University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

Author:
Adam J. Gray

Student Number:
3329798

Question 1

Part a

Use the comparison test for series to prove that for $s > 11$, (s real),

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}.$$

Part b

Deduce that $\lim_{s \longrightarrow 1^+}(s-1)\zeta(s) = 1$.

Part c

Further show that $\frac{\log(\zeta(s))}{\log(\frac{1}{s-1})} \longrightarrow 1$ as $s \longrightarrow 1^+$, and explain what this means in terms of the Dirichlet density of the set of all primes.

Solution

Part a

First note that

$$\int_1^\infty \frac{1}{t^s} dt = \frac{1}{s-1}.$$

So we wish to show that

$$\int_1^\infty \frac{1}{t^s} dt \leq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n^s} \leq \int_1^\infty \frac{dt}{t^s}$$

which is equivelent to showing that

$$\sum_{n=1}^{\infty} \int_n^{n+1} \frac{dt}{t^s} \leq \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dt}{\lceil t \rceil^s} \tag{1}$$

and

$$\sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{dt}{\lceil t \rceil^s} \leq \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{dt}{t^s}. \tag{2}$$

By the integral estimation lemma we see that each term in (1)

$$\int_n^{n+1} \frac{dt}{t^s} \leq \frac{1}{n^s}$$

and we also have that
$$\int_n^{n+1} \frac{dt}{\lceil t \rceil^s} = \frac{1}{n^s}$$
so clearly
$$\int_n^{n+1} \frac{dt}{t^s} \leq \int_n^{n+1} \frac{dt}{\lceil t \rceil^s}$$
and hence (1) holds.
In a similar manner for (2) we have that
$$\int_{n-1}^{n} \frac{dt}{t^s} \geq \frac{1}{n^s}$$
so we have that
$$\int_{n-1}^{n} \frac{dt}{\lceil t \rceil^s} \leq \int_{n-1}^{n} \frac{dt}{t^s}$$
and (2) follows.

$\square$

Part b

By multiplying all terms of the result of part a we get that
$$1 \leq (s-1)\zeta(s) \leq 1 + (s-1)$$
and so
$$1 \leq \lim_{s \longrightarrow 1^+} (s-1)\zeta(s) \leq \lim_{s \longrightarrow 1^+} 1 + (s-1)$$
which immidiatly implies
$$\lim_{s \longrightarrow 1^+} (s-1)\zeta(s) = 1.$$

$\square$

Part c

Taking the log of the result of part a we get that
$$\log(\frac{1}{s-1}) \leq \log(\zeta(s)) \leq \log(1 + \frac{1}{s-1}).$$
which means that
$$1 \leq \frac{\log(\zeta(s))}{\log\left(\frac{1}{s-1}\right)} \leq \frac{\log\left(1 + \frac{1}{s-1}\right)}{\log\left(\frac{1}{s-1}\right)}. \tag{3}$$

We now evalutate

$$\begin{aligned}
\lim_{s \longrightarrow 1^+} \frac{\log\left(1+\frac{1}{s-1}\right)}{\log\left(\frac{1}{s-1}\right)} &= \lim_{s \longrightarrow 1^+} \frac{\frac{1}{1+\frac{1}{s-1}}\left(\frac{-1}{(s-1)^2}\right)}{\frac{1}{\frac{1}{s-1}}\left(\frac{-1}{(s-1)^2}\right)} \\
&= \lim_{s \longrightarrow 1^+} \frac{\frac{(s-1)}{s}}{(s-1)} \\
&= 1
\end{aligned}$$

so by taking limits of (3) we get that

$$\lim_{s \longrightarrow 1^+} \frac{\log(\zeta(s))}{\log(\frac{1}{s-1})} = 1$$

This is completely unsuprising as it means that the set of all primes has Dirichlet density 1.

□

Question 2

Part a

Prove that if p is prime and $s > 2$ then

$$1 + \frac{\phi(p)}{p^s} + \frac{\phi(p^2)}{p^{2s}} + \cdots = \frac{1-p^{-s}}{1-p^{1-s}}.$$

Part b

Deduce that for $s > 2$ that $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$.

