
E[f(X_n)] &= \sum_{k=0}^{n} f(k)\mu_n(k) &(21)\\
&= \sum_{k=0}^{n} f(k)\mu_n(k) &(22)\\
&= \sum_{k=0}^{n} f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k} &(23)
\end{aligned}
$$

then

$$
\lim_{n\longrightarrow\infty} E[f(X_n)] = \lim_{n\longrightarrow\infty}\sum_{k=0}^{n} f(k)\binom{n}{k}p_n^k(1-p_n)^{n-k} \tag{24}
$$

and

$$
\begin{aligned}
\mathbb{E}[f(Y)] &= f(x)\nu(x) &(25)\\
&= \sum_{k=0}^{\infty} f(k)\frac{\lambda^k}{k!}e^{-\lambda} &(26)\\
& &(27)
\end{aligned}
$$

5

5.1

We wish to show that $\mathbb{P}(B_n) = 1/2$ for every $n \geq 1$. Note that

$$
B_n = \bigcup_{k=0}^{2^{n-1}-1}\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \tag{28}
$$

and thus

$$
\begin{aligned}
\mathbb{P}(B_n) &= \mathbb{P}\left(\bigcup_{k=0}^{2^{n-1}-1}\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) &(29)\\
&= \sum_{k=0}^{2^{n-1}-1}\mathbb{P}\left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) &(30)\\
&= \sum_{k=0}^{2^{n-1}-1}\frac{1}{2^n} &(31)\\
&= 2^{n-1}\frac{1}{2^n} &(32)\\
&= \frac{1}{2}. &(33)
\end{aligned}
$$

5.2

We now wish to show that the sequence of events B_n form an infinite sequence of independent events. Take a finite subset $J \subset \mathbb{N}$ with $|J| = m$ and $\max J = r$ then

$$\mathbb{P}\left(\bigcap_{n\in J} B_n\right) = \mathbb{P}\left(\bigcap_{n\in J} \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) \tag{34}$$

$$= \mathbb{P}\left(\bigcup_{k=0}^{2^{r-m}-1} \left[\frac{2k}{2^r}, \frac{2k+1}{2^r}\right)\right) \tag{35}$$

$$= \sum_{k=0}^{2^{r-m}-1} \mathbb{P}\left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) \tag{36}$$

$$= 2^{r-m}\frac{1}{2^r} \tag{37}$$

$$= \frac{1}{2^m} \tag{38}$$

$$= \prod_{n\in J} \mathbb{P}\left(B_n\right) \tag{39}$$

and so the sequence of events B_n form an infinite sequence of independent events.

5.3