Chapter 0

Mathematical Tools

0.1 Introduction

This chapter reviews a variety of mathematical tools that are important when studying light and optics. Students may be expected to bring many of these tools to bear on homework and exam problems without difficulty. This chapter is not intended to be a comprehensive review of all necessary mathematical principles. For example, a basic understanding of differentiation, integration, and standard trigonometric and algebraic manipulation is assumed. Section 0.2 is a review of complex arithmetic, and students *need to know this material by heart.* Section 0.3 is an overview of vector calculus and related theorems, which are used extensively in electromagnetic theory. It is not essential to be well versed in all of the material presented in section 0.3 (since it is only occasionally needed in homework problems). Nevertheless, vector calculus is invoked frequently throughout this book, as electromagnetic principles are linked to optical phenomena. This connection is made as much as possible to help students appreciate the connection between the two topics. Section 0.4 is an introduction to Fourier theory. Fourier transforms are used extensively in this course beginning with chapter 7. The presentation below is sufficiently comprehensive for the student who encounters Fourier transforms here for the first time, and such a student is strongly advised to study this section before starting chapter 7. Please note that a short table of formulas (mostly integrals) is given in Appendix 0.B, which are useful for various problems encountered in the text.

0.2 Complex Numbers

In optics, it is often convenient to represent electromagnetic wave phenomena as a superposition of sinusoidal functions having the form $A\cos(x+\phi)$, where x represents a variable, and A and ϕ represent parameters. We see that the sine is intrinsically present in this function through the trigonometric identity

$$\cos(x+\phi) = \cos x \cos\phi - \sin x \sin\phi. \tag{0.2.1}$$

The student of optics should retain this formula in memory as well as the frequently used identity

$$\sin(x+\phi) = \sin x \cos\phi + \sin\phi \cos x\,. \tag{0.2.2}$$

With a basic familiarity with trigonometry, one is prepared to approach many optical problems including those involving the addition of multiple waves. Nevertheless, the manipulation of trigonometric functions via identities (0.2.1) and (0.2.2) can be rather cumbersome and tedious. Fortunately, complex notation offers an equivalent approach with far less busy work. The convenience of complex notation has its origins in Euler's formula:

$$\exp(i\theta) = \cos\theta + i\sin\theta, \tag{0.2.3}$$

where $i = \sqrt{-1}$. Euler's formula can be proven using Taylor's expansion:

$$f(x) = f(x_0) + \frac{1}{1!}(x - x_0)\left.\frac{df}{dx}\right|_{x=x_0} + \frac{1}{2!}(x - x_0)^2\left.\frac{d^2 f}{dx^2}\right|_{x=x_0} + \frac{1}{3!}(x - x_0)^3\left.\frac{d^3 f}{dx^3}\right|_{x=x_0} + \ldots \tag{0.2.4}$$

Let each function appearing in (0.2.3) be expanded in a Taylor's series about the origin:

$$\begin{aligned}
\exp(i\theta) &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots, \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots, \\
i\sin\theta &= i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots
\end{aligned} \tag{0.2.5}$$

The first line is seen to be the sum of the other two lines, from which Euler's formula directly follows.

It is left as an exercise to show that the inversion of Euler's formula (0.2.3) yields

$$\begin{aligned}
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\end{aligned} \tag{0.2.6}$$

This representation of the cosine and sine shows that they are intimately related to the hyperbolic cosine and hyperbolic sine. If θ happens to be imaginary such that $\theta = i\phi$ where ϕ is real, then we have

$$\begin{aligned}
\sin i\phi &= \frac{e^{-\phi} - e^{\phi}}{2i} = i\sinh\phi, \\
\cos i\phi &= \frac{e^{-\phi} + e^{\phi}}{2} = \cosh\phi.
\end{aligned} \tag{0.2.7}$$

There are several situations in optics where one is interested in a complex angle, $\theta = \alpha + i\beta$ where α and β are real numbers. An example of this is in the solution to the wave equation when absorption or amplification takes place. In this case, the imaginary part of θ introduces exponential decay or growth as is apparent upon examination of (0.2.6). Another important situation occurs when one attempts to calculate the transmission angle for light incident upon a surface beyond the critical angle for total internal reflection. In this case, it is necessary to compute the arcsine of a number greater than one in an effort to satisfy Snell's law. Even though such an angle does not exist in the usual sense, a complex value for θ can be found which satisfies (0.2.6). The complex value for the angle is useful in computing the characteristics of the evanescent wave on the transmitted side of the surface.

One can avoid the use of complex notation in the study of optics, and this may seem appealing to the student who is unfamiliar with its use. Such a student might opt to pursue all problems using sines, cosines, and real exponents, together with large quantities of trigonometric identities. This, however, would be far more effort than the modest investment needed to become comfortable with the use of complex notation. Optics problems can become cumbersome enough even with the complex notation, so keep in mind that it could be far more messy!

As was mentioned previously, we will be interested in waves having the form $A\cos(x+\phi)$. Complex notation is used to represent this wave simply by writing

$$A\cos(x+\phi) = \mathrm{Re}\left\{\tilde{A}e^{ix}\right\}, \tag{0.2.8}$$

where the phase ϕ is conveniently contained within the complex factor $\tilde{A} \equiv Ae^{i\phi}$. The operation $\mathrm{Re}\{\ \}$ means to retain only the real part of the argument without regard for the imaginary part. As an example, we have $\mathrm{Re}\{1+2i\} = 1$. The expression (0.2.8) is an obvious result of Euler's equation (0.2.3).

