University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

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Question 1

Part a

Use the character table given in lectures for $\mathbb{Z}_5$, extended to a Dirichlet character,to evaluate

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)}, \qquad \text{for each } b \in \mathbb{U}_5.$$

Part b

Use the results of (a) to prove, in detail, that there are infinitely many primes congruent to 1 mod 5, 2 mod 5, and 3 mod 5 and 4 mod 5.

Solution

For this question we use the following character table:

	1	3	4	2
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	-i
χ_4	1	-i	-1	i

Part a

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)} = \begin{cases} 0 & \text{if } n \not\equiv b \mod 5 \\ 4 & \text{if } n \equiv b \mod 5 \end{cases}$$

This follows immediately from the orthogonality relation proved in lectures. (With the subtle difference that this is a Dirichlet character, but because $b \in \{1, 2, 3, 4\}$ this makes no difference.)

Part b

Firstly see that

$$\begin{aligned} L(s, \chi_i) &= \sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{\chi_i(p)}{p^s}\right)^{-1} \end{aligned}$$

and so

$$\begin{aligned}
\log L(s,\chi_i) &= -\sum_{n=1}^{\infty} \log\left(1-\frac{\chi_i(p)}{p^s}\right)\\
&= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \cdots\\
&= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + R_i(s)
\end{aligned}$$

where

$$R_i(s) = \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \frac{1}{4}\left(\frac{\chi_i(p)}{p^s}\right)^4 + \cdots.$$

See that

$$\begin{aligned}
|R_i(s)| &\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{|\chi_i(p)|}{p^s}\right)^2 + \frac{1}{3}\left(\frac{|\chi_i(p)|}{p^s}\right)^3 + \frac{1}{4}\left(\frac{|\chi_i(p)|}{p^s}\right)^4 + \cdots\\
&\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{1}{p^s}\right)^2 + \frac{1}{2}\left(\frac{1}{p^s}\right)^3 + \frac{1}{2}\left(\frac{1}{p^s}\right)^4 + \cdots\\
&\text{by the geometric sum formula}\\
&= \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p^s(p^s-1)}\\
&< \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p(p-1)}\\
&< \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n-1)}\\
&= \frac{1}{2}
\end{aligned}$$

and so $R_i(s)$ is bounded as $s \longrightarrow 1^+$.
Now see that

$$L(s,\chi_1) = \sum_{b=1}^{4}\sum_{n=0}^{\infty} \frac{1}{(5n+b)^s} \longrightarrow \infty \text{ as } s \longrightarrow 1^+.$$

For χ_2 see that

$$\begin{aligned}
\Re(L(s,\chi_2)) &= \underbrace{\frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s}}_{>\frac{1}{6}} + \underbrace{\frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{8^s} - \frac{1}{9^s}}_{>0} + \underbrace{\cdots}_{>0}\\
&> \frac{1}{6}
\end{aligned}$$

and

$$\Re(L(s,\chi_2)) = \frac{1}{1^s} - \underbrace{\left(\frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} - \frac{1}{6^s}\right)}_{>0} - \underbrace{\left(\frac{1}{7^s} + \frac{1}{8^s} - \frac{1}{9^s} - \frac{1}{11^s}\right)}_{>0} - \underbrace{\cdots}_{>0}$$
$$< 1$$

so $\frac{1}{6} < \Re(L(s,\chi_2)) < 1$ for all $s > 1$. Also note that

$$\Im(L(s,\chi_2)) = 0.$$

For χ_3 see that

$$\Re(L(s,\chi_3)) = \underbrace{\frac{1}{1^s} - \frac{1}{4^s}}_{>\frac{3}{4}} + \underbrace{\frac{1}{6^s} + \frac{1}{9^s}}_{>0} + \underbrace{\cdots}_{>0}$$
$$> \frac{3}{4}$$

and

$$\Re(L(s,\chi_3)) = \frac{1}{1^s} - \underbrace{\left(\frac{1}{4^s} - \frac{1}{6^s}\right)}_{>0} - \underbrace{\left(\frac{1}{9} - \frac{1}{11}\right)}_{>0} - \underbrace{\cdots}_{>0}$$
$$< 1$$

so $\frac{3}{4} < \Re(L(s,\chi_3)) < 1$ for $s > 0$. We can also see that

$$\Re(L(s,\chi_4)) = \Re(L(s,\chi_3))$$

and so $\frac{3}{4} < \Re(L(s,\chi_4)) < 1$ for $s > 0$.
Also note that

$$\Im(L(s,\chi_3)) = -\frac{1}{2} + \underbrace{\frac{1}{3} - \frac{1}{7}}_{>0} + \underbrace{\frac{1}{8} - \frac{1}{12}}_{>0} + \underbrace{\cdots}_{>0}$$
$$> -\frac{1}{2}$$

and

$$\Im(L(s,\chi_3)) = \underbrace{-\frac{1}{2} + \frac{1}{3}}_{<0} + \underbrace{-\frac{1}{7} + \frac{1}{8}}_{<0} + \underbrace{\cdots}_{<0}$$
$$< 0$$

so $-\frac{1}{2} < \Im(L(s,\chi_3)) < 0$.
Also $\Im(L(s,\chi_4)) = -\Im(L(s,\chi_3))$ so $0 < \Im(L(s,\chi_4)) < \frac{1}{2}$.

