

We wish to consider the existence and uniqueness of solutions to a fractional differential equation. This generalises a result of Tisdell [What do I reference? ]

**Theorem 1** (Existence and Uniqueness). *Define*

$$S := \{(t, p) \in \mathbb{R}^2 : t \in [0, a], p \in \mathbb{R}\}$$

Let  $f : S \rightarrow \mathbb{R}$  be continuous. If there is a positive constant  $L$  such that

$$|f(t, u) - f(t, v)| \leq L|u - v|, \text{ for all } (t, u), (t, v) \in S$$

and a set of constants  $\{\alpha_j\}_{j=1}^N, \{\beta_j\}_{j=1}^N$  such that

$$\sum_{j=2}^N \left| \frac{\beta_j}{\beta_1} \right| \frac{a^{\alpha_1 - \alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} < 1$$

then the following initial value problem has a unique solution on  $[0, a]$ .

$$\sum_{j=1}^N \beta_j \left( {}^C \mathcal{D}_0^{\alpha_j} x \right) (t) = f(t, x(t)) \quad (1)$$

$$x(0) = A_0, x_1(0) = A_1, \dots, x_{n_N}(0) = A_{n_N} \quad (2)$$

where  $\alpha_1 > \alpha_2 > \dots > \alpha_N$  and  $n_j = \lceil \alpha_j \rceil - 1$ .

To do this we will need several lemmas.

**Lemma 1.** *The IVP defined (1), (2) is equivalent to the integral equation*

$$\begin{aligned} x(t) = & \sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1} \left( \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} f(s, x(s)) ds \right. \\ & \left. - \sum_{j=2}^N \beta_j \frac{1}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} \left( x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!} \right) ds \right) \end{aligned}$$

*Proof.* Apply  $(I_0^\alpha)$  to both sides of (1) and recognize that

$$\left( I_0^\alpha \left( {}^C \mathcal{D}_0^\alpha x \right) \right) (t) = x(t) + \sum_{k=0}^n \frac{x^{(k)}(0) t^k}{k!}$$

where  $n = \lceil \alpha \rceil - 1$ . □

**Lemma 2.**

$$\left( I_0^\xi E_\alpha(\gamma t^\alpha) \right) \leq \frac{t^\xi}{\Gamma(\xi)} E_\alpha(\gamma t^\alpha)$$

*Proof.* See that

$$\begin{aligned}
 \left(I_0^\xi E_\alpha(\gamma t^\alpha)\right) &= \frac{1}{\Gamma(\xi)} \int_0^t E_\alpha(\gamma s^\alpha)(t-s)^{\xi-1} ds \\
 &= \frac{1}{\Gamma(\xi)} \int_0^t \sum_{k=0}^{\infty} \frac{\gamma^k s^{\alpha k}}{\Gamma(\alpha k + 1)} (t-s)^{\xi-1} ds \\
 &= \frac{1}{\Gamma(\xi)} \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma(\alpha k + 1)} \underbrace{\int_0^t s^{\alpha k} (t-s)^{\xi-1} ds}_{\circledast}.
 \end{aligned}$$

Letting  $\tau = \frac{s}{t}$  we have that

$$\begin{aligned}
 \circledast &= \int_0^1 (t\tau)^{\alpha k} (t-t\tau)^{\xi-1} t d\tau \\
 &= t^{\alpha k + \xi} \int_0^1 (\tau)^{\alpha k} (1-\tau)^{\xi-1} d\tau \\
 &= t^{\alpha k + \xi} B(\alpha k + 1, \xi) \\
 &= t^{\alpha k + \xi} \frac{\Gamma(\alpha k + 1) \Gamma(\xi)}{\Gamma(\alpha k + \xi + 1)}.
 \end{aligned}$$

This means that

$$\begin{aligned}
 \left(I_0^\xi E_\alpha(\gamma t^\alpha)\right) &= \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha k + \xi}}{\Gamma(\alpha k + \xi + 1)} \\
 &= t^\xi \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k + \xi + 1)} \\
 &\leq \frac{t^\xi}{\Gamma(\xi)} \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k + 1)} \\
 &= \frac{t^\xi}{\Gamma(\xi)} E_\alpha(\gamma t^\alpha).
 \end{aligned}$$

□

**Lemma 3.**

$$(I_0^\alpha E_\alpha(\gamma t^\alpha)) = \frac{1}{\gamma} (E_\alpha(\gamma t^\alpha) - 1)$$

*Proof.* See that

$$\begin{aligned}
 (I_0^\alpha E_\alpha(\gamma t^\alpha)) &= \frac{1}{\Gamma(\alpha)} \int_0^t E_\alpha(\gamma s^\alpha)(t-s)^{\alpha-1} ds \\
 &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\gamma^k}{\Gamma(\alpha k + 1)} \underbrace{\int_0^t s^{\alpha k} (t-s)^{\alpha-1} ds}_{\circledast}.
 \end{aligned}$$

