University of New South Wales

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# Assignment 1

Measure Theory

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# 1

## 1.1

Define

$$\ell_n = \sum_{k=1}^{N_n} \alpha_k \chi_{C_k}$$
$$u_n = \sum_{k=1}^{N_n} \beta_k \chi_{C_k}$$

where $\alpha_k := \inf\{f(x) : x \in C_k\}$ and $\beta_k := \sup\{f(x) : x \in C_k\}$ and $C_k \in \mathcal{P}_n$ where $\mathcal{P}$ is defined in the question. We wish to show that

$$\lim_{n \longrightarrow \infty} |\ell - f| = 0 = \lim_{n \longrightarrow \infty} |f - u_n|$$

$\lambda$ a.e.
It is obvious that $\ell \leq f \leq u_n$ and so proving

$$\lim_{n \longrightarrow \infty} |u_n - \ell_n| = 0$$

is sufficient. Define

$$\phi_n := u_n - \ell_n = \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k}.$$

Firstly we must show $\lim_{n \longrightarrow \infty} \phi_n$ exists $\lambda$ a.e. This follows from the fact that $\ell_n$ ( $u_n$ ) is a non-decreasing (non-increasing) sequence which is bounded above (below) by $f$ which is also bounded.
We now wish to establish that $\lim_{n \longrightarrow \infty} \phi_n = 0$. Note that

$$\phi_n \leq \sup\{f(x) - f(y) : x, y \in S\} \leq K \chi_S$$

for some $K$ because $f$ is bounded. Now because $S$ is bounded in $\mathbb{R}^d$ we have that $K\chi_S \in \mathcal{L}^1(S)$. The dominated convergence theorem therefore allows us to write

$$\begin{aligned}
\int \lim_{n \longrightarrow \infty} \underbrace{\sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k}}_{=\phi_n} d\lambda &= \lim_{n \longrightarrow \infty} \int \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k} d\lambda \\
&= \lim_{n \longrightarrow \infty} \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \lambda(C_k) \text{ because } \phi_n \text{ is a simple function} \\
&= 0 \text{ because } f \text{ is Riemann integrable .}
\end{aligned}$$

This means that $\lim_{n \longrightarrow \infty} \phi_n = 0$ and hence $\lim_{n \longrightarrow \infty} \ell_n = f = \lim_{n \longrightarrow \infty} \ell_n$.

We now show that the Riemann integral and the Lebesgue integral coincide. We have that

$$|\ell_n| \leq M\chi_S$$

for some $M$ because $f$ is bounded. By the same argument as above $M\chi_S \in \mathcal{L}^1(S)$. $\lim_{n\longrightarrow\infty} \ell_n$ exists and equals $f$ (this was established above), so by the dominated convergence theorem,

$$\begin{aligned}\underbrace{\lim_{n\longrightarrow\infty}\int \ell_n d\lambda}_{\int_S f(x)dx} &= \int \underbrace{\lim_{n\longrightarrow\infty}\ell_n}_{\circledast} d\lambda \\ &= \underbrace{\int f d\lambda}_{\text{Lebesgue integral}}\end{aligned}$$