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LECTURE NOTES 3

TRACELESS SYMMETRIC TENSOR APPROACH TO LEGENDRE POLYNOMIALS AND SPHERICAL HARMONICS, PART III

9. CONSTRUCTION OF THE LEGENDRE POLYNOMIALS:

In Section 6, at the end of Lecture Notes 1, we discussed traceless symmetric tensor constructions in the case of azimuthal symmetry. We said that an arbitrary function of θ can be expanded as in Eq. (1.49), and we said that the function appearing in this expansion are called Legendre polynomials $P_\ell(\cos\theta)$, where

$$P_\ell(\cos\theta) = N(\ell)\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ , \tag{3.1}$$

where the normalization factor $N(\ell)$ is defined by the convention that

$$P_\ell(\cos\theta{=}1) = 1\ . \tag{3.2}$$

I said there that $N(\ell)$ would be evaluated in Lecture Notes 2, which turned out not to be the case. Instead you were asked to evaluate $N(\ell)$ as Problem 3 of Problem Set 4.

To summarize the argument, note that $\cos\theta = 1$ implies that $\theta = 0$; since θ is the angle between $\hat{\boldsymbol{n}}$ and the z axis, it follows that $\cos\theta = 1$ implies that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$. (To see this in equations rather than words, look at Eq. (1.9) of Lecture Notes 1.) Thus, the normalization convention becomes

$$N(\ell)\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell} = 1\ . \tag{3.3}$$

But at the end of Lecture Notes 2, we showed that for any unit vector $\hat{\boldsymbol{n}}$,

$$\boxed{\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}\ .} \tag{2.50}$$

In Problem 3 of Problem Set 4, you are asked to show that $\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}$ can be evaluated by iteratively using Eq. (2.50) to sum over i_ℓ, then $i_{\ell-1}$, etc., through i_1. The result becomes

$$N(\ell) = \frac{2\ell-1)!!}{\ell!}\ . \tag{3.4}$$

or equivalently

$$N(\ell) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\ , \tag{3.5}$$

so finally we have

$$\boxed{P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ .} \tag{3.6}$$

10. INTEGRATION OVER LEGENDRE POLYNOMIALS:

It will be useful to be able to integrate over traceless symmetric tensor expressions such

$$\int d\Omega \left[C^{(\ell)}_{i_1 \ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \right] \left[C'^{(\ell')}_{j_1 \ldots j_{\ell'}} \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\text{TS}} \right] , \tag{3.7}$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$ is the standard measure over solid angles. (This is related to the fact that in spherical polar coordinates, the volume element is $dV = r^2 \, dr \, \sin\theta \, d\theta \, d\phi$.) Integrations of this sort will allow us to construct an orthonormal basis for these functions, which will give us the standard definition of the spherical harmonics.

We begin by evaluating the basic integral within the expression (3.7), which can be defined as

$$I_{i_1 \ldots i_{2\ell}} \equiv \int d\Omega \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{2\ell}} , \tag{3.8}$$

where of course $\hat{\boldsymbol{n}}$ represents the unit vector in the (θ, ϕ) direction,

$$\hat{\boldsymbol{n}} = \sin\theta \cos\phi \, \hat{\boldsymbol{e}}_{\boldsymbol{1}} + \sin\theta \sin\phi \, \hat{\boldsymbol{e}}_{\boldsymbol{2}} + \cos\theta \, \hat{\boldsymbol{e}}_{\boldsymbol{3}} . \tag{3.9}$$

Note that if the number of $\hat{\boldsymbol{n}}_i$ factors is odd, the integral of the form of Eq. (3.8) would vanish, since the integrand would change sign under the transformation $\hat{\boldsymbol{n}} \to -\hat{\boldsymbol{n}}$, while the range of integration is symmetric under $\hat{\boldsymbol{n}} \to -\hat{\boldsymbol{n}}$. Thus we lose no information by denoting the number of factors by 2ℓ.

One convenient way of evaluating $I_{i_1 \ldots i_{2\ell}}$ is to first evaluate

$$I_1(\vec{\boldsymbol{J}}) \equiv \int d^3x \, e^{-\frac{|\vec{r}\,|^2}{2} + \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{r}}} , \tag{3.10}$$

where as usual $\vec{\boldsymbol{r}} \equiv x_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$. If we can evaluate this integral, it is easy to see that

$$\frac{\partial^{2\ell}}{\partial J_{i_1} \ldots \partial J_{i_{2\ell}}} I_1(\vec{\boldsymbol{J}}) = \int d^3x \, x_{i_1} \ldots x_{i_{2\ell}} \, e^{-\frac{|\vec{r}\,|^2}{2} + \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{r}}} , \tag{3.11}$$

