Solution to a Simple FDE

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t) \tag{1}$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \tag{2}$$

has the solution $y(t) = E_{\alpha}\left(\beta t^{\alpha}\right)$. Where E_{α} is the one parameter Mittag-Lefler function.

This solution is arrived at by a Laplace transform method. We will then go on to show that this solution is unique. The proof of that fact, will be presented via the proof of a more general theorem, which broadly follows a Banach fixed point method.

Definition 1 (Fractional Derivatives and Integrals)**.** *For $\alpha > 0$ we define*

$$\begin{aligned} (I_{a+}^{\alpha} f)(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \\ (\mathcal{D}_{a+}^{\alpha} f)(x) &:= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \\ ({}^{C}\mathcal{D}_{a+}^{\alpha} f)(x) &:= I_0^{n-\alpha} \frac{d^n}{dx^n} f(x) \end{aligned}$$

where $n = \lceil \alpha \rceil - 1$. We will refer to $I_{a+}^{\alpha} f$ as the (Riemann Louiville) integral f of over α (based at a). Likewise we refer to $\mathcal{D}_{a+}^{\alpha} f$ as the (Riemann Louiville) derivative of order α (based at a). We also refer to ${}^{C}\mathcal{D}_{a+}^{\alpha} f$ as the Caputo derivative of order α (based at a).

The motivation for these definitions are based of the Cauchy formula for repeated integration, and in the case of the Caputo derivative, practical considerations. [3, 2]

For the rest of our considerations we will take $a = 1$ (based at 0).

We now consider the Laplace transform of the fractional integration and differentiation operators.

Lemma 1.

$$\mathcal{L}\left\{I_0^{\alpha} f\right\} = s^{-\alpha} \mathcal{L}\left\{f\right\}$$

Proof. Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^x f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of f with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{f(t)\right\} \underbrace{\mathcal{L}\left\{t^{\alpha-1}\right\}}_{=s^{-\alpha}\Gamma(\alpha)} \\
&= s^{-\alpha} \mathcal{L}\left\{f\right\}.
\end{aligned}$$

$\square$

Lemma 2.

$$\mathcal{L}\left\{\mathcal{D}_0^\alpha f\right\} = s^\alpha \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{k-1} \left(\mathcal{D}_0^{\alpha-k} f\right)(0)$$

Proof. See that

$$\begin{aligned}
\mathcal{L}\left\{(\mathcal{D}_0^\alpha f)\right\} &= \mathcal{L}\left\{\frac{d^n}{dt^n}\left(I_0^{n-\alpha} f\right)\right\} \\
&= s\mathcal{L}\left\{\left(I_0^{n-\alpha} f\right)\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dt^{n-k-1}} \left(I_0^{n-\alpha} f\right)(0) \\
&= s\mathcal{L}\left\{\left(I_0^{n-\alpha} f\right)\right\} - \sum_{k=0}^{n-1} s^{k-1} \left(\mathcal{D}_0^{\alpha-k} f\right)(0).
\end{aligned}$$

$\square$

Lemma 3.

$$\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1} \left(\frac{d^k f}{dt^k}\right)(0)\right]$$

Proof. See that

$$\begin{aligned}
\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} &= \mathcal{L}\left\{\frac{1}{\Gamma(n-\alpha)}\left(I_0^{n-\alpha} \frac{d^n f}{dt^n}\right)\right\} \\
&= \frac{1}{\Gamma(n-\alpha)} \mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1} \frac{d^n f}{dt^n} du\right\}
\end{aligned}$$

which is the Laplace transform of a convolution so

$$\begin{aligned}
\Gamma(n-\alpha)\mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\} &= \mathcal{L}\left\{t^{n-\alpha-1}\right\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\
&= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right)\left(s^n\mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\
&= s^{\alpha-n}\left[s^n\mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].
\end{aligned}$$

□

We now define the Mittag-Lefler function and calculate its Laplace transform.

Definition 2. *The one parameter Mittag-Lefler E_α function is defined by its power series.*

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$$

It is clear to see the definition of this function is inspired by the exponential function. Before we can calculate the Laplace transform of the Mittag-Lefler function we have to prove a simple lemma about the convergence of the series which is used in its definition.

Lemma 4. *The series*

$$\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$$

converges absolutely for all $t \in \mathbb{R}$.

Proof. Let $a_k = \frac{t^k}{\Gamma(\alpha k+1)}$ and see that

$$|\frac{a_{k+1}}{a_k}| = |t|\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}$$

and that hence

$$\lim_{k\longrightarrow\infty} |\frac{a_{k+1}}{a_k}| = 0$$

for all $t \in \mathbb{R}$ so by the ratio test, the series $\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$ converges for all $t \in \mathbb{R}$. □

Lemma 5.

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

Proof. See that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \int_0^\infty e^{-st} \sum_{k=0}^{\infty} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ (lemma 4) we may interchange the integral and the sum to get

$$\begin{aligned}\int_0^\infty e^{-st} \sum_{k=0}^{\infty} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt &= \sum_{k=0}^{\infty} \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt \\ &= \sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt.\end{aligned}$$

By performing the change of variables $x = st$ we get that

$$\begin{aligned}\sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt &= \sum_0^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k+1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k+1)} \\ &= \sum_{k=0}^{\infty} \beta^k s^{-(\alpha k+1)} \\ &= \frac{s^{\alpha-1}}{s^\alpha - \beta}.\end{aligned}$$

So we have that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

as required. □

We now have sufficient tools to attack the original problem, that is finding a solution to (1), (2).

