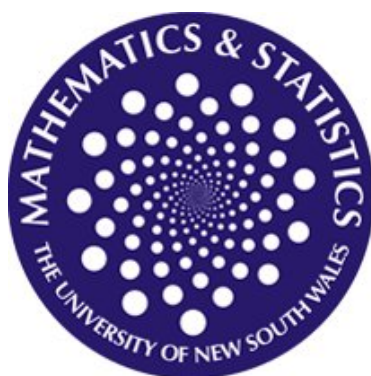




UNSW  
A U S T R A L I A



UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

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## Assignment

Number Theory

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*Author:*  
Adam J. Gray

*Student Number:*  
3329798

## Question 1

Suppose  $x > 2$  and let  $m$  be the largest integer such that  $2^m \leq x$ .

### Part a

Use the definition of  $\psi(x)$  to deduce that  $\psi(x) \geq \vartheta(x)$  and conclude from Tutorial problem 1 that  $\vartheta(x) \leq 2x$ .

### Part c

Show that  $\frac{\log(x)}{x^\alpha}$  has a maximum of  $\frac{1}{\alpha e}$

### Part d

Deduce that  
 $\psi(x) - \vartheta(x) \leq 9x^{\frac{1}{2}}.$

### Part e

Conclude that, as  $x \rightarrow \infty$ ,  $\frac{\psi(x)}{x} \rightarrow 1 \Leftrightarrow \frac{\vartheta(x)}{x} \rightarrow 1$ .

## Solution

### Part a

We have that

$$\begin{aligned}\psi(x) &= \sum_{m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) \\ &= \vartheta(x) + \underbrace{\sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}})}_{\geq 0}\end{aligned}$$

so it is obvious that  $\psi(x) \geq \vartheta(x)$ .

From the tutorial problems we have that  $\psi(x) \leq 2x$  so it must also be that  $\vartheta(x) \leq 2x$ .

**Part b**

See that

$$\begin{aligned}\frac{d}{dx} \frac{\log(x)}{x^\alpha} &= \frac{\frac{1}{x}x^\alpha - \alpha \log(x)x^{\alpha-1}}{x^{2\alpha}} \\ &= \underbrace{\frac{x^{\alpha-1}(1 - \alpha \log(x))}{x^{2\alpha}}}_{\circledast}.\end{aligned}$$

Now setting  $\circledast = 0$  we have that

$$\begin{aligned}\log(x) &= \frac{1}{\alpha} \\ x &= e^{\frac{1}{\alpha}}.\end{aligned}$$

Now checking

$$\begin{aligned}\frac{d^2}{dx^2} \frac{\log(x)}{x^\alpha} \Big|_{x=e^{\frac{1}{\alpha}}} &= e^{\frac{-2-\alpha}{2}}(1-\alpha)(1-1) + e^{\frac{-1-\alpha}{\alpha}} \left( \frac{-\alpha}{e^{-\alpha}} \right) \\ &\leq 0\end{aligned}$$

so there must be a maximum at  $x = e^{\frac{1}{\alpha}}$ . Then evaluating we get that

$$\frac{\log(x)}{x^\alpha} \Big|_{x=e^{\frac{1}{\alpha}}} = \frac{1}{e\alpha}.$$

□

**Part c**

We can write

$$\begin{aligned}\psi(x) - \vartheta(x) &= \sum_{1 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) - \vartheta(x) \\ &= \sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}).\end{aligned}$$

Now using the result of **part a** we can write

$$\begin{aligned}\sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) &\leq \sum_{2 \leq m \leq \log_2(x)} 2x^{\frac{1}{m}} \\ &\leq 2x^{\frac{1}{2}} + \sum_{3 \leq m \leq \log_2(x)} 2x^{\frac{1}{m}} \\ &\leq 2x^{\frac{1}{2}} + \sum_{3 \leq m \leq \log_2(x)} 2x^{\frac{1}{3}} \\ &\leq 2x^{\frac{1}{2}} + \frac{2}{\log(2)} \frac{\log(x)}{x^{\frac{1}{6}}} x^{\frac{1}{2}}.\end{aligned}$$

