University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

Author:
Adam J. Gray

Student Number:
3329798

Question 1

Here is another proof that $\sum \frac{1}{p_i}$ diverges. For a contradiction, suppose that there is an integer k such that

$$\sum_{m=k+1}^{\infty} \frac{1}{p_m} < \frac{1}{2}.$$

Part a

Let p_k denote, as usual, denote the kth prime and let $\alpha_k(x)$ be the number of positive integers not exceeding x, all of whose primes factors are less or equal to p_k. Show that there can be no more than 2^k such square-free integers and then prove that $\alpha_k(x) \leq 2^k \sqrt{(x)}$.

Part b

By noting that the number of positive integers less than a given x that are divisible by the prime p is no more that $\frac{x}{p}$, show that $x - \alpha_k(x) < \frac{x}{2}$. Deduce that $x < 2^{2k+2}$ and arrive at a contradiction.

Solution

Part a

We can write every positive integer x as $x = s^2 r$ where r is a square-free integer. Now if we take x_k to be a positive integer whose prime factors are less than or equal to p_k, and let M_k be the set of such intengers then $\alpha_k(x) = |M_k|$. Writing $x_k = s_k^2 r_k$ with $x_k \in M_k$ then each of $p_1, \ldots, p_k$ can appear in the prime factorization of r_k at most once. Then there is at most 2^k ways to choose some subset of these k primes. That is to say there there are not more than 2^k square-free integers whose prime factors are all less than of equal to p_k.
Now it also clear that there are at most $\sqrt{x_k}$ possible values for s_k, because if $s_k \geq \sqrt{x_k}$ then $s_k^2 \nmid x_k$.
This says that there are at most 2^k valid choices for r_k and at most $\sqrt{x}$ valid choices for s_k which means that $\alpha_k(x) = |M_k| \leq 2^k \sqrt{x}$.

Part b

If we let $N_{i,x}$ be the set of all integers less than or equal to x which are divisible by p_i. Then let N_x be the set of all integers less than or equal to x which are divisible by a prime greater than p_k. Then it is clear that

$$N_x = \bigcup_{i=k+1}^{\infty} N_{i,x}.$$

Now clearly $|N_{i,x}| \leq \frac{x}{p_i}$. Note in particular that when $p_i > x$ then $|N_{i,x}| = 0$. Now because $|N_x| = x - \alpha_k(x)$ it follows that

$$x - \alpha_k(x) \leq \sum_{i=k+1}^{\infty} |N_{i,x}| < \sum_{i=k+1}^{\infty} \frac{x}{p_i}.$$

Now by the assumption laid out at the beginning

$$\sum_{i=k+1}^{\infty} \frac{1}{p_i} < \frac{1}{2}$$

so

$$\sum_{i=k+1}^{\infty} \frac{x}{p_i} < \frac{x}{2}$$

and thus $x - \alpha_k(x) < \frac{x}{2}$ or $\alpha_k(x) > \frac{x}{2}$.
So we now have an upper and lower bound on $\alpha_k(x)$ given by

$$\frac{x}{2} < \alpha_k(x) < \frac{2^k}{\sqrt{x}}$$

but say $x = 2^{2k+4}$ then $2^{2k+3} < \alpha_k(x) < 2^{2k+2}$ which is clearly a contradiction, so our assumption that there exists a k such that

$$\sum_{m=k+1}^{\infty} \frac{1}{p_m} < \frac{1}{2}$$

is wrong and hence $\sum \frac{1}{p_i}$ must diverge.

Question 2

Part a

Show that if p is prime and $p|x^2+2$ then $p \equiv 1$ or $3 \mod 8$.

Part b

By considering $N = (p_1 \dots p_r)^2 + 2$, where $p_i \equiv 3 \mod 8$, prove that there are infinitely many primes congruent to $3 \mod 8$.

