

A Common Model, Separated by Two Disciplines

Bayesian Factorization Machines with Stan and R

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Welcome

- ▶ Talk: Factorization machines:
 - ▶ What are they?
 - ▶ How do we fit them?
- ▶ Stan code + simulations!
 - ▶ Start with simple model
 - ▶ Next, hierarchical model
- ▶ Finally, real data analysis:
 - ▶ Plotting model results

Who Am I?

- ▶ Senior Data Scientist, JUST Capital¹
- ▶ Ph.D. in Political Science (The Ohio State University); B.A. in Political Science (Grinnell College)
- ▶ Word Embeddings, Causal Inference, Bayesian Statistics, Discrete Choice Models

¹All views are mine alone, and do not represent JUST Capital, any of its funders, or anyone other than me.

Why We Model

- ▶ Descriptive Inference: is there an association between X and y ?
- ▶ Prediction: with X and y , what is y for a new X ?
- ▶ Causal Inference: does changing X , change y ?
- ▶ Shameless plug: Book Chapter

[https://www.dropbox.com/s/cmwd15wmad4coy9/
Statistics%20and%20International%20Security.pdf](https://www.dropbox.com/s/cmwd15wmad4coy9/Statistics%20and%20International%20Security.pdf)

Latent Variable Models/Unsupervised models

- ▶ Learning parameters to reconstruct observed data
 - ▶ CS: For prediction, Social Science: for inference
- ▶ Ex: Principal Components Analysis, Factor Analysis, Matrix Factorization, Ideal Point Models, Word2vec
- ▶ Data $\mathbf{X}_{N \times J}$ is decomposed into two low rank matrices:
 $\boldsymbol{\gamma}_{N \times K}$ and $\boldsymbol{\delta}_{K \times J}$
- ▶ Various assumptions about the structure of $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$

Factorization Machines

- ▶ Combine regression on observables with a latent variable model on the residuals
- ▶ Can plug and play with any generalized linear model

But Why?

Factorization Machines

- ▶ Regression Model, for one observation
- ▶ Categorical predictors $\mathbf{x}_{n \in N}, \mathbf{x}_{j \in J}$
- ▶ Outcome y
- ▶ Parameters β
- ▶ Error ε_{nj}

$$y_{nj} = \mathbf{x}_n \beta_1 + \mathbf{x}_j \beta_2 + \varepsilon_{nj}$$

```
m1 <- lm(y ~ factor(group_1) + factor(group_2))
```

- ▶ With Interactions

$$y_{nj} = \mathbf{x}_n \beta_1 + \mathbf{x}_j \beta_2 + \mathbf{x}_n \times \mathbf{x}_j \beta_3 + \varepsilon_{nj}$$

```
m1 <- lm(y ~ factor(group_1) + factor(group_2) + factor(group_1) * factor(group_2))
```

Factorization Machines

Problems:

- ▶ We can only estimate β_3 for observed interactions
- ▶ As N and J grow, β_3 increases with size $N * J$

Factorization Machines

Solution!

- ▶ Replace β_3 with the dot product of low-rank latent factors:
- ▶ $\gamma_{N \times K}$
- ▶ $\delta_{J \times K}$
- ▶ β_3 is now $\gamma_n \cdot \delta_j^\top$

Factorization Machines

- ▶ Interaction model:

$$y_{nj} = \mathbf{x}_n \beta_1 + \mathbf{x}_j \beta_2 + \boldsymbol{\gamma}_n \cdot \boldsymbol{\delta_j}^\top + \varepsilon_{nj}$$

Factorization Machines

- ▶ Low-rank structure regularizes interaction term
- ▶ Don't need to observe interaction to predict

Factorization Machines

Basic model - Each element of δ_j and γ_n is distributed standard normal: $\mathcal{N}(0, 1)$

$$y_{nj} \sim \mathcal{N}(\mathbf{x}_n \beta_1 + \mathbf{x}_j \beta_2 + \boldsymbol{\gamma}_n \cdot \boldsymbol{\delta}_j^\top, \sigma_y)$$

$$\boldsymbol{\beta} \sim \mathcal{N}(0, \sigma_\beta)$$

$$\gamma_{n,k} \sim \mathcal{N}(0, 1)$$

$$\delta_{j,k} \sim \mathcal{N}(0, 1)$$

Simulating Data:

