UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Assignment

Number Theory

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Question 1

Part a

Use the character table given in lectures for $\mathbb{Z}_5$, extended to a Dirichlet character,to evaluate

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)}, \qquad \text{for each } b \in \mathbb{U}_5.$$

Part b

Use the results of (a) to prove, in detail, that there are infnitely many primes congruent to 1 mod 5, 2 mod 5, and 3 mod 5 and 4 mod 5.

Solution

Part a

$$\sum_{i=1}^{4} \chi_i(n)\overline{\chi_i(b)} = \begin{cases} 0 & \text{if } n \not\equiv b \mod 5 \\ 4 & \text{if } n \equiv b \mod 5 \end{cases}$$

This follows immediatly from the orthogonality relation proved in lectures.

Part b

Firstly see that

$$\begin{aligned} L(s, \chi_i) &= \sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{\chi_i(p)}{p^s}\right)^{-1} \end{aligned}$$

and so

$$\begin{aligned} \log L(s, \chi_i) &= -\sum_{n=1}^{\infty} \log\left(1 - \frac{\chi_i(p)}{p^s}\right) \\ &= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \cdots \\ &= \sum_{p \text{ prime}} \frac{\chi_i(p)}{p^s} + R_i(s) \end{aligned}$$

where

$$R_i(s) = \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{\chi_i(p)}{p^s}\right)^2 + \frac{1}{3}\left(\frac{\chi_i(p)}{p^s}\right)^3 + \frac{1}{4}\left(\frac{\chi_i(p)}{p^s}\right)^4 + \cdots.$$

See that

$$\begin{aligned}
|R_i(s)| &\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{|\chi_i(p)|}{p^s}\right)^2 + \frac{1}{3}\left(\frac{|\chi_i(p)|}{p^s}\right)^3 + \frac{1}{4}\left(\frac{|\chi_i(p)|}{p^s}\right)^4 + \cdots \\
&\leq \sum_{p \text{ prime}} \frac{1}{2}\left(\frac{1}{p^s}\right)^2 + \frac{1}{2}\left(\frac{1}{p^s}\right)^3 + \frac{1}{2}\left(\frac{1}{p^s}\right)^4 + \cdots \\
&\text{by the geometric sum formula} \\
&= \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p^s(p^s-1)} \\
&< \frac{1}{2}\sum_{p \text{ prime}} \frac{1}{p(p-1)} \\
&< \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n-1)} \\
&= \frac{1}{2}
\end{aligned}$$

and so $R_i(s)$ is bounded as $s \longrightarrow 1^+$.