A Geometric Theory of Préchac Transformations

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1 Abstract

This must be the way.

Here is a video of a great passing pattern executed by Jorge Vilchis and Santiago Malabari. This pattern is fast, it's lively, and it has some really interesting symmetry. Part of what makes this pattern so lively is that the two of them are never throwing at the same time, Jorge is throwing right in between Santi's throws. There's the visual element of the one blue club being passed back and forth. Plus there's a pleasant symmetry that Jorge and Santi are making all the same throws, but staggered in time from each other. The overall effect is similar to a feeling of watching popcorn pop.

Passing patterns like this, where the two jugglers are doing the same thing but at different times, arise from solo patterns via a process called the Préchac transformation, named after the French juggler and mathematician Christophe Préchac who discovered them in 1999. This document aims to investigate the beautiful math that makes these patterns possible, and to answer some natural questions that show up. For jugglers, I have personally found that understanding the theory of these patterns has made it much easier for me to juggle them, I find more joy in watching them, and certainly it is certainly easier for me to generate new patterns designed to my preferences. The basic idea is that you take an individual throw, and share its flight time among multiple jugglers. For mathematicians, I think there is some very satisfying math going on as well. After all, how can you watch prechac with my friends and not want to juggle and understand these patterns?

Some familiarity with solo siteswaps is helpful, but we'll build the theory from the ground up, with a slightly different perspective than usual that will be amenable to where we're going. We'll start with some mathematical preliminaries to aid us in our discussion of these patterns.

2 Notation and mathematical preliminaries

We will be using functions a lot, so we recall some of their most important properties. We recall that if we have two sets A and B a function $f : A \to B$ is an association that associates to every element of A a unique element of B. The element associated to an element a in A is written as $f(a)$. So if a is associated to b, then we have $f(a) = b$. For instance, if P is the set of all people who have ever lived, and T is the set of all moments in time there have ever been, we get a function $f : P \to T$ sending a person to the exact time of their birth. So maybe f(my friend John) = 3:55pm eastern standard time on March 15th, 1995.

The main property of functions that interests us is that they can compose. That is, if we have two functions $f : A \to B$ and $g : B \to C$, then we get a new function $g \circ f : A \to C$ (read aloud as "g composed with f") defined by $g \circ f(a) = g(f(a))$. This makes sense as $f(a)$ is an element of B, and so g associates to it an element of C.

For instance, if we let Y denote the collection of all years, and again let T

denote the collection of all moments in time, we have a function $g : T \to Y$ sending a moment in time to the year that moment occurred. So for instance g(3:55pm eastern standard time on March 15th, 1995) = 1995. Considering the function $f : P \to T$ sending a person to the moment they were born, we have the new function $g \circ f : P \to Y$ sending a person to their birth year, and in particular we have

$$g \circ f(\text{my friend John}) = g(\text{3:55pm eastern standard time on March 15th, 1995}) = 1995,$$

saying that John was born in 1995.

There is a special class of functions we will work with a lot, the so-called "invertible" functions. Namely, a function $f : A \to B$ is said to be invertible precisely when it yields a perfect translation between A and B. That is, if we think of f as translating from A to B, then we can translate back from B to A via another function $g : B \to A$. This translation means that if we translate from A, then to B, then back to A, we should have done nothing. That is, we should have $g \circ f(a) = a$ for all a in A, and similarly $f \circ g(b) = b$ for all b in B. This relation is expressed by saying that g and f are inverses to each other, often written as $g = f^{-1}$.

Indeed, if we consider two languages like say English and Spanish then the goal of an interpreter is to build translation functions $f : \text{English} \to \text{Spanish}$ and $g : \text{Spanish} \to \text{English}$. How can we tell if the translator is doing a good job? Say a person says a sentence S in Spanish, and that gets interpreted to a sentence $f(S) = E$ in English. If the interpreter was to translate that same sentence back into Spanish, we should have $g(E) = S$. That is, $g \circ f(S) = S$, and similarly $f \circ g(E) = E$.[1]

We can get a more scientific example if we consider the conversion between temperatures measured in Celsius or Fahrenheit. Indeed, if we have a temperature C in degrees celsius, that's the same temperature as $T(C) = \frac{9}{5}C + 32$ in Fahrenheit. This can be undone by the translations $T^{-1}(F) = \frac{5}{9}(F - 32)$ sending a temperature in Fahrenheit to the temperature in Celsius, and again the key feature is that if we translate from Celsius to Fahrenheit, then back to Celsius, we get the same temperature we started with. See if you can prove this!

If we consider the identity function $\mathrm{id} : A \to A$ defined by $\mathrm{id}(a) = a$ for all a in A (so named as each element maintains its identity), then the statement that functions f and g are inverses is the statement that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$.

We will have occasion to write diagrams of functions between sets as follows:

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "h"] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

[1] A linguistic aside. This condition is not quite enough to tell if a translation is correct, merely enough to tell if it is consistent. That isn't always the highest priority, as words and sentences change meaning in context. Still, this should be approximately true for real world translation.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"] & B \arrow[d, "h"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

where A, B, C, D are sets, and e, f, g, h are functions between these sets. In these diagrams, we move along arrows, composing the functions that label the arrows. We will say that a diagram commutes when any path we take between two sets, composing the functions along the way, yields the same overall function. The first diagram pictured commutes exactly when $h = g \circ f$, that is, for any a in A we'd have $h(a) = g(f(a))$. The second diagram commutes exactly when $g \circ e = h \circ f$, that is, when $g(e(a)) = h(f(a))$ for all a in A.

If you haven't seen this notation, here are two commutative diagrams expressing facts about arithmetic. Let $\mathbb{Z}$ denote the set of all the integers, so $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Let $+2 : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $+2(n) = 2 + n$, let $+3 : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $+3(n) = 3 + n$, and let $+5 : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $+5(n) = 5 + n$. The statement that $5 = 2 + 3$ is expressed in the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "+3"'] \arrow[dr, "+5"] & \\ \mathbb{Z} \arrow[r, "+2"'] & \mathbb{Z} \end{tikzcd}$$

That is, if you add three to something, then add 2 to the result, it's the same as adding 5 to what you started with. Or in symbols $2 + (3 + n) = 5 + n$. Similarly, the statement that $2+3 = 3+2$ can be expressed in the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "+2"] \arrow[d, "+3"] & \mathbb{Z} \arrow[d, "+3"] \\ \mathbb{Z} \arrow[r, "+2"] & \mathbb{Z} \end{tikzcd}$$

These diagrams are useful as they convey equalities of functions in a visual manner that keeps things organized and engages the brain's vast wealth of visual processing power.

Throughout, we will write the letter p after a given throw in a juggling pattern to denote that that throw is a pass, all other throws are assumed selves. We will let $\mathbb{Z}$ denote the integers, $\mathbb{N}$ denote the non-negative integers (so in particular $\mathbb{N}$ includes 0), and for any integer n we will let $\mathbb{Z}/n$ denote the the group of rotations of the n-gon sending vertices to vertices. Usually n will be the period of the juggling pattern in question. We have the function $C : \mathbb{Z} \to \mathbb{Z}/n$ where $C(i)$ is the rotation by i turns clockwise, or if you prefer by an angle of $\frac{i}{n} \cdot 2\pi$ radians, or $\frac{i}{n} \cdot 360$ degrees clockwise, or a fraction i/n of a full clockwise rotation. We will also on occasion write $C(i) = \bar{i}$, don't worry if this function C is meaningless to you write now, understanding it will be the project of this upcoming section.

3 Keeping time on a juggler's clock

Many juggling patterns have a periodic nature to them, where after a certain amount of time they loop back to where they started and run again. The key to our analysis will be to capture this periodic behavior as geometrically as possible. To do this, we consider one of the most common periodic phenomena we all observe in our day to day lives, the marking of time. Every 24 hours we're back at the same time of day, every 7 days we're back at the same day of the week, usually every 365 days we're back at the same day of the year, and every 12 hours we're back at the same time on the clock, time always looping back on itself.

For the most geometric interpretation, we'll focus on the clock, though we'll slightly renumber things so that the clock runs from 0 to 11 o'clock, rather than 1 to 12 o'clock (this is more similar to 24-hour time), so that 12 o'clock is now 0 o'clock. Here's where this is going: Say a juggler is juggling a 3-beat pattern where it repeats every third throw. Then we can imagine that juggler as juggling to the tick of a clock with 3 marked points, as opposed to our 12-hour clock with its 12 tick-points. Then the statement that the pattern repeats every third beat is saying that whenever the juggler's clock comes back around to the same point, they make the same throw.

Our goal here is to be more precise about the nature of arithmetic on clocks, so that we can eventually plot out juggling patterns as diagrams on clocks, building up the geometric foundation for the Préchac transformation. This visual representation of juggling patterns serves a similar purpose to sheet music for musicians. Both situations share the same phenomenon of the means of counting time looping back on itself, like counting in 4/4 time as 1, 2, 3, 4, 1, 2, 3, 4, ...

As a first question, we ask: How do we figure out where on the clock we are at any given time? For instance, if we're currently at 3 o'clock, what time will it be 145 hours from now?

To be a little more formal about this, suppose that we have agreed that time 0 is reprsented as 0 o'clock. Then for any integer $0 \leq i \leq 11$, let $\overline{i}$ be the tick-mark for i o'clock, thought of as a position on the clock. Let $T = \{\overline{0}, \overline{1}, \ldots, \overline{11}\}$ denote the set of all hour tick-marks. Then we should have a function $C : \mathbb{Z} \to T$ (think C for clock and T for tick-marks), where any time that's an integer number of hours from time 0 should land at some tick-mark on the clock [2].

Let's make some observations about this function C. First off, by construction we have $\overline{i} = C(i)$ for $0 \leq i < 11$, and $C(12) = \overline{0}$, 12 hours later we're back to the top of the clock. Since integers like to be added together, and are built by successively adding many copies of 1, to figure out where the rest of the numbers go, it will be helpful to figure out how this function C interacts with addition.

For instance, suppose you know $C(x) = \overline{5}$, and you want to know the value

[2] This notation is non-standard. Most authors use either a greek letter like μ or π to denote this function, (either evoking m for modular arithmetic, or p for projection), or simply write $C(i) = \overline{i}$ for all times, not just $0 \leq i \leq 11$

of $C(1 + x)$. This is the time it will be one hour later, which is $\overline{6}$. Similarly if you knew $C(x) = \overline{11}$, then we'd get $C(1 + x) = \overline{0}$. In general, $C(1 + x)$ is the next tickmark, read clockwise around the clock. That is, if you take any input starting time, the time it will be in one hour is the time you land on if you rotate one turn of the clock from the starting time, or 1/12th of the way around. That is, the act of allowing one hour of time to pass is represented on the clock as rotating one unit clockwise. Here is an animation of the process that might clear things up.

Similarly, if we were to allow time to move forwards by 4 hours, then we'd be interested in the value of $C(4+x)$. Then this would be equivalent to rotating the clock 4 units clockwise, or 4/12th of the way around, as in this gif .

