University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

Author:
Adam J. Gray

Student Number:
3329798

Question 1

Suppose $x > 2$ and let m be the largest integer such that $2^m \leq x$.

Part a

Use the definition of $\psi(x)$ to deduce that $\psi(x) \geq \vartheta(x)$ and conclude from Tutorial problem 1 that $\vartheta(x) \leq 2x$.

Part c

Show that $\frac{\log(x)}{x^\alpha}$ has a maximum of $\frac{1}{\alpha e}$

Part d

Deduce that
$psi(x) - \vartheta(x) \leq 9x^{\frac{1}{2}}$.

Part e

Conclude that, as $x \longrightarrow \infty$, $\frac{\psi(x)}{x} \longrightarrow 1 \Leftrightarrow \frac{\vartheta(x)}{x} \longrightarrow 1$.

Solution

Part a

We have that

$$\begin{aligned}\psi(x) &= \sum_{m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) \\ &= \vartheta(x) + \underbrace{\sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}})}_{\geq 0}\end{aligned}$$

so it is obvious that $\psi(x) \geq \vartheta(x)$.
From the tutorial problems we have that $\psi(x) \leq 2x$ so it must also be that $\vartheta(x) \leq 2x$.

Part b

See that

$$\begin{aligned}\frac{d}{dx}\frac{log(x)}{x^\alpha} &= \frac{\frac{1}{x}x^\alpha - \alpha\log(x)x^{\alpha-1}}{x^{2\alpha}} \\ &= \underbrace{\frac{x^{\alpha-1}(1-\alpha\log(x)}{x^{2\alpha}}}_{\circledast}.\end{aligned}$$

Now setting $\circledast = 0$ we have that

$$\begin{aligned}\log(x) &= \frac{1}{\alpha} \\ x &= e^{\frac{1}{\alpha}}.\end{aligned}$$

Now checking

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\log(x)}{x^\alpha}\bigg|_{x=e^{\frac{1}{\alpha}}} &= e^{\frac{-2-\alpha}{2}}(1-\alpha)(1-1) + e^{\frac{-1-\alpha}{\alpha}}\left(\frac{-\alpha}{e^{-\alpha}}\right) \\ &\leq 0\end{aligned}$$

so there must be a maximum at $x = e^{\frac{1}{\alpha}}$. Then evaluating we get that

$$\frac{\log(x)}{x^\alpha}\bigg|_{x=e^{\frac{1}{\alpha}}} = \frac{1}{e\alpha}.$$

$\square$

Question 2

Part a

Assuming that

$$\lim_{x\longrightarrow\infty}\frac{\pi(x)\log(x)}{x} = 1$$

show that

$$\lim_{x\longrightarrow\infty}\frac{\log(\pi(x))}{\log(x)} = 1.$$

Part b

Deduce that

$$\lim_{x\longrightarrow\infty}\frac{\pi(x)\log(pi(x))}{x} = 1.$$

Part c

If p_n denotes that nth prime, show that the PNT implies

$$\lim_{n \longrightarrow \infty} \frac{n \log(n)}{p_n} = 1.$$

(This says that the nth primes is 'roughly' $n \log(n)$ for large n.)

Solutions

Part a

Given

$$\lim_{x \longrightarrow \infty} \frac{\pi(x) \log(x)}{x} = 1$$

we take the *log* of both sides to get

$$\lim_{x \longrightarrow \infty} \log(\pi(x)) + \log(\log(x)) - \log(x) = 0$$

and dividing by $\log(x)$ we get that

$$\lim_{x \longrightarrow \infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} = 0.$$

Now see that

$$\begin{aligned} \lim_{x \longrightarrow \infty} \frac{\log(\log(x))}{\log(x)} &= \lim_{x \longrightarrow \infty} \frac{\frac{1}{x \log(x)}}{\frac{1}{x}} \\ &= \lim_{x \longrightarrow \infty} \frac{1}{\log(x)} \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} \lim_{x \longrightarrow \infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} &= \lim_{x \longrightarrow \infty} \frac{\log(\pi(x))}{\log(x)} - 1 \\ &= 0 \end{aligned}$$

and so the result follows. □

Part b

The assumption of **part a** is essentially that

$$\pi(x) \sim \frac{x}{\log(x)}$$

or equivelently

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}$$

so we can deduce that from **part a** that

$$\lim_{x \longrightarrow \infty} \frac{\log(\pi(x))}{\log(x)} = \lim_{x \longrightarrow \infty} \frac{\pi(x)\log(\pi(x))}{x}$$

and so

$$\lim_{x \longrightarrow \infty} \frac{\pi(x)\log(\pi(x))}{x} = 1.$$

$\square$

Part c

The prime number theorem asserts that the assumption in **part a** is infact correct.
We can write the result of **part b** as

$$\lim_{p_n \longrightarrow \infty} \frac{n\log(n)}{p_n} = 1$$

or equivelently as

$$\lim_{n \longrightarrow \infty} \frac{n\log(n)}{p_n} = 1$$

$\square$