MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II December 8, 2019
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PROBLEM SET 9 SