Chapter 1

Electromagnetic Phenomena

1.1 Introduction

In the mid 1800's, James Maxwell assembled the various known relationships of electricity and magnetism into a concise set of equations. (In his notation they would hardly fit on a T-shirt.) After introducing a key component into one of the relationships, Maxwell realized that all together the equations comprised a complete self-consistent theory of electromagnetic phenomena. Most remarkably, the equations implied the existence of electromagnetic waves, which travel at the speed of light. Since the speed of light had already been measured, Maxwell immediately realized (as was already suspected) that light is simply a high-frequency manifestation of the same phenomena that govern the influence of currents and charges upon each other. Previously, optics was considered to be a topic quite separate from electricity and magnetism.

In this chapter, we will briefly review the physical principles and the derivations leading to Maxwell's equations. Students studying optics are expected to remember and appreciate Maxwell's equations, but they don't necessarily need to remember the details of the derivations given here. Nevertheless, the main intent is to help students appreciate the connection between electricity and magnetism and light.

1.2 Coulomb's and Gauss' Laws

The force on a charge q located at $\vec{r}$ that is exerted by another charge q' located at $\vec{r}\,'$ is

$$\vec{F} = q\vec{E}, \text{ where } \vec{E}(\vec{r}) = \frac{q'}{4\pi\varepsilon_o}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}. \tag{1.2.1}$$

This is known as Coulomb's law, which we have written in a slightly sophisticated form, emphasizing the concept of an electric field $\vec{E}(\vec{r})$, defined throughout space (regardless of whether a second charge q is actually present). This is the familiar inverse square law directed along the unit vector $(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|$ (see Fig. 1.1). The parameter $\varepsilon_o = 8.854\times 10^{-12}\ \mathrm{C^2/N\cdot m^2}$ called the permittivity amounts to a proportionality constant.

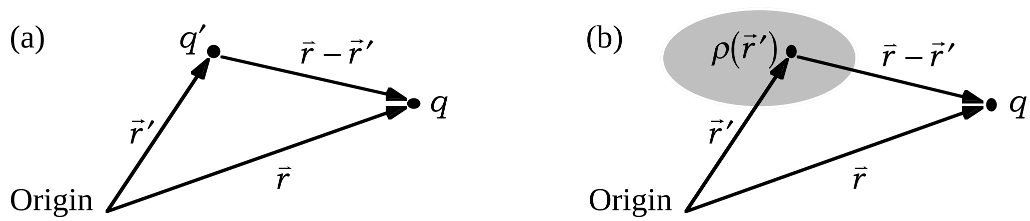

Fig. 1.1. The geometry of Coulomb's law for (a) a point charge and (b) a charge distribution.

The total force from a collection of charges is found by summing over (1.2.1) for each charge q_n' with its specific location $\vec{r}_n'$. If the charge distribution is continuous, having density $\rho(\vec{r}\,')$ (units of charge per volume), this linear superposition is performed by integration:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int_V \rho(\vec{r}\,') \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dv'. \tag{1.2.2}$$

This three-dimensional integral gives the net contribution of the charge distributed throughout volume V.

Gauss' law is derived from Coulomb's law. There is no new physical phenomenon introduced by doing this. The new law is nothing more than a mathematical interpretation of Coulomb's law, utilizing the concept of the electric field. To derive Gauss' law, take the divergence of (1.2.2):

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int_V \rho(\vec{r}\,') \vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dv'. \tag{1.2.3}$$

The subscript on $\vec{\nabla}_{\vec{r}}$ indicates that it operates on $\vec{r}$ while treating $\vec{r}\,'$ as a constant. As messy as this integral appears, it contains a remarkable mathematical property that can be exploited, even without specifying the form of the charge distribution $\rho(\vec{r}\,')$. In modern mathematical language, the vector expression in the integral is a three-dimensional delta function:

$$\vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \equiv 4\pi\delta^3(\vec{r}\,' - \vec{r}) = 4\pi\delta(x' - x)\delta(y' - y)\delta(z' - z). \tag{1.2.4}$$

A derivation of this formula and a description of its properties are addressed in problem P0.3.17. In short, the delta function allows the integral in (1.2.3) to be performed, and the relation becomes simply

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_o}. \tag{1.2.5}$$

This is the differential form of Gauss' law.

