Chapter 4

Polarization

4.1 Introduction – Linear, Circular, and Elliptical Polarization

In this chapter, we continue our study of the usual plane-wave solution to Maxwell's equations written as $\vec{E}(\vec{r},t)=\vec{E}_o \exp\left[i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right]$ (see (2.2.7)), where the wave vector $\vec{k}$ specifies the direction of propagation. We neglect absorption so that the refractive index is real (i.e. $k=n\omega/c=2\pi n/\lambda_{vac}$; see (2.3.20)-(2.3.23)).

In an isotropic medium, $\vec{k}$ and $\vec{E}_o$ are perpendicular. Hence, if the direction of $\vec{k}$ is specified, there remain only two dimensions wherein the vector $\vec{E}_o$ is confined. To make our analysis easier, let us orient our coordinate system with the z-axis in the direction of $\vec{k}$. The wave vector then reduces to $\vec{k}=k\hat{z}$, and our plane wave becomes

$$\vec{E}(z,t)=\vec{E}_o e^{i(kz-\omega t)}. \tag{4.1.1}$$

The amplitude of the electric field can be written as

$$\vec{E}_o=E_{ox}\hat{x}+E_{oy}\hat{y}. \tag{4.1.2}$$

The relationship between E_{ox} and E_{oy} describes the *polarization* of the light. If, for example, the y-component of the field E_{oy} is zero, the plane wave is said to be *linearly polarized* along the x-dimension. Linearly polarized light can have any orientation in the x-y plane, and it occurs whenever E_{ox} and E_{oy} have the same complex phase (plus or minus an integer times π). We often take the x-dimension to be horizontal and the y-dimension to be vertical.

Only the real part of (4.1.1) is physically relevant. The complex amplitudes of the vector components of $\vec{E}_o$ keep track of the phase of the oscillating field. In general the complex phases of E_{ox} and E_{oy} can be different. To the extent that this is the case, the wave in one of the dimensions lags or leads the wave in the other dimension.

As an example, suppose $E_{oy}=iE_{ox}$, where E_{ox} is real. The y-component of the field is then out of phase with the x-component by the factor $i=e^{i\pi/2}$. Taking the real part of the field (4.1.1) we get

$$\begin{aligned}\vec{E}(z,t)&=\mathrm{Re}\left[E_{ox}e^{i(kz-\omega t)}\right]\hat{x}+\mathrm{Re}\left[e^{i\pi/2}E_{ox}e^{i(kz-\omega t)}\right]\hat{y}\\&=E_{ox}\cos(kz-\omega t)\hat{x}+E_{ox}\cos(kz-\omega t+\pi/2)\hat{y}\\&=E_{ox}\left[\cos(kz-\omega t)\hat{x}-\sin(kz-\omega t)\hat{y}\right].\ \text{(left circular)}\end{aligned} \tag{4.1.3}$$

In this example, the field in the y-dimension lags the field in the x-dimension by a quarter cycle. That is, the behavior seen in the x-dimension happens in the y-dimension a quarter cycle later. The field never goes to zero simultaneously in both dimensions. In fact, in this example the strength of the electric field is constant, and it rotates in a circular pattern in the x-y dimensions. For this reason, this type of field is called *circularly polarized*.

If we view the field throughout space at a frozen instant in time, the electric field vector spirals as we move along the z-dimension. With time frozen, if the sense of the spiral matches that of a common wood screw oriented along the z-axis, then the polarization is called *right handed*. (It makes no difference whether the screw is flipped end for end.) If instead the field spirals in the opposite sense, then the polarization is called *left handed*. The field in (4.1.3) is an example of left-handed circularly polarized light.

An equivalent way to view the handedness convention is to imagine the light impinging on a screen as a function of time. The field of a right-handed circularly polarized wave rotates counter clockwise at the screen, when looking in the $\vec{k}$ direction (towards the front side of the screen). The field rotates clockwise for a left-handed circularly polarized wave.

In the next section, we develop a convenient way for keeping track of polarization in terms of a two-dimensional vector, called a Jones vector. In section 4.3, we introduce polarizing filters and demonstrate how their effect on a light field can be represented as a 2×2 matrix operating on the polarization vector. We continue our analysis in subsequent sections and show how to deal with polarizers oriented at arbitrary angles with respect to the coordinate system. The analysis applies also to wave plates, devices that retard one field component with respect to the other. A wave plate is used to create, for example, circularly polarized light starting with linearly polarized light.

Beginning in section 4.6, we investigate how the reflection and transmission at a material interface influences field polarization. The Fresnel coefficients studied in the previous chapter can be conveniently incorporated into the 2×2 matrix formulation for handling polarization. As we saw, the amount of light reflected from a surface depends on the type of polarization, s or p. In addition, upon reflection, s-polarized light can acquire a phase lag or phase advance relative to p-polarized light. This is especially true at metal surfaces, which have complex indices of refraction (i.e. highly absorptive).

As we shall see, linear polarized light can become circularly or, in general, *elliptically* polarized after reflection from a metal surface if the incident light has both s and p-polarized components. Every good experimentalist working with light needs to know this. For reflections involving materials with real indices such as glass, the situation is less complicated and linearly polarized light remains linear. However, even if the index is real, there can be interesting phase shifts (different for s and p components) for total internal reflection, examined in section 4.8. We also examine the occurrence of *evanescent* waves, which appear on the transmitted side of the boundary where total internal reflection takes place. In section 4.9 we briefly discuss *ellipsometry*, which is the science of characterizing optical properties of materials by observing the polarization of light reflected from surfaces.

4.2 Jones Vectors for Representing Polarization

In 1941, R. Clark Jones introduced a two-dimensional matrix algebra that is useful for keeping track of light polarization and the effects of optical elements that influence polarization. The algebra deals with light having a definite polarization, such as plane waves. It does not apply to un-polarized or partially polarized light (e.g. sunlight). For partially polarized light, a four-dimensional algebra known as Stokes calculus is used, not discussed here.

In preparation for writing Jones vectors to characterize the state of polarization, we combine (4.1.1) and (4.1.2):

$$\vec{E}(z,t) = \left(E_{ox}\hat{x} + E_{oy}\hat{y}\right)e^{i(kz-\omega t)}. \tag{4.2.1}$$

For convenience we factor out the effective strength of the electric field — call it E_{eff}. This field in general is neither E_{ox} nor E_{oy}, but it is the strength of an effective linearly polarized field that would give the same intensity that (4.2.1) would yield. We can use E_{eff} to write the intensity (2.6.7) as

$$I = \langle S \rangle_t = \frac{1}{2} n c \varepsilon_o \vec{E}_o \cdot \vec{E}_o^* = \frac{1}{2} n c \varepsilon_o \left|E_{\text{eff}}\right|^2. \tag{4.2.2}$$

This expression essentially defines what we mean by E_{eff}. Let us adopt the convention of giving E_{eff} the same phase as the x-component of the field, so we have $E_{ox} = \left|E_{ox}\right| e^{i\delta_x}$ and $E_{eff} = \left|E_{eff}\right| e^{i\delta_x}$. Therefore, if we factor out E_{eff} from (4.2.1), the residual coefficient of $\hat{x}$ is real and non-negative.