Notice in particular that 4 hours after 8 o'clock is 12 o'clock, which we've represented as 0 o'clock, and similarly that 4 hours after 11 o'clock is 15 o'clock = 3 o'clock.

And finally, we could also ask for any given time on the clock, what time was it 2 hours ago, the value of $C(-2 + x)$? This is equivalent to rotating all our times counterclockwise by 2 units and noting where they land, as in this gif

Thus, we have a correspondence where allowing any amount of time to pass and asking what that does to times on the clock is equivalent to rotating the clock by that amount. That is, if we let R_{12} be the collection of all possible rotations of this 12-sided clock which send hour tick-marks to hour tick-marks, then we have a function $C_{\text{rot}} : \mathbb{Z} \to R_{12}$ where $C(n + x) = C_{\text{rot}}(n)\big(C(x)\big)$, where $C_{\text{rot}}(n)$ is the rotation we get by allowing n hours to pass on the clock, and $C_{\text{rot}}(-n)$ is the rotation we get by sending times on the clock to the time it was n hours ago. We could have just as well started from C_{rot} and built C, as

$$C(x) = C(x + 0) = C_{\text{rot}}(x)(C(0)) = C_{\text{rot}}(x)(\overline{0}).$$

That is, knowing what time on the clock it is at time x is the same as knowing what time on the clock it is x hours after time 0.

There are a number of features of this function C_{rot} that are worth highlighting. Philosophically, the function C thinks of integers as individual points on the number line, and we are sending points on the number line to tick-marks on the clock. The function C_{rot} doesn't output points, it outputs rotational actions, so it would be nice if we had a corresponding way to think of the integers as actions, so that C_{rot} sends actions to actions, and C sends points to points.

To do this, let $\mathbb{Z}_{\text{slide}}$ be the collection of all sliding actions on the number line that send integers to integers. We have a function point : $\mathbb{Z}_{\text{slide}} \to \mathbb{Z}$ sending a sliding action s to the point $s(0)$. This function is invertible, with inverse sending a point i to the sliding action that is addition by i. This is the unique sliding action that sends 0 to i. Here is a gif of the correspondence for the number 3.

Note for instance that the statement that $3 + 2 = 5$ says that after sliding over by 3 units, the number 2 lands on top of the number 5, and indeed for any number x, after sliding over by 3 units it lands on top of $3 + x$ [3]. The juggler's

[3] The same correspondence holds for any real number, where any real number can be

relationship to this duality is that the we think of the beats in time at which we throw like the points on a number line, and a throw that starts at some time and lands at another time as corresponding to the sliding action that takes the first time to the second. The arc of a ball in the air is equated to the active passage of time, and the moments when we throw or catch a ball are points on the timeline.

Now we build a similar dual perspective on the clock, between rotations on the clock and the hour tick-marks. To do this, note that since the rotations in R_{12} by definition send hour tick-marks to hour tick-marks, we get a function $\text{tick} : R_{12} \to T$ where for any rotation r in R_{12} we have that $\text{tick}(r) = r(\overline{0})$. That is, we send any rotation to where it moves time zero. In fact, this function is invertible. That is, for any tick-mark on the clock, there is a unique rotation that sends $\overline{0}$ to that tick-mark, the rotation clockwise by that many units. You can visualize this as you rotate the clock by dragging the top tick-mark around, holding onto this point that starts at the top, and the rest of the points follow along. For this point to end at an hour tick-mark, there are 12 possibilities. Here is a gif of the correspondence for the tick-mark $\overline{5}$.

One possibility for where $\overline{0}$ lands is back where it started, corresponding to the identity rotation, leaving the clock unchanged. Otherwise it lands at one of the other 11 hour tick-marks, and so we get 12 possible rotations in total, the rotations $C_{\text{rot}}(i)$ for $0 \leq i \leq 11$ [4].

Now it's time to compare these two perspectives. Let $T = \{\overline{0}, \overline{1}, \ldots, \overline{11}\}$ is the set of hour tick-marks on the clock. Let $\text{point} : \mathbb{Z} \to \mathbb{Z}$ be the function which takes a sliding action and outputs the point it corresponds to, sending a sliding action s to the point $s(0)$. Let $\text{tick} : R_{12} \to T$ be the function which takes a rotation on the clock and outputs the corresponding hour tick-mark. That is, for any rotation r on the clock we have $\text{tick}(r) = r(\overline{0})$. Then we have a commutative diagram, which we will call the action diagram. [5]

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\text{point}"] \arrow[d, "C_{\text{rot}}"'] & \mathbb{Z} \arrow[d, "C"] \\ R_{12} \arrow[r, "\text{tick}"'] & T \end{tikzcd} \tag{1}$$

This diagram says that the clock function turns sliding actions on the line, the active passage of time, into rotational actions on the clock. Further, if you look at a sliding action on the integers, watch where 0 goes, and then go to

thought of as either a point on the number line, or a sliding action on that number line, where we recover the point from the action as the point where 0 lands.

[4]The same correspondence holds for points on the circle and rotations. Any rotation yields a point on the circle by asking where does the topmost vertex go, and any point determines a unique such rotation. It is more standard to focus on the rightmost point on the circle and ask where does it go, we choose the top point for greater similarity with the clock.

[5]We get a similar relationship between real numbers as sliding actions/points on the number line and rotations/points on the circle. In that context, the resulting map from the line to the circle, wrapping time around the circle, is called the universal cover of the circle, for reasons topologists care about. The same reasoning as here shows that this map turns sliding actions into rotation actions, in a way that respects composition of those actions.

the corresponding time on the clock, that's the same as if you watch where the tick-mark $\overline{0}$ goes under the corresponding rotation action for the same amount of time, which is true and demonstrated in this gif this gif.

Alternately, if you prefer to work more algebraically, then the commutativity of this diagram asserts that for all integers i we have

$$\begin{aligned}\mathrm{tick}(C_{\mathrm{rot}}(i)) &= C(\mathrm{point}(i))\\ C_{\mathrm{rot}}(i)(\overline{0}) &= C(i),\end{aligned}$$

that is, the time that it reads on the clock at hour i is the time we obtain by letting i hours pass from the top tickmark at time 0, which is true.

There are some special arithmetic properties of this function C_{rot} as well. To study these, we give a special name to the rotation that does nothing to the clock, corresponding to allowing no extra time to pass. We call this special rotation the identity rotation, often shortened to just id (so named as each point maintains its identity). First of all, we have that $C_{\mathrm{rot}}(12) = \mathrm{id}$, in that no matter what time on the clock it is, 12 hours later it will be the same time. This equality is highlighted in this gif.

Furthermore, suppose we let 3 hours pass, then let 2 hours pass. On the number line, we've slid over 3 units, then 2 units, overall sliding for 5 units. That is, addition of numbers corresponds to composition of these sliding actions, which is a beautiful result in and of itself. On the clock rotating by these 5 units is decomposed as rotating 3 units, then rotating 2 units. Symbolically, we have an operation on R_{12} where if we have two rotations r, s, we get a new rotation $r \circ s$, read aloud as "r composed with s", where we first apply s, then apply r. Then we get the fundamental property

$$C_{\mathrm{rot}}(i + j) = C_{\mathrm{rot}}(i) \circ C_{\mathrm{rot}}(j).$$

Here is a gif that expresses this relationship visually.

The fact that the horizontal arrows in the tick-diagram are are invertible mean that we have a way of combining tick-marks as well. That is, if you have two tickmarks a, b, these are the tick-marks associated to two unique rotations r and s, that is $a = \mathrm{tick}(r)$ and $b = \mathrm{tick}(s)$. Then we can define $a + b = \mathrm{tick}(r \circ s)$, an operation which was not so obvious thinking of these as just points on a clock.

Combining these properties yields some powerful results.

For instance, if r, s are any two rotations on the clock, say $\mathrm{tick}(r) = \overline{i}$ and $\mathrm{tick}(s) = \overline{j}$. Then we have $\mathrm{tick}(r) = C(i)$ and $\mathrm{tick}(s) = C(j)$, and so we have that $r = C_{\mathrm{rot}}(i)$ and $s = C_{\mathrm{rot}}(j)$. We then have

$$\begin{aligned}r \circ s &= C_{\mathrm{rot}}(i) \circ C_{\mathrm{rot}}(j)\\ &= C_{\mathrm{rot}}(i + j)\\ &= C_{\mathrm{rot}}(j + i)\\ &= C_{\mathrm{rot}}(j) \circ C_{\mathrm{rot}}(i)\\ &= s \circ r.\end{aligned}$$

That is, if we have any two rotations, regardless of what order we compose them in, we get the same overall rotation, a property which was not immediately obvious.

For another useful property, note that

$$\begin{aligned} C_{\text{rot}}(17) &= C_{\text{rot}}(12+5) \\ &= C_{\text{rot}}(12) \circ C_{\text{rot}}(5) \\ &= \operatorname{id} \circ C_{\text{rot}}(5) \\ &= C_{\text{rot}}(5) \end{aligned}$$

So that the time on the clock 17 hours later is the same as the time 5 hours later.

Similarly, we get

$$\begin{aligned} C_{\text{rot}}(28) &= C_{\text{rot}}(24+4) \\ &= C_{\text{rot}}(12+12+4) \\ &= C_{\text{rot}}(12) \circ C_{\text{rot}}(12) \circ C_{\text{rot}}(4) \\ &= \operatorname{id} \circ \operatorname{id} \circ C_{\text{rot}}(4) \\ &= C_{\text{rot}}(4), \end{aligned}$$

so that the time on the clock 28 hours later is the same as the time 4 hours later. In general, when we perform division with remainder, we may write any positive number i as $i = 12 \cdot q + r$ with $0 \leq r < 12$ and $q \geq 0$ an integer (that is we get $12q$ is the largest multiple of 12 still smaller than i, and $r = i - 12q$, e.g. $36 = 3 \cdot 12$ is the largest multiple of 12 that's still smaller than 39, and so we get $39 = 3 \cdot 12 + 3$) so that

$$\begin{aligned} C_{\text{rot}}(i) &= C_{\text{rot}}(12 \cdot q + r) \\ &= C_{\text{rot}}(12 \cdot q) \circ C_{\text{rot}}(r) \\ &= C_{\text{rot}}(\underbrace{12+12+\cdots+12}_{q \text{ times}}) \circ C_{\text{rot}}(r) \\ &= \underbrace{C_{\text{rot}}(12) \circ C_{\text{rot}}(12) \circ \cdots \circ C_{\text{rot}}(12)}_{q \text{ times}} \circ C_{\text{rot}}(r) \\ &= \underbrace{\operatorname{id} \circ \operatorname{id} \circ \cdots \circ \operatorname{id}}_{q \text{ times}} \circ C_{\text{rot}}(r) \\ &= C_{\text{rot}}(r) \end{aligned}$$

This also works for negative numbers, as $C_{\text{rot}}(-12) = \operatorname{id}$ as well (12 hours ago, it was the same time on the clock as it is now). Because of this connection with division with remainder, we will denote this group of rotations $\mathbb{Z}/12$, to be read aloud as "Z mod 12", where mod is short for modular, as in modular arithmetic, another word for working with these remainders.

