University of New South Wales

School of Mathematics and Statistics

Tutorial 2

Measure Theory

Author:
Adam J. Gray

Student Number:
3329798

1

Consider a line ℓ restricted to $[0,1]^d$. This line has length at most $\sqrt{d}$ and thus can be covered by $N = \lceil \frac{\sqrt{d}}{\varepsilon} \rceil$ *boxes* of size ε.
Where $d > 1$, ℓ is closed and hence Borel, and hence measurable and so $\mu(\ell)$ is well defined.
Then

$$\begin{aligned} \mu(\ell) &= \lim_{\varepsilon \longrightarrow 0} \varepsilon^d \times \lceil \frac{\sqrt{d}}{\varepsilon} \rceil \\ &= 0 && \text{so long as } d > 1. \end{aligned}$$

This result extends to a line in $\mathbb{R}^d$ by countable additivity. □

2

2.1

2.1.1

$$\begin{aligned} A^c \Delta B^c &= (A^c \setminus B^c) \cup (B^c \setminus A^c) \\ &= (A^c \cap B) \cup (B^c \cap A) \\ &= (B \setminus A) \cup (A \setminus B) \\ &= A \Delta B \end{aligned}$$

□

2.1.2

Suppose $x \in A$ but $x \notin C$ then
if $x \in B$, $x \notin A\Delta B$ but $X \in B\Delta C$ because $x \in C$.
If $x \in B$ then $x \in A\Delta B$.
The argument is symmetric if $x \in C$ but $x \notin A$.
Thus $A\Delta C \subseteq (A\Delta B) \cup (B\Delta C)$. □

2.2

Suppose $B \in G$, then for all $\varepsilon > 0$ $\exists$ $B_\varepsilon \in \mathcal{A}$ s.t. $\mu(B_\varepsilon \Delta B) < \varepsilon$.
Because $(B_\varepsilon \Delta B) = (B_\varepsilon^c \Delta B^c)$ then $\mu(B_\varepsilon^c \Delta B^c) = \mu(B_\varepsilon \Delta B) < \varepsilon$. Note that ε was arbitrary so it holds for all $\varepsilon > 0$ and thus $B^c \in \mathcal{G}$.
Note

$$(B \in \mathcal{G} \Longrightarrow B^c \in \mathcal{G}) \Longrightarrow \left(B^C \in \mathcal{G} \Longrightarrow B \in \mathcal{G}\right)$$

and thus the result follows. □

2.3

Since $A = \bigcup_{j=1} A_j$ and A_j is an increasing sequence

$$A = \dot{\bigcup_{j=0}} (A_j \Delta A_{j+1}) \qquad \text{where } A_0 = \emptyset$$

and thus

$$\mu(A) = \sum_{j=1} \mu(A_j \Delta A_{j+1}).$$

Because $\mu(A)$ must be finite there exists an N such that

$$\sum_{j=N} \mu(A_j \Delta A_{j+1}) < \varepsilon$$

but

$$A_n \Delta A \subseteq \dot{\bigcup_{j=N}} A_j \Delta A_{j+1}$$

and so

$$\mu(A_N \Delta A) \leq \sum_{j=N} \mu(A_j \Delta A_{j+1}) < \varepsilon$$

□

2.4

The ε used in the above argument was arbitrary so it holds for *all* $\varepsilon > 0$. Then by definition $A \in \mathcal{G}$. (because in each case $B_\varepsilon = A_N$) □

2.5

The following is clear

- $\mathcal{A} \subseteq \mathcal{G}$
- $\sigma(\mathcal{G}) = \mathcal{F}$
- $\mathcal{G}$ is a d-class (proved above)
- $\mathcal{G}$ is an algebra (and hence π-class)

thus

$$\mathcal{F} = \sigma(\mathcal{F}) = \sigma(\mathcal{G}) \underbrace{=}_{\text{by the monotone class theorem}} d(\mathcal{G}) = \mathcal{G}.$$