When we factor out E_{eff} from (4.2.1), the equation can be written as

$$\vec{E}(z,t) = E_{\text{eff}}\left(A\hat{x} + Be^{i\delta}\hat{y}\right)e^{i(kz-\omega t)}, \text{ where} \tag{4.2.3}$$

$$E_{\text{eff}} = \sqrt{\left|E_{ox}\right|^2 + \left|E_{oy}\right|^2}\, e^{i\delta_x}, \tag{4.2.4}$$

$$A \equiv \frac{\left|E_{ox}\right|}{\sqrt{\left|E_{ox}\right|^2 + \left|E_{oy}\right|^2}}, \tag{4.2.5}$$

$$B \equiv \frac{\left|E_{oy}\right|}{\sqrt{\left|E_{ox}\right|^2 + \left|E_{oy}\right|^2}}, \tag{4.2.6}$$

$$\delta \equiv \delta_y - \delta_x, \tag{4.2.7}$$

$$\delta_y \equiv \tan^{-1}\frac{\operatorname{Im}\left(E_{oy}\right)}{\operatorname{Re}\left(E_{oy}\right)}, \text{ (add } \pi \text{ if } \operatorname{Re}\left(E_{oy}\right) < 0\text{)} \tag{4.2.8}$$

$$\delta_x \equiv \tan^{-1}\frac{\operatorname{Im}\left(E_{ox}\right)}{\operatorname{Re}\left(E_{ox}\right)}. \text{ (add } \pi \text{ if } \operatorname{Re}\left(E_{ox}\right) < 0\text{)} \tag{4.2.9}$$

If the x-component of the field happens to be zero, then we let $E_{\text{eff}} = E_{oy}$, $B = 1$, and $\delta = 0$. Although the above expressions may seem a bit messy, please notice that we have created a kind of unit vector inside the parentheses of (4.2.3). Explicitly, we have $\left(A\hat{x} + Be^{i\delta}\hat{y}\right)\cdot\left(A\hat{x} + Be^{i\delta}\hat{y}\right)^* = 1$, where the asterisk represents the complex conjugate. (If (4.2.8) and (4.2.9) seem mysterious, please review section 0.2 on how to transform a complex number $a+ib$ into the form $\rho e^{i\theta}$.) Please notice that A and B are real non-negative dimensionless numbers.

We are now ready to introduce the Jones vector, which amounts to nothing more than the vector $A\hat{x}+Be^{i\delta}\hat{y}$. This vector contains the essential information regarding the field polarization; the overall field strength E_{eff} is set aside in the discussion of polarization. Also note that the phase of E_{eff} represents an overall phase shift that one can trivially adjust by moving the light source (a laser, say) forward or backward by a fraction of a wavelength. This is often unimportant.

When writing the Jones vector, instead of using the standard $\hat{x}$ and $\hat{y}$ vector notation, we organize its components in a column vector for later use in matrix algebra. The general expression for the Jones vector is

$$\begin{pmatrix} A \\ Be^{i\delta} \end{pmatrix}. \tag{4.2.10}$$

This vector can describe the polarization state of any plane wave field. The following is a list of Jones vectors representing various polarization states:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ (linearly polarized along x-dimension)} \tag{4.2.11}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ (linearly polarized along y-dimension)} \tag{4.2.12}$$

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \text{ (linearly polarized at an angle } \alpha \text{ from the x-axis)} \tag{4.2.13}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \text{ (right circularly polarized)} \tag{4.2.14}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}. \text{ (left circularly polarized)} \tag{4.2.15}$$

The last Jones vector in the list corresponds to the example given in (4.1.3). These vectors are all special cases of the general Jones vector (4.2.10).

In general, (4.2.10) represents a polarization state in between linear and circular. This ‘in-between’ state is known as *elliptically polarized* light. As the wave travels, it undergoes a spiral motion while tracing out an ellipse. One of the axes (major or minor) of the ellipse occurs at the angle (see P4.4.3)

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2AB\cos\delta}{A^2-B^2}\right] \tag{4.2.16}$$

with respect to the x-axis. This angle sometimes corresponds to the minor axis and sometimes to the major axis of the ellipse, depending on the exact values of A, B, and δ. Whichever is the case, the other axis of the ellipse (major or minor) then occurs at $\alpha \pm \pi/2$.

We can deduce whether (4.2.16) corresponds to the major axis or to the minor axis of the ellipse. The strength of the electric field when it spirals through the direction specified by α is found to be (see P4.4.3)

$$E_\alpha = |E_{\text{eff}}|\sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha + AB\cos\delta\sin 2\alpha}\,,\ (E_{\max} \text{ or } E_{\min}) \quad (4.2.17)$$

and the strength of the electric field when it spirals through the orthogonal direction ($\alpha \pm \pi/2$) is

$$E_{\alpha\pm\pi/2} = |E_{\text{eff}}|\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha - AB\cos\delta\sin 2\alpha}\,.\ (E_{\max} \text{ or } E_{\min}) \quad (4.2.18)$$

We could predict in advance which of (4.2.17) and (4.2.18) corresponds to the major axis and which corresponds to the minor axis. However, the rule for making such a prediction is as complicated as simply evaluating (4.2.17) and (4.2.18) and then comparing the results to determine which is greater. Based on the comparison, we make the assignment according to

$$E_{\max} \geq E_{\min}. \quad (4.2.19)$$

Elliptically polarized light is often characterized by the ratio of the minor axis to the major axis. This ratio is called the *ellipticity,* which is a dimensionless number:

$$e \equiv \frac{E_{\min}}{E_{\max}}. \quad (4.2.20)$$

The ellipticity e ranges between zero (corresponding to linearly polarized light) and one (corresponding to circularly polarized light). Finally, the *helicity* or handedness of elliptically polarized light is as follows (see P4.2.2):

$$0 < \delta < \pi\,,\ \text{(left-handed helicity)} \quad (4.2.21)$$

$$\pi < \delta < 2\pi\,.\ \text{(right-handed helicity)} \quad (4.2.22)$$

Exercises

P4.2.1 Show that $\left(A\hat{x} + Be^{i\delta}\hat{y}\right)\cdot\left(A\hat{x} + Be^{i\delta}\hat{y}\right)^* = 1$, as defined in connection with (4.2.3).

P4.2.2 Prove that if $0 < \delta < \pi$, the helicity is left-handed, and if $\pi < \delta < 2\pi$ the helicity is right-handed. HINT: Write the relevant real field associated with (4.2.3) $\vec{E}(z,t) = |E_{\text{eff}}|\left[\hat{x}A\cos(kz - \omega t + \phi) + \hat{y}B\cos(kz - \omega t + \phi + \delta)\right]$, where ϕ is the phase of E_{eff}. Freeze time at, say, $t = \phi/\omega$. Determine the field at $z = 0$ and at $z = \lambda/4$ (a quarter cycle), say. If $\vec{E}(0,t)\times\vec{E}(\lambda/4,t)$ points in the direction of $\vec{k}$, then the helicity matches that of a wood screw.

P4.2.3 For the following cases, what is the orientation of the major axis, and what is the ellipticity of the light? Case I: $A = B = 1/\sqrt{2}$, $\delta = 0$; Case II: $A = B = 1/\sqrt{2}$, $\delta = \pi/2$; Case III: $A = 1/\sqrt{2}$, $B = 1/\sqrt{2}$, $\delta = \pi/4$.