so

$$\begin{aligned}
\left. \frac{\partial^{2\ell}}{\partial J_{i_1} \ldots J_{i_{2\ell}}} I_1(\vec{\boldsymbol{J}}) \right|_{\vec{\boldsymbol{J}}=0} &= \int d^3x \, x_{i_1} \ldots x_{i_{2\ell}} \, e^{-\frac{|\vec{r}\,|^2}{2}} \\
&= \int_0^\infty r^2 \, dr \int d\Omega \, x_{i_1} \ldots x_{i_{2\ell}} \, e^{-\frac{|\vec{r}\,|^2}{2}} \\
&= \int_0^\infty r^{2\ell+2} \, e^{-r^2/2} \, dr \int d\Omega \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{2\ell}} \\
&= I_{i_1 \ldots i_{2\ell}} \int_0^\infty r^{2\ell+2} \, e^{-r^2/2} \, dr ,
\end{aligned} \tag{3.12}$$

where we used the fact that $\hat{\boldsymbol{n}}_i = x_i/r$. To evaluate Eq. (3.10), rewrite it by completing the square,

$$I_1(\vec{\boldsymbol{J}}) = e^{\vec{\boldsymbol{J}}^2/2} \int d^3x \, e^{-\frac{1}{2}(\vec{\boldsymbol{r}}+\vec{\boldsymbol{J}})^2} \ , \tag{3.13}$$

which is a simple consequence of $(\vec{\boldsymbol{r}}+\vec{\boldsymbol{J}})^2 = \vec{\boldsymbol{r}}^2 + \vec{\boldsymbol{J}}^2 + 2\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{r}}$. Now notice that the integral can be drastically simplified by changing the variable of integration to $\vec{\boldsymbol{r}}\,' \equiv \vec{\boldsymbol{r}} + \vec{\boldsymbol{J}}$, which gives $d^3x' = d^3x$, so

$$I_1(\vec{\boldsymbol{J}}) = e^{\vec{\boldsymbol{J}}^2/2} \int d^3x' \, e^{-\frac{1}{2}\vec{\boldsymbol{r}}\,'^2} = e^{\vec{\boldsymbol{J}}^2/2} \int d^3x \, e^{-\frac{1}{2}\vec{\boldsymbol{r}}^2} \ , \tag{3.14}$$

where we dropped the prime on the newly defined variable of integration. Now switch to polar coordinates, giving

$$I_1(\vec{\boldsymbol{J}}) = 4\pi e^{\vec{\boldsymbol{J}}^2/2} \int_0^\infty r^2 \, dr \, e^{-r^2/2} \ , \tag{3.15}$$

where we used $\int d\Omega = 4\pi$.

To evaluate the radial integrals that appear in Eqs. (3.12) and (3.15), we can first evaluate

$$I_2(\lambda) \equiv \int_0^\infty dr \, e^{-\lambda r^2} \ , \tag{3.16}$$

where we see that

$$\int_0^\infty r^{2\ell+2} \, e^{-\lambda r^2} \, dr = (-1)^{\ell+1} \frac{d^{\ell+1} I_2(\lambda)}{d\lambda^{\ell+1}} \tag{3.17}$$

The integral $I_2(\lambda)$ can be evaluated by using the (famous) trick of evaluating its square, renaming the variables of integration for the two factors as x and y. Then

$$I_2^2(\lambda) = \int_0^\infty dx \int_0^\infty dy \, e^{-\lambda(x^2+y^2)} \ . \tag{3.18}$$

We can now treat x and y as if they are Cartesian coordinates, and we can switch to polar coordinates, recognizing that the range of integration includes only one quadrant. Thus,

$$I_2^2(\lambda) = \int_0^{\pi/2} d\phi \int_0^\infty r \, dr \, e^{-\lambda r^2} \ . \tag{3.19}$$

The integral over ϕ gives $\pi/2$, and the r-integral can be carried out by the change of variables $u = r^2$, $du = 2r\,dr$, so

$$I_2^2(\lambda) = \frac{\pi}{2} \int_0^\infty \frac{du}{2} \, e^{-\lambda u} = \frac{\pi}{4\lambda} \ . \tag{3.20}$$

Thus

$$I_2(\lambda) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \tag{3.21}$$

and then

$$\begin{aligned}
\int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\, dr &= (-1)^{\ell+1}\, \frac{d^{\ell+1}}{d\lambda^{\ell+1}} I_2(\lambda)\bigg|_{\lambda=\frac{1}{2}} \\
&= \frac{\sqrt{\pi}}{2}\frac{(2\ell+1)!!}{2^{\ell+1}}\left(\frac{1}{2}\right)^{-(2\ell+3)/2} \\
&= \sqrt{\frac{\pi}{2}}(2\ell+1)!! \\
&= \sqrt{\frac{\pi}{2}}\frac{(2\ell+1)!}{2^\ell \ell!}\ ,
\end{aligned} \tag{3.22}$$

where in the last line we used the identity of Eq. (2.22), $(2\ell-1)!! = (2\ell)!/(2^\ell\ell!)$, rewritten for $(2\ell+1)!!$.

