A Stochastic Approach to Fractional Diffusion

help I'm trapped in a LaTeX compiler

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Outline

- Standard diffusion
- Super diffusion
- Sub diffusion
- Solving a fractional differential equation

Central Limit Theorem and Random Walks

Random Walk

Let $X_1, \dots X_n$ be a sequence of random variables, then

$$S_n = \sum_{k=1}^{n} X_n \tag{1}$$

represents the position after n steps.

If $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 2$ then the central limit theorem gives us that

$$\frac{S_n}{\sqrt{n}} \longrightarrow Z \tag{2}$$

weakly as $n \longrightarrow \infty$, where $Z \sim \mathcal{N}(0, 2)$

Central Limit Theorem and Random Walks

We can extend this idea by introducing a scaling paramter γ.

$$S_{\lfloor \gamma t \rfloor} = \sum_{k=1}^{\lfloor \gamma t \rfloor} X_k. \tag{3}$$

γ has the effect of changing the *timescale* that we are considering the process running over.
We can calculate the characteristic function of

$$\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}}. \tag{4}$$

By the convolution theorems we can say that it is

$$\left(1 - \frac{k^2}{\gamma} + o(\gamma^{-1})\right)^{\lfloor \gamma t \rfloor} \tag{5}$$

Long Time Limit

We can rearrange

$$\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\lfloor\gamma t\rfloor}=\left[\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\gamma}\right]^{\frac{\lfloor\gamma t\rfloor}{\gamma}} \tag{6}$$

and then take $\gamma\longrightarrow\infty$ to get that

$$\left[\underbrace{\left(1-\frac{k^2}{\gamma}+o(\gamma^{-1})\right)^{\gamma}}_{\circledast}\right]^{\frac{\lfloor\gamma t\rfloor}{\gamma}}\longrightarrow e^{-tk^2} \tag{7}$$

by using the well known result

$$\lim_{n\longrightarrow\infty}\left(1+\frac{x}{n}\right)^n=e^x. \tag{8}$$

Fourier Transform and a PDE

Notice that e^{-tk^2} is just the CF of $Z_t \sim \mathcal{N}(0, 2t)$. Further we can say that $\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}} \longrightarrow Z_t$ by LCT.

We can also regard e^{-tk^2} as the FT of the density of Z_t.

Rather usefully we have also have that e^{-tk^2} is a solution to

$$\frac{d\hat{u}}{dt} = -k^2 \hat{u} \tag{9}$$

We can actually use a analogous result from last week's homework to invert the Fourier transform of u to get

$$\frac{\partial u}{dt} = \frac{\partial^2 u}{\partial x^2} \tag{10}$$

A More General Result

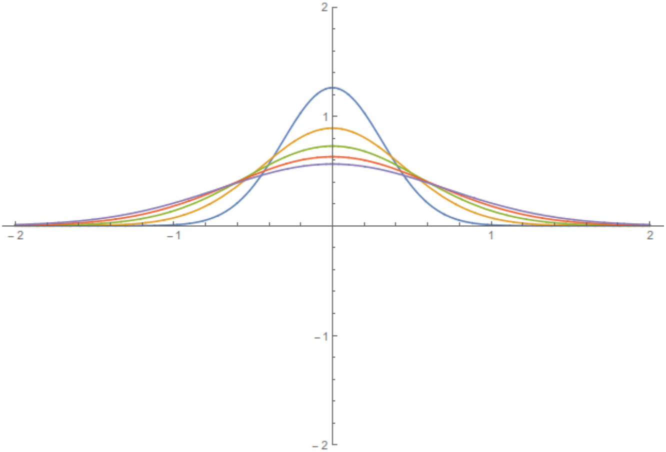
We can generalise this result a bit and say that if $\mathbb{E}[X_i^2] = \sigma^2$ then $\frac{S_{\lfloor \gamma t \rfloor}}{\sqrt{\gamma}} \longrightarrow Z_t$, but now $Z_t \sim \mathcal{N}(0, 2\sigma^2 t)$.

Further the Fourier transform of the density of Z_t is $e^{-t\sigma^2 k^2}$ which means the density is now a solution to

$$\frac{\partial u}{\partial t} = \underbrace{\frac{\sigma^2}{2}}_{D} \frac{\partial^2 u}{\partial x^2}. \tag{11}$$

What this says is that as particles jump around more, they diffuse more rapidly.

A Plot



Recap

- All we have done to get this result is require that the random variables that represent the *jumps* fullfill the requirements of the CLT.
- What if we relax the *finite second moment* condition?