Now we ask the question, when do do two different amounts of time passing give the same rotation on the clock? That is, when do we have $C_{\text{rot}}(i) = C_{\text{rot}}(j)$?

Well, if we write $i = q \cdot 12 + r$ and $j = t \cdot 12 + s$ with $0 \leq r, s < 12$, then we have $C_{\text{rot}}(i) = C_{\text{rot}}(r)$ and $C_{\text{rot}}(j) = C_{\text{rot}}(s)$, so $C_{\text{rot}}(i) = C_{\text{rot}}(j)$ if and only if $C_{\text{rot}}(r) = C_{\text{rot}}(s)$. I claim that this is true if and only if $r = s$. Indeed, if $r = s$ then obviously $C_{\text{rot}}(r) = C_{\text{rot}}(s)$. And furthermore, since $0 \leq r, s < 12$, we have that $C_{\text{rot}}(r)$ and $C_{\text{rot}}(s)$ send the 0-hour tick-mark to the tick-marks labeled r and s, respectively. For these two rotations to be the same, we must then have that $r = s$.

Thus, we have that two different amounts of time give the same rotation on the clock if and only if they have the same remainder mod 12. That is, we can simultaneously write

$$\begin{aligned} i &= q \cdot 12 + r \\ j &= t \cdot 12 + r. \end{aligned}$$

Subtracting these two equations yields

$$\begin{aligned} i - j &= q \cdot 12 + r - (t \cdot 12 + r) \\ &= q \cdot 12 + \cancel{r} - t \cdot 12 - \cancel{r} \\ &= q \cdot 12 - t \cdot 12 \\ &= (q - t) \cdot 12, \end{aligned}$$

so that $i - j$ is divisible by 12. Similarly, if $i - j$ is divisible by 12, say $i - j = 12 \cdot k$ then $i = 12 \cdot k + j$ and

$$C_{\text{rot}}(i) = C_{\text{rot}}(12 \cdot k + j) = C_{\text{rot}}(12 \cdot k) \circ C_{\text{rot}}(j) = C_{\text{rot}}(j),$$

as before. Thus, we have that

$$C_{\text{rot}}(i) = C_{\text{rot}}(j) \iff i - j \text{ is divisible by } 12.$$

The commutativity of the action diagram means that all of these results hold for C as well. For instance, suppose we write $i = 12q + r$ with $0 \leq r \leq 11$, then we have

$$\begin{aligned} C(i) &= C(12q + r) \\ &= C(\text{point}(12q + r)) \\ &= \text{tick}(C_{\text{rot}}(12q + r) \\ &= C_{\text{rot}}(12q + r)(\overline{0})) && \text{(commutativity of the action diagram)} \\ &= C_{\text{rot}}(12q) \circ C_{\text{rot}}(r)(\overline{0}) \\ &= \text{id} \circ C_{\text{rot}}(r)(\overline{0}) \\ &= C_{\text{rot}}(r)(\overline{0}) \\ &= \text{tick} \circ C_{\text{rot}}(r) \\ &= C(r) \end{aligned}$$

Because of this, from here on out we will stop being so formal about the difference between C and C_{rot}, between points on the line and sliding actions, and between hour tick-marks and rotations on the clock, where which interpretation we mean should be clear from context.

The final insight here is that there was nothing special about 12. For any positive integer n, we could consider measuring time on an n-hour clock, which is best represented as an n-gon, a polygon with n sides connecting the hour tickmarks. Then again we'd have the group $\mathbb{Z}/n$ of rotations of this n-gon which send hour tickmarks to hour tickmarks, i.e. those which send the vertices of the n-gon to other vertices of the n-gon. And again we'd have a function $C : \mathbb{Z} \to \mathbb{Z}/n$ with $C(i)$ being the time on the clock it is at time i. Again we'd have the properties

1. $C(0) = \text{id}$
2. $C(i + j) = C(i) \circ C(j)$
3. If $i = nq + r$ then $C(i) = C(r)$
4. $C(i) = C(j) \iff i - j$ is divisible by n.

   For the rest of the paper we will not be so careful about distinguishing between C and C_{rot} and call them both C, just as we are often not careful about distinguishing between integers as sliding actions or as points on the number line.

4 Review of solo siteswaps

The main idea of a solo siteswap starts with imagining a juggler making throws and catches evenly spaced out in time. Assigning an integer to every beat, we get an association where the elements of $\mathbb{Z}$ corresponds to beats in a juggling pattern. At any given beat, we can track when the ball thrown at that beat is thrown again. This gives us what we'll define as a *Juggling function* $S : \mathbb{Z} \to \mathbb{Z}$ defined by $S(i) = j$ means the ball thrown at beat i is thrown again at beat j (using the letter S for siteswap). To be totally precise, it is entirely possible for a juggler to make no throws at a given beat (maybe they've flashed all the balls high to make room to clap or spin around). So at any beat i where no throw occurs, we define $S(i) = i$.

If we study the properties of this function, a few rise up to greet us:

(1) S is *rightward-moving*, in the sense that $S(i) \geq i$ for all integers i.

(2) If we assume that the pattern has no collisions, then no two balls thrown at different beats land at the same time. That is, if $i \neq j$ then $S(i) \neq S(j)$, though we'll more commonly use the formulation that if $S(i) = S(j)$ then $i = j$. In math lingo, this says that the function S is **one to one/injective**

(3) In order to make a throw at beat i, the juggler must have caught a ball from some other throw in the past. That is, for any integer i where a

throw is made, there must be some integer j with $j < i$ and $S(j) = i$. And at any beat where no throw is made, we have $S(i) = i$, regardless, every integer i is S(something). In math lingo, this says that the function S is **onto/surjective**

Combining these properties yields that S shuffles the integers around somehow, but that's all it does, re-ordering the integers. In particular we can define an inverse function $S^{-1} : \mathbb{Z} \to \mathbb{Z}$ where we send an integer j to the unique integer i with $S(i) = j$. Such an i exists because S is onto, and is unique because S is one to one. That is, the equations $S(i) = j$ and $i = S^{-1}(j)$ are equivalent to each other. In math lingo, this is saying that S is a rightward-moving permutation of the integers.

Furthermore, property (1) allows us to define an associated function $T : \mathbb{Z} \to \mathbb{N}$ given by $T(i) = S(i) - i$ called the *throw height* function. That is, at any beat i, $T(i)$ tells you how many beats later the juggler will throw the ball at that beat. If you keep the amount of time a ball spends in your hands before it is thrown (called dwell time) constant, this is equivalent to specifying how high you throw the ball at beat i. Indeed, as may be familiar to jugglers, T(i) = 5 often means you throw the ball at beat i to the height of your 5 ball pattern, though there is certainly room to fudge these heights. The function T is best thought of as instructions to the juggler, "throw the ball this height at this beat", and the function S is best thought of as observations of the juggler – recording the results of executing those instructions, where the balls go over time.

The throw height function and the juggling function come hand in hand. Above we've shown how to recover T from S, but you can also build S from T by defining $S(i) = i + T(i)$, and indeed this is usually the order jugglers prefer. [6].

Many juggling tricks have a periodic nature to them, where after a certain number of throws they're back to where they started. In the juggling diagram above, we can see that shifting the whole diagram over by 3 units leaves the picture unchanged (though the colors get swapped around), which you can verify by just noting that any two successive peaks of arches of the same height are distance 3 apart. So we add one extra condition to these patterns to capture this periodicity. This extra condition can be stated in multiple equivalent ways, as given in the following lemma.

Lemma 1. *Fix a positive integer n, and let $C : \mathbb{Z} \to \mathbb{Z}/n$ be the function sending an integer to the rotation of the n-gon by i turns. For any pair of*

[6]For the mathematician: In general if you have a group G acting on a set X, then for any function $T : X \to G$ you always get a function $S : X \to X$ via $S(x) = T(x) \cdot x$. If you start from S, the existence of a corresponding T requires the action be transitive, and the statement that T is uniquely determined requires that the action be free. Both together means that $X \cong G$ as a G-set, i.e. X is a G-torsor. We can also intepret this via the Yoneda lemma. The action map $G \times X \xrightarrow{\sigma} X$ is equivalent to a map $\mathrm{Hom}(Y, G) \times \mathrm{Hom}(Y, X) \xrightarrow{\sigma_*} \mathrm{Hom}(Y, X)$ for any set Y (natural in Y), and in the case $Y = X$, if you fix the map $\mathrm{id} : X \to X$ as the second coordinate in the product, then you get a map $\mathrm{Hom}(X, G) \to \mathrm{Hom}(X, X)$ via $T \mapsto \sigma_*(T, \mathrm{id})$

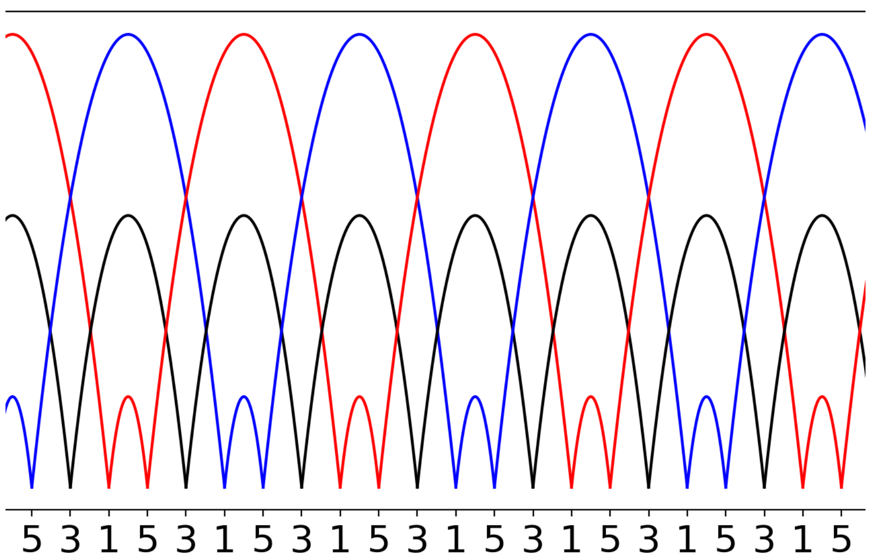

Figure 1: A portion of a juggling function, where the labels on the horizontal axis are the throw heights made at that beat, and the heights of the arches are suggestive of the heights of throws. This picture can be loosely thought of as the result of a long exposure photograph taken from the side as a juggler juggles this pattern while walking forwards. At any beat labeled 5, the arch that goes up there comes down 5 beats later, and similar for the other throws. The three different colored trajectories are the paths of the three balls in this pattern. Note that the average of the throw heights is $\frac{5+3+1}{3} = 3$, the number of balls in the pattern.

functions $S : \mathbb{Z} \to \mathbb{Z}$ *and* $T : \mathbb{Z} \to \mathbb{N}$ *with* $S(i) = i + T(i)$*, the following are equivalent.*

(1) $T(i + qn) = T(i)$ *for all integers* i, q

(2) $S(i + qn) = S(i) + qn$ *for all integers* i, q

(3) There exists a unique function $\widetilde{T} : \mathbb{Z}/n \to \mathbb{N}$ *such that the following diagram commutes*

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "C"] \arrow[dr, "T"] & \\ \mathbb{Z}/n \arrow[r, dashed, "\widetilde{T}"'] & \mathbb{N} \end{tikzcd}$$

meaning $\widetilde{T} \circ C = T$.

A juggling function satisfying any of these three conditions is called ***periodic with period* n** [7]

You can verify that all of these conditions hold in the figure above. That is, if you take any two throws made 3 beats apart, they also land 3 beats apart, which is the second condition. And furthermore the throw heights repeat every third throw, which is the first condition, so they can be recorded on a 3-hour clock, which is the third condition.

Proof. The first condition in this lemma with $q = 1$ is maybe the simplest definition of periodicity. No matter what throw you're making now, n beats later you'll be making it again. The last condition in the lemma says that whenever your n-hour clock winds back around to the same tick-mark, you make the same throw, also a natural definition of periodicity. These are the same condition, as we saw in section 3 that $C(i) = C(j)$ if and only if $j = i + qn$ for some integer q, so we make the same throw whenever we hit a given tick-mark on the clock, if and only if we make the same throw whenever the times differ by a multiple of n.

If you want to get fancy with it, condition (2) is stating that S is equivariant with respect to the action of the (sub)group $n\mathbb{Z}$, a very nice kind of mathematical symmetry indeed. Rearranging (2) to $S(i+qn) - S(i) = qn$ and plugging in the definition of throw heights gives

$$i + qn + T(i + qn) - (i + T(i)) = qn \iff T(i + qn) - T(i) = 0,$$

where the symbol $\iff$ means the equality to the left of the arrow holds if and only if the equality to the right arrow holds, they are equivalent to each other. The second equality is the equality in (1) rearranged, so conditions (1) and (2) are equivalent. Heuristically, this says that if two throws differ in time by some number of loops around the clock, then so do the catches, which makes sense as the catches are both the same distance forward from the throws, so the distance between the catches equals the distance between the throws.

□

An natural question one might ask is how can we detect the number of balls in a juggling pattern? For that we have the following beautiful result whose proof we will leave to the appendix.

[7] For the mathematician: We can again look at the map induced by the action σ_* : $\mathrm{Hom}(Y, G) \times \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, X)$, for any other set Y. The question is if Y is also a G-set, when is $\sigma_*(T, f)$ equivariant with respect to a subgroup $H \subseteq G$? Here we're looking at the case $Y = X$ and $f = \mathrm{id}$, noting that f is already equivariant. If the action of G on X is free, and f is equivariant, then we get that $\sigma_*(T, f)$ is equivariant if and only if $T(h \cdot y) = hT(y)h^{-1}$ for all $h \in H, y \in Y$. In the case that G is abelian, or just $H \subset Z(G)$, this reduces to the statement here, that $T(h \cdot y) = T(y)$. And vice versa, if this equality holds for T, then $\sigma_*(T, f)$ is equivariant if and only if f is equivariant. See if you can prove this!

Theorem 2. *Let T be the throw heights of a juggling function of period n. Then the number of balls in the pattern is the average of the throw heights. That is*

$$\#balls = \frac{\sum_{i=0}^{n-1} T(i)}{n}$$

So for example the period 3 pattern with throw heights $6, 4, 5$ has

$$\#\text{balls} = \frac{6+4+5}{3} = \frac{15}{3} = 5.$$

Periodic juggling patterns also have an additional property that will turn out to be very useful for the Préchac transformation.

Corollary 3. *Let S be a juggling function with period n. Then there is a unique function $\widetilde{S} : \mathbb{Z}/n \to \mathbb{Z}/n$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "S"] \arrow[d, "C"] & \mathbb{Z} \arrow[d, "C"] \\
\mathbb{Z}/n \arrow[r, dashed, "\widetilde{S}"] & \mathbb{Z}/n.
\end{tikzcd}$$

Furthermore, if $\widetilde{T} : \mathbb{Z}/n \to \mathbb{N}$ are the periodic throw heights, then $\widetilde{S}(\bar{i}) = \bar{i} + C(\widetilde{T}(\bar{i}))$, with addition using modular arithmetic.

Proof. Suppose such a function $\widetilde{S}$ existed. Let $\bar{i}$ in $\mathbb{Z}/n$ be some tick-mark on the clock for $0 \leq i \leq n-1$. Then since $\bar{i} = C(i)$, we're forced to define

$$\begin{aligned}
\widetilde{S}(\bar{i}) &= \widetilde{S} \circ C(i) \\
&= C \circ S(i) \\
&= C(i + T(i)) \\
&= C(i) + C(T(i)) \\
&= \bar{i} + C(T(\bar{i})),
\end{aligned}$$

so if $\widetilde{S}$ existed making the diagram commute, it must have the given formula.

Thus, we define $\widetilde{S}$ by the given formula, and check that the diagram commutes. To do this, note that by the periodicity assumption, whenever the clock winds around to a given tick-mark $\bar{i}$, we throw a ball for the same height $\widetilde{T}(\bar{i})$, and so it lands at the desired position, $\widetilde{T}(\bar{i})$ ticks of the clock later. This shows that the diagram commutes, concluding the proof.

□

The key observation is that $\widetilde{S}$ is a function between two finite sets, which makes it simpler to study than S. And furthermore, you can reconstruct S from $\widetilde{S}$ as follows.

Theorem 4. *Let $\widetilde{S} : \mathbb{Z}/n \to \mathbb{Z}/n$ be any function. Let $\widetilde{T} : \mathbb{Z}/n \to \mathbb{N}$ be any function such that*

$$C\left[\widetilde{T}(i)\right] = \widetilde{S}(i) - i.$$

Define $S : \mathbb{Z} \to \mathbb{Z}$ by

$$S(i) = i + \widetilde{T}(C(i)).$$

Then S is a juggling function if and only if $\widetilde{S}$ is a permutation of $\mathbb{Z}/n$.

In this theorem, the statement that $\widetilde{S}$ is a permutation of $\mathbb{Z}/n$ means that there is some other function $B : \mathbb{Z}/n \to \mathbb{Z}/n$ such that $\widetilde{S} \circ B = B \circ \widetilde{S} = \mathrm{id}$. That is, $\widetilde{S}$ shuffles the elements of $\mathbb{Z}/n$ around, in such a way that the can be put back in order by B.

What this is saying is that most of the data in determining a juggling function lies in picking a permutation of $\mathbb{Z}/n$. That is, any two juggling functions with the same underlying permutation of $\mathbb{Z}/n$ have all their throw heights differing by a multiple of n. It's a little more challenging to show that if $\widetilde{S}$ is a permutation, then S is a juggling function, so we'll leave that to the appendix at the end. However, we can show that if S is a juggling function, then $\widetilde{S}$ is a permutation.

Proof. Let $S : \mathbb{Z} \to \mathbb{Z}$ be a juggling function of period n. By our reasoning at the beginning of this section there is a function $S^{-1} : \mathbb{Z} \to \mathbb{Z}$ such that $S \circ S^{-1} = S^{-1} \circ S = \mathrm{id}$. I claim that S^{-1} is also periodic of period n. That is, we want to show that for any i in $\mathbb{Z}$ that

$$S^{-1}(i + n) = S^{-1}(i) + n.$$

To see this, we compute

$$\begin{aligned} i + n &= S(S^{-1}(i)) + n \\ &= S(S^{-1}(i) + n), \end{aligned}$$

where the second equality holds because S has period n. Applying S^{-1} on both sides yields

$$S^{-1}(i + n) = S^{-1} \circ S(S^{-1}(i) + n) = S^{-1}(i) + n,$$

where the second equality holds as $S^{-1} \circ S(\text{blah}) = \text{blah}$. This shows that S^{-1} has period n as well.

Thus, applying the previous result to S^{-1} means there exists a unique function $\widetilde{S^{-1}}$ such that the following diagram commutes.

I claim that $\widetilde{S^{-1}} = \widetilde{S}^{-1}$. That is, the values of the inverse of S on the clock are exactly the inverse to the values of S on the clock, which is a reasonable claim. We will prove this in two different ways, by direct computation and by appeal to diagrams.

To show this by direct computation, note that for any tick-mark $\bar{i}$ with $0 \leq i < n$ we have that $\bar{i} = C(i)$, and so we compute

$$\begin{aligned}
\widetilde{S^{-1}} \circ \widetilde{S}(\bar{i}) &= \widetilde{S^{-1}} \circ \widetilde{S}(C(i)) \\
&= \widetilde{S^{-1}}(C(S(i))) && \text{(definition of } \widetilde{S}) \\
&= C \circ S^{-1}(S(i)) && \text{(definition of } \widetilde{S^{-1}}) \\
&= C(i) \\
&= \bar{i}.
\end{aligned}$$

Since this holds for all tick-marks on the clock, we have that $\widetilde{S^{-1}} \circ \widetilde{S} = \mathrm{id}$, and we can similarly show that $\widetilde{S} \circ \widetilde{S^{-1}} = \mathrm{id}$, which is the desired result.

For a more diagram-focused proof, note that we can combine the diagrams defining $\widetilde{S}$ and $\widetilde{S^{-1}}$ to get a larger commutative diagram.

Ignoring the middle term and just looking at the outer commuting rectangle gives the diagram.

But by definition we have $S^{-1} \circ S = \mathrm{id}$, and so this is actually the diagram

However, we also evidently have the commuting diagram

as the identity arrow doesn't do anything to C, expressed more formally as

$$\mathrm{id} \circ C(i) = C(i) = C \circ \mathrm{id}(i)$$

for all i in $\mathbb{Z}$. But then by the uniqueness in the previous corollary, we get that

$$\mathrm{id} = \widetilde{S^{-1}} \circ \widetilde{S}.$$

Similar arguments show that

$$\mathrm{id} = \widetilde{S} \circ \widetilde{S^{-1}},$$

and so

$$\widetilde{S^{-1}} = \widetilde{S}^{-1},$$

as desired. □

If one is interested in verifying whether or not a given juggling pattern is a valid siteswap or not, we get the following incredibly practical result.

Corollary 5. *Let $\widetilde{T} : \mathbb{Z}/n \to \mathbb{N}$ be any function, and define $\widetilde{S} : \mathbb{Z}/n \to \mathbb{Z}/n$ as $\widetilde{S}(i) = i + C(T(i))$. Then $\widetilde{T}$ describes the throw heights of a juggling function if and only if $\widetilde{S}$ has no collisions, that is, it is one to one.*

Proof. Applying the previous result, we just need to show that $\widetilde{S}$ is a permutation (one to one and onto) if and only if it is one to one. We have the easy implication that being a permutation implies being one to one by definition, so we just need to prove the reverse implication.

So now we have to how that if $\widetilde{S}$ is one to one, it is also onto. We instead prove the contrapositive, that if $\widetilde{S}$ is not onto, then it is not one to one. We imagine the function $\widetilde{S}$ as some way of taking n balls, labeled with the numbers 1 to n, and placing them in n buckets, also labeled with the numbers 1 to n, where $\widetilde{S}(i)$ is the label of the bucket we place ball i in.

If $\widetilde{S}$ is not onto, then we are placing n balls into fewer than n buckets. Thus, there must be some bucket that has more than one ball in it, an argument often referred to in mathematics as the pigeonhole principle. For example, if you are trying to place 5 balls into 3 buckets, then some bucket must have more than one ball, as otherwise there'd only be space to place at most 3 balls.

$\square$

To get to the theory of Préchac transformations, we suggest a helpful way of visualizing permutations of $\mathbb{Z}/n$. As we've been doing, we view $\mathbb{Z}/n$ as a collection of symmetries of a regular n-gon, where the action of adding i is identified with rotating the n-gon by an angle of $2\pi\frac{i}{n}$ radians or $360\frac{i}{n}^\circ$, or rotating the n-gon by i turns, where 1 turn moves every vertex over by 1 counterclockwise. Note that in either case that adding n is associated with rotating by 2π radians or 360° or n full turns of the n-gon. That is, adding n means doing nothing, which is what adding $n = 0$ should do in $\mathbb{Z}/n$. We can identify the elements of $\mathbb{Z}/n$ by labeling the vertices of the n-gon, so that addition by i is the rotation sending the vertex labeled 0 to the vertex labeled i.

Then we picture a permutation $\widetilde{S} : \mathbb{Z}/n \to \mathbb{Z}/n$ as a decoration on the n-gon, drawing an arrow pointing from vertex i to vertex j if $\widetilde{S}(i) = j$. The statement that $\widetilde{S}$ is a permutation means exactly that every vertex on the n-gon has exactly one arrow pointing out of it to $\widetilde{S}(i)$, and exactly one arrow pointing into it, from $\widetilde{S}^{-1}(i)$. The visceral meaning of the arrows is that the arrow at vertex i represents the flight path of the ball thrown at beat i, as recorded on the clock.

In these pictures, we'll have the n-gon oriented clockwise, so that vertex $i+1$ is immediately clockwise from vertex i. Instead of labeling the vertices corresponding to its element in $\mathbb{Z}/n$, we'll label it with the throw height at that beat, so at vertex i we'll write $\widetilde{T}(i)$, some integer j chosen that moving clockwise around the n-gon j edges lands at where the arrow from that vertex is pointing. Or alternately, so that $\widetilde{S}(i) = C(\widetilde{T}(i))(i)$, which is the rotational version of $S(i) = T(i) + i$. Most geometrically, we have that mod n, $\widetilde{T}(i)$ is the oriented distance from the tail to the tip of the arrow starting at vertex i.

Figure 2: The siteswap 333. We could also have chosen 000 or 600 or 630, for example

Figure 3: The siteswap 531, could also have chosen 231 or 501, for example

Figure 4: The siteswap 6451, could also have chosen 2451 or 6411, for example

Figure 5: The siteswap 97531.

There's an interesting property about patterns that these pictures make clear. These pictures break up the vertices of the polygon into little cycles. For instance in the pattern 531, the vertices labeled 5 and 1 are in one cycle, while the vertex labeled 3 is in its own cycle. This means that we can color code this pattern, so that all the balls throwing 5s and 1s are one color, while the balls throwing 3s are another color. In fact there are two balls throwing the 5s and 1s, while one ball throws the 3, as you can verify by looking at the juggling function diagram towards the beginning of this section.

Similarly the pattern 6451 breaks up into two cycles, one among the throws 6, 5, and 1, and the other being the 4, so we can again color code this pattern into 2 colors, say juggled with 3 red balls throwing 6, 5, 1 and 1 blue ball throwing 4s. Or in the pattern 97531, we can see three cycles, and we can color code this pattern with 3 colors.

Another property that these pictures make clear is that if you take one of these pictures and then perform any symmetry of the n-gon, you'll end up with another picture of the same type, a decoration of the n-gon with arrows forming a permutation of the vertices. Furthermore, if you perform a rotation, then the throw heights will also be unchanged, just counted in a different cycle, as the throw heights are read as the oriented distance between the tail and tip of arrows. Or alternately, if we take two times that are say 3 hours apart on the clock and let any amount of time pass, those two times will remain 3 hours apart. Or in fancy language if $y = C(3)(x)$, then we get

$$\begin{aligned} C(801)(y) &= C(801) \circ C(3)(x) \\ &= C(801+3)(x) \\ &= C(3+801)(x) \\ &= C(3) \circ C(801)(y), \end{aligned}$$

so if $y' = C(801)(y)$ and $x' = C(801)(x)$ we still have $y' = C(3)(x')$.

This process is expressed visually in this gif, where the siteswap 6451 rotates three turns clockwise to become 4516. There's an interesting phenomenon here, that the new first throw is the throw that originally occurred three beats before the end. Similarly if you wanted to say start some pattern two beats later, that would correspond to rotating the picture two turns counterclockwise.

The two types of symmetries of the n-gon are rotations and reflections. Rotations don't change throw heights for solo patterns, as we've seen above. Reflections have a more complicated effect on the throw heights, which while interesting, won't lead us towards the Préchac transformation. We'll discuss reflections in an appendix at the end for those interested.

It's now time to use this geometric intuition to build the Préchac transformation.

5 Building the Préchac transformation

Let's imagine we have two jugglers juggling the same pattern side by side. Here I will identify the juggler with their decorated n-gon. Let's say we have to jugglers juggling the 3-club pattern 5313 at the same time as each other. I'll represent this as having one square drawn above the other, with the vertices matching up to mean they're throwing at the same time.

Figure 6: Two jugglers juggling 5313

To turn one of these throws into a pass means to take an arrow with tail in one square, and move its tip to the other square. Our first alteration to the pattern is to choose one throw in the pattern, and turn it into a pass by having both jugglers make that throw at the same height, but to the other person. This corresponds in the diagram to picking an arrow from each pattern, and swapping the locations of their tips. Here is the result from turning the 5 into a pass.

Figure 7: Two jugglers passing 5p313

What we've stumbled upon here is symmetric same time passing. Two jugglers juggle the same solo pattern at the same time, and on the same beat pass an object that they were going throw as a self at the original throw height. Standard 6 club 2-count, 3-count, ultimates, or any other similar pattern all follow this form, as do 2-count patterns with even numbers of clubs.

To see why this always produces a valid juggling pattern, we just need to verify that at every vertex in the picture, there is exactly one arrow pointing in, and one arrow pointing out. Swapping the locations of the tips of two arrows doesn't change the number of arrows pointing out anywhere. Say we swap the tips of arrows A and B landing at positions x and y, respectively. Before the swap, A was x's inward-pointing arrow, and B was y's inward-pointing arrow. After the swap A is y's inward pointing arrow, and B is x's inward-pointing arrow. So these two vertices still have one arrow in and one arrow out, and since we haven't changed the arrows going in anywhere else, we still have a valid juggling pattern.

If we're in the interest of making new patterns from old ones, we might be lead to the idea of performing a symmetry of the n-gon on one of these two patterns. Understanding how these patterns are changed with rotation is not well suited for written prose, so instead I've made a youtube video describing the Préchac transformation. This video contains the core of the whole discussion, feel free to pause and rewind to think about it.

Here is the interactive animation I go through in the video, where you can explore the operations of turning these selves into a pass, and rotating the top juggler's pattern, effectively changing when the top juggler starts juggling their throws relative to the bottom juggler. If you open that link and look to the algebra view to the left, there are two sliders, one labeled "pass", and the other labeled "angle". The first slider lets you switch between throwing the 5s as selves, when pass is 0, and throwing them as passes, when pass is 1. The second slider lets you rotate the top juggler, from an angle of 0, where both patterns are lined up, to an angle of 180°, where the top juggler has been rotated halfway around, all the way back around to an angle of 360°. You can click and drag to pan the camera, or scroll on the image to zoom in and out.

Here is an animation of the pattern I go through in the video, 3p313. This is such a fun little pattern, it's a 5 club pattern where each juggler is doing a slightly easier version of a 6 club every others/4 count, but shifted off from each other by 2 beats.

Note that at the exact same moment one juggler is throwing a pass, the other juggler is throwing the 1 as a zip across. Not also that the throw heights are all the same as for solo juggling, a 3p goes about as high as a regular 3, which is one of the main advantages of Préchac notation from a practical standpoint. The color coding is also interesting. Note that the white clubs are never passed, and the red clubs are always looped as a 3 from one juggler to themselves, a 3p to the other juggler, then the same loop for the other juggler.

I made this animation using JoePass, a great piece of software for animating all kinds of juggling patterns with many jugglers. If you want to watch videos of jugglers doing Préchac patterns I highly recommend Jorge and Santi's instagram.

They post a lot of great patterns, usually include the Préchac notation in the description, and often color code the patterns to make the underlying math clearer.

Let's be a little more systematic about the process that I described in the video. We start with two jugglers juggling the same pattern of period n at the same time as each other, represented as two decorated n-gons stacked vertically one above the other. The we start to rotate the top n-gon clockwise, corresponding to the top juggler starting their pattern after the bottom juggler. After making t turns, the top juggler is now making their pass t beats later, and so their pass's throw height decreases by t.

Similarly the beat the bottom juggler is aiming for in the top juggler's pattern now happens t beats later, and so their pass's throw height increases by t.

Without doing any further alterations, this is a great way for generating relatively simple asymmetric passing patterns, where the two jugglers have the same selves but different passes. The idea for the Préchac transformation is that if we rotate juggler B's pattern half of the way around, then we can ensure that both jugglers have the same passes and selves, but are thrown at different times. Mathematically, this is the statement that $C(n/2) = C(-n/2)$, which is true as $n - n/2 = n/2$, so

$$C(n/2) = C(n - n/2) = C(n) \circ C(-n/2) = \mathrm{id} \circ C(-n/2) = C(-n/2).$$

That is, rotating juggler B by $n/2$ turns means we increase the value of juggler A's pass by $n/2$, and decrease juggler B's pass by $-n/2$. Since $C(n/2) = C(-n/2)$, we can choose these throw heights to be the same.

The most natural choices for symmetric throw heights we get by either adding $n/2$ to both passes, or subtracting $n/2$ from both passes (the same operation on the clock). These are the staggered symmetric passing patterns that have been growing in popularity among jugglers worldwide, arguably most famously with the Gandini jugglers.

One beautiful thing about this is that the process works exactly the same way if n is odd. Take two copies of a permutation on the n-gon, swap the landing sites of a corresponding pair of throws, then rotate one n-gon halfway around. Below is an interactive animation you can walk through in the odd case, turning the pattern 423 into the Préchac pattern 2.5p23 (here the period is $n = 3$, and so subtracting half the period is subtracting 1.5 from 4).

Here is an animation of the pattern. Note that one juggler throws their self a half beat after the other juggler throws their pass, corresponding to the fact that when you rotate a triangle halfway around, the vertices now live exactly on the midpoints of where the edges used to be. The same is true for any odd polygon, a kind of juggler's version of a musical canon. Rotating an even polygon halfway around lands back aligned with itself, so jugglers juggling an even period Préchac pattern have their metronomes perfectly synced up. True to form, notice that these 2.5p throws are a little lower and faster than a 3p, often called a "zap".

