

Chapter 3

Reflection and Refraction

3.1 Introduction

In the previous chapter, we considered a plane wave propagating in a homogeneous isotropic medium. In this chapter, we examine what happens when such a wave propagates from one material (characterized by index n or even by complex index \mathcal{N}) to another material. As we know from everyday experience, some light reflects and some light transmits from the interface between materials. We will derive expressions for the amount of reflection and transmission. The results depend on the angle of incidence (i.e. the angle between \vec{k} and the normal to the surface) as well as on the orientation of the electric field (called polarization --- not to be confused with \vec{P} , also called polarization).

As we develop the connection between incident, reflected, and transmitted light waves, many familiar relationships will emerge naturally (e.g. Snell's law, Brewster's angle). The formalism also describes polarization-dependent phase shifts upon reflection (especially interesting in the case of total internal reflection or in the case of reflections from absorbing surfaces such as metals), which we will study in chapter 4.

For simplicity, we neglect in this chapter the imaginary part of refractive index. Each plane wave is thus characterized by a real wave vector \vec{k} . We will write each plane wave in the form $\vec{E}(\vec{r}, t) = \vec{E}_o \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$ (2.2.7), where, as usual, only the real part of the field corresponds to the physical field. The restriction to real indexes is not as serious as it might seem since the results can be easily extended to include complex indexes, and we will do this in chapter 4. The use of the letters k and n instead of \mathcal{K} and \mathcal{N} hardly matters. The math is all the same, which demonstrates the power of the complex notation.

In an isotropic medium, the electric field amplitude \vec{E}_o is confined to a plane perpendicular to \vec{k} . Therefore, \vec{E}_o can always be broken into two orthogonal polarization components within that plane. The two vector components of \vec{E}_o contain the individual phase information for each dimension. If the phases of the two components of \vec{E}_o are the same, then the polarization of the electric field is said to be linear. If the components of the vector \vec{E}_o differ in phase, then the electric field polarization is said to be elliptical (or circular) as will be studied in chapter 4.

The information presented here constitutes an essential foundation for the study of optics, and students are responsible for its contents.

3.2 Refraction at an Interface

To study the reflection and transmission of light at a material interface, we will examine three distinct waves traveling in the directions \vec{k}_i , \vec{k}_r , and \vec{k}_t as depicted in the Fig. 3.1. In the upcoming development, we will refer to Fig. 3.1 often. We assume a planar boundary between the two

materials. The index n_i characterizes the material on the left, and the index n_t characterizes the material on the right. \vec{k}_i specifies an incident plane wave making an angle θ_i with the normal to the interface. \vec{k}_r specifies a reflected plane wave making an angle θ_r with the interface normal. These two waves exist only to the left of the interface. \vec{k}_t specifies a transmitted plane wave making an angle θ_t with the interface normal. The transmitted wave exists only to the right of the material interface.

We choose the y - z plane to be the *plane of incidence*, containing all three k -vectors (the plane of the page). (By symmetry, all k -vectors must lie in a single plane, assuming an isotropic material.) We are free to orient our coordinate system in many different ways (as every textbook seems to do differently). We choose the normal incidence on the interface to be along the z -direction. The x -axis points into the page.