The integral form of Gauss' law (perhaps more familiar) is obtained by integrating (1.2.5) over a volume V and by applying the divergence theorem (0.3.8) to the left-hand side:

$$\oint_S \vec{E}(\vec{r}) \cdot \hat{n}\, da = \frac{1}{\varepsilon_o} \int_V \rho(\vec{r})\, dv. \tag{1.2.6}$$

This formula says that the total electric field flux extruding through a closed surface S is proportional to the net charge contained within it (i.e. within volume V contained by S).

In the study of light and optics, we seldom consider the propagation of electromagnetic waveforms through charged materials. One might be tempted therefore to immediately set the right-hand side of (1.2.5) to zero. However, this would be wrong because neutral materials can become polarized. This polarization can vary within a material, leading to local concentrations of positive or negative charge even though on average the material is neutral. This polarization of the material is described by a dipole distribution field $\vec{P}(\vec{r})$, called the *polarization* (in units of dipoles per volume, or charge times length per volume).

To better understand the effects of medium polarization, consider the divergence theorem (0.3.8) applied to $\vec{P}(\vec{r})$ in a neutral medium:

$$\oint_S \vec{P}(\vec{r}) \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{P}(\vec{r}) dv. \tag{1.2.7}$$

The left-hand side of (1.2.7) is a surface integral, which upon completion has units of charge. Physically, it is the sum of the charges touching the inside of surface S (multiplied by a minus since dipole vectors point from the negatively charged end of a molecule to the positively charged end). The situation is depicted in Fig. 1.2. Keep in mind that $\vec{P}(\vec{r})$ is a continuous function so that Fig. 1.2 depicts crudely an infinite number of infinitely tiny dipoles (no fair drawing a surface that avoids cutting the dipoles; cut through them at random). When $\vec{\nabla} \cdot \vec{P}$ is zero, evidently there are equal numbers of positive and negative charges touching S from within. When $\vec{\nabla} \cdot \vec{P}$ is not zero, the positive and negative charges touching S are not balanced. Essentially, excess charge ends up within the volume because the non-uniform alignment of dipoles causes them to be cut preferentially at the surface. Since either side of (1.2.7) is equal to the excess charge inside the volume, $-\vec{\nabla} \cdot \vec{P}$ may be interpreted as a charge density (it certainly has the right units --- charge per volume). To make this interpretation, we let the volume V be sufficiently small so that $\vec{\nabla} \cdot \vec{P}$ is uniform throughout.

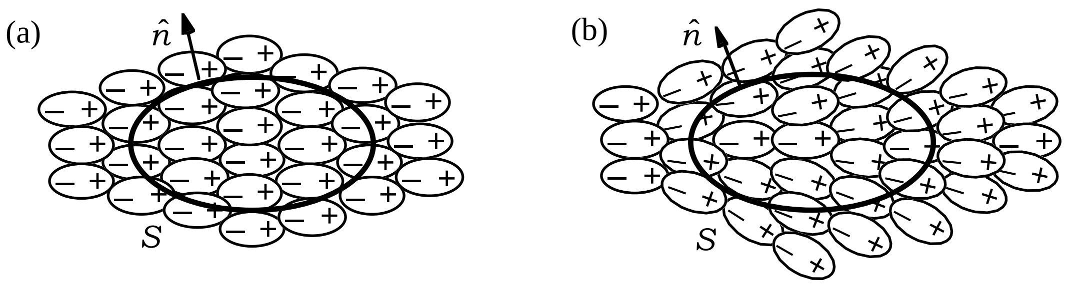

Fig. 1.2. A polarized medium with (a) $\vec{\nabla} \cdot \vec{P} = 0$ and with (b) $\vec{\nabla} \cdot \vec{P} \neq 0$.

The charge density then derives from two distinct components:

$$\rho(\vec{r}) = \rho_{\text{free}}(\vec{r}) - \vec{\nabla} \cdot \vec{P}(\vec{r}). \tag{1.2.8}$$

The first term is a charge distribution in the usual sense, and the second term arises due to variations in the polarization of a neutral medium. This second type of charge density is sometimes referred to as a bound distribution since positive and negative charges come bound in pairs. Again, the negative sign in (1.2.8) arises since when $\vec{P}$ points out of the surface S, negative charges are left inside.