4.3 Jones Matrices

In 1928, Edwin Land at the age of nineteen invented Polaroid. He did it by stretching a polymer sheet and infusing it with iodine. The stretching causes the polymer chains to align along a common direction, whereupon the sheet is cemented to a substrate. The infusion of iodine causes the individual chains to become conductive. When light impinges upon the Polaroid sheet, the component of electric field that is parallel to the polymer chains causes a current $\vec{J}_{free}$ to oscillate in that dimension. The resistance to the current quickly dissipates the energy (i.e. the refractive index is complex) and the light is absorbed. The thickness of the Polaroid sheet is chosen sufficiently large to ensure that virtually none of the electric field component oscillating along the chains makes it through the device.

The component of electric field that is orthogonal to the polymer chains encounters electrons that are essentially bound, unable to leave their polymer chains. For this polarization component, the wave passes through the material like it does through typical dielectrics such as glass (i.e. the refractive index is real). Today, there are a wide variety of technologies for making polarizers, many very different from Polaroid.

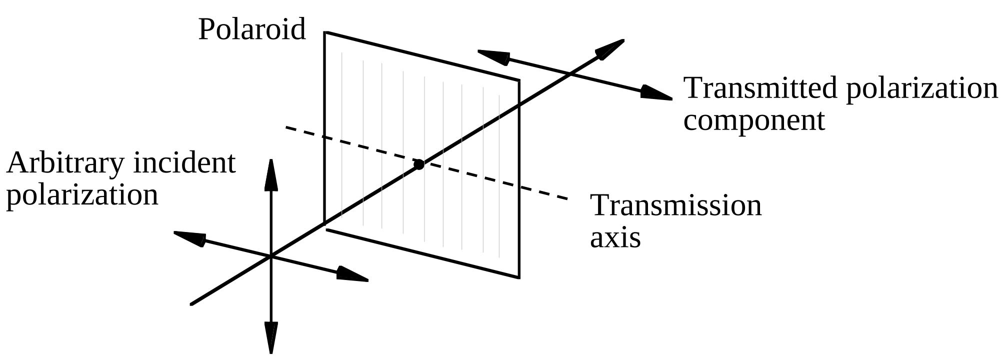


Fig. 4.1 Light transmitting through a Polaroid sheet.

A polarizer can be represented as a 2×2 matrix that operates on Jones vectors. The function of a polarizer is to pass only the component of electric field that is oriented along the polarizer transmission axis (perpendicular to the polymer chains). Thus, if a polarizer is oriented with its transmission axis along the x-dimension, then only the x-component of polarization transmits; the y-component is killed. If the polarizer is oriented with its transmission axis along the y-dimension, then only the y-component of the field transmits, and the x-component is killed. These two scenarios can be represented with the following Jones matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{(polarizer with transmission along } x\text{-axis)} \tag{4.3.1}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{(polarizer with transmission along } y\text{-axis)} \tag{4.3.2}$$

These matrices operate on any Jones vector representing the polarization of incident light. The result gives the Jones vector for the light exiting the polarizer. As an example, consider a horizontally polarized plane wave traversing a polarizer with its transmission axis oriented also horizontally (x-dimension). Then we have (4.3.1) operating on (4.2.11):

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \text{ (horizontal polarizer on horizontally polarized field)} \qquad (4.3.3)$$

As expected, the light is unaffected by the polarizer.

Now consider vertically polarized light traversing the same horizontal polarizer. In this case, we have (4.3.1) operating on (4.2.12):

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \text{ (horizontal polarizer on vertical linear polarization)} \qquad (4.3.4)$$

As expected, the polarizer extinguishes the light. When a horizontally oriented polarizer operates on light with an arbitrary Jones vector (4.2.13), we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ Be^{i\delta} \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix}. \text{ (horizontal polarizer on arbitrary polarization)} \qquad (4.3.5)$$

Only the horizontal component of polarization is transmitted through the polarizer.

Exercises

P4.3.1 (a) Suppose that linearly polarized light is oriented at an angle α with respect to the horizontal axis (x-axis) (see (4.2.13)). What fraction of the original <u>intensity</u> emerges from a vertically oriented polarizer?

(b) If the original light is right-circularly polarized, what fraction of the original <u>intensity</u> emerges from the same polarizer?

4.4 Jones Matrix for Polarizers at Arbitrary Angles

While students will readily agree that the matrices given in (4.3.1) and (4.3.2) can be used to get the right result for light traversing a horizontal or a vertical polarizer, the real advantage of the matrix formulation as well as the justification for its use has yet to be demonstrated. The usefulness of the formalism becomes clear as we consider the problem of passing a plane wave with arbitrary polarization through a polarizer with its transmission axis aligned at angle θ with the x-axis.

We will analyze this problem in a general context so that we can take advantage of our work when we discuss wave plates in the next section. To help keep things on a more conceptual level, let us revert back to (4.2.1). We will make the connection with Jones calculus at a later point. The electric field of our plane wave is

$$\vec{E}(z,t) = E_x\hat{x} + E_y\hat{y}\text{, where} \qquad (4.4.1)$$

$$E_x \equiv E_{ox}e^{i(kz-\omega t)}\text{, and} \qquad (4.4.2)$$

$$E_y \equiv E_{oy}e^{i(kz-\omega t)}. \qquad (4.4.3)$$

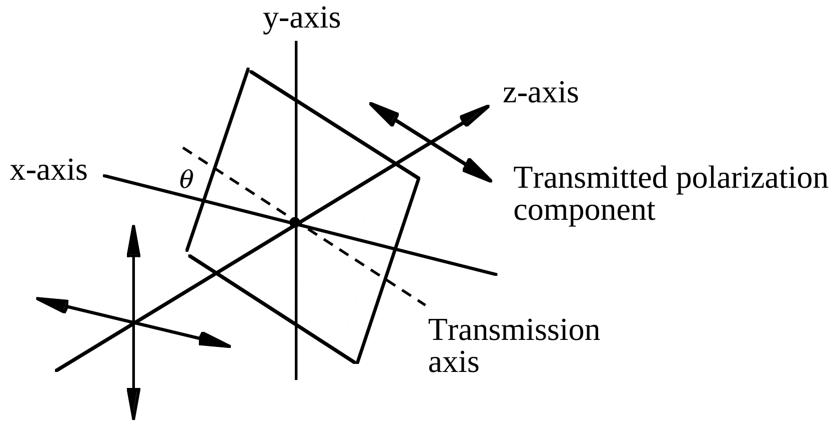


Fig. 4.2 Polarizer oriented with transmission axis at angle θ from x-axis.

