University of New South Wales

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Fractional Differential Equations

Thesis Introduction

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Useful Results in ODE Theory

Lemma 1 (Gronwell-Bellman Lemma). *Let $\rho : [a, b] \longrightarrow [0, \infty)$ be a piecewise continuous function and let $\tau \in [a, b]$. If there exists a set of non-negative constants $\{K_0, K_1, K_2\}$ such that for all $t \in [a, b]$*

$$\rho(t) \leq K_0 + K_1|t - \tau| + |\int_\tau^t K_2\rho(s)ds|$$

then for all $t \in [a, b]$ we have that

$$\rho(t) \leq \begin{cases} K_0 \exp(K_2|t - \tau|) + \frac{K_1}{K_2} [\exp(K_2|t - \tau|) - 1] & \text{if } K_2 > 0 \\ K_0 + K_1|t - \tau| & \text{if } K_2 = 0 \end{cases}$$

Proof. We prove this result in two parts, first for $t \in [\tau, b]$ and then for $t \in [a, \tau]$.

Let $t \in [\tau, b]$. In the case where $K_2 = 0$ then it is trivial to see that $\rho(t) \leq K_0 + K_1|t - \tau|$ so assume that $K_2 > 0$.

Set

$$h(t) = K_0 + K_1|t - \tau| + |\int_\tau^t K_2\rho(s)ds|$$

then note that $\rho(t) \leq h(t)$ and $h(\tau) = K_0$.

Now see that

$$h'(t) = K_1 + K_2\rho(t)$$
$$h'(t) \leq K_1 + K_2h(t).$$

We therefore solve the differential inequality

$$h'(t) - K_2h(t) \leq K_1.$$

By multiplying by the integrating factor $\exp(-K_2t)$ and applying the product rule we get that

$$\frac{d}{dt} [\exp(-K_2t)h(t)] \leq K_1 \exp(-K_2t)$$

and integrating both sides from τ to t yields,

$$\exp(-K_2t)h(t) - \exp(-K_2\tau)h(\tau) \leq K_1 \int_\tau^t \exp(-K_2s)ds.$$

By evaluating the integral we have that

$$\exp(-K_2t)h(t) - \exp(-K_2\tau)h(\tau) \leq \frac{-K_1}{K_2} [\exp(-K_2t) - \exp(-K_2\tau)]$$

and then rearranging to get $h(t)$ as the subject we get

$$h(t) \leq \exp(K_2(t-\tau))h(\tau) + \frac{K_1}{K_2}\left[\exp(K_2|t-\tau|) - 1\right]. \tag{1}$$

Now consider the case when $t \in [a, \tau]$. Again when $K_2 = 0$, we can immediately see that $\rho(t) \leq K_0 + K_1|t-\tau|$, so we assume that $K_2 > 0$. Again we let

$$\begin{aligned} h(t) &= K_0 + K_1|t-\tau| + |\int_\tau^t K_2\rho(s)ds| \\ &= K_0 + K_1(\tau - t) + \int_t^\tau K_2\rho(s)ds \end{aligned}$$

and note that $\rho(t) \leq h(t)$. Differentiating $h(t)$ we get

$$h'(t) = -K_1 + K_2\rho(t)$$

with $h(\tau) = K_0$.

By substituting $h(t)$ for $\rho(t)$ and rearranging we get that $h'(t) + K_2h(t) \geq -K_1$.

Multiplying both sides by the integrating factor $\exp(K_2t)$ and applying the product rule we get

$$\frac{d}{dt}\left[\exp(K_2t)h(t)\right] \geq K_1\exp(K_2t)$$

and by integrating both sides from t to τ we get

$$\exp(\tau K_2)h(\tau) - \exp(tK_2)h(t) \geq \int_t^\tau K_1\exp(K_2s)ds.$$

Evaluating the integral and rearranging to get $h(t)$ the subject we get

$$h(t) \leq K_0\exp(K_2|t-\tau|) + \frac{K_1}{K_2}\left[\exp(K_2|t-\tau|) - 1\right]. \tag{2}$$

Now in both cases we have that $\rho(t) \leq h(t)$ so from (1) and (2) the result immediately follows. □

We will use this corollary in other results so we put it here for completeness.