By combining Eqs. (3.12), (3.15), and (3.21), we find

$$I_{i_1\ldots i_{2\ell}} = 4\pi\frac{2^\ell\ell!}{(2\ell+1)!}\ \frac{\partial^{2\ell}}{\partial J_{i_1}\ldots\partial J_{i_{2\ell}}}\, e^{\vec{\boldsymbol{J}}^2/2}\bigg|_{\vec{\boldsymbol{J}}=0}\ . \tag{3.23}$$

To evaluate the partial derivative in the above equation, we begin by looking at the first few terms:

$$\begin{aligned}
\frac{\partial^{2\ell}}{\partial J_{i_1}\ldots J_{i_{2\ell}}}\, e^{\vec{\boldsymbol{J}}^2/2} &= \frac{\partial^{2\ell-1}}{\partial J_{i_1}\ldots\partial J_{i_{2\ell-1}}}\ \left[J_{i_{2\ell}} e^{\vec{\boldsymbol{J}}^2/2}\right] \\
&= \frac{\partial^{2\ell-2}}{\partial J_{i_1}\ldots\partial J_{i_{2\ell-2}}}\ \left[\left(\delta_{2\ell-1,2\ell} + J_{i_{2\ell-1}}J_{i_{2\ell}}\right) e^{\vec{\boldsymbol{J}}^2/2}\right] \\
&= \frac{\partial^{2\ell-2}}{\partial J_{i_1}\ldots\partial J_{i_{2\ell-2}}}\ \Big[\big(J_{i_{2\ell-1}}\delta_{2\ell-1,2\ell} + J_{i_{2\ell-1}}J_{i_{2\ell-1}}J_{i_{2\ell}} \\
&\qquad +\ \delta_{i_{2\ell-2},i_{2\ell-1}}J_{i_{2\ell}} + \delta_{i_{2\ell-2},i_{2\ell}}J_{i_{2\ell-1}}\big)\, e^{\vec{\boldsymbol{J}}^2/2}\Big]
\end{aligned} \tag{3.24}$$

Each derivative will act on all the factors present after the earlier derivatives have acted. The factor $e^{\vec{\boldsymbol{J}}^2/2}$ is always present, so the derivative with respect to J_k will always produce a term that is equal to J_k times all the previous terms. In addition, the derivative with respect to J_{i_k} will act on all the factors of J_{i_m} already present, where

$$\frac{\partial J_{i_m}}{\partial J_{i_k}} = \delta_{i_k,i_m}\ . \tag{3.25}$$

Eq. (3.23) tells us that after carrying out all 2ℓ derivatives, we should set $\vec{\boldsymbol{J}} = 0$, which means that all terms proportional to any factors of J_{i_m} will vanish, and we are left only with the terms proportional to products of Kronecker δ-functions. Each possible pairing of indices on the Kronecker δ-functions arises once, when the derivative with respect to the J_{i_k} on the left of the list $J_{i_1} \ldots J_{i_{2\ell}}$ acts the factor of J_{i_m} that was further to the right. The final result can be written as

$$I_{i_1 \ldots i_{2\ell}} = 4\pi \frac{2^\ell \ell!}{(2\ell+1)!} \sum_{\text{all pairings}} \delta_{i_1,i_2}\, \delta_{i_3,i_4} \ldots \delta_{i_{2\ell-1},i_{2\ell}} \ . \tag{3.26}$$

In itemizing the pairings in the sum, remember that a pairing does not depend on the order in which the δ-function factors are written, and it does not depend on the ordering of the two indices on each δ. For example, for $\ell = 2$,

$$\sum_{\text{all pairings}} \delta_{i_1,i_2}\delta_{i_3,i_4} = \delta_{i_1,i_2}\delta_{i_3,i_4} + \delta_{i_1,i_3}\delta_{i_2,i_4} + \delta_{i_1,i_4}\delta_{i_1,i_3} \ . \tag{3.27}$$

Now we can apply this result to our target calculation, Eq. (3.7), finding

$$\begin{aligned} &\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \right] \left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}, \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\text{TS}} \right] \\ &\qquad \propto C^{(\ell)}_{i_1\ldots i_\ell} C'^{(\ell')}_{j_1\ldots j_{\ell'}} \sum_{\text{all pairings}} \delta_{k_1,k_2} \ldots \delta_{k_{\ell+\ell'-1},k_{\ell+\ell'}} \ , \end{aligned} \tag{3.28}$$

where the sum is over all pairings of the full set of indices $\{i_1, \ldots, i_\ell, j_1, \ldots, j_{\ell'}\}$. Whenever two of the i_m indices are paired, or whenever two of the j_m indices are paired, the resulting δ-function will take a trace when it acts on $C^{(\ell)}$ or $C'^{(\ell')}$, and the term will vanish, since the C's are traceless. Thus, a term will be nonzero only if ever pair involves one of the indices on $C^{(\ell)}$ and one of the indices on $C'^{(\ell')}$. This is possible only if $\ell' = \ell$, so we immediately see that

$$\boxed{\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \right] \left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}, \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\text{TS}} \right] = 0 \text{ if } \ell' \neq \ell.} \tag{3.29}$$

