Question 5

Let A denote the set of all integers of the form, $2^r 3^s 5^t$. Evaluate $\sum_{n \in A} \frac{1}{n}$ and $\sum_{n \in A}$.

Solution

Let $\mathbb{N}^0$ denote the set of natural numbers including 0. We can clearly write

$$\sum_{n \in A} \frac{1}{n} = \sum_{r,s,t \in \mathbb{N}^0} \frac{1}{2^r 3^s 5^t}.$$

Now we have that $\sum_{k=0}^{\infty} |\ell^{-k}|$ converges for all $\ell > 1$, so we can write

$$\begin{aligned}
\sum_{r,s,t \in \mathbb{N}^0} \frac{1}{2^r 3^s 5^t} &= \left(\sum_{r \in \mathbb{N}^0} \frac{1}{2^r} \right) \left(\sum_{s \in \mathbb{N}^0} \frac{1}{3^s} \right) \left(\sum_{t \in \mathbb{N}^0} \frac{1}{5^t} \right) \\
&= \left(\frac{1}{1 - \frac{1}{2}} \right) \left(\frac{1}{1 - \frac{1}{3}} \right) \left(\frac{1}{1 - \frac{1}{5}} \right) \\
&= \frac{15}{4}.
\end{aligned}$$

In a similar manner we get that

$$\begin{aligned}
\sum_{n \in A} \frac{1}{n^2} &= \sum_{r,s,t \in \mathbb{N}^0} \frac{1}{(2^r 3^s 5^t)^2} \\
&= \sum_{r,s,t \in \mathbb{N}^0} \frac{1}{4^r 9^s 25^t} \\
&= \left(\sum_{r \in \mathbb{N}^0} \frac{1}{4^r} \right) \left(\sum_{s \in \mathbb{N}^0} \frac{1}{9^s} \right) \left(\sum_{t \in \mathbb{N}^0} \frac{1}{25^t} \right) \\
&= \left(\frac{1}{1 - \frac{1}{4}} \right) \left(\frac{1}{1 - \frac{1}{9}} \right) \left(\frac{1}{1 - \frac{1}{25}} \right) \\
&= \frac{25}{16}.
\end{aligned}$$

Question 6

A slightly generalised version of Bertrand's postulate states that, for $n \geq 6$, there are at least 2 primes between n and $2n$. Use this to prove that $p_{k+2} \leq p_k + p_{k+1}$.

Solution

By this extended Bertrand's postulate we have that

$$\begin{aligned}
p_{k+2} &< 2p_k \\
&< p_{k+1} + p_k
\end{aligned}$$

We just have to tidy up cases when $p_k < 6$. See that

$$\begin{aligned} 2+3 &\leq 5 \\ 3+5 &< 11 \\ 5+7 &< 13 \end{aligned}$$

So we have that $p_{k+2} \leq p_k + p_{k+1}$ which can be strengthened to $p_{k+2} < p_k + p_{k+1}$ when for $k > 1$. □

Question 7

a

Use Bertrand's Postulate to prove that for $m \geq 2$, if $m! = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ then $\alpha_i = 1$ for at least one value of i.

b

Deduce that $m!$ is never a kth power for any $k \geq 2$.

Solution

a

Write $m = 2k+1$ or $m = 2k$ for some $k \in \mathbb{N}$. In either case there must exist a prime $k < p < 2k$. Just considering $m > 4$ we have that in the factorization $m! = m \cdot (m-1) \cdot (m-2) \cdots p \cdots 2$ there is only one power of p because for $p > 2$ we have that $p^2 > 2p > m > p$. In the language used above, that is simply to say that there exists an i such that $\alpha_i = 1$.
To tidy up the remaining cases it is clear to see that

$$\begin{aligned} 4! &= 1 \cdot 2 \cdot 3 \cdot 2^2 \\ 3! &= 1 \cdot 2 \cdot 3 \\ 2! &= 1 \cdot 2 \\ 1! &= 1 \end{aligned}$$

and so the result holds. □

b

It is clear that for any number of the form $s = x^k$ with $x, k \in \mathbb{N}$, s must have a prime factorization with $s = p_1^{k\alpha_1} \dots p_n^{k\alpha_n}$. For $m!$, however, we have proven that there are prime factors which only occur once (not $k > 1$ times). This is to day $m!$ cannot be writen in the form x^k. □

Question 8

Use Bertrand's postulate to prove that for every integer $k > 2$, there is a prime p such that $p < k < 2p$.

Solution

Consider $r = \left\lfloor \frac{k}{2} \right\rfloor$ with $k > 2$. We know that there must exist a prime p such that $r < p < 2r$.
Then we have that $2r < 2p$.
If $k = 2r$ then we are finished.
If $k = 2r + 1$ then $p \geq r + 1$ so $2p \geq 2r + 2$ which implies $2p > 2r + 1$.
In conclusion we have that for all $k > 2$ there exists a prime p such that $p < k < 2p$. $\square$