In the upcoming discussion, let the transmission axis of the polarizer be called axis 1 and the absorption axis of the polarizer be called axis 2 (orthogonal to axis 1) as depicted in Fig. 4.2. Axis 1 is oriented at an angle θ from the x-axis. We need to write the electric field components in terms of the new basis specified by the unit vectors $\hat{e}_1$ and $\hat{e}_2$ as shown in Fig. 4.3. These new unit vectors are connected to the original ones via

$$\hat{x} = \cos\theta\hat{e}_1 - \sin\theta\hat{e}_2, \text{ and} \tag{4.4.4}$$

$$\hat{y} = \sin\theta\hat{e}_1 + \cos\theta\hat{e}_2. \tag{4.4.5}$$

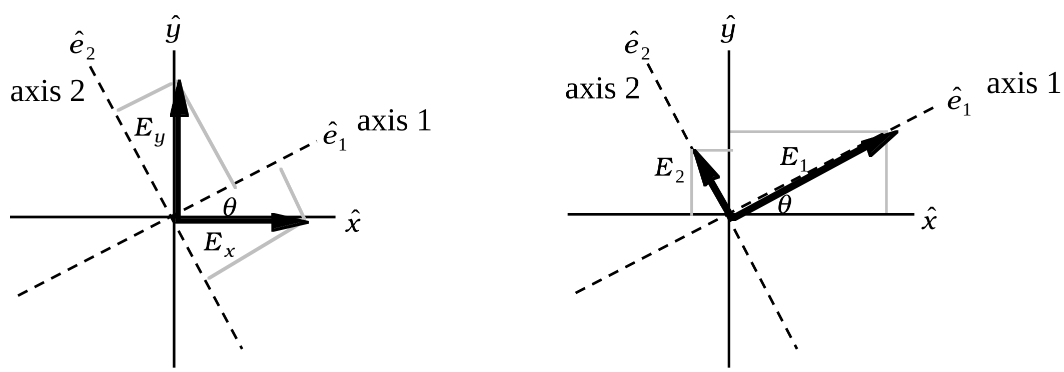


Fig. 4.3 Electric field components written in either the $\hat{x}$-$\hat{y}$ basis or the $\hat{e}_1$-$\hat{e}_2$ basis.

By direct substitution of (4.4.4) and (4.4.5) into (4.4.1), the electric field can be written as

$$\vec{E}(z,t) = E_1\hat{e}_1 + E_2\hat{e}_2, \text{ where} \tag{4.4.6}$$

$$E_1 \equiv E_x\cos\theta + E_y\sin\theta, \text{ and} \tag{4.4.7}$$

$$E_2 \equiv -E_x\sin\theta + E_y\cos\theta. \tag{4.4.8}$$

At this point, we can easily introduce the effect of the polarizer on the field: E_1 is transmitted unaffected, and E_2 is killed. Let us multiply E_2 by a parameter ξ to signify the effect of the device. In the case of the polarizer, ξ is zero, but in the next section we will consider other values for ξ. After traversing the polarizer, the field becomes

$$\vec{E}_{\substack{\text{after}\\\text{device}}}(z,t) = E_1\hat{e}_1 + \xi E_2\hat{e}_2. \tag{4.4.9}$$

This completes the job since we now have the field after the polarizer. However, it would be nice to rewrite it in terms of the original x-y basis. By inverting (4.4.4) and (4.4.5), or by inspection of Fig. 4.3, if preferred, we see that

$$\hat{e}_1 = \cos\theta\hat{x} + \sin\theta\hat{y}\text{, and} \tag{4.4.10}$$

$$\hat{e}_2 = -\sin\theta\hat{x} + \cos\theta\hat{y}. \tag{4.4.11}$$

Substitution of these relationships into (4.4.9) together with the definitions for E_1 (4.4.7) and E_2 (4.4.8) yields

$$\begin{aligned}\vec{E}_{\substack{\text{after}\\\text{device}}}(z,t) &= \left(E_x\cos\theta + E_y\sin\theta\right)\left(\cos\theta\hat{x} + \sin\theta\hat{y}\right) + \xi\left(-E_x\sin\theta + E_y\cos\theta\right)\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right)\\ &= \left[E_x\left(\cos^2\theta + \xi\sin^2\theta\right) + E_y\left(\sin\theta\cos\theta - \xi\sin\theta\cos\theta\right)\right]\hat{x}\\ &\quad + \left[E_x\left(\sin\theta\cos\theta - \xi\sin\theta\cos\theta\right) + E_y\left(\sin^2\theta + \xi\cos^2\theta\right)\right]\hat{y}.\end{aligned} \tag{4.4.12}$$

Notice that if $\xi = 1$ (i.e. no polarizer), then we get back exactly what we started with (i.e. (4.4.12) reduces to (4.4.1)). There remains only to recognize that (4.4.12) is a linear mixture of E_x and E_y, used to express $\vec{E}^{\text{after}}_{\text{device}}(z,t)$. This type of linear mixture can be represented with matrix algebra. If we represent $\vec{E}^{\text{after}}_{\text{device}}(z,t)$ as a two dimensional column vector with its x-component in the top and its y-component in the bottom (like a Jones vector), then we can rewrite (4.4.12) as

$$\vec{E}_{\substack{after\\device}}(z,t) = \begin{bmatrix}\cos^2\theta + \xi\sin^2\theta & \sin\theta\cos\theta - \xi\sin\theta\cos\theta\\ \sin\theta\cos\theta - \xi\sin\theta\cos\theta & \sin^2\theta + \xi\cos^2\theta\end{bmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}. \tag{4.4.13}$$

The matrix here is a Jones matrix, appropriate for operating on Jones vectors (although the vector here is not a properly normalized Jones vector). We used the full representation of the electric field to make things easier to visualize, but we could have done the derivation using matrix and vector notation. We are now ready to write down the Jones matrix for a polarizer (with $\xi = 0$):

$$\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}. \text{ (polarizer with transmission axis at angle } \theta\text{)} \tag{4.4.14}$$

Notice that when $\theta = 0$ this matrix reduces to that of a horizontal polarizer (4.3.1), and when $\theta = \pi/2$, it reduces to that of a vertical polarizer (4.3.2).

To the extent that part of the light is absorbed by the polarizer, the Jones vector of the exiting wave is no longer normalized to magnitude one. The Jones vector dotted with its complex conjugate gives the factor by which the <u>intensity</u> of the light decreases. In accordance with (4.2.2), the intensity of the exiting light is

$$I = \frac{1}{2} nc\varepsilon_o \left|E_{eff}\right|^2 \begin{pmatrix} A' & B'e^{i\delta'} \end{pmatrix}^* \begin{pmatrix} A' \\ B'e^{i\delta'} \end{pmatrix} = \frac{1}{2} nc\varepsilon_o \left|E_{eff}\right|^2 \left[\left|A'\right|^2 + \left|B'\right|^2\right], \qquad (4.4.15)$$

where $\begin{pmatrix} A' \\ B'e^{i\delta'} \end{pmatrix}$ represent the Jones vector that emerges from the polarizer (or some other devices), and $\begin{pmatrix} A' & B'e^{i\delta'} \end{pmatrix}^*$ is the complex conjugate, or rather the Hermitian conjugate, written in a format conducive to vector multiplication resulting in a scalar.

The intensity is attenuated by the factor $|A'|^2 + |B'|^2$. Recall that E_{eff} represents the effective strength of the field before it enters the polarizer (or other device), so that the initial Jones vector is normalized to one (see (4.2.2)). By convention we normally remove an overall phase factor from the Jones vector so that A' is real and non-negative, and we choose δ' so that B' is real and non-negative. However, if we don't bother doing this, the absolute value signs on A' and B' in (4.4.15) ensure that we get the correct value for intensity.

A product of Jones matrices can represent a sequence of polarizers (with varying orientations). The matrices operate on the Jones vector in the order that the light encounters the devices. Therefore, the matrix for the first device is written on the *right*, and so on until the last device encountered, which is written on the *left*, farthest from the Jones vector.

Exercises

P4.4.1 Horizontally polarized light ($\alpha = 0$) is sent through two polarizers, the first oriented at $\theta_1 = 45^\circ$ and the second at $\theta_2 = 90^\circ$. What fraction of the original intensity emerges? What if the ordering of the polarizers is reversed?