First, the regression component:

```
seed_to_use = 123
set.seed(seed_to_use)
# number of levels for first covariate
N <- 100
group_1 <- paste0("i", 1:N)
# number of levels for second covariate
J <- 20
group_2 <- paste0("j", 1:J)
# number of latent dimensions
K <- 5

# observed data ----
predictors <- expand.grid(group_1 = group_1, group_2 = group_2)
X_mat <- sparse.model.matrix(~ factor(group1) + factor(group2) - 1, data = predictors)

# for sparsity, since here, we're assuming we have only dummies
# creating numeric values for each individual FE
predictors_as_numeric <- cbind(
  as.numeric(factor(predictors[, 1])), as.numeric(factor(predictors[, 2])))

# the regression part of the equation
betas <- matrix(rnorm(n = ncol(X_mat), 0, 2))
linear_predictor <- X_mat %*% betas
```

Simulate a Factorization Machine:

Next, latent factors:

```
gamma_omega <- rlkjcorr(n = 1, K = K, eta = gamma_omega_prior)
delta_omega <- rlkjcorr(n = 1, K = K, eta = delta_omega_prior)

gammas <- mvrnorm(n = N, mu = rep(0, K), Sigma = gamma_omega)

# group 2 factors are deltas
deltas <- mvrnorm(n = J, mu = rep(0, K), Sigma = delta_omega)

factor_terms <-matrix(
  NA, nrow = nrow(linear_predictor), ncol = 1)

for (i in 1:nrow(predictors)) {
  g1 <- as.character(predictors[i, 1])
  g1 <- as.numeric(substr(g1, 2, nchar(g1)))

  g2 <- as.character(predictors[i, 2])
  g2 <- as.numeric(substr(g2, 2, nchar(g2)))

  factor_terms[i,] <- matrix(gammas[g1,], nrow = 1) %%%
    matrix(deltas[g2,], ncol = 1)
}

y <- linear_predictor + factor_terms + rnorm(n = nrow(linear_predictor), 0, y_sigma)
```

Factorization Machines in Stan

Use Stan to fit FMs:

- ▶ Data:

```
data{  
    int<lower = 0> N ; // number of group 1 observations  
    int<lower = 0> J ; // number of group 2 observations  
    int<lower = 0> K ; // number of latent dimensions  
    int X[(N*J), 2] ; // covariate matrix  
    vector[(N*J)] y ; // outcome  
    real<lower = 0> beta_sigma ; // sd on regression coefficients  
    real<lower = 0> y_sigma ; // sd on the outcome, y  
}
```

Factorization Machines in Stan

Use Stan to fit FMs:

- ▶ Parameters:

```
parameters{
  vector[N] group_1_betas; // non-interacted coefficients
  vector[J] group_2_betas; // non-interacted coefficients
  matrix[N, K] gammas; // group 1 factors
  matrix[J, K] deltas; // group 2 factors
}

transformed parameters{
  real linear_predictor[(N*J)] ;
  for(i in 1:(N*J)){
    linear_predictor[i] =
      group_1_betas[X[i, 1]] + group_2_betas[X[i, 2]] +
      (gammas[X[i, 1], ] * deltas[ X[i, 2], ]');
  }
}
```

Factorization Machines in Stan

Use Stan to fit FMs:

► Model:

```
model{

    // regression coefficients
    group_1_betas ~ normal(0, beta_sigma) ;
    group_2_betas ~ normal(0, beta_sigma) ;

    // latent factors
    for(n in 1:N){
        gammas[n, ] ~ normal(rep_vector(0, K), 1) ;
    }

    for(j in 1:J){
        deltas[j, ] ~ normal(rep_vector(0, K), 1) ;
    }

    // outcome
    y ~ normal(linear_predictor, y_sigma) ;
}
```

Factorization Machines in Stan

Use Stan to fit FMs:

