University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

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Question 1

Suppose $x > 2$ and let m be the largest integer such that $2^m \leq x$.

Part a

Use the definition of $\psi(x)$ to deduce that $\psi(x) \geq \vartheta(x)$ and conclude from Tutorial problem 1 that $\vartheta(x) \leq 2x$.

Part c

Show that $\frac{\log(x)}{x^\alpha}$ has a maximum of $\frac{1}{\alpha e}$

Part d

Deduce that
$psi(x) - \vartheta(x) \leq 9x^{\frac{1}{2}}$.

Part e

Conclude that, as $x \longrightarrow \infty$, $\frac{\psi(x)}{x} \longrightarrow 1 \Leftrightarrow \frac{\vartheta(x)}{x} \longrightarrow 1$.

Solution

Part a

We have that

$$\begin{aligned}
\psi(x) &= \sum_{m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) \\
&= \vartheta(x) + \underbrace{\sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}})}_{\geq 0}
\end{aligned}$$

so it is obvious that $\psi(x) \geq \vartheta(x)$.
From the tutorial problems we have that $\psi(x) \leq 2x$ so it must also be that $\vartheta(x) \leq 2x$.

Part b

See that

$$\begin{aligned}\frac{d}{dx}\frac{log(x)}{x^\alpha} &= \frac{\frac{1}{x}x^\alpha - \alpha\log(x)x^{\alpha-1}}{x^{2\alpha}}\\ &= \underbrace{\frac{x^{\alpha-1}(1-\alpha\log(x)}{x^{2\alpha}}}_{\circledast}.\end{aligned}$$

Now setting $\circledast = 0$ we have that

$$\begin{aligned}\log(x) &= \frac{1}{\alpha}\\ x &= e^{\frac{1}{\alpha}}.\end{aligned}$$

Now checking

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\log(x)}{x^\alpha}\bigg|_{x=e^{\frac{1}{\alpha}}} &= e^{\frac{-2-\alpha}{2}}(1-\alpha)(1-1) + e^{\frac{-1-\alpha}{\alpha}}\left(\frac{-\alpha}{e^{-\alpha}}\right)\\ &\leq 0\end{aligned}$$

so there must be a maximum at $x = e^{\frac{1}{\alpha}}$. Then evaluating we get that

$$\frac{\log(x)}{x^\alpha}\bigg|_{x=e^{\frac{1}{\alpha}}} = \frac{1}{e\alpha}.$$

$\square$