Here is a video of Jorge and Santi jugging 421.5p, where they've color coded it so the blue club is always thrown as a 1.5p, and the rest are alternating between 4 and 2. Notice how Jorge's throws occur right in between Santi's throws. This is a Préchac transformation of the 3 object pattern 423 where we subtracted half the period from the 3. Notice that the number of clubs is $5 = 2 \cdot 3 - 1$. Or alternately we could think of this as a Préchac transformation of the 2 object pattern 420 where we add half the period to the 0. Then the number of clubs is $5 = 2 \cdot 3 + 1$. This happens in general, that transforming up means you get one more than twice the number of props, and transforming down means you have one less than twice the number of props. We'll go into this in more detail later.

Here is a video of them juggling the pattern 5122p. Again the 2p is color coded as its own color. Notice here that their throws are made exactly at the same time, and whenever one juggler is throwing the 2p, the other juggler is throwing the throw 2 beats earlier/later, the 1 as a quick zip across. This is a Préchac transformation of the 3 object pattern 5124 where we subtract half the period from the 4, and again the number of clubs is $2 \cdot 3 - 1 = 5$.

Now we take a moment to discuss color coding in the Préchac patterns. First, here's a video walking through the case of the pattern 3p313.

This same process will always happen with Préchac patterns. Say you transform the first throw in the pattern. When juggled as a solo pattern, on the clock let the beats this throw loops through be $0, i_1, i_2, \ldots, i_\ell, 0$ for some values i_j, where i_ℓ is thrown back to beat 0 on the clock. Turning both juggler's throws at beat 0 into a pass at the same height, my throw at beat 0 now lands in your beat i_1. Then it loops around to your beats $i_2, i_3, \ldots, i_\ell, 0$. Then since your clock has ticked back around to 0, you throw it to my beat i_1, which then cycles around my beats $i_2, i_3, \ldots, i_\ell, 0$. So in total this throw travels its cycle around each of our patterns before coming back to where it started, for a cycle that's twice as long as it originally was, and again these throws can all be made with one color. This property is maintained after rotating, as it just stretches out the arrows, not changing the overall cycle, as can be seen in the video above.

5.1 Timing in odd vs. even Préchac patterns

5.2 What throw to start on

Let's say we perform a Préchac transformation on an even length pattern. Then juggler B's pattern is rotated $n/2$ turns away from juggler A, and the throws sync up in time. So juggler B's first throw has the same value as the throw which occurs $n/2$ beats before juggler A's first throw, which is the same as $n/2$ beats later, by periodicity.

Now let's consider performing a Préchac transformation on an odd length pattern. Then again juggler B's pattern is rotated clockwise $n/2$ turns away from juggler A, and the throws are separated by exactly half a beat. In particular, $n/2$ beats after I make some throw A, you make the same throw A. Thus, $1/2$ of of a beat after I make my first throw, you will make the throw that occurs

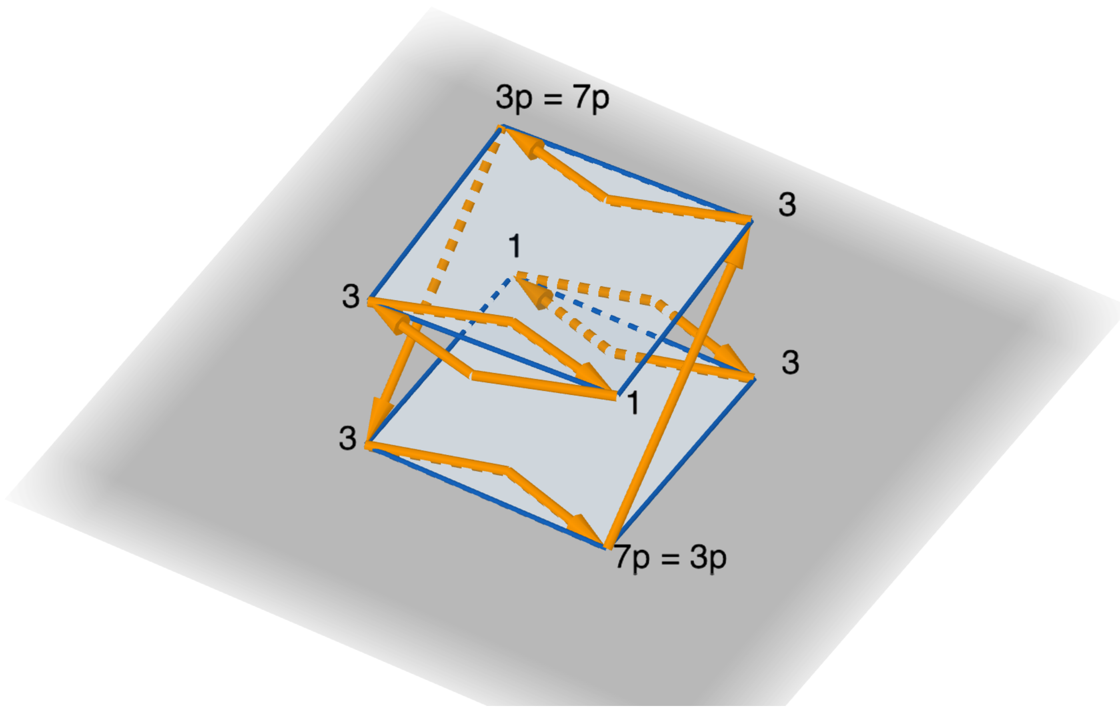

Figure 8: Here the top juggler's pattern is shifted off from the bottom juggler's pattern by $4/2 = 2$ beats. In the pattern 3p313, when I'm throwing the 3p, you're throwing the 1.

in your pattern $\frac{n}{2} - \frac{1}{2} = \frac{n-1}{2}$ beats before throw A, or $\frac{n+1}{2}$ beats after throw A, which are the same throw because

$$\frac{-(n-1)}{2} + n = \frac{-n+1}{2} + \frac{2n}{2} = \frac{2n-n+1}{2} = \frac{n+1}{2},$$

and two throws made n beats apart are the same throw.

6 Préchac patterns with many jugglers

The theory with more jugglers proceeds along mostly the same lines. We start with $J > 2$ jugglers juggling the same pattern at the same time. We'll label these jugglers with the integers $0, 1, 2, \ldots, J-1$. Then we first take a given beat j_0 in the pattern and have all jugglers pass the throw at beat j_0 at the same height as before.

There are now many choices of who to pass to. Recording for every juggler who they pass to gives us a function $P : \{0, 1, \ldots, J-1\} \to \{0, 1, \ldots, J-1\}$ where

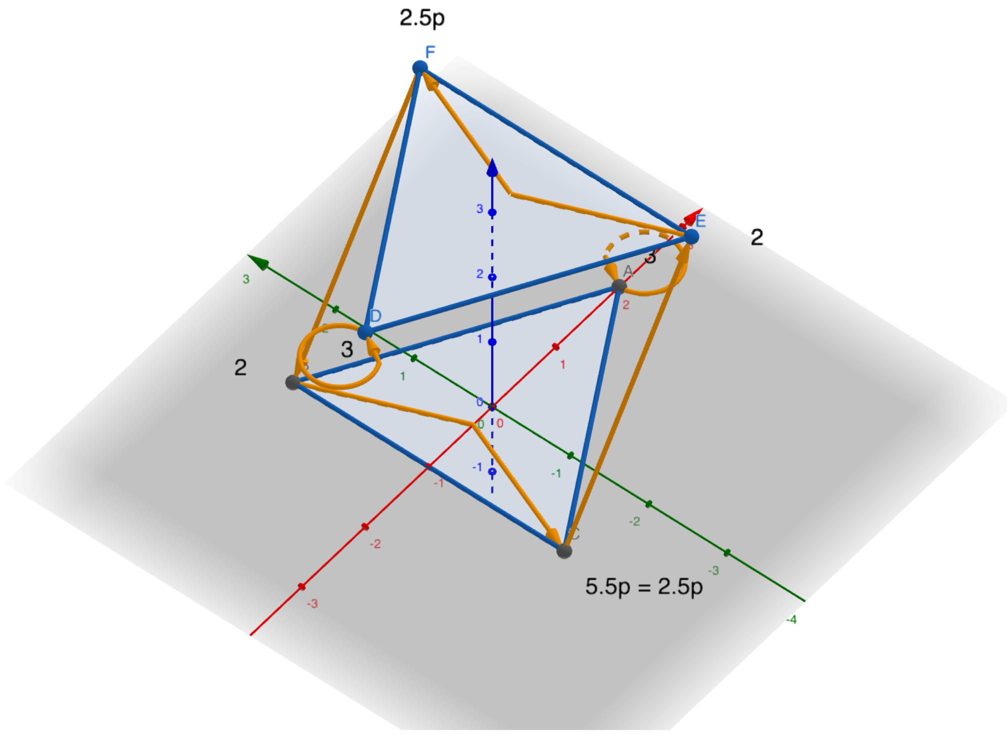

Figure 9: Here the top juggler's pattern is shifted off from the bottom juggler's pattern by $3/2 = 1.5$ beats. In the pattern 2.5p23, 1.5 beats after I throw my 2.5p, you throw your 2.5p, so 0.5 beats after I throw my 2.5p, you're throwing the throw that is $\frac{3-1}{2} = 1$ beats before the 2.5p, which is the 3.

if juggler a passes to juggler b then we set $P(a) = b$. To ensure that everyone receives an object on the beat where throw j_0 lands means that P is onto. To ensure that no juggler receives two objects when they were expecting one, we must have that P is one to one, so altogether we have that P is a permutation. If we have some small cycle like $P(0) = 1$, $P(1) = 2$, and juggler 2 looping back around with $P(2) = 0$, then those three jugglers are passing amongst themselves separate from the other jugglers, which begs the question of why we gathered all J jugglers together in the first place (and mathematically they'll be doing a Préchac transformation with 3 jugglers). Thus, we may assume that P cycles through every juggler, and so after renumbering the jugglers we may assume that P is the cyclic permutation $P(i) = i + 1$ for $0 \leq i < J - 1$ and $P(J - 1) = 0$, written more compactly as $P(i) = i + 1 \pmod{J}$.

Thus, we have as before a symmetric same-time passing pattern by turning one throw into a pass for every juggler. Then we can rotate any of these juggler's patterns to get new passing patterns as a result, many of them asymmetric. If we rotate juggler i clockwise a fraction A/B of a full rotation, then the incoming throw from juggler $i-1$ takes $n * A/B$ beats longer to arrive (mod n), and the outgoing throw from juggler i to juggler $i+1$ is in the air for $n * A/B$ fewer beats. The most symmetric choice comes from evenly spreading the rotation around these jugglers. With two jugglers, we saw that the most symmetric result came from having the second juggler rotated halfway around relative to the first juggler. Here we get the best result by following the same idea all the way around the passing cycle, each juggler rotated a fraction $1/J$ of the way around relative to the previous juggler (who they're receiving passes from). Adding up these successive rotations means that juggler i is rotated clockwise i/J of the whole way around, for a total of $n(i/J)$ turns. Practically speaking, I think of this as any juggler starts throwing n/J beats after the juggler passing to them, as $n(1/J) = n/J$.