Pareto Distribution

- Consider a random variable P with density $Cx^{-\alpha-1}$ for some normalising constant C.
- If we require $1 < \alpha < 2$ then we have $\mathbb{E}[P]$ exists but $\mathbb{E}[P^2] = \infty$ does not.
- It can be shown (with very lengthy computation) that the FT of the density of P is $1 + (ik)^\alpha + O(k^2)$.
- The idea is to setup a sequence of random variables $Y_1, \dots Y_n$, all iid with Pareto distribution with parameter α and use these as the *jumps* in the random walk.

Pareto Distribution

Setup $S_n = \sum_{k=1}^{n} Y_n$ as before.
By the convolution theorems for FTs we can calculate the FT of the density of $\frac{S_n}{n^{\frac{1}{\alpha}}}$ to be

$$\left(1 + \frac{(ik)^\alpha}{n} + O(n^{-\frac{2}{\alpha}})\right)^n \tag{12}$$

Notice that as $n \longrightarrow \infty$ this has limit $e^{(ik)^\alpha}$.
The LCT implies that $\frac{S_n}{n^{\frac{1}{\alpha}}} \longrightarrow Z$ where Z has FT $e^{(ik)^\alpha}$.
Notice that in some way this is an *Extended Central Limit Theorem*.

Long Time Limit

Like before we can introduce a time scale paramter γ and write

$$S_{\lfloor \gamma t \rfloor} = \sum_{k=1}^{\lfloor \gamma t \rfloor} Y_k. \tag{13}$$

Again by considering the FT of

$$\frac{S_{\lfloor \gamma t \rfloor}}{\gamma^{\frac{1}{\alpha}}} \tag{14}$$

which is

$$\left(1 + \frac{(ik)^{\alpha}}{\gamma} + o(\gamma^{\frac{-2}{\alpha}})\right)^{\lfloor \gamma t \rfloor}. \tag{15}$$

Long Time Limit

$$\left(1 + \frac{(ik)^\alpha}{\gamma} + o(\gamma^{\frac{-2}{\alpha}})\right)^{\lfloor \gamma t \rfloor} \tag{16}$$

By taking $\gamma \longrightarrow \infty$ we get that the Fourier transform converges to

$$e^{t(ik)^\alpha} \tag{17}$$

and by the LCT this gives us that

$$\frac{S_{\lfloor \gamma t \rfloor}}{\gamma^{\frac{1}{\alpha}}} \longrightarrow Z_t \tag{18}$$

where Z_t has FT $\hat{u}(k) = e^{t(ik)^\alpha}$.
Notice that this is a stable distribution.

It is clear that $\hat{u}(t) = e^{t(ik)^\alpha}$ is a solution to

$$\frac{d\hat{u}}{dt} = (ik)^\alpha \hat{u}. \tag{19}$$

Unfortunately we can't use *last week's homework* to invert this FT and this is where we introduce fractional calculus.

Motivations

Cauchy Formula for Repeated Integration

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n) dy_n \cdots dy_2 dy_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt$$

Motivations

Cauchy Formula for Repeated Integration

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The idea is to replace the factorials with gamma functions to define an integral of arbitrary order

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The idea is to replace the factorials with gamma functions to define an integral of arbitrary order

Riemann-Liouville Fractional Integral

$$\left(I_a^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

Motivations (Derivatives)

Riemann-Liouville Fractional Derivative

$$
\begin{aligned}
\left(\mathcal{D}_a^{\alpha} f\right)(x) &= \frac{d^{\lceil \alpha \rceil}}{dx^{\lceil \alpha \rceil}} \left(I_a^{\lceil \alpha \rceil - \alpha} f \right)(x) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha - n + 1}}
\end{aligned}
$$

where $n = \lfloor \alpha \rfloor + 1$.

Motivations (Derivatives)

Caputo Fractional Derivative

$$\left({}^{C}\mathcal{D}_a^{\alpha} f\right)(x) = \left(I_a^{\lceil\alpha\rceil-\alpha} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} f\right)(x)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\frac{d^t}{dt^n} f(t)dt}{(x-t)^{\alpha-n+1}}$$

where $n = \lfloor\alpha\rfloor + 1$.

Riemann-Liouville vs Caputo Derivative

- The Caputo derivative is often used in fractional differential equations because it can be coupled with integer order initial conditions, whereas often the Riemann-Liouville derivative can't be coupled with integer order initial conditions.
- When we set $a = -\infty$ for a large class of functions these derivatives are the same.

