Homework #4 STP 427

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1) If $X_1, \dots X_n$ constitute a random sample of size n, from a gamma population with parameter $\alpha=3$, rate parameter $\beta>0$

a) method of moment estimator β

$X_1, \dots X_n \sim Gamma(\alpha=3)$

$$f(x;\beta) = \frac{\beta^3}{\Gamma(3)} x^{3-1} e^{-\beta x} = \frac{\beta^3}{2} x^2 e^{-\beta x} : \beta > 0$$

$\Gamma(3)=2$
$\Gamma(4)=6$

Using Gamma Integral $\Gamma(n) = \int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma(n)}{a^n}$

$$E(X) = \int x f(x) dx = \int x f(x;\beta) dx$$
$$= \frac{\beta^3}{2} \underbrace{\int_0^\infty x^3 e^{-\beta x} dx}$$
$$= \frac{\beta^3}{2} \frac{\Gamma(4)}{\beta^4} \leftarrow 6$$
$$= \frac{3}{\beta}$$

$E(X) = \frac{3}{\beta} \Rightarrow \bar{X} = \frac{3}{\beta} \Rightarrow \beta = \frac{3}{\bar{X}}$ use sample mean

$\therefore \hat{\beta}_{mom} = \frac{3}{\bar{x}}$

b) maximum likelihood

$$L(\beta;x) = \prod_{i=1}^n f(x_i;\beta)$$
$$= \prod_{i=1}^n \frac{\beta^3}{2} x_i^2 e^{-\beta x_i}$$
$$= \frac{\beta^{3n}}{2n} \prod_{i=1}^n x_i^2 e^{-\sum_{i=1}^n \beta x_i}$$

$$\ell(\beta;x) = \ln(L(\beta;x))$$
$$= \ln\left(\frac{\beta^{3n}}{2n} \prod_{i=1}^n x_i^2 e^{-\sum_{i=1}^n \beta x_i}\right)$$
$$= 3n\ln(\beta) - n\ln(2) + \sum_{i=1}^n \ln(x_i^2) - \sum_{i=1}^n \beta x_i$$

$$\frac{\partial \ell}{\partial \beta} = \frac{3n}{\beta} - \sum_{i=1}^n x_i = 0$$ solve for β

$$\Rightarrow \frac{3n}{\beta} - \sum_{i=1}^n x_i = 0$$

$$\Rightarrow \beta = \frac{3n}{\sum x_i} = \frac{3n}{\bar{x}}$$

and

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{3n}{\beta} < 0$$

$\therefore \ell(\beta)$ is maximum when $\beta = \frac{3}{\bar{x}} = \hat{\beta}_{MLE}$

2) Given a random sample of size n from the Poisson population with parameter λ use the method of max. likelihood to obtain estimator for the parameter λ.

$$f(x,\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$\Rightarrow L = \prod_{i=1}^n f(x_i,\lambda) = \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$
$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \dots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$
$$= \frac{e^{-n\lambda}}{x_1! \dots x_n!} \lambda^{\sum_{i=1}^n x_i}$$

$$\therefore \ell = \ln(L) = \ln\left(\frac{e^{-n\lambda}}{x_1! \dots x_n!} \lambda^{\sum_{i=1}^n x_i}\right)$$
$$= -n\lambda + \sum_{i=1}^n x_i \cdot \ln(\lambda) - \ln(x_1! x_2! \dots x_n!)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left[-n\lambda + \sum_{i=1}^n x_i \cdot \ln(\lambda) - \ln(x_1! x_2! \dots x_n!)\right]$$
$$= -n + \frac{1}{\lambda}\sum_{i=1}^n x_i$$
$$0 = -n + \frac{1}{\lambda}\sum_{i=1}^n x_i$$
$$\lambda = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$$

$\therefore \hat{\lambda}_{MLE} = \bar{x}$

3) If $X_1, \dots X_n$ constitute a random sample from a normal population $\mu=0$

