We wish to consider the existence an uniqueness of solutions to a fractional differential equation. This generalizes a result of Tisdell [What do I reference?]

Theorem 1 (Existence and Uniqueness)**.** *Define*

$$S := \{(t,p) \in \mathbb{R}^2 : t \in [0,a], p \in \mathbb{R}\}$$

Let $f : S \longrightarrow \mathbb{R}$ *be continuous. If there is a positive constant* L *such that*

$$|f(t,u) - f(t,v)| \leq L|u - v|, \text{ for all } (t,u),(t,v) \in S$$

and a set of constants $\{\alpha_j\}_{j=1}^N$, $\{\beta_j\}_{j=1}^N$ *such that*

$$\sum_{j=2}^{N} \left|\frac{\beta_j}{\beta_1}\right| \frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1 - \alpha_j)} < 1$$

then the following initial value problem has a unique solution on $[0,a]$.

$$\sum_{j=1}^{N} \beta_j \left({}^C\mathcal{D}_0^{\alpha_j} x\right)(t) = f(t, x(t)) \tag{1}$$

$$x(0) = A_0, x_1(0) = A_1, \ldots, x^{n_N}(0) = A_{n_N} \tag{2}$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_N$ *and* $n_j = \lceil \alpha_j \rceil - 1$.

To do this we will need several lemmas.

Lemma 1. *The IVP defined* (1), (2) *is equivalent to the integral equation*

$$\begin{aligned} x(t) = \sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1}\Big(&\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} f(s,x(s))ds \\ &- \sum_{j=2}^{N} \beta_j \frac{1}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1-\alpha_j-1} \left(x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!}\right) ds\Big) \end{aligned}$$

Proof. Apply (I_0^α) to both sides of (1) and recognize that

$$\left(I_0^\alpha \left({}^C\mathcal{D}_0^\alpha x\right)\right)(t) = x(t) + \sum_{k=0}^{n} \frac{x^{(k)}(0)t^k}{k!}$$

where $n = \lceil \alpha \rceil - 1$. □

Lemma 2.

$$\left(I_0^\xi E_\alpha(\gamma t^\alpha)\right) \leq \frac{t^\xi}{\Gamma(\xi)} E_\alpha(\gamma t^\alpha)$$

Proof. See that

$$\begin{aligned}
\left(I_0^\xi E_\alpha(\gamma t^\alpha)\right) &= \frac{1}{\Gamma(\xi)}\int_0^t E_\alpha(\gamma s^\alpha)(t-s)^{\xi-1}ds \\
&= \frac{1}{\Gamma(\xi)}\int_0^t \sum_{k=0}^{\infty}\frac{\gamma^k s^{\alpha k}}{\Gamma(\alpha k+1)}(t-s)^{\xi-1}ds \\
&= \frac{1}{\Gamma(\xi)}\sum_{k=0}^{\infty}\frac{\gamma^k}{\Gamma(\alpha k+1)}\underbrace{\int_0^t s^{\alpha k}(t-s)^{\xi-1}ds}_{\circledast}.
\end{aligned}$$

Letting $\tau = \frac{s}{t}$ we have that

$$\begin{aligned}
\circledast &= \int_0^1 (t\tau)^{\alpha k}(t-t\tau)^{\xi-1}td\tau \\
&= t^{\alpha k+\xi}\int_0^1 (\tau)^{\alpha k}(1-1\tau)^{\xi-1}d\tau \\
&= t^{\alpha k+\xi}B(\alpha k+1,\xi) \\
&= t^{\alpha k+\xi}\frac{\Gamma(\alpha k+1)\Gamma(\xi)}{\Gamma(\alpha k+\xi+1)}.
\end{aligned}$$

This means that

$$\begin{aligned}
\left(I_0^\xi E_\alpha(\gamma t^\alpha)\right) &= \sum_{k=0}^{\infty}\frac{\gamma^k t^{\alpha k+\xi}}{\Gamma(\alpha k+\xi+1)} \\
&= t^\xi\sum_{k=0}^{\infty}\frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k+\xi+1)} \\
&\le \frac{t^\xi}{\Gamma(\xi)}\sum_{k=0}^{\infty}\frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= \frac{t^\xi}{\Gamma(\xi)}E_\alpha(\gamma t^\alpha).
\end{aligned}$$

□

Lemma 3.

$$(I_0^\alpha E_\alpha(\gamma t^\alpha)) = \frac{1}{\gamma}\left(E_\alpha(\gamma t^\alpha)-1\right)$$

Proof. See that

$$\begin{aligned}
(I_0^\alpha E_\alpha(\gamma t^\alpha)) &= \frac{1}{\Gamma(\alpha)}\int_0^t E_\alpha(\gamma s^\alpha)(t-s)^{\alpha-1}ds \\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\gamma^k}{\Gamma(\alpha k+1)}\underbrace{\int_0^t s^{\alpha k}(t-s)^{\alpha}ds}_{\circledast}.
\end{aligned}$$