It is conventional in the study of optics to omit the explicit writing of $\mathrm{Re}\{\ \}$. Thus, physicists agree that $\tilde{A}e^{ix}$ actually means $A\cos(x+\phi)$ (or $A\cos\phi\cos x - A\sin\phi\sin x$ via (0.2.1)). This laziness is permissible because it is possible to perform linear operations on $\mathrm{Re}\{f\}$ such as addition, differentiation, or integration while procrastinating the taking of the real part until the end. Notice that

$$\mathrm{Re}\{f\} + \mathrm{Re}\{g\} = \mathrm{Re}\{f+g\},\ \frac{d}{dx}\mathrm{Re}\{f\} = \mathrm{Re}\left\{\frac{df}{dx}\right\}, \text{and} \int \mathrm{Re}\{f\}dx = \mathrm{Re}\left\{\int f dx\right\}. \tag{0.2.9}$$

As an example, we have $\mathrm{Re}\{1+2i\} + \mathrm{Re}\{3+4i\} = \mathrm{Re}\{(1+2i)+(3+4i)\} = 4$. Nevertheless, one must be careful when performing other operations such as multiplication. In this case, it is essential to take the real parts before performing the operation. Notice that

$$\mathrm{Re}\{f\} \times \mathrm{Re}\{g\} \neq \mathrm{Re}\{f \times g\}. \tag{0.2.10}$$

As an example, we see $\mathrm{Re}\{1+2i\} \times \mathrm{Re}\{3+4i\} = 3$, but $\mathrm{Re}\{(1+2i)(3+4i)\} = -5$.

When dealing with complex numbers it is often advantageous to transform between a Cartesian representation and a polar representation. With the aid of Euler's formula, it is possible to transform any complex number $a+ib$ into the form $\rho e^{i\theta}$, where a, b, ρ, and θ are real. From (0.2.3), the required connection between (ρ,θ) and (a,b) is

$$\rho e^{i\theta} = \rho\cos\theta + i\rho\sin\theta = a+ib. \tag{0.2.11}$$

The real and imaginary parts of this equation must separately be equal. Thus, we have

$$\begin{aligned} a &= \rho\cos\theta, \\ b &= \rho\sin\theta. \end{aligned} \tag{0.2.12}$$

These equations can be inverted to yield

$$\rho = \sqrt{a^2 + b^2}\ ,$$
$$\theta = \tan^{-1}\frac{b}{a} \quad (a > 0)\ . \tag{0.2.13}$$

When $a < 0$, we must adjust θ by π since the arctangent has a range only from $-\pi/2$ to $\pi/2$.

The transformations in (0.2.12) and (0.2.13) have a clear geometrical interpretation in the complex plane, and this makes it easier to remember them. They are just the usual connections between Cartesian and polar coordinates. As seen in Fig. 0.1, ρ is the hypotenuse of a right triangle having legs with lengths a and b, and θ is the angle that the hypotenuse makes with the x-axis. Again, students should be careful when a is negative since the arctangent is defined in quadrants I and IV. An easy way to deal with the situation of a negative a is to factor the minus sign out before proceeding (i.e. $a + ib = -(-a - ib)$). Then the transformation is made on $-a - ib$ where $-a$ is positive. The minus sign out in front is just carried along unaffected and can be factored back in at the end. Notice that $-\rho e^{i\theta}$ is the same as $\rho e^{i(\theta \pm \pi)}$.

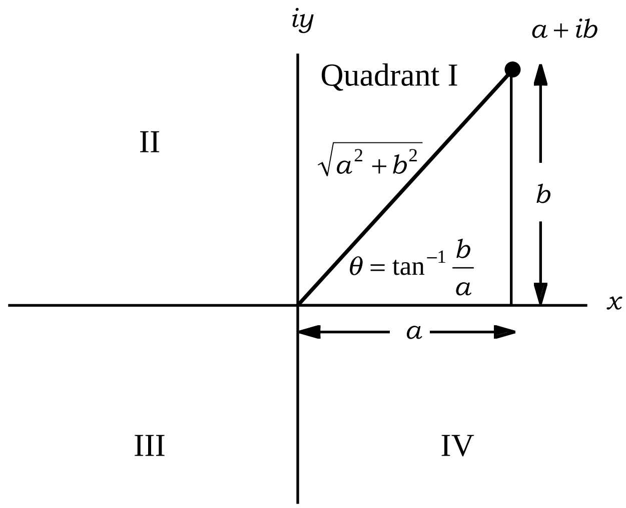

Fig. 0.1. A number in the complex plane can be represented either by Cartesian or polar coordinates.

Finally, we consider the concept of a complex conjugate. The conjugate of a complex number $z = a + ib$ is denoted with an asterisk and amounts to changing the sign on the imaginary part of the number:

$$z^* = (a + ib)^* \equiv a - ib. \tag{0.2.14}$$

The complex conjugate is useful when computing the magnitude ρ as defined in (0.2.13):

$$|z| = \sqrt{z^* z} = \sqrt{(a - ib)(a + ib)} = \sqrt{a^2 + b^2} = \rho. \tag{0.2.15}$$

The complex conjugate is also useful for eliminating complex numbers from the denominator of expressions:

$$\frac{a + ib}{c + id} = \frac{(a + ib)}{(c + id)}\frac{(c - id)}{(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2}. \tag{0.2.16}$$

No matter how complicated an expression, the complex conjugate is taken simply by inserting a minus sign in front of all occurrences of i in the expression and by placing an asterisk on all complex variables in the expression. For example, the complex conjugate of $\rho e^{i\theta}$ is $\rho e^{-i\theta}$, as can be easily seen from Euler's formula (0.2.3). As another example consider $\left[E \exp\{i(\mathcal{K}z - \omega t)\}\right]^*$ $= E^* \exp\left\{-i\left(\mathcal{K}^* z - \omega t\right)\right\}$, assuming z, ω, and t are real, but E and $\mathcal{K}$ are complex.