P4.4.2 (a) Suppose that linearly polarized light is oriented at an angle α with respect to the horizontal or x-axis. What fraction of the original <u>intensity</u> emerges from a polarizer oriented with its transmission at angle θ from the x-axis? ANSWER: $\cos^2(\theta - \alpha)$; see P4.3.1.

(b) If the original light is right circularly polarized, what fraction of the original <u>intensity</u> emerges from the same polarizer?

P4.4.3 Derive (4.2.16), (4.2.17), and (4.2.18).

HINT: Analyze the Jones vector just as you would analyze light in the laboratory. Put a polarizer in the beam and observe the intensity of the light as a function of polarizer angle. Compute the intensity via (4.4.15). Then find the polarizer angle (call it α) that gives a maximum (or a minimum) of intensity. The angle then corresponds to an axis of the ellipse followed by the E-field as it spirals. When taking the arctangent, remember that it is defined only over a range of π. You can add π for another valid result (which corresponds to the second ellipse axis).

4.5 Jones Matrices for Wave Plates

The other device for influencing polarization that we will consider is called a wave plate (or retarder). A wave plate is made from a non-isotropic material such as a crystal. Such materials have different indices of refraction, depending on the orientation of the electric field polarization. A wave plate has the appearance of a thin window through which the light passes. However, it has a fast and a slow axis, which are 90° apart in the plane of the window. If the light is polarized along the fast axis, it experiences an index of refraction n_{fast}. This index is less than an index n_{slow} that light experiences when polarized along the orthogonal (slow) axis.

When a plane wave passes through a wave plate, the component of the electric field oriented along the fast axis travels faster than its orthogonal counterpart. The speed of the fast wave component is $v_{\text{fast}} = c/n_{\text{fast}}$ while the speed of the other component is $v_{\text{slow}} = c/n_{\text{slow}}$. Evidently, the fast component gets ahead, and this introduces a relative phase between the two polarization components.

By adjusting the thickness of the wave plate, we can introduce any desired phase difference between the two components. From (2.3.21) and (2.3.23), we have for the $\vec{k}$-vectors within the wave plate (associated with the two electric field components)

$$k_{\text{slow}} = \frac{2\pi n_{\text{slow}}}{\lambda_{\text{vac}}}, \text{ and} \tag{4.5.1}$$

$$k_{\text{fast}} = \frac{2\pi n_{\text{fast}}}{\lambda_{\text{vac}}} \tag{4.5.2}$$

As light passes through a wave plate of thickness d, the phase difference in (4.1.1) that accumulates between the fast and the slow polarization components is

$$k_{\text{slow}} d - k_{\text{fast}} d = \frac{2\pi d}{\lambda_{\text{vac}}}\left(n_{\text{slow}} - n_{\text{fast}}\right). \tag{4.5.3}$$

The most common types of wave plates are the quarter-wave plate and the half-wave plate. The quarter-wave plate introduces a phase difference between the two polarization components equal to

$$k_{\text{slow}} d - k_{\text{fast}} d = \pi/2 + 2\pi m, \text{ (quarter-wave plate)} \tag{4.5.4}$$

where m is an integer. This means that the polarization component along the slow axis is delayed spatially by one quarter of a wavelength (or five quarters, etc.). The half-wave plate introduces a phase delay between the two polarization components equal to

$$k_{\text{slow}} d - k_{\text{fast}} d = \pi + 2\pi m, \text{ (half-wave plate)} \tag{4.5.5}$$

where m is an integer. This means that the polarization component along the slow axis is delayed spatially by half of a wavelength (or three halves, etc.).

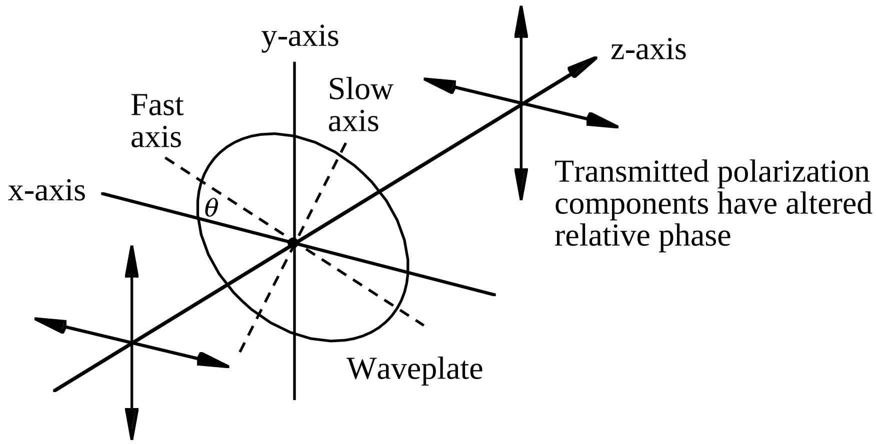


Fig. 4.4 Wave plate interacting with a plane wave.

The derivation of the Jones matrix for these two wave plates is essentially the same as the derivation for the polarizer in the previous section. Let axis 1 correspond to the fast axis, and let axis 2 correspond to the slow axis. We proceed as before. However, instead of setting ξ equal to zero in (4.4.13), we must choose values for ξ appropriate for each wave plate. Since nothing is absorbed, ξ should have a magnitude equal to one. The important feature is the phase of ξ. As seen in (4.5.3), the field component along the slow axis accumulates excess phase relative to the component along the fast axis, and we let ξ account for this. In the case of the quarter-wave plate, the appropriate factor from (4.5.4) is

$$\xi = e^{i\pi/2} = i. \text{ (quarter-wave plate)} \tag{4.5.6}$$

For the polarization component along the slow axis, the term $-i\omega t$ in (4.1.1) is able to counteract this added phase only at a later time t. Thus, there is a *relative delay* for the light emerging with polarization along the slow axis. We are not concerned with the overall delay of both polarization components relative to travel through vacuum. What concerns us is the difference between the two components. For the half-wave plate, the appropriate factor is

$$\xi = e^{i\pi} = -1. \text{ (half-wave plate)} \tag{4.5.7}$$

We can now write the Jones matrices (4.4.13) for the quarter-wave and half-wave plates:

$$\begin{bmatrix} \cos^2\theta + i\sin^2\theta & \sin\theta\cos\theta - i\sin\theta\cos\theta \\ \sin\theta\cos\theta - i\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix}, \text{ (quarter-wave plate)} \tag{4.5.8}$$

$$\begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \text{ (half-wave plate)} \tag{4.5.9}$$

Again, θ refers to the angle that the fast axis makes with respect to the x-axis.

These two matrices are especially interesting at $\theta = 45^\circ$, where the Jones matrix for the quarter-wave plate reduces to

$$\frac{e^{i\pi/4}}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}. \text{ (quarter-wave plate, fast axis at } \theta = 45^\circ\text{)} \tag{4.5.10}$$

The factor $e^{i\pi/4}$ in front is not important since it merely accompanies the overall phase of the beam, which can be adjusted arbitrarily by moving the light source forwards or backwards through a fraction of a wavelength. The Jones matrix for the half-wave plate reduces to

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \text{ (half-wave plate, fast axis at } \theta = 45^\circ\text{)} \tag{4.5.11}$$

As an example, consider the effect of the two wave plates (oriented at $\theta = 45^\circ$) operating on horizontally polarized light. For the quarter-wave plate, we get

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{4.5.12}$$

Notice that the quarter-wave plate (properly oriented) turns linearly polarized light into right-circularly polarized light (see (4.2.14)). The half-wave plate operating on horizontally polarized light gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.5.13}$$

The half-wave plate (when properly oriented) transforms horizontally polarized light into vertically polarized light (see (4.2.12)).