When $\ell' = \ell$, Eq. (3.26) tells us that

$$\begin{aligned} &\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \right] \left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}, \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\text{TS}} \right] \\ &= 4\pi \frac{2^\ell \ell!}{(2\ell+1)!} C^{(\ell)}_{i_1\ldots i_\ell} C'^{(\ell)}_{j_1\ldots j_\ell} \sum_{\text{all pairings}} \delta_{i_1,j_1} \ldots \delta_{i_\ell,j_\ell} \ , \end{aligned} \tag{3.30}$$

where the sum is over all pairings which pair every index on $C^{(\ell)}$ with an index on $C'^{(\ell)}$. Since $C^{(\ell)}$ and $C'^{(\ell)}$ are both symmetric, every pairing will give the same result, which will be equal to the result of one pairing, times the number of pairings. How many pairings are there? The index i_1 can be paired with any of ℓ different j's, then the index i_2 can be paired with any of the $\ell - 1$ remaining j's, and so on, so the total number of pairings is $\ell!$. Finally then,

$$\boxed{\begin{aligned}\int d\Omega \left[C^{(\ell)}_{i_1 \ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\rm TS}\right] \left[C'^{(\ell')}_{j_1 \ldots j_{\ell'}}, \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\rm TS}\right] \\ = 4\pi \frac{2^\ell \ell!^2}{(2\ell+1)!} C^{(\ell)}_{i_1 \ldots i_\ell} C'^{(\ell)}_{i_1 \ldots i_\ell} \quad \text{if } \ell' = \ell.\end{aligned}} \tag{3.31}$$

On Problem 1 of Problem Set 5, you are asked to use this identity, along with the definition of Eq. (3.6) of the Legendre polynomials, to show that

$$\boxed{\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{2\ell+1} \delta_{\ell,\ell'} \ .} \tag{3.32}$$

11. CONSTRUCTION OF THE SPHERICAL HARMONICS:

We have learned that a traceless symmetric tensor of rank ℓ that is azimuthally symmetric can be constructed as $\{ \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \}_{\rm TS}$, but now we would like to generalize this to obtain a complete basis for the traceless symmetric tensors of rank ℓ. Clearly, in addition to $\hat{\boldsymbol{z}}$, we will want to make use of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. There are a number of reasonable ways to choose such a basis, but the standard convention involves forming complex linear combinations of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, namely

$$\begin{aligned}\hat{\boldsymbol{u}}^{(1)} &\equiv \hat{\boldsymbol{u}}^{+} \equiv \frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} + i \hat{\boldsymbol{e}}_{\boldsymbol{y}}\right) \ , \\ \hat{\boldsymbol{u}}^{(2)} &\equiv \hat{\boldsymbol{u}}^{-} \equiv \frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} - i \hat{\boldsymbol{e}}_{\boldsymbol{y}}\right) \ ,\end{aligned} \tag{3.33}$$

where in this context we also define

$$\hat{\boldsymbol{u}}^{(3)} \equiv \hat{\boldsymbol{z}} \ . \tag{3.34}$$

This is then an orthonormal basis in the sense that

$$\hat{\boldsymbol{u}}^{(i)*} \cdot \hat{\boldsymbol{u}}^{(j)} = \delta_{ij} \ . \tag{3.35}$$

The standard $Y_{\ell m}(\theta,\phi)$ functions have a ϕ-dependence proportional to $e^{im\phi}$, and we can see that this can be obtained in our formalism through the use of $\hat{\boldsymbol{u}}^+$, since

$$\hat{\boldsymbol{u}}^+ \cdot \hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}} \sin\theta e^{i\phi} \ . \tag{3.36}$$

Note that

$$\hat{\boldsymbol{u}}^+_i \hat{\boldsymbol{u}}^+_i = 0 \ , \tag{3.37}$$

which makes the behavior of these vectors rather counterintuitive. In particular it means that $\hat{\boldsymbol{u}}^+_i \hat{\boldsymbol{u}}^+_j$ is a traceless symmetric tensor, without any need for subtracting off terms containing Kronecker δ-functions. If we consider the quantity

$$\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m} \, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \tag{3.37}$$

for $m \geq 0$, we can ask what results when the traceless symmetric part is fully expanded, and all summations of repeated indices are carried out. In the final expression every $\hat{\boldsymbol{u}}^+$ will be dotted with one of the other vectors: either $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{n}}$, or another $\hat{\boldsymbol{u}}^+$. Of these three choices, the only nonzero option is $\hat{\boldsymbol{u}}^+ \cdot \hat{\boldsymbol{n}}$, as given by Eq. (3.36). Thus we can see that

$$\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m} \, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \propto e^{im\phi} \ , \tag{3.38}$$

exactly as wanted to match the standard conventions for $Y_{\ell,m}$. For negative values of m, $m = -|m|$, we can similarly see that

$$\{\, \hat{\boldsymbol{u}}^-_{i_1} \ldots \hat{\boldsymbol{u}}^-_{i_{|m|}} \, \hat{\boldsymbol{z}}_{i_{|m|+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \propto e^{-i|m|\phi} \ , \tag{3.39}$$

which is exactly what we want to match the standard convention for $Y_{\ell,m}$ for negative m.

