A Solution to a Fractional Differential Equation

The Laplace Transform Method

Adam J. Gray

School of Mathematics and Statistics
University of New South Wales

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The Goal

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases}$$

Motivations

Cauchy Formula for Repeated Integration

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n)dy_n \cdots dy_2 dy_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt$$

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The idea is to replace the factorials with gamma functions to define an integral of arbitrary order

Riemann-Liouville Fractional Integral

$$\left(I_a^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

Motivations (Derivatives)

Riemann-Liouville Fractional Derivative

$$\left(\mathcal{D}_a^\alpha f\right)(x) = \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}}\left(I_a^{\lceil\alpha\rceil-\alpha} f\right)(x)$$
$$= \frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)dt}{\underline{(x-t)^{\alpha-n+1}}}$$

where $n-1 < \alpha \leq n$

Motivations (Derivatives)

Caputo Fractional Derivative

$$\left({}^{C}\mathcal{D}_a^{\alpha} f\right)(x) = \left(I_a^{\lceil\alpha\rceil-\alpha} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} f\right)(x)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\frac{d^t}{dt^n} f(t) dt}{(x-t)^{\alpha-n+1}}$$

where $n-1 < \alpha \leq n$

Riemann-Liouville vs Caputo Derivative

Note!

The Caputo derivative and the and the Riemann-Liouville derivatives are note the same. In general

$$\left({}^{C}\mathcal{D}_a^\alpha f\right)(x) \neq \left(\mathcal{D}_a^\alpha f\right)(x).$$

The reason is exactly the same reason that in general

$$f(x) \neq \int_a^x f'(t)dt.$$

In some sense if you differentiate first you "lose information" about the function.

Riemann-Liouville vs Caputo Derivative

The Caputo derivative is often used in fractional differential equations because it can be coupled with integer order initial conditions, whereas often the Riemann-Liouville derivative can't be coupled with integer order initial conditions.

A Quick Note on the Laplace Transform

Definition

We the define the Laplace transform of a function f to be the function F given by

$$F(s) := \int_0^\infty e^{-st} f(t) dt$$

We often write $F(s) = \mathcal{L}\{f(t)\}$.

A Quick Note on the Laplace Transform

The Laplace transform is particularly useful as it allows us to transform a differential equation into an "algebraic" equation. Lerch's theorem guarantees, with minor caveats, that the Laplace transform of a function is unique.

Basic Idea of the Laplace Transform Method

- Apply the Laplace transform to both sides of the differential equation to get and "algebraic" equation.
- Apply the Laplace transform to the initial conditions.
- Sub the transformed initial conditions into the transformed equation.
- Rearrange to get an expression for the Laplace transform of the function of interest.
- Invert. (This is possible, and guaranteed with minor caveats by Lerch's theorem)

The Laplace Transform of the Riemann-Liouville Integral

Lemma

The Laplace transform of the Riemann-Liouville integral of a function f is given by

$$\mathcal{L}\left\{I_0^{\alpha} f\right\} = s^{-\alpha} \mathcal{L}\left\{f\right\}.$$

The Laplace Transform of the Riemann-Liouville Integral[Proof]

Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of f with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{f(t)\right\} \underbrace{\mathcal{L}\left\{t^{\alpha-1}\right\}}_{=s^{-\alpha}\Gamma(\alpha)} \\
&= s^{-\alpha} \mathcal{L}\left\{f\right\}.
\end{aligned}$$

The Laplace Transform of the Caputo Derivative

The Laplace transform of the Caputo derivative of a function f is given by

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].$$

The Laplace Transform of the Caputo Derivative [Proof]

See that

$$\begin{aligned}\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} &= \mathcal{L}\left\{\left(I_0^{n-\alpha}\frac{d^n f}{dt^n}\right)\right\} \\ &= \underbrace{\frac{1}{\Gamma(n-\alpha)}\mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\}}_{\circledast}\end{aligned}$$

The Laplace Transform of the Caputo Derivative [Proof]

$\circledast$ is just the Laplace transform of a convolution so

$$\begin{aligned}\circledast &= \mathcal{L}\left\{t^{n-\alpha-1}\right\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\ &= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right) \\ &\quad \times \left(s^n\mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\ &= s^{\alpha-n}\left[s^n\mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].\end{aligned}$$

$\square$

One Parameter Mittag-Lefler Function

Definition

The one parameter Mittag-Lefler E_α function is defined by its power series.

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$

Lemma

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

See that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ we may interchange the integral and the sum to get

$$\begin{aligned}\int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt &= \sum_{k=0}^\infty \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} dt \\ &= \sum_{0}^\infty \frac{\beta^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-st} t^{\alpha k} dt.\end{aligned}$$

Laplace Transform of $E_\alpha(\beta t^\alpha)$ [Proof]

By performing the change of variables $x = st$ we get that

$$\sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt = \sum_0^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k+1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k+1)}$$

$$= \sum_{k=0}^\infty \beta^k s^{-(\alpha k+1)}$$

$$= \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

$\square$

Summary of Important Results

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha} f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right]$$

$$\mathcal{L}\{E_\alpha(\beta t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

The Solution to the Differential Equation

Lemma

The fractional differential equation,

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t) \tag{1}$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \tag{2}$$

has solution $y(t) = E_{\alpha}\left(\beta t^{\alpha}\right)$

Proof of Proposed Solution

Taking the Laplace transform of both sides of (1) yields

$$\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^{\alpha}y\right)\right\} = \beta\mathcal{L}\left\{y\right\}$$

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - \sum_{k=0}^{n-1} s^{n-k-1}y^{(k)}(0)\right] = \beta\mathcal{L}\left\{y\right\}$$

Proof of Proposed Solution

Then taking into account (2) (the initial conditions) we get

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - s^{n-1}\right] = \beta\mathcal{L}\left\{y\right\}$$

and so

$$\mathcal{L}\left\{y\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

Proof of Proposed Solution

By by noticing that

$$\mathcal{L}\left\{y\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\beta}.$$

is the Laplace transform of $E_{\alpha}(\beta t^{\alpha})$ we have that

$$y(t) = E_{\alpha}(\beta t^{\alpha})$$

$\square$