Accelerating Numerical Solutions to Fractional Differential Equations

Adams Moulton Bashforth Method

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Problem

$$\left({}_{0}^{C}\mathcal{D}_{x}^{\alpha} y\right)(x) = f(x, y)$$

$$y^{(k)}(0) = a_k \qquad\qquad 0 \leq \alpha \leq \lceil k \rceil$$

Similar Problems

$$\frac{dy}{dx} = f(x, y)$$

$$y(0) = y_0$$

Explanation of (Explicit) Euler Method

The Euler method is designed to solve problems of the form

$$y'(x) = f(x, y(x)) \qquad\qquad y(x_0) = y_0.$$

From first principles

$$y'(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

therefore

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f(x_0, y(x_0))$$

and by rearranging and choosing some small but non-zero h,

$$y(x_0 + h) = hf(x_0, y(x_0)) + y(x_0).$$

Explanation of (Explicit) Euler Method (cont)

Restating this iteratively we get

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Computational Complexity

Scheme	Complexity
Euler	$O(n)$
Runge-Kutta	$O(n)$
Adams Moulton	$O(n)$
Adams Bashforth	$O(n)$
etc	O(n)

Truncation Errors

Local Truncation Error

This is the amount of error built up in each step. From a Taylor series expansion we have,

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{1}{2}h^2 y''(x_0) + O(h^3)$$

and thus

$$e_{local} = \frac{1}{2}h^2 y''(x_0) + O(h^3)$$

so long as y has bounded third derivative.

Truncation Errors

Global Truncation Error

The number of steps is

$$n = \frac{x - x_0}{h}$$

and the local truncation error is $O(h^2)$ so we can expect the global truncation error to be $O(h)$.

This result holds for *sufficiently nice* f and y.

Computational Complexity and Truncation Errors

Scheme	Complexity	Global Truncation Error
Euler	$O(n)$	$O(h)$
Runge-Kutta (RK4)	$O(n)$	$O(h^4)$