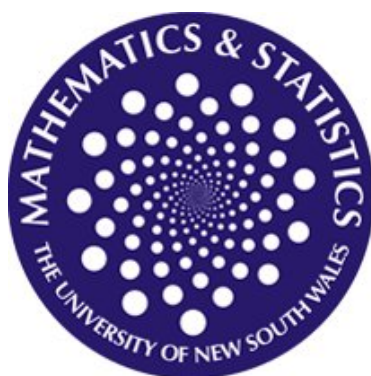




UNSW  
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UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

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## Assignment

Number Theory

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## Question 1

### Part a

Use the character table given in lectures for  $\mathbb{Z}_5$ , extended to a Dirichlet character, to evaluate

$$\sum_{i=1}^4 \chi_i(n) \overline{\chi_i(b)}, \quad \text{for each } b \in \mathbb{U}_5.$$

### Part b

Use the results of (a) to prove, in detail, that there are infinitely many primes congruent to 1 mod 5, 2 mod 5, and 3 mod 5 and 4 mod 5.

## Solution

### Part a

$$\sum_{i=1}^4 \chi_i(n) \overline{\chi_i(b)} = \begin{cases} 0 & \text{if } n \not\equiv b \pmod{5} \\ 4 & \text{if } n \equiv b \pmod{5} \end{cases}$$

This follows immediately from the orthogonality relation proved in lectures.

### Part b

Firstly see that

$$\begin{aligned} L(s, \chi_i) &= \sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s} \\ &= \prod_{p \text{ prime}} \left( 1 - \frac{\chi_i(p)}{p^s} \right)^{-1} \end{aligned}$$

and so

$$\begin{aligned} \log L(s, \chi_i) &= - \sum_{n=1}^{\infty} \log \left( 1 - \frac{\chi_i(p)}{p^s} \right) \\ &= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + \frac{1}{2} \left( \frac{\chi_i(p)}{p^s} \right)^2 + \frac{1}{3} \left( \frac{\chi_i(p)}{p^s} \right)^3 + \dots \\ &= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + R_i(s) \end{aligned}$$

where

$$R_i(s) = \sum_{p \text{ prime}} \frac{1}{2} \left( \frac{\chi_i(p)}{p^s} \right)^2 + \frac{1}{3} \left( \frac{\chi_i(p)}{p^s} \right)^3 + \frac{1}{4} \left( \frac{\chi_i(p)}{p^s} \right)^4 + \dots.$$

See that

$$\begin{aligned} |R_i(s)| &\leq \sum_{p \text{ prime}} \frac{1}{2} \left( \frac{|\chi_i(p)|}{p^s} \right)^2 + \frac{1}{3} \left( \frac{|\chi_i(p)|}{p^s} \right)^3 + \frac{1}{4} \left( \frac{|\chi_i(p)|}{p^s} \right)^4 + \dots \\ &\leq \sum_{p \text{ prime}} \frac{1}{2} \left( \frac{1}{p^s} \right)^2 + \frac{1}{2} \left( \frac{1}{p^s} \right)^3 + \frac{1}{2} \left( \frac{1}{p^s} \right)^4 + \dots \end{aligned}$$

by the geometric sum formula

$$\begin{aligned} &= \frac{1}{2} \sum_{p \text{ prime}} \frac{1}{p^s(p^s - 1)} \\ &< \frac{1}{2} \sum_{p \text{ prime}} \frac{1}{p(p - 1)} \\ &< \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(n - 1)} \\ &= \frac{1}{2} \end{aligned}$$

and so  $R_i(s)$  is bounded as  $s \rightarrow 1^+$ .