

Chapter 2

Plane Waves and Refractive Index

2.1 Introduction

As was discussed in the previous chapter, we are primarily interested in sinusoidal solutions to Maxwell's equations. This allows us to deal with individual frequencies, one at a time. When considering the propagation of light within a material medium, we are essentially unable to do anything else. Any waveform propagating in a material medium invariably experiences distortion (called dispersion) unless that waveform is a pure sinusoidal wave. It should come as no surprise then, that physicists and engineers choose to work with sinusoidal waves. This is not as limiting as it might seem, since any complicated waveform can be decomposed into a linear superposition of sinusoidal waves, as will be addressed in chapter 7.

In this chapter we consider the interaction between matter and sinusoidal waves called plane waves. We also consider the energy carried by such waves. Poynting's theorem, which governs the flow of energy carried by electromagnetic fields, is introduced in section 2.5. This leads to the concept of irradiance (or intensity), which we discuss in the plane-wave context in section 2.6.

When describing light, it is convenient to use complex number notation. This is particularly true for problems involving absorption of light such as that which takes place inside metals and, to a lesser degree (usually), inside dielectrics (e.g. glass). In such materials, oscillatory fields decay as they travel, owing to absorption. When the electric field is represented using complex notation, we can accommodate decay conveniently by allowing the phase parameter $\vec{k} \cdot \vec{r}$ (associated with distance) to become a complex number. The imaginary part then controls the rate that the field decays while the real part governs the familiar oscillatory behavior. In chapter 4, we shall see that this absorption rate plays an important role in the reflectance of light from metal surfaces. In this chapter we introduce the complex index of refraction $\mathcal{N} \equiv n + i\kappa$ (section 2.3). The complex index only makes sense when using complex notation for the electric field, so don't be alarmed at this point if this seems puzzling. We will introduce complex electric field waves in section 2.2.

To compute the index of refraction in either a dielectric or a conducting material, we require a model to describe the response of electrons in the material to the passing electric field wave. That is, we need to know χ in $\vec{P} = \epsilon_0 \chi \vec{E}$, equation (1.8.1). The model in turn influences how the electric field propagates, which in turn influences the model, which in turn influences the field, and so on. The model therefore must be solved together with the propagating field in a self-consistent manner. A very successful model developed by Lorentz in the late 1800's treats each (active) electron in the medium as a classical particle obeying Newton's second law ($F = ma$). In the case of a dielectric medium, electrons are subject to an elastic restoring force (keeping the electrons in their respective atoms) in addition to a dragging force, which dissipates energy and gives rise to absorption. In the case of a conducting medium, electrons are free to move outside of atoms but are subject to a dragging force (due to collisions), which removes energy and gives rise to absorption.

2.2 Plane Wave Solutions to the Wave Equation

Consider the wave equation for an electric field waveform propagating in vacuum (1.7.5):

$$\nabla^2 \bar{E} - \mu_o \epsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} = 0. \quad (2.2.1)$$

A magnetic wave accompanies this electric wave, but in most optics problems we can ignore it. The influence of the magnetic field only becomes important (in comparison to the electric field) for particles moving near the speed of light. This typically only takes place for extremely intense lasers (intensities near 10^{18} W/cm^2 , see P2.6.5) where the electric field is sufficiently strong to cause electrons to oscillate with velocities near the speed of light.

We are interested in solutions to (2.2.1) that have the functional form

$$\bar{E}(\bar{r}, t) = \bar{E}_o \cos(\bar{k} \cdot \bar{r} - \omega t + \phi). \quad (2.2.2)$$

Here ϕ represents an arbitrary (constant) phase term. Note that in vacuum any function (e.g. Gaussian or triangle waveform) can satisfy (2.2.1) if its argument is $\hat{u} \cdot \bar{r} - ct$, where \hat{u} is a unit vector pointing in the direction that the wave propagates, taking place at speed $c \equiv 1/\sqrt{\epsilon_o \mu_o}$.

	Frequency $\nu = \omega/2\pi$	Wavelength λ_{vac}
AM Radio	10^6 Hz	300 m
FM Radio	10^8 Hz	3 m
Radar	10^{10} Hz	0.03 m
Microwave	$10^9 - 10^{12} \text{ Hz}$	$0.3 \text{ m} - 3 \times 10^{-4} \text{ m}$
Infrared	$10^{12} - 4 \times 10^{14} \text{ Hz}$	$3 \times 10^{-4} - 7 \times 10^{-7} \text{ m}$
Light (red)	$4.6 \times 10^{14} \text{ Hz}$	$6.5 \times 10^{-7} \text{ m}$
Light (yellow)	$5.5 \times 10^{14} \text{ Hz}$	$5.5 \times 10^{-7} \text{ m}$
Light (blue)	$6.7 \times 10^{14} \text{ Hz}$	$4.5 \times 10^{-7} \text{ m}$
Ultraviolet	$10^{15} - 10^{17} \text{ Hz}$	$4 \times 10^{-7} - 3 \times 10^{-9} \text{ m}$
X-rays	$10^{17} - 10^{20} \text{ Hz}$	$3 \times 10^{-9} - 3 \times 10^{-12} \text{ m}$
Gamma rays	$10^{20} - 10^{23} \text{ Hz} \rightarrow$	$3 \times 10^{-12} - 3 \times 10^{-15} \text{ m} \rightarrow$

Table 2.1 The electromagnetic spectrum.

In (2.2.2) the argument $\hat{u} \cdot \bar{r} - ct$ is multiplied by a constant (which we are free to do) to make it dimensionless, and it is written as $\bar{k} \cdot \bar{r} - \omega t$. The constant that we have chosen is $2\pi/\lambda_{\text{vac}}$ (multiplying both \hat{u} and c). Then λ_{vac} is identified as the length by which \bar{r} must vary (in the direction of \hat{u}) to cause the cosine to go through a complete cycle. This distance is known as the (vacuum) wavelength. In summary we have

$$\bar{k} \equiv \frac{2\pi}{\lambda_{\text{vac}}} \hat{u}, \text{ (vacuum)} \quad (2.2.3)$$

and

$$\omega = \frac{2\pi c}{\lambda_{\text{vac}}}. \quad (2.2.4)$$

Notice that k and ω are not independent of each other but form a pair. k is called the wave number, and when written in vector form, we call it the wave vector. Typical values for λ_{vac} are given in table 2.1

We next check our solution (2.2.2) in the wave equation. First, however, we will adopt complex number notation. (For a review of complex notation, see section 0.2.) Although this change in notation will not make the task at hand any easier, it will save considerable labor in sections 2.3 and 2.4. Using complex notation we rewrite (2.2.2) as

$$\bar{E}(\vec{r}, t) = \text{Re} \left\{ \tilde{\bar{E}}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\}, \quad (2.2.5)$$

where we have hidden the phase term ϕ inside of $\tilde{\bar{E}}_o$ as follows:

$$\tilde{\bar{E}}_o \equiv \bar{E}_o e^{i\phi}. \quad (2.2.6)$$

The next step we take is to become intentionally sloppy. Physicists throughout the world have conspired to avoid writing $\text{Re}\{ \}$ (in an effort, or a lack of effort if you prefer, to make expressions less cluttered). Nevertheless, only the real part of the field is physically relevant even though expressions and calculations contain both real and imaginary terms. This sloppy notation is okay since the real and imaginary parts of complex numbers never intermingle when adding and subtracting, or differentiating and integrating. We can delay taking the real part of the expression until the end of the calculation. Also, when hiding a phase ϕ inside of the field amplitude as in (2.2.6), we drop the tilde since we are already being sloppy; when using complex notation, we will automatically assume that the complex field amplitude contains phase information.

