University of New South Wales

School of Mathematics and Statistics

Assignment 2

Measure Theory

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2.1

Suppose u and ν are σ-finite *positive* measures on $(\Omega, \mathcal{F})$. Then suppose that $\mu << \nu$ and $\nu << \mu$. Then for $A \in \mathcal{F}$ we have that $\mu(a) = 0 \Rightarrow \nu(A) = 0$ and $\nu(A) = 0 \Rightarrow \mu(A)$. That is to say that $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$. That is to say that ν and μ have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.
We now wish to show that there is an $\mathcal{F}$-measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$.

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3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u\rangle)] \tag{1}$$

and so for the random vector cX with $c \in \mathbb{R}$ we have that

$$\begin{aligned} \hat{\mu}_{cX}(u) &= \mathbb{E}[\exp(i\langle cX, u\rangle)] && (2) \\ &= \mathbb{E}[\exp(i\langle X, cu\rangle)] && (3) \\ &= \hat{\mu}_X(cu) && (4) \end{aligned}$$

3.2

By definition we have that

$$\hat{\mu}(\mathbf{u}) = \int \exp(i\langle \mathbf{x}, \mathbf{u}\rangle) d\mathbb{P}_X(\mathbf{x}). \tag{5}$$

We can say that

$$\frac{d^\alpha \hat{\mu}(\mathbf{u})}{d\mathbf{u}^\alpha} = \frac{d^\alpha}{d\mathbf{u}^\alpha} \int \exp(i\langle \mathbf{x}, \mathbf{u}\rangle) d\mathbb{P}_X(\mathbf{x}). \tag{6}$$

We wish to justify taking this derivative through the integral sign. To do this we use an extension of corollary 2.28 (2) from the notes.

Claim
If the partial derivative

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \tag{7}$$

exists for all $(\mathbf{x}, \mathbf{u}) \in X \times [a, b]^d$ and if there is a function $g \in L^1(\mu)$ such that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) \right| \leq g \tag{8}$$

for every $\mathbf{x} \in X$ and $\mathbf{u} \in (a, b)^d$ then

$$\frac{d^\alpha}{d\mathbf{u}^\alpha} \int f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) = \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} f(\mathbf{x}, \mathbf{u}) d\mu(\mathbf{x}) \qquad \text{for } \mathbf{u} \in (a, b)^d. \tag{9}$$

The proof follows from induction on the dimension of $\mathbf{x}$ and $\mathbf{u}$ and the order of the derivative. The base case is specifically the statement of corollary 2.28 (2) from the notes.

We have that

$$\frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) = i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \tag{10}$$

and that

$$\left| \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) \right| \leq \underbrace{\prod_{k=1}^d |x_k|^{\alpha_k}}_{\circledast}. \tag{11}$$

Now because

$$\mathbf{E}\left(\prod_{k=1}^d |X_k|^{\alpha_k} \right) = \int \prod_{k=1}^d |x_k|^{\alpha_k} \, d\mathbb{P}_X(\mathbf{x}) < \infty \tag{12}$$

then $\circledast \in L^1(\mathbb{P}_X)$ and so we can apply our claim (the DCT) to get

$$\begin{aligned}
\frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha} &= \int \frac{\partial^\alpha}{\partial \mathbf{u}^\alpha} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) && (13)\\
&= \int i^{|\alpha|} \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) && (14)\\
&= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbb{P}_X(\mathbf{x}) && (15)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial^\alpha \hat{\mu}(\mathbf{u})}{\partial \mathbf{u}^\alpha}\Big|_{\mathbf{u}=\mathbf{0}} &= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} \exp(i\langle \mathbf{x}, \mathbf{0} \rangle) d\mathbb{P}_X(\mathbf{x}) && (16)\\
&= i^{|\alpha|} \int \prod_{k=1}^d x_k^{\alpha_k} d\mathbb{P}_X(\mathbf{x}) && (17)\\
&= i^{|\alpha|} \mathbb{E}(X^\alpha) && (18)
\end{aligned}$$

3.3

Let $d = 1$ and let μ have the Lebesgue density,

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \qquad x \in \mathbb{R}. \tag{19}$$

We wish to show that $E[X]$ is not defined but $\hat{\mu}(u)$ is differentiable at 0. Firstly we show that $E[X]$ is not defined.

$$E[X] = \int_{-\infty}^{\infty} \frac{Cx}{(1+x^2)\log(e+x^2)} dx \tag{20}$$

$$= \int_{-\infty}^{\infty} \frac{C}{(x^{-1}+x)\log(e+x^2)} \tag{21}$$

note that

$$\frac{C}{(x^{-1}+x)\log(e+x^2)} \sim \frac{C}{2x\log(x)} \tag{22}$$

and

$$\int_{a}^{\infty} \frac{C}{2x\log(x)} dx, \text{ and } \int_{-\infty}^{b} \frac{C}{2x\log(x)} dx \tag{23}$$

do not converge so $\mathbb{E}[X]$ does not exist.
We now just have to show that $\hat{mu}(u)$ is differentiable at $u = 0$.

