Abel's Integral Equation

We wish to consider a simple integral equation of the form

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\phi(t)dt}{(x-t)^\alpha} = f(x) \qquad\qquad x \geq 0, 0 \leq \alpha \leq 1 \tag{1}$$

We call this integral equation an Abel integral equation. It is worth noting that there are many forms of Abel's integral equation and we are just considering one form here.

We wish to layout a simple method for solving Abel's integral equation.

Solution Method

Firstly let's consider the integral

$$I(x) := \int_a^x \frac{f(s)ds}{(x-s)^{1-\alpha}}. \tag{2}$$

Now by substituting (1) into (2) we get

$$\begin{aligned} I(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-s)^{1-\alpha}}\left(\int_a^s \frac{\phi(t)dt}{(s-t)^\alpha}\right)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\int_a^s \frac{\phi(t)dt}{(x-s)^{1-\alpha}(s-t)^\alpha}\right)ds \end{aligned}$$

Now noting that the region of integration in $\mathbb{R}^2$ is just

$$\begin{aligned} a \leq s \leq x \\ a \leq t \leq s \end{aligned}$$

which is equivalent to

$$\begin{aligned} t \leq s \leq x \\ a \leq t \leq x \end{aligned}$$

we can write

$$\begin{aligned} \frac{1}{\Gamma(\alpha)}\int_a^x \left(\int_a^s \frac{\phi(t)dt}{(x-s)^{1-\alpha}(s-t)^\alpha}\right)ds &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\int_t^x \frac{\phi(t)ds}{(x-s)^{1-\alpha}(s-t)^\alpha}\right)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi(t)\left(\int_t^x (x-s)^{\alpha-1}(s-t)^{-\alpha}ds\right)dt. \end{aligned} \tag{3}$$

Now performing the substitution $\tau = \frac{s-t}{x-t}$ yields

$$\begin{aligned} \int_t^x (x-s)^{\alpha-1}(s-t)^{-\alpha}ds &= \int_0^1 \tau^{-\alpha}(1-\tau)^{\alpha-1}d\tau \\ &= B(1-\alpha,\alpha) \\ &= \Gamma(1-\alpha)\Gamma(\alpha) \end{aligned}$$

and so (3) becomes

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_a^x \phi(t)\left(\int_t^x (x-s)^{\alpha-1}(s-t)^{-\alpha}ds\right)dt &= \frac{1}{\Gamma(\alpha)}\int_a^x \phi(t)\Gamma(\alpha)\Gamma(1-\alpha)dt \\ &= \Gamma(1-\alpha)\int_a^x \phi(t)dt.\end{aligned}$$

So we have that

$$\int_a^x \frac{f(s)ds}{(x-s)^{1-\alpha}} = \Gamma(1-\alpha)\int_a^x \phi(t)dt$$

and by differentiating we get

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(s)ds}{(x-s)^{1-\alpha}}$$

□

References

[1] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives.* Gordon and Breach Science Publishers, 1993.