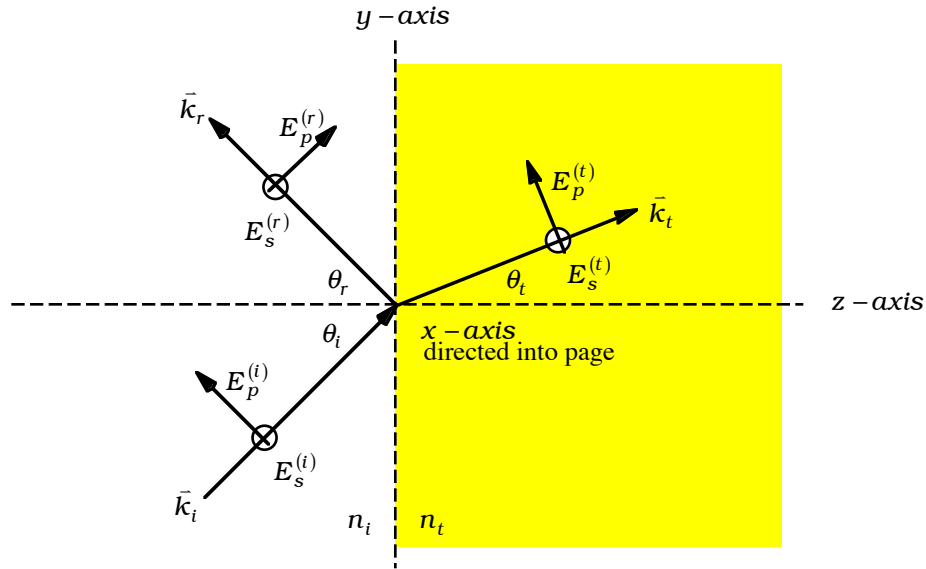


Fig. 3.1 Incident, reflected, and transmitted plane wave fields at a material interface.

For a given \vec{k}_i , the electric field vector \vec{E}_i can be decomposed into arbitrary components as long as they are perpendicular to \vec{k}_i . For convenience, we choose one of the electric field vector components to be that which lies within the plane of incidence as depicted in Fig. 3.1. $E_p^{(i)}$ denotes this component, represented by an arrow in the plane of the page. The remaining electric field vector component, denoted by $E_s^{(i)}$, is directed normal to the plane of incidence. The subscript s stands for *senkrecht*, a German word meaning perpendicular. In Fig. 3.1, $E_s^{(i)}$ is represented by the tail of an arrow pointing into the page, or the x -direction, by our convention. The other fields \vec{E}_r and \vec{E}_t are similarly split into s and p components as indicated in Fig. 3.1. (Our choice of coordinate system orientation is motivated in part by the fact that it is easier to draw arrow tails rather than arrow tips to represent the electric field in the s -direction.) All field components align with their respective arrows when they are positive.

By inspection of Fig. 3.1, we can write the various k -vectors in terms of the \hat{y} and \hat{z} unit vectors:

$$\begin{aligned}
\bar{k}_i &= k_i(\hat{y}\sin\theta_i + \hat{z}\cos\theta_i), \\
\bar{k}_r &= k_r(\hat{y}\sin\theta_r - \hat{z}\cos\theta_r), \\
\bar{k}_t &= k_t(\hat{y}\sin\theta_t + \hat{z}\cos\theta_t).
\end{aligned} \tag{3.2.1}$$

Also by inspection of Fig. 3.1 (following the conventions for the electric fields depicted by the arrows), we can write the incident, reflected, and transmitted fields in terms of \hat{x} , \hat{y} , and \hat{z} :

$$\begin{aligned}
\bar{E}_i &= \left[E_p^{(i)}(\hat{y}\cos\theta_i - \hat{z}\sin\theta_i) + \hat{x}E_s^{(i)} \right] e^{i[k_i(y\sin\theta_i + z\cos\theta_i) - \omega_i t]}, \\
\bar{E}_r &= \left[E_p^{(r)}(\hat{y}\cos\theta_r + \hat{z}\sin\theta_r) + \hat{x}E_s^{(r)} \right] e^{i[k_r(y\sin\theta_r - z\cos\theta_r) - \omega_r t]}, \\
\bar{E}_t &= \left[E_p^{(t)}(\hat{y}\cos\theta_t - \hat{z}\sin\theta_t) + \hat{x}E_s^{(t)} \right] e^{i[k_t(y\sin\theta_t + z\cos\theta_t) - \omega_t t]}.
\end{aligned} \tag{3.2.2}$$

Each field has the form (2.2.7), and we have utilized the k-vectors (3.2.1) in the exponents of (3.2.2).

At this point we are ready to apply a boundary condition on the fields. The tangential component of \bar{E} (parallel to the surface) must be identical on either side of the plane $z = 0$, as explained in appendix 3.A (see (3.A.4)). This means that the parallel components (in the \hat{x} and \hat{y} directions only) of the combined incident and reflected fields must match the parallel components of the transmitted field (with z set to zero):

$$\begin{aligned}
&\left[E_p^{(i)}\hat{y}\cos\theta_i + \hat{x}E_s^{(i)} \right] e^{i(k_i y \sin\theta_i - \omega_i t)} + \left[E_p^{(r)}\hat{y}\cos\theta_r + \hat{x}E_s^{(r)} \right] e^{i(k_r y \sin\theta_r - \omega_r t)} \\
&= \left[E_p^{(t)}\hat{y}\cos\theta_t + \hat{x}E_s^{(t)} \right] e^{i(k_t y \sin\theta_t - \omega_t t)}.
\end{aligned} \tag{3.2.3}$$