Fractional Derivative Fourier Transform

It can be shown that

$$\mathcal{F}\left\{{}_{-\infty}\mathcal{D}^{\alpha} f(x)\right\} = (ik)^{\alpha}\mathcal{F}\{f(x)\} \tag{20}$$

and it is precisely this result that we use to *invert* the Fourier transform we had before. That is we can say that

$$\frac{\partial u}{\partial t}(x,t) = {}_{-\infty}\mathcal{D}_x^{\alpha} u(x,t) \tag{21}$$

where u is the density of $Z_t = \lim_{\gamma \longrightarrow \infty} \frac{Y_1+Y_2+\cdots+Y_{\lfloor \gamma t \rfloor}}{\gamma^{\frac{1}{\alpha}}}$.

It can be shown that $u(x,t) = Ax^{-\alpha-1} + o(x^{-\alpha-1})$ as $x \longrightarrow \infty$ with A depending on t and α.

A Quick Note on the Laplace Transform

Definition

We the define the Laplace transform of a function f to be the function F given by

$$F(s) := \int_0^\infty e^{-st} f(t) dt$$

We often write $F(s) = \mathcal{L}\{f(t)\}$.

A Quick Note on the Laplace Transform

The Laplace transform is particularly useful as it allows us to transform a differential equation into an "algebraic" equation. Lerch's theorem guarantees, with minor caveats, that the Laplace transform of a function is unique.

Basic Idea of the Laplace Transform Method

- Apply the Laplace transform to both sides of the differential equation to get and "algebraic" equation.
- Apply the Laplace transform to the initial conditions.
- Sub the transformed initial conditions into the transformed equation.
- Rearrange to get an expression for the Laplace transform of the function of interest.
- Invert. (This is possible, and guaranteed with minor caveats by Lerch's theorem)

The Laplace Transform of the Riemann-Liouville Integral

Lemma

The Laplace transform of the Riemann-Liouville integral of a function f is given by

$$\mathcal{L}\left\{I_0^\alpha f\right\} = s^{-\alpha}\mathcal{L}\left\{f\right\}.$$

The Laplace Transform of the Riemann-Liouville Integral[Proof]

Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of f with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{f(t)\right\} \underbrace{\mathcal{L}\left\{t^{\alpha-1}\right\}}_{=s^{-\alpha}\Gamma(\alpha)} \\
&= s^{-\alpha} \mathcal{L}\left\{f\right\}.
\end{aligned}$$

The Laplace Transform of the Caputo Derivative

The Laplace transform of the Caputo derivative of a function f is given by

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].$$

The Laplace Transform of the Caputo Derivative [Proof]

See that

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = \mathcal{L}\left\{\left(I_0^{n-\alpha}\frac{d^n f}{dt^n}\right)\right\}$$

$$= \underbrace{\frac{1}{\Gamma(n-\alpha)}\mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\}}_{\circledast}$$

The Laplace Transform of the Caputo Derivative [Proof]

$\circledast$ is just the Laplace transform of a convolution so

$$
\begin{aligned}
\circledast &= \mathcal{L}\left\{t^{n-\alpha-1}\right\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\
&= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right) \\
&\quad \times \left(s^n \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\
&= s^{\alpha-n}\left[s^n \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].
\end{aligned}
$$

$\square$

One Parameter Mittag-Lefler Function

Definition

The one parameter Mittag-Lefler E_α function is defined by its power series.

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$

Lemma

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

See that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ we may interchange the integral and the sum to get

$$\begin{aligned}\int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt &= \sum_{k=0}^\infty \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt \\ &= \sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-st} t^{\alpha k} dt.\end{aligned}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

By performing the change of variables $x = st$ we get that

$$\sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt = \sum_0^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k+1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k+1)}$$

$$= \sum_{k=0}^\infty \beta^k s^{-(\alpha k+1)}$$

$$= \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

$\square$

Summary of Important Results

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right]$$

$$\mathcal{L}\{E_\alpha(\beta t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

The Solution to the Differential Equation

Lemma

The fractional differential equation,

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t) \tag{22}$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \tag{23}$$

has solution $y(t) = E_{\alpha}\left(\beta t^{\alpha}\right)$

Proof of Proposed Solution

Taking the Laplace transform of both sides of (22) yields

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} y\right)\right\} = \beta\mathcal{L}\left\{y\right\}$$

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - \sum_{k=0}^{n-1} s^{n-k-1}y^{(k)}(0)\right] = \beta\mathcal{L}\left\{y\right\}$$

Proof of Proposed Solution

Then taking into account (23) (the initial conditions) we get

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - s^{n-1}\right] = \beta\mathcal{L}\left\{y\right\}$$

and so

$$\mathcal{L}\left\{y\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

Proof of Proposed Solution

By by noticing that

$$\mathcal{L}\{y\} = \frac{s^{\alpha-1}}{s^{\alpha} - \beta}.$$

is the Laplace transform of $E_\alpha(\beta t^\alpha)$ we have that

$$y(t) = E_\alpha(\beta t^\alpha)$$

$\square$