University of New South Wales

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Assignment

Number Theory

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Question 1

Write $\omega(n)$ for the number of (distinct) prime divisors of n, $\Omega(n)$ for the number of prime factors of n, counted with repetition. Thus if, $n = \prod_{j=1}^{m} p_j^{k_j}$, then $\omega(n) = m$, and $\Omega(n) = \sum_{j=1}^{m} k_j$.

Part a

Prove that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$ for $n \geq 2$.

Part b

When does $\tau(n) = 2^{\omega(n)}$.

Solution

Part b

Firstly we prove that $2^{\omega}(n) \leq \tau(n)$. From the lecture notes we have that if $n = \prod_{j=1}^{m} p_j^{k_j}$ then $\tau(n) = \prod_{j=1}^{m}(k_j + 1)$ so we can say

$$\begin{aligned}
\tau(n) &= \prod_{j=1}^{m} \underbrace{(k_j + 1)}_{\geq 2} \\
&\leq \prod_{j=1}^{m} 2 \qquad (1) \\
&= 2^m \\
&= 2^{\omega(n)}
\end{aligned}$$

so $2^{\omega(n)} \leq \tau(n)$.
We now show that $\tau(n) \leq 2^{\Omega(n)}$. See that

$$\begin{aligned}
2^{\Omega(n)} &= 2^{\sum_{j=1}^{m} k_j} \\
&= \prod_{j=1}^{m} 2^{k_j}
\end{aligned}$$

and because for all $k_j \geq 1$, $k_j + 1 \leq 2^{k_j}$. then

$$\prod_{j=1}^{m} 2^{k_j} \geq \prod_{j=1}^{m}(k_j + 1)$$

and thus

$$\tau(n) \leq 2^{\Omega(n)}.$$

It remains to show that $2^{\Omega(n)} \leq n$. Because we have that

$$n = \prod_{j=1}^{m} p_j^{k_j}$$

and

$$2^{\Omega(n)} = \prod_{j=1}^{m} 2^{k_j}$$

then it is clear because $2 \leq k_j$ for all j.
So we have shown that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$.

Part a

Clearly just when n is square-free.

Question 2

Define the **Jordan totient function** by

$$n^k \prod_{p|n} (1 - p^- k),$$

where, as usual, the product is taken over the primes. This is a generalization of Euler's totient function.

Part a

Prove that J is multiplicative.

Part b

Show that $J_k(n) = \sum_{d|n} \mu(d) (\frac{n}{d})^k$.

Part c

Find a simple expression (as a product over primes) for $J^{-1}(n)$.

Solution

Part a

For $(n, m) = 1$ see that

$$J_k(nm) = (nm)^k \prod_{p|nm} (1 - p^{-k})$$

but as $(n, m) = 1$ then $p|n$ or $p|m$ but not both so

$$\begin{aligned}
J_k(nm) &= n^k m^k \prod_{p|n}(1 - p^{-k}) \prod_{p|m}(1 - p^{-k}) \\
&= \left(n^k \prod p|n(1 - p^{-k})\right) \left(n^k \prod q|m(1 - q^{-k})\right) \\
&= J_k(n)J_k(m)
\end{aligned}$$

□

Part b

Let p be a prime and observe that

$$J_k(p^\alpha) = p^{\alpha k}(1 - p^{-k})$$

and note in addition that $J_k(1) = 1$. Now define

$$\gamma_k(n) := \sum_{d|n} J_k(d)$$

and note that since J_k is multiplicative, so is γ_k.
Therefore we only need to consider $\gamma_k(p^\alpha)$, where p is prime, to characterize γ_k.
Therefore observe that

$$\begin{aligned}
\gamma_k(p^\alpha) &= \sum_{j=0}^{\alpha} J_k(p^\alpha) \\
&= \sum_{j=1}^{\alpha} p^{jk}(1 - p^k) + 1 \\
&= (1 - p^{-k}) \sum_{j=1}^{\alpha} p^{jk} + 1 \\
&= (1 - p^{-k}) \left(\frac{p^k - p^{k(\alpha+1)}}{1 - p^k}\right) + 1 \\
&= p^{\alpha k}.
\end{aligned}$$