Exercises

L4.5.1 Create a source of unknown elliptical polarization by reflecting a linearly polarized HeNe laser beam from a metal mirror with a large incident angle (i.e. $\theta_i \geq 80^\circ$). Arrange to have roughly equal amounts of s and p-polarized light incident on the mirror. Use a quarter-wave plate and a polarizer to determine the Jones vector of the reflected beam. Give the ellipticity, the helicity (right or left handed), and the orientation of the major axis.
HINT: Use a quarter-wave plate to convert the unknown elliptically polarized light into linearly polarized light. When the wave plate is properly oriented, a subsequent polarizer will be able to extinguish the light. The final polarizer also allows you to read out the orientation of the linearly polarized light (Jones vector (4.2.13)) created by the wave plate. This equals the Jones matrix for the wave plate (4.5.8) operating on the unknown polarization Jones vector (4.2.10). This matrix equation is then solved to find the unknown Jones vector. You can check your answer using only the polarizer (with no wave plate) to determine the direction of the major axis and the ellipticity. This polarizer alone, however, does not reveal the helicity.

L4.5.2 Determine how much right-handed circularly polarized light ($\lambda_{\text{vac}} = 633\,\text{nm}$) is delayed (or advanced) with respect to left-handed circularly polarized light as it goes through approximately 3 cm of Karo syrup (the neck of the bottle). This phenomenon is called optical activity. All life on planet Earth seems to be associated with a definite handedness of complex molecules. The other handedness, while viable, is biologically incompatible. People from an identical planet with molecules of the opposite handedness would not be able to utilize food from Earth and visa versa. Because of the handedness of the molecules in the syrup, right- and left-handed polarized light experience slightly different refractive indices.

HINT: Linearly polarized light contains equal amounts of right and left circularly polarized light. Consider $\frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix}+\frac{e^{i\phi}}{2}\begin{pmatrix}1\\ -i\end{pmatrix}$, where ϕ is the phase delay of the right circular polarization. Show that this can be written as $e^{i\delta}\begin{pmatrix}\cos\phi/2\\ \sin\phi/2\end{pmatrix}$. Compare this with $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$, where α is the angle through which the polarization is rotated, beginning with horizontally polarized light. The overall phase δ is unimportant.

P4.5.3 What is the minimum thickness (called zero-order thickness) of a quartz plate made to operate as a quarter-wave plate for $\lambda_{\text{vac}} = 500\,\text{nm}$? The indices of refraction are $n_{\text{fast}} = 1.54424$ and $n_{\text{slow}} = 1.55335$.

4.6 Polarization Effects of Reflection and Transmission

When light encounters a material interface, the amount of reflected and transmitted light depends on the polarization. The Fresnel coefficients (3.3.1)-(3.3.4) dictate how much of each polarization is reflected and how much is transmitted. In addition, the Fresnel coefficients keep track of phases intrinsic in the reflection phenomenon. To the extent that the s and p components of the field behave differently, the overall polarization state is altered. For example, a linearly-polarized field upon reflection can become elliptically polarized (see L4.5.1). Even when a wave reflects at normal incidence so that the s and p components are indistinguishable, right-circularly polarized light becomes left circularly polarized. This is the same effect that causes a right-handed person to appear left-handed when viewed in a mirror.

We can use Jones calculus to keep track of how reflection and transmission influences polarization. However, before proceeding, we emphasize that in this context we do not strictly adhere to the coordinate system depicted in Fig. 3.1. (Please refer to Fig. 3.1 right now.) For purposes of examining polarization, we consider each of the three plane waves separately. We treat each plane wave as though traveling in its own z-direction, regardless of the incident angle in the figure. This loose manner of defining coordinate systems has a great advantage. The individual x and y dimensions for each of the three separate plane waves can all be aligned parallel to their respective s and p field component. Let us adopt the convention that p-polarized light in all cases is associate the x-dimension (horizontal). The s-polarized component then lies along the y-dimension (vertical). These are quite different from the absolute coordinate system defined in Fig. 3.1.

We are now in a position to see why there is a handedness inversion upon reflection from a mirror. Notice that for the incident light, the s-component of the field crossed (vector cross product) into the p-component yields that beam's z-direction. However, for the reflected light, the s-component crossed into the p-component yields that beam's negative z-direction.

Within our conveniently selected coordinates, the Jones matrix corresponding to reflection from a surface is simply

$$\begin{bmatrix} -r_p & 0 \\ 0 & r_s \end{bmatrix}. \quad \text{(Jones matrix for reflection)} \tag{4.6.1}$$

The minus sign on the coefficient r_p takes care of the handedness inversion. The Fresnel coefficients specify the ratios of the exiting fields to the incident ones. When (4.6.1) operates on an arbitrary Jones vector such as (4.2.10), $-r_p$ multiplies the horizontal component of the field, and r_s multiplies the vertical component of the field. These factors are just what is needed to determine the final relative proportions of the x and y field components (including relative phases).

Transmission through a material interface can also influence the polarization of the field. However, there is no handedness inversion, since the light continues on in a forward sense. Nevertheless, the relative amplitudes (and phases) of the field components are modified by the Fresnel transmission coefficients. The Jones matrix for this effect is

$$\begin{bmatrix} t_p & 0 \\ 0 & t_s \end{bmatrix}. \quad \text{(Jones matrix for transmission)} \tag{4.6.2}$$

If a beam of light encounters a series of mirrors, the final polarization is determined by multiplying the sequence of appropriate Jones matrices (4.6.1) onto the initial polarization. This procedure is straightforward if the normals to all of the mirrors lie in a single plane (parallel to the surface of an optical bench). However, if the beam path deviates from this plane (due to vertical tilt on the mirrors), then we necessarily must reorient our coordinate system before each mirror to have a new 'horizontal' (p-polarized dimension) and the new 'vertical' (s-polarized dimension). We have already examined the rotation of a coordinate system through an angle θ in (4.4.7) and (4.4.8). This rotation can be accomplished by multiplying the following matrix to the incident Jones vector:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad \text{(rotation of coordinates through an angle } \theta\text{)} \tag{4.6.3}$$

When such a reorientation of coordinates is necessary, evidently the two orthogonal field components in the initial coordinate system are stirred together to form the field components in the new system. This does not change the elliptical characteristics of the polarization, but it does change the designations for the orientations of the minor and major axes.

4.7 Reflection from Metallic or other Absorptive Surfaces

Our primary motivation for studying complex indices of refraction $\mathcal{N} \equiv n + i\kappa$ back in chapter 2 was to prepare for the discussion in this section. As a reminder, the imaginary part of the index controls absorption of a wave as it propagates within a material. The real part of the index governs the oscillatory nature of the wave. It turns out that both the imaginary and real parts of the index strongly influence the reflection of light from a surface. The reader may be grateful that there is no need to re-derive the Fresnel coefficients (3.3.1)-(3.3.4) for the case of complex indices. The coefficients remain perfectly valid whether the index is real or complex. We just need to be a bit careful when applying them.