- ▶ Model Checking

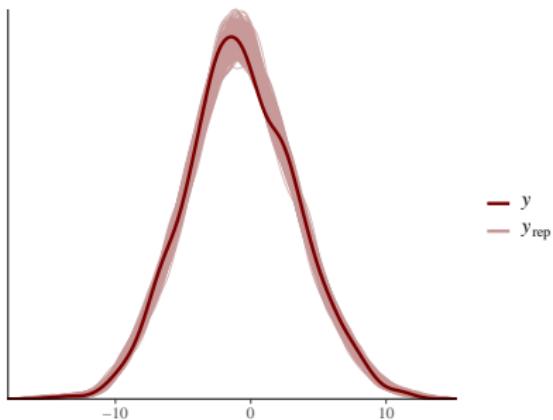
```
generated quantities{  
    real y_pred[(N*J)] ;  
    y_pred = normal_rng(linear_predictor, y_sigma) ;  
}
```

Fitting a Model to Simulated Data

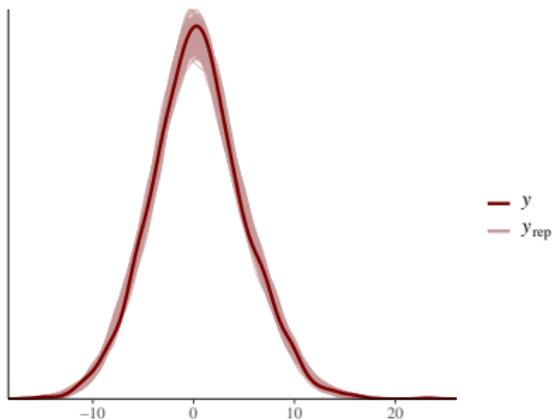
- ▶ Using `simulate_data` in `simulation_function.R`.
- ▶ Two versions:
 - ▶ 100 members of group 1, 20 members of group 2
 - ▶ 20 members of group 1, 100 members of group 2
- ▶ Fit model in `stan`, using NUTS

Fitting a Model to Simulated Data

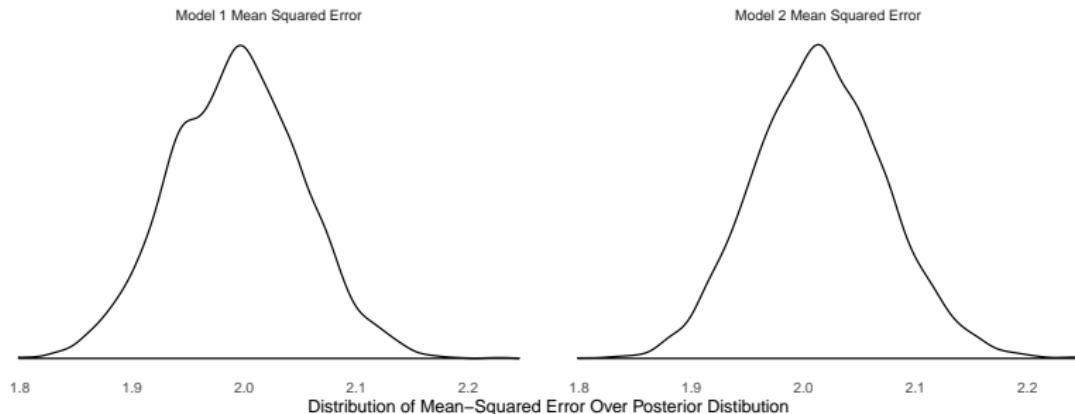
Model 1 Posterior Predictive Check



Model 2 Posterior Predictive Check



Fitting a Model to Simulated Data



Now, Let's Get Hierarchical!

Hierarchical Factorization Machines

- ▶ Basic FM implementation assumes all parameters are independent
- ▶ Hierarchical: Sharing parameters within groups
 - ▶ Explicitly modeling correlation between members
- ▶ Sharing parameters = sharing information!
- ▶ *Caveat:* More parameters = more computation time