□

3

3.1

Note that

- $\emptyset, X \in \mathcal{A}_\mu$ because $\emptyset, X \in \mathcal{A}$
- If $E, F \in \mathcal{A}$ such that $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$ for some $A \in \mathcal{A}_\mu$, then $F^c \subseteq A^c \subseteq E^c$ and $\mu(E^c \setminus F^c) = 0$ and so $A^c \in \mathcal{A}_\mu$.
- If $E_i, F_i \in \mathcal{A}$ such that $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i)$ for some countable collection $\{A_i\}_{i\in\mathbb{Z}} \subseteq \mathcal{A}_\mu$ then

$$\underbrace{\bigcup_{i\in\mathbb{Z}} E_i}_{\circledast} \subseteq \bigcup_{i\in\mathbb{Z}} A_i \subseteq \underbrace{\bigcup_{i\in\mathbb{Z}} F_i}_{\circledast\circledast}$$

and because $\mathcal{A}$ is a σ-algebra, $\circledast$ and $\circledast\circledast \in \mathcal{A}$.

Also

$$\begin{aligned}
\mu\left(\bigcup_{i\in\mathbb{Z}} F_i \setminus \bigcup_{i\in\mathbb{Z}} F_i\right) &= \mu\left(\bigcup_{i\in\mathbb{Z}} F_i \cap \left(\bigcup_{i\in\mathbb{Z}} E_i\right)^c\right) \\
&= \mu\left(\bigcup_{i\in\mathbb{Z}} F_i \cap \left(\bigcap_{i\in\mathbb{Z}} E_i^c\right)\right) \\
&\leq \mu\left(\bigcup_{i\in\mathbb{Z}} (F_i \cap E_i^c)\right) \\
&= \mu\left(\bigcup_{i\in\mathbb{Z}} (F_i \setminus E_i^c)\right) \\
&\leq \sum_{i\in\mathbb{Z}} \mu(F_i \setminus E_i) \\
&= 0.
\end{aligned}$$

Thus $\mathcal{A}_\mu$ is closed with respect to countable unions and hence $\mathcal{A}_\mu$ is a σ-algebra.

$\square$

3.2

Suppose $\{A_i\}_{i\in\mathbb{Z}} \subseteq \mathcal{A}_u$ such that $A_i \cap A_j = \emptyset$ when $j \neq i$ with $E_i, F_i \in \mathcal{A}$ such that $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i) = 0$. Then

$$\begin{aligned}\overline{\mu}\left(\bigcup_{i\in\mathbb{Z}} A_i\right) &= \mu\left(\bigcup_{i\in\mathbb{Z}} E_i\right) \\ &= \sum_{i\in\mathbb{Z}} \mu(E_i) && \text{because the } E_i \text{ must be disjoint} \\ &= \sum_{i\in\mathbb{Z}} \overline{\mu}(A_i).\end{aligned}$$

This means that $\overline{\mu}$ is countably additive and thus $\overline{\mu}$ is a measure on $\mathcal{A}_\mu$. $\square$

3.3

Suppose $A \in \mathcal{A}$ with $E \subseteq A \subseteq F$ such that $\mu(F \setminus E) = 0$ and $E, F \in \mathcal{A}$. Then $\mu(E) \leq \mu(A) \leq \mu(F)$ but $\mu(E) = \mu(F)$ and hence $\mu(A) = \mu(E) = \overline{\mu}(A)$. $\square$

3.4

Suppose $D \in 2^X$ and $D \subseteq A \in \mathcal{A}_\mu$ such that $\overline{\mu}(A) = 0$. Then $\emptyset \subseteq D \subseteq A$ and $\mu(A \setminus \emptyset) = \mu(A) - \mu(\emptyset) = 0$. Thus $D \in \mathcal{A}_\mu$ and so $(X, \mathcal{A}_\mu, \overline{\mu})$ is complete. $\square$