

Solution to a Simple FDE

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^C\mathcal{D}_0^\alpha y\right)(t) = \beta y(t) \quad (1)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \quad (2)$$

has the solution $y(t) = E_\alpha(\beta t^\alpha)$. Where E_α is the one parameter Mittag-Leffler function.

This solution is arrived at by a Laplace transform method.

Definition 1 (Fractional Derivatives and Integrals). *For $\alpha > 0$ we define*

$$\begin{aligned} (I_{a+}^\alpha f)(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \\ (\mathcal{D}_{a+}^\alpha f)(x) &:= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \\ ({}^C\mathcal{D}_{a+}^\alpha f)(x) &:= I_0^{n-\alpha} \frac{d^n}{dx^n} f(x) \end{aligned}$$

where $n = \lceil \alpha \rceil - 1$. We will refer to $I_{a+}^\alpha f$ as the (Riemann-Liouville) integral f of order α (based at a). Likewise we refer to $\mathcal{D}_{a+}^\alpha f$ as the (Riemann-Liouville) derivative of order α (based at a). We also refer to ${}^C\mathcal{D}_{a+}^\alpha f$ as the Caputo derivative of order α (based at a).

The motivation for these definitions are based on the Cauchy formula for repeated integration, and in the case of the Caputo derivative, practical considerations. [4, 3]

For the rest of our considerations we will take $a = 0$ (based at 0).

We now consider the Laplace transform of the fractional integration and differentiation operators.

Lemma 1.

$$\mathcal{L}\{I_0^\alpha f\} = s^{-\alpha} \mathcal{L}\{f\}$$

Proof. Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of f with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned} \mathcal{L}\{I_0^\alpha f\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\ &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\{f(t)\} \underbrace{\mathcal{L}\{t^{\alpha-1}\}}_{=s^{-\alpha}\Gamma(\alpha)} \\ &= s^{-\alpha} \mathcal{L}\{f\}. \end{aligned}$$

□

Lemma 2.

$$\mathcal{L}\{\mathcal{D}_0^\alpha f\} = s^\alpha \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^k (\mathcal{D}_0^{\alpha-k-1} f)(0)$$

Proof. See that

$$\begin{aligned} \mathcal{L}\{(\mathcal{D}_0^\alpha f)\} &= \mathcal{L}\left\{\frac{d^n}{dt^n} (I_0^{n-\alpha} f)\right\} \\ &= s \mathcal{L}\{(I_0^{n-\alpha} f)\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dt^{n-k-1}} (I_0^{n-\alpha} f)(0) \\ &= s \mathcal{L}\{(I_0^{n-\alpha} f)\} - \sum_{k=0}^{n-1} s^k (\mathcal{D}_0^{\alpha-k-1} f)(0). \end{aligned}$$

□

Lemma 3.

$$\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} = s^{\alpha-n} \left[s^n \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1} \left(\frac{d^k f}{dt^k}\right)(0) \right]$$

Proof. See that

$$\begin{aligned} \mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} &= \mathcal{L}\left\{\left(I_0^{n-\alpha} \frac{d^n f}{dt^n}\right)\right\} \\ &= \frac{1}{\Gamma(n-\alpha)} \mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1} \frac{d^n f}{dt^n} du\right\} \end{aligned}$$

which is the Laplace transform of a convolution so

$$\begin{aligned}\Gamma(n-\alpha)\mathcal{L}\left\{\int_0^t(t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\} &= \mathcal{L}\{t^{n-\alpha-1}\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\ &= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right)\left(s^n\mathcal{L}\{f\}-\sum_{k=0}^{n-1}s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\ &= s^{\alpha-n}\left[s^n\mathcal{L}\{f\}-\sum_{k=0}^{n-1}s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].\end{aligned}$$

□

We now define the Mittag-Leffler function and calculate its Laplace transform.

Definition 2. The one parameter Mittag-Leffler E_α function is defined by its power series.

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$$

It is clear to see the definition of this function is inspired by the exponential function. Before we can calculate the Laplace transform of the Mittag-Leffler function we have to prove a simple lemma about the convergence of the series which is used in its definition.

Lemma 4. The series

$$\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$$

converges absolutely for all $t \in \mathbb{R}$.

Proof. Let $a_k = \frac{t^k}{\Gamma(\alpha k + 1)}$ and see that

$$\left|\frac{a_{k+1}}{a_k}\right| = |t| \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)}$$

and that hence

$$\lim_{k \rightarrow \infty} \left|\frac{a_{k+1}}{a_k}\right| = 0$$

for all $t \in \mathbb{R}$ so by the ratio test, the series $\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$ converges for all $t \in \mathbb{R}$. □

Lemma 5.

$$\mathcal{L}\{E_\alpha(\beta t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

Proof. See that

$$\mathcal{L}\{E_\alpha(\beta t^\alpha)\} = \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ (lemma 4) we may interchange the integral and the sum to get

$$\begin{aligned} \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt &= \sum_{k=0}^\infty \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt \\ &= \sum_{k=0}^\infty \frac{\beta^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-st} t^{\alpha k} dt. \end{aligned}$$

By performing the change of variables $x = st$ we get that

$$\begin{aligned} \sum_{k=0}^\infty \frac{\beta^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-st} t^{\alpha k} dt &= \sum_{k=0}^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k + 1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k + 1)} \\ &= \sum_{k=0}^\infty \beta^k s^{-(\alpha k + 1)} \\ &= \frac{s^{\alpha-1}}{s^\alpha - \beta}. \end{aligned}$$

So we have that

$$\mathcal{L}\{E_\alpha(\beta t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

as required. □

We now have sufficient tools to attack the original problem, that is finding a solution to (1), (2).

Lemma 6. *The FDE defined in (1) and (2), restated here for completeness*

$$\left({}^C\mathcal{D}_0^\alpha y\right)(t) = \beta y(t)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases}$$

has solution $y(t) = E_\alpha(\beta t^\alpha)$.

Proof. Taking the Laplace transform of both sides of (1) yields

$$\begin{aligned}\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha y\right)\right\} &= \beta \mathcal{L}\{y\} \\ s^{-(n+\alpha)}\left[s^n \mathcal{L}\{y\} - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0)\right] &= \beta \mathcal{L}\{y\}\end{aligned}$$

by the result of lemma 3. Then taking into account (2) we get

$$s^{-(n+\alpha)}\left[s^n \mathcal{L}\{y\} - s^{n-1}\right] = \beta \mathcal{L}\{y\}$$

and so

$$\mathcal{L}\{y\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

By using the result of lemma 5 we have that

$$y(t) = E_\alpha(\beta t^\alpha)$$

□

References

- [1] K. Diethelm and N.J. Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265:229–248, 2002.
- [2] A.N. Kolmogorov and S.V. Fomin. *Introductory Real Analysis*. Dover, 1975.
- [3] I. Podlubny. *Fractional Differential Equations*. Academic Press, 1999.
- [4] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives*. Breach Science Publishers, 1993.