In particular, the throw leaving from juggler i is changed by rotating closer a fraction i/J, so it's decreased by this rotation, but it's also increased by the next juggler rotating away a fraction $(i+1)/J$, so the overall number change to the throw value is

$$n\left(\frac{i+1}{J} - \frac{i}{J}\right) = \frac{n}{J}.$$

Alternately, to get the same value on the clock we could have some or all of the jugglers decrease their throw value by $\frac{(J-1)n}{J}$, as mod n we have

$$\frac{n}{J} + \frac{n(J-1)}{J} = \frac{n+n(J-1)}{k} = \frac{n(1+(J-1))}{J} = \frac{nJ}{J} = n$$

so

$$C\left(\frac{n}{J}\right) = C\left(n - \frac{n(J-1)}{J}\right) = C(n) \circ C\left(-\frac{n(J-1)}{J}\right) = C\left(-\frac{n(J-1)}{J}\right).$$

6.1 What throw to start on

Each juggler is rotated by n/J beats relative to the partner they're receiving passes from. That means that when juggler i makes a throw A, n/J beats later, juggler $i+1$ will make the same throw. Letting k be the smallest integer so that $\frac{n-k}{J}$ is an integer, we get that the throw juggler $i+1$ makes at beat $\frac{k}{J}$ after juggler i is the throw

$$\frac{n}{J} - \frac{k}{J} = \frac{n-k}{J}$$

beats before throw A.

6.2 Timing

There is an interesting timing phenomenon that comes up here, analogous to how odd period patterns with two jugglers end up being thrown off time, and even period patterns with two jugglers end up being thrown in sync. Thus, we should expect some sort of timing result based on whether or not the number of jugglers divides the period, and we can be a little more precise using their greatest common divisor.

Lemma 6. *Let d be the greatest common divisor of n and J, let $n' = n/d$, and let $J' = J/d$, so that n' and J' are relatively prime (have no factors in common). Then jugglers i and j in the passing order are throwing at the same time if and only if J' divides $j - i$. No two jugglers are ever passing at the same time.*

Before we prove this result, let's prove some corollaries

Corollary 7. *Suppose that n and J are relatively prime (like odd length patterns with two jugglers). Then no two jugglers ever throw at the same time.*

Proof. Since n and J are relatively prime, their greatest common divisor is 1, and so $J = J'$. Thus, we must show that for integers i, j with $0 \leq i, j < J$ and $i \neq j$, that we have J does not divide $j - i$. But indeed, we know that $j - i \neq 0$, and the restrictions on the size of i and j yield that $|j - i| < J$. But if we had $j - i = Jr$ for some integer r we would have that

$$J > |j - i| = |Jr| = J|r|,$$

Dividing by J, we get that $1 > |r|$, so we must have $r = 0$. But then $j - i = J \cdot 0 = 0$, a contradiction.

□

Corollary 8. *Suppose that J divides n (like even length patterns with two jugglers). Then all jugglers are throwing at the same time.*

Proof. Since J divides n, their greatest common divisor is J, and so $J' = J/J = 1$. But then $J' = 1$ always divides $j - i$, no matter the values of j and i, so all jugglers throw at the same time.

□

So now all that remains is to prove the lemma.

Proof. Jugglers i and j are offset from juggler zero by ni/J and nj/J turns. These two jugglers are throwing at the same time in the pattern (not necessarily throwing their passes at the same time) if

$$\frac{nj}{J} - \frac{ni}{J} = \frac{n(j-i)}{J}$$

is an integer. Furthermore, these two jugglers will also be throwing their passes at the same time exactly when this fraction is an integer divisible by n. To see that the latter situation is always impossible, note that if this fraction is an integer divisible by n, then we have

$$\frac{n(j-i)}{J} = nr \iff \frac{j-i}{J} = r,$$

with r an integer. This is impossible, as for any i, j with $0 \le i, j < J$ we get that $|j - i| < J$, so $\left|\frac{j-i}{J}\right| < 1$, so for this fraction to be an integer, it must be zero, which implies $j = i$.

So all that remains is to show that $\frac{n(j-i)}{J}$ is an integer exactly when J' divides $j - i$. To do this, we compute

$$\frac{n(j-i)}{J} = \frac{(j-i)n/d}{J/d} = \frac{n'(j-i)}{J'}$$

Since n' and J' are relatively prime (have no factors in common), this is an integer exactly when J' divides $j - i$. So two jugglers are synced up in time if and only if they differ by a multiple of J' in the throwing order.

$\square$

7 Relationship to four-handed siteswap

Let's say first we're juggling a Préchac-transformed odd-length pattern with two jugglers. As we've discussed previously, this means we have two n-gons pictured one above the other, with arrows pointing from the throw origin to the throw destination. Since the pattern is of odd length, the two jugglers never throw at the same time, and indeed all the throws are spaced out exactly half a beat apart.

We can build a solo pattern on a $2n$-gon that captures this data by simply projecting both n-gons down onto a $2n$-gon placed below each. The image here is if we were to shine a light down from above either n-gon, we draw their shadow on the $2n$-gon below. Or alternately you can think about this as merging the two n-gons together. Here is an interactive animation of the process for the pattern 421.5p. There is a slider labeled height, and you can drag that value around to move the two triangles down toward the hexagon, or hit the play button to watch an animation. In addition, you can click and drag around the picture to move the camera, and scroll to zoom in and out. In this picture we're oriented so that time proceeds forward when moving clockwise, viewed from above the hexagon.

To understand what happens to throw heights, note that traversing two edges in this $2n$-gon corresponds to traversing one edge in the n-gon it came from. So if the throw values on the n-gon give an oriented number of edges traveled around the n-gon, in this new pattern that same throw has to traverse twice as many edges. That is, all the throw values are doubled, as you can verify in the example above.

The process is similar for a Préchac pattern with J jugglers and n and J coprime. There we get J distinct rotations of the n-gon, which project down onto an nJ-gon, and traversing J edges on this larger nJ-gon corresponds to traversing 1 edges in any of the jugglers' n-gons, and so throw values all get multiplied by J.

8 Further properties of the Préchac Transformation

First let's get the number of balls in the pattern.

Theorem 9. *In any two-person Préchac pattern with U passes shifted up and D passes shifted down from a b-ball solo juggling pattern, we have that*

$$\#balls = 2b + U - D$$

Or more generally, if we perform Préchac transformations with J jugglers on a b-ball solo pattern, with U shifts up and D shifts down, then we have

$$\#balls = Jb + U - D(J-1)$$

Proof. We prove this here only in the case of odd period patterns with two jugglers, which readily generalizes to patterns with J jugglers of period relatively prime to J. The appendix has a proof for the other cases.

Every one of the U throws that gets shifted up increases the throw value by $\frac{n}{2}$, and every one of the D throws that gets shifted down decreases its throw value by $\frac{n}{2}$. Thus, we have that

$$\sum_{i=0}^{n-1} T^{pre}(i) = \frac{Un}{2} - \frac{Dn}{2} + \sum_{i=0}^{n-1} T(i).$$

For a two-person Préchac pattern, when thought of as a one-person siteswap, each throw gets thrown twice at twice its throw value. So in computing the average of throw heights for the one-person pattern, each throw contributes 4 times its value to the sum (both jugglers make throws of twice the value), and the total number of throws, which we're dividing by, is $2n$. Combining these pieces of reasoning, we compute

$$\begin{aligned}
\#\text{balls} &= \frac{\sum_{i=0}^{n-1} 4T^{pre}(i)}{2n} \\
&= \frac{2}{n} \sum_{i=0}^{n-1} T^{pre}(i) \\
&= \frac{2}{n}\Big(\frac{Un}{2} - \frac{Dn}{2} + \sum_{i=0}^{n-1} T(i)\Big) \\
&= U - D + \frac{2\sum_{i=0}^{n-1} T(i)}{n} \\
&= U - D + 2b,
\end{aligned}$$

via the average theorem for solo juggling patterns. This is the desired result.

For even period patterns, the jugglers are throwing at the same time, and as is customary in synchronous patterns, we add in an extra beat between pairs of throws (corresponding to the fact that synchronous patterns tend to be juggled at half the speed of asynchronous patterns), and double the throw value, and the average theorem is expressed the same way as above, as adding in this extra beat in between throws doubles the period in much the same way interweaving the two odd-length patterns did above.

□

9 Proof Appendix

Theorem 10. *Let $\widetilde{S} : \mathbb{Z}/n \to \mathbb{Z}/n$ be any function. Let $\widetilde{T} : \mathbb{Z}/n \to \mathbb{N}$ be any function such that*

$$\left(C \circ \widetilde{T}(\bar{i})\right) + \bar{i} = \widetilde{S}(\bar{i})$$

for all $\bar{i}$ in $\mathbb{Z}/n$. Then define $S : \mathbb{Z} \to \mathbb{Z}$ by

$$S(i) = i + \widetilde{T}(C(i)).$$

Then S is a juggling function if and only if $\widetilde{S}$ is a permutation of $\mathbb{Z}/n$. [8]

Proof. First off, note that

$$\begin{aligned} C(S(i)) &= C(i) + C\left[\widetilde{T}(C(i))\right] \\ &= \widetilde{S}(C(i)) \end{aligned}$$

Thus, we still have a diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{S} & \mathbb{Z} \\ \downarrow C & & \downarrow C \\ \mathbb{Z}/n & \xrightarrow{\widetilde{S}} & \mathbb{Z}/n. \end{array}$$

First let's suppose that S is a juggling function (that is, it is one to one and onto), and show that $\widetilde{S}$ is a permutation.

To show that $\widetilde{S}$ is one to one, suppose that we two times on the clock $\bar{i}$ and $\bar{j}$ in $\mathbb{Z}/n$ such that $\widetilde{S}(\bar{i}) = \widetilde{S}(\bar{j})$. This means that for any times a and b with $\bar{i} = C(a)$ and $\bar{j} = C(b)$ on the clock, that $S(a)$ and $S(b)$ yield the same time on the clock. Then we have that $S(b) = S(a) + nq$ for some integer q. But then by periodicity we have

$$S(a + nq) = S(a) + nq = S(b).$$

[8] For the mathematician: The more general form of this is that if you have a group G acting on sets X and Y, and an equivariant map $S : X \to Y$, then we get an associated map $\widetilde{S} : X/G \to Y/G$ fitting into the same type of diagram. Then the statement is that S is surjective if and only if $\widetilde{S}$ is, $\widetilde{S}$ is injective if S is, and this last implication can be reversed when the action of G on Y is free. The proof is exactly the same as what we give here.

Since S is one to one, we get that $b = a + nq$, and so

$$\bar{i} = C(a) = C(b) = \bar{j}$$

To show that $\widetilde{S}$ is onto, let $\bar{i} \in \mathbb{Z}/n$. Then we have that $\bar{i} = C(i)$. Since S is onto, we have that $i = S(j)$ for some integer j. Thus we have that

$$\begin{aligned}\bar{i} &= C(i) \\ &= C(S(j)) \\ &= \widetilde{S}(C(j)),\end{aligned}$$

thus, we have that $\bar{i}$ is hit by $C(j)$ under $\widetilde{S}$. Since r was arbitrary, we have that $\widetilde{S}$ is onto.