Since this equation must hold for all conceivable values of t and y , we are compelled to set all exponential factors equal to each other. This immediately requires the frequency of all waves to be the same:

$$\omega_i = \omega_r = \omega_t \equiv \omega. \tag{3.2.4}$$

We could have guessed that all frequencies would be the same; otherwise, some wave fronts would annihilate (or be created) at the interface. Equating the terms in the exponents of (3.2.3) also requires

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t. \tag{3.2.5}$$

Next, we use the conditions $k_i = k_r = n_i\omega/c$ and $k_t = n_t\omega/c$ from (2.3.20). From this we obtain the law of reflection

$$\theta_r = \theta_i \tag{3.2.6}$$

and Snell's law

$$n_i \sin\theta_i = n_t \sin\theta_t. \tag{3.2.7}$$

Evidently, the three angles θ_i , θ_r , and θ_t are not independent. Not surprisingly, the reflected angle matches the incident angle, and the transmitted angle obeys Snell's law. The phenomenon of refraction refers to the fact that θ_i and θ_t are different.

Because the exponents are all identical, (3.2.3) reduces to two relatively simple equations (one for each dimension, \hat{x} and \hat{y}):

$$E_s^{(i)} + E_s^{(r)} = E_s^{(t)} \text{ and} \quad (3.2.8)$$

$$(E_p^{(i)} + E_p^{(r)}) \cos \theta_i = E_p^{(t)} \cos \theta_t. \quad (3.2.9)$$

We have derived these equations from the simple boundary condition (3.A.4) on the parallel component of the electric field. We have yet to use the boundary condition (3.A.8) on the parallel component of the magnetic field, from which we can derive two similar but distinct equations.

From Maxwell's equation (1.6.3), we have for a plane wave

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{n}{c} \hat{u} \times \vec{E}, \quad (3.2.10)$$

where $\hat{u} \equiv \vec{k}/k$ is a unit vector in the direction of \vec{k} . We have also utilized (2.3.20). This expression is useful to obtain expressions for \vec{B}_i , \vec{B}_r , and \vec{B}_t in terms of the electric field components that we have already introduced. Introducing (3.2.1) and (3.2.2) into (3.2.10), the incident, reflected, and transmitted magnetic fields are

$$\begin{aligned} \vec{B}_i &= \frac{n_i}{c} \left[-\hat{x} E_p^{(i)} + E_s^{(i)} (-\hat{z} \sin \theta_i + \hat{y} \cos \theta_i) \right] e^{i[k_i(y \sin \theta_i + z \cos \theta_i) - \omega_i t]}, \\ \vec{B}_r &= \frac{n_r}{c} \left[\hat{x} E_p^{(r)} + E_s^{(r)} (-\hat{z} \sin \theta_r - \hat{y} \cos \theta_r) \right] e^{i[k_r(y \sin \theta_r - z \cos \theta_r) - \omega_r t]}, \\ \vec{B}_t &= \frac{n_t}{c} \left[-\hat{x} E_p^{(t)} + E_s^{(t)} (-\hat{z} \sin \theta_t + \hat{y} \cos \theta_t) \right] e^{i[k_t(y \sin \theta_t + z \cos \theta_t) - \omega_t t]}. \end{aligned} \quad (3.2.11)$$

Next, we apply the boundary condition (3.A.8), which require the parallel components of \vec{B} (i.e. the components in the \hat{x} and \hat{y} directions) to be the same on either side of the plane $z = 0$. Since we already know that the exponents are all equal and that $\theta_r = \theta_i$ and $n_i = n_r$, the boundary condition gives

$$\begin{aligned} &\frac{n_i}{c} \left[-\hat{x} E_p^{(i)} + E_s^{(i)} \hat{y} \cos \theta_i \right] + \frac{n_i}{c} \left[\hat{x} E_p^{(r)} - E_s^{(r)} \hat{y} \cos \theta_i \right] \\ &= \frac{n_t}{c} \left[-\hat{x} E_p^{(t)} + E_s^{(t)} \hat{y} \cos \theta_t \right]. \end{aligned} \quad (3.2.12)$$