Solution

Part a

This statement is actually false because $2 \equiv 0^2 + 2 \mod 8$, however, for $x > 0$ the statement is true.
Note that for any integer x we have that $x^2 + 2 \equiv 2, 3, 6 \mod 8$, so if $pk = x^2 + 2$ for some integer k and prime p then we must have $pk \equiv 2, 3, 6 \mod 8$. Clearly $p \not\equiv 2, 6 \mod 8$ because then p would be divisible by 2 and hence not a prime, unless $p = 2$ and this case was dealt with at the start of the proof.
We are therefore left to consider the following 3 cases as such:

- So if $pk \equiv 3 \mod 8$ then we could have $p \equiv 1 \mod 8$, $k \equiv 3 \mod 8$ or $p \equiv 3 \mod 8$ and $k \equiv 1 \mod 8$.
- If $pk \equiv 2 \mod 8$ then we could have $p \equiv 1 \mod 8$ and $k \equiv 2 \mod 8$.
- If $pk \equiv 6 \mod 8$ then we could have $p \equiv 1 \mod 8$, $k \equiv 6 \mod 8$ or $p \equiv 3 \mod 8$ and $k \equiv 2 \mod 8$.

In each case $p \equiv 1$ or $3 \mod 8$. $\square$

Part b

Suppose there are finitely many primes of the form $p \equiv 3 \mod 8$. Then write $N = (p_1 \dots p_r) + 2$ where $(p_1 \dots p_r)$ is the product of all such primes. Note that $N \equiv 3 \mod 8$. Now consider the prime factorization of $N = (q_1^{\alpha_1} \dots q_j^{\alpha_j})$. From part a we have that each q_i must be either $q_i \equiv 1$ or $3 \mod 8$, but as $N \equiv 3 \mod 8$ there must be at least one q_k such that $q_k \equiv 3 \mod 8$, but this q_k cannot appear in the product $(p_1 \dots p_r)$ because $(p_1 \dots p_r)^2 + 2 \mod p_i \equiv 2 \mod p_i$ for all choices of p_i, whereas $q_k | (p_1 \dots p_r)^2 + 2$. This is a contradiction because $q_k \equiv 3 \mod 8$ but we assumed $(p_1 \dots p_r)$ was the product of all $3 \mod 8$ primes. So our assumption must be wrong and there must be an infinite number of primes congruent to $3 \mod 8$. $\square$

Question 3

Write $\pi^*(x)$ for the number of integers not greater than x, that are of the form p^k for some prime p and some integer k. (Hence $\pi^*(x)$ counts primes and prime powers.)

Part a

Explain why

$$\pi^*(x) = \pi(x) + \pi(x^{\frac{1}{2}}) + \pi(x^{\frac{1}{3}}) + \ldots + \pi(x^{\frac{1}{m}}),$$

where m is the largest integer such that $2^m \leq x$.

Part b

Suppose C is a constant such that $\pi(x) \leq \frac{Cx}{\log(x)} forall x \geq 2$. Explain why $\pi^*(x) - \pi(x) \leq \frac{Cx}{\log(x)}$ for all $x \geq 2$. Explain why $\pi * (x) - \pi(x) \leq \pi(x^{\frac{1}{2}}) + m\pi(x^{\frac{1}{3}})$, with m as in (a), and hence prove that

$$\pi * (x) = \pi(x) \leq 12C \frac{x^{\frac{1}{2}}}{\log(x)}$$

for all $x \geq 2$. (Hint: the inequality $x^{\frac{1}{3}} \log(x) \leq 6e^{-1} x^{\frac{1}{2}}$ may be useful.)

Part c

What does this tell us about prime powers? (Hint: plot $\frac{x^{\frac{1}{2}}}{\log(x)}$)

Solution

Part a

Note that

$$\begin{aligned} \pi(x^{\frac{1}{k}}) &= |\{p : p \leq x^{\frac{1}{k}}, p \text{ prime } \}| \\ &= |\{p : p^k \leq x, p \text{ prime } \}| \end{aligned}$$

so clearly

$$\pi^*(x) = \sum_{k=1}^{\infty} \pi(x^{\frac{1}{k}}) \tag{1}$$

but when $2^m > x$ then any prime p is such that $p^m > x$ and hence

$$\begin{aligned}\pi(x^{\frac{1}{m}}) &= |\{p : p \leq x^{\frac{1}{m}}\}| \\ &= |\{p : p^k \leq x\}| \\ &= 0\end{aligned}$$