Lemma 6. *The FDE defined in* (1) *and* (2)*, restated here for completness*

$$\left({}^C\mathcal{D}_0^\alpha y\right)(t) = \beta y(t)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \le k \le \lfloor \alpha \rfloor - 1 \end{cases}$$

has solution $y(t) = E_\alpha\left(\beta t^\alpha\right)$.

Proof. Taking the Laplace transform of both sides of (1) yields

$$\begin{aligned}\mathcal{L}\left\{\left({}^{C}\mathcal{D}_0^\alpha y\right)\right\} &= \beta\mathcal{L}\left\{y\right\} \\ s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - \sum_{k=0}^{n-1} s^{n-k-1}y^{(k)}(0)\right] &= \beta\mathcal{L}\left\{y\right\}\end{aligned}$$

by the result of lemma 3. Then taking into account (2) we get

$$s^{-(n+\alpha)}\left[s^n\mathcal{L}\left\{y\right\} - s^{n-1}\right] = \beta\mathcal{L}\left\{y\right\}$$

and so

$$\mathcal{L}\left\{y\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}.$$

By using the result of lemma 5 we have that

$$y(t) = E_\alpha(\beta t^\alpha)$$

□

An obvious question to ask now, is whether the solution to (1), (2) is unique. To answer this in affermative we will prove a result about the existance and uniqueness of solutions to non-linear Voltera integral equations of the second kind then show that a more general FDE is equivelent to such a Voltera integral equation and hence arrive at the desired result. This technique follows that in [1]. This is more general than what is required here, but it lays the groundwork for future results.

Lemma 7. *If the function f is continuous, then the initial value problem*

$$\left({}^{C}\mathcal{D}_0^\alpha y\right)(t) = f(t, y(t)) \tag{3}$$

along with

$$y^{(k)}(0) = \gamma_k \qquad\qquad k = 0, 1, \ldots, n-1 \tag{4}$$

where $n = \lceil\alpha\rceil$ is equivelent to the non-linear Voltera equation of the second kind,

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\gamma_k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u, y(u))du.$$

Proof. Apply I_0^α to both sides of (3) to get

$$\begin{aligned}\left(I_0^\alpha\left(\mathcal{D}_0^\alpha y\right)\right)(t) &= \left(I_0^\alpha f(t, y(t))\right) \\ \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t\int_0^x (t-x)^{\alpha-1}(x-u)^{n-\alpha-1}y^{(n)}(u)dudx &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u, f(u))du\end{aligned} \tag{5}$$

then considering just the left hand side we have that

$$\left(I_0^{\alpha}\left(\mathcal{D}_0^{\alpha}y\right)\right)(t) = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t\int_0^x (t-x)^{\alpha-1}(x-u)^{n-\alpha-1}y^{(n)}(u)dudx.$$

This integral is over the region

$$R := \begin{cases} 0 \le u \le x \\ 0 \le x \le t \end{cases}$$

which is equivelent to

$$R' := \begin{cases} 0 \le u \le t \\ u \le x \le t \end{cases}$$

so we can change the order of integration to get

$$\left(I_0^{\alpha}\left(\mathcal{D}_0^{\alpha}y\right)\right)(t) = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t y^{(n)}(u)\underbrace{\left(\int_u^t (t-x)^{\alpha-1}(x-u)^{n-\alpha-1}dx\right)du}_{\circledast}.$$

Focusing just on $\circledast$ and by performing the change of variables $\tau = \frac{x-u}{t-u}$ we get that

$$\begin{aligned} \circledast &= (t-u)^{n-1}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha}d\tau \\ &= (t-u)^{n-1}B(\alpha, n-\alpha) \\ &= (t-u)^{n-1}\frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)}. \end{aligned}$$

So we have that

$$\left(I_0^{\alpha}\left(\mathcal{D}_0^{\alpha}y\right)\right)(t) = \underbrace{\frac{1}{\Gamma(n)}\int_0^t (t-u)^{n-1}y^{(n)}(u)du}_{\circledast\circledast}.$$

Now by considering the Cauchy formula for repeated integration we can see that $\circledast\circledast$ is just the $n-$fold integral of f based at 0 and so

$$\begin{aligned} \left(I_0^{\alpha}\left(\mathcal{D}_0^{\alpha}y\right)\right)(t) &= \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{n-1} y^{(n)}(u)dudt_{n-1}dt_{n-2}\cdots dt_1 \\ &= \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{n-2}\left(y^{(n-1)}(t_{n-1}) - y^{(n)}(0)\right)dt_{n-1}dt_{n-2}\cdots dt_1 \\ &= \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{n-3}\left(y^{(n-2)}(t_{n-2}) - y^{(n)}(0) - ty^{(n)}(0)\right)dt_{n-1}dt_{n-2}\cdots dt_1 \\ &= y(t) - \sum_{k=0}^{n-1}\frac{t^k f^{(k)}(0)}{k!}. \end{aligned}$$

Applying the initial conditions in (**??**) we get that

$$\left(I_0^\alpha \left(\mathcal{D}_0^\alpha y\right)\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}\gamma_k$$

and by subsituting back into (5) and rearranging we have

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\gamma_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u)) du.$$

□

References

[1] K. Diethelm and N.J. Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265:229–248, 2002.

[2] I. Podlubny. *Fractional Differential Equations.* Academic Press, 1999.

[3] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives.* Breach Science Publishers, 1993.