With (1.2.8), the differential form of Gauss' law (1.2.5) becomes

$$\vec{\nabla} \cdot \left(\varepsilon_o \vec{E} + \vec{P}\right) = \rho_{\text{free}}, \tag{1.2.9}$$

where the combination $\varepsilon_o\vec{E}+\vec{P}$ is often called the displacement field, denoted by $\vec{D}$. This is one of Maxwell's equations (one down; three to go).

1.3 Biot-Savart and Ampere's Laws

The Biot-Savart law (in conjunction with the Lorentz force) is similar to Coulomb's law except that it describes the force on a charged particle that comes about from a magnetic field instead of an electric field. In this case, in order to experience the force, the charge q must move with a velocity $\vec{\upsilon}$. The magnetic field arises from charges that are also in motion. We will consider a moving charge distribution that forms a current distributed throughout space. The effect of the moving charge distribution is described by a continuous current density $\vec{J}(\vec{r}\,')$ in units of charge times velocity per volume (or current per cross sectional area). Analogous to (1.2.1) and (1.2.2), the Biot-Savart law is

$$\vec{F}=q\vec{\upsilon}\times\vec{B},\ \text{where}\ \ \vec{B}(\vec{r})=\frac{\mu_o}{4\pi}\int_V \vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\upsilon'. \tag{1.3.1}$$

(The latter equation is referred to as the Biot-Savart law; the first equation is the Lorentz force.)

As before, we will apply mathematics to the Biot-Savart law to obtain another of Maxwell's equations. With the formula in P0.3.7, we can rewrite (1.3.1) as

$$\vec{B}(\vec{r})=-\frac{\mu_o}{4\pi}\int_V \vec{J}(\vec{r}\,')\times\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}\,'|}\,d\upsilon'=\frac{\mu_o}{4\pi}\vec{\nabla}_{\vec{r}}\times\int_V\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d\upsilon'. \tag{1.3.2}$$

Taking the divergence of this expression immediately gives (see P0.3.9)

$$\vec{\nabla}\cdot\vec{B}=0, \tag{1.3.3}$$

which is another of Maxwell's equations (two down; two to go). The similarity between this equation and Gauss' law for electric fields (1.2.5) is immediately apparent. In fact, (1.3.3) is known as Gauss' law for magnetic fields. If one were to imagine the existence of magnetic charges (monopoles), then the right-hand side of (1.3.3) would not be zero. However, since magnetic charges have yet to be discovered (if ever), there is no point in introducing a magnetic charge distribution into the equation. In integral form, Gauss' law for magnetic fields looks like that for electric fields (1.2.6), with zero on the right hand side. The law implies that the total magnetic flux extruding through any closed surface is zero (i.e. there will be as many field lines pointing inwards as pointing outwards).

Although we will not derive another of Maxwell's equations in this section, it is interesting to show that the Biot-Savart law implies Ampere's law. Ampere's law is obtained when the Biot-Savart law (1.3.1) is inverted so that $\vec{J}$ appears by itself, unfettered by integrals or the like. This is accomplished through mathematics, so no new physical phenomena are introduced, only a new interpretation. We carry out this exercise because we will refer to this derivation in section 1.5 to obtain the last of Maxwell's equations.

The curl of (1.3.1) yields

$$\vec{\nabla}\times\vec{B}(\vec{r})=\frac{\mu_o}{4\pi}\int_V \vec{\nabla}_{\vec{r}}\times\left[\vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]dv'. \tag{1.3.4}$$

We next apply the differential vector rule P0.3.11 while noting that $\vec{J}(\vec{r}\,')$ does not depend on $\vec{r}$ so that only two terms survive. The curl of $\vec{B}(\vec{r})$ then becomes

$$\vec{\nabla}\times\vec{B}(\vec{r})=\frac{\mu_o}{4\pi}\int_V\left(\vec{J}(\vec{r}\,')\left[\vec{\nabla}_{\vec{r}}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]-\left[\vec{J}(\vec{r}\,')\cdot\vec{\nabla}_{\vec{r}}\right]\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)dv'. \tag{1.3.5}$$