and so we can rewrite 1 as

$$\pi^*(x) = \sum_{k=1}^{\lfloor \log_2(x) \rfloor} \pi(x^{\frac{1}{k}})$$

as required. $\square$

Part b

Let $m = \lfloor \log_2(x) \rfloor$ as it was (implicitly) in part (a). Then see that

$$\pi * (x) - \pi(x) = \pi(x^{\frac{1}{2}}) + \underbrace{\pi(x^{\frac{1}{3}}) \ldots + \pi(x^{\frac{1}{m}})}_{m-1 \text{ terms}}$$

so

$$\pi^*(x) - \pi(x) \leq \pi(x^{\frac{1}{2}}) + m\pi(x^{\frac{1}{3}})$$

and by the assumption $\pi(x) \leq \frac{Cx}{\log(x)}$ for all $x > 2$ then we have

$$\begin{aligned}\pi * (x) - \pi(x) &\leq C\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}\log(x)} + \frac{mx^{\frac{1}{3}}}{\frac{1}{3}\log(x)}\right) \\ &= C\left(\frac{2x^{\frac{1}{2}}}{\log(x)} - \frac{3\lfloor \log_2(x) \rfloor x^{\frac{1}{3}}}{\log(x)}\right) \\ &\leq C\left(\frac{2x^{\frac{1}{2}}}{\log(x)} - \frac{3\log_2(x) x^{\frac{1}{3}}}{\log(x)}\right) \\ &= C\left(\frac{2x^{\frac{1}{2}}}{\log(x)} - \frac{3x^{\frac{1}{3}}}{\log(2)}\right) \\ &= C\left(\frac{2x^{\frac{1}{2}}\log(2)}{\log(x)\log(2)} - \frac{3x^{\frac{1}{3}}\log(x)}{\log(2)\log(x)}\right)\end{aligned}$$

and by the inequality mentioned in the question we have

$$\leq \frac{Cx^{\frac{1}{2}}}{\log(x)}\left(\frac{2 + 18e^{-1}}{\log(2)}\right)$$

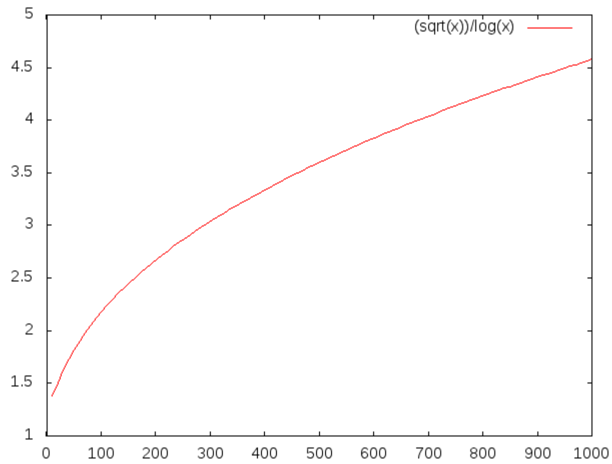
and by numerical evaluation of $\left(\frac{2+18e^{-1}}{\log(2)}\right) \approx 12.439$ we must have that

$$\frac{Cx^{\frac{1}{2}}}{\log(x)}\left(\frac{2 + 18e^{-1}}{\log(2)}\right) \leq 12C\frac{x^{\frac{1}{2}}}{\log(x)}$$

so

$$\pi^*(x) - \pi(x) \leq 12C\frac{x^{\frac{1}{2}}}{\log(x)}.$$

Part c



This tells as that as $x \longrightarrow \infty$ prime powers get "less common", in particular because

$$\frac{x^{\frac{1}{2}}}{\log(x)} < \frac{x}{\log(x)}$$

for sufficiently large x it tells us that in some sense prime powers are "less common" than primes.

Acknowledgements

The proof from question 1 is based off a proof by Erdos, and a similar proof can be found on Wikipedia at `http://en.wikipedia.org/wiki/Divergence_of_the_sum_of_the_reciprocals_of_the_primes`. This was pointed out to me by Adrian Miranda.