University of New South Wales

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# Assignment

Number Theory

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# Question 1

Write $\omega(n)$ for the number of (distinct) prime divisors of $n$, $\Omega(n)$ for the number of prime factors of $n$, counted with repetition. Thus if, $n = \prod_{j=1}^{m} p_j^{k_j}$, then $\omega(n) = m$, and $\Omega(n) = \sum_{j=1}^{m} k_j$.

### Part a

Prove that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$ for $n \geq 2$.

### Part b

When does $\tau(n) = 2^{\omega(n)}$.

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## Solution

### Part a

Firstly we prove that $2^{\omega}(n) \leq \tau(n)$. From the lecture notes we have that if $n = \prod_{j=1}^{m} p_j^{k_j}$ then $\tau(n) = \prod_{j=1}^{m}(k_j + 1)$ so we can say

$$\begin{aligned}
\tau(n) &= \prod_{j=1}^{m} \underbrace{(k_j + 1)}_{\geq 2} \\
&\leq \prod_{j=1}^{m} 2 \qquad (1) \\
&= 2^m \\
&= 2^{\omega(n)}
\end{aligned}$$

so $2^{\omega(n)} \leq \tau(n)$.
We now show that $\tau(n) \leq 2^{\Omega(n)}$. See that

$$\begin{aligned}
2^{\Omega(n)} &= 2^{\sum_{j=1}^{m} k_j} \\
&= \prod_{j=1}^{m} 2^{k_j}
\end{aligned}$$

and because for all $k_j \geq 1$, $k_j + 1 \leq 2^{k_j}$. then

$$\prod_{j=1}^{m} 2^{k_j} \geq \prod_{j=1}^{m} (k_j + 1)$$

and thus

$$\tau(n) \leq 2^{\Omega(n)}.$$

It remains to show that $2^{\Omega(n)} \leq n$. Because we have that

$$n = \prod_{j=1}^{m} p_j^{k_j}$$

and

$$2^{\Omega(n)} = \prod_{j=1}^{m} 2^{k_j}$$

then it is clear because $2 \leq k_j$ for all $j$.
So we have shown that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$.

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# Question 2

Define the **Jordan totient function** by

$$n^k \prod_{p|n} (1 - p^- k),$$

where, as usual, the product is taken over the primes. This is a generalization of Euler's totient function.

### Part a

Prove that $J$ is multiplicative.

### Part b

Show that $J_k(n) = \sum_{d|n} \mu(d) (\frac{n}{d})^k$.

### Part c

Find a simple expression (as a product over primes) for $J^{-1}(n)$.

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## Solution

### Part a

For $(n, m) = 1$ see that

$$J_k(nm) = (nm)^k \prod_{p|nm} (1 - p^{-k})$$

but as $(n, m) = 1$ then $p|n$ or $p|m$ but not both so

$$\begin{aligned} J_k(nm) &= n^k m^k \prod_{p|n}(1 - p^{-k}) \prod_{p|m}(1 - p^{-k}) \\ &= \left(n^k \prod p|n(1 - p^{-k})\right) \left(n^k \prod q|m(1 - q^{-k})\right) \\ &= J_k(n) J_k(m) \end{aligned}$$

□

## Part b

Let $p$ be a prime and observe that

$$J_k(p^\alpha) = p^{\alpha k}(1 - p^{-k})$$

and note in addition that $J_k(1) = 1$. Now define

$$\gamma_k(n) := \sum_{d|n} J_k(d)$$

and note that since $J_k$ is multiplicative, so is $\gamma_k$.
Therefor it we only need to see consider $\gamma_k(p^\alpha)$, where $p$ is prime, to characterize $\gamma_k$.
Therefore observe that

$$\begin{aligned} \gamma_k(p^\alpha) &= \sum_{j=0}^{\alpha} J_k(p^\alpha) \\ &= \sum_{j=1}^{\alpha} p^{jk}(1 - p^k) + 1 \\ &= (1 - p^{-k}) \sum_{j=1}^{\alpha} p^{jk} + 1 \\ &= (1 - p^{-k}) \left(\frac{p^k - p^{k(\alpha+1)}}{1 - p^k}\right) + 1 \\ &= p^{\alpha k}. \end{aligned}$$

So if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ then

$$\begin{aligned} \gamma_k(n) &= \gamma_k(p_1^{\alpha_1}) \dots \gamma_k(p_r^{\alpha_r}) \\ &= p_1^{\alpha_1 k} \dots p_r^{\alpha_r k} \\ &= n^k. \end{aligned}$$

So we have that

$$\sum_{d|n} J_k(d) = n^k$$

and hence by Mobius inversion

$$J_k(d) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k.$$

$\square$

### Part c

Let $N^k(n) = n^k$. It is clear that $N^k$ is completely multiplicative. Also see that $J_k = \mu * N^k$. So by the result in the notes, since $N^k$ is completely multiplicative $J_k^{-1} = N^k$. $\square$

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# Question 3

Write $M_2(x) = \sum n \leq x(\mu(n))^2$.
Hence $M_2(x)$ counts the number of square free integers $\leq x$.

### Part a

Explain why $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$.

### Part b

Proce that $M_2(x) = x \sum_{m\leq\sqrt{x}} \frac{\mu(n)}{m^2} - \sum_{m\leq\sqrt{x}} \mu(m) \left\{\frac{x}{m^2}\right\}$.

### Part c

Deduce that

$$M_2(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

### Part d

Interpret this result in terms of the proportion of square-free numbers in the interval $[1, x]$.

### Part e

Use MAPLE (or otherwise) to count the number of square-free numbers between 1 and 1000. (Comment!)

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## Solution

### Part a