Thus we can define the standard spherical harmonic basis functions as

$$Y_{\ell,m}(\theta,\phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \tag{3.40}$$

where

$$C^{(\ell,m)}_{i_1 \ldots i_\ell} = \begin{cases} N(\ell,m)\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m} \, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} & \text{for } m \geq 0 \ , \\ N(\ell,m)\{\, \hat{\boldsymbol{u}}^-_{i_1} \ldots \hat{\boldsymbol{u}}^-_{i_{|m|}} \, \hat{\boldsymbol{z}}_{i_{|m|+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} & \text{for } m \leq 0 \ , \end{cases} \tag{3.41}$$

for some normalization factors $N(\ell,m)$, which still have to be specified.

One might wonder why we can't find more independent tensors by forming products that involve both $\hat{\boldsymbol{u}}^+$ and $\hat{\boldsymbol{u}}^-$. Eqs. (3.38) and (3.39) suggest that such combinations give nothing new, since factors of $e^{i\phi}$ and $e^{-i\phi}$ will cancel each other. In terms of the tensors, it is instructive to expand

$$\begin{aligned}\operatorname*{Sym}_{ij}\left[\hat{\boldsymbol{u}}^+_i\hat{\boldsymbol{u}}^-_j\right] &= \frac{1}{4}\left[(\hat{\boldsymbol{x}}_i + i\hat{\boldsymbol{y}}_i)(\hat{\boldsymbol{x}}_j - i\hat{\boldsymbol{y}}_j) + (\hat{\boldsymbol{x}}_i - i\hat{\boldsymbol{y}}_i)(\hat{\boldsymbol{x}}_j + i\hat{\boldsymbol{y}}_j)\right] \\ &= \frac{1}{2}\left[\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j + \hat{\boldsymbol{y}}_i\hat{\boldsymbol{y}}_j\right] \ .\end{aligned} \tag{3.42}$$

Notice, however, that

$$\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j + \hat{\boldsymbol{y}}_i\hat{\boldsymbol{y}}_j + \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j = \delta_{ij} \ , \tag{3.43}$$

an identity that can be verified by considering the various cases. Thus,

$$\operatorname*{Sym}_{ij}\left[\hat{\boldsymbol{u}}^+_i\hat{\boldsymbol{u}}^-_j\right] = \frac{1}{2}\left[\delta_{ij} - \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\right] \ , \tag{3.44}$$

which makes it very clear that products involving factors of both $\hat{\boldsymbol{u}}^+$ and $\hat{\boldsymbol{u}}^-$ can always be rewritten in terms of products of $\hat{\boldsymbol{z}}$'s and either $\hat{\boldsymbol{u}}^+$ or $\hat{\boldsymbol{u}}^-$, but not both.

The standard spherical harmonics are normalized so that

$$\int d\Omega\, Y^*_{\ell',m'}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm} \ . \tag{3.45}$$

In Problem 2 of Problem Set 5, you are asked to show that the above equation is satisfied by the definition of Eqs. (3.40) and (3.41) whenever $\ell' \neq \ell$, or $m' \neq m$, in which case both sides are zero. You are also asked to calculate the normalization, determining $|N(\ell, m)|$, given the formula

$$\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{z}}_{i_\ell} = \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)} \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \,\}_{\rm TS} \ . \tag{3.46}$$

Here we want to derive the formula above.

This calculation is a generalization to arbitrary values of m of the $m = 0$ calculation of Eq. (2.50) of Lecture Notes 2. As with the original derivation, we start with Eq. (2.35),

$$\begin{aligned}\{\, S_{i_1\ldots i_\ell} \,\}_{\rm TS} = S_{i_1\ldots i_\ell} + \operatorname*{Sym}_{i_1\ldots i_\ell}\Big[& a_{1,\ell}\delta_{i_1 i_2}\delta^{j_1 j_2} S_{j_1 j_2 i_3\ldots i_\ell} \\ & + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4}\delta^{j_1 j_2}\delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big] \ ,\end{aligned} \tag{2.35}$$

which gives the traceless symmetric part of any symmetric tensor $S_{i_1\ldots i_\ell}$, where we have found in Eq. (2.47) that

$$a_{1,\ell} = -\frac{\ell(\ell-1)}{2(2\ell-1)} \ . \tag{2.47}$$

We wish to apply this formula to

$$S_{i_1\ldots i_\ell} = \underset{i_1\ldots i_\ell}{\mathrm{Sym}}\left[\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\right] \tag{3.47}$$