A common way of obtaining the real part of an expression is simply by adding the complex conjugate and dividing the result by 2:

$$\mathrm{Re}\{z\} = \frac{1}{2}\left(z + z^*\right). \tag{0.2.17}$$

Notice that the expression for $\cos\theta$ in (0.2.6) is an example of this formula. Sometimes when a complicated expression is added to its complex conjugate, we let 'C.C.' represent the complex conjugate in order to avoid writing the expression twice.

Exercises

P0.2.1 Show that $-3+4i$ can be written as $5\exp\left\{-i\tan^{-1} 4/3 + i\pi\right\}$.

P0.2.2 Show $(a-ib)/(a+ib) = \exp\left\{-2i\tan^{-1} b/a\right\}$ regardless of the sign of a, assuming a and b are real.

P0.2.3 Invert (0.2.3) to get both formulas in (0.2.6).

P0.2.4 Show $\mathrm{Re}\{A\} \times \mathrm{Re}\{B\} = \left(AB + A^*B\right)/4 + C.C.$

P0.2.5 If $E = |E|e^{i\phi_E}$ and $B = |B|e^{i\phi_B}$, and if k, z, ω, and t are all real, prove

$$\mathrm{Re}\left\{Ee^{i(kz-\omega t)}\right\} \times \mathrm{Re}\left\{Be^{i(kz-\omega t)}\right\} = \frac{E^* \times B + E \times B^* + 2|E| \times |B|\cos\left[2(kz-\omega t) + \phi_E + \phi_B\right]}{4}.$$

P0.2.6 If $\sin\theta = 2$, show that $\cos\theta = i\sqrt{3}$. HINT: Use $\sin^2\theta + \cos^2\theta = 1$.

P0.2.7 Show that the angle θ in P0.2.6 is $\pi/2 - i\ln\left(2+\sqrt{3}\right)$.

0.3 Vector Calculus

The many theorems given in this section appear in proofs of various principles throughout this book. They are less often employed in homework problems (except for basic vector multiplication and differentiation). We will be concerned primarily with electromagnetic fields that are defined throughout space. Each position in space corresponds to a unique vector $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors of length one, pointing respectively along the major axes. Electric and magnetic fields are vectors and depend on position as denoted by $\vec{E}(\vec{r})$ or $\vec{B}(\vec{r})$. An example of such a field is $\vec{E}(\vec{r}) = q(\vec{r} - \vec{r}_o)/4\pi\varepsilon_o|\vec{r} - \vec{r}_o|^3$, which is the static electric field surrounding a point charge located at position $\vec{r}_o$. The absolute value brackets indicate the magnitude (length) of the vector given by

$$|\vec{r} - \vec{r}_o| = |(x - x_o)\hat{x} + (y - y_o)\hat{y} + (z - z_o)\hat{z}| = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}. \tag{0.3.1}$$

An example of a time-dependent field common in optics is $\vec{E}(\vec{r},t) = \vec{E}_o \exp\left\{i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right\}$, where, as discussed above, physicists have the agreement in advance that only the real part of this expression corresponds to the actual field. The dot product in this expression signifies the following operation:

$$\vec{k}\cdot\vec{r} = \left(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right)\cdot\left(x\hat{x} + y\hat{y} + z\hat{z}\right) = k_x x + k_y y + k_z z. \tag{0.3.2}$$

The dot product results in a scalar, which is the product of the vector magnitudes times the cosine of the angle between them (see P0.3.4).

Another type of vector multiplication is the cross product, which is accomplished in the following manner:

$$\vec{E}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & E_z \\ B_x & B_y & B_z \end{vmatrix} = \left(E_y B_z - E_z B_y\right)\hat{x} - \left(E_x B_z - E_z B_x\right)\hat{y} + \left(E_x B_y - E_y B_x\right)\hat{z}. \tag{0.3.3}$$

At this point, we introduce the multidimensional derivative called the gradient:

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}. \tag{0.3.4}$$

In addition we have the divergence

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}, \tag{0.3.5}$$

and the curl

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}. \tag{0.3.6}$$

The Laplacian of a function $f(x,y,z)$ is given by

$$\nabla^2 f(x,y,z) \equiv \vec{\nabla}\cdot\vec{\nabla} f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{0.3.7}$$

Finally, we mention several integral theorems involving vector functions. The first is the divergence theorem:

$$\oint_S \vec{f}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{f}\,dv. \tag{0.3.8}$$

The integration on the left-hand side is over the closed surface S, which contains the volume V associated with the integration on the right hand side. The unit vector $\hat{n}$ points normal to the surface. The divergence theorem, which is a general mathematical relationship, is especially useful in connection with Gauss' law. The left hand side is interpreted as the number of field lines exiting a closed surface.