As before, (3.2.12) reduces to two relatively simple equations (one for each dimension, \hat{x} and \hat{y}):

$$n_i (E_p^{(i)} - E_p^{(r)}) = n_t E_p^{(t)} \text{ and} \quad (3.2.13)$$

$$n_i (E_s^{(i)} - E_s^{(r)}) \cos \theta_i = n_t E_s^{(t)} \cos \theta_t. \quad (3.2.14)$$

These two equations together with (3.2.8) and (3.2.9) give a complete description of how the fields on each side of the boundary relate to each other. If we choose an incident field \vec{E}_i , these equations can be used to predict \vec{E}_r and \vec{E}_t . To use these equations, we must break the fields into their respective s and p polarization components. However, (3.2.8), (3.2.9), (3.2.13), and (3.2.14) are not yet in their most convenient form.

3.3 The Fresnel Coefficients

Augustin Fresnel (1788-1829) first developed the equations derived in the previous section. However, at the time he did not have the benefit of Maxwell's equations, since he lived well before Maxwell's time. Instead, Fresnel thought of light as transverse mechanical waves propagating within materials. (We can easily see why Fresnel was a great proponent of the now-discredited luminiferous ether.) Instead of relating the parallel components of the electric and magnetic fields across the boundary between the materials, Fresnel used the principle that, as a transverse mechanical wave propagates from one material to the other, the two materials should not slip past each other at the interface. This 'gluing' of the materials at the interface also forbids the possibility of the materials detaching from one another or passing through one another as they experience the wave vibration. This mechanical approach to light worked splendidly and explained polarization effects along with the variations in reflectance and transmission as a function of the incident angle of the light.

Fresnel wrote the relationships between the various plane waves depicted in Fig. 3.1 in terms of coefficients that compare the reflected and transmitted field amplitudes to the amplitude of the incident field. This is done separately for each polarization. It is left as a homework exercise (see P3.3.1 and P3.3.2) to solve (3.2.8), (3.2.9), (3.2.13), and (3.2.14) together with Snell's law (3.2.5). The ratio of the reflected and transmitted field components to the incident field components are specified by the following coefficients (called Fresnel coefficients):

$$r_s \equiv \frac{E_s^{(r)}}{E_s^{(i)}} = \frac{\sin \theta_t \cos \theta_i - \sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i + \sin \theta_i \cos \theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t}, \quad (3.3.1)$$

$$t_s \equiv \frac{E_s^{(t)}}{E_s^{(i)}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin \theta_t \cos \theta_i + \sin \theta_i \cos \theta_t} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_i + \theta_t)} = \frac{2 n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t}, \quad (3.3.2)$$

$$r_p \equiv \frac{E_p^{(r)}}{E_p^{(i)}} = \frac{\cos \theta_t \sin \theta_t - \cos \theta_i \sin \theta_i}{\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{n_i \cos \theta_t - n_t \cos \theta_i}{n_i \cos \theta_t + n_t \cos \theta_i}, \quad (3.3.3)$$

$$t_p \equiv \frac{E_p^{(t)}}{E_p^{(i)}} = \frac{2 \cos \theta_i \sin \theta_t}{\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)} = \frac{2 n_i \cos \theta_i}{n_i \cos \theta_t + n_t \cos \theta_i}. \quad (3.3.4)$$

The Fresnel coefficients are traditionally written as we have done here, in terms of the incident and transmitted angles θ_i and θ_t . These angles of course are subject to Snell's law (3.2.5).

Exercises

P3.3.1 Derive the Fresnel coefficients (3.3.1) and (3.3.2) for *s*-polarized light.

SOLUTION: We use (3.2.8) $E_s^{(i)} + E_s^{(r)} = E_s^{(t)}$ and (3.2.14), which with the help of Snell's law is written $E_s^{(i)} - E_s^{(r)} = \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i} E_s^{(t)}$.

Add equations: $2E_s^{(i)} = \left[1 + \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i} \right] E_s^{(t)}$.

Subtract equations: $2E_s^{(r)} = \left[1 - \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i} \right] E_s^{(t)}$.