According to (1.2.4), the first term in the integral is $4\pi\vec{J}(\vec{r}\,')\delta^3(\vec{r}\,'-\vec{r})$ so that it is easily integrated. To make progress on the second term, we observe that the gradient can be changed to operate on the primed variables without affecting the final result (i.e. $\vec{\nabla}_{\vec{r}}\to-\vec{\nabla}_{\vec{r}\,'}$). In addition, we take advantage of the vector integral theorem (0.3.10) to arrive at

$$\vec{\nabla}\times\vec{B}(\vec{r})=\mu_o\vec{J}(\vec{r})-\frac{\mu_o}{4\pi}\int_V\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\left[\vec{\nabla}_{\vec{r}\,'}\cdot\vec{J}(\vec{r}\,')\right]dv'+\frac{\mu_o}{4\pi}\oint_S\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\left[\vec{J}(\vec{r}\,')\cdot\hat{n}\right]da'. \tag{1.3.6}$$

The last term in (1.3.6) vanishes if we assume that the current density $\vec{J}$ is completely contained within the volume V so that it is zero at the surface S. Thus, the expression for the curl of $\vec{B}(\vec{r})$ reduces to

$$\vec{\nabla}\times\vec{B}(\vec{r})=\mu_o\vec{J}(\vec{r})-\frac{\mu_o}{4\pi}\int_V\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\left[\vec{\nabla}_{\vec{r}\,'}\cdot\vec{J}(\vec{r}\,')\right]dv'. \tag{1.3.7}$$

The latter term in (1.3.7) vanishes if $\nabla\cdot\vec{J}=0$, which is the case if the current density is steady in time, as will be discussed in Sect. 1.5. Maxwell was the first to realize that this term in general does not vanish for dynamic situations. However, for steady currents, (1.3.7) reduces to

$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J}, \tag{1.3.8}$$

which is the differential form of Ampere's law.

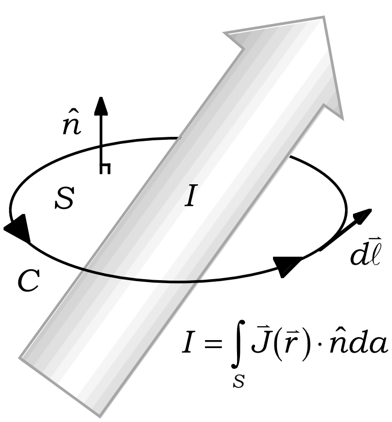

Fig. 1.3 Ampere's law.

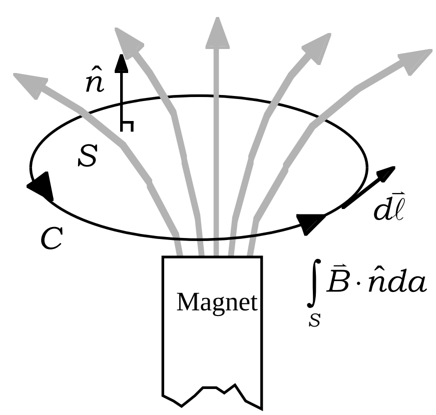

Fig. 1.4 Faraday's law.

The integral form of Ampere's law (perhaps more familiar) can be obtained by integrating both sides of (1.3.8) over an open surface S, contained by contour C. Stokes' theorem (0.3.9) is applied to the left-hand side:

$$\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell} = \mu_o \int_S \vec{J}(\vec{r})\cdot \hat{n}da \,. \tag{1.3.9}$$

This law says that the line integral of $\vec{B}$ around a closed loop C is proportional to the total current flowing through the loop (see Fig. 1.3). The proportionality constant called the permeability is $\mu_o = 4\pi \times 10^{-7}\ \mathrm{T\cdot m/A}$ (same as $\mathrm{kg\cdot m/C^2}$). Recall that the units of $\vec{J}$ are current per area. In summary, we have shown that the physics in Ampere's law is already present in the Biot-Savart law. The laws are connected through mathematics.

1.4 Faraday's Law

Michael Faraday was very remarkable in that his knowledge of mathematics was limited, yet he was able to characterize the relationship between changing magnetic fluxes and induced electric fields. Faraday showed that a change in magnetic flux through the area of a circuit loop induces an electromotive force in the loop according to (see Fig. 1.4)

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot \hat{n}da \,. \tag{1.4.1}$$

Faraday's law is one of Maxwell's equations. However, we would like to rewrite it in differential form. To do this, we apply Stokes' theorem to the left-hand side and obtain

$$\int_S \vec{\nabla}\times\vec{E}\cdot \hat{n}da = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot \hat{n}da, \text{ or } \int_S \left(\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot \hat{n}da = 0 \,. \tag{1.4.2}$$

This implies

$$\vec{\nabla}\times\vec{\mathrm{E}} + \frac{\partial \vec{\mathrm{B}}}{\partial \mathrm{t}} = 0, \tag{1.4.3}$$

which is the differential form of Faraday's law (three of Maxwell's equations down; one to go).

