University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

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Question 1

Write $\omega(n)$ for the number of (distinct) prime divisors of n, $\Omega(n)$ for the number of prime factors of n, counted with repetition. Thus if, $n = \prod_{j=1}^{m} p_j^{k_j}$, then $\omega(n) = m$, and $\Omega(n) = \sum_{j=1}^{m} k_j$.

Part a

Prove that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$ for $n \geq 2$.

Part b

When does $\tau(n) = 2^{\omega(n)}$.

Solution

Part a

Firstly we prove that $2^{\omega}(n) \leq \tau(n)$. From the lecture notes we have that if $n = \prod_{j=1}^{m} p_j^{k_j}$ then $\tau(n) = \prod_{j=1}^{m}(k_j + 1)$ so we can say

$$\begin{aligned}
\tau(n) &= \prod_{j=1}^{m} \underbrace{(k_j + 1)}_{\geq 2} \\
&\leq \prod_{j=1}^{m} 2 \qquad (1) \\
&= 2^m \\
&= 2^{\omega(n)}
\end{aligned}$$

so $2^{\omega(n)} \leq \tau(n)$.
We now show that $\tau(n) \leq 2^{\Omega(n)}$. See that

$$\begin{aligned}
2^{\Omega(n)} &= 2^{\sum_{j=1}^{m} k_j} \\
&= \prod_{j=1}^{m} 2^{k_j}
\end{aligned}$$

and because for all $k_j \geq 1$, $k_j + 1 \leq 2^{k_j}$. then

$$\prod_{j=1}^{m} 2^{k_j} \geq \prod_{j=1}^{m} (k_j + 1)$$

and thus

$$\tau(n) \leq 2^{\Omega(n)}.$$

It remains to show that $2^{\Omega(n)} \leq n$. Because we have that

$$n = \prod_{j=1}^{m} p_j^{k_j}$$

and

$$2^{\Omega(n)} = \prod_{j=1}^{m} 2^{k_j}$$

then it is clear because $2 \leq k_j$ for all j.
So we have shown that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$.