After dividing through we have $t_s \equiv \frac{E_s^{(t)}}{E_s^{(i)}} = \frac{2 \sin \theta_t \cos \theta_i}{\sin \theta_t \cos \theta_i + \sin \theta_i \cos \theta_t}$ and

$$r_s \equiv \frac{E_s^{(r)}}{E_s^{(i)}} = \frac{\sin \theta_t \cos \theta_i - \sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i + \sin \theta_i \cos \theta_t}.$$

P3.3.2 Derive the Fresnel coefficients (3.3.3) and (3.3.4) for *p*-polarized light.

P3.3.3 Verify the first alternative form given in each of (3.3.1)-(3.3.4).

P3.3.4 Verify the second alternative form given in each of (3.3.1)-(3.3.4). Show that at normal incidence (i.e. $\theta_i = \theta_t = 0$) the Fresnel coefficients reduce to

$$\lim_{\theta_i \rightarrow 0} r_s = \lim_{\theta_i \rightarrow 0} r_p = -\frac{n_t - n_i}{n_t + n_i} \text{ and } \lim_{\theta_i \rightarrow 0} t_s = \lim_{\theta_i \rightarrow 0} t_p = \frac{2n_i}{n_t + n_i}. \text{ HINT: Use Snell's law.}$$

3.4 Reflectance and Transmission

The Fresnel coefficients connect the electric field amplitudes on the two sides of the boundary. However, we are often interested in knowing the fraction of intensity that transmits and reflects. Since intensity is proportional to the square of the amplitude of the electric field, we can write the fraction of the light reflected from the surface (called *reflectance*) as

$$R_s \equiv |r_s|^2 \text{ and } R_p \equiv |r_p|^2. \quad (3.4.1)$$

These expressions are applied individually to each polarization component (to r_s or r_p). The intensity reflected for each of these orthogonal polarizations is additive because the two electric fields are orthogonal and do not interfere with each other. The total reflected intensity is therefore

$$I_{total}^{(r)} = I_s^{(r)} + I_p^{(r)} = R_s I_s^{(i)} + R_p I_p^{(i)}, \quad (3.4.2)$$

where the incident intensity is given by (2.6.7):

$$I_{total}^{(i)} = I_s^{(i)} + I_p^{(i)} = \frac{1}{2} n_i \epsilon_0 c \left[|E_s^{(i)}|^2 + |E_p^{(i)}|^2 \right]. \quad (3.4.3)$$

From the requirement that power be conserved, we can write the fraction of the light that transmits (called *transmission*) for either polarization as

$$T_s \equiv 1 - R_s \text{ and } T_p \equiv 1 - R_p. \quad (3.4.4)$$

Again, the transmitted intensity is separated into the individual polarizations:

$$I_{total}^{(t)} = I_s^{(t)} + I_p^{(t)} = T_s I_s^{(i)} + T_p I_p^{(i)}. \quad (3.4.5)$$

It might be surprising at first to learn that

$$T_s \neq |t_s|^2 \text{ and } T_p \neq |t_p|^2. \quad (3.4.6)$$

Actually, this should not be surprising in view of the fact that the transmitted intensity (in terms of the transmitted fields) depends also on the refractive index. Recall that t_s and t_p relate the bare electric fields to each other, whereas the transmitted intensity (similar to (3.4.3)) is

$$I_{total}^{(t)} = I_s^{(t)} + I_p^{(t)} = \frac{1}{2} n_t \epsilon_0 c \left[|E_s^{(t)}|^2 + |E_p^{(t)}|^2 \right]. \quad (3.4.7)$$

Therefore, we expect T_s and T_p to depend on the ratio of the refractive indexes n_t and n_i as well as on the squares of t_s and t_p . However, there is another more subtle reason for the inequalities in (3.4.6).

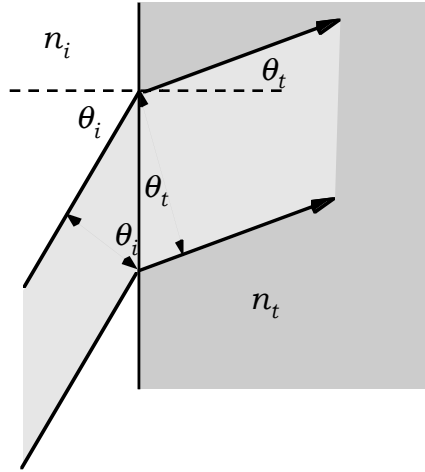


Fig. 3.2 Light refracting into a surface.