1.5 Continuity Equation

We now return to the discussion of Ampere's law in section 1.3 and consider the possibility of a current density $\vec{J}$ that varies dynamically in time. Consider a volume of space enclosed by a surface S wherein current is flowing. The total current exiting the volume is

$$I = \oint_S \vec{J}\cdot \hat{n}da \,. \tag{1.5.1}$$

The units on this equation are that of current, or charge per time, leaving the volume. Note the similarity with the right-hand side of Ampere's law (1.3.9).

Since we have considered a *closed* surface S, the current leaving the enclosed volume V can also be expressed in terms of the rate at which the net of charge within the volume changes in time:

$$I = -\int_V \frac{\partial \rho}{\partial t} dv. \tag{1.5.2}$$

Upon combining (1.5.1) and (1.5.2) as well as applying the divergence theorem to the former, we get

$$\int_V \vec{\nabla} \cdot \vec{J} dv = -\int_V \frac{\partial \rho}{\partial t} dv, \text{ or } \int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) dv = 0. \tag{1.5.3}$$

This implies

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \tag{1.5.4}$$

which is a statement of the conservation of charge. This equation implies that if there is a net current flowing out of a volume, then the net charge inside the volume must decrease in time. This is not a concern for the traditional Ampere's law since a steady current has an equal amount of charge flowing out-of and into a particular volume.

Maxwell's main contribution (aside from organizing other people's formulas) was the injection of the continuity equation (1.5.4) into (1.3.7) to make Ampere's law appropriate for time-varying currents:

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \frac{\mu_o}{4\pi} \frac{\partial}{\partial t} \int_V \rho(\vec{r}\,') \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dv'. \tag{1.5.5}$$

A substitution of (1.2.2) into the final term yields

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} = \vec{J} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}. \tag{1.5.6}$$

The final term is known as the 'displacement current', which exists even in the absence of any actual charge density ρ. Evidently, a changing electric field behaves like a current. Notice the similarity with (1.4.3), which no doubt was in part what motivated Maxwell's work.

We are essentially finished with the fourth of Maxwell's equations except for a brief examination of the current density $\vec{J}$. The current density can be decomposed into three categories. The first category is associated with charges that are free (not dipoles), such as electrons in metals. We will denote this type of current density by $\vec{J}_{\text{free}}$. The second category is associated with minute currents within individual atoms that give rise to paramagnetic and diamagnetic effect. These have very little importance to most optics problems, and so we will neglect these types of currents. The third type of current occurs when a distribution of dipoles, represented by $\vec{P}$, varies in time. Physically, if the dipoles depicted in Fig. 1.2 change their orientation as a function of time in some coordinated fashion, an effective current density can result. To see this, we substitute (1.2.8) into (1.5.4) to obtain

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_{\text{free}}(\vec{r})}{\partial t} + \vec{\nabla} \cdot \frac{\partial \vec{P}(\vec{r})}{\partial t}. \tag{1.5.7}$$

We then associate $\vec{J}_{\text{free}}$ with $\rho_{\text{free}}(\vec{r})$, so that the total current density is written as

$$\vec{J} = \vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t}. \tag{1.5.8}$$

The term $\partial \vec{P}/\partial t$ is therefore the effective current density (having appropriate units) due to a time variations in $\vec{P}$. With this, the last of Maxwell's equations (1.5.6) becomes

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} - \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t}. \tag{1.5.9}$$

Note that $\vec{B}/\mu_o$ is often written as $\vec{H}$, which can be generalized to include the magnetic effects of materials (ignored here).

1.6 Maxwell's Equations

In summary, Maxwell's equations in electrically neutral materials (i.e. $\rho_{\text{free}} = 0$) are

$$\vec{\nabla} \cdot \vec{E} = -\frac{\vec{\nabla} \cdot \vec{P}}{\varepsilon_o} \qquad \text{(Coulomb's law} \Rightarrow \text{Gauss' law)} \tag{1.6.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(Biot-Savart law} \Rightarrow \text{Gauss' law)} \tag{1.6.2}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{(Faraday's law)} \tag{1.6.3}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} - \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} \qquad \text{(Ampere's law; fixed by Maxwell)} \tag{1.6.4}$$

These equations form a complete description of (classical) electromagnetic phenomena (superceded by quantum electrodynamics). The equations are relativistically invariant, and therefore embody the principle of relativity. This means that when the Lorentz transformations are applied to Maxwell's equations, a new set of equations result that have the identical form. After all, that's how Lorentz came up with his transformations.