Another important theorem is Stokes' theorem:

$$\int_S \vec{\nabla} \times \vec{f} \cdot \hat{n} da = \oint_C \vec{f} \cdot d\vec{\ell}. \tag{0.3.9}$$

The integration on the left hand side is over an open surface S (not enclosing a volume). The integration on the right hand side is around the edge of the surface. Again, $\hat{n}$ is a unit vector that always points normal to the surface. The vector $d\vec{\ell}$ points along the curve C containing the area. If the fingers of your right hand point in the direction of integration around C, then your thumb points in the direction of $\hat{n}$. Stokes' theorem, which is a general mathematical relationship, is especially useful in connection with Ampere's law and Faraday's law. The right-hand side is an integration of a field around a loop.

The following vector integral theorem is also of interest:

$$\int_V \left[\vec{f}(\nabla \cdot \vec{g}) + (\vec{g} \cdot \nabla)\vec{f}\right] dv = \oint_S \vec{f}(\vec{g} \cdot \hat{n}) da. \tag{0.3.10}$$

Exercises

P0.3.1 Find the magnitude of $\vec{r} = (\hat{x} + 2\hat{y} - 3\hat{z})\text{m}$. ANSWER: $r = \sqrt{14}\text{m}$.

P0.3.2 Find $\vec{r} - \vec{r}_o$ where $\vec{r}_o = (-\hat{x} + 3\hat{y} + 2\hat{z})\text{m}$ and $\vec{r}$ is defined above.

P0.3.2 Find the angle between $\vec{r}$ and $\vec{r}_o$ defined above. ANSWER: 94°.

P0.3.4 Prove that the dot product between two vectors is the product of the magnitudes of the two vectors multiplied by the cosine of the angle between them.

SOLUTION: Consider the plane containing the two vectors in (0.3.2). Call it the xy-plane. In this coordinate system, the two vectors can be written as $\vec{k} = k\cos\theta\hat{x} + k\sin\theta\hat{y}$ and $\vec{r} = r\cos\phi\hat{x} + r\sin\phi\hat{y}$, where θ and ϕ are the respective angles that the two vectors make with the x-axis. The dot product gives $\vec{k} \cdot \vec{r} = kr(\cos\theta\cos\phi + \sin\theta\sin\phi)$. From (0.2.1) we have $\vec{k} \cdot \vec{r} = kr\cos(\theta - \phi)$. Obviously $\theta - \phi$ is the angle between the vectors.

P0.3.5 Prove that the cross product between two vectors is the product of the magnitudes of the two vectors multiplied by the sine of the angle between them. The result is a vector directed perpendicular to the plane containing the original two vectors in accordance with the right hand rule.

P0.3.6 Verify the "BAC-CAB" rule: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$.

P0.3.7 Prove $\vec{\nabla}_{\vec{r}} \dfrac{1}{|\vec{r} - \vec{r}'|} = -\dfrac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$, where $\vec{\nabla}_{\vec{r}}$ operates only on $\vec{r}$, treating $\vec{r}'$ as a constant vector.

P0.3.8 Prove $\vec{\nabla}_{\vec{r}} \cdot \dfrac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$ is zero, except at $\vec{r} = \vec{r}'$ where a singularity situation occurs.

P0.3.9 Verify $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0$ for any vector function $\vec{f}$.

P0.3.10 Verify $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{f}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{f}\right)-\nabla^2\vec{f}$.

SOLUTION: From (0.3.6), we have

$$\vec{\nabla}\times\vec{f}=\left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right)\hat{x}-\left(\frac{\partial f_z}{\partial x}-\frac{\partial f_x}{\partial z}\right)\hat{y}+\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)\hat{z} \text{ and}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{f}\right)=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right) & -\left(\frac{\partial f_z}{\partial x}-\frac{\partial f_x}{\partial z}\right) & \left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)\end{vmatrix}=\left[\frac{\partial}{\partial y}\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f_z}{\partial x}-\frac{\partial f_x}{\partial z}\right)\right]\hat{x}$$

$$-\left[\frac{\partial}{\partial x}\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right)\right]\hat{y}+\left[-\frac{\partial}{\partial x}\left(\frac{\partial f_z}{\partial x}-\frac{\partial f_x}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right)\right]\hat{z}.$$

After rearranging, we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{f}\right)=\left[\frac{\partial^2 f_x}{\partial x^2}+\frac{\partial^2 f_y}{\partial x\partial y}+\frac{\partial^2 f_z}{\partial x\partial z}\right]\hat{x}+\left[\frac{\partial^2 f_x}{\partial x\partial y}+\frac{\partial^2 f_y}{\partial y^2}+\frac{\partial^2 f_z}{\partial y\partial z}\right]\hat{y}+\left[\frac{\partial^2 f_x}{\partial x\partial z}+\frac{\partial^2 f_y}{\partial y\partial z}+\frac{\partial^2 f_z}{\partial z^2}\right]\hat{z}$$

$$-\left[\frac{\partial^2 f_x}{\partial x^2}+\frac{\partial^2 f_x}{\partial y^2}+\frac{\partial^2 f_x}{\partial z^2}\right]\hat{x}-\left[\frac{\partial^2 f_y}{\partial x^2}+\frac{\partial^2 f_y}{\partial y^2}+\frac{\partial^2 f_y}{\partial z^2}\right]\hat{y}-\left[\frac{\partial^2 f_z}{\partial x^2}+\frac{\partial^2 f_z}{\partial y^2}+\frac{\partial^2 f_z}{\partial z^2}\right]\hat{z},$$

where we have added and subtracted $\frac{\partial^2 f_x}{\partial x^2}+\frac{\partial^2 f_y}{\partial y^2}+\frac{\partial^2 f_z}{\partial z^2}$.