Hierarchical Factorization Machines

$$y_{nj} \sim \mathcal{N}(\boldsymbol{x}_n \beta_1 + \boldsymbol{x}_j \beta_2 + \boldsymbol{\gamma}_n \cdot \boldsymbol{\delta}_j^\top, \sigma_y)$$

$$\boldsymbol{\beta} \sim \mathcal{N}(0, \sigma_\beta)$$

$$\boldsymbol{\gamma}_n \sim \mathcal{MVN}(\boldsymbol{\mu}_\gamma, \boldsymbol{\Sigma}_\gamma)$$

$$\boldsymbol{\delta}_j \sim \mathcal{MVN}(\boldsymbol{\mu}_\delta, \boldsymbol{\Sigma}_\delta)$$

$$\boldsymbol{\mu}_\gamma \sim \mathcal{N}(0, 1)$$

$$\boldsymbol{\mu}_\delta \sim \mathcal{N}(0, 1)$$

$$\boldsymbol{\Sigma}_\gamma = \sigma_\gamma \boldsymbol{\Omega}_\gamma$$

$$\boldsymbol{\Sigma}_\delta = \sigma_\delta \boldsymbol{\Omega}_\delta$$

$$\boldsymbol{\Omega}_\gamma \sim \mathcal{LKJ}(\Omega_0)$$

$$\boldsymbol{\Omega}_\delta \sim \mathcal{LKJ}(\Omega_0)$$

$$\sigma_\gamma \sim \mathcal{T}\mathcal{N}(0, \sigma_0)$$

$$\sigma_\delta \sim \mathcal{T}\mathcal{N}(0, \sigma_0)$$

Hierarchical Factorization Machines in Stan

► Data

```
data{
  int<lower = 0> N ; // number of group 1 observations
  int<lower = 0> J ; // number of group 2 observations
  int<lower = 0> K ; // number of latent dimensions
  int X[(N*J), 2] ; // covariate matrix
  vector[(N*J)] y ; // outcome
  real<lower = 0> beta_sigma ; // sd on regression coefficients
  real<lower = 0> y_sigma ; // sd on the outcome, y
  real<lower = 0> gamma_sigma_prior ; //sd on gamma factors
  real<lower = 0> delta_sigma_prior ; //sd on delta factors
  real<lower = 0> gamma_omega_prior ; // prior on omega value for gamma
  real<lower = 0> delta_omega_prior ; // prior on omega value for delta
}
```

Hierarchical Factorization Machines in Stan

► Parameters

```
parameters{
  vector[N] group_1_betas; // non-interacted coefficients
  vector[J] group_2_betas; // non-interacted coefficients

  matrix[N, K] gamma_mu ; //gamma prior mean
  vector<lower=0>[K] gamma_sigma ; // embedding SD
  cholesky_factor_corr[K] gamma_omega; // correlation matrix
  matrix[K, N] gamma_a ; // for non-centered parameterization

  matrix[J , K] delta_mu ; //delta prior mean
  vector<lower=0>[K] delta_sigma ; // embedding SD
  cholesky_factor_corr[K] delta_omega; // correlation matrix
  matrix[K, J] delta_a ; // for non-centered parameterization
}
```

Hierarchical Factorization Machines in Stan

► Transformed Parameters

```
transformed parameters{
  real linear_predictor[(N*J)] ;
  matrix[N, K] gammas ;
  matrix[J, K] deltas ;

  gammas = gamma_mu + (diag_pre_multiply(gamma_sigma, gamma_omega) * gamma_a)' ;
  deltas = delta_mu + (diag_pre_multiply(delta_sigma, delta_omega) * delta_a)' ;

  for(i in 1:(N*J)){
    linear_predictor[i] =
      group_1_betas[X[i, 1]] + group_2_betas[X[i, 2]] +
      (gammas[X[i, 1], ] * deltas[ X[i, 2], ]');
  }
}
```

