University of New South Wales

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Assignment 1

Measure Theory

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1.1

Define

$$\ell_n = \sum_{k=1}^{N_n} \alpha_k \chi_{C_k}$$
$$u_n = \sum_{k=1}^{N_n} \beta_k \chi_{C_k}$$

where $\alpha_k := \inf\{f(x) : x \in C_k\}$ and $\beta_k := \sup\{f(x) : x \in C_k\}$ and $C_k \in \mathcal{P}_n$ where $\mathcal{P}$ is defined in the question. We wish to show that

$$\lim_{n \longrightarrow \infty} |\ell - f| = 0 = \lim_{n \longrightarrow \infty} |f - u_n|$$

λ a.e.
It is obvious that $\ell \leq f \leq u_n$ and so proving

$$\lim_{n \longrightarrow \infty} |u_n - \ell_n| = 0$$

is sufficient. Define

$$\phi_n := u_n - \ell_n = \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k}.$$

Firstly we must show $\lim_{n \longrightarrow \infty} \phi_n$ exists λ a.e. This follows from the fact that ℓ_n (u_n) is a non-decreasing (non-increasing) sequence which is bounded above (below) by f which is also bounded.
We now wish to establish that $\lim_{n \longrightarrow \infty} \phi_n = 0$. Note that

$$\phi_n \leq \sup\{f(x) - f(y) : x, y \in S\} \leq K\chi_S$$

for some K because f is bounded. Now because S is bounded in $\mathbb{R}^d$ we have that $K\chi_S \in \mathcal{L}^1(S)$. The dominated convergence theorem therefore allows us to write

$$\begin{aligned}
\int \lim_{n \longrightarrow \infty} \underbrace{\sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k}}_{=\phi_n} d\lambda &= \lim_{n \longrightarrow \infty} \int \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \chi_{C_k} d\lambda \\
&= \lim_{n \longrightarrow \infty} \sum_{k=1}^{N_n} (\beta_k - \alpha_k) \lambda(C_k) \text{ because } \phi_n \text{ is a simple function} \\
&= 0 \text{ because } f \text{ is Riemann integrable .}
\end{aligned}$$

This means that $\lim_{n \longrightarrow \infty} \phi_n = 0$, λ a.e. and hence $\lim_{n \longrightarrow \infty} \ell_n = f = \lim_{n \longrightarrow \infty} \ell_n$, λ a.e.

We now show that the Riemann integral and the Lebesgue integral coincide. We have that

$$|\ell_n| \leq M\chi_S$$

for some M because f is bounded. By the same argument as above $M\chi_S \in \mathcal{L}^1(S)$. $\lim_{n\longrightarrow\infty} \ell_n$ exists and equals f (this was established above), so by the dominated convergence theorem,

$$\begin{aligned}\underbrace{\lim_{n\longrightarrow\infty}\int \ell_n d\lambda}_{\int_S f(x)dx} &= \int \underbrace{\lim_{n\longrightarrow\infty} \ell_n}_{\circledast} d\lambda \\ &= \underbrace{\int f d\lambda}_{\text{Lebesgue integral}}\end{aligned}$$

1.2

Let $N = \{x : \lim_{n\longrightarrow\infty} \ell_n(x) \neq \lim_{n\longrightarrow\infty} u_n(x)\}$ and let $X = S/(\partial S \cup N)$. We wish to show that for all $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup\{f(x_1) - f(x_2) : ||x - x_1|| < \delta, ||x - x_2|| < \delta\} < \varepsilon.$$

Fix x and ε. As $\lim_{n\longrightarrow\infty}(u_n - \ell_n) = 0$ there exists an L such that $n > L$ implies $u_n - \ell_n < \varepsilon$ which is to say that

$$\sum_{k=1}^{N_n}(\beta_k - \alpha_k)\chi_{C_k} < \varepsilon$$

which implies that

$$(\beta_k - \alpha_k) < \varepsilon$$

where $x \in C_k$. Therefore chosing $\delta = \inf\{||y - x|| : y \notin C_k\}$ guarentees $\{y : ||y - x|| < \delta\} \subseteq C_k$ and hence

$$\sup\{f(x_1) - f(x_2) : ||x - x_1|| < \delta, ||x - x_2|| < \delta\} < \varepsilon.$$

We now need to prove that $\delta > 0$. To do this we show that there exits a partitioning of S such that

$$\underbrace{\inf\{||x_1 - x|| : x_1 \notin C_k\} > 0}_{\circledast}$$

for all $x \in X$ so long as $x \in C_k/(\partial C_k)$. We then show that there always exists a partitioning of X such that $x \notin \partial C_k$ and the result will follow.

A partitioning of S such that $\circledast$ holds for all n is given by

$$C_k = \Big(\frac{k(b_1 - a_1)}{q^n}, \frac{(k+1)(b_1 - a_1)}{q^n}\Big] \times \cdots \times \Big(\frac{k(b_d - a_d)}{q^n}, \frac{(k+1)(b_d - a_d)}{q^n}\Big]$$

where $q = 2$. Now suppose that $x \in \partial C_k$ then the $\circledast$ would still hold when $q = 3$ but $x \notin \partial C_k$.

