Solution to a Simple FDE

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^{C}\mathcal{D}_0^{\alpha}y\right)(t) = \beta y(t)$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \le k \le \lfloor \alpha \rfloor - 1 \end{cases}$$

has the solution $y(t) = E_\alpha(\beta t^\alpha)$. Where E_α is the one parameter Mittag-Lefler function.

This solution is arrived at by a Laplace transform method. We will then go on to show that this solution is unique. The proof of that fact, will be presented via the proof of a more general theorem, which broadly follows a Banach fixed point method.

Definition 1 (Fractional Derivatives and Integrals)**.** *For $\alpha > 0$ we define*

$$(I_{a+}^{\alpha}f)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt$$
$$(\mathcal{D}_{a+}^{\alpha}f)(x) := \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt$$
$$({}^{C}\mathcal{D}_{a+}^{\alpha}f)(x) := I_0^{n-\alpha}\frac{d^n}{dx^n}f(x)$$

where $n = \lceil \alpha \rceil - 1$. We will refer to $I_{a+}^{\alpha}f$ as the (Riemann Louiville) integral f of over α (based at a). Likewise we refer to $\mathcal{D}_{a+}^{\alpha}f$ as the (Riemann Louiville) derivative of order α (based at a). We also refer to ${}^{C}\mathcal{D}_{a+}^{\alpha}f$ as the Caputo derivative of order α (based at a).

The motivation for these definitions are based of the Cauchy formula for repeated integration, and in the case of the Caputo derivative, practical considerations. [**?**, **?**]

For the rest of our considerations we will take $a = 1$ (based at 0).

We now consider the Laplace transform of the fractional integration and differentiation operators.

Lemma 1.

$$\mathcal{L}\{I_0^{\alpha}f\} = s^{-\alpha}\mathcal{L}\{f\}$$

Proof. Since

$$(I_0^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^x f(u)(t-u)^{\alpha-1} du$$

is just $\frac{1}{\Gamma(\alpha)}$ times the convolution of f with $t^{\alpha-1}$ then by the convolution theorem for Laplace transforms we have that

$$\begin{aligned}
\mathcal{L}\left\{I_0^\alpha f\right\} &= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{\int_0^t f(u)(t-u)^{\alpha-1} du\right\} \\
&= \frac{1}{\Gamma(\alpha)} \mathcal{L}\left\{f(t)\right\} \underbrace{\mathcal{L}\left\{t^{\alpha-1}\right\}}_{=s^{-\alpha}\Gamma(\alpha)} \\
&= s^{-\alpha} \mathcal{L}\left\{f\right\}.
\end{aligned}$$

□

Lemma 2.

$$\mathcal{L}\left\{\mathcal{D}_0^\alpha f\right\} = s^\alpha \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{k-1} \left(\mathcal{D}_0^{\alpha-k} f\right)(0)$$

Proof. See that

$$\begin{aligned}
\mathcal{L}\left\{(\mathcal{D}_0^\alpha f)\right\} &= \mathcal{L}\left\{\frac{d^n}{dt^n}\left(I_0^{n-\alpha} f\right)\right\} \\
&= s\mathcal{L}\left\{\left(I_0^{n-\alpha} f\right)\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dt^{n-k-1}} \left(I_0^{n-\alpha} f\right)(0) \\
&= s\mathcal{L}\left\{\left(I_0^{n-\alpha} f\right)\right\} - \sum_{k=0}^{n-1} s^{k-1} \left(\mathcal{D}_0^{\alpha-k} f\right)(0).
\end{aligned}$$

□

Lemma 3.

$$\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} = s^{\alpha-n}\left[s^n \mathcal{L}\left\{f\right\} - \sum_{k=0}^{n-1} s^{n-k-1} \left(\frac{d^k f}{dt^k}\right)(0)\right]$$

Proof. See that

$$\begin{aligned}
\mathcal{L}\left\{\left({}^C\mathcal{D}_0^\alpha f\right)\right\} &= \mathcal{L}\left\{\frac{1}{\Gamma(n-\alpha)}\left(I_0^{n-\alpha} \frac{d^n f}{dt^n}\right)\right\} \\
&= \frac{1}{\Gamma(n-\alpha)} \mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1} \frac{d^n f}{dt^n} du\right\}
\end{aligned}$$

which is the Laplace transform of a convolution so

$$
\begin{aligned}
\Gamma(n-\alpha)\mathcal{L}\left\{\int_0^t (t-u)^{n-\alpha-1}\frac{d^n f}{dt^n}du\right\} &= \mathcal{L}\left\{t^{n-\alpha-1}\right\}\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} \\
&= \frac{1}{n-\alpha}\left(s^{-(n-\alpha)}\Gamma(n-\alpha)\right)\left(s^n\mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right) \\
&= s^{\alpha-n}\left[s^n\mathcal{L}\{f\} - \sum_{k=0}^{n-1} s^{n-k-1}\left(\frac{d^k f}{dt^k}\right)(0)\right].
\end{aligned}
$$

□

We now define the Mittag-Lefler function and calculate its Laplace transform.

Definition 2. *The one parameter Mittag-Lefler E_α function is defined by its power series.*

$$E_\alpha(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}$$

It is clear to see the definition of this function is inspired by the exponential function. Before we can calculate the Laplace transform of the Mittag-Lefler function we have to prove a simple lemma about the convergence of the series which is used in its definition.

Lemma 4. *The series*

$$\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}$$

converges absolutely for all $t \in \mathbb{R}$.

Proof. Let $a_k = \frac{t^k}{\Gamma(\alpha k+1)}$ and see that

$$|\frac{a_{k+1}}{a_k}| = |t|\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}$$

and that hence

$$\lim_{k\longrightarrow\infty}|\frac{a_{k+1}}{a_k}| = 0$$

for all $t \in \mathbb{R}$ so by the ratio test, the series $\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}$ converges for all $t \in \mathbb{R}$. □

Lemma 5.

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha-\beta}$$

Proof. See that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt$$

and because the series converges absolutely for all $t \in \mathbb{R}$ we may interchange the integral and the derivitive to get

$$\begin{aligned}\int_0^\infty e^{-st} \sum_{k=0}^\infty \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt &= \sum_{k=0}^\infty \int_0^\infty e^{-st} \frac{(\beta t^\alpha)^k}{\Gamma(\alpha k+1)} dt \\ &= \sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt.\end{aligned}$$

By performing the change of variables $x = st$ we get that

$$\begin{aligned}\sum_0^\infty \frac{\beta^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-st} t^{\alpha k} dt &= \sum_0^\infty \frac{\beta^k s^{-(k+1)}}{\Gamma(\alpha k+1)} \underbrace{\int_0^\infty e^{-x} x^{\alpha k} dx}_{\Gamma(\alpha k+1)} \\ &= \sum_{k=0}^\infty \beta^k s^{-(\alpha k+1)} \\ &= \frac{s^{\alpha-1}}{s^\alpha - \beta}.\end{aligned}$$

So we have that

$$\mathcal{L}\left\{E_\alpha(\beta t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \beta}$$

as required. □

References

[1] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives.* Gordon and Breach Science Publishers, 1993.

[2] I. Podlubny. *Fractional Differential Equations.* Academic Press, 1999.