Notice that we have dismissed the possibility of a free charge density ρ_{free} while we have retained the possibility of free current density $\vec{J}_{\text{free}}$. This is not a contradiction. In a neutral material, some charges may move differently than their oppositely charged counterparts, such as electrons versus ions in a metal. This gives rise to currents without the requirement of a net charge.

Exercises

P1.6.1 Memorize Maxwell's Equations. Be prepared to reproduce them on an exam.

P1.6.2 Suppose that the electric field is given by $\vec{E}(\vec{r},t) = \vec{E}_o \cos\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)$, where $\vec{k} \perp \vec{E}_o$ and ϕ is a constant phase. Show that $\vec{B}(\vec{r},t) = \dfrac{\vec{k}\times\vec{E}_o}{\omega}\cos\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)$ is consistent with (1.6.3).

P1.6.3 $\vec{E}$ and $\vec{B}$ in the previous problem specify waves that travel in phase and which are perpendicular to each other and to the direction of propagation $\vec{k}$. What are the implications for $\vec{J}_{\text{free}}$ and $\partial\vec{P}/\partial t$ in (1.6.4)?

1.7 The Wave Equation

When Maxwell unified electromagnetic theory, he noticed that waves are a solution to his set of equations. No doubt his desire to find a set of equations that predicted waves partly directed his line of reasoning. After all, it was already known that light traveled as waves and there was reason to suspect that light was connected with electricity and magnetism. Kirchhoff had previously noticed that $1/\sqrt{\varepsilon_o\mu_o}$ gives the correct speed of light $c = 3\times 10^8\ \text{m/s}$, which had already been measured. Faraday and Kerr had observed that strong magnetic and electric fields affect light propagating in crystals. At first glance, Maxwell's equations might not immediately suggest (to the inexperienced eye) that waves are solutions. However, we can manipulate the equations (first order differential equations coupling $\vec{E}$ and $\vec{B}$) into the familiar wave equation (second order differential equations for either $\vec{E}$ or $\vec{B}$, decoupled).

We derive the wave equation for $\vec{E}$. The derivation of the wave equation for $\vec{B}$ is very similar, and we defer it to problem P1.7.1. We begin our derivation of the wave equation by taking the curl of (1.6.3), from which we obtain

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) + \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) = 0. \tag{1.7.1}$$

The equation can be simplified with the differential vector identity (see P0.3.10):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E}. \tag{1.7.2}$$

In addition, we can make a substitution for $\vec{\nabla}\times\vec{B}$ from (1.6.4). Together, these substitutions give

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} + \frac{\partial}{\partial t}\left(\varepsilon_o\mu_o\frac{\partial\vec{E}}{\partial t} + \mu_o\vec{J}_{\text{free}} + \mu_o\frac{\partial\vec{P}}{\partial t}\right) = 0. \tag{1.7.3}$$

Employing (1.6.1) on the first term and after rearranging, we get

$$\nabla^2\vec{E} - \mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2} = \mu_o\frac{\partial\vec{J}_{\text{free}}}{\partial t} + \mu_o\frac{\partial^2\vec{P}}{\partial t^2} - \frac{1}{\varepsilon_o}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{P}\right). \tag{1.7.4}$$

The left-hand side of (1.7.4) is the familiar wave equation. However, the right-hand side contains many complicated source terms, which arise when various currents and polarizations are present. Nevertheless, in vacuum all of the terms on the right-hand side are zero, in which case the equation reduces to

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{1.7.5}$$

For most problems in optics, a some of the terms on the right-hand side of (1.7.4) are zero. However, typically at least one of the terms must be retained, depending on the nature of the medium.