After some factorization, we obtain

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{f}\right)=\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\left[\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right]-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\left[f_x\hat{x}+f_y\hat{y}+f_z\hat{z}\right].$$

$$=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{f}\right)-\nabla^2\vec{f}.$$

where on the final line we invoked (0.3.4), (0.3.5), and (0.3.7).

P0.3.11 Verify $\vec{\nabla}\times\left(\vec{f}\times\vec{g}\right)=\vec{f}\left(\vec{\nabla}\cdot\vec{g}\right)-\vec{g}\left(\vec{\nabla}\cdot\vec{f}\right)+\left(\vec{g}\cdot\vec{\nabla}\right)\vec{f}-\left(\vec{f}\cdot\vec{\nabla}\right)\vec{g}$.

P0.3.12 Verify $\vec{\nabla}\cdot\left(\vec{f}\times\vec{g}\right)=\vec{g}\cdot\left(\vec{\nabla}\times\vec{f}\right)-\vec{f}\cdot\left(\vec{\nabla}\times\vec{g}\right)$.

P0.3.13 Verify $\vec{\nabla}\cdot\left(g\vec{f}\right)=\vec{f}\cdot\vec{\nabla}g+g\vec{\nabla}\cdot\vec{f}$.

P0.3.14 Verify $\vec{\nabla}\times\left(g\vec{f}\right)=\left(\vec{\nabla}g\right)\times\vec{f}+g\vec{\nabla}\times\vec{f}$.

P0.3.15 Verify the divergence theorem (0.3.8) for $\vec{f}(x,y,z)=y^2\hat{x}+xy\hat{y}+x^2z\hat{z}$. Take as the volume a cube contained by the six planes $|x|=\pm1$, $|y|=\pm1$, and $|z|=\pm1$.

SOLUTION:

$$\oint_S \vec{f}\cdot\hat{n}da=\int_{-1}^{1}\int_{-1}^{1}dxdy\left(x^2z\right)_{z=1}-\int_{-1}^{1}\int_{-1}^{1}dxdy\left(x^2z\right)_{z=-1}+\int_{-1}^{1}\int_{-1}^{1}dxdz(xy)_{y=1}-\int_{-1}^{1}\int_{-1}^{1}dxdz(xy)_{y=-1}$$

$$+\int_{-1}^{1}\int_{-1}^{1}dydz\left(y^2\right)_{x=1}-\int_{-1}^{1}\int_{-1}^{1}dydz\left(y^2\right)_{x=-1}$$

$$=2\int_{-1}^{1}\int_{-1}^{1}dxdyx^2+2\int_{-1}^{1}\int_{-1}^{1}dxdzx=4\left.\frac{x^3}{3}\right|_{-1}^{1}+4\left.\frac{x^2}{2}\right|_{-1}^{1}=\frac{8}{3}.$$

$$\int_V \vec{\nabla}\cdot\vec{f}dv = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dxdydz\left[x+x^2\right] = 4\int_{-1}^{1} dx\left[x+x^2\right] = 4\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{3}.$$

P0.3.16 Verify Stokes' theorem (0.3.9) for the function given in P0.3.15 Take the surface to be a square in the xy- plane contained by $|x| = \pm 1$ and $|y| = \pm 1$.

P0.3.17 Use the divergence theorem to show that the function in P0.3.8 is 4π times the three-dimensional delta function.

SOLUTION: We have by the divergence theorem $\oint_S \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot\hat{n}da = \int_V \vec{\nabla}_{\vec{r}}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dv$. From P0.3.8, the argument in the integral on the right-hand side is zero except at $\vec{r}=\vec{r}'$. Therefore, if the volume V does not contain the point $\vec{r}=\vec{r}'$, then the result of both integrals must be zero. Let us construct a volume between an arbitrary surface S_1 containing $\vec{r}=\vec{r}'$ and S_2, the surface of a tiny sphere centered on $\vec{r}=\vec{r}'$. Since the point $\vec{r}=\vec{r}'$ is excluded by the tiny sphere, the result of either integral in the divergence theorem is still zero. However, we have on the tiny sphere

$\oint_{S_2}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot\hat{n}da = -\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{r_\varepsilon^2}\right)r_\varepsilon^2\sin\theta d\theta d\phi = -4\pi$. Therefore, for the outer surface S_1 (containing $\vec{r}=\vec{r}'$) we must have the equal and opposite result: $\oint_{S_1}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot\hat{n}da = 4\pi$. This implies $\int_V \vec{\nabla}_{\vec{r}}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dv = \begin{cases}4\pi & \text{if } V \text{ contains } \vec{r}' \\ 0 & \text{otherwise}\end{cases}$.

The argument of this integral exhibits the same characteristics as the delta function $\delta^3(\vec{r}'-\vec{r}) \equiv \delta(x'-x)\delta(y'-y)\delta(z'-z)$. Namely, $\int_V \delta^3(\vec{r}'-\vec{r})dv = \begin{cases}1 & \text{if } V \text{ contains } \vec{r}' \\ 0 & \text{otherwise}\end{cases}$.

Therefore, $\vec{\nabla}_{\vec{r}}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = 4\pi\delta^3(\vec{r}-\vec{r}')$. The delta function is defined below in (0.4.13).