Our solution (2.2.2) or (2.2.5) is written simply as

$$\bar{E}(\vec{r}, t) = \bar{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad (2.2.7)$$

and this is known as a plane wave. Any electromagnetic disturbance can be expressed as a linear superposition of such waves. The name ‘plane wave’ is given since the argument in (2.2.7) at any moment is constant (and hence the electric field is uniform) throughout any given plane perpendicular to \vec{k} . A plane wave fills all space and may be thought of as a continuum of infinite sheets of uniform electric field moving in the \vec{k} direction. Note that in vacuum the electric field amplitude \bar{E}_o is always perpendicular to \vec{k} in order to satisfy Maxwell’s equations (see P1.6.2 and P1.8.2).

Finally, we verify (2.2.7) as a solution to the wave equation (2.2.1). The first term gives

$$\begin{aligned} \nabla^2 \bar{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} &= \bar{E}_o \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] e^{i(k_x x + k_y y + k_z z - \omega t)} \\ &= -\bar{E}_o (k_x^2 + k_y^2 + k_z^2) e^{i(\vec{k} \cdot \vec{r} - \omega t)} = -k^2 \bar{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \end{aligned} \quad (2.2.8)$$

and the second term gives

$$\frac{1}{c^2} \frac{\partial^2 \bar{E}_o e^{i(\bar{k} \cdot \bar{r} - \omega t)}}{\partial t^2} = \frac{(-i\omega)^2}{c^2} \bar{E}_o e^{i(\bar{k} \cdot \bar{r} - \omega t)} = -\frac{\omega^2}{c^2} \bar{E}_o e^{i(\bar{k} \cdot \bar{r} - \omega t)}. \quad (2.2.9)$$

Upon insertion into (2.2.1) we have

$$\bar{k} = \frac{\omega}{c}. \quad (\text{vacuum}) \quad (2.2.10)$$

This final relation is consistent with (2.2.3) and (2.2.4). The relation (2.2.10) demonstrates, as was mentioned, that \bar{k} and ω are not independently chosen. Different frequencies correspond to different wavelengths.

2.3 Dielectric Model of Refractive Index and Absorption

The index of refraction (1.8.5) is the ratio of the speed the light in vacuum to the speed that light propagates in a material medium. The index varies with the material and with the frequency of the light. In this section, we develop a simple *linear* model for describing optical index. The model provides the required connection between the electric field \bar{E} and the polarization \bar{P} (i.e. susceptibility χ in (1.8.1)). Lorentz introduced this model well before the development of quantum mechanics. Even though the model pays no attention to quantum physics, it works surprisingly well at describing frequency-dependent optical index and absorption of light. As it turns out, the Schroedinger equation applied to two levels in an atom reduces in mathematical form to the Lorentz model in the limit of low-intensity light. Quantum mechanics also explains a fudge factor (called the oscillator strength) in the Lorentz model, which before the development of quantum mechanics had to be inserted *ad hoc* to make the model agree with experiments.

We assume an isotropic, homogeneous, and non-conducting medium (i.e. $\vec{J}_{\text{free}} = 0$). In this case, we expect \bar{E} and \bar{P} to be parallel to each other, and the general wave equation (1.7.4) for the electric field reduces to

$$\nabla^2 \bar{E} - \epsilon_o \mu_o \frac{\partial^2 \bar{E}}{\partial t^2} = \mu_o \frac{\partial^2 \bar{P}}{\partial t^2}. \quad (2.3.1)$$

We assume (for simplicity) that all atoms (or molecules) in the medium are identical, each with one (or a few) active electrons responding to the external field. The atoms are uniformly distributed throughout space with N identical active electrons per volume (units of number per volume). The polarization of the material is then

$$\bar{P} = q_e N \bar{r}_{\text{micro}}. \quad (2.3.2)$$

Recall that polarization has units of dipoles per volume. Each dipole has strength $q_e \bar{r}_{\text{micro}}$, where \bar{r}_{micro} is a microscopic displacement of the electron from equilibrium. In our modern quantum-mechanical viewpoint, \bar{r}_{micro} corresponds to an average displacement of the electronic cloud, which surrounds the nucleus (see Fig. 2.1). (At the time of Lorentz, atoms were thought to be clouds of positive charge wherein point-like electrons sit at rest unless stimulated by an applied electric field.)

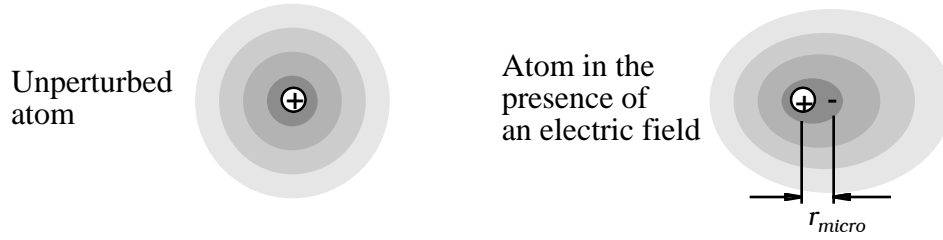


Fig. 2.1 A distorted electronic cloud becomes a dipole.

The displacement \vec{r}_{micro} of the electron charge in an individual atom depends on the *local* strength of the applied electric field \vec{E} . By *local*, we mean the position of the atom. Since the diameter of the electronic cloud is tiny compared to a wavelength of (visible) light, we may consider the electric field to be uniform across any individual atom.

The Lorentz model uses Newton's equation of motion to describe an electron displacement from equilibrium within an atom. In accordance with the classical laws of motion, the electron mass m_e times its acceleration is equal to the sum of the forces on the electron:

$$m_e \ddot{\vec{r}}_{\text{micro}} = q_e \vec{E} - m_e \gamma \dot{\vec{r}}_{\text{micro}} - k_{\text{Hooke}} \vec{r}_{\text{micro}}. \quad (2.3.3)$$

The electric field pulls on the electron with force $q_e \vec{E}$. A dragging force $-m_e \gamma \dot{\vec{r}}_{\text{micro}}$ opposes the electron motion and accounts for absorption of energy. Without this term, it is only possible to describe optical index at frequencies far from where absorption takes place. Finally, $-k_{\text{Hooke}} \vec{r}_{\text{micro}}$ is a force accounting for the fact that the electron is bound to the nucleus. To a good approximation this restoring force can be thought of as a spring that pulls the displaced electron back in proportion to the amount of displacement. Mathematically, this term resembles the familiar Hooke's law.