Corollary 1 (Corollary to the Gronwell-Bellman Lemma)**.** *Let $\rho : [a, b] \longrightarrow [0, \infty)$ be a continuous function and let $\tau \in [a, b]$. If there is a non-negative constant K_2 such that for all $t \in [a, b]$*

$$\rho(t) \leq |\int_\tau^t K_2\rho(s)ds|$$

then $\rho(t) \equiv 0$ for all $t \in [a, b]$.

Proof. Applying lemma 1 with $K_0 = K_1 = 0$ we see that $\rho(t) \leq 0$ for all $t \in [a, b]$, but since by assumption $\rho(t) \geq 0$ for all $t \in [a, b]$ it must be that $\rho(t) \equiv 0$ on $[a, b]$. □

In some sense the Gronwell Bellman lemma put an a-priori bound on the solution given an integral inequality.

Naturally if α and β are piecewise continuous, non-negative functions on $[a, b]$ and we have the following inequality

$$\rho(t) \leq \alpha(t) + C|t - \tau| + |\int_\tau^t \beta(s)\rho(s)ds|$$

then we could set $K_0 = \max_{t\in[a,b]}\{\alpha(t)\}, K_1 = C, K_2 = \max_{t\in[a,b]}\{\beta(t)\}$ and apply lemma 1, however we can improve this bound by replicating the techniques used the proof of 1. This leads us to the following lemma.

Lemma 2 (Extended Gronwell Bellman Lemma). *Let $\rho : [a, b] \longrightarrow [0, \infty)$ be a piecewise continuous function and let $\tau \in [a, b]$. If there exists a non-negative constant K and non-negative piecewise continuous functions $\alpha \in C^2[a, b]$ and $\beta \in C^1[a, b]$ such that for all $t \in [a, b]$*

$$\rho(t) \leq \alpha(t) + K|t - \tau| + |\int_\tau^t \beta(t)\rho(s)ds|$$

then for all $t \in [a, b]$ we have that

$$\rho(t) \leq \frac{\int_\tau^t \alpha'(s)\left(\exp(-\int_\tau^s \beta(r)dr)\right)ds + K|t - \tau| + \alpha(\tau)}{\exp(-\int_\tau^t \beta(s)ds)}$$

Proof. TODO □

Now lemma 1 is a sepecial case of lemma 2 although in most cases lemma 1 will suffice.

Lemma 3 (Non-multiplicity of Solutions). *Let $D \subseteq \mathbb{R}$ and let $f : D \longrightarrow \mathbb{R}$ be continuous. Let $(\tau, A) \in D$ and consider the IVP*

$$x'(t) = f(t, x(t)) \tag{3}$$
$$x(\tau) = A \tag{4}$$

If there exists a constant $L > 0$ such that

$$|f(t, u) - f(t, v)| \leq L|u - v|$$

for all $(t, u), (t, v) \in D$ then the IVP given in (3) *and* (4) *has at most one solution, whose graph lies in D.*

Proof. Let $x = x(t)$ and $y = y(t)$ be two solutions to (3) and (4). Then we have the equivalent integral representations

$$x(t) = A + \int_\tau^t f(s, x(s))ds$$
$$y(t) = A + \int_\tau^t f(s, y(s))ds.$$

Now consider $\rho(t) = |x(t) - y(t)|$ and note that it would suffice to show that $\rho(t) \equiv 0$. See that,

$$\begin{aligned}\rho(t) &= |\int_\tau^t f(s, x(s)) - f(s, y(s))ds| \\ &\leq \int_\tau^t |f(s, x(s)) - f(s, y(s))|ds \\ &\leq \int_\tau^t L|x(s) - y(s)|ds \\ &= \int_\tau^t L|\rho(s)|ds.\end{aligned}$$

Corollary 1 then implies that $\rho(t) \equiv 0$ and hence $x(t) \equiv y(t)$, which is to say that there is at most one solution to (3) and (4). □

For future theorems and lemmas we shall use the following definitions, unless otherwise stated.

