A Solution to a Fractional Differential Equation

The Laplace Transform Method

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The Goal

We aim to get a solution to the following fractional differential equation (in terms of Caputo derivatives)

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t) \tag{1}$$

along with the initial conditions

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases} \tag{2}$$

Motivations

Cauchy Formula for Repeated Integration

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n) dy_n \cdots dy_2 dy_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt$$

Motivations

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The idea is to replace the factorials with gamma functions to define an integral of arbitrary order

Motivations

Cauchy Formula for Repeated Integration

$$\int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n)\,dy_n \cdots dy_2\,dy_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$$

The idea is to replace the factorials with gamma functions to define an integral of arbitrary order

Riemann-Liouville Fractional Integral

$$\left(I_a^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

Motivations (Derivatives)

Riemann-Liouville Fractional Derivative

$$\left(\mathcal{D}_a^{\alpha} f\right)(x) = \frac{d^{\lceil \alpha \rceil}}{dx^{\lceil \alpha \rceil}} \left(I_a^{\lceil \alpha \rceil - \alpha} f \right)(x)$$
$$= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)dt}{\underline{(x-t)^{\alpha-n+1}}}$$

where $n - 1 < \alpha \leq n$

Motivations (Derivatives)

Caputo Fractional Derivative

$$\left({}^{C}\mathcal{D}_a^{\alpha} f\right)(x) = \left(I_a^{\lceil\alpha\rceil-\alpha} \frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} f\right)(x)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{\frac{d^t}{dt^n} f(t) dt}{(x-t)^{\alpha-n+1}}$$

where $n-1 < \alpha \leq n$

Riemann-Liouville vs Caputo Derivative

Note!

The Caputo derivative and the and the Riemann-Liouville derivatives are note the same. In general

$$\left({}^{C}\mathcal{D}_a^\alpha f\right)(x) \neq \left(\mathcal{D}_a^\alpha f\right)(x).$$

The reason is exactly the same reason that in general

$$f(x) \neq \int_a^x f'(t)dt.$$

In some sense if you differentiate first you "loose information" about the function.

Riemann-Liouville vs Caputo Derivative

The Caputo derivative is often used in fractional differential equations because it can be coupled with integer order initial conditions, whereas often the Riemann-Liouville derivative can't be coupled with integer order initial conditions.

A Quick Note on the Laplace Transform

Definition

We the define the Laplace transform of a function f to be the function F given by

$$F(s) := \int_0^{\infty} e^{-st} f(t) dt$$

We often write $F(s) = \mathcal{L}\{f(t)\}$.

A Quick Note on the Laplace Transform

The laplace is particularly useful as it allows us to transform a differential equation into an "algebraic" equation.
Lerch's theorem guarantees, with minor caveats, that the Laplace transform of a function is unique.

Basic Idea of the Laplace Transform Method

- Apply the Laplace transform to both sides of the differential equation to get and "algebraic" equation.
- Apply the Laplace transform to the initial conditions.
- Sub the transformed intial conditions into the transformed equation.
- Rearange to get an expression for the Laplace transform of the function of interest.
- Invert. (This is possible, and guarenteed with minor caveats by Lerch's theorem)

The Differential Equation

$$\left({}^{C}\mathcal{D}_0^{\alpha} y\right)(t) = \beta y(t)$$

$$y^{(k)}(0) = \begin{cases} 1 & k = 0 \\ 0 & 1 \leq k \leq \lfloor \alpha \rfloor - 1 \end{cases}$$