0.4 Fourier Theory

Fourier analysis is an important part of optics. We often decompose complicated light fields into a superposition of pure sinusoidal waves. This enables us to consider one frequency at a time (important since, for example, the optical index is different for different frequencies). After determining how individual sine waves move through an optical system (say a piece of glass), we can reassemble the sinusoidal waves to see the effect of the system on the overall waveform. Fourier transforms are used for this purpose. In fact, it will be possible to work simultaneously with an infinitude of sinusoidal waves, where the frequencies comprising a light field are spread over a continuous range. Fourier transforms are also used in diffraction problems where a single frequency is associated with a superposition of many waves propagating in differing spatial directions. We begin with a derivation of the Fourier integral theorem.

A *periodic* function can be represented in terms of the sine and the cosine in the following manner:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\Delta\omega t) + b_n \sin(n\Delta\omega t). \tag{0.4.1}$$

This is called a Fourier expansion. It is similar in idea to a Taylor's series (0.2.4), which rewrites a function as a polynomial. In either case, the goal is to represent one function in terms of a linear combination of other functions (requiring a complete basis set).

The expansion (0.4.1) is possible even if $f(t)$ is complex (requiring a_n and b_n to be complex). By inspection, we see that all terms in (0.4.1) repeat with a maximum period of $2\pi/\Delta\omega$. Thus, the expansion is limited in its use to periodic functions. Apparently, the period of the function is such that $f(t) = f(t + 2\pi/\Delta\omega)$.

We can rewrite the sines and cosines in the above expansion using (0.2.6) as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{e^{in\Delta\omega t} + e^{-in\Delta\omega t}}{2} + b_n \frac{e^{in\Delta\omega t} - e^{-in\Delta\omega t}}{2i} = a_o + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\Delta\omega t} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-in\Delta\omega t}. \tag{0.4.2}$$

Thus, there is no difference between writing (0.4.1) or

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\Delta\omega t}, \tag{0.4.3}$$

where

$$c_{n<0} \equiv \frac{a_{-n} - ib_{-n}}{2},\ c_{n>0} \equiv \frac{a_n + ib_n}{2}, \text{ and } c_o \equiv a_o. \tag{0.4.4}$$

Notice that if $c_{-n} = c_n^*$ for all n, then $f(t)$ is real (i.e. real a_n and b_n); otherwise $f(t)$ is complex. The real parts of the c_n coefficients are connected with the cosine terms in (0.4.1), and the imaginary parts of the c_n coefficients are connected with the sine terms in (0.4.1).

Given a known function $f(t)$, we can compute the various coefficients c_n. There is a trick for doing this. We multiply both sides of (0.4.3) by $e^{im\Delta\omega t}$, where m is an integer, and integrate over the function period $2\pi/\Delta\omega$:

$$\begin{aligned} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(t) e^{im\Delta\omega t} dt &= \sum_{n=-\infty}^{\infty} c_n \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} e^{i(m-n)\Delta\omega t} dt = \sum_{n=-\infty}^{\infty} c_n \left[\frac{e^{i(m-n)\Delta\omega t}}{i(m-n)\Delta\omega} \right]_{-\pi/\Delta\omega}^{\pi/\Delta\omega} \\ &= \sum_{n=-\infty}^{\infty} \frac{2\pi c_n}{\Delta\omega} \left[\frac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{2i(m-n)\pi} \right] = \sum_{n=-\infty}^{\infty} \frac{2\pi c_n}{\Delta\omega} \frac{\sin[(m-n)\pi]}{(m-n)\pi}. \end{aligned} \tag{0.4.5}$$

The function $\sin[(m-n)\pi]/[(m-n)\pi]$ is equal to zero for all $n \neq m$, and it is equal to one when $n = m$ (to see this, use L'Hospital's rule on the zero-over-zero situation). Thus, only one term contributes to the summation in (0.4.5). We now have

$$c_m = \frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(t) e^{im\Delta\omega t} dt, \tag{0.4.6}$$

from which the coefficients c_n can be computed, given a function $f(t)$. (Note that m is a dummy index so we can change it back to n if we like.)

This completes the circle. If we know the function $f(t)$, we can find the coefficients c_n via (0.4.6), and, if we know the coefficients c_n, we can generate the function $f(t)$ via (0.4.3). If we are feeling a bit silly, we can combine these into a single identity:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(t) e^{in\Delta\omega t} dt \right] e^{-in\Delta\omega t}. \tag{0.4.7}$$

We start with a function $f(t)$ followed by a lot of computation and obtain the function back again! This is not as foolish as it first appears, as we will see later.

As mentioned above, we are restricted to functions $f(t)$ that are periodic over the interval $2\pi/\Delta\omega$. This is disappointing since many optical waveforms do not repeat (e.g. a single short laser pulse). Nevertheless, we can represent a function $f(t)$ that is not periodic if we let the period $2\pi/\Delta\omega$ become infinitely long. In other words, we can accommodate non-periodic functions if we take the limit as $\Delta\omega$ goes to zero so that the spacing of terms in the series becomes very fine:

$$f(t) = \frac{1}{2\pi} \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \left[e^{-in\Delta\omega t} \int_{-\infty}^{\infty} f(t') e^{in\Delta\omega t'} dt' \right] \Delta\omega. \tag{0.4.8}$$