With some rearranging, (2.3.3) can be written as

$$\ddot{\vec{r}}_{\text{micro}} + \gamma \dot{\vec{r}}_{\text{micro}} + \omega_o^2 \vec{r}_{\text{micro}} = \frac{q_e}{m_e} \vec{E}, \quad (2.3.4)$$

where $\omega_o \equiv \sqrt{k_{\text{Hooke}}/m_e}$ is the natural oscillation frequency (or resonant frequency) associated with the electron mass and the 'spring constant'.

We assume a single plane wave in the medium with frequency ω . In this case, we can write the time dependence of the electric field explicitly as

$$\vec{E}(\vec{r}, t) = \vec{E}'(\vec{r}) e^{-i\omega t}. \quad (2.3.5)$$

$\vec{E}'(\vec{r})$ indicates the electric field at the location of the atom (at location \vec{r}), but with the time dependence factored out. When this is inserted into (2.3.4) we obtain

$$\ddot{\vec{r}}_{\text{micro}} + \gamma \dot{\vec{r}}_{\text{micro}} + \omega_o^2 \vec{r}_{\text{micro}} = \frac{q_e}{m_e} \vec{E}'(\vec{r}) e^{-i\omega t}. \quad (2.3.6)$$

Note that within a given atom the excursions of \vec{r}_{micro} are so small that \vec{r} remains essentially unchanged (i.e. in the differential equation on \vec{r}_{micro} , the atom position \vec{r} is a constant). The inhomogeneous solution to (2.3.6) is (see P2.3.1)

$$\bar{r}_{\text{micro}} = \frac{q_e}{m_e} \frac{\bar{E}'(\bar{r})e^{-i\omega t}}{\omega_o^2 - i\omega\gamma - \omega^2} = \frac{q_e}{m_e} \frac{\bar{E}(\bar{r}, t)}{\omega_o^2 - i\omega\gamma - \omega^2}. \quad (2.3.7)$$

The electron position \bar{r}_{micro} oscillates (not surprisingly) with the same frequency ω as the driving electric field. This solution illustrates the convenience of the complex notation. The imaginary part in the denominator implies that the electron oscillates with a different phase from the electric field oscillations; the dragging term γ (the imaginary part in the denominator) causes the two to be out of phase somewhat. The complex algebra in (2.3.7) accomplishes what would otherwise require trigonometric manipulation.

We are now able to write the polarization in terms of the electric field. By substituting (2.3.7) into (2.3.2), we obtain

$$\bar{P} = \frac{q_e^2 N}{m_e} \frac{\bar{E}}{\omega_o^2 - i\omega\gamma - \omega^2}. \quad (2.3.8)$$

To this point the electric field is still unknown, except that we have assumed that it oscillates at single frequency ω . We must now search for our self-consistent solution by placing the expression for polarization (2.3.8) into the wave equation (2.3.1). This substitution gives

$$\nabla^2 \bar{E} - \epsilon_o \mu_o \left[1 + \frac{q_e^2 N}{m_e \epsilon_o} \frac{1}{\omega_o^2 - i\omega\gamma - \omega^2} \right] \frac{\partial^2 \bar{E}}{\partial t^2} = 0. \quad (2.3.9)$$

In summary, we have used a classical atomic model to specify how the electric field \bar{E} (yet unknown) influences the polarization. By writing \bar{P} in terms of \bar{E} , we can now solve for \bar{E} so that the entire procedure is self-consistent.

A comparison with (1.8.2) reveals that the (complex) susceptibility evidently is

$$\chi = \frac{q_e^2 N}{\epsilon_o m_e} \frac{1}{\omega_o^2 - i\omega\gamma - \omega^2} \quad (2.3.10)$$

In terms of the susceptibility, the index is specified through

$$\mathcal{N}^2 \equiv 1 + \chi, \quad (2.3.11)$$

as in (1.8.5). Since the susceptibility is complex, so is the index of refraction:

$$\mathcal{N} \equiv n + i\kappa, \quad (2.3.12)$$

where n and κ are respectively the real and imaginary parts of the index. (Note that κ is not k .) The real and imaginary parts of the index are solved by equating separately the real and imaginary parts of (2.3.11), namely

$$(n + i\kappa)^2 = 1 + \frac{q_e^2 N}{m_e \epsilon_o} \frac{1}{\omega_o^2 - i\omega\gamma - \omega^2}. \quad (2.3.13)$$

A graph of n and κ is given in Fig. 2.2. In actuality, materials usually have more than one species of active electron. For example, the medium may contain a mixture of molecular species.

In addition, different active electrons within a given molecule also behave differently. The generalization of (2.3.13) in this case is

$$(n + i\kappa)^2 = 1 + \sum_j \frac{f_j e^2 N_j}{m_e \epsilon_o} \frac{1}{\omega_{oj}^2 - i\omega\gamma_j - \omega^2}, \quad (2.3.14)$$

where f_j is the so-called oscillator strength for the j^{th} species of active electrons. Keep in mind that (2.3.14) applies to a low-density gas. In a solid, where the dipoles are packed much tighter, the dipoles influence neighboring dipoles and a further modification to the formula must be made (i.e. Clausius-Mossotti).

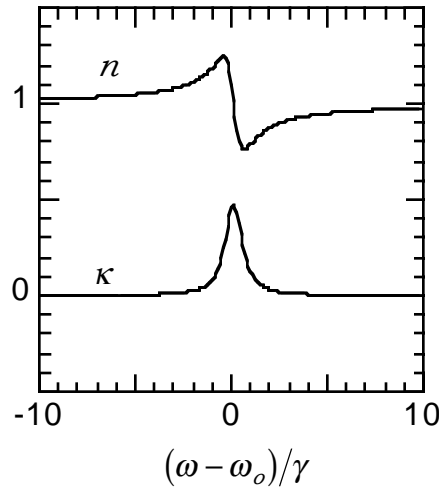


Fig. 2.2 Real and imaginary part of index for $\frac{e^2 N}{m_e \epsilon_o} = 100\gamma^2$.

In terms of the refractive index, the wave equation becomes

$$\nabla^2 \vec{E} - \frac{\mathcal{N}^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \quad (2.3.15)$$

This looks very much like the wave equation in vacuum (2.2.1). Since the solution to (2.2.1) is $\vec{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}$, where $k = \omega/c$ (see (2.2.7) and (2.2.10)), we know immediately that the solution to (2.3.15) is

$$\vec{E}(\vec{r}, t) = \vec{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \text{ where} \quad (2.3.16)$$

$$\mathcal{K} = \frac{\mathcal{N}\omega}{c} = \frac{(n + i\kappa)\omega}{c} \quad (2.3.17)$$

is a complex wave number. This solution again illustrates the convenience of the complex number notation. We need not employ a different procedure in finding (2.3.16) than we did in finding the solution (2.2.7) to the wave equation in vacuum.