The first term on the right-hand side of (1.7.4) arises due to electric currents, important for determining the reflection of light from a metallic surface or the propagation of light within a plasma. The second term on the right-hand side of (1.7.4) arises from dipole oscillations, which behave similar to currents. In a non-conducting optical material such as glass, the free current is zero, but $\partial^2 \vec{P} / \partial t^2$ is not zero, as the medium polarization responds to the light field. This polarization current determines the refractive index of the material (discussed in chapter 2). The final term on the right-hand side of (1.7.4) is important in non-isotropic media such as a crystal. In this case, the polarization $\vec{P}$ responds to the electric field along a direction not necessarily parallel to $\vec{E}$, due to the influence of the crystal lattice. In the case of non-isotropic media (addressed in chapter 5), we will find it more convenient to work directly with Maxwell's equations rather than with the wave equation (1.7.4).

After solving (1.7.4) for $\vec{E}$, one may obtain $\vec{B}$ through an application of (1.6.3), one of Maxwell's equations. Even though the magnetic field $\vec{B}$ satisfies a wave equation similar to (1.7.5), decoupled from $\vec{E}$, the two wave equations must not be considered to be independent. The fields for $\vec{E}$ and $\vec{B}$ are still required to be consistent with each other through Maxwell's equations.

Exercises

P1.7.1 Derive the wave equation for the magnetic field $\vec{B}$ in vacuum (i.e. $\vec{J}_{\text{free}} = 0$ and $\vec{P} = 0$).

P1.7.2 Show that the electric field defined in P1.6.2 satisfies the wave equation in vacuum (1.7.5) if a special relation between k and ω holds

P1.7.3 Show that the magnetic field in P1.6.2 is consistent with the wave equation derived in P1.7.1.

1.8 Wave Equation in Isotropic Media

If a material is isotropic, then the medium polarization and the electric field can be connected as follows:

$$\vec{P} = \varepsilon_o \chi \vec{E}. \tag{1.8.1}$$

This is called a constitutive relation, and χ is called the linear susceptibility. Again, this relation is not valid in anisotroptic crystals where the lattice can skew the electronic response such that the polarization is not necessarily parallel to the electric field that induces it. In addition, (1.8.1) assumes that the polarization is proportional to the electric field, and the susceptibility χ is treated as a constant in time. Strictly speaking, this approximation of a time-independent χ is valid only if the electric field is a monochromatic wave (sinusoidal with a single frequency). In chapter 7, we examine how to decompose arbitrary waveforms into individual frequency components, each frequency component experiencing its respective value of χ for a given material.

In an isotropic medium, we also have $\vec{\nabla} \cdot \vec{P} = 0$ (together with $\vec{\nabla} \cdot \vec{E} = 0$), seen upon substitution of (1.8.1) into (1.6.1), since in general $\chi + 1 \neq 0$. If we also take the medium to be non-conducting (i.e. $\vec{J}_{\text{free}} = 0$), the wave equation (1.7.4) reduces to

$$\nabla^2 \vec{E} - \varepsilon_o \mu_o (1+\chi) \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \qquad \text{(single frequency)} \tag{1.8.2}$$

While this equation is strictly true only for single-frequency waves, this restriction can be slightly relaxed to the extent that χ is approximately constant for all frequencies involved (which together form the temporal structure of a pulse). Analysis (see P1.8.4) shows that the waves propagate with speed

$$v = \frac{1}{\sqrt{\varepsilon_o \mu_o (1+\chi)}}. \qquad \text{(phase velocity, single frequency)} \tag{1.8.3}$$

In vacuum where $\chi = 0$, speed of the wave is

$$c \equiv 1 \big/ \sqrt{\varepsilon_o \mu_o} = 2.9979 \times 10^8 \text{ m/s}, \qquad \text{(vacuum)} \tag{1.8.4}$$

independent of frequency. In a vacuum, any waveform satisfies the wave equation as long as it caries the dependence $\vec{E}(\hat{u} \cdot \vec{r} - ct)$, where $\hat{u}$ is a unit vector specifying the direction of propagation. By checking this solution in (1.7.5), one effectively verifies that the speed of propagation is c (see P1.8.4).

The optical index of refraction for a material is defined to be the speed of light in vacuum divided by the speed of the wave in the material:

$$n \equiv \frac{c}{v} = \sqrt{1+\chi}. \tag{1.8.5}$$

Again, it should be stressed that n depends on frequency through χ (as will be studied in chapter 2). The wave equation in a medium (1.8.2) should only be used for single-frequency sinusoidal waves. More complicated waveforms inside a material must first be decomposed into their various frequency components and (1.8.2) applied to each component individually.