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2.1

Note that for $E \in \mathcal{B}$

$$\begin{aligned}\left(f^{-1}(E)\right)^c &= (\{x \in X : f(x) \in E\})^c \\ &= \{x \in X : f(x) \notin E\} \\ &= \{x \in X : f(x) \in E^c\} \\ &= f^{-1}(E^c).\end{aligned}$$

So if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.
Also note that for $E_1, E_2, \ldots$ we have that

$$\begin{aligned}\bigcup_i f^{-1}(E_i) &= \bigcup_i \{x \in X : f(x) \in E_i\} \\ &= \left\{x \in X : f(x) \in \bigcup_i E_i\right\} \\ &= f^{-1}\left(\bigcup_i E_i\right).\end{aligned}$$

which means that if $F_1, F_2, \ldots$ is some countable collection of sets in $\mathcal{A}$ then their union is also in $\mathcal{A}$.
Finnaly note that $f^{-1}(Y) = X$. Therefore $\mathcal{A} := \{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra.
Clearly if we remove any set from $\mathcal{A}$ then f would not be $\mathcal{A}$ measurable because there would exist a set $B \in \mathcal{B}$ such that $f^{-1}(B) \notin \mathcal{A}$.

2.2

Define

$$\begin{aligned}g &: Y \longrightarrow Z \\ g &:= a_n \mathbb{1}_{B_n}\end{aligned}$$

Now suppose $E \in \mathcal{C}$ and note that

$$\begin{aligned} h^{-1}E &= f \circ h^{-1}E \\ &= f \bigcup_{n \in I} \underbrace{h^{-1}a_n}_{\circledast} \end{aligned}$$

where $I = \{n : a_n \in E\}$.
$\circledast$ is because if $E \in \mathcal{C}$ then if $U := \{a_n : n \in I, I \subseteq \mathbb{Z}\} \subseteq E$ then

$$h^{-1}E = h^{-1}\left(U \cup (E \setminus U)\right) = h^{-1} \bigcup_{n \in I} a_n = \bigcup_{n \in I} h^{-1}a_n.$$

Now because h is measurable then

$$D := \bigcup_{n \in I} h^{-1}a_n$$

is the *countable* union of measurable sets, hence measurable. Then we just have to argue that

$$fD \in Y.$$

This is clear because $D = f^{-1}E$ for some $E \in \mathcal{B}$ and therefore $fD = ff^{-1}E = E \in \mathcal{B}$.

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3.1

$$\begin{aligned} x \in \liminf_n A_n &\Leftrightarrow x \in \bigcup_n \bigcap_{k \geq n} A_k && (1) \\ &\Leftrightarrow \exists N \text{ such that } x \in \bigcap_{k \geq N} A_k && (2) \\ &\Leftrightarrow \exists N \text{ such that } k \geq N \implies x \in A_k && (3) \\ &\Leftrightarrow \exists N \text{ such that } k \geq N \implies \chi_{A_k}(x) = 1 && (4) \\ &\Leftrightarrow \lim_{N \longrightarrow \infty} \inf_{n \geq N} \chi_{A_n}(x) = 1 && (5) \\ &\Leftrightarrow \liminf_n \chi_{A_n}(x) = 1 && (6) \end{aligned}$$

Note that (3) is completely the same as saying that $x \in A_k$ for all but finitely many A_k ($x \notin A_k$ for at most N A_k).

3.2

$$\begin{aligned} x \in \limsup_n A_n \\ \limsup_n \chi_{A_n}(x) = 1 \\ x \in A_n \text{ for infinitely many } n \end{aligned}$$

Clearly the third condition here is less restrictive than the third contion in 3.1 and so $\liminf_n A_n \subseteq \limsup_n A_n$.

4

4.1

Clearly

$$\emptyset \in \mathcal{A} \Longrightarrow \emptyset \in \mathcal{A}_c$$

also if we let $A \in \mathcal{A}_c$

$$A^c := A^c \cap C$$

and $A^c \cap C \in \mathcal{A}_c$ because $A^c \in \mathcal{A}$.
Now let $\{A_n\}_n \subseteq \mathcal{A}_c$ where $A_n = A_n' \cap C$ with $A_n' \in \mathcal{A}$ then

$$\begin{aligned} \bigcup_n A_n &= \bigcup_n (A_n' \cap C) \\ &= \bigcup_n (A_n' \cap C) \\ &= C \cap \underbrace{\bigcup_n (A_n')}_{\in \mathcal{A}} \end{aligned}$$

and thus

$$\bigcup_n A_n \in \mathcal{A}_c.$$

4.2

Let $\{A_n\}_n \subseteq \mathcal{A}$ be a sequence of sets such that $A_n \subseteq A_{n+1}$, $A_n \subseteq B$ for all n and

$$\lim_{n \longrightarrow \infty} \mu(A_n) = \sup_n \{\mu(A) : A \subseteq B\}.$$