$$\hat{\mu}(u) = \int_{-\infty}^{\infty} \frac{e^{iux}C}{(1+x^2)\log(e+x^2)} dx \tag{24}$$

and

$$\frac{d}{du}\hat{\mu}(u)\Big|_{u=0} = \int_{-\infty}^{\infty} \frac{ixe^{iux}C}{(1+x^2)\log(e+x^2)} dx \tag{25}$$

4

Let μ be the binomial distribution with n trials and probability of success p, that is $\mu = \text{Bin}(n,p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

4.1

We wish to verify that $\hat{\mu}(u) = (1-p+pe^{iu})^n$. Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that $(1-p+pe^{iu}$ is the characteristic function for $Bernoulli(p)$.
If ν is the Bernoulli measure and X has law ν then

$$\hat{\nu}(u) = \mathbb{E}[\exp(iuX)] \tag{26}$$

$$= \sum_{k\in\{0,1\}} e^{iuk}\nu_X(k) \tag{27}$$

$$= pe^{iu} + (1-p). \tag{28}$$

Then by repeated application of the convolution theorem we get that $\hat{\mu}(u) = (1-p+pe^{iu})^n$.

4.2

We wish to verify that $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$. The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!} e^{-\lambda} \tag{29}$$

and thus

$$\begin{aligned}
\mathbb{E}[\exp(iuX)] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} && (30)\\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (e^{iu})^k && (31)\\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} && (32)\\
&= e^{-\lambda} e^{\lambda e^{iu}} && (33)\\
&= e^{\lambda(e^{iu}-1)} && (34)
\end{aligned}$$

4.3

We wish to show that if p_n is a sequence in $[0, 1]$ such that $p_n \downarrow 0$ and $np_n \longrightarrow \lambda$ then $\mu_n \longrightarrow \nu$ in the weak sense where $\mu_n = \text{Bin}(n, p_n)$.
Let $f \in C_b$ then

$$\begin{aligned}
E[f(X_n)] &= \sum_{k=0}^{n} f(k)\mu_n(k) && (35)\\
&= \sum_{k=0}^{n} f(k)\mu_n(k) && (36)\\
&= \sum_{k=0}^{n} f(k)\binom{n}{k} p_n^k (1 - p_n)^{n-k} && (37)
\end{aligned}$$

then

$$\lim_{n \longrightarrow \infty} E[f(X_n)] = \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} f(k)\binom{n}{k} p_n^k (1 - p_n)^{n-k} \tag{38}$$

and

$$\begin{aligned}
\mathbb{E}[f(Y)] &= f(x)\nu(x) && (39)\\
&= \sum_{k=0}^{\infty} f(k)\frac{\lambda^k}{k!} e^{-\lambda} && (40)\\
& && (41)
\end{aligned}$$

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5.1

We wish to show that $\mathbb{P}(B_n) = 1/2$ for every $n \geq 1$. Note that

$$B_n = \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \tag{42}$$

and thus

$$\begin{aligned}
\mathbb{P}(B_n) &= \mathbb{P}\left(\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) && (43)\\
&= \sum_{k=0}^{2^{n-1}-1} \mathbb{P}\left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) && (44)\\
&= \sum_{k=0}^{2^{n-1}-1} \frac{1}{2^n} && (45)\\
&= 2^{n-1}\frac{1}{2^n} && (46)\\
&= \frac{1}{2}. && (47)
\end{aligned}$$

5.2

We now wish to show that the sequence of events B_n form an infinite sequence of independent events. Take a finite subset $J \subset \mathbb{N}$ with $|J| = m$ and $\max J = r$ then

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{n\in J} B_n\right) &= \mathbb{P}\left(\bigcap_{n\in J}\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) && (48)\\
&= \mathbb{P}\left(\bigcup_{k=0}^{2^{r-m}-1} \left[\frac{2k}{2^r}, \frac{2k+1}{2^r}\right)\right) && (49)\\
&= \sum_{k=0}^{2^{r-m}-1} \mathbb{P}\left(\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)\right) && (50)\\
&= 2^{r-m}\frac{1}{2^r} && (51)\\
&= \frac{1}{2^m} && (52)\\
&= \prod_{n\in J} \mathbb{P}\left(B_n\right) && (53)
\end{aligned}$$

and so the sequence of events B_n form an infinite sequence of independent events.

5.3

We wish to show / argue that the probability that a randomly sampled number ω will have the sequence 5825 occur infinitely often in its decimal expansion is 1.

We use the Borel-Cantelli lemma. Ignoring possible overlaps (on the 5s) we can see that we can break any decimal expansion of ω up into blocks of 4 digits.
Then by we can define E_i as the probability of obtaining 5285 in the i-th block possition. By the same argument as above these events are independent.
The for any i we have that $\mathbb{P}(E_i) = \frac{1}{10000}$ (the same argument as above applied to a decimal expansion). Then clearly

$$\mathbb{P}(E_i) = \infty. \tag{54}$$

By the Borel-Cantelli lemma this implies

$$\mathbb{P}(\limsup_n (E_n) = 1. \tag{55}$$

Now

$$\limsup_n (E_n) = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j \tag{56}$$

can be intuatively read as E_j happens infinitely often. Which is to say that 5285 occurs *blockwise* in the expansion of ω infinitely often. Clearly as allowing for overlaps allows for more configurations then the probability is 1 (it can be no more).