The complex index \mathcal{N} takes account of the absorption rate as well as the usual oscillatory behavior of the wave. We see this by explicitly placing (2.3.17) into (2.3.16):

$$\vec{E}(\vec{r}, t) = \vec{E}_o e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}} e^{i\left(\frac{n\omega}{c}\hat{u}\cdot\vec{r} - \omega t\right)}. \quad (2.3.18)$$

Here again \hat{u} is a *real* unit vector specifying the direction of \vec{K} .

As a reminder, when looking at (2.3.18), by special agreement in advance, we should just think of the real part, namely

$$\vec{E}(\vec{r}, t) = \vec{E}_o e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}} \cos(\vec{k}\cdot\vec{r} - \omega t + \phi), \quad (2.3.19)$$

where the phase ϕ was formerly held in the complex vector \vec{E}_o . Fig. 2.3 shows a graph of the exponent and cosine factor in (2.3.19). For convenience in plotting, the direction of propagation is chosen to be in the z direction (i.e. $\hat{u} = \hat{z}$). The imaginary part of the index κ causes the wave to decay as it travels. In writing (2.3.19), we have generalized the context for the wave number \vec{k} from that in vacuum (2.2.10) to be the real part of \mathcal{K} :

$$\vec{k} = \frac{n\omega}{c}. \quad (2.3.20)$$

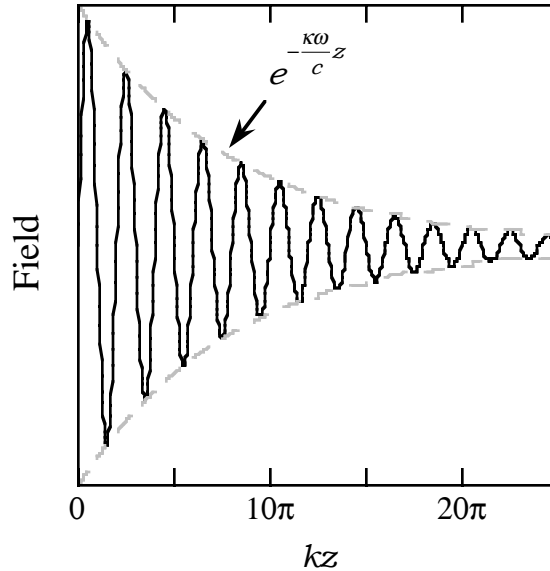


Fig. 2.3 Electric field of a decaying plane wave.

The connection with the wavelength should look the same as in (2.2.3), since (2.3.19) has a similar cosine part. However, in order to be consistent with (2.3.20), the equations (2.2.3) and (2.2.4) must be modified to read

$$\vec{k} \equiv \frac{2\pi}{\lambda} \hat{u}, \text{ and} \quad (2.3.21)$$

$$\omega = \frac{2\pi c}{\lambda n}, \text{ where} \quad (2.3.22)$$

$$\lambda \equiv \lambda_{\text{vac}}/n. \quad (2.3.23)$$

While the frequency ω is the same, whether in a material or in vacuum, the wavelength is different.

As a final note, for the sake of simplicity in writing (2.3.19) from (2.3.18) we assumed linearly polarized light (described in chapter 4). That is, all vector components of \vec{E}_o were assumed to have the same complex phase ϕ . The expression would be somewhat more complicated, for example, in the case of circularly polarized light (also described in chapter 4).

Exercises

P2.3.1 Verify that (2.3.7) is a solution to (2.3.6).

P2.3.2 Derive the Sellmeier equation $n^2 = 1 + \frac{A\lambda_{\text{vac}}^2}{\lambda_{\text{vac}}^2 - \lambda_{o,\text{vac}}^2}$ from (2.3.13) for a gas with negligible absorption (i.e. $\gamma \equiv 0$, valid far from resonance ω_o), where $\lambda_{o,\text{vac}}$ corresponds to frequency ω_o . Many materials (e.g. glass, air) have strong resonances in the ultraviolet. In such materials, do you expect the index of refraction for blue light to be greater than that for red light? Make a sketch of n as a function of wavelength for visible light down to the ultraviolet (where $\lambda_{o,\text{vac}}$ is located).

2.4 Conductor Model of Refractive Index and Absorption

The details of the conductor model are very similar to those of the dielectric model. We will go through the derivation quickly since the procedure so closely parallels the previous section. In this model, we will ignore polarization:

$$\vec{P} = 0. \quad (2.4.1)$$

However, we take the current density \vec{J}_{free} to be non-zero. The wave equation then becomes

$$\nabla^2 \vec{E} - \epsilon_o \mu_o \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_o \frac{\partial \vec{J}_{\text{free}}}{\partial t}. \quad (2.4.2)$$

In a manner similar to (2.3.2), we assume that the current is made up of individual electrons traveling with velocity \vec{v}_{micro} :

$$\vec{J}_{\text{free}} = q_e N \vec{v}_{\text{micro}}. \quad (2.4.3)$$

Again, N is the number density of free electrons (in units of number per volume). Recall that current density \vec{J}_{free} has units of charge times velocity per volume (or current per cross sectional area), so (2.4.3) may be thought of as a definition of current density in a fundamental sense.

As before, we use Newton's equation of motion on a representative electron. Mass times acceleration equals the sum of the forces on the electron:

$$m_e \dot{\bar{v}}_{\text{micro}} = q_e \bar{E} - m_e \gamma \bar{v}_{\text{micro}}. \quad (2.4.4)$$

The electric field pulls on the electron with force $q_e \bar{E}$. A dragging force $-m_e \gamma \bar{v}_{\text{micro}}$ opposes the motion in proportion to the speed (identical to the dielectric model, see (2.3.3)). Physically, the dragging term arises due to collisions between electrons and lattice sites in a metal. Such collisions give rise to resistance in a conductor.

When a DC field is applied, we may take the acceleration on average to be zero where the other two forces balance (i.e. $\dot{\bar{v}} = 0$). Then by combining (2.4.3) and (2.4.4) we get Ohm's law $\bar{J} = \sigma \bar{E}$, where $\sigma = Ne^2/m_e \gamma$ is the conductivity. Although our model relates the dragging term γ to the DC conductivity σ , the connection matches poorly with experimental observations made for visible frequencies. This is because the collision rate actually varies somewhat with frequency. Nevertheless, the qualitative behavior of the model is useful.

