# A Stochastic Approach to Fractional Diffusion

## help I'm trapped in a LaTeX compiler


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# Outline

- Standard diffusion
- Super diffusion
- Sub diffusion
- Solving a fractional differential equation

# Central Limit theorem and Random Walks

## Random Walk

Let $X_1, \dots X_n$ be a sequence of random variables, then

$$S_n = \sum_{k=1}^{n} X_n \tag{1}$$

represents the position after $n$ steps.

If $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 2$ then the central limit theorem gives us that

$$\frac{S_n}{\sqrt{n}} \longrightarrow Z \tag{2}$$

*weakly* as $n \longrightarrow \infty$, where $Z \sim \mathcal{N}(0, 2)$

# Central Limit theorem and Random Walks

We can extend this idea by introducing a scaling paramter $\gamma$.

$$S_{\lfloor \gamma t \rfloor} = \sum_{k=1}^{\lfloor \gamma t \rfloor} X_k. \tag{3}$$

$\gamma$ has the effect of changing the *timescale* that we are considering the process running over.
We can calculate the characteristic function of

$$\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}}. \tag{4}$$

By the convolution theorems we can say that it is

$$\left(1 - \frac{k^2}{\gamma} + o(\gamma^{-1})\right)^{\lfloor \gamma t \rfloor} \tag{5}$$

# Long Time Limit

We can rearrange

$$\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\lfloor\gamma t\rfloor}=\left[\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\gamma}\right]^{\frac{\lfloor\gamma t\rfloor}{\gamma}} \tag{6}$$

and then take $\gamma \longrightarrow \infty$ to get that

$$\left[\underbrace{\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\gamma}}_{\circledast}\right]^{\frac{\lfloor\gamma t\rfloor}{\gamma}} \longrightarrow e^{-tk^2} \tag{7}$$

by using the well known result

$$\lim_{n\longrightarrow\infty}\left(1+\frac{x}{n}\right)^n=e^x. \tag{8}$$

# Fourier Transform and a PDE

Notice that $e^{-tk^2}$ is just the CF of $Z_t \sim \mathcal{N}(0, 2t)$. Further we can say that $\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}} \longrightarrow Z_t$ by LCT.

We can also regard $e^{-tk^2}$ as the FT of the density of $Z_t$.

Rather usefully we have also have that $e^{-tk^2}$ is a solution to

$$\frac{d\hat{u}}{dt} = -k^2 \hat{u} \tag{9}$$

We can actually use a analogous result from last week's homework to invert the Fourier transform of $u$ to get

$$\frac{\partial u}{dt} = \frac{\partial^2 u}{\partial x^2} \tag{10}$$

# A More General Result

We can generalise this result a bit and say that if $\mathbb{E}[X_i^2] = \sigma^2$ then $\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}} \longrightarrow Z_t$, but now $Z_t \sim \mathcal{N}(0, 2\sigma^2 t)$.
Further the Fourier transform of the density of $Z_t$ is $e^{-t\sigma^2 k^2}$ which means the density is now a solution to

$$\frac{\partial u}{\partial t} = \underbrace{\frac{\sigma^2}{2}}_{D} \frac{\partial^2 u}{\partial x^2}. \tag{11}$$

What this says is that as particles jump around more, they diffuse more rapidly.

# Riemann-Liouville vs Caputo Derivative

Note!

The Caputo derivative and the and the Riemann-Liouville derivatives are note the same. In general

$$\left({}^{C}\mathcal{D}_a^{\alpha} f\right)(x) \neq \left(\mathcal{D}_a^{\alpha} f\right)(x).$$

The reason is exactly the same reason that in general

$$f(x) \neq \int_a^x f'(t)dt.$$

In some sense if you differentiate first you "lose information" about the function.

# Riemann-Liouville vs Caputo Derivative

The Caputo derivative is often used in fractional differential equations because it can be coupled with integer order initial conditions, whereas often the Riemann-Liouville derivative can't be coupled with integer order initial conditions.

# A Quick Note on the Laplace Transform

**Definition**

We the define the Laplace transform of a function $f$ to be the function $F$ given by

$$F(s) := \int_0^\infty e^{-st} f(t) dt$$

We often write $F(s) = \mathcal{L}\{f(t)\}$.