We restrict our discussion to reflections from a metallic surface (as opposed to transmission through the surface). To use the Fresnel reflection coefficients (3.3.1) and (3.3.3), we actually do not need to know the transmitted angle θ_t. We need only acquire expressions for $\cos\theta_t$ and $\sin\theta_t$, and we can obtain these from Snell's law (3.2.7). To minimize complications, let the incident refractive index be $n_i = 1$ (which is often the case). Let the index on the transmitted side be written simply as $\mathcal{N}_t = \mathcal{N}$. Then by Snell's law the sine of the transmitted angle is

$$\sin\theta_t = \frac{\sin\theta_i}{\mathcal{N}}. \tag{4.7.1}$$

This expression is of course complex since $\mathcal{N}$ is complex, but that does not concern us. The cosine of the same angle is

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{\mathcal{N}^2 - \sin^2\theta_i}\Big/\mathcal{N}. \tag{4.7.2}$$

The positive sign in front of the square root is appropriate since it is clearly the right choice if the imaginary part of the index approaches zero.

Upon substitution of these expressions, the Fresnel reflection coefficients (3.3.1) and (3.3.3) become

$$r_s = \frac{\cos\theta_i - \sqrt{\mathcal{N}^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\mathcal{N}^2 - \sin^2\theta_i}}, \text{ and} \tag{4.7.3}$$

$$r_p = \frac{\sqrt{\mathcal{N}^2 - \sin^2\theta_i} - \mathcal{N}^2\cos\theta_i}{\sqrt{\mathcal{N}^2 - \sin^2\theta_i} + \mathcal{N}^2\cos\theta_i}. \tag{4.7.4}$$

These expressions are tedious to evaluate. Nevertheless, they can be placed into the Jones matrix (4.6.1) to determine the effect on polarization. We can better appreciate their significance if we work (see P4.7.1) to put them into the form

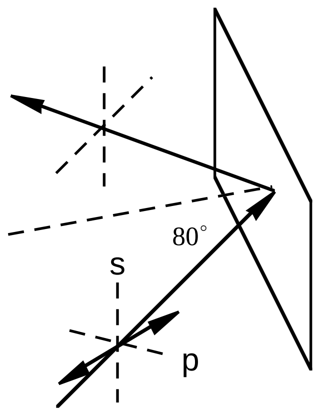
$$r_s = |r_s|e^{i\delta_{r_s}} \text{ and} \tag{4.7.5}$$

$$r_p = |r_p|e^{i\delta_{r_p}}. \tag{4.7.6}$$

We refrain from putting the general expression (4.7.3) and (4.7.4) into this form; we would get a big mess. Nevertheless, when numbers corresponding to specific data are inserted into (4.7.3) and (4.7.4), it is convenient to put the expressions into the simplified form (i.e. (4.7.5) and (4.7.6)) for purposes of evaluating the Jones calculus. It is a good idea to let a computer do it. The main point to notice here is that even when the reflectivity is high (i.e., $|r_s|$ and $|r_p|$ on the order of unity), the phases upon reflection can be very different for s and p-polarization components (i.e. δ_{r_p} and δ_{r_s} are different). Of course, this can dramatically alter the polarization when the incident light has both s and p-components.

Exercises

P4.7.1 The complex index for silver is given by $n = 0.2$ and $\kappa = 3.4$ (see P2.4.3). Light that is linearly polarized at $\alpha = 45^\circ$ has a Jones vector according to (4.2.13). Suppose this light is reflected from a vertical silver mirror with an incident angle $\theta_i = 80^\circ$. Find the Jones vector representation for the polarization of the reflected light (see L.4.5.1).



HINT: The following expressions will help. (You can see why it is better to have a computer do this.)

$$\sqrt{\mathcal{N}^2 - \sin^2\theta_i} = \sqrt{(0.2 + i3.4)^2 - \sin^2 80^\circ} = \sqrt{-12.49 + i1.36} = i\sqrt{12.49 - i1.36}$$

$$= i\sqrt{\sqrt{12.49^2 + 1.36^2}\,e^{-i\tan^{-1}\frac{1.36}{12.49}}} = i3.54e^{-i0.054} = 0.192 + i3.53$$

$$r_s = \frac{\cos 80^\circ - (0.192 + i3.53)}{\cos 80^\circ + 0.192 + i3.53} = \frac{-0.018 - i3.53}{0.366 + i3.53} = \frac{-\sqrt{0.018^2 + 3.53^2}\,e^{i\tan^{-1}\frac{3.53}{0.018}}}{\sqrt{0.366^2 + 3.53^2}\,e^{i\tan^{-1}\frac{3.53}{0.366}}} = -0.994e^{i0.10}$$

$$r_p = \frac{0.192 + i3.53 - (0.2 + i3.4)^2\cos 80^\circ}{0.192 + i3.53 + (0.2 + i3.4)^2\cos 80^\circ} = \frac{2.19 + i3.29}{-1.81 + i3.77} = \frac{\sqrt{2.19^2 + 3.29^2}\,e^{i\tan^{-1}\frac{3.29}{2.19}}}{-\sqrt{1.81^2 + 3.77^2}\,e^{-i\tan^{-1}\frac{3.77}{1.81}}} = -0.945e^{i2.11}$$

ANSWER: $\begin{pmatrix} 0.668 \\ 0.702e^{1.13i} \end{pmatrix}$.

NOTE: This may be somewhat different than the result measured in L4.5.1. For one thing, we have not even considered that a silver mirror inevitably has a thin oxide layer.

4.8 Polarization and Total Internal Reflection

In this section, we consider the index of refraction to be real. As we saw in section 3.6, if $n_i > n_t$, there exists a critical angle for θ_i beyond which we cannot find a physical value for θ_t. Beyond the critical angle θ_C defined in (3.6.2), Snell's law imposes the requirement $\sin\theta_t > 1$, and total internal reflection occurs. We demonstrate this fact by showing $|r_s| = 1$ and $|r_p| = 1$ in this case (see P3.6.1).

Again, to compute the Fresnel reflection coefficients (3.3.1) and (3.3.3) we need only find adequate expressions for $\sin\theta_t$ and $\cos\theta_t$. These can be obtained from Snell's law and are given in (3.6.3) and (3.6.4). These formulas are essentially the same as those given in the previous section: (4.7.1) and (4.7.2). The angle θ_t is a complex number, but we do not assign physical significance to it in terms of a direction.