Let $D \subseteq \mathbb{R}^2$ and let

$$f : D \longrightarrow \mathbb{R} \tag{5}$$

be a continuous.

Let A and τ be constants and define the IVP

$$x'(t) = f(t, x(t)) \tag{6}$$

$$x(\tau) = A. \tag{7}$$

Define the sequence of functions

$$\{\phi_k(t)\}_{k=0}^{\infty} \tag{8}$$

by

$$\begin{aligned}\phi_0(t) &= A \\ \phi_k(t) &= A + \int_\tau^t f(s, \phi_{k-1}(s))ds\end{aligned}$$

Let $B > 0$ be some constant then define the rectable

$$\mathcal{R}_{[a,b]} = \{(t, p) \in \mathbb{R}^2 : t \in [a, b], |p - A| \leq B\}. \tag{9}$$

Let

$$\begin{aligned}\alpha &= \min\{\tau - a, \frac{B}{M}\} \\ \beta &= \min\{b - \tau, \frac{B}{M}\}\end{aligned}$$

and define the interval

$$I = [\tau - \alpha, \tau + \beta] \tag{10}$$

Lemma 4. *If $f : \mathcal{R}_{[a,b]} \longrightarrow \mathbb{R}$ then each $\phi_k(t)$ is is continuous.*

Proof. Consider

$$\begin{aligned}
|\phi_k(t_1) - \phi_k(t_2)| &= |\int_\tau^{t_1} f(s, \phi_{k-1}(s))ds - \int_\tau^{t_2} f(s, \phi_{k-1}(s))ds| \\
&\leq |\int_{t_2}^{t_1} |f(s, \phi_{k-1}(s))|ds| \\
&\leq |\int_{t_1}^{t_2} Mds| \\
&= M|t_1 - t_2|
\end{aligned}$$

where $M = \sup\{f(u,v)|(u,v) \in \mathcal{R}_{[a,b]}\}$. We therefore have that for any $\varepsilon > 0$ there exists a $\delta = \frac{\varepsilon}{M}$ such that $|\phi_k(t_1) - \phi_k(t_2)| \leq \varepsilon$ whenever $|t_1 - t_2| < \delta$. ☐

Lemma 5 (Existence: Picard - Lindelof)**.** *Let $f : \mathcal{R}_{[a,b]} \longrightarrow \mathbb{R}$ be a continuous function. If there exists a constant $L > 0$ satisfying $|f(t,u) - f(t,v) \leq L|u - v|$ for all $(t,u),(t,v) \in \mathcal{R}_{[a,b]}$ then the IVP defined by* (6) *and* (7) *has a unique solution on I, with $(t, x(t)) \in \mathcal{R}_{[\tau-\alpha,\tau+\beta]}$ for all $t \in I$.*

Proof. Since f is Lipschitz there is at most on solution to (6) and (7). This is guarenteed by lemma 3. We also know, from lemma 5, that $\phi_k(t)$ are all continuous and it is not hard to

Now suppose $\phi_k \longrightarrow \phi$ uniformly in I, then we have that

- ϕ is continuous on I
- $(t, \phi(t)) \in \mathcal{R}_{[\tau-\alpha,\tau+\beta]}$
- $$\begin{aligned}
\lim_{k \longrightarrow \infty} \phi_k(t) &= \lim_{k \longrightarrow \infty} \left(A + \int_\tau^t f(s, \phi_{k-1}(s))ds \right) \\
\phi(t) &= A + \int_\tau^t f(s, \phi(s))ds
\end{aligned}$$