Firstly we show such a sequence exists.
Suppose such a sequence did not exist. Then for all sequences $\{E_n\}_n \subseteq \mathcal{A}$ there exists a $\varepsilon > 0$ such that

$$\lim_{n \longrightarrow \infty} \mu(E_n) \leq \sup_n \{\mu(A) : A \subseteq B\} - \varepsilon.$$

Now because the sup must exist because this is a finite measure space there must exist an $A' \in \mathcal{A}$ such that

$$\mu(A') - \sup\{\mu(A) : A \subseteq B\} < \frac{\varepsilon}{2}.$$

Then we can construct a sequence $\{E_n \cup A'\}_n \subseteq \mathcal{A}$ such that

$$\lim_{n \longrightarrow \infty} \mu(E_n \cup A') > \sup_n \{\mu(A) : A \subseteq B\} - \varepsilon$$

which is a contradiction.
Returning to the sequence $\{A_n\}_n$ note that

$$\bigcup_n A_n \in \mathcal{A}$$

and that

$$\lim_{n \longrightarrow \infty} \mu(A_n) = \mu\left(\bigcup_n A_n\right)$$

and hence there exists an $A_0 := \cup_n A_n$ such that $\mu(A_0) = \mu_*(B)$.
Using a symetric argument to that above we can argue that there exists a sequence $\{A_n\}_n \subseteq \mathcal{A}$ such that $A_{n+1} \subseteq A_n$, $A_n \supseteq B$ for all n and that

$$\lim_{n \longrightarrow \infty} \mu(A_n) = \inf\{\mu(A) : A \supseteq B\}.$$

Now note that

$$\bigcap_n A_n \in \mathcal{A}$$

and so

$$\lim_{n \longrightarrow \infty} \mu(A_n) = \mu\left(\bigcap_n A_n\right)$$

and thus there exists $A_1 = \cap_n A_n$ such that $\mu(A_1) = \mu^*(B)$.

4.3

A_1 and A_2 must be $\mathcal{M}(\mu*)$ measurable which means that for C we must have

$$\mu^*(C) = \mu^*(C \cap A_1) + \mu^*(C \setminus A_1)$$

and

$$\mu^*(C) = \mu^*(C \cap A_2) + \mu^*(C \setminus A_2)$$

It is not hard to see that

$$\mu^*(C \setminus A_1) \leq \mu(C_1 \setminus A_1).$$

Clearly $(C_1 \setminus A_1) \supseteq (C \setminus A_1)$ and $(C_1 \setminus A_1) \in \mathcal{A}$ so

$$\inf\{\mu(E) : E \supseteq (C \setminus A_1), E \in \mathcal{A}\}$$

thus $\mu^*(C \setminus A_1) \leq \mu(C_1 \setminus A_1)$. By the same reasoning $\mu^*(C \setminus A_2) \leq \mu(C_1 \setminus A_2)$. We can then see that

$$\mu(C_1) = \mu^*(C) \leq \mu^*(C \cap A_1) + \mu(C_1 \setminus A_1)$$

and so

$$\mu(C_1 \cap A_1) \leq \mu^*(C \cap A_1)$$

and likewise

$$\mu(C_1 \cap A_2) \leq \mu^*(C \cap A_2)$$

The reverse inequality holds because $(C_1 \cap A_1) \in \mathcal{A}$ and

$$\inf\{\mu(E) : E \supseteq (C \cap A_1), E \in \mathcal{A}\} \leq \mu(C_1 \cap A_1).$$

Likewise for A_2. So $\mu^*(C \cap A_1) = \mu(C_1 \cap A_1)$. Now because $A_1 \cap C = A_2 \cap C$ the it is clear to see that $\mu^*(C \cap A_1) = \mu^*(C \cap A_2)$ and hence $\mu(C_1 \cap A_1) = \mu(C_1 \cap A_2)$.

4.4

In the arguments above we show that for $A \in \mathcal{A}$ we have $\mu^*(A \cap C) = \mu(A \cap C_1)$ and C_1 was arbitrary, so long as $\mu(C_1) = \mu^*(C)$. In 4.2 we showed at least one such C_1 must exist and so μ_C is well defined and $\mu_C = \mu^*$.

4.5

Let $\{A_{C_n}\}_n \subseteq \mathcal{A}_C$ such that the A_{C_n} are all mutually disjoint. Then

$$\begin{aligned}
\mu_C\left(\bigcup_n A_{C_n}\right) &= \mu\left(\left(\bigcup_n A_{C_n}\right) \cap C_1\right) \\
&= \mu\left(\bigcup_n (A_{C_n} \cap C_1)\right) \\
&= \sum_n \mu(A_{C_n} \cap C_1) \text{ because the } (A_{C_n} \cap C) \text{ must be mutually disjoint} \\
&= \sum_n \mu_C(A_{C_n})
\end{aligned}$$

and so μ_C is a measure.