University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

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Question 1

Part a

Use the character table given in lectures for $\mathbb{Z}_5$, extended to a Dirichlet character,to evaluate

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)}, \qquad \text{for each } b \in \mathbb{U}_5.$$

Part b

Use the results of (a) to prove, in detail, that there are infnitely many primes congruent to 1 mod 5, 2 mod 5, and 3 mod 5 and 4 mod 5.

Solution

For this question we use the following character table:

	1	3	4	2
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	-i
χ_4	1	-i	-1	i

Part a

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)} = \begin{cases} 0 & \text{if } n \not\equiv b \mod 5 \\ 4 & \text{if } n \equiv b \mod 5 \end{cases}$$

This follows immediatly from the orthogonality relation proved in lectures.

Part b

Firstly see that

$$\begin{aligned} L(s, \chi_i) &= \sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{\chi_i(p)}{p^s}\right)^{-1} \end{aligned}$$

and so

$$\begin{aligned}
\log L(s,\chi_i) &= -\sum_{n=1}^{\infty} \log\left(1 - \frac{\chi_i(p)}{p^s}\right) \\
&= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \cdots \\
&= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + R_i(s)
\end{aligned}$$

where

$$R_i(s) = \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \frac{1}{4}\left(\frac{\chi_i(p)}{p^s}\right)^4 + \cdots.$$

See that

$$\begin{aligned}
|R_i(s)| &\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{|\chi_i(p)|}{p^s}\right)^2 + \frac{1}{3}\left(\frac{|\chi_i(p)|}{p^s}\right)^3 + \frac{1}{4}\left(\frac{|\chi_i(p)|}{p^s}\right)^4 + \cdots \\
&\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{1}{p^s}\right)^2 + \frac{1}{2}\left(\frac{1}{p^s}\right)^3 + \frac{1}{2}\left(\frac{1}{p^s}\right)^4 + \cdots \\
&\quad \text{by the geometric sum formula} \\
&= \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p^s(p^s-1)} \\
&< \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p(p-1)} \\
&< \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n-1)} \\
&= \frac{1}{2}
\end{aligned}$$

and so $R_i(s)$ is bounded as $s \longrightarrow 1^+$.
Now see that

$$L(s,\chi_1) = \sum_{b=1}^{4}\sum_{n=0}^{\infty} \frac{1}{(5n+b)^s} \longrightarrow \infty \text{ as } s \longrightarrow 1^+.$$

For χ_2 see that

$$\begin{aligned}
\Re(L(s,\chi_2)) &= \underbrace{\frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s}}_{>\frac{1}{6}} + \underbrace{\frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{8^s} - \frac{1}{9^s}}_{>0} + \underbrace{\cdots}_{>0} \\
&> \frac{1}{6}
\end{aligned}$$

and

$$\begin{aligned}\Re(L(s,\chi_2)) &= \frac{1}{1^s} - \underbrace{\left(\frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} - \frac{1}{6^s}\right)}_{>0} - \underbrace{\left(\frac{1}{7^s} + \frac{1}{8^s} - \frac{1}{9^s} - \frac{1}{11^s}\right)}_{>0} - \underbrace{\cdots}_{>0} \\ &< 1\end{aligned}$$

so $\frac{1}{6} < \Re(L(s,\chi_2)) < 1$ for all $s > 1$.
For χ_3 see that

$$\begin{aligned}\Re(L(s,\chi_3)) = \underbrace{\frac{1}{1^s} - \frac{1}{4^s}}_{>\frac{3}{4}} + \underbrace{\frac{1}{6^s} + \frac{1}{9^s}}_{>0} + \underbrace{\cdots}_{>0} \\ > \frac{3}{4}\end{aligned}$$

and

$$\begin{aligned}\Re(L(s,\chi_3)) &= \frac{1}{1^s} - \underbrace{\left(\frac{1}{4^s} - \frac{1}{6^s}\right)}_{>0} - \underbrace{\left(\frac{1}{9} - \frac{1}{11}\right)}_{>0} - \underbrace{\cdots}_{>0} \\ &< 1\end{aligned}$$

so $\frac{3}{4} < \Re(L(s,\chi_3)) < 1$ for $s > 0$. We can also see that

$$\Re(L(s,\chi_4)) = \Re(L(s,\chi_3))$$

and so $\frac{3}{4} < \Re(L(s,\chi_4)) < 1$ for $s > 0$.
Now see that (by appling the result of part a)

$$\begin{aligned}\sum_{i=1}^{4} \chi_i(b)\log(L(s,\chi_i)) &= \sum_{p \text{ prime}} \frac{\sum_{i=1}^{4} \chi_i(b)\chi_i(p)}{p^s} + \underbrace{\sum_{i=1}^{4} \chi_i(b) R_i(s)}_{\text{bounded as } s \longrightarrow 1^+} \\ &= 4 \sum_{\substack{p \text{ prime} \\ p \equiv b \mod 5}} \frac{1}{p^s} + \underbrace{\sum_{i=1}^{4} \chi_i(b) R_i(s)}_{\text{bounded as } s \longrightarrow 1^+} .\end{aligned}$$

Now as only $\log(L(s,\chi_1))$ is unbounded as $s \longrightarrow 1^+$ then

$$\sum_{\substack{p \text{ prime} \\ p \equiv b \mod 5}} \frac{1}{p^s} \longrightarrow \infty \text{ as } s \longrightarrow 1^+$$

which implies that there are infinitly many primes $p \equiv b \mod 5$ for $b \in \{1, 2, 3, 4\}$.