At this point, a review of the definition of an integral is helpful. (Welcome to freshman calculus.) This short calculus review will help us to better understand the next step that we administer to (0.4.8). Recall that an integral is really a summation of rectangles under a curve with finely spaced steps:

$$\int_a^b g(\omega) d\omega \equiv \lim_{\Delta\omega\to 0} \sum_{n=0}^{\frac{b-a}{\Delta\omega}} g(a + n\Delta\omega)\Delta\omega = \lim_{\Delta\omega\to 0} \sum_{n=-\frac{b-a}{2\Delta\omega}}^{\frac{b-a}{2\Delta\omega}} g\left(\frac{a+b}{2} + n\Delta\omega \right) \Delta\omega. \tag{0.4.9}$$

The final expression has been manipulated so that the index ranges through both negative and positive numbers. If we set $a = -b$ and take the limit $b \to \infty$, then (0.4.9) becomes

$$\int_{-\infty}^{\infty} g(\omega) d\omega = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} g(n\Delta\omega)\Delta\omega. \tag{0.4.10}$$

This concludes our short review of calculus.

We can use (0.4.10) in connection with (0.4.8) (where $g(n\Delta\omega)$ represents everything in the square brackets). The result is the Fourier integral theorem:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \left[\int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt' \right] d\omega. \tag{0.4.11}$$

The piece in brackets is called the Fourier transform (except for a factor of $1/\sqrt{2\pi}$), and the rest of the operation is called the inverse Fourier transform. The Fourier integral theorem (0.4.11) is often written with the following (potentially confusing) notation:

$$f(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \text{ and } f(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega . \tag{0.4.12}$$

The transform and inverse transform are also sometimes written as $f(\omega) \equiv F\{f(t)\}$ and $f(t) \equiv F^{-1}\{f(\omega)\}$. Note that the functions $f(t)$ and $f(\omega)$ are entirely different, even taking on different units (e.g. the latter having extra units of per frequency). The two functions are distinguished by their arguments, which also have different units (e.g. time vs. frequency). Nevertheless, it is customary to use the same letter to denote either function since they form a transform pair. (Be aware that it is arbitrary which is called the transform and which is called the inverse transform. In other words, the signs in the exponents of (0.4.12) may be interchanged. The convention varies in published works. Also, the factor 2π may be placed on either the transform or the inverse transform, or divided onto each has been done here.)

As was previously mentioned, it would seem rather pointless to perform a Fourier transform on the function $f(t)$ followed by an inverse Fourier transform, just to end up with $f(t)$ again. Nevertheless, we are interested in this because we want to know the effect of an optical system on a waveform (represented by $f(t)$). It turns out that in many cases, the effect of the optical system can only be applied to $f(\omega)$ (if the effect is frequency dependent). Thus, we perform a Fourier transform on $f(t)$, then apply the frequency-dependent effect on $f(\omega)$, and finally perform an inverse Fourier transform on the result. The final function will be different from $f(t)$. Keep in mind that $f(\omega)$ is the continuous analog of the discrete coefficients c_n (or the a_n and b_n). The real part of $f(\omega)$ indicates the amplitudes of the cosine waves necessary to construct the function $f(t)$. The imaginary part of $f(\omega)$ indicates the amplitudes of the sine waves necessary to construct the function $f(t)$.

Finally, we note that a remarkable attribute of the delta function can be seen from the Fourier integral theorem. The delta function $\delta(t'-t)$ is defined indirectly through

$$f(t) = \int_{-\infty}^{\infty} f(t')\delta(t'-t)dt' . \tag{0.4.13}$$

Evidently $\delta(t'-t)$ is zero everywhere except at $t'=t$, since the result of the integration only pays attention to the value of $f(t')$ at that point. At $t'=t$, the delta function is infinite in such a way as to make the integral take on the value of the function $f(t)$. [One can consider $\delta(t'-t)dt'$ with $t'=t$ to be the dimensions of an infinitely tall and infinitely thin rectangle with an area unity.] After rearranging the order of integration, the Fourier integral theorem (0.4.11) can be written as

$$f(t) = \int_{-\infty}^{\infty} f(t') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega \right] dt' . \tag{0.4.14}$$

A comparison of (0.4.13) and (0.4.14) reveals the delta function to be a uniform superposition of all frequency components:

$$\delta(t'-t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(t'-t)}d\omega. \tag{0.4.15}$$

This representation of the delta function comes in handy when proving Parseval's theorem (see P0.4.8), which is used extensively in the study of light and optics.

Exercises

P0.4.1 Prove linear superposition of Fourier Transforms:
$F\{ag(t)+bh(t)\}=ag(\omega)+bh(\omega)$, where $g(\omega)\equiv F\{g(t)\}$ and $h(\omega)\equiv F\{h(t)\}$.

P0.4.2 Prove $F\{g(at)\}=\frac{1}{|a|}g\left(\frac{\omega}{a}\right)$.

P0.4.3 Prove $F\{g(t-\tau)\}=g(\omega)e^{i\omega\tau}$.

P0.4.4 Find the Fourier transform of $E(t)=E_o e^{-(t/\tau)^2}\cos\omega_o t$.

ANSWER: $E(\omega)=\frac{\tau E_o}{\sqrt{2}}\frac{e^{-\frac{(\omega+\omega_o)^2}{4/\tau^2}}+e^{-\frac{(\omega-\omega_o)^2}{4/\tau^2}}}{2}$.

P0.4.5 Take the inverse Fourier transform of the result in P0.4.4. Check that it returns exactly the original function.

