

A Solution to a Fractional Differential Equation

The Laplace Transform Method

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May 10, 2014

The Goal

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^C\mathcal{D}_0^\alpha y\right)(t) = \beta y(t) \quad (1)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq [\alpha] - 1 \end{cases} \quad (2)$$

Motivations

Cauchy Formula for Repeated Integration

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n) dy_n \cdots dy_2 dy_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt$$

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Riemann-Liouville Fractional Integral

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

Motivations (Derivatives)

Riemann-Liouville Fractional Derivative

$$\begin{aligned}(\mathcal{D}_a^\alpha f)(x) &= \frac{d^{[\alpha]}}{dx^{[\alpha]}} \left(I_a^{[\alpha]-\alpha} f \right)(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}\end{aligned}$$

where $n-1 < \alpha \leq n$

Motivations (Derivatives)

Caputo Fractional Derivative

$$\begin{aligned} \left({}^C\mathcal{D}_a^\alpha f\right)(x) &= \left(I_a^{[\alpha]-\alpha} \frac{d^{[\alpha]}}{dx^{[\alpha]}} f\right)(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\frac{d^t}{dt^n} f(t) dt}{(x-t)^{\alpha-n+1}} \end{aligned}$$

where $n-1 < \alpha \leq n$

Riemann-Liouville vs Caputo Derivative

Note!

The Caputo derivative and the Riemann-Liouville derivatives are not the same. In general

$$\left({}^C\mathcal{D}_a^\alpha f\right)(x) \neq \left(\mathcal{D}_a^\alpha f\right)(x).$$

The reason is exactly the same reason that in general

$$f(x) \neq \int_a^x f'(t)dt.$$

In some sense if you differentiate first you "lose information" about the function.

Riemann-Liouville vs Caputo Derivative

The Caputo derivative is often used in fractional differential equations because it can be coupled with integer order initial conditions, whereas often the Riemann-Liouville derivative can't be coupled with integer order initial conditions.