CS 1510 Quiz 2
Fall 2020

Instructions: Answer at most one of the following two questions.

1. (20 points) Consider the following problem. The input consists of a positive integer L and a collection

$$\mathcal{I} = \{I_1 = (a_1, b_1), I_2 = (a_2, b_2), \ldots I_n = (a_n, b_n)\}$$

consisting of n open intervals over the real line. Further each a_i and each b_i is an integer in the range between 0 and L inclusive, and $a_i < b_i$. So no interval starts before 0 and no interval ends after L, and all intervals are of positive length. The intervals being open means that they do not include their end points. So neither a_i or b_i is in the open interval (a_i, b_i). A feasible solution is subset $\mathcal{C}$ of $\mathcal{I}$ such that no pair of intervals in $\mathcal{C}$ intersect (contain a point in common). The objective is the minimum average squared length of the gaps formed by the intervals in $\mathcal{C}$. Note that the output for the problem is the optimal objective value, not the set $\mathcal{C}$.

To formally define this objective let $\mathcal{C} = \{I_{\sigma(1)}, \ldots I_{\sigma(k)}\}$, where

$$a_{\sigma(1)} < b_{\sigma(1)} \leq a_{\sigma(2)} < b_{\sigma(2)} \leq \ldots \leq a_{\sigma(k)} < b_{\sigma(k)}.$$

Then the objective value for $\mathcal{C}$ is

$$\frac{a_{\sigma(1)}^2 + \sum_{i=2}^{k} (a_{\sigma(i)} - b_{\sigma(i-1)})^2 + (L - b_{\sigma(k)})^2}{k+1}$$

Consider for example the input where $L = 10$ and

$$\mathcal{I} = \{(0,4), (3,7), (4,6), (9,10), (5,9), (1,7)\}$$

Then $\mathcal{C}_1 = \{(3,7), (0,4), (9,10)\}$ is infeasible because $(3,7)$ and $(0,4)$ intersect. Then $\mathcal{C}_2 = \{(0,4), (4,6), (9,10)\}$ is feasible and has objective value $(0^2 + 0^2 + 3^2 + 0^2)/4 = 9/4$. Then $\mathcal{C}_3 = \{(1,7), (9,10)\}$ is feasible and has objective value $(1^2 + 2^2 + 0^2)/3 = 4/3$. So $\mathcal{C}_3$ is a better answer than $\mathcal{C}_2$.

   (a) Either give a recursive algorithm for this problem (which may run in exponential time), or give a pruning rule for a tree that enumerates all feasible solutions (this includes explaining the tree). Pseudo-code is fine, but it is not sufficient in and of itself, you must also have some explanation of your algorithm.

   (b) Using your answer for the above subproblem give an $O(n^3)$-time dynamic programming algorithm. Don't forget to explain how to compute the final answer from the filled in array. Pseudo-code is fine, but it is not sufficient in and of itself, you must also have some explanation of your algorithm.

   (c) Define intended meaning of the entries in your array in your dynamic programming algorithm.

2. (10 points) Consider the following problem. The input consists of a positive integer L and a collection

$$\mathcal{I} = \{I_1 = (a_1, b_1), I_2 = (a_2, b_2), \ldots I_n = (a_n, b_n)\}$$

consisting of n open intervals over the real line. Further each a_i and each b_i is an integer in the range between 0 and L inclusive, and $a_i < b_i$. So no interval starts before 0 and no interval ends after L, and all intervals are of positive length. The intervals being open means that they do not include their end points. So neither a_i or b_i is in the open interval (a_i, b_i). A feasible solution is subset $\mathcal{C}$ of $\mathcal{I}$ such that no pair of intervals in $\mathcal{C}$ intersect (contain a point in common). The objective is the minimum aggregate squared length of the gaps formed by the intervals in $\mathcal{C}$. Note that the output for the problem is the optimal objective value, not the set $\mathcal{C}$.

To formally define this objective let $\mathcal{C} = \{I_{\sigma(1)}, \ldots I_{\sigma(k)}\}$, where

$$a_{\sigma(1)} < b_{\sigma(1)} \leq a_{\sigma(2)} < b_{\sigma(2)} \leq \ldots \leq a_{\sigma(k)} < b_{\sigma(k)}.$$

Then the objective value for $\mathcal{C}$ is

$$a_{\sigma(1)}^2 + \sum_{i=2}^{k} (a_{\sigma(i)} - b_{\sigma(i-1)})^2 + (L - b_{\sigma(k)})^2$$

Consider for example the input where $L = 10$ and

$$\mathcal{I} = \{(0,4), (3,7), (4,6), (9,10), (5,9), (1,7)\}$$

Then $\mathcal{C}_1 = \{(3,7), (0,4), (9,10)\}$ is infeasible because $(3,7)$ and $(0,4)$ intersect. Then $\mathcal{C}_2 = \{(0,4), (4,6), (9,10)\}$ is feasible and has objective value $(0^2 + 0^2 + 3^2 + 0^2) = 9$. Then $\mathcal{C}_3 = \{(1,7), (9,10)\}$ is feasible and has objective value $(1^2 + 2^2 + 0^2) = 4$. So $\mathcal{C}_3$ is a better answer than $\mathcal{C}_2$.

(a) Either give a recursive algorithm for this problem (which may run in exponential time), or give a pruning rule for a tree that enumerates all feasible solutions (this includes explaining the tree). Pseudo-code is fine, but it is not sufficient in and of itself, you must also have some explanation of your algorithm.

(b) Using your answer for the above subproblem give an $O(n^2)$-time dynamic programming algorithm. Don't forget to explain how to compute the final answer from the filled in array. Pseudo-code is fine, but it is not sufficient in and of itself, you must also have some explanation of your algorithm.

(c) Define intended meaning of the entries in your array in your dynamic programming algorithm.