University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

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Question 1

Suppose $x > 2$ and let m be the largest integer such that $2^m \leq x$.

Part a

Use the definition of $\psi(x)$ to deduce that $\psi(x) \geq \vartheta(x)$ and conclude from Tutorial problem 1 that $\vartheta(x) \leq 2x$.

Part c

Show that $\frac{\log(x)}{x^\alpha}$ has a maximum of $\frac{1}{\alpha e}$

Part d

Deduce that $\psi(x) - \vartheta(x) \leq 9x^{\frac{1}{2}}$.

Part e

Conclude that, as $x \longrightarrow \infty$, $\frac{\psi(x)}{x} \longrightarrow 1 \Leftrightarrow \frac{\vartheta(x)}{x} \longrightarrow 1$.

Solution

Part a

We have that

$$\begin{aligned}
\psi(x) &= \sum_{m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) \\
&= \vartheta(x) + \underbrace{\sum_{2 \leq m \leq \log_2(x)} \vartheta(x^{\frac{1}{m}})}_{\geq 0}
\end{aligned}$$

so it is obvious that $\psi(x) \geq \vartheta(x)$.
From the tutorial problems we have that $\psi(x) \leq 2x$ so it must also be that $\vartheta(x) \leq 2x$.

Part b

See that

$$\begin{aligned}
\frac{d}{dx}\frac{log(x)}{x^\alpha} &= \frac{\frac{1}{x}x^\alpha - \alpha\log(x)x^{\alpha-1}}{x^{2\alpha}} \\
&= \underbrace{\frac{x^{\alpha-1}(1-\alpha\log(x)}{x^{2\alpha}}}_{\circledast}.
\end{aligned}$$

Now setting $\circledast = 0$ we have that

$$\begin{aligned}\log(x) &= \frac{1}{\alpha}\\ x &= e^{\frac{1}{\alpha}}.\end{aligned}$$

Now checking

$$\begin{aligned}\frac{d^2}{dx^2}\frac{\log(x)}{x^\alpha}\Big|_{x=e^{\frac{1}{\alpha}}} &= e^{\frac{-2-\alpha}{2}}(1-\alpha)(1-1) + e^{\frac{-1-\alpha}{\alpha}}\left(\frac{-\alpha}{e^{-\alpha}}\right)\\ &\leq 0\end{aligned}$$

so there must be a maximum at $x = e^{\frac{1}{\alpha}}$. Then evaluating we get that

$$\frac{\log(x)}{x^\alpha}\Big|_{x=e^{\frac{1}{\alpha}}} = \frac{1}{e\alpha}.$$

$\square$

Part c

We can write

$$\begin{aligned}\psi(x) - \vartheta(x) &= \sum_{1\leq m\leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) - \vartheta(x)\\ &= \sum_{2\leq m\leq \log_2(x)} \vartheta(x^{\frac{1}{m}}).\end{aligned}$$

Now using the result of **part a** we can write

$$\begin{aligned}\sum_{2\leq m\leq \log_2(x)} \vartheta(x^{\frac{1}{m}}) &\leq \sum_{2\leq m\leq \log_2(x)} 2x^{\frac{1}{m}}\\ &\leq 2x^{\frac{1}{2}} + \sum_{3\leq m\leq \log_2(x)} 2x^{\frac{1}{m}}\\ &\leq 2x^{\frac{1}{2}} + \sum_{3\leq m\leq \log_2(x)} 2x^{\frac{1}{3}}\\ &\leq 2x^{\frac{1}{2}} + \frac{2}{\log(2)}\frac{\log(x)}{x^{\frac{1}{6}}}x^{\frac{1}{2}}.\end{aligned}$$

Now by applying the result of **part b** we can see that

$$\frac{\log(x)}{x^{\frac{1}{6}}} \leq \frac{6}{e}$$

so we have that

$$2x^{\frac{1}{2}} + \frac{2}{\log(2)}\frac{\log(x)}{x^{\frac{1}{6}}}x^{\frac{1}{2}} \leq x^{\frac{1}{2}}\left(2 + \frac{2}{\log(2)}\frac{6}{e}\right).$$

