We wish to consider the existence an uniqueness of solutions to a fractional differential equation. This generalises a result of Tisdell [What do I reference?]

Theorem 1 (Existance and Uniqueness)**.** *Define*

$$S := \{(t,p) \in \mathbb{R}^2 : t \in [0,a], p \in \mathbb{R}\}$$

Let $f : S \longrightarrow \mathbb{R}$ be continuous. If there is a positive constant L such that

$$|f(t,u) - f(t,v)| \leq L|u-v|, \text{ for all } (t,u),(t,v) \in S$$

and a set of constants $\{\alpha_j\}_{j=1}^N$, $\{\beta_j\}_{j=1}^N$ such that

$$\sum_{j=2}^{N} \left|\frac{\beta_j}{\beta_1}\right| \frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)} < 1$$

then the following initial value problem has a unique solution on $[0,a]$.

$$\sum_{j=1}^{N} \beta_j \left({}^C\mathcal{D}_0^{\alpha_j} x\right)(t) = f(t,x(t)) \tag{1}$$

$$x(0) = A_0, x_1(0) = A_1, \ldots, x^{n_N}(0) = A_{n_N} \tag{2}$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_N$ and $n_j = \lceil \alpha_j \rceil - 1$.

To do this we will need several lemmas.

Lemma 1. *The IVP defined* (1), (2) *is equivelent to the integral equation*

$$\begin{aligned} x(t) = &\sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1} f(s,x(s))ds \\ &- \sum_{j=2}^{N} \beta_j \frac{1}{\Gamma(\alpha_1-\alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1}\left(x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!}\right) ds\Bigg) \end{aligned}$$

Proof. Apply (I_0^α) to both sides of (1) and recognize that

$$\left(I_0^\alpha \left({}^C\mathcal{D}_0^\alpha x\right)\right)(t) = x(t) + \sum_{k=0}^{n} \frac{x^{(k)}(0)t^k}{k!}$$

where $n = \lceil \alpha \rceil - 1$. □

Proof of theorem 1. To arrive at this we only have to prove that the map

$$\begin{aligned} [Fx](t) := &\sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1}\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1} f(s,x(s))ds \\ &- \sum_{j=2}^{N} \frac{\beta_j}{\Gamma(\alpha_1-\alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1}\left(x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!}\right) ds\Bigg) \end{aligned}$$

is contractive in the metric space $\left(C[0,a], d_\gamma^{\alpha_1}\right)$ where

$$d_\gamma^{\alpha_1}(x,y) = \max_{t\in[0,a]} \frac{|x(t)-y(t)|}{E_{\alpha_1}(\gamma t^{\alpha_1})}.$$

To see this note that

$$\begin{aligned}
d_\gamma^{\alpha_1}(Fx,Fy) &= \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})} \left|\frac{1}{\beta_1}\right| \Big| \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}(f(s,x(s)) - f(s,y(s))ds \\
&\quad - \sum_{j=2}^{N} \frac{\beta_j}{\Gamma(\alpha_1-\alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1}(x(s)-y(s))ds\Big| \\
&\leq \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|} \Big(\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}|f(s,x(s)) - f(s,y(s))|ds \\
&\quad + \sum_{j=2}^{N} \frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1}|x(s)-y(s))|ds\Big).
\end{aligned}$$

By exploiting the Lipshitz condition we can further write that

$$
\begin{aligned}
d_\gamma^{\alpha_1}(Fx, Fy) &\leq \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|x(s)-y(s)|ds \\
&\quad + \sum_{j=2}^{N}\frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}|x(s)-y(s))|ds\Big) \\
&= \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\frac{|x(s)-y(s)|}{E_{\alpha_1}(\gamma s^{\alpha_1})}E_{\alpha_1}(\gamma s^{\alpha_1})ds \\
&\quad + \sum_{j=2}^{N}\frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}\frac{|x(s)-y(s))|}{E_{\alpha_1}(\gamma s^{\alpha_1})}E_{\alpha_1}(\gamma s^{\alpha_1})ds\Big) \\
&\leq d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(\gamma s^{\alpha_1})ds \\
&\quad + \sum_{j=2}^{N}\frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}E_{\alpha_1}(\gamma s^{\alpha_1})ds\Big) \\
&= d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(L\,(I_0^{\alpha_1}E_{\alpha_1}(\gamma t^{\alpha_1}) \\
&\quad + \sum_{j=2}^{N}|\beta_j|\left(I_0^{\alpha_1-\alpha_j}E_{\alpha_1}(\gamma t^{\alpha_1})\right)\Big) \\
&\leq d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\gamma}\,(E_{\alpha_1}(\gamma t^{\alpha_1})-1) \\
&\quad + \sum_{j=2}^{N}|\beta_j|\frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}E_{\alpha_1}(\gamma t^{\alpha_1})\Big) \\
&= d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{|\beta_1|}\Big(\frac{L}{\gamma}\left(1-\frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})}\right) \\
&\quad + \sum_{j=2}^{N}|\beta_j|\frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}\Big) \\
&\leq d_\gamma^{\alpha_1}(x,y)\frac{1}{|\beta_1|}\left(\frac{L}{\gamma}+\sum_{j=2}^{N}|\beta_j|\frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}\right)
\end{aligned}
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$\square$