Hierarchical Factorization Machines in Stan

► Model

```
model{

    // regression coefficients
    group_1_betas ~ normal(0, beta_sigma) ;
    group_2_betas ~ normal(0, beta_sigma) ;

    // factor sd: half-normal distribution
    gamma_sigma ~ normal(0, gamma_sigma_prior) ;
    delta_sigma ~ normal(0, delta_sigma_prior) ;

    // correlation matrices
    gamma_omega ~ lkj_corr_cholesky(gamma_omega_prior) ;
    delta_omega ~ lkj_corr_cholesky(delta_omega_prior) ;

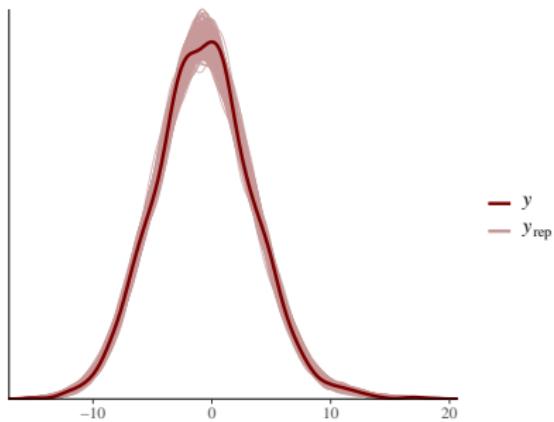
    // for non-centered parameterization
    to_vector(gamma_a) ~ std_normal() ;
    to_vector(delta_a) ~ std_normal() ;

    // hierarchical means
    for(n in 1:N){
        gamma_mu[n, ] ~ normal(0, 1) ;
    }
    for(j in 1:J){
        delta_mu[j, ] ~ normal(0, 1) ;
    }

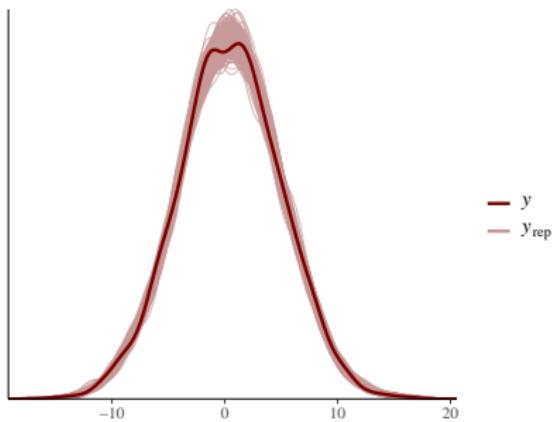
    // outcome
    y ~ normal(linear_predictor, y_sigma) ;
}
```

Fitting a (Hierarchical) Model to Simulated Data

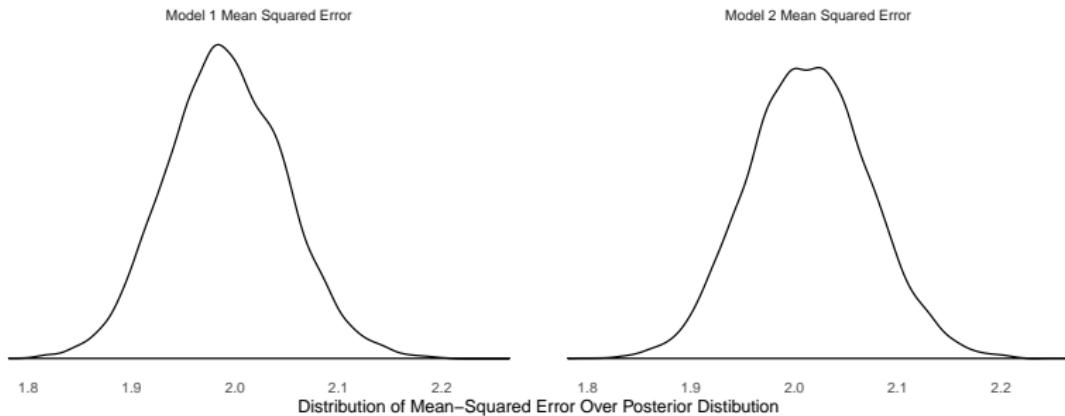
Hierarchical Model 1 Posterior Predictive