Show that $\sum_{i=1}^n \frac{X_i^2}{n}$ is an unbiased estimator of σ^2.

$$E\left(\sum \frac{x_i^2}{n}\right) = \frac{1}{n}E\left(\sum x_i^2\right)$$
$$= \frac{1}{n}\sum E(x_i^2)$$
$$= \frac{1}{n} \text{ since } \mu=0 \ \forall_i$$

Know $Var(X_i) = \sigma^2 \because \sim N(0,\sigma^2)$

$$E(X_i^2) = Var(X_i) + [E(X_i)]^2 = \sigma^2 + 0 = \sigma^2$$

Thus, $\frac{1}{n}\sum E(x_i^2) = \frac{1}{n}\sum \sigma^2 = \frac{n}{n}\sigma^2 = \sigma^2$

$$E\left(\sum \frac{x_i^2}{n}\right) = \sigma^2$$

$\therefore \sum \frac{x_i^2}{n}$ is an unbiased estimator

#4) If X is r.v. having Binomial dist with parameters n, θ

Show that $n\frac{X}{n}\left(1-\frac{X}{n}\right)$ is a biased estimator of X.

let $X \sim Binom(n,\theta) \Rightarrow f(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x} : x=0,1,\dots,n$

we know $E(X) = n\theta$, $Var(X) = n\theta(1-\theta)$

Thus, $E(X^2) = n\theta(1-\theta) + n\theta$

Given $n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)$

Need to show that X is a biased estimator for the variance of X

WTS: $E\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right] \neq n\theta(1-\theta)$

$$E\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right] = \frac{1}{n}\left(n E\left[1-\frac{X}{n}\right]\right)$$
$$= \frac{1}{n}E\left(n - \frac{nX}{n}\right)$$
$$= E\left(\frac{n}{n} - \frac{nX}{n^2}\right)$$
$$= E\left(1 - \frac{X}{n}\right)$$
$$= 1 - E\left(\frac{X}{n}\right)$$
$$= 1 - \frac{\theta}{n}$$
$$\neq n\theta(1-\theta)$$

$\therefore n\frac{X}{n}\left(1-\frac{X}{n}\right)$ is a biased estimator of X.

#5) Show that the mean of a random sample of size n, is the minimum variance unbiased estimator of the parameter λ of poisson population.

Sps $Y_1, \dots Y_n$ denote random sample from Poisson(λ)

we know

$$f(y;\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^y}{y!}, & y=0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$$

Begin by using the factorization criterion to find sufficient statistic that best summarizes information of λ.

$$L(y_1,y_2,\dots y_n|\ell) = f(y_1,\dots,y_n|\lambda)$$
$$= \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \cdot \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \dots \frac{e^{-\lambda}\lambda^{y_n}}{y_n!}$$
$$= \frac{e^{-n\lambda}}{y_1! \dots y_n!} \lambda^{\sum_{i=1}^n y_i}$$

Thus $\sum_{i=1}^n y_i$ is the minimal sufficient statistic for λ.

Now show unbiased

$E(Y_i) = \lambda$ $Var(Y_i) = \lambda$,

and sample mean

$$\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$$

Consider

$$E(\bar{Y}) = E\left(\frac{1}{n}\sum_{i=1}^n y_i\right) = \frac{1}{n}\sum_{i=1}^n E(y_i) = \frac{n\lambda}{n} = \lambda$$

$\therefore \bar{y}$ is unbiased

$\therefore \bar{y}$ is unbiased estimator of λ that is a function of the sufficient statistic $\sum_{i=1}^n y_i$

$\therefore$ mean random sample of size n from Poisson population is a min. variance unbiased estimator of parameter λ.