Now see that (by applying the result of part a)

$$\begin{aligned}\sum_{i=1}^{4} \chi_i(b)\log(L(s,\chi_i)) &= \sum_{p \text{ prime}} \frac{\sum_{i=1}^{4} \chi_i(b)\chi_i(p)}{p^s} + \underbrace{\sum_{i=1}^{4} \chi_i(b) R_i(s)}_{\text{bounded as } s \longrightarrow 1^+} \\ &= 4 \sum_{\substack{p \text{ prime} \\ p \equiv b \mod 5}} \frac{1}{p^s} + \underbrace{\sum_{i=1}^{4} \chi_i(b) R_i(s)}_{\text{bounded as } s \longrightarrow 1^+} .\end{aligned}$$

Now as only $\log(L(s,\chi_1))$ is unbounded as $s \longrightarrow 1^+$ then

$$\sum_{\substack{p \text{ prime} \\ p \equiv b \mod 5}} \frac{1}{p^s} \longrightarrow \infty \text{ as } s \longrightarrow 1^+$$

which implies that there are infinitely many primes $p \equiv b \mod 5$ for $b \in \{1,2,3,4\}$.

Question 2

Let χ be any Dirichlet character. Then, for $s > 1$, prove that

$$\frac{1}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s}$$

Solution

This is almost obvious.
Firstly we know that

$$L(s,\chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

so

$$\begin{aligned}
\frac{1}{L(s,\chi)} &= \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right) \\
&= \left(1 - \frac{\chi(2)}{2^s}\right)\left(1 - \frac{\chi(3)}{3^s}\right)\left(1 - \frac{\chi(5)}{5^s}\right)\cdots \\
&\text{Peter Brown: Deep breath...} \\
&= 1 - \frac{\chi(2)}{2^s} - \frac{\chi(3)}{3^s} - \frac{\chi(5)}{5^s} - \cdots \\
&\quad + \frac{\chi(2)\chi(3)}{2^s3^s} + \frac{\chi(2)\chi(5)}{2^s5^s} + \cdots + \frac{\chi(3)\chi(5)}{3^s5^s} + \cdots \\
&\quad - \frac{\chi(2)\chi(3)\chi(5)}{2^s3^s5^s} - \cdots \\
&\quad \vdots
\end{aligned}$$

and because χ is completely multiplicative

$$\begin{aligned}
\frac{1}{L(s,\chi)} &= 1 - \frac{\chi(2)}{2^s} - \frac{\chi(3)}{3^s} - \frac{\chi(5)}{5^s} - \cdots \\
&\quad + \frac{\chi(2\times 3)}{(2\times 3)^s} + \frac{\chi(2\times 5)}{(2\times 5)^s} + \cdots + \frac{\chi(3\times 5)}{(3\times 5)^s} + \cdots \\
&\quad - \frac{\chi(2\times 3\times 5)}{(2\times 3\times 5)^s} - \cdots \\
&\quad \vdots
\end{aligned}$$

It's clear from the definition of μ that this can be rewritten as

$$\frac{1}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s}.$$

Question 3

Suppose χ_4 and χ_6 are the (unique) non-principle characters modulo 4 and 6 respectively. Show that $L(1,\chi_4) = \frac{\pi}{4}$ and $L(1,\chi_6) = \frac{\pi}{2\sqrt{3}}$.

Solution

We have that for a non-principle character modulo k

$$L(1,\chi) = \int_0^1 \frac{\lambda(t)}{1-t^k} dt$$

where $\lambda(t) = \sum_{n=1}^{k} \chi(n) t^{n-1}$.
In the case of χ_4 we have that $\lambda(t) = 1 - t^2$, and so we have the evaluate

$$\begin{aligned}\int_0^1 \frac{1-t^2}{1-t^4} dt &= \int_0^1 \frac{(1-t^2)}{(1-t^2)(1+t^2)} dt \\ &= \left[\tan^{-1}(t)\right]_1^1 \\ &= \frac{\pi}{4}\end{aligned}$$

so $L(1, \chi_4) = \frac{\pi}{4}$.
In the case of χ_6 we have that $\lambda(t) = 1 - t^4$, and so we evaluate

$$\begin{aligned}\int_0^1 \frac{1-t^4}{1-t^6} dt &= \int_0^1 \frac{(1-t^2)(1+t^2}{(1-t^2)(1+t^2+t^4)} dt \\ &= \int_0^1 \frac{(1+t^2)}{(1+t^2+t^4)} \\ &= \int_0^1 \frac{(1+\frac{1}{t^2}}{(\frac{1}{t^2}+1+t^2)} dt.\end{aligned}$$

Let $x = t - \frac{1}{t}$ and note that $dx = (1 + \frac{1}{t^2}) dt$ and note that as $t \longrightarrow 0+$, $x \longrightarrow -\infty$ and when $t = 1$, $x = 0$. So we have that

$$\begin{aligned}\int_0^1 \frac{(1+\frac{1}{t^2}}{(\frac{1}{t^2}+1+t^2)} dt &= \int_{-\infty}^0 \frac{1}{x^2+3} dx \\ &= \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_{x \longrightarrow -\infty}^0 \\ &= \frac{\pi}{2\sqrt{3}}\end{aligned}$$

so $L(1, \chi_6) = \frac{\pi}{2\sqrt{3}}$.

Acknowledgements

Thank you to Roberto for reminding me to bound the imaginary parts of the L functions in question 1 part b.