Hierarchical Model 2 Posterior Predictive



Fitting a (Hierarchical) Model to Simulated Data



Fitting a Model to Real Data

Dataset

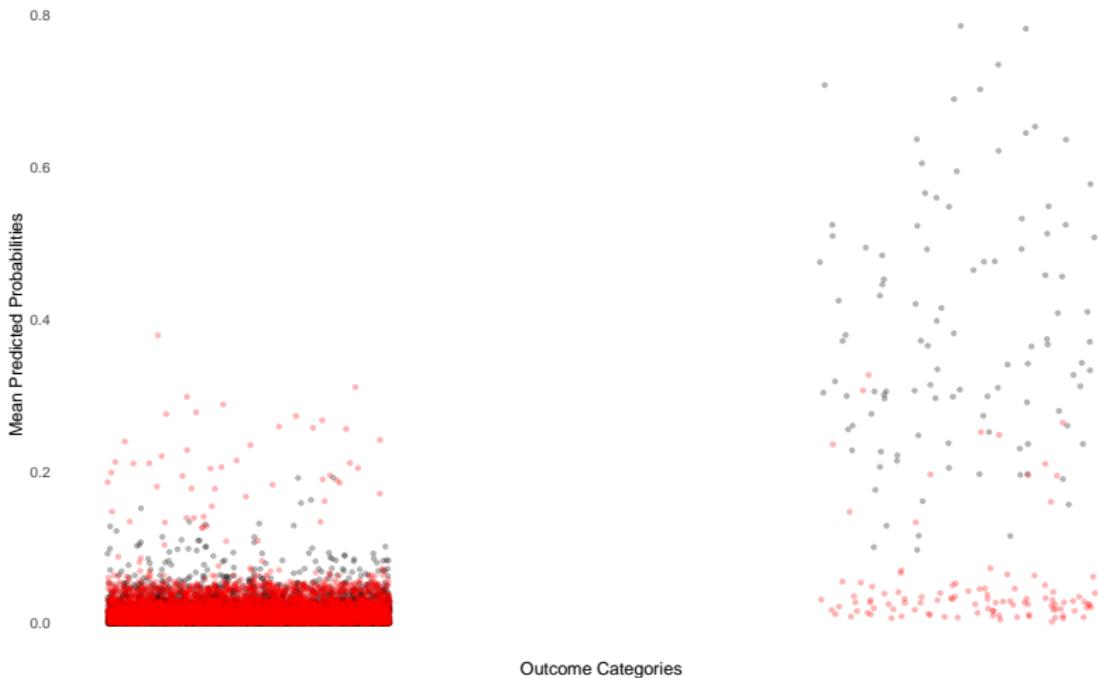
- ▶ Fearon and Laitin, APSR 2003
- ▶ Ethnicity, Insurgency, and Civil War
- ▶ Goal: Understand relationship between ethnic fractionalization and the onset of Civil War
- ▶ Country-year data: All countries in the world, 1946-1999
- ▶ Data and Code: <https://web.stanford.edu/group/ethnic/publicdata/publicdata.html>
- ▶ Fit model with logistic regression

Factorization Machine

- ▶ Modify Existing FM Code:
 - ▶ Logistic Regression
 - ▶ Additional Covariates
- ▶ Note: Rare events outcome (1% positive outcomes)