#6) If X_1, X_2, X_3 constitute a random sample of size $n=3$ from normal population of mean μ, variance σ^2 find the efficiency of $\frac{X_1+2X_2+X_3}{4}$ relative to $\frac{X_1+X_2+X_3}{3}$ as estimates of μ.

$$T_1 = \frac{X_1+2X_2+X_3}{4}, \quad T_2 = \frac{X_1+X_2+X_3}{3}$$

$$Var(T_1) = Var\left(\frac{X_1+2X_2+X_3}{4}\right)$$
$$= \frac{1}{16}Var(X_1+2X_2+X_3)$$
$$= \frac{1}{16}\left[Var(X_1) + 4Var(X_2) + Var(X_3)\right]$$ assuming $cov(X_i,X_j)=0$
$$= \frac{1}{16}(\sigma^2 + 4\sigma^2 + \sigma^2)$$
$$= \frac{6\sigma^2}{16} = \frac{3\sigma^2}{8}$$

$$Var(T_2) = Var\left(\frac{X_1+X_2+X_3}{3}\right)$$
$$= \frac{1}{9}\left[Var(X_1+X_2+X_3)\right]$$
$$= \frac{1}{9}\left[Var(X_1) + Var(X_2) + Var(X_3)\right]$$
$$= \frac{3\sigma^2}{9} = \frac{\sigma^2}{3}$$

relative efficiency T_1 relative to $T_2 \Rightarrow \frac{Var(T_2)}{Var(T_1)}$

$$\therefore \frac{\frac{\sigma^2}{3}}{\frac{3\sigma^2}{8}} \Rightarrow \frac{\cancel{\sigma^2}}{3}\left(\frac{8}{3\cancel{\sigma^2}}\right) = \frac{8}{9}$$

#7) If $X_1, \dots X_n$ constitute a random sample of size n, from exponential population show that $\bar{X}$ is a consistent estimator of parameter θ

$$f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, \quad E(X)=\theta, \ Var(X)=\theta^2$$

WTS: $\bar{X} = \frac{1}{n}\sum_{i=1}^n x_i$ is a consistent estimator of parameter θ

WTS: $Var(\bar{X}) \to 0$ as $n \to \infty$

Unbiasedness:

$$E(\bar{X}) = \frac{1}{n}\sum E(x_i)$$
$$= \frac{1}{n}\left[E(X_1) + E(X_2) + \dots + E(X_n)\right]$$
$$= \frac{1}{n}\left[\theta + \theta + \theta + \dots + \theta\right] \quad \because E(X_i) = \theta$$
$$= \frac{n\theta}{n}$$
$$= \theta$$

and

$$V(\bar{X}) = V\left(\frac{1}{n}\sum x_i\right)$$
$$= \frac{1}{n^2}\left[V(X_1) + V(X_2) + \dots + V(X_n)\right]$$
$$= \frac{1}{n^2}\left[\theta^2 + \theta^2 + \dots + \theta^2\right]$$
$$= \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

finally

$$\lim_{n\to\infty} V(\bar{X}) = \lim_{n\to\infty}\frac{\theta^2}{n} = 0$$

#8) If $X_1, \dots X_n$ constitute a random sample of size n, from a geometric population, show that $Y = X_1 + X_2 + \dots + X_n$ is a sufficient estimator of parameter θ

$$f(x) = P(X=x) = p^x(1-p)^{x-1}$$

$$L = \prod_{i=1}^n f(x_i)$$
$$= \prod_{i=1}^n p^x(1-p)^{x-1}$$
$$= p^x(1-p)^{\sum_{i=1}^n x_i - n} \quad (1)$$

By Factorization $t(x)$ is sufficient for $\theta=p$

iff

$$L = g_{\theta=p}(t(x), \theta=p)\,h(x) \quad (2)$$

equating (1), (2)

$$g_{\theta=p}(t(x),\theta=p) = \left(p^x(1-p)^{\sum_{i=1}^n x_i - n}\right) h(x)$$

$\Rightarrow h(x) = 1$

$\therefore t(x) = \sum_{i=1}^n x_i$ is sufficient for estimating p