Exercises

L1.8.1 Measure the speed of light using a rotating mirror (see Fig. 1.5). Provide an estimate of the experimental uncertainty in your answer (not the percentage error from the known value).

P1.8.2 Ole Roemer (1644-1710) made the first successful measurement of the speed of light by observing the orbital period of Io, a moon of Jupiter (period 42.5 hours). What is the maximum amount of time that the orbit "gets ahead" (or "gets behind") during a half Earth year? Take the Earth's orbital radius to be 1.5×10^{11} m, and take the position of Jupiter to be roughly fixed during an Earth half-year.

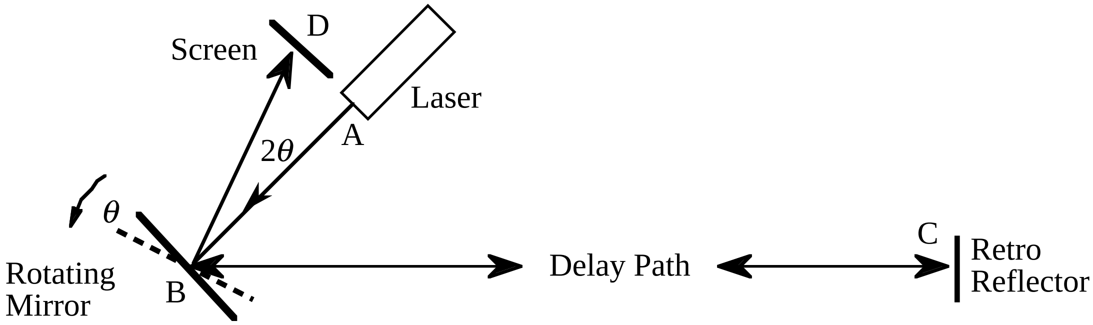

Fig. 1.5 Setup for measuring the speed of light. Laser light from A reflects from a rotating mirror at B towards C. The light returns to B, where the mirror has rotated, sending the light to point D. Additional optics compensate for beam divergence in the delay path.

P1.8.3 In an isotropic medium, the constitutive relation can be written as $\vec{P} = \varepsilon_o \chi(E)\vec{E}$, where $\chi(E) = \chi_1 + \chi_2 E + \chi_3 E^2 \ldots$. This is an important expansion used in the field of nonlinear optics, where, if the light field becomes very large, the polarization responds nonlinearly to the electric field, leading to phenomena such as harmonic generation. At low intensity, only the first term in the expansion is important.

Starting with Maxwell's equations, derive the wave equation for nonlinear optics in an isotropic medium:

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o (1+\chi_1)\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_o \varepsilon_o \frac{\partial^2\left(\chi_2 E + \chi_3 E^2 \ldots\right)\vec{E}}{\partial t^2} + \mu_o \frac{\partial \vec{J}}{\partial t}.$$

We have retained the possibility of current here since, for example, in a gas some of the molecules might ionize in the presence of a strong field, giving rise to a current.

P1.8.4

(a) Show that $\vec{E}(\vec{r},t) = \vec{E}_o \cos\left(\omega\sqrt{\varepsilon_o \mu_o (1+\chi)}\,\hat{u}\cdot\vec{r} - \omega t + \phi\right)$ is a solution to (1.8.2) where $\hat{u}$ is an arbitrary unit vector.

(b) Show that each wave front is a plane by setting the cosine argument to an arbitrary constant. In what direction do the wave fronts move? NOTE: A wave front can be any part of the wave.

(c) Determine the speed $v = \Delta r/\Delta t$ of a wave front, again by setting the cosine argument to a constant. Verify (1.8.5) in this case.

(d) Determine the wavelength λ in terms of n and c. HINT: Find the distance between identical wave fronts by changing the cosine argument by 2π at a given instant in time.

(e) If $\omega\sqrt{\varepsilon_o \mu_o (1+\chi)}\,\hat{u}$ is written as $\vec{k}$, find k in terms of c, n, and ω. Also find k in terms of the wavelength λ.

P1.8.5 Show that $\vec{E}_o$ and $\vec{k}$ in the wave $\vec{E}(\vec{r},t) = \vec{E}_o \cos\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)$ must be perpendicular to each other in an isotropic medium. HINT: Use (1.6.1) and (1.8.1).