University of New South Wales

School of Mathematics and Statistics

Assignment 2

Measure Theory

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2.1

Suppose u and ν are σ-finite *positive* measures on $(\Omega, \mathcal{F})$. Then suppose that $\mu << \nu$ and $\nu << \mu$. Then for $A \in \mathcal{F}$ we have that $\mu(a) = 0 \Rightarrow \nu(A) = 0$ and $\nu(A) = 0 \Rightarrow \mu(A)$. That is to say that $\nu(A) = 0 \Leftrightarrow \mu(A) = 0$. That is to say that ν and μ have the same null sets. This argument is symmetric so it is clear that the reverse implication also holds.
We now wish to show that there is an $\mathcal{F}$-measurable function g that satisfies $0 < g(\omega) < +\infty$ at each $\omega \in \Omega$ and is such that $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{F}$.

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3.1

Firstly note that the definition of the characteristic function is

$$\hat{\mu}_X(u) = \mathbb{E}[\exp(i\langle X, u\rangle)] \tag{1}$$

and so for the random vector cX with $c \in \mathbb{R}$ we have that

$$\begin{aligned} \hat{\mu}_{cX}(u) &= \mathbb{E}[\exp(i\langle cX, u\rangle)] &(2)\\ &= \mathbb{E}[\exp(i\langle X, cu\rangle)] &(3)\\ &= \hat{\mu}_X(cu) &(4) \end{aligned}$$

3.2

3.3

Let $d = 1$ and let μ have the Lebesgue density,

$$f(x) = \frac{C}{(1+x^2)\log(e+x^2)}, \qquad x \in \mathbb{R}. \tag{5}$$

We wish to show that $E[X]$ is not defined but $\hat{\mu}(u)$ is differentiable at 0. Firstly we show that $E[X]$ is not defined.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} \frac{Cx}{(1+x^2)\log(e+x^2)} dx &(6)\\ &= \int_{-\infty}^{\infty} \frac{C}{(x^{-1}+x)\log(e+x^2)} &(7) \end{aligned}$$

note that

$$\frac{C}{(x^{-1}+x)\log(e+x^2)} \sim \frac{C}{2x\log(x)} \tag{8}$$

and

$$\int_a^\infty \frac{C}{2x\log(x)}dx, \text{ and } \int_{-\infty}^b \frac{C}{2x\log(x)}dx \tag{9}$$

do not converge so $\mathbb{E}[X]$ does not exist.
We now just have to show that $\hat{mu}(u)$ is differentiable at $u = 0$.

$$\hat{\mu}(u) = \int_{-\infty}^\infty \frac{e^{iux}C}{(1+x^2)\log(e+x^2)}dx \tag{10}$$

and

$$\frac{d}{du}\hat{\mu}(u)\Big|_{u=0} = \int_{-\infty}^\infty \frac{ixe^{iux}C}{(1+x^2)\log(e+x^2)}dx \tag{11}$$

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Let μ be the binomial distribution with n trials and probability of success p, that is $\mu = \text{Bin}(n, p)$, and let ν be the Poisson distribution with mean $\lambda > 0$.

4.1

We wish to verify that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$. Because the binomial distribution is just the convolution of identical independent Bernoulli distributions then we just have to verify that $(1 - p + pe^{iu}$ is the characteristic function for $Bernoulli(p)$.
If ν is the Bernoulli measure then

$$\begin{aligned} \mathbb{E}[\exp(iuX)] &= \sum_{k\in\{0,1\}} e^{iuk}\nu_X(k) &(12)\\ &= pe^{iu} + (1-p). &(13)\end{aligned}$$

Then by repeated application of the convolution theorem we get that $\hat{\mu}(u) = (1 - p + pe^{iu})^n$.

4.2

We wish to verify that $\hat{\nu}(u) = \exp(\lambda(e^{iu} - 1))$. The probability mass function of the Poisson distribution is

$$\frac{\lambda^k}{k!}e^{-\lambda} \tag{14}$$

and thus

$$\mathbb{E}[\exp(iuX)] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} \tag{15}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (e^{iu})^k \tag{16}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} \tag{17}$$

$$= e^{-\lambda} e^{\lambda e^{iu}} \tag{18}$$

$$= e^{\lambda(e^{iu}-1)} \tag{19}$$