# A Quick Note on the Laplace Transform

The Laplace transform is particularly useful as it allows us to transform a differential equation into an "algebraic" equation. Lerch's theorem guarantees, with minor caveats, that the Laplace transform of a function is unique.

# Basic Idea of the Laplace Transform Method

- Apply the Laplace transform to both sides of the differential equation to get and "algebraic" equation.
- Apply the Laplace transform to the initial conditions.
- Sub the transformed initial conditions into the transformed equation.
- Rearrange to get an expression for the Laplace transform of the function of interest.
- Invert. (This is possible, and guaranteed with minor caveats by Lerch's theorem)

# The Laplace Transform of the Riemann-Liouville Integral

**Lemma**

*The Laplace transform of the Riemann-Liouville integral of a function $f$ is given by*

$$\mathcal{L}\left\{I_0^\alpha f\right\} = s^{-\alpha}\mathcal{L}\left\{f\right\}.$$

# The Laplace Transform of the Riemann-Liouville Integral[Proof]

Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of $f$ with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{f(t)\right\} \underbrace{\mathcal{L}\left\{t^{\alpha-1}\right\}}_{=s^{-\alpha}\Gamma(\alpha)} \\
&= s^{-\alpha} \mathcal{L}\left\{f\right\}.
\end{aligned}$$

# The Laplace Transform of the Caputo Derivative

The Laplace transform of the Caputo derivative of a function $f$ is given by

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].$$

# The Laplace Transform of the Caputo Derivative [Proof]

See that

$$\begin{aligned}\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} &= \mathcal{L}\left\{\left(I_0^{n-\alpha}\frac{d^n f}{dt^n}\right)\right\}\\ &= \underbrace{\frac{1}{\Gamma(n-\alpha)}\mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\}}_{\circledast}\end{aligned}$$

# The Laplace Transform of the Caputo Derivative [Proof]

$\circledast$ is just the Laplace transform of a convolution so

$$\begin{aligned}
\circledast &= \mathcal{L}\left\{t^{n-\alpha-1}\right\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\
&= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right) \\
&\quad\times\left(s^n\mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\
&= s^{\alpha-n}\left[s^n\mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].
\end{aligned}$$

$\square$

# One Parameter Mittag-Lefler Function

### Definition

The one parameter Mittag-Lefler $E_\alpha$ function is defined by its power series.

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$$

# Laplace Transform of $E_\alpha(\beta t^\alpha)$

Lemma

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

## Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

See that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \int_0^\infty e^{-st} \sum_{k=0}^{\infty} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ we may interchange the integral and the sum to get

$$\begin{aligned}
\int_0^\infty e^{-st} \sum_{k=0}^{\infty} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt &= \sum_{k=0}^{\infty} \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt \\
&= \sum_{0}^{\infty} \frac{\beta^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-st} t^{\alpha k} dt.
\end{aligned}$$

# Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

By performing the change of variables $x = st$ we get that

$$\sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt = \sum_0^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k+1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k+1)}$$

$$= \sum_{k=0}^\infty \beta^k s^{-(\alpha k+1)}$$

$$= \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

□

# Summary of Important Results

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right]$$

$$\mathcal{L}\left\{E_{\alpha}(\beta t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\beta}$$

## The Solution to the Differential Equation

**Lemma**

*The fractional differential equation,*

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t) \tag{12}$$

*along with the initial conditions*

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \tag{13}$$

*has solution* $y(t) = E_{\alpha}\left(\beta t^{\alpha}\right)$

# Proof of Proposed Solution

Taking the Laplace transform of both sides of (12) yields

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} y\right)\right\} = \beta\mathcal{L}\left\{y\right\}$$

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - \sum_{k=0}^{n-1} s^{n-k-1}y^{(k)}(0)\right] = \beta\mathcal{L}\left\{y\right\}$$

# Proof of Proposed Solution

Then taking into account (13) (the initial conditions) we get

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\{y\} - s^{n-1}\right] = \beta\mathcal{L}\{y\}$$

and so

$$\mathcal{L}\{y\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

## Proof of Proposed Solution

By by noticing that

$$\mathcal{L}\{y\} = \frac{s^{\alpha-1}}{s^{\alpha} - \beta}.$$

is the Laplace transform of $E_{\alpha}(\beta t^{\alpha})$ we have that

$$y(t) = E_{\alpha}(\beta t^{\alpha})$$

$\square$