Letting $\tau = \frac{s}{t}$ we have that

$$\begin{aligned}
\circledast &= \int_0^1 (t\tau)^{\alpha k}(t - t\tau)^{\alpha-1} t d\tau \\
&= t^{\alpha(k+1)} \int_0^1 \tau^{\alpha k}(1-\tau)^{\alpha-1} d\tau \\
&= t^{\alpha(k+1)} B(\alpha k + 1, \alpha) \\
&= t^{\alpha(k+1)} \frac{\Gamma(\alpha k+1)\Gamma(\alpha)}{\Gamma(\alpha(k+1)+1)}.
\end{aligned}$$

This then means that

$$\begin{aligned}
(I_0^\alpha E_\alpha(\gamma t^\alpha)) &= \sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)} \\
&= \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= \frac{1}{\gamma} \left(\sum_{k=0}^{\infty} \frac{\gamma^k t^{\alpha k}}{\Gamma(\alpha k+1)} - 1 \right) \\
&= \frac{1}{\gamma} \left(E_\alpha(\gamma t^\alpha) - 1 \right).
\end{aligned}$$

□

Proof of theorem 1. To arrive at this we only have to prove that the map

$$\begin{aligned}
[Fx](t) := \sum_{k=1}^{n_1} \frac{A_k t^k}{k!} + \frac{1}{\beta_1} \Big(&\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} f(s, x(s)) ds \\
&- \sum_{j=2}^{N} \frac{\beta_j}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} \left(x(s) - \sum_{k=1}^{n_j} \frac{A_k s^k}{k!} \right) ds \Big)
\end{aligned}$$

is contractive in the metric space $\left(C[0,a], d_\gamma^{\alpha_1}\right)$ where

$$d_\gamma^{\alpha_1}(x,y) = \max_{t \in [0,a]} \frac{|x(t) - y(t)|}{E_{\alpha_1}(\gamma t^{\alpha_1})}.$$

To see this note that

$$\begin{aligned}
d_\gamma^{\alpha_1}(Fx, Fy) = &\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})} \left| \frac{1}{\beta_1} \right| \Big| \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} (f(s,x(s)) - f(s,y(s))) ds \\
&- \sum_{j=2}^{N} \frac{\beta_j}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} (x(s) - y(s)) ds \Big| \\
\leq &\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1}) |\beta_1|} \Big(\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} |f(s,x(s)) - f(s,y(s))| ds \\
&+ \sum_{j=2}^{N} \frac{|\beta_j|}{\Gamma(\alpha_1 - \alpha_j)} \int_0^t (t-s)^{\alpha_1 - \alpha_j - 1} |x(s) - y(s))| ds \Big).
\end{aligned}$$

By exploiting the Lipshitz condition we can further write that

$$\begin{aligned}
d_\gamma^{\alpha_1}(Fx,Fy) &\leq \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|x(s)-y(s)|ds \\
&\quad + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}|x(s)-y(s))|ds\Big) \\
&= \max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\frac{|x(s)-y(s)|}{E_{\alpha_1}(\gamma s^{\alpha_1})}E_{\alpha_1}(\gamma s^{\alpha_1})ds \\
&\quad + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}\frac{|x(s)-y(s))|}{E_{\alpha_1}(\gamma s^{\alpha_1})}E_{\alpha_1}(\gamma s^{\alpha_1})ds\Big) \\
&\leq d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(\gamma s^{\alpha_1})ds \\
&\quad + \sum_{j=2}^N \frac{|\beta_j|}{\Gamma(\alpha_1-\alpha_j)}\int_0^t (t-s)^{\alpha_1-\alpha_j-1}E_{\alpha_1}(\gamma s^{\alpha_1})ds\Big) \\
&= d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(L\,(I_0^{\alpha_1}E_{\alpha_1}(\gamma t^{\alpha_1}) \\
&\quad + \sum_{j=2}^N |\beta_j|\left(I_0^{\alpha_1-\alpha_j}E_{\alpha_1}(\gamma t^{\alpha_1})\right)\Big).
\end{aligned}$$

We can now use the results of lemmas 2 and 3 to write

$$\begin{aligned}
d_\gamma^{\alpha_1}(Fx,Fy) &\leq d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})|\beta_1|}\Big(\frac{L}{\gamma}\left(E_{\alpha_1}(\gamma t^{\alpha_1})-1\right) \\
&\quad + \sum_{j=2}^N |\beta_j|\frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}E_{\alpha_1}(\gamma t^{\alpha_1})\Big) \\
&= d_\gamma^{\alpha_1}(x,y)\max_{t\in[0,a]} \frac{1}{|\beta_1|}\Big(\frac{L}{\gamma}\left(1-\frac{1}{E_{\alpha_1}(\gamma t^{\alpha_1})}\right) \\
&\quad + \sum_{j=2}^N |\beta_j|\frac{t^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}\Big)
\end{aligned}$$

and finally we get that

$$d_\gamma^{\alpha_1}(Fx,Fy) \leq d_\gamma^{\alpha_1}(x,y)\frac{1}{|\beta_1|}\left(\frac{L}{\gamma}+\sum_{j=2}^N |\beta_j|\frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}\right).$$

By choosing γ sufficiently large we get that

$$\frac{1}{|\beta_1|}\left(\frac{L}{\gamma}+\sum_{j=2}^N |\beta_j|\frac{a^{\alpha_1-\alpha_j}}{\Gamma(\alpha_1-\alpha_j)}\right) < 1$$

and so F is a contractive mapping and thus the IVP defined in (1), (2) has a unique solution on $[0, a]$. $\square$