So we just have to show that $\phi_k \longrightarrow \phi$ uniformly on I. Now note that $\phi_k(t) = \phi_0(t) + \sum_{i=1}^k (\phi_i(t) - \phi_{i-1}(t))$ so to show that ϕ_k converges uniformly we apply the Weierstrass M Test. So we wish to consider $\phi(t) - \phi_{i-1}(t)|$ and we claim that

$$|\phi(t) - \phi_{i-1}(t)| \leq \frac{ML^{i-1}|t - \tau|^i}{i!} \tag{11}$$

We prove this by induction. In the case $i = 0$,

$$\begin{aligned}
|\phi_1(t) - \phi_0(t)| &= |A + \int_\tau^t f(s, \phi_0(s))ds - A| \\
&= |\int_\tau^t f(s, A)ds| \\
&\leq |\int_\tau^t |f(s, A)|ds|
\end{aligned}$$
$$\&leqM|t - \tau|.$$

So the claim holds in the base case. Assuming that (11) holds for some $i = n \geq 1$ then

$$\begin{aligned}
|\phi_{n+1}(t) - \phi_n(t)| &= |\left(A + \int_\tau^t f(s, \phi_{n+1}(s))ds\right) - \left(A + \int_\tau^t f(s, \phi_{n-1}(s))ds\right)| \\
&= |\int_\tau^t f(s, \phi_n(s)) - f(s, \phi_{n-1}(s))ds| \\
&\leq |\int_\tau^t f(s, \phi_n(s))) - f(s, \phi_{n-1}(s))|ds \\
&\leq L|\int_\tau^t |\phi_n(s) - \phi_{n-1}(s)|ds| \\
&\leq |\int_\tau^t \frac{L^n M(s-\tau)^n}{n!} ds| \\
&\leq \frac{ML^n(t-\tau)^{n+1}}{(n+1)!}.
\end{aligned}$$

So we have that (11) holds for $i = n + 1$ and so it holds for all $i \geq 0$. Now it is clear to see that

$$\sum_{i=0}^{\infty} \frac{ML^i - 1(t-\tau)^i}{(i)!} < \infty$$

so by the Wierstrass M Test $\phi_k \longrightarrow \phi$ uniformly on $[\tau, \tau + \beta]$, with ϕ a solution to (6) and (7) on $[\tau, \tau + \beta]$. TODO: Extend. $\square$

We now wish to introduce some other definitions and theorems which are useful when considering the existence the uniqueness of ODEs.

Definition 1. *Let (M_1, d_1), (M_2, d_2) be some metric spaces then a map $T : M_1 \longrightarrow M_2$ is a contraction mapping if there exists some real number γ such that for all $x, y \in M_1$*

$$d_2(T(x), T(y)) \leq \gamma d_1(x, y)$$

.

Lemma 6 (Continuity of Contraction Mappings)**.** *If (M_1, d_1), (M_2, d_2) are some metric spaces then a contraction mapping $T : M_1 \longrightarrow M_2$ is uniformly continuous.*

Proof. Given any $\varepsilon > 0$ there exists a $\delta = \varepsilon$ such that whenever $d_1(x, y) \leq \delta$, we have that $d_2(x, y) \leq \varepsilon$. So T must be uniformly continuous. $\square$

Theorem 1 (Banach Fixed Point Theorem)**.** *Let (M, d) be a non-empty complete metric space with a contraction mapping $T : M \longrightarrow M$. Then there exists a unique point $x^* \in M$ such that $T(x) = x$. Furthermore the the sequence*

$$\{x_n\}_n^\infty \tag{12}$$

defined by

$$x_0 \in M \tag{13}$$
$$x_k = T(x_{k-1}) \tag{14}$$

is such that $x_n \longrightarrow x^*$.

Proof. We first show that $d(x_{n+1}, x_n) \leq \gamma^n d(x_1, x_0)$. We prove this by indution so consider the base case

$$\begin{aligned} d(x_2, x_1) &= d(T(x_1), T(x_0)) \\ d(x_2, x_1) &\leq \gamma d(x_1, x_0). \end{aligned}$$

Now assume that this is true for n and consider the case $n + 1$ and see that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(T(x_{n+1}, T(x_n)) \\ &\leq \gamma d(x_{n+1}, x_n) \\ &\leq \gamma\gamma^n d(x_1, x_0) \\ &\leq \gamma^{n+1} d(x_1, x_0). \end{aligned}$$

So it is true for $n + 1$ and so we have the result by indution.