So if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ then

$$\begin{aligned}
\gamma_k(n) &= \gamma_k(p_1^{\alpha_1}) \dots \gamma_k(p_r^{\alpha_r}) \\
&= p_1^{\alpha_1 k} \dots p_r^{\alpha_r k} \\
&= n^k.
\end{aligned}$$

So we have that

$$\sum_{d|n} J_k(d) = n^k$$

and hence by Mobius inversion

$$J_k(d) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k.$$

□

Part c

Let $N^k(n) = n^k$. We can then write that $J_k = N^k * \mu$. So

$$\begin{aligned} J_k^{-1} &= (N^k * \mu)^{-1} \\ &= \mu^{-1} * N^{k-1}. \end{aligned}$$

As it is clear that N_k is completely multiplicative so $N^{k-1}(n) = \mu(n)N^k$, and hence

$$\begin{aligned} J_k^{-1}(n) &= \sum_{d|n} u(d)\mu\left(\frac{n}{d}\right) N^k\left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mu(d)N^k(d) \end{aligned}$$

Now as μN^k is multiplicative J_k^{-1} is the Dirichlet product of two multiplicative functions, and is hence multiplicative. For p prime we have that

$$\begin{aligned} J_k^{-1}(p^\alpha) &= \sum_{j=0}^{\alpha} \mu(p^j)p^{jk} \\ &= (1 - p^k) \end{aligned}$$

So if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where the p_j are distinct primes, then

$$J_k^{-1}(n) = \prod_{j=1}^{r} \left(1 - p^k\right).$$

□

Question 3

Write $M_2(x) = \sum n \leq x(\mu(n))^2$.
Hence $M_2(x)$ counts the number of square free integers $\leq x$.

Part a

Explain why $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$.

Part b

Prove that

$$M_2(x) = x \sum_{m \le \sqrt{x}} \frac{\mu(n)}{m^2} - \sum_{m \le \sqrt{x}} \mu(m) \left\{ \frac{x}{m^2} \right\}.$$

Part c

Deduce that

$$M_2(x) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

Part d

Interpret this result in terms of the proportion of square-free numbers in the interval $[1, x]$.

Part e

Use MAPLE (or otherwise) to count the number of square-free numbers between 1 and 1000. (Comment!)

Solution

Part a

Write $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where p_j is prime and all p_j are distinct. Without loss of generality assume we can reorder this factorization such that there exists a $1 \le t \le r$ such that when $j < t$, $2 \le \alpha_j$. Then it can be easily seen that

$$\begin{aligned}
\sum_{m^2|n} \mu(m) &= 1 + \mu(p_1) + \mu(p_2) + \dots + \mu(p_{t-1}) \\
&\quad + \mu(p_1 p_2) + \mu(p_1 p_3) + \dots + \mu(p_{t-2} p_{t-1}) \\
&\quad + \mu(p_1 p_2 p_3) + \dots + \mu(p_{t-3} p_{t-2} p_{t-1}) \\
&\quad + \dots \\
&\quad + \mu(p_1 p_2 \dots p_{t-1}) \\
&= \binom{t-1}{0}(-1)^0 + \binom{t-1}{1}(-1)^1 + \dots + \binom{t-1}{t-1}(-1)^{t-1} \\
&= \begin{cases} (1-1)^{t-1} = 0 & \text{where } n \text{ is not square-free} \\ 1 & \text{where } n \text{ is square-free} \end{cases}
\end{aligned}$$

It should be noted that higher powers of primes do not appear in the sum because they are not square-free and hence Mobius function would be 0 for these numbers.

Since it is clear that

$$(\mu(n))^2 = \begin{cases} (1-1)^{t-1} = 0 & \text{where } n \text{ is not square-free} \\ 1 & \text{where } n \text{ is square-free} \end{cases}$$

we have shown that $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$. □

Part b

If we prove two simple lemmas (which are just DSIs), the solution to this question is very straight-forward.

