University of New South Wales

School of Mathematics and Statistics

Assignment

Number Theory

Author:
Adam J. Gray

Student Number:
3329798

Question 1

Write $\omega(n)$ for the number of (distinct) prime divisors of n, $\Omega(n)$ for the number of prime factors of n, counted with repetition. Thus if, $n = \prod_{j=1}^{m} p_j^{k_j}$, then $\omega(n) = m$, and $\Omega(n) = \sum_{j=1}^{m} k_j$.

Part a

Prove that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$ for $n \geq 2$.

Part b

When does $\tau(n) = 2^{\omega(n)}$.

Solution

Part a

Firstly we prove that $2^{\omega}(n) \leq \tau(n)$. From the lecture notes we have that if $n = \prod_{j=1}^{m} p_j^{k_j}$ then $\tau(n) = \prod_{j=1}^{m}(k_j + 1)$ so we can say

$$\begin{aligned} \tau(n) &= \prod_{j=1}^{m} \underbrace{(k_j + 1)}_{\geq 2} \\ &\leq \prod_{j=1}^{m} 2 \qquad (1) \\ &= 2^m \\ &= 2^{\omega(n)} \end{aligned}$$

so $2^{\omega(n)} \leq \tau(n)$.
We now show that $\tau(n) \leq 2^{\Omega(n)}$. See that

$$\begin{aligned} 2^{\Omega(n)} &= 2^{\sum_{j=1}^{m} k_j} \\ &= \prod_{j=1}^{m} 2^{k_j} \end{aligned}$$

and because for all $k_j \geq 1$, $k_j + 1 \leq 2^{k_j}$. then

$$\prod_{j=1}^{m} 2^{k_j} \geq \prod_{j=1}^{m}(k_j + 1)$$

and thus

$$\tau(n) \leq 2^{\Omega(n)}.$$

It remains to show that $2^{\Omega(n)} \leq n$. Because we have that

$$n = \prod_{j=1}^{m} p_j^{k_j}$$

and

$$2^{\Omega(n)} = \prod_{j=1}^{m} 2^{k_j}$$

then it is clear because $2 \leq k_j$ for all j.
So we have shown that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)} \leq n$.

Question 2

Define the **Jordan totient function** by

$$n^k \prod_{p|n} (1 - p^- k),$$

where, as usual, the product is taken over the primes. This is a generalization of Euler's totient function.

Part a

Prove that J is multiplicative.

Part b

Show that $J_k(n) = \sum_{d|n} \mu(d)(\frac{n}{d})^k$.

Part c

Find a simple expression (as a product over primes) for $J^{-1}(n)$.

Solution

Part a

For $(n, m) = 1$ see that

$$J_k(nm) = (nm)^k \prod_{p|nm} (1 - p^{-k})$$

but as $(n, m) = 1$ then $p|n$ or $p|m$ but not both so

$$\begin{aligned}
J_k(nm) &= n^k m^k \prod_{p|n}(1 - p^{-k}) \prod_{p|m}(1 - p^{-k}) \\
&= \left(n^k \prod p|n(1 - p^{-k})\right) \left(n^k \prod q|m(1 - q^{-k})\right) \\
&= J_k(n)J_k(m)
\end{aligned}$$

□

Part b

Let p be a prime and observe that

$$J_k(p^\alpha) = p^{\alpha k}(1 - p^{-k})$$

and note in addition that $J_k(1) = 1$. Now define

$$\gamma_k(n) := \sum_{d|n} J_k(d)$$

and note that since J_k is multiplicative, so is γ_k.
Therefor it we only need to see consider $\gamma_k(p^\alpha)$, where p is prime, to characterize γ_k.
Therefore observe that

$$\begin{aligned}
\gamma_k(p^\alpha) &= \sum_{j=0}^{\alpha} J_k(p^\alpha) \\
&= \sum_{j=1}^{\alpha} p^{jk}(1 - p^k) + 1 \\
&= (1 - p^{-k}) \sum_{j=1}^{\alpha} p^{jk} + 1 \\
&= (1 - p^{-k}) \left(\frac{p^k - p^{k(\alpha+1)}}{1 - p^k}\right) + 1 \\
&= p^{\alpha k}.
\end{aligned}$$