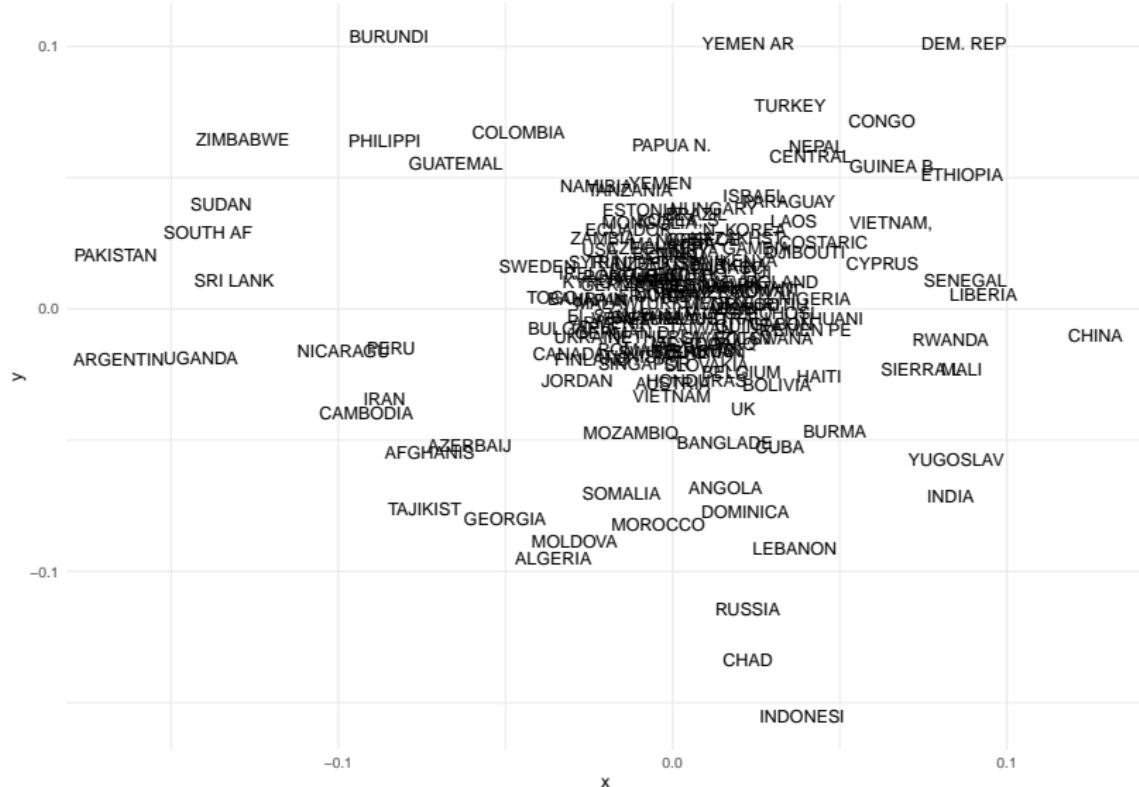
Model Fit

Predicted Probabilities for Each Model;
Red is Logistic Regression, Black is Factorization Machine
Logistic Regression Log-loss: 0.084 Factorization Machine Log-loss: 0.033