Now by numerical evaluation we see that

$$\left(2+\frac{2}{\log(2)}\frac{6}{e}\right) < 9$$

and so

$$\psi(x)-\vartheta(x) \leq 9x^{\frac{1}{2}}.$$

□

Part d

By exploiting the result of **part c** we see that

$$\begin{aligned} 0 \leq \lim_{x\longrightarrow\infty}\left(\frac{\psi(x)}{x}-\frac{\vartheta(x)}{x}\right) &\leq \lim_{x\longrightarrow\infty}\frac{9x^{\frac{1}{2}}}{x} \\ &= 0 \end{aligned}$$

so

$$\lim_{x\longrightarrow\infty}\frac{\psi(x)}{x} = \lim_{x\longrightarrow\infty}\frac{\vartheta(x)}{x}$$

which means that

$$\lim_{x\longrightarrow\infty}\frac{\psi(x)}{x} = 1 \Leftrightarrow \lim_{x\longrightarrow\infty}\frac{\vartheta(x)}{x} = 1.$$

□

Question 2

Part a

Assuming that

$$\lim_{x\longrightarrow\infty}\frac{\pi(x)\log(x)}{x} = 1$$

show that

$$\lim_{x\longrightarrow\infty}\frac{\log(\pi(x))}{\log(x)} = 1.$$

Part b

Deduce that

$$\lim_{x\longrightarrow\infty}\frac{\pi(x)\log(pi(x))}{x} = 1.$$

Part c

If p_n denotes that nth prime, show that the PNT implies

$$\lim_{n\longrightarrow\infty} \frac{n\log(n)}{p_n} = 1.$$

(This says that the nth primes is 'roughly' $n\log(n)$ for large n.)

Solutions

Part a

Given

$$\lim_{x\longrightarrow\infty} \frac{\pi(x)\log(x)}{x} = 1$$

we take the *log* of both sides to get

$$\lim_{x\longrightarrow\infty} \log(\pi(x)) + \log(\log(x)) - \log(x) = 0$$

and dividing by $\log(x)$ we get that

$$\lim_{x\longrightarrow\infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} = 0.$$

Now see that

$$\begin{aligned}
\lim_{x\longrightarrow\infty} \frac{\log(\log(x))}{\log(x)} &= \lim_{x\longrightarrow\infty} \frac{\frac{1}{x\log(x)}}{\frac{1}{x}} \\
&= \lim_{x\longrightarrow\infty} \frac{1}{\log(x)} \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\longrightarrow\infty} \frac{\log(\pi(x)) + \log(\log(x)) - \log(x)}{\log(x)} &= \lim_{x\longrightarrow\infty} \frac{\log(\pi(x))}{\log(x)} - 1 \\
&= 0
\end{aligned}$$

and so the result follows. □

Part b

The assumption of **part a** is essentially that

$$\pi(x) \sim \frac{x}{\log(x)}$$

or equivalently

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}$$

so we can deduce that from **part a** that

$$\lim_{x \longrightarrow \infty} \frac{\log(\pi(x))}{\log(x)} = \lim_{x \longrightarrow \infty} \frac{\pi(x)\log(\pi(x))}{x}$$

and so

$$\lim_{x \longrightarrow \infty} \frac{\pi(x)\log(\pi(x))}{x} = 1.$$

$\square$

Part c

The prime number theorem asserts that the assumption in **part a** is in fact correct.
We can write the result of **part b** as

$$\lim_{p_n \longrightarrow \infty} \frac{n\log(n)}{p_n} = 1$$

or equivalently as

$$\lim_{n \longrightarrow \infty} \frac{n\log(n)}{p_n} = 1$$

$\square$

Question 3

Use the PNT to show heuristically that there should be about $\pi(n)$ primes between n^2 and $(n+1)^2$.

Solution

The prime number theorem states that

$$\pi(x) \sim \frac{x}{\log(x)}$$

so we can write

$$\pi(n^2) \sim \frac{n^2}{2\log(n)}$$

and

$$\pi((n+1)^2) \sim \frac{(n+1)^2}{2\log(n+1)}$$

so we can say that "heuristically"

$$\pi((n+1)^2) - \pi(n^2) \sim \frac{(n+1)^2}{2\log(n+1)} - \frac{n^2}{2\log(n)}.$$

Now as $\log(n) \sim \log(n+1)$ we can write

$$\frac{(n+1)^2}{2\log(n+1)} \sim \frac{(n+1)^2}{2\log(n)}$$

and so

$$\begin{aligned} \pi((n+1)^2) - \pi(n^2) &\sim \frac{(n+1)^2}{2\log(n)} - \frac{n^2}{2\log(n)} \\ &= \frac{2n+1}{2\log(n)} \\ &\sim \frac{n}{\log(n)} \\ &\sim \pi(n). \end{aligned}$$