Consider a lateral strip of the energy associated with a plane wave incident upon the material interface in Fig. 3.2. Upon refraction into the second medium, the strip is seen to change its width by the factor $\cos\theta_t/\cos\theta_i$. This is a geometrical artifact, owing to the change in propagation direction at the interface. The variation in the refractive index together with this geometrical effect leads to the transmission coefficients,

$$T_s = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_s|^2 \text{ and } T_p = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_p|^2 \text{ (not valid if total internal reflection occurs).} \quad (3.4.8)$$

Note that (3.4.8) is valid only if a real angle θ_t exists; it does not hold when the incident angle exceeds the critical angle for total internal reflection, discussed in section 3.6. In that situation, we must stick with (3.4.4).

Exercises

P3.4.1 Show *analytically* for *p*-polarized light that $R_p + T_p = 1$, where R_p is given by (3.4.1) and T_p is given by (3.4.8).

SOLUTION: From (3.3.3) we have

$$R_p = \left| \frac{\cos \theta_t \sin \theta_t - \cos \theta_i \sin \theta_i}{\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i} \right|^2 = \frac{\cos^2 \theta_t \sin^2 \theta_t - 2 \cos \theta_i \sin \theta_i \cos \theta_t \sin \theta_t + \cos^2 \theta_i \sin^2 \theta_i}{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2}.$$

From (3.3.4) we have

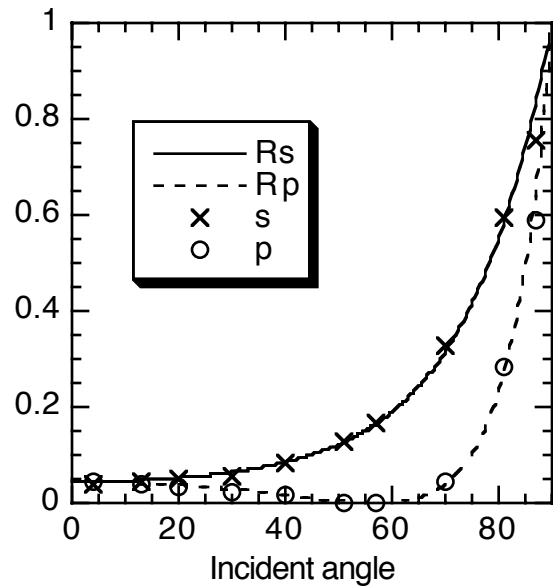
$$\begin{aligned} T_p &= \frac{n_t \cos \theta_t}{n_i \cos \theta_i} \left[\frac{2 \cos \theta_i \sin \theta_t}{\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i} \right]^2 = \frac{\sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i} \frac{4 \cos^2 \theta_i \sin^2 \theta_t}{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2} \\ &= \frac{4 \cos \theta_i \sin \theta_t \sin \theta_i \cos \theta_t}{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2}. \end{aligned}$$

Then

$$\begin{aligned} R_p + T_p &= \frac{\cos^2 \theta_t \sin^2 \theta_t + 2 \cos \theta_i \sin \theta_i \cos \theta_t \sin \theta_t + \cos^2 \theta_i \sin^2 \theta_i}{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2} \\ &= \frac{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2}{(\cos \theta_t \sin \theta_t + \cos \theta_i \sin \theta_i)^2} = 1. \end{aligned}$$

P3.4.2 Show *analytically* for *s*-polarized light that $R_s + T_s = 1$, where R_s is given by (3.4.1) and T_s is given by (3.4.8).

L3.4.3 Use a computer to calculate the theoretical air-to-glass reflectance as a function of incident angle (i.e. plot R_s and R_p as a function of θ_i). Take the index of refraction for glass to be $n_t = 1.54$ and the index for air to be one. Plot this theoretical calculation as a smooth line on a graph. In the laboratory, measure this for both *s* and *p* polarized light at about 10 points, and plot the points on your graph (not points connected by lines). You can normalize the detector by placing it in the incident beam of light before the glass surface. Especially watch for Brewster's angle (described in section 3.5).