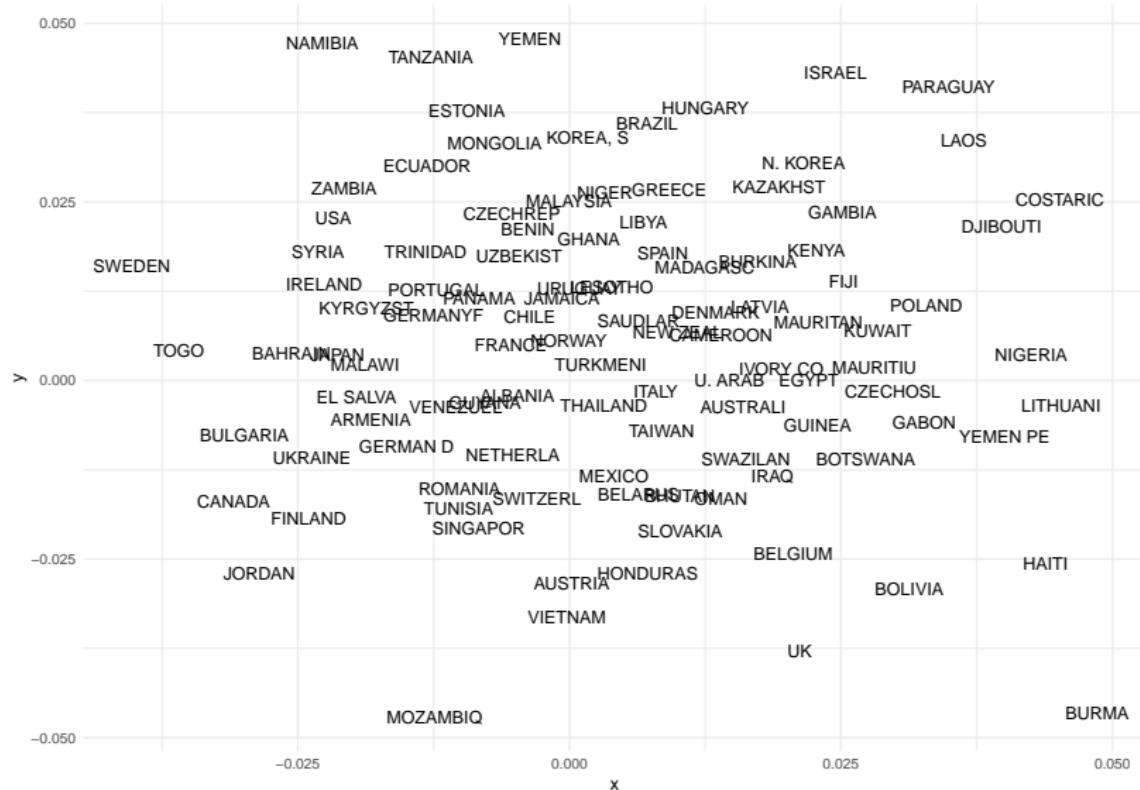
Clustering Factors

Use Sammon Mapping to cluster factors



Clustering Factors

Zoom in on hairball



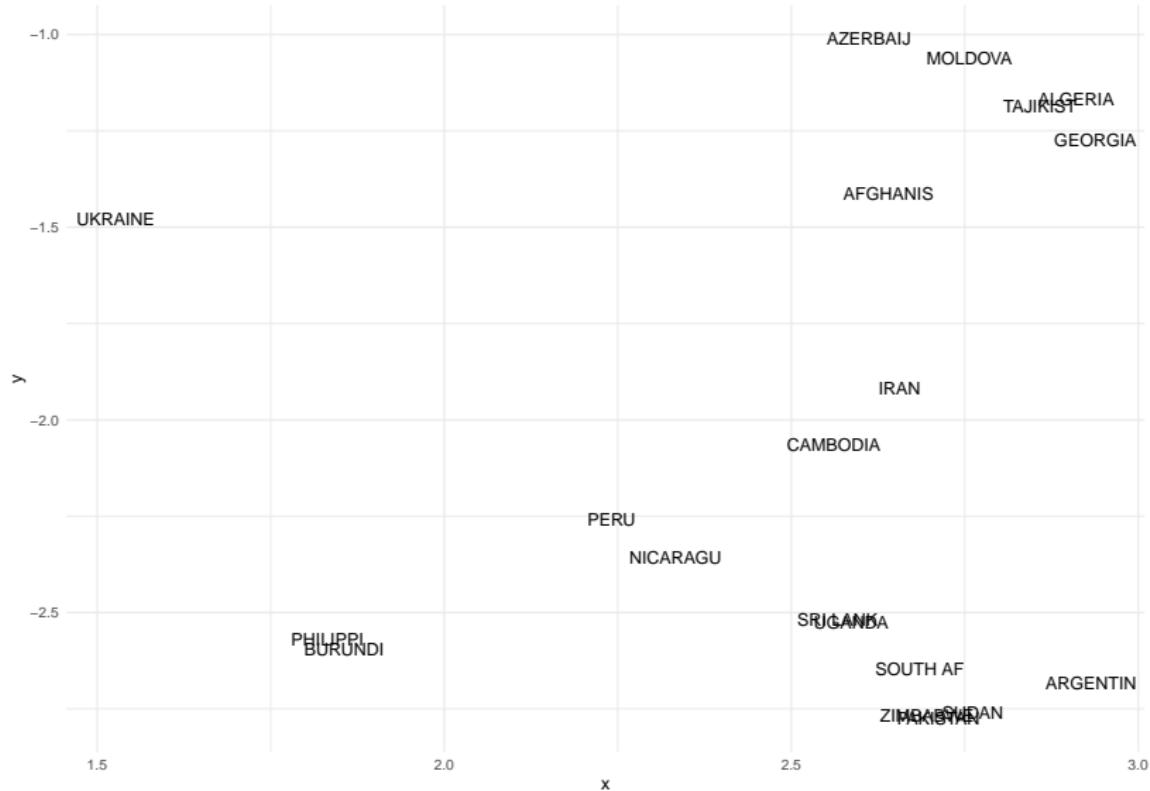
Clustering Factors

UMAP



Clustering Factors

Zoom in on bottom right



Wrap Up

- ▶ Factorization Machines: Extend GLM models with matrix factorization on residuals
- ▶ Two versions:
 - ▶ Factors distributed $\mathcal{N}(0, 1)$
 - ▶ Hierarchical Model
- ▶ Use dimensionality reduction to plot factors
- ▶ What if there are more than two interactions?
 - ▶ Tensor Factorization/Exponential Machines

Thank You!

Code and Slides: https://github.com/adaml Lauretig/ny_r_talk
Twitter: @Lauretig

Thank you!