As before, we assume a frequency ω and factor it from the field explicitly as in (2.3.5). Then (2.4.4) becomes

$$\dot{\bar{v}}_{\text{micro}} + \gamma \bar{v}_{\text{micro}} = \frac{q_e}{m_e} \bar{E} e^{-i\omega t}. \quad (2.4.5)$$

The solution to this equation is (see P2.4.1)

$$\bar{v} = \frac{q_e}{m_e} \frac{\bar{E}}{\gamma - i\omega}. \quad (2.4.6)$$

We are now able to find an expression for the current density (2.4.3) in terms of the electric field:

$$\bar{J}_{\text{free}} = \frac{Nq_e^2}{m_e} \frac{\bar{E}}{\gamma - i\omega}. \quad (2.4.7)$$

The electric field is still unknown, but we can now find it by substituting (2.4.7) back into (2.4.2). The substitution yields

$$\nabla^2 \bar{E} - \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} - \frac{\mu_o Nq_e^2}{m_e} \frac{1}{\gamma - i\omega} \frac{\partial \bar{E}}{\partial t} = 0. \quad (2.4.8)$$

The solution (2.3.16) written in the previous section also satisfies this wave equation as long as the following holds (see P2.4.2):

$$\mathcal{K}^2 = \frac{\omega^2}{c^2} - \frac{\mu_o Nq_e^2}{m_e} \frac{\omega}{i\gamma + \omega}. \quad (2.4.9)$$

Thus, for the conductor model, instead of (2.3.13) we have

$$(n + i\kappa)^2 = 1 - \frac{\mu_o c^2 Nq_e^2}{m_e \omega^2} \frac{\omega}{i\gamma + \omega}. \quad (2.4.10)$$

Equations (2.3.16) through (2.3.23) hold as before.

Here we have introduced a complex refractive index for the conductor model just as we did for the dielectric model. The similarity is not surprising since both models include oscillating electrons. In the one case the electrons are free, and in the other case they are tethered to their atoms. In either model, the damping term removes energy from the electron oscillations. In the complex notation for the field, the damping term gives rise to an imaginary part of the index. Again, the imaginary part of the index causes an exponential attenuation of the plane wave as it propagates.

Exercises

P2.4.1 Verify that (2.4.6) is a solution to (2.4.5).

P2.4.2 Verify that (2.3.16) is a solution to (2.4.8).

P2.4.3 For silver, the complex refractive index is characterized by $n = 0.2$ and $\kappa = 3.4$. Find the distance that light travels inside of silver before the field is reduced by a factor of $1/e$. Assume a wavelength of $\lambda_{vac} = 633\text{nm}$. What is the speed of the wave crests in the silver (written as a number times c)? Are you surprised?

P2.4.4 Show that the dielectric model and the conductor model give identical results for n in the case of a low-density plasma where there is no restoring force (i.e. $\omega_o = 0$) and no dragging term (i.e., $\gamma = 0$). Write n in terms of $\omega_p \equiv \sqrt{\frac{q_e^2 N}{m_e \epsilon_o}}$, called the plasma frequency.

P2.4.5 Use the result from P2.4.4.

(a) If the index of refraction of the ionosphere is $n = 0.9$ for an FM station at $\nu = \omega/2\pi = 100\text{MHz}$, calculate the number of free electrons per cubic meter.

(b) What is the complex refractive index for KSL radio at 1160kHz ? Assume the same density of free electrons as in part (a). For your information, AM radio reflects better than FM radio from the ionosphere (like visible light from a metal mirror). At night, the lower layer of the ionosphere goes away so that AM radio waves reflect from a higher layer.

2.5 Poynting's Theorem

We next turn our attention to the detection and measurement of light. Until now, we have described light as the propagation of an electromagnetic disturbance. However, we typically observe light by detecting absorbed energy rather than the field amplitude directly. In this section we examine the connection between propagating electromagnetic fields (such as the plane waves discussed above) and the energy transported by such fields.

John Henry Poynting (1852-1914) developed (from Maxwell's equations) the theoretical foundation that describes light energy transport. In this section we examine its development, which is surprisingly concise. Students should concentrate mainly on the ideas involved (rather than the details of the derivation), especially the definition and meaning of the Poynting vector, describing energy flow in an electromagnetic field.

Poynting's theorem derives from just two of Maxwell's Equations: (1.6.3) and (1.6.4). We take the dot product of \vec{B}/μ_o with the first equation and the dot product of \vec{E} with the second equation. Then by subtracting the second equation from the first we obtain

$$\frac{\vec{B}}{\mu_o} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot \left(\vec{\nabla} \times \frac{\vec{B}}{\mu_o} \right) + \epsilon_o \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{\vec{B}}{\mu_o} \cdot \frac{\partial \vec{B}}{\partial t} = -\vec{E} \cdot \left(\vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} \right). \quad (2.5.1)$$

The first two terms can be simplified using the vector identity P0.3.12. The next two terms are the time derivatives of $\epsilon_o E^2/2$ and $B^2/2\mu_o$, respectively. The relation (2.5.1) then becomes

$$\vec{\nabla} \cdot \left(\vec{E} \times \frac{\vec{B}}{\mu_o} \right) + \frac{\partial}{\partial t} \left(\frac{\epsilon_o E^2}{2} + \frac{B^2}{2\mu_o} \right) = -\vec{E} \cdot \left(\vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} \right). \quad (2.5.2)$$

This is Poynting's theorem. Each term in this equation has units of power per volume.

The conventional way of writing Poynting's theorem is as follows:

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u_{\text{field}}}{\partial t} = -\frac{\partial u_{\text{exchange}}}{\partial t}, \text{ where} \quad (2.5.3)$$

$$\vec{S} \equiv \vec{E} \times \frac{\vec{B}}{\mu_o}, \quad (2.5.4)$$

$$u_{\text{field}} \equiv \frac{\epsilon_o E^2}{2} + \frac{B^2}{2\mu_o}, \text{ and} \quad (2.5.5)$$

$$\frac{\partial u_{\text{exchange}}}{\partial t} = \vec{E} \cdot \left(\vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} \right). \quad (2.5.6)$$

\vec{S} is called the Poynting vector and has units of power per area, called irradiance. The quantity u_{field} is the energy per volume stored in the electric and magnetic fields. Derivations of the electric field energy density and the magnetic field energy density are given in Appendixes 2.A and 2.B. (See (2.A.7) and (2.B.7).) The term $\partial u_{\text{exchange}}/\partial t$ is the power per volume delivered to the medium. Equation (2.5.6) is reminiscent of the familiar circuit power law, *Power=Voltage×Current*. Power is delivered when a charged particle traverses a distance while experiencing a force. This happens when currents flow in the presence of electric fields. Recall that $\partial \vec{P}/\partial t$ is a current density similar to \vec{J}_{free} , with units of charge times velocity per volume.