P0.4.6 Prove the convolution theorem: $F\left\{\int_{-\infty}^{\infty}g(t)h(\tau-t)dt\right\}=\sqrt{2\pi}g(\omega)h(\omega)$.

SOLUTION:

$$F\left\{\int_{-\infty}^{\infty}g(t)h(\tau-t)dt\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}g(t)h(\tau-t)dt\right\}e^{i\omega\tau}d\tau \quad (\text{Let } \tau=t'+t)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}g(t)h(t')dt\right\}e^{i\omega(t'+t)}dt'=\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{i\omega t}dt\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(t')e^{i\omega t'}dt'$$

$$=\sqrt{2\pi}g(\omega)h(\omega).$$

P0.4.7 Prove the autocorrelation theorem: $F\left\{\int_{-\infty}^{\infty}h(t)h^*(t-\tau)dt\right\}=\sqrt{2\pi}\,|h(\omega)|^2$.

P0.4.8 Prove Parseval's theorem: $\int_{-\infty}^{\infty}|f(\omega)|^2 d\omega=\int_{-\infty}^{\infty}|f(t)|^2 dt$.

Appendix 0.A Sylvester's Theorem

If $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin\theta}\begin{bmatrix} A\sin N\theta - \sin(N-1)\theta & B\sin N\theta \\ C\sin N\theta & D\sin N\theta - \sin(N-1)\theta \end{bmatrix}, \quad (0.A.1)$$

where

$$\cos\theta = \frac{1}{2}(A+D). \quad (0.A.2)$$

This is known as Sylvester's theorem. It is useful when a 2×2 matrix (with a determinate of unity) is raised to a high power. This situation appears for periodic multilayer mirror coatings and for light rays trapped in a laser cavity.

We now prove the theorem by induction. When $N=1$, the equation is seen to be correct. Next we assume that the theorem holds for arbitrary N, and we check to see if it holds for $N+1$:

$$\begin{aligned}
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1} &= \frac{1}{\sin\theta}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A\sin N\theta - \sin(N-1)\theta & B\sin N\theta \\ C\sin N\theta & D\sin N\theta - \sin(N-1)\theta \end{bmatrix} \\
&= \frac{1}{\sin\theta}\begin{bmatrix} \left(A^2+BC\right)\sin N\theta - A\sin(N-1)\theta & (AB+BD)\sin N\theta - B\sin(N-1)\theta \\ (AC+CD)\sin N\theta - C\sin(N-1)\theta & \left(D^2+BC\right)\sin N\theta - D\sin(N-1)\theta \end{bmatrix} \\
&= \frac{1}{\sin\theta}\begin{bmatrix} \left(A^2+AD-1\right)\sin N\theta - A\sin(N-1)\theta & B[(A+D)\sin N\theta - \sin(N-1)\theta] \\ C[(A+D)\sin N\theta - \sin(N-1)\theta] & \left(D^2+AD-1\right)\sin N\theta - D\sin(N-1)\theta \end{bmatrix}.
\end{aligned}$$

The expression $AD-BC=1$ for the determinant has been applied in the diagonal elements. Further rearrangement gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1}$$
$$= \frac{1}{\sin\theta}\begin{bmatrix} A[(A+D)\sin N\theta - \sin(N-1)\theta] - \sin N\theta & B[(A+D)\sin N\theta - \sin(N-1)\theta] \\ C[(A+D)\sin N\theta - \sin(N-1)\theta] & D[(A+D)\sin N\theta - \sin(N-1)\theta] - \sin N\theta \end{bmatrix}.$$

In each matrix element, the expression

$$(A+D)\sin N\theta = 2\cos\theta\sin N\theta = \sin(N+1)\theta + \sin(N-1)\theta$$

occurs, which we have rearranged using $\cos\theta = \frac{1}{2}(A+D)$ while twice invoking (0.2.2). The result is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1} = \frac{1}{\sin\theta}\begin{bmatrix} A\sin(N+1)\theta - \sin N\theta & B\sin(N+1)\theta \\ C\sin(N+1)\theta & D\sin(N+1)\theta - \sin N\theta \end{bmatrix},$$

which completes the proof.

Appendix 0.B Integrals and Formulas

$$\int_{-\infty}^{\infty} e^{-Ax^2+Bx+C}dx = \sqrt{\frac{\pi}{A}}e^{B^2/4A+C} \text{ if } \operatorname{Re}\{A\} > 0. \tag{0.B.1}$$

$$\int_{0}^{\infty} \frac{e^{iax}}{1+x^2/b^2}dx = \frac{\pi|b|}{2}e^{-|ab|} \text{ if } b^2 > 0. \tag{0.B.2}$$

$$J_o(\alpha) = \frac{1}{2\pi}\int_{0}^{2\pi} e^{\pm i\alpha\cos(\theta-\theta')}d\theta. \tag{0.B.3}$$

$$\int_{0}^{a} J_o(bx)xdx = \frac{a}{b}J_1(ab). \tag{0.B.4}$$

$$\int_{0}^{\infty} e^{-\alpha x^2}J_o(bx)xdx = \frac{e^{-b^2/4\alpha}}{2\alpha}. \tag{0.B.5}$$

$$\sum_{n=1}^{N} r^n = r\frac{r^N-1}{r-1}. \tag{0.B.6}$$

$$\int_{0}^{\infty} \frac{\sin^2\alpha x}{(\alpha x)^2}dx = \frac{\pi}{2\alpha}. \tag{0.B.7}$$