Letting  $\tau = \frac{s}{t}$  we have that

$$\begin{aligned}
 \circledast &= \int_0^1 (t\tau)^{\alpha k} (t - t\tau)^{\alpha-1} t d\tau \\
 &= t^{\alpha(k+1)} \int_0^1 \tau^{\alpha k} (1 - \tau)^{\alpha-1} d\tau \\
 &= t^{\alpha(k+1)} B(\alpha k + 1, \alpha) \\
 &= t^{\alpha(k+1)} \frac{\Gamma(\alpha k + 1) \Gamma(\alpha)}{\Gamma(\alpha(k+1) + 1)}.
 \end{aligned}$$

This then means that

$$\begin{aligned}
 (I_0^\alpha E_\alpha(\gamma t^\alpha)) &= \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha(k+1)}}{\Gamma(\alpha(k+1) + 1)} \\
 &= \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k + 1)} \\
 &= \frac{1}{\gamma} \left( \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k + 1)} - 1 \right) \\
 &= \frac{1}{\gamma} (E_\alpha(\gamma t^\alpha) - 1).
 \end{aligned}$$

□

*Proof of theorem 1.* To arrive at this we only have to prove that the map

$$\begin{aligned}
 [Fx](t) &:= \sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1} \left( \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} f(s, x(s)) ds \right. \\
 &\quad \left. - \sum_{j=2}^N \frac{\beta_j}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} \left( x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!} \right) ds \right)
 \end{aligned}$$

is contractive in the metric space  $(C[0, a], d_\gamma^{\alpha_1})$  where

$$d_\gamma^{\alpha_1}(x, y) = \max_{t \in [0, a]} \frac{|x(t) - y(t)|}{E_{\alpha_1}(\gamma t^{\alpha_1})}.$$

To see this note that

$$\begin{aligned}
 d_\gamma^{\alpha_1}(Fx, Fy) &= \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})} \left| \frac{1}{\beta_1} \left| \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} (f(s, x(s)) - f(s, y(s))) ds \right. \right. \\
 &\quad \left. \left. - \sum_{j=2}^N \frac{\beta_j}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} (x(s) - y(s)) ds \right| \right| \\
 &\leq \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1}) |\beta_1|} \left( \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} |f(s, x(s)) - f(s, y(s))| ds \right. \\
 &\quad \left. + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} |x(s) - y(s)| ds \right).
 \end{aligned}$$

By exploiting the Lipschitz condition we can further write that

$$\begin{aligned}
d_{\gamma}^{\alpha_1}(Fx, Fy) &\leq \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \left( \frac{L}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} |x(s) - y(s)| ds \right. \\
&\quad \left. + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1} |x(s) - y(s)| ds \right) \\
&= \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \left( \frac{L}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \frac{|x(s) - y(s)|}{E_{\alpha_1}(\gamma s^{\alpha_1})} E_{\alpha_1}(\gamma s^{\alpha_1}) ds \right. \\
&\quad \left. + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1} \frac{|x(s) - y(s)|}{E_{\alpha_1}(\gamma s^{\alpha_1})} E_{\alpha_1}(\gamma s^{\alpha_1}) ds \right) \\
&\leq d_{\gamma}^{\alpha_1}(x, y) \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \left( \frac{L}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} E_{\alpha_1}(\gamma s^{\alpha_1}) ds \right. \\
&\quad \left. + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1} E_{\alpha_1}(\gamma s^{\alpha_1}) ds \right) \\
&= d_{\gamma}^{\alpha_1}(x, y) \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \left( L (I_0^{\alpha_1} E_{\alpha_1}(\gamma t^{\alpha_1})) \right. \\
&\quad \left. + \sum_{j=2}^N |\beta_j| \left( I_0^{\alpha_1-\alpha_j} E_{\alpha_1}(\gamma t^{\alpha_1}) \right) \right).
\end{aligned}$$

We can now use the results of lemmas 2 and 3 to write

$$\begin{aligned}
d_{\gamma}^{\alpha_1}(Fx, Fy) &\leq d_{\gamma}^{\alpha_1}(x, y) \max_{t \in [0, a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \left( \frac{L}{\gamma} (E_{\alpha_1}(\gamma t^{\alpha_1}) - 1) \right. \\
&\quad \left. + \sum_{j=2}^N |\beta_j| \frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} E_{\alpha_1}(\gamma t^{\alpha_1}) \right) \\
&= d_{\gamma}^{\alpha_1}(x, y) \max_{t \in [0, a]} \frac{1}{|\beta_1|} \left( \frac{L}{\gamma} \left( 1 - \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})} \right) \right. \\
&\quad \left. + \sum_{j=2}^N |\beta_j| \frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} \right)
\end{aligned}$$

and finally we get that

$$d_{\gamma}^{\alpha_1}(Fx, Fy) \leq d_{\gamma}^{\alpha_1}(x, y) \frac{1}{|\beta_1|} \left( \frac{L}{\gamma} + \sum_{j=2}^N |\beta_j| \frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} \right).$$

By choosing  $\gamma$  sufficiently large we get that

$$\frac{1}{|\beta_1|} \left( \frac{L}{\gamma} + \sum_{j=2}^N |\beta_j| \frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} \right) < 1$$

and so  $F$  is a contractive mapping and thus the IVP defined in (1), (2) has a unique solution on  $[0, a]$ .  $\square$