3.5 Brewster's Angle

Notice that the Fresnel coefficient (3.3.3) for r_p can be written with $\tan(\theta_i + \theta_t)$ in the denominator. Since the tangent ‘blows up’ at $\pi/2$, the reflection coefficient evidently goes to zero when

$$\theta_i + \theta_t = \frac{\pi}{2} \quad (\text{requirement for zero } p\text{-polarized reflection}). \quad (3.5.1)$$

By inspecting Fig. 3.1, we see that this condition occurs when the reflected and transmitted \mathbf{k} -vectors, \vec{k}_r and \vec{k}_t , are perpendicular to each other. If we insert (3.5.1) into Snell's law (3.2.7), we can solve for the incident angle θ_i that gives rise to this special circumstance:

$$n_i \sin \theta_i = n_t \sin \left(\frac{\pi}{2} - \theta_i \right) = n_t \cos \theta_i. \quad (3.5.2)$$

The special incident angle that satisfies this equation, in terms of the refractive indexes, is found to be

$$\theta_B = \tan^{-1} \frac{n_t}{n_i}. \quad (3.5.3)$$

We have replaced the specific θ_i with θ_B in honor of Sir David Brewster (1781-1868) who first discovered the phenomenon. The angle θ_B is called Brewster's angle. At Brewster's angle, no p -polarized light reflects (see L3.4.3). Physically, the p -polarized light cannot reflect because \vec{k}_r and \vec{k}_t are perpendicular. A reflection would require the microscopic dipoles at the surface of the second material to radiate along their axes, which they cannot do. Maxwell's equations ‘know’ about this, and so everything is nicely consistent.

Exercises

P3.5.1 Find Brewster's angle for glass $n = 1.5$.

3.6 Total Internal Reflection

From Snell's law (3.2.7), we can easily compute the transmitted angle in terms of the incident angle:

$$\theta_t = \sin^{-1} \left(\frac{n_i}{n_t} \sin \theta_i \right). \quad (3.6.1)$$

The angle θ_t is real only if the argument of the sine is less than or equal to one. If $n_i > n_t$, we can find a critical angle at which the argument begins to exceed one:

$$\theta_C \equiv \sin^{-1} \frac{n_t}{n_i}. \quad (3.6.2)$$

When $\theta_i > \theta_C$, then there is total internal reflection and we can directly show that $R_s = 1$ and $R_p = 1$ (see P3.6.1). To demonstrate this, one computes the Fresnel coefficients (3.3.1) and (3.3.3) while employing the following substitutions:

$$\sin \theta_t = \frac{n_i}{n_t} \sin \theta_i \quad (\text{Snell's law}) \quad \text{and} \quad (3.6.3)$$

$$\cos \theta_t = i \sqrt{\frac{n_i^2}{n_t^2} \sin^2 \theta_i - 1} \quad (\text{see P0.2.6}). \quad (3.6.4)$$

In this case, θ_t is a complex number. However, we do not assign geometrical significance to it in terms of any direction. Actually, we don't even need to know the value for θ_t ; we need only the values for $\sin \theta_t$ and $\cos \theta_t$, as specified in (3.6.3) and (3.6.4). Even though $\sin \theta_t$ is greater than one and $\cos \theta_t$ is imaginary, we can use their values to compute r_s , r_p , t_s , and t_p . (Complex notation is wonderful!)

By using (3.6.3) and (3.6.4) in the case of total internal reflection, we obtain non-zero coefficients t_s and t_p . They correspond to decaying fields called evanescent waves on the transmitted side of the boundary, even though no actual power is transmitted (and we rightly say that total internal reflection takes place). We also obtain complex values for r_s and r_p , (each with amplitude one), indicating a phase shift in the field upon reflection. We will discuss this more in chapter 4.

Exercises

P3.6.1 Prove that if $\theta_i > \theta_C$, then $R_s = 1$ and $R_p = 1$.

P3.6.2 Use a computer to plot the air-to-water transmission as a function of incident angle (i.e. plot (3.4.4) as a function of θ_i). Also plot the water-to-air transmission on a separate graph. Plot both T_s and T_p on each graph. The index of refraction for water is $n = 1.33$. Take the index of air to be one.

Appendix 3.A Boundary Conditions For Fields at an Interface

We are interested in the continuity of fields across a boundary from one medium with index n_1 to another medium with index n_2 . We can make some statements that are surprisingly general. We will show that the components of electric field parallel to the interface surface must be the same on the two sides of the surface (next to the interface only). This result is independent of the refractive index of the materials. We will also show that the component of magnetic field parallel to the interface surface is the same on the two sides.