Solution

Part a

Note that for any prime p we have that

$$\begin{aligned}
\frac{\phi(p^0)}{p^0} + \frac{\phi(p^1)}{p^1} + \frac{\phi(p^2)}{p^2} + \cdots &= 1 + \frac{p}{p^2} - \frac{1}{p^s} + \frac{p^2}{p^{2s}} - \frac{p}{p^{2s}} + \cdots \\
&\text{and by reordering we get} \\
&= 1 + \left(\frac{p}{p^2}\right)^1 + \left(\frac{p}{p^2}\right)^2 + \left(\frac{p}{p^2}\right)^3 + \cdots \\
&\quad - \frac{1}{p^s} - \frac{p}{p^{2s}} - \frac{p^2}{p^{3s}} - \cdots \\
&= (p^{1-s})^0 + (p^{1-s})^1 + (p^{1-s})^2 + \cdots \\
&\quad - \frac{1}{p}\left((p^{1-s})^1 + (p^{1-s})^2 + \cdots\right) \\
&\text{and by the geometric sum formula we get} \\
&= \frac{1}{1-p^{1-s}} + \frac{1}{p}\left(\frac{p^{1-s}}{1-p^{1-s}}\right) \\
&= \frac{1-p^{-s}}{1-p^{1-s}}.
\end{aligned}$$

□

Part b

Since ϕ is a multiplicative function we can apply the result in the lecture notes to say that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} &= \prod_p \left(\frac{\phi(p^0)}{p^0} + \frac{\phi(p^1)}{p^1} + \frac{\phi(p^2)}{p^2} + \cdots\right) \\
&\text{and by the above result} \\
&= \prod_p \left(\frac{1-p^{-s}}{1-p^{s-1}}\right) \\
&= \frac{\zeta(s-1)}{\zeta(s)}
\end{aligned}$$

□

Question 3

Let $M(x) = \sum_{n \le x} \mu(n)$.

Part a

Use the fact that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma = (s) > 1$, to deduce that

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} M(x) x^{-s-1} dx.$$

Part b

Assume that $M(x) = O(x^{\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$ is true and deduce that the integral on the right convergess for $\sigma > \frac{1}{2} + \epsilon$.

Part c

Hence explain why $M(x) = O(x^{\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$, implies the Riemann Hypothesis.

Solutions

Part a

Firstly note that

$$\begin{aligned}
\int_n^{n+1} \frac{d}{dx}\left(\frac{1}{x^s}\right) dx &= -\int_n^{n+1} \frac{s}{x^{s+1}} dx \\
&= \left[\frac{1}{x^{s+1}}\right]_{x=n}^{x=n+1} \\
&= \left(\frac{1}{(n+1)^{s+1}} - \frac{1}{(n^s)}\right).
\end{aligned}$$

Also note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{M(n)-M(n-1)}{n^s}\\ &= \sum_{n=1}^{\infty}M(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\\ &\text{and by using the observation above}\\ &= \sum_{n=1}^{\infty}M(n)\int_{n-1}^{n}\frac{s}{x^{s+1}}dx\\ &= \sum_{n=1}^{\infty}\int_{n-1}^{n}\frac{sM(x)}{x^{s+1}}dx\\ &= s\int_{1}^{\infty}\frac{M(x)}{x^{s+1}}dx.\end{aligned}$$

□

Part b

Let $K(x) = \frac{M(x)}{x^{s+1}}$ then

$$\begin{aligned}K(x) &= \frac{O(x^{\frac{1}{2}+\epsilon})}{x^{s+1}}\\ &= O(x^{-s-\frac{1}{2}+\epsilon})\end{aligned}$$

then we have that

$$\begin{aligned}\int_1^{\infty}K(x)dx &= \int_1^{\infty}O(x^{-s-\frac{1}{2}+\epsilon})dx\\ &< \infty\end{aligned}$$

so long as if $s = \sigma + it$, then $\sigma > \frac{1}{2} + \epsilon$, by the p test.
We don't need to consider the imaginary component because, $|x^s|$ is independent of its imaginary component, i.e.

$$|x^{\sigma+it}| = |e^{\ln(x)(\sigma+it)}| = |e^{\ln(x)\sigma}|.$$

□

Part c

If you can prove that $M(x) = O(x^{\frac{1}{2}+\epsilon})$ then the integral converges for all $\sigma > \frac{1}{2} + \epsilon$ and hence $\zeta(s) = \zeta(\sigma + it) \neq 0$ for any choice of $\epsilon > 0$. This would mean that the only non-trivial zeros must have $\sigma \leq \frac{1}{2}$. The functional equation strengthens this by symetry to say $\sigma = \frac{1}{2}$ for any non-trivial zero of the Riemann-Zeta function.

□