So if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ then

$$\begin{aligned}
\gamma_k(n) &= \gamma_k(p_1^{\alpha_1}) \dots \gamma_k(p_r^{\alpha_r}) \\
&= p_1^{\alpha_1 k} \dots p_r^{\alpha_r k} \\
&= n^k.
\end{aligned}$$

So we have that

$$\sum_{d|n} J_k(d) = n^k$$

and hence by Mobius inversion

$$J_k(d) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k .$$

$\square$

Part c

Let $N^k(n) = n^k$. We can then write that $J_k = N^k * \mu$. So

$$\begin{aligned} J_k^{-1} &= (N^k * \mu)^{-1} \\ &= \mu^{-1} * N^{k-1}. \end{aligned}$$

As it is clear that N_k is completely multiplicative so $N^{k-1}(n) = \mu(n)N^k$, and hence

$$\begin{aligned} J_k^{-1}(n) &= \sum_{d|n} u(d)\mu\left(\frac{n}{d}\right) N^k \left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mu(d) N^k(d) \end{aligned}$$

Now as $\mu(n)N^k$ is multiplicative so J_k^{-1} is the Dirichlet product of two multiplicative functions, and is hence multiplicative. For p prime we have that

$$\begin{aligned} J_k^{-1}(p^\alpha) &= \sum_{j=0}^{\alpha} \mu(p^j) p^{jk} \\ &= (1 - p^k) \end{aligned}$$

So if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where the p_j are distinct primes, then

$$J_k^{-1}(n) = \prod_{j=1}^{r} \left(1 - p^k\right) .$$

$\square$

Question 3

Write $M_2(x) = \sum n \leq x (\mu(n))^2$.
Hence $M_2(x)$ counts the number of square free integers $\leq x$.

Part a

Explain why $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$.

Part b

Proce that $M_2(x) = x \sum_{m \leq \sqrt{x}} \frac{\mu(n)}{m^2} - \sum_{m \leq \sqrt{x}} \mu(m) \left\{ \frac{x}{m^2} \right\}$.

Part c

Deduce that

$$M_2(x) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

Part d

Interpret this result in terms of the proportion of square-free numbers in the interval $[1, x]$.

Part e

Use MAPLE (or otherwise) to count the number of square-free numbers between 1 and 1000. (Comment!)

Solution

Part a

Write $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where p_j is prime and all p_j are distinct. Without loss of generality assume we can reorder this factorization such that there exists a $1 \leq t \leq r$ such that when $j < t$, $2 \leq \alpha_j$. Then it can be easily seen that

$$\begin{aligned}
\sum_{m^2|n} \mu(m) &= 1 + \mu(p_1) + \mu(p_2) + \ldots + \mu(p_{t-1}) \\
&\quad + \mu(p_1 p_2) + \mu(p_1 p_3) + \ldots + \mu(p_{t-2} p_{t-1}) \\
&\quad + \mu(p_1 p_2 p_3) + \ldots + \mu(p_{t-3} p_{t-2} p_{t-1}) \\
&\quad + \ldots \\
&\quad + \mu(p_1 p_2 \ldots p_{t-1}) \\
&= \binom{t-1}{0}(-1)^0 + \binom{t-1}{1}(-1)^1 + \ldots \binom{t-1}{t-1}(-1)^{t-1} \\
&= \begin{cases} (1-1)^{t-1} = 0 & \text{where } n \text{ is not squarefree} \\ 1 & \text{where } n \text{ is squarefree} \end{cases}
\end{aligned}$$

It should be noted that higher powers of primes do not appear in the sum because they are not squarefree and hence mobuis function would be 0 for these numbers.
Since it is clear that

$$(\mu(n))^2 = \begin{cases} (1-1)^{t-1} = 0 & \text{where } n \text{ is not squarefree} \\ 1 & \text{where } n \text{ is squarefree} \end{cases}$$

we have shown that $(\mu(n))^2 = \sum_{m^2|n} \mu(m)$. $\square$