Consider a surface S (a rectangle) that is *perpendicular* to the interface between the two media and which extends into both media, as depicted in Fig. 3.3.

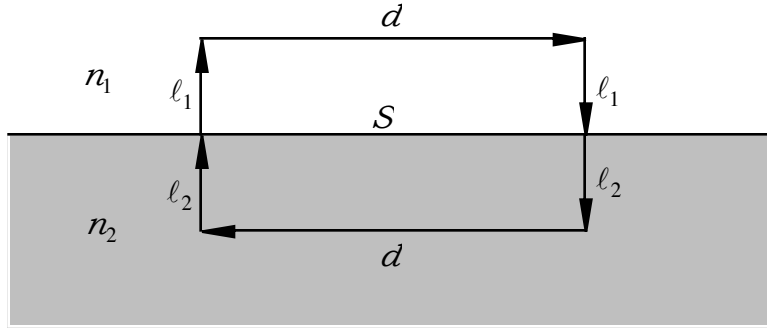


Fig. 3.3 Interface of two materials.

First we examine the implications of Faraday's law (1.4.1):

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \hat{n} da. \quad (3.A.1)$$

We apply Faraday's law to the rectangular contour depicted in Fig. 3.3. We can perform the contour integration on the left-hand side of (3.A.1). The integration around the loop gives

$$\oint \vec{E} \cdot d\vec{\ell} = E_{1\parallel}d - E_{1\perp}\ell_1 - E_{2\perp}\ell_2 - E_{2\parallel}d + E_{2\perp}\ell_2 + E_{1\perp}\ell_1 = (E_{1\parallel} - E_{2\parallel})d. \quad (3.A.2)$$

Here, $E_{1\parallel}$ refers to the component of the electric field in the material with index n_1 that is parallel to the interface. $E_{1\perp}$ refers to the component of the electric field in the material with index n_1 which is perpendicular to the interface. Similarly, $E_{2\parallel}$ and $E_{2\perp}$ are the parallel and perpendicular components of the electric field in the material with index n_2 . We have assumed that the rectangle is small enough that the fields are uniform within the half rectangle on either side of the boundary.

We can continue to shrink the loop down until it has zero surface area by letting the lengths ℓ_1 and ℓ_2 go to zero. In this situation, the right-hand side of Faraday's law goes to zero

$$\int_S \vec{B} \cdot \hat{n} da \rightarrow 0, \quad (3.A.3)$$

and we are left with

$$E_{1\parallel} = E_{2\parallel}. \quad (3.A.4)$$

This simple relation is a general boundary condition, which is met at any material interface. The component of the electric field that lies in the plane of the interface must be the same on both sides of the interface.

We now derive a similar boundary condition for the magnetic field. Maxwell's equation (1.6.4), upon integration over the surface S in Fig. 3.3 and after applying Stokes' theorem (0.3.9) to the magnetic field term, can be written as

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o \int_S \left(\vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \right) \cdot \hat{n} da. \quad (3.A.5)$$

As before, we are able to perform the contour integration on the left-hand side for the geometry depicted in the figure. When we integrate around the loop we get

$$\oint \vec{B} \cdot d\vec{\ell} = B_{1\parallel}d - B_{1\perp}\ell_1 - B_{2\perp}\ell_2 - B_{2\parallel}d + B_{2\perp}\ell_2 + B_{1\perp}\ell_1 = (B_{1\parallel} - B_{2\parallel})d. \quad (3.A.6)$$

The notation for parallel and perpendicular components on either side of the interface is similar to that used in (3.A.2).

Again, we can continue to shrink the loop down until it has zero surface area by letting the lengths ℓ_1 and ℓ_2 go to zero. In this situation, the right-hand side of (3.A.5) goes to zero (not considering the possibility of surface currents):

$$\int_S \left(\vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} + \epsilon_o \frac{\partial \vec{E}}{\partial t} \right) \cdot \hat{n} da \rightarrow 0, \quad (3.A.7)$$

and again we are left with

$$B_{1\parallel} = B_{2\parallel}. \quad (3.A.8)$$

This is a general boundary condition that must be satisfied at the material interface.