Upon substitution of (3.6.3) and (3.6.4), the Fresnel reflection coefficients (3.3.1) and (3.3.3) become

$$r_s = \frac{\frac{n_i}{n_t}\cos\theta_i - i\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}{\frac{n_i}{n_t}\cos\theta_i + i\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}, \text{ and} \tag{4.8.1}$$

$$r_p = -\frac{\cos\theta_i - i\frac{n_i}{n_t}\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}{\cos\theta_i + i\frac{n_i}{n_t}\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}. \tag{4.8.2}$$

These expressions both contain the form

$$\frac{a - ib}{a + ib} = \frac{\sqrt{a^2 + b^2}\, e^{-i\tan^{-1}\frac{b}{a}}}{\sqrt{a^2 + b^2}\, e^{i\tan^{-1}\frac{b}{a}}} = \frac{e^{-i\tan^{-1}\frac{b}{a}}}{e^{i\tan^{-1}\frac{b}{a}}} = e^{-2i\tan^{-1}\frac{b}{a}}, \tag{4.8.3}$$

where a is always positive since θ_i does not exceed $\pi/2$. With this simplification, the Fresnel coefficients become

$$r_s = \exp\left\{-2i\tan^{-1}\left[\frac{n_t}{n_i\cos\theta_i}\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}\right]\right\}, \text{ and} \tag{4.8.4}$$

$$r_p = -\exp\left\{-2i\tan^{-1}\left[\frac{n_i}{n_t\cos\theta_i}\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}\right]\right\}. \tag{4.8.5}$$

From these expressions, we obviously have $|r_s| = 1$ and $|r_p| = 1$ even though the phases upon reflection are very different in the two cases. We conclude that 100% of the light is reflected. Even so, the boundary conditions from Maxwell's equations (see appendix A.2) require that the fields are non-zero on the transmitted side of the boundary. If we bothered to compute the remaining Fresnel coefficients (3.3.2) and (3.3.3), we would find out that $t_s \neq 0$ and $t_p \neq 0$. This may seem surprising, but it does not contradict our assertion that 100% of the light is reflected.

The coefficients t_s and t_p give the strength of an evanescent wave on the transmitted side of the boundary. The evanescent wave travels *parallel* to the boundary so that no energy is actually carried away on the transmitted side. In the direction perpendicular to the boundary, the strength of the evanescent wave decays in strength. The transmitted energy is zero as dictated by (3.4.4). For total internal reflection, one may not employ (3.4.8).

To compute the explicit form of the evanescent wave, we plug (3.6.3) and (3.6.4) into the transmitted field (3.2.2):

$$\vec{E}_t = \left[E_p^{(t)}\left(\hat{y}\cos\theta_t - \hat{z}\sin\theta_t\right) + \hat{x}E_s^{(t)}\right]e^{i\left[k_t\left(y\sin\theta_t + z\cos\theta_t\right)-\omega t\right]}$$

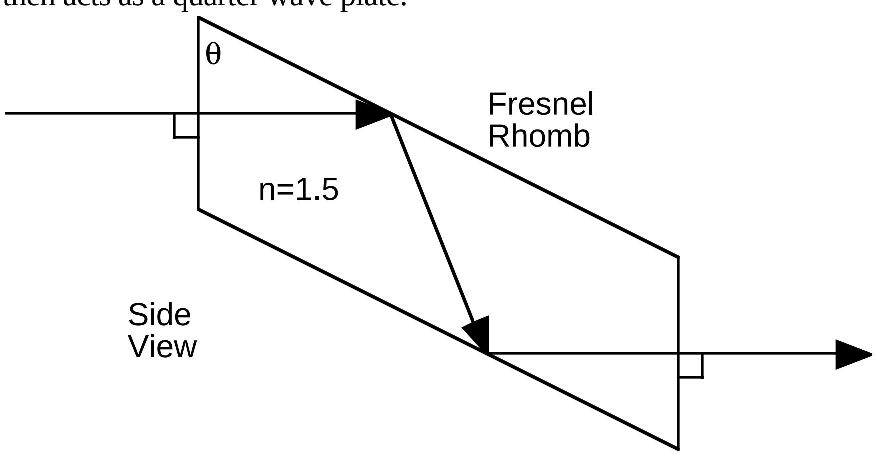
$$= \left[t_p E_p^{(i)}\left(\hat{y}i\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1} - \hat{z}\frac{n_i}{n_t}\sin\theta_i\right) + \hat{x}t_s E_s^{(i)}\right]e^{i\left[k_t\left(y\frac{n_i}{n_t}\sin\theta_i + zi\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}\right)-\omega t\right]}$$

$$= \left[t_p E_p^{(i)}\left(\hat{y}i\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1} - \hat{z}\frac{n_i}{n_t}\sin\theta_i\right) + \hat{x}t_s E_s^{(i)}\right]e^{-k_t z\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}\, e^{i\left[k_t y\frac{n_i}{n_t}\sin\theta_i - \omega t\right]}. \quad (4.8.6)$$

Although we do not know the strength of the field without finding t_s and t_p, we can see that the evanescent wave propagates parallel to the boundary (in the y-dimension) and its strength diminishes away from the boundary (in the z-dimension).

Exercises

P4.8.1 Calculate the angle θ to cut the glass in a Fresnel rhomb such that after the two internal reflections there is a phase difference of $\pi/2$ between the two polarization states. The rhomb then acts as a quarter wave plate.



HINT: You need to find the phase difference between (4.8.4) and (4.8.5). Set the difference equal to $\pi/4$ for each bounce. You will get a transcendental equation that you can solve in a few minutes by trial and error with your calculator.
ANSWER: There are two angles that work: $\theta \cong 50^\circ$ and $\theta \cong 53^\circ$.

P4.8.2 Light ($\lambda_{\text{vac}} = 500\,\text{nm}$) reflects internally from a glass surface ($n = 1.5$) surrounded by air. The incident angle is $\theta_i = 45^\circ$. An evanescent wave travels parallel to the surface on the air side. At what distance from the surface is the amplitude of the evanescent wave $1/e$ of its value at the surface?

4.9 Ellipsometry

In this final section we consider how measuring the polarization of light reflected from a surface can yield information regarding the optical constants of that surface (i.e. n and κ). As seen in L4.5.1, it is possible to characterize the polarization of a beam of light using a quarter-wave plate and a polarizer. However, we often want to know n and κ at a range of frequencies, and this would require a different quarter-wave plate thickness d for each wavelength used (see (4.5.4)). Therefore, many commercial ellipsometers do not try to extract the helicity of the light, but only the ellipticity. In this case only polarizers are used, which can be made to work over a wide range of wavelengths.

Inasmuch as most commercial ellipsometers do not determine directly the helicity of the reflected light, the measurement is usually made for a variety of different incident angles on the sample. This adds enough redundancy that n and κ can be pinned down (allowing a computer to take care of the busy work). If many different incident angles are measured at many different wavelengths, it is possible to extract detailed information about the optical constants and the thicknesses of possibly many layers of materials influencing the reflection. (We will learn to deal with multilayer coatings in chapter 6.)

Commercial ellipsometers typically employ two polarizers, one before and one after the sample, where s and p-polarized reflections take place. The first polarizer ensures that linearly polarized light arrives at the test surface (polarized at angle α to give both s and p-components). The Jones matrix for the test surface reflection is given by (4.6.1), and the Jones matrix for the analyzing polarizer oriented at angle θ is given by (4.4.14). The Jones vector for the light arriving at the detector is

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} -r_p & 0 \\ 0 & r_s \end{bmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} -r_p\cos\alpha\cos^2\theta + r_s\sin\alpha\sin\theta\cos\theta \\ -r_p\cos\alpha\sin\theta\cos\theta + r_s\sin\alpha\sin^2\theta \end{pmatrix}. \qquad (4.9.1)$$

In an ellipsometer, the angle θ of the analyzing polarizer often rotates at a high speed, and the time dependence of the light reaching a detector is analyzed and correlated with the polarizer orientation. From the measurement of the intensity where θ and α are continuously varied, it is possible to extract the values of n and κ (with the aid of a computer!).