Next we wish to show that the sequence $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence.

Consider x_n, x_m with $m > n$ and note that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leq \gamma^{m-1} d(x_1, x_0) + \gamma^{m-2} d(x_1, x_0) + \ldots + \gamma n d(x_0, x_1) \\ &= \gamma^n d(x_1, x_0) \sum_{k=0}^{m-n-1} \gamma^k \\ &\leq \gamma^n d(x_1, x_0) \sum_{k=0}^{\infty} \gamma^k \\ &= \gamma^n d(x_1, x_0) \left(\frac{1}{1-\gamma} \right). \end{aligned} \tag{15}$$

Now for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, say,

$$N = \left\lceil \frac{\log\left(\frac{\varepsilon(1-\gamma)}{d(x_1, x_0)}\right)}{\log(\gamma)} \right\rceil$$

such that $n > N$ implies

$$\gamma^n d(x_1, x_0)\left(\frac{1}{1-\gamma}\right) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have the fact that the sequence is Cauchy. The fact that the metric space M is complete guarentees that the limit of the Cauchy sequence exists.

We now wish to prove that $\lim_{n\longrightarrow\infty} x_n = x^*$ is a fixed point of T. To do this consider

$$\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} T(x_{n-1})$$

but since T is a contraction mapping, lemma 6 guarentees that T is uniformly continuous, and hence

$$\lim_{n\longrightarrow\infty} T(x_{n-1}) = T\left(\lim_{n\longrightarrow\infty} x_{n-1}\right)$$

which is to say that $x^* = T(x^*)$.

It now only remains to prove the uniqueness claim. Suppose y^* is another fixed point of T then we must have that $d(x^*, y^*) = d(T(x^*), T(y^*))$ and hence

$$0 \leq d(x^*, y^*) \leq \gamma d(x^*, y^*),$$

but $\gamma \in [0, 1)$ which implies $0 \leq d(x^*, y^*) \leq 0$ and hence $x^* = y^*$. So we have show that there exists a unique point $x^* \in M$ such that $T(x) = x$ and that this point is the limit of sequence (12). □

Lemma 7 (Alternate Picard - Lindelof)**.** *Given a constant K and compact $D subseteq \mathbb{R}$ and the continuous function $f : D\mathbb{R}$ which satisfies a Lipschitz condition of the form*

$$|f(t, u) - f(t, v)| \leq K|u - v|,$$

the IVP defined by 6 and 7 has a unique solution.

Proof. The IVP 6, 7 is equivelent to the integral equation

$$x(t) - x(\tau) = \int_\tau^t f(s, x(s))ds \tag{16}$$

Now as f is continuous and D is compact there must exist a constant M such that $|f(t, u)| \leq M$ and a neighbourhood $D' \subseteq D$ such that $(\tau, A) \in D'$.

We can choose δ such that

- $(t, u) \in D'$ if $|t - \tau| \leq$ and $|x(t) - A| \leq M\delta$
- $K\delta < 1$.

FINISH PROOF

□

We now present a general statement of Dini's theorem.

Theorem 2 (Dini's Theorem)**.** *Let X be a compact topological space, and let $\{f_n\}$ be a sequence of continuous real valued functions such that $f_n(x) \leq f_{n+1}(x)$ for all n and $x \in X$ and such that $f_n \underset{pointwise}{\longrightarrow} f$ where f is continuous. In this case $f_n \underset{uniformly}{\longrightarrow} f$.*

Proof. Let $\varepsilon > 0$. Let $g_n = f - f_n$ and $E_n = \{x \in X : g_n(x) < \varepsilon\}$. Now as each g_n is continuous then each E_n must be open. Now as $f_n \underset{\text{pointwise}}{\longrightarrow} f$ E_n must form an open over of X , but by compactness there must exist an integer N such that $E_N = X$ which is to say that if $n \geq N$ then $|f(x) - f_n(x)| \leq \varepsilon$. So $f_n \underset{\text{uniformly}}{\longrightarrow} f$. □

