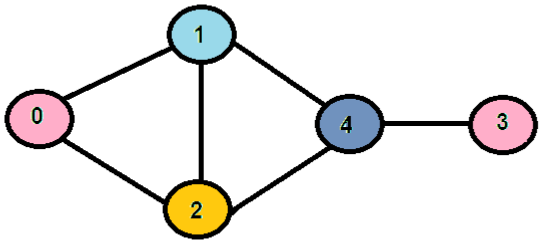
32. (2 points) A subset S of vertices in a graph G is triad free if for all triples of distinct vertices x, y and z in S, at least one of the edges (x, y), (x, z) or (y, z) is not present in G. So for example, in the following graph:



the set $\{0, 1, 3, 4\}$ is triad free, but the set $\{0, 1, 2, 4\}$ is not triad free. Show that the problem of, given a graph G and an integer k, of determining whether G has a triad free set S of size k is NP-hard using the fact the the independent set problem is NP-complete. Recall that the independent set problem is to determine whether a graph has a independent set of a particular size, where an independent set is a collection of mutually nonadjacent vertices.

33. (4 points) Consider the following problem. The input is a graph $G = (V, E)$, a subset R of vertices of G, and a positive integer k. The problem is to determine if there is a subset U of V such that

   - All the vertices in R are contained in U, and
   - the number of vertices in U is at most k, and
   - for every pair of vertices x and y in R, one can walk from x to y in G only traversing vertices that are in U.

   Show that this problem is NP-hard using a reduction from Vertex Cover. Recall that the input for the vertex cover problem is a graph H and an integer ℓ, and the problem is to determine whether H has a vertex cover of size ℓ or not. A vertex cover S is a collection of vertices with the property that every edge is incident on at least one vertex in S.

   HINT: Let (H, ℓ) be the graph and integer pair instance for vertex cover. Have R consist of a single source vertex plus one vertex for each edge in H. Now have one more vertex in G, but not in R, for each vertex in H. Now add edges to G so that you can connect the vertices in R using few additional vertices if and only if H has a vertex cover of size ℓ.

34. (6 points) In the disjoint paths problem the input is a directed graph G and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices. The problem is to determine if there exist a collection of vertex disjoint paths between the pairs of vertices (from each s_i to each t_i). Show that this problem is NP-hard by a reduction from the 3SAT problem. Note that this problem is not easy.

   HINT: Construct one pair (s_i, t_i) for each variable x_i in your formula F. Intuitively there will be two possible paths between s_i and t_i depending on whether x_i is true or false. There will be a component/subgraph D_j of G for each clause C_j in F. There will be three possible paths between the (s_i, t_i)'s pairs for each D_j. You want that it is possible to route any two of these paths (but not all three) through D_j.