The interpretation of the Poynting vector is straightforward when we recognize Poynting's theorem as a statement of the conservation of power. \vec{S} describes the flow of energy. To see this more clearly, consider Poynting's theorem (2.5.3) integrated over a volume V (enclosed by surface S). If we also apply the divergence theorem (0.3.8) to the term involving $\vec{\nabla} \cdot \vec{S}$ we obtain

$$\oint_S \vec{S} \cdot \hat{n} da = -\frac{\partial}{\partial t} \int_V (u_{\text{field}} + u_{\text{exchange}}) dv. \quad (2.5.7)$$

Notice that the volume integral over energy densities u_{field} and u_{exchange} gives the total energy stored in V , whether in the form of electromagnetic field energy density or as energy density that has been given to the medium. The integration of the Poynting vector over the surface gives the net

Poynting vector flux directed outward. Equation (2.5.7) indicates that the outward Poynting vector flux matches the rate that total energy disappears from the interior of V . Conversely, if the Poynting vector is directed inward (negative), then the net inward flux matches the rate that energy increases within V . *Evidently, \vec{S} defines the flow of energy through space.* Its units of *power per area* are just what are needed to describe the brightness of light impinging on a surface.

2.6 Irradiance of a Plane Wave

Consider the electric field wave described by (2.3.16). The magnetic field that accompanies this electric field can be found from Maxwell's equation (1.6.3), and it turns out to be

$$\vec{B}(\vec{r}, t) = \frac{\vec{K} \times \vec{E}_o}{\omega} e^{i(\vec{K} \cdot \vec{r} - \omega t)}. \quad (2.6.1)$$

When \vec{K} is complex, evidently \vec{B} is out of phase with \vec{E} , and this occurs when absorption takes place. When there is no absorption, then $\vec{K} \rightarrow \vec{k}$ is real, and \vec{B} and \vec{E} carry the same complex phase.

Before computing the Poynting vector (2.5.4), which involves multiplication, we must remember our unspoken agreement that only the real parts of the fields are relevant. We necessarily remove the imaginary parts before multiplying (see (0.2.10)). We could rewrite \vec{B} and \vec{E} like in (2.3.19), imposing the assumption that the complex phase for each vector component of \vec{E}_o is the same. However, we can defer making this assumption by taking the real parts of the field in the following manner: Obtain the real parts of the fields by adding their respective complex conjugates and dividing the result by 2 (see (0.2.17)). The real field associated with (2.3.16) is

$$\vec{E}(\vec{r}, t) = \frac{1}{2} \left[\vec{E}_o e^{i(\vec{K} \cdot \vec{r} - \omega t)} + \vec{E}_o^* e^{-i(\vec{K}^* \cdot \vec{r} - \omega t)} \right], \quad (2.6.2)$$

and the real field associated with (2.6.1) is

$$\vec{B}(\vec{r}, t) = \frac{1}{2} \left[\frac{\vec{K} \times \vec{E}_o}{\omega} e^{i(\vec{K} \cdot \vec{r} - \omega t)} + \frac{\vec{K}^* \times \vec{E}_o^*}{\omega} e^{-i(\vec{K}^* \cdot \vec{r} - \omega t)} \right]. \quad (2.6.3)$$

By writing (2.6.2) and (2.6.3), we have merely exercised our previous agreement that only the real parts (2.3.16) and (2.6.1) are to be retained.

The Poynting vector (2.5.4) associated with the plane wave is then computed as follows:

$$\begin{aligned} \vec{S} &\equiv \vec{E} \times \frac{\vec{B}}{\mu_o} = \frac{1}{2} \left[\vec{E}_o e^{i(\vec{K} \cdot \vec{r} - \omega t)} + \vec{E}_o^* e^{-i(\vec{K}^* \cdot \vec{r} - \omega t)} \right] \times \frac{1}{2\mu_o} \left[\frac{\vec{K} \times \vec{E}_o}{\omega} e^{i(\vec{K} \cdot \vec{r} - \omega t)} + \frac{\vec{K}^* \times \vec{E}_o^*}{\omega} e^{-i(\vec{K}^* \cdot \vec{r} - \omega t)} \right] \\ &= \frac{1}{4\mu_o} \left[\frac{\vec{E}_o \times (\vec{K} \times \vec{E}_o)}{\omega} e^{2i(\vec{K} \cdot \vec{r} - \omega t)} + \frac{\vec{E}_o^* \times (\vec{K} \times \vec{E}_o)}{\omega} e^{i(\vec{K} - \vec{K}^*) \cdot \vec{r}} \right. \\ &\quad \left. + \frac{\vec{E}_o \times (\vec{K}^* \times \vec{E}_o^*)}{\omega} e^{i(\vec{K} - \vec{K}^*) \cdot \vec{r}} + \frac{\vec{E}_o^* \times (\vec{K}^* \times \vec{E}_o^*)}{\omega} e^{-2i(\vec{K}^* \cdot \vec{r} - \omega t)} \right] \end{aligned}$$

$$= \frac{1}{4\mu_o} \left[\frac{\mathcal{K}}{\omega} \bar{\mathbf{E}}_o \times (\hat{\mathbf{u}} \times \bar{\mathbf{E}}_o) e^{2i(\bar{\mathbf{K}} \cdot \bar{\mathbf{r}} - \omega t)} + \frac{\mathcal{K}}{\omega} \bar{\mathbf{E}}_o^* \times (\hat{\mathbf{u}} \times \bar{\mathbf{E}}_o) e^{-2\frac{\kappa\omega}{c} \hat{\mathbf{u}} \cdot \bar{\mathbf{r}}} + C.C. \right]. \quad (2.6.4)$$

The letters “C.C.” stand for the complex conjugate of what precedes. The direction of $\bar{\mathbf{K}}$ is specified with the real unit vector $\hat{\mathbf{u}}$. We have also used (2.3.17) to rewrite $i(\bar{\mathbf{K}} - \bar{\mathbf{K}}^*)$ as $-2(\kappa\omega/c)\hat{\mathbf{u}}$.

In an isotropic medium (not a crystal) we have from Maxwell’s equations the requirement $\bar{\nabla} \cdot \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) = 0$ (see (1.6.1)), or in other words $\hat{\mathbf{u}} \cdot \bar{\mathbf{E}}_o = 0$. We can use this together with the BAC-CAB rule P0.3.6 to replace the above expression with

$$\bar{\mathbf{S}} = \frac{\hat{\mathbf{u}}}{4\mu_o} \left[\frac{\mathcal{K}}{\omega} (\bar{\mathbf{E}}_o \cdot \bar{\mathbf{E}}_o) e^{2i(\bar{\mathbf{K}} \cdot \bar{\mathbf{r}} - \omega t)} + \frac{\mathcal{K}}{\omega} (\bar{\mathbf{E}}_o \cdot \bar{\mathbf{E}}_o^*) e^{-2\frac{\kappa\omega}{c} \hat{\mathbf{u}} \cdot \bar{\mathbf{r}}} + C.C. \right]. \quad (2.6.5)$$

This expression shows that in an isotropic medium the flow of energy is in the direction of $\hat{\mathbf{u}}$ (or $\bar{\mathbf{K}}$). This agrees with our intuition that energy flows in the direction that the wave propagates.