The calculation will be similar to our calculation of $\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}$ in Eq. (2.50), but will be slightly more complicated. The new complication comes in evaluating the quantity $\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell}$ on the right-hand side of Eq. (2.35) above. In the previous calculation we were interested in $S_{i_1\ldots i_\ell} = \hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}$, so it was trivial to take its trace. In that case, $\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell} = \delta^{j_1j_2}\hat{\boldsymbol{n}}_{j_1}\hat{\boldsymbol{n}}_{j_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell} = \hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}$. In this case we need to evaluate

$$\begin{aligned}\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell} &= \delta^{j_1j_2}\underset{j_1,j_2,i_3,\ldots,i_\ell}{\mathrm{Sym}}\left[\hat{\boldsymbol{u}}^+_{j_1}\hat{\boldsymbol{u}}^+_{j_2}\hat{\boldsymbol{u}}^+_{i_3}\ldots\hat{\boldsymbol{u}}^+_{i_m}\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\right]\\ &= \delta^{j_1j_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } j_1j_2i_3\ldots i_\ell}}\left[\hat{\boldsymbol{u}}^+_{j_1}\hat{\boldsymbol{u}}^+_{j_2}\hat{\boldsymbol{u}}^+_{i_3}\ldots\hat{\boldsymbol{u}}^+_{i_m}\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\right]\,.\end{aligned} \tag{3.48}$$

After symmetrization the indices j_1 and j_2 can find themselves in different places within the quantity in square brackets [], so we need to consider all possible cases, as we did when we derived Eq. (2.47) at the end of Lecture Notes 2. In this calculation, the relevant cases are:

Case I: Both j_1 and j_2 contract with the indices on the $\hat{\boldsymbol{z}}$ factors. There are $\ell-m$ factors of $\hat{\boldsymbol{z}}$, so j_1 can have any of $\ell-m$ locations, and then j_2 can have any of $\ell-m-1$ locations. The remaining $\ell-2$ indices can then appear in any order, which means that there are $(\ell-2)!$ possibilities for these assignments. So, the multiplicity for this case is $(\ell-m)(\ell-m-1)(\ell-2)!$. The contraction of two $\hat{\boldsymbol{z}}$ vectors, $\delta^{j_1j_2}\hat{\boldsymbol{z}}_{j_1}\hat{\boldsymbol{z}}_{j_2}$, gives a factor of 1, since $\hat{\boldsymbol{z}}$ is a unit vector. Thus, the value of the contribution for each ordering in Case I is $\frac{1}{\ell!}\,\mathrm{Sym}[\hat{\boldsymbol{u}}^+_{i_3}\ldots\hat{\boldsymbol{u}}^+_{i_{m+2}}\hat{\boldsymbol{z}}_{i_{m+3}}\ldots\hat{\boldsymbol{z}}_{i_\ell}]$. Note that the number of factors of $\hat{\boldsymbol{u}}^+$ must remain m, so since since the indices in this expression run from i_3 to i_ℓ, the first m indices are $i_3\ldots i_{m+2}$. Summarizing,

$$\begin{aligned}\text{Case I:}\qquad &\text{Multiplicity} = (\ell-m)(\ell-m-1)(\ell-2)!\\ &\text{Value} = \frac{1}{\ell!}\underset{i_3\ldots i_\ell}{\mathrm{Sym}}[\hat{\boldsymbol{u}}^+_{i_3}\ldots\hat{\boldsymbol{u}}^+_{i_{m+2}}\hat{\boldsymbol{z}}_{i_{m+3}}\ldots\hat{\boldsymbol{z}}_{i_\ell}]\,.\end{aligned} \tag{3.49}$$

Case II: One of the pair j_1, j_2 contracts with one of the $\hat{\boldsymbol{z}}$ vectors, while the other contracts with one of the $\hat{\boldsymbol{u}}^+$ vectors. There are m choices of which $\hat{\boldsymbol{u}}^+$ vector to contract with, and $\ell-m$ choices of which $\hat{\boldsymbol{z}}$ vector. The index that contracts with a $\hat{\boldsymbol{z}}$ vector can be either j_1 or j_2, giving a factor of 2. Finally, the remaining $\ell-2$ indices can appear in any order, giving a factor of $(\ell-2)!$. The value of these terms will be zero, since $\delta^{j_1 j_2} \hat{\boldsymbol{z}}_{j_1} \hat{\boldsymbol{u}}^+_{j_2} = 0$. We could ignore this case, but I will tabulate it so that I can check that I have the right total multiplicity. Summarizing,

$$\text{Case II:} \qquad \text{Multiplicity} = 2m(\ell - m)(\ell - 2)!\ , \qquad \text{Value} = 0\ . \tag{3.50}$$