Lemma 1. *If f and g are arithmetic functions then*

$$\sum_{n\leq x} \sum_{m^2|n} = \sum_{m\leq\sqrt{x}} g(m) \sum_{j\leq \frac{x}{m^2}} f(m^2 j).$$

Proof. See that

$$\begin{aligned}\sum_{n\leq x} f(n) \sum_{m^2|n} g(m) = f(1)g(1) + f(2)g(1) + \ldots \\ &+ f(4)[g(1)+g(2)] + f(5)g(1) + \ldots \\ &+ f(8)[g(1)+g(2)] + \ldots \\ &\text{by reordering} \\ &= g(1)[f(1)+f(2)+f(3)+\ldots]+ \\ &+ g(2)[f(4)+f(8)+f(12)+\ldots] + \ldots \\ &= \sum_{m\leq\sqrt{x}} g(m) \sum_{jm^2\leq x} f(jm^2).\end{aligned}$$

□

Lemma 2. *If g is an arithmetic function then*

$$\sum_{n\leq x} \sum_{m^2|n} g(m) = \sum_{m\leq\sqrt{x}} g(m) \left\lfloor \frac{x}{m^2} \right\rfloor.$$

Proof. Setting $f = u$ in lemma 1 yields

$$\begin{aligned}\sum_{n\leq x} \sum_{m^2|n} g(m) &= \sum_{n\leq\sqrt{x}} u(n) \sum_{m^2|n} g(m) \\ &= \sum_{m\leq\sqrt{x}} g(m) \sum_{jm^2\leq x} u(m)\end{aligned}$$

and since it is clear to see that

$$\begin{aligned}\sum_{jm^2\leq x} u(m) &= \sum_{j\leq\frac{x}{m^2}} u(m) \\ &= \left\lfloor \frac{x}{m^2} \right\rfloor\end{aligned}$$

it follows that

$$\sum_{n \leq x} \sum_{m^2 | n} g(m) = \sum_{m \leq \sqrt{x}} g(m) \left\lfloor \frac{x}{m^2} \right\rfloor .$$

$\square$

We now return to the original problem.
Letting $g = \mu$ in lemma 2 we get that

$$\sum_{n \leq x} \sum_{m^2 | n} \mu(m) = \sum_{m \leq \sqrt{x}} \mu(m) \left\lfloor \frac{x}{m^2} \right\rfloor .$$

Its then clear to see that

$$\sum_{m \leq \sqrt{x}} \mu(x) \left\lfloor \frac{x}{m^2} \right\rfloor = \sum_{m \leq \sqrt{x}} \mu(x) \left(\frac{x}{m^2} - \left\{ \frac{x}{m^2} \right\} \right) \quad = x \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m^2} - \sum_{m \leq \sqrt{x}} \mu(m) \left\{ \frac{x}{m^2} \right\}$$

So we have shown that

$$M_2(x) = x \sum_{m \leq \sqrt{x}} \frac{\mu(n)}{m^2} - \sum_{m \leq \sqrt{x}} \mu(m) \left\{ \frac{x}{m^2} \right\} .$$

$\square$

Part c

Since $\zeta(2) = \frac{\pi^2}{6}$ and

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{6}{\pi^2}$$

then

$$\begin{aligned}
x \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m^2} &= x \frac{6}{\pi^2} - x \left(\sum_{m > \sqrt{x}} \frac{\mu(m)}{m^2} \right) \\
&= x \frac{6}{\pi^2} + xO \left(\sum_{m > \sqrt{x}} \frac{1}{m^2} \right) \\
&= x \frac{6}{\pi^2} + xO \left(\frac{1}{\sqrt{x}} \right) \\
&= x \frac{6}{\pi^2} + O \left(\sqrt{x} \right)
\end{aligned}$$

Additionally it is clear that

$$O \left(\sum_{m \leq \sqrt{x}} \mu(m) \left\{ \frac{x}{m^2} \right\} \right) = O(1)$$

and so

$$M_2(x) = \frac{6}{\pi^2} + O(\sqrt{x}).$$

$\square$

Part d

There are x natural numbers in the interval $[1, x]$ so the proportion of square-free integers in the interval $[1, x]$ would be given by

$$\frac{M_2(x)}{x}$$

which is equal to

$$\frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right).$$

So as $x \longrightarrow \infty$ then the proportion of square free integers in the interval becomes "closer" to $\frac{6}{\pi^2}$.

Part e

There are 608 square free integers between 1 and 1000 and $\frac{6}{\pi^2} \times 1000 \approx 607.9$ which is a very close approximation.