Very often, we are interested in the time-average of the Poynting vector, denoted by $\langle \bar{\mathbf{S}} \rangle_t$. Under the time averaging, the first term in (2.6.5) vanishes since it oscillates positive and negative by the same amount. Note that \mathcal{K} is the only factor in the second term that is not real. The time-averaged Poynting vector becomes

$$\langle \bar{\mathbf{S}} \rangle_t = \frac{\hat{\mathbf{u}}}{4\mu_o} \frac{\mathcal{K} + \mathcal{K}^*}{\omega} (\bar{\mathbf{E}}_o \cdot \bar{\mathbf{E}}_o^*) e^{-2\frac{\kappa\omega}{c} \hat{\mathbf{u}} \cdot \bar{\mathbf{r}}} = \hat{\mathbf{u}} \frac{n\epsilon_o c}{2} \left(|E_{ox}|^2 + |E_{oy}|^2 + |E_{oz}|^2 \right) e^{-2\frac{\kappa\omega}{c} \hat{\mathbf{u}} \cdot \bar{\mathbf{r}}}. \quad (2.6.6)$$

We have used (2.3.17) to rewrite $\mathcal{K} + \mathcal{K}^*$ as $2(n\omega/c)$. We have also used (1.8.4) to rewrite $1/\mu_o c$ as $\epsilon_o c$.

The expression (2.6.6) is called irradiance (with the direction $\hat{\mathbf{u}}$ included). However, we often speak of the intensity of a field I , which amounts to the same thing, but without regard for the direction $\hat{\mathbf{u}}$. The definition of intensity is thus less specific, and it can be applied, for example, to standing waves where the net irradiance is technically zero (i.e. counter-propagating plane waves with zero net energy flow). Nevertheless, atoms in standing waves ‘feel’ the oscillating field. In general, the intensity is written as

$$I = \frac{n\epsilon_o c}{2} \bar{\mathbf{E}}_o \cdot \bar{\mathbf{E}}_o^* = \frac{n\epsilon_o c}{2} \left(|E_{ox}|^2 + |E_{oy}|^2 + |E_{oz}|^2 \right), \quad (2.6.7)$$

where in this case we have ignored absorption (i.e. $\kappa \equiv 0$), or, alternatively, we could have considered $|E_{ox}|^2$, $|E_{oy}|^2$, and $|E_{oz}|^2$ to possess the factor $\exp\{-2(\kappa\omega/c)\hat{\mathbf{u}} \cdot \bar{\mathbf{r}}\}$ already.

Exercises

P2.6.1 In the case of a linearly-polarized plane wave, where the phase of each vector component of $\bar{\mathbf{E}}_o$ is the same, re-derive (2.6.6) directly from the real field (2.3.19). For simplicity, you may ignore absorption (i.e. $\kappa \equiv 0$). HINT: The time-average of $\cos^2(\bar{\mathbf{k}} \cdot \bar{\mathbf{r}} - \omega t + \phi)$ is $1/2$.

P2.6.2 (a) Find the intensity (in W/cm^2) produced by a short laser pulse (linearly polarized) with duration $\Delta t = 2.5 \times 10^{-14} \text{ s}$ and energy $\mathcal{E} = 100 \text{ mJ}$, focused in vacuum to a round spot with radius $r = 5 \mu\text{m}$.

(b) What is the peak electric field (in $\text{V}/\text{\AA}$)? HINT: The units of electric field are $\text{N}/\text{C} = \text{V}/\text{m}$.

(c) What is the peak magnetic field (in $\text{T} = \text{kg}/(\text{s} \cdot \text{C})$)?

P2.6.3 What is the intensity (in W/cm^2) on the retina when looking directly at the sun? Assume that the eye's pupil has a radius $r_{\text{pupil}} = 1 \text{ mm}$. Take the Sun's irradiance at the earth's surface to be $1.4 \text{ kW}/\text{m}^2$, and neglect refractive index (i.e. set $n = 1$). HINT: The Earth-Sun distance is $d_o = 1.5 \times 10^8 \text{ km}$ and the pupil-retina distance is $d_i = 22 \text{ mm}$. The radius of the Sun $r_{\text{sun}} = 7.0 \times 10^5 \text{ km}$ is de-magnified on the retina according to the ratio d_i/d_o .

P2.6.4 What is the intensity at the retina when looking directly into a 1 mW HeNe laser? Assume that the smallest radius of the laser beam is $r_{\text{beam waist}} = 0.5 \text{ mm}$ positioned $d_o = 2 \text{ m}$ in

front of the eye, and that the entire beam enters the pupil. Compare with P2.6.3 (see HINT).

P2.6.5 Show that the magnetic field of an intense laser pulse becomes important for a free electron oscillating in the field at intensities above $10^{18} \text{ W}/\text{cm}^2$. This marks the transition to relativistic physics. Nevertheless, for convenience, use classical physics in making the estimate. HINT: At lower intensities, the oscillating electric field dominates, so the electron motion can be thought of as arising solely from the electric field. Use this motion to calculate the magnetic force on the moving electron, and compare it to the electric force. The forces become comparable at $10^{18} \text{ W}/\text{cm}^2$.

Appendix 2.A Energy Density of Electric Fields

In this appendix and the next, we prove that the term $\epsilon_o E^2/2$ in (2.5.5) corresponds to the energy density of an electric field. The electric potential $\phi(\vec{r})$ (in units of energy per charge, or in other words volts) describes each point of an electric field in terms of the potential energy that a charge would experience if placed in that field. The electric field and the potential are connected through

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}). \quad (2.A.1)$$

The energy U necessary to assemble a distribution of charges (owing to attraction or repulsion) can be written in terms of a summation over all of the charges (or charge density $\rho(\vec{r})$) located within the potential:

$$U = \frac{1}{2} \int_V \phi(\vec{r}) \rho(\vec{r}) dV. \quad (2.A.2)$$

The factor $1/2$ is necessary to avoid double counting. To appreciate this factor consider two charges: We need only count the energy due to one charge in the presence of the other's potential to obtain the energy required to bring the charges together.