Case III: Both j_1 and j_2 contract with the indices on the $\hat{\boldsymbol{u}}^+$ factors. The value is zero, since $\delta^{j_1 j_2} \hat{\boldsymbol{u}}^+_{j_1} \hat{\boldsymbol{u}}^+_{j_2} = 0$, but I'll calculate the multiplicity anyway, so that I can check that my total multiplicity calculation gives $\ell!$. Since there are m factors of $\hat{\boldsymbol{u}}^+$, j_1 can be one of m choices, j_2 can be one of the remaining $m-1$ choices, and then the other $\ell - 2$ indices can appear in any order, so

$$\text{Case III:} \qquad \text{Multiplicity} = m(m-1)(\ell - 2)!\ , \qquad \text{Value} = 0\ . \tag{3.51}$$

The total multiplicity is then

$$(\ell - 2)!\,[(\ell - m)(\ell - m - 1) + 2m(\ell - m) + m(m-1)] = (\ell - 2)!\ell(\ell - 1) = \ell!\ , \tag{3.52}$$

so the multiplicities give the right sum.

The final result comes entirely from Case I, so

$$\begin{aligned} \delta^{j_1 j_2} S_{j_1 j_2 i_3 \ldots i_\ell} &= \frac{(\ell - m)(\ell - m - 1)(\ell - 2)!}{\ell!} \operatorname*{Sym}_{i_3 \ldots i_\ell} [\hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell}] \\ &= \frac{(\ell - m)(\ell - m - 1)}{\ell(\ell - 1)} \operatorname*{Sym}_{i_3 \ldots i_\ell} [\hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell}]\ . \end{aligned} \tag{3.53}$$

Using this result with Eq. (2.35), we can write

$$\begin{aligned} \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}} &= \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \\ &\quad - \frac{\ell(\ell-1)}{2(2\ell - 1)} \frac{(\ell - m)(\ell - m - 1)}{\ell(\ell - 1)} \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\delta_{i_1, i_2} \operatorname*{Sym}_{i_3 \ldots i_\ell} [\hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell}] \right] + \ldots\ , \end{aligned} \tag{3.54}$$

or, after simplifying,

$$\boxed{\begin{aligned} \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}} &= \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \\ &\quad - \frac{(\ell - m)(\ell - m - 1)}{2(2\ell - 1)} \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\delta_{i_1, i_2} \hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \\ &\quad + \ldots \end{aligned}} \tag{3.55}$$

Note that I omitted the operation

$$\operatorname*{Sym}_{i_3 \dots i_\ell}$$

that appeared on the right-hand side of Eq. (3.54), since it is rendered unnecessary by the

$$\operatorname*{Sym}_{i_1 \dots i_\ell}$$

operator that acts on the same indices and more. The … at the end of Eq. (3.55) refers to a sum of terms, each of which has one more Kronecker δ-function than the previous term.

Now we can consider the quantity $\{\,\hat{\boldsymbol{u}}^+_{i_1} \dots \hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}} \dots \hat{\boldsymbol{z}}_{i_\ell}\,\}_{\text{TS}}\hat{\boldsymbol{z}}_{i_\ell}$. This quantity is clearly traceless and symmetric on the indices $i_1 \dots i_{\ell-1}$, since $\{\,\hat{\boldsymbol{u}}^+_{i_1} \dots \hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}} \dots \hat{\boldsymbol{z}}_{i_\ell}\,\}_{\text{TS}}$ has these properties, and they are not affected by multiplying by $\hat{\boldsymbol{z}}_{i_\ell}$ and summing over i_ℓ. We also know, from Sec. 5 of Lecture Notes 1, that the number of linearly independent traceless symmetric tensors of rank ℓ is $2\ell+1$. The tensors described by Eq. (3.41) are therefore a complete basis for the space of traceless symmetric tensors of rank ℓ, since there are $2\ell+1$ of them, and they can be shown to be linearly independent.

To prove linear independence, consider a linear sum

$$S_{i_1 \dots i_\ell} \equiv \sum_{m=-\ell}^{\ell} a_m C^{(\ell,m)}_{i_1 \dots i_\ell} \ . \tag{3.56}$$

Then from Eqs. (3.38) and (3.39),

$$S_{i_1 \dots i_\ell}\hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_\ell} = \sum_{m=-\ell}^{\ell} a_m g_m e^{im\phi} \ , \tag{3.57}$$

where the coefficients g_m are the constants of proportionality in Eqs. (3.38) and (3.39), which can be seen to be nonzero. It then follows that

$$\int_0^{2\pi} d\phi\, e^{-i\bar{m}\phi}\, S_{i_1 \dots i_\ell}\hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_\ell} = 2\pi a_{\bar{m}} g_{\bar{m}} \ . \tag{3.58}$$

Thus, if $S_{i_1 \dots i_\ell} = 0$, it follows that each a_m must be zero, which is the definition of linear independence.