Lemma 8 (Bihari's Inequality)**.** *Let $f : [0, \infty) \longrightarrow [0, \infty)$ and $u : [0, infty) \longrightarrow [0, \infty)$ be continuous and let $w : [0, \infty) \longrightarrow [0, \infty)$ be a non-decreasing continuous function such that $w(u) > 0$ for $u \in (0, \infty)$. Let α be a non-negative constant. If u satisfies the following inequality*

$$u(t) \leq \alpha + \int_0^\infty f(s)w(u(s))ds \qquad t \in [0, \infty)$$

then

$$u(t) \leq G^{-1}\left(G(\alpha) + \int_0^t f(s)ds\right) \qquad t \in [0, T]$$

where

Proof. □

Useful Results in FDE Theory

Since the gamma function will come up often when dealing with fractional differential equations we provide a definition of the gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$

We also present without proof (due to triviality) some important results relating to the gamma function.

$$\Gamma(n+1) = n!$$
$$\Gamma(z) = (z-1)\Gamma(z-1)$$

A less trivial result we will prove is that

Proposition 1.

$$\Gamma(z) = \lim_{n \longrightarrow \infty} \frac{n! n^z}{\prod_{k=0}^{n}(z+k)}.$$

Proof. Consider the sequence of functions

$$f_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$$

and observe that if we permit the limit to pass through the integral [1] then

$$\lim_{n \longrightarrow \infty} f_n(z) = \Gamma(z).$$

Also if we perform the substitution $\tau = \frac{t}{n}$ and perform repeated integration by parts we obtain

$$\begin{aligned} f_n(z) &= \int_0^1 (1-\tau)^n \tau^z - 1 d\tau \\ &= \frac{n^z}{z} n \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \\ &= \frac{n^z n!}{\prod_{k=0}^{n-1}(z+k)} \int_0^1 \tau^{z+n-1} d\tau \\ &= \frac{n^z}{\prod_{k=0}^{n}(z+k)} \end{aligned}$$

□

We would also like to introduce the beta function, as it will often be more convenient to use the beta function than several combinations of gamma functions.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

Proposition 2 (Relationship Between the Beta and Gamma Functions)**.**

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Proof. Consider the integral

$$h_{x,y}(t) = \int_0^t = \tau^{x-1}(1-\tau)^y - 1 d\tau. \tag{17}$$

[1] It is possible to prove that this is justified and the interested reader is encouraged to consult FIXME

Clearly (17) is the convolution of t^{x-1} and t^{y-1} and therefore

$$\mathcal{L}\left\{h_{x,y}(t)\right\} = \mathcal{L}\left\{t^{x-1}\right\}\mathcal{L}\left\{t^{y-1}\right\}$$

but clearly we have $\mathcal{L}(t^{x-1}) = \frac{\Gamma(x)}{s^x}$ and likewise for t^{y-1} so we have

$$\mathcal{L}\left\{h_{x,y}(t)\right\} = \frac{\Gamma(z)\Gamma(y)}{s^{x+y}}$$

The applying the inverse Laplace transform we arrive at

$$h_{x,y}(t) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}t^{x+y-1}$$

but $h_{x,y}(1) = B(x,y)$ so we have that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

□

PUT IN COMPLEX EVALUATION OF THE GAMMA FUNCTION

We now wish to define the parameter Mittag-Leffler functions

Definition 2.

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \tag{18}$$

$$M_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \tag{19}$$

The one and two paramter Mittag-Leffler functions are extremly general and many elementry functions can be written in terms of either of these functions.

For example it is clear that

$$\begin{aligned}
M_{0,1}(t) &= \frac{1}{1-t} \\
M_{1,1}(t) &= \exp(t) \\
M_{2,1}(t) &= \cosh(t) \\
M_{2,2}(t) &= \frac{\sinh(t)}{t}
\end{aligned}$$