Now by applying the result of **part b** we can see that

$$\frac{\log(x)}{x^{\frac{1}{6}}} \leq \frac{6}{e}$$

so we have that

$$2x^{\frac{1}{2}} + \frac{2}{\log(2)} \frac{\log(x)}{x^{\frac{1}{6}}} x^{\frac{1}{2}} \leq x^{\frac{1}{2}} \left( 2 + \frac{2}{\log(2)} \frac{6}{e} \right).$$

Now by numerical evaluation we see that

$$\left( 2 + \frac{2}{\log(2)} \frac{6}{e} \right) < 9$$

and so

$$\psi(x) - \vartheta(x) \leq 9x^{\frac{1}{2}}.$$

□

### Part d

By exploiting the result of **part c** we see that

$$0 \leq \lim_{x \rightarrow \infty} \left( \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \right) \leq \lim_{x \rightarrow \infty} \frac{9x^{\frac{1}{2}}}{x} = 0$$

so

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = \lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x}$$

which means that

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1 \Leftrightarrow \lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} = 1.$$

□

## Question 2

### Part a

Assuming that

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log(x)}{x} = 1$$

show that

$$\lim_{x \rightarrow \infty} \frac{\log(\pi(x))}{\log(x)} = 1.$$

**Part b**

Deduce that

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log(\pi(x))}{x} = 1.$$

**Part c**

If  $p_n$  denotes the  $n$ th prime, show that the PNT implies

$$\lim_{n \rightarrow \infty} \frac{n \log(n)}{p_n} = 1.$$

(This says that the  $n$ th prime is ‘roughly’  $n \log(n)$  for large  $n$ .)

**Solutions****Part a**

Given

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log(x)}{x} = 1$$

we take the  $\log$  of both sides to get

$$\lim_{x \rightarrow \infty} \log(\pi(x)) + \log(\log(x)) - \log(x) = 0$$

and dividing by  $\log(x)$  we get that

$$\lim_{x \rightarrow \infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} = 0.$$

Now see that

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\log(\log(x))}{\log(x)} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x \log(x)}}{\frac{1}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\log(x)} \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} &= \lim_{x \rightarrow \infty} \frac{\log(\pi(x))}{\log(x)} - 1 \\ &= 0 \end{aligned}$$

and so the result follows.  $\square$

**Part b**

The assumption of **part a** is essentially that

$$\pi(x) \sim \frac{x}{\log(x)}$$

or equivalently

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}$$

so we can deduce that from **part a** that

$$\lim_{x \rightarrow \infty} \frac{\log(\pi(x))}{\log(x)} = \lim_{x \rightarrow \infty} \frac{\pi(x) \log(\pi(x))}{x}$$

and so

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log(\pi(x))}{x} = 1.$$

□

**Part c**

The prime number theorem asserts that the assumption in **part a** is in fact correct.

We can write the result of **part b** as

$$\lim_{p_n \rightarrow \infty} \frac{n \log(n)}{p_n} = 1$$

or equivalently as

$$\lim_{n \rightarrow \infty} \frac{n \log(n)}{p_n} = 1$$

□

**Question 3**

Use the PNT to show heuristically that there should be about  $\pi(n)$  primes between  $n^2$  and  $(n+1)^2$ .

**Solution**

The prime number theorem states that

$$\pi(x) \sim \frac{x}{\log(x)}$$

so we can write

$$\pi(n^2) \sim \frac{n^2}{2 \log(n)}$$

and

$$\pi((n+1)^2) \sim \frac{(n+1)^2}{2 \log(n+1)}$$

so we can say that “heuristically”

$$\pi((n+1)^2) - \pi(n^2) \sim \frac{(n+1)^2}{2 \log(n+1)} - \frac{n^2}{2 \log(n)}.$$

Now as  $\log(n) \sim \log(n+1)$  we can write

$$\frac{(n+1)^2}{2 \log(n+1)} \sim \frac{(n+1)^2}{2 \log(n)}$$

and so

$$\begin{aligned} \pi((n+1)^2) - \pi(n^2) &\sim \frac{(n+1)^2}{2 \log(n)} - \frac{n^2}{2 \log(n)} \\ &= \frac{2n+1}{2 \log(n)} \\ &\sim \frac{n}{\log(n)} \\ &\sim \pi(n). \end{aligned}$$