Now let's suppose that $\widetilde{S}$ is a permutation and show that S is a juggling function (one to one and onto). To show that S is one to one, suppose that $S(i) = S(j)$. Then we have that $C(S(i)) = C(S(j))$, and so we have that $\widetilde{S}(C(i)) = \widetilde{S}(C(j))$. Since $\widetilde{S}$ is one to one, we have that $C(i) = C(j)$, and thus $i = j + qn$ for some integer q. Thus, we have

$$\begin{aligned}S(j) &= S(i) \\ &= S(j + qn) \\ &= S(j) + qn\end{aligned}$$

and so $qn = 0$, and thus $i = j + qn = j + 0 = j$.

Now let's show that S is onto. To see this, let $i \in Z$. Since $\widetilde{S}$ is onto we have that at least there is an element $\bar{j} \in \mathbb{Z}/n$ so that $\widetilde{S}(C(j)) = C(i)$. But then we have by commutativity of the diagram that $C(i) = C(S(j))$. Thus, we have that $i = S(j) + qn$ for some integer q. But by periodicity we have $i = S(j + qn)$, and so i is hit by $j + qn$ under S. Since i was arbitrary, we have that S is onto. □

We also record here my personal favorite proof of the average theorem, that the number of balls in a pattern equals the average of the throws in the pattern.

Theorem 11. *Let $S : \mathbb{Z} \to \mathbb{Z}$ be a juggling function of period n, with throw heights $T : \mathbb{Z}/n \to \mathbb{N}$. Then then number of balls in the pattern is the average of the throw heights. In particular, this average is always an integer. But symbolically, if we let b denote the number of balls, we have*

$$b = \frac{\sum_{i=0}^{n-1} T(i)}{n}.$$

Proof. We will instead prove the equivalent equality

$$nb = \sum_{i=0}^{n-1} T(i),$$

by showing that both sides of this equation count the total airtime of the balls in the pattern over the course of a given period.

Let's consider the interval of time between the first throw at beat 0, and the $n+1$th throw at beat n. We will obtain the desired result by counting the total airtime of balls during this interval in two different ways. To define airtime, we imagine the ball thrown at a given beat as having landed at that beat and then immediately being launched into the air. Or in other words, we ignore dwell time, the amount of time a ball spends in the hand. So then the airtime of a ball in this given window is the amount of time in this interval that the ball is not in a hand. Since the ball is only in a hand for an instant at a time, this means the airtime of a ball in this interval is just the length of this interval, which is n. Thus, if we let b denote the total number of balls in the pattern, the airtime in this interval is simply nb.

On the other hand, the ball thrown at beat i is launched into the air for a total time of $T(i)$, by definition of throw heights. So then counting the airtime of balls launched in this interval we get $\sum_{i=0}^{n-1} T(i)$ (the ball thrown at beat n spends all its airtime outside the interval, so isn't counted). In spirit, equating these two views of airtime shows that these two are equal and gives us the result, and this line of thinking is how I remember this proof. However, there are a couple of issues that need to be addressed.

First off, adding up the throw heights only counts the airtime of balls launched during this interval, so the airtime of balls thrown before this interval that land during or after this interval have the portion of their airtime that remains in this interval ignored. Similarly, some of the balls thrown during this interval land after this interval, and so counting their full throw height overcounts their airtime during this interval.

It turns out that the missing airtime from balls thrown before the window that land during the window is exactly cancelled out by the overcounted airtime from balls thrown during the window that land after the window. We'll prove this below. However, this still leaves the problem of balls thrown before the window that land after the window, which haven't entered our count at all. To be clear, this absolutely happens, like in the pattern 64 of period 2, that pattern has 5 balls, but the 4 thrown immediately before the 6 lands after this cycle, as does the 6 thrown before that 4.

To resolve this last issue, we extend our window of observation so that no ball is thrown higher than the length of our window. In particular, we select an integer k such that nk is larger than the highest throw in our pattern. Then looking at the window from 0 to nk, the length of this window is nk, and so the total airtime is nkb. Furthermore, since no ball is thrown at or higher than nk, no ball can be thrown before this interval and land after it. This also shows that every ball in the pattern is thrown at least once during this window, and is a proof that the number of balls is finite. Furthermore, this window is k copies of a full period of the pattern. Thus, all the throw heights are repeated k times, and so the sum of throw heights over this interval is

$$k \sum_{i=0}^{n-1} T(i)$$

Thus, our mission is to show by counting airtime that

$$nkb = \sum_{i=0}^{nk-1} T(i),$$

which we have just remarked equals $k \sum_{i=0}^{n-1} T(i)$, and dividing on both sides by k (or nk if you prefer) yields the desired result.

So now the balls in the air during this window can be sorted into three distinct classes. Let B denote the throws made before the window that land during the window, let D denote the throws made during the window that land during the window, and let A denote the throws made during the window that land after the window. If a beat i is in D it yield no problem, and contributes the expected $T(i)$ to the airtime during this interval. The two sets A and B are where the problems lie, but it turns out their problems exactly cancel each other out.

I claim that there is a natural correspondence between A and B. I claim that we have a function $f : A \to B$ sending a beat $i \in A$ to the beat $i - nk$, and a function $G : B \to A$ sending a beat i in B to the beat $i + nk$. If these two functions do actually send points in A to points in B and vice versa, then they are inverse to each other, as $[i+nk]-nk = i = [i-nk]+nk$. Furthermore, we have by periodicity that $T(f(i)) = T(i - nk) = T(i)$, and similarly $T(g(i)) = T(i)$. So all that needs to be done is to show that these two functions do have outputs in the prescribed places, and that the combined airtime of two paired beats is their common throw value.

Let's get cracking! First let's suppose that $i \in A$, so that i lands after the window, but i is thrown during the window. That is, we have that $S(i) > nk$ but $0 \leq i < nk$. In terms of throw heights, the first condition means that $i + T(i) > nk$. Since we have chosen nk to be larger than the maximum throw height, we also get $i+T(i) < i+nk$. Subtracting nk from all of these inequalities yields

$$\begin{aligned} 0 < i - nk + T(i) < i \\ -nk \leq i - nk < 0. \end{aligned}$$

The second condition yields that $f(i) = i - nk$ occurs before the window. For the first condition, we note that $T(i) = T(i - nk)$ by periodicity, and so

$$i - nk + T(i) = i - nk + T(i - nk) = S(i - nk),$$

by definition of throw heights. Thus, the first pair of inequalities yields

$$0 < S(f(i)) < i,$$

so $f(i)$ lands during the window, as 0 and i are in the window. That is $f(i) \in A$. Similar reasoning shows that $g : B \to A$ actually outputs beats in A.

To get the combined airtime of these balls, let $i \in A$, so that the ball thrown at beat i lands after the window. Then the airtime that this throw spends in the window just the length of the interval from i to nk, which is $nk - i$. If we consider the throw at beat $f(i) = i - nk$, then the amount of airtime this ball spends in the window is just the length of the interval from 0 to $S(i - nk)$, which is $S(i - nk)$. So the in-window airtime from these two throws is

$$\begin{aligned} S(i-nk) + nk - i &= i - nk + T(i-nk) + nk - i \\ &= T(i-nk) \\ &= T(i), \end{aligned}$$

as desired. Thus, we have

$$\begin{aligned} \text{total airtime} &= \text{airtime from B} + \text{airtime from D} + \text{airtime from A} \\ &= \text{airtime from D} + \sum_{i \in A} \left[\text{airtime}(i) + \text{airtime}(f(i)) \right] \\ &= \sum_{j \in D} T(j) + \sum_{i \in A} T(i) && \text{(by the above reasoning)} \\ &= \sum_{i=0}^{nk-1} T(i) \\ &= \sum_{q=0}^{k-1} \sum_{r=0}^{n-1} T(nq + r) \\ &= \sum_{q=0}^{k-1} \sum_{r=0}^{n-1} T(r) \\ &= k \sum_{r=0}^{n-1} T(r), \end{aligned}$$

which is the desired result.

□

I'll pause for some remarks on this proof and why I like it in addition to the more standard argument you get by swapping throw heights to end up at the base pattern for a given number of objects. First of all, the argument generalizes immediately to multiplex patterns, as the fact that multiple balls are thrown at once is irrelevant from the perspective of airtime, and there it's not so clear on how to do the standard siteswapping. This proof also generalizes to the context of Préchac patterns. Indeed there you have J jugglers making all n throws in an interval of length n, either the throws are occurring synchronously or not, but always they're at rational points in time, and we still have period n, in the sense that even for fractional inputs i we have $T(i+n) = T(i)$. Thus, we get the total airtime as $J\sum_{i=0}^{n-1} T(i)$, and so the number of balls is $\frac{J\sum_{i=0}^{n-1} T(i)}{n}$. Like in the pattern 421.5p for two jugglers we get the number of balls is $\frac{2*(4+2+1.5)}{3} = 5$, which is correct (it's a Préchac transformation down from 423, which has 3 balls). This form also makes it clear why the Préchac pattern has as many balls as it does. Say we Préchac transform all the J jugglers' passes up by $\frac{n}{J}$ Then the sum of throw heights is increased by $J(n/J) = n$, and so the average is increased by 1. Or similarly if we Préchac transform all of the jugglers' passes down by $\frac{n(J-1)}{J}$, then the sum of the throw heights is decreased by $\frac{Jn(J-1)}{J} = n(J-1)$, and the average is decreased by $J-1$. In general say we transform U of the the jugglers' throws up by n/J, and $D = J - U$ of the jugglers' down by $\frac{n(J-1)}{J}$, then the sum of throw heights is changed by

$$\begin{aligned} \frac{Un}{J} - \frac{Dn(J-1)}{J} &= \frac{n}{J}\Big(U - D(J-1)\Big) \\ &= \frac{n}{J}(U - DJ + D) \\ &= \frac{n}{J}(J - DJ) \\ &= n(1-D), \end{aligned}$$

so the average is changed by $1 - D$, with $D = 0$ we get one extra object, with $D = 1$ we get no change to the number of objects, for $D = 2$ we get one fewer object, and in general increasing D by some number decreases the number of objects by the same amount.

10 References and Examples

- Christophe Prechac's original post in which the theory was first described.
- `https://www.instagram.com/clubpassing/` for great Préchac patterns
- `https://www.youtube.com/watch?v=GC_fLpkLye8` for a really great video of lots of these patterns with many different props
- Joepass for animating partner patterns `http://westerboer.net/w/?page_id=151`
- Sean Gandini's document which is incredibly useful from a juggler's perspective, and outlined much of the theory. `https://open-source-juggling-project.github.io/Staggered-Symmetric-Passing/SymmetricPatternsClassified_SG_2008-02-24.pdf`
- Préchac This is a great tool for generating Préchac patterns, and it automatically calculates things like causal diagrams and who starts with which clubs. `http://www.prechacthis.org/`