Getting back to the main thread, the quantity $\{\,\hat{\boldsymbol{u}}^+_{i_1} \dots \hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}} \dots \hat{\boldsymbol{z}}_{i_\ell}\,\}_{\text{TS}}\hat{\boldsymbol{z}}_{i_\ell}$ must be expressible as a linear sum of these basis tensors (the tensors of Eq. (3.41)), for rank $\ell - 1$. Furthermore, by multiplying by $\hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}}$ and observing the ϕ-dependence of the form $e^{im\phi}$, one can see that the only basis tensor that can contribute is $\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \,\}_{\mathrm{TS}}$. Thus,

$$\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} \hat{\boldsymbol{z}}_{i_\ell} \propto \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \,\}_{\mathrm{TS}} \ , \tag{3.59}$$

so all we need to figure out is the constant of proportionality. The constant of proportionality can be found by imagining that the left- and right-hand sides are each expanded as in Eq. (2.35), and then knowledge of the leading term on each side would be enough to determine the constant of proportionality.

From Eq. (3.55), we see that

$$\begin{aligned} \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}}\, \hat{\boldsymbol{z}}_{i_\ell} &= \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} \\ &\quad - \frac{(\ell - m)(\ell - m - 1)}{2(2\ell - 1)} \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\delta_{i_1, i_2} \hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} \\ &\quad + \ldots \ . \end{aligned} \tag{3.60}$$

To evaluate

$$\operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} \tag{3.61}$$

and

$$\operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\delta_{i_1, i_2} \hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} \ , \tag{3.62}$$

we need to keep track of the different possibilities for the placement of the index i_ℓ when the symmetrization is carried out. We could list all cases as in Eqs. (3.49)–(3.51), but the situation here is simpler, so we will not need to be so thorough. We are looking for the contribution to the leading term in Eq. (3.60), which means the term that includes no Kronecker δ-functions. In Eq. (3.61), the nonzero contributions arise when the index i_ℓ appears on one of the $\ell - m$ factors of $\hat{\boldsymbol{z}}$, while the contribution vanishes when i_ℓ appears on one of the m factors of $\hat{\boldsymbol{u}}^+$. Consequently,

$$\operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} = \frac{\ell - m}{\ell} \operatorname*{Sym}_{i_1 \ldots i_{\ell-1}} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \right] . \tag{3.63}$$

In Eq. (3.62), the index i_ℓ can appear in any of ℓ different locations, but contributions to the leading term arise only when i_ℓ is an index of the Kronecker δ-function, which causes it to disappear. There are 2 indices on the Kronecker δ-function, so the leading term will result in a fraction $2/\ell$ of the orderings:

$$\begin{aligned} &\operatorname*{Sym}_{i_1 \ldots i_\ell} \left[\delta_{i_1, i_2} \hat{\boldsymbol{u}}^+_{i_3} \ldots \hat{\boldsymbol{u}}^+_{i_{m+2}} \hat{\boldsymbol{z}}_{i_{m+3}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \right] \hat{\boldsymbol{z}}_{i_\ell} \\ &\qquad = \frac{2}{\ell} \operatorname*{Sym}_{i_1 \ldots i_{\ell-1}} \left[\hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m} \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \right] + \ldots \ . \end{aligned} \tag{3.64}$$

Inserting these results into Eq. (3.60), we find

$$\begin{aligned}\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_\ell} &= \left[\frac{\ell-m}{\ell}-\frac{(\ell-m)(\ell-m-1)}{2(2\ell-1)}\,\frac{2}{\ell}\right]\\ &\quad\times\mathop{\rm Sym}_{i_1\ldots i_{\ell-1}}\left[\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\right]+\ldots\\ &= \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\mathop{\rm Sym}_{i_1\ldots i_{\ell-1}}\left[\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\right]+\ldots\;.\end{aligned}\tag{3.65}$$

Knowing that Eq. (3.59) is valid, it follows that

$$\boxed{\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_\ell} = \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\,\}_{\rm TS}\;.}\tag{3.66}$$

In Problem 2 of Problem Set 5, you are asked to use Eqs. (3.66) and (3.31) to compute the magnitude of the normalization factors $N(\ell,m)$ that appear in Eq. (3.41). The final formula for $N(\ell,m)$, including a sign factor $(-1)^m$ that is part of the standard conventions, is given for $m\geq 0$ by

$$\boxed{N(\ell,m)=\frac{(-1)^m(2\ell)!}{2^\ell\ell!}\sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}\;.}\tag{3.67}$$

For $m<0$, the factor of $(-1)^m$ is not included. With these conventions

$$Y_{\ell,-m}(\theta,\phi)=(-1)^mY^*_{\ell,m}(\theta,\phi)\tag{3.68}$$

for any m. Spherical harmonics with $m<0$ can be evaluated either by using Eq. (3.67) without the factor of $(-1)^m$, along with Eqs. (3.40) and (3.41), or by using Eq. (3.68).