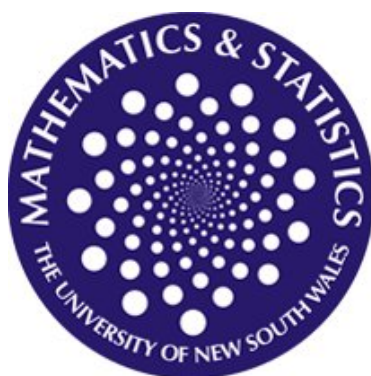




UNSW  
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UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

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## Assignment

Number Theory

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### Question 3

Let  $M(x) = \sum_{n \leq x} \mu(n)$ .

#### Part a

Use the fact that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for  $\sigma = (s) > 1$ , to deduce that

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} M(x) x^{-s-1} dx.$$

#### Part b

Assume that  $M(x) = O(x^{\frac{1}{2}+\epsilon})$ , for any  $\epsilon > 0$  is true and deduce that the integral on the right converges for  $\sigma > \frac{1}{2} + \epsilon$ .

#### Part c

Hence explain why  $M(x) = O(x^{\frac{1}{2}+\epsilon})$ , for any  $\epsilon > 0$ , implies the Riemann Hypothesis.

### Solutions

#### Part a

Firstly note that

$$\begin{aligned} \int_n^{n+1} \frac{d}{dx} \left( \frac{1}{x^s} \right) dx &= - \int_n^{n+1} \frac{s}{x^{s+1}} dx \\ &= \left[ \frac{1}{x^{s+1}} \right]_{x=n}^{x=n+1} \\ &= \left( \frac{1}{(n+1)^{s+1}} - \frac{1}{(n^s)} \right). \end{aligned}$$

Also note that

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^s} \\
 &= \sum_{n=1}^{\infty} M(n) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\
 &\text{and by using the observation above} \\
 &= \sum_{n=1}^{\infty} M(n) \int_{n-1}^n \frac{s}{x^{s+1}} dx \\
 &= \sum_{n=1}^{\infty} \int_{n-1}^n \frac{sM(x)}{x^{s+1}} dx \\
 &= s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx
 \end{aligned}$$

### Part b

Let  $K(x) = \frac{M(x)}{x^{s+1}}$  then

$$\begin{aligned}
 K(x) &= \frac{O(x^{\frac{1}{2}+\epsilon})}{x^{s+1}} \\
 &= O(x^{-s-\frac{1}{2}+\epsilon})
 \end{aligned}$$

then we have that

$$\begin{aligned}
 \int_1^{\infty} K(x) dx &= \int_1^{\infty} O(x^{-s-\frac{1}{2}+\epsilon}) dx \\
 &< \infty
 \end{aligned}$$

so long as if  $s = \sigma + it$ , then  $\sigma > \frac{1}{2} + \epsilon$ , by the  $p$  test.

We don't need to consider the imaginary component because,  $|x^s|$  is independent of its imaginary component, i.e.

$$|x^{\sigma+it}| = |e^{\ln(x)(\sigma+it)}| = |e^{\ln(x)\sigma}|.$$

### Part c

If you can prove that  $M(x) = O(x^{\frac{1}{2}+\epsilon})$  then the integral converges for all  $\sigma > \frac{1}{2} + \epsilon$  and hence  $\zeta(s) = \zeta(\sigma + it) \neq 0$  for any choice of  $\epsilon > 0$ . This would mean that the only non-trivial zeros must have  $\sigma \leq \frac{1}{2}$ . The functional equation strengthens this by symmetry to say  $\sigma = \frac{1}{2}$  for any zero of the Riemann-Zeta function.