A substitution of (1.2.5) for $\rho(\vec{r})$ into (2.A.2) gives

$$U = \frac{\epsilon_o}{2} \int_V \phi(\vec{r}) \bar{\nabla} \cdot \vec{E}(\vec{r}) d\nu. \quad (2.A.3)$$

Next, we use the vector identity P0.3.13 and get

$$U = \frac{\epsilon_o}{2} \int_V \bar{\nabla} \cdot [\phi(\vec{r}) \vec{E}(\vec{r})] d\nu - \frac{\epsilon_o}{2} \int_V \vec{E}(\vec{r}) \cdot \bar{\nabla} \phi(\vec{r}) d\nu. \quad (2.A.4)$$

An application of the Divergence theorem (0.3.8) on the first integral and a substitution of (2.A.1) into the second integral yields

$$U = \frac{\epsilon_o}{2} \oint_S \phi(\vec{r}) \vec{E}(\vec{r}) \cdot \hat{n} da + \frac{\epsilon_o}{2} \int_V \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r}) d\nu. \quad (2.A.5)$$

Finally, we consider the volume V (enclosed by S) to be extremely large so that all charges are contained well within it. If we choose a large enough volume, say a sphere of radius R , the surface integral over S vanishes. The integrand of the surface integral becomes negligibly small $\phi \sim 1/R$ and $E \sim 1/R^2$, whereas $da \sim R^2$. Therefore, the energy associated with an electric field in a region of space is

$$U = \int_V u_E(\vec{r}) d\nu, \text{ where} \quad (2.A.6)$$

$$u_E(\vec{r}) \equiv \frac{\epsilon_o E^2}{2} \quad (2.A.7)$$

is interpreted as the energy density of the electric field.

Appendix 2.B Energy Density of Magnetic Fields

In a derivation similar to that in appendix 2.A, we consider the energy associated with magnetic fields. The magnetic vector potential $\vec{A}(\vec{r})$ (in units of energy per charge \times velocity) describes the potential energy that a charge moving with velocity \vec{v} would experience if placed in the field. The magnetic field and the vector potential are connected through

$$\vec{B}(\vec{r}) = \bar{\nabla} \times \vec{A}(\vec{r}). \quad (2.B.1)$$

The energy U necessary to assemble a distribution of current can be written in terms of a summation over all of the currents (or current density $\vec{J}(\vec{r})$) located within the vector potential field:

$$U = \frac{1}{2} \int_V \vec{J}(\vec{r}) \cdot \vec{A}(\vec{r}) d\nu. \quad (2.B.2)$$

As in (2.A.2), the factor $1/2$ is necessary to avoid double counting the influence of the currents on each other.

Under the assumption of steady currents (no variations in time), we may substitute Ampere's law (1.3.8) into (2.B.2), which yields

$$U = \frac{1}{2\mu_o} \int_V [\vec{\nabla} \times \vec{B}(\vec{r})] \cdot \vec{A}(\vec{r}) d\nu. \quad (2.B.3)$$

Next we employ the vector identity P0.3.12 from which the previous expression becomes

$$U = \frac{1}{2\mu_o} \int_V \vec{B}(\vec{r}) \cdot [\vec{\nabla} \times \vec{A}(\vec{r})] d\nu - \frac{1}{2\mu_o} \int_V \vec{\nabla} \cdot [\vec{A}(\vec{r}) \times \vec{B}(\vec{r})] d\nu. \quad (2.B.4)$$

Upon substituting (2.B.1) into the first equation and applying the Divergence theorem (0.3.8) on the second integral, this expression for total energy becomes

$$U = \frac{1}{2\mu_o} \int_V \vec{B}(\vec{r}) \cdot \vec{B}(\vec{r}) d\nu - \frac{1}{2\mu_o} \oint_S [\vec{A}(\vec{r}) \times \vec{B}(\vec{r})] \cdot \hat{n} da. \quad (2.B.5)$$

As was done in connection with (2.A.5), if we choose a large enough volume (a sphere with radius R), the surface integral vanishes because $A \sim 1/R$ and $B \sim 1/R^2$, whereas $da \sim R^2$. The total energy (2.B.5) then reduces to

$$U = \int_V u_B(\vec{r}) d\nu, \text{ where} \quad (2.B.6)$$

$$u_B(\vec{r}) \equiv \frac{B^2}{2\mu_o} \quad (2.B.7)$$

is the energy density for a magnetic field.

Appendix 2.C Radiometry Versus Photometry

Photometry refers to the characterization of light sources in the context of the spectral response of the human eye. However, physicists most often deal with *radiometry*, which treats light of any wavelength on equal footing. Table 2.2 lists several concepts important in radiometry. The last two entries are associated with the average Poynting flux described in section 2.6.

The concepts used in photometry are similar, except that the radiometric quantities are multiplied by the spectral response of the human eye, a curve that peaks at $\lambda_{\text{vac}} = 555 \text{ nm}$ and drops to near zero for wavelengths longer than $\lambda_{\text{vac}} = 700 \text{ nm}$ or shorter than $\lambda_{\text{vac}} = 400 \text{ nm}$. Photometric units, which may seem a little obscure, were first defined in terms of an actual candle with prescribed dimensions made from whale tallow. The basic unit of luminous power is called the lumen, defined to be $(1/683) \text{ W}$ of light with wavelength $\lambda_{\text{vac}} = 555 \text{ nm}$, the peak of the eye's response. More radiant power is required to achieve the same number of lumens for wavelengths away from the center of the eye's spectral response. Photometric units are often used to characterize room lighting as well as photographic, projection, and display equipment. Table 2.3 gives the names of the various photometric quantities, which parallel the entries in table 2.2. We include a variety of units that are sometimes encountered.

Name	Concept	Typical Units
Radiant Power (of a source)	Electromagnetic energy emitted per time from a source	$W = J/s$
Radiant Solid-Angle Intensity (of a source)	Radiant power per steradian emitted from a point-like source (4π steradians in a sphere)	W/Sr
Radiance or Brightness (of a source)	Radiant solid-angle intensity per unit (projected) area of an extended source. (The <i>projected</i> area foreshortens by $\cos\theta$, where θ is the observation angle relative to the surface normal.)	$\frac{W}{Sr \cdot cm^2}$
Radiant Emittance or Exitance (from a source)	Radiant Power emitted per unit surface area of an extended source: Poynting flux leaving.	W/cm^2
Irradiance (to a receiver) * Often called intensity	Electromagnetic power delivered per area to a receiver: Poynting flux arriving.	W/cm^2

Table 2.2 Radiometric quantities and units.

Name	Concept	Typical Units
Luminous Power (of a source)	Visible light energy emitted per time from a source: lumen (lm).	$lm = (1/683)W$ @ 555 nm
Luminous Solid-Angle Intensity (of a source)	Luminous power per steradian emitted from a point-like source: candela (cd).	$cd = lm/Sr$
Luminance (of a source)	Luminous solid-angle intensity per (projected) area of an extended source. (The <i>projected</i> area foreshortens by $\cos\theta$, where θ is the observation angle relative to the surface normal.)	$cd/cm^2 = stilb$ $cd/m^2 = nit$ $= 3183 \text{ lamberts}$ $= 3.426 \text{ footlamberts}$
Luminous Emittance or Exitance (from a source)	Luminous Power emitted per unit surface area of an extended source	lm/cm^2
Illuminance (to a receiver)	Incident luminous power delivered per area to a receiver: lux .	$lm/m^2 = lux$ $lm/cm^2 = phot$ $lm/ft^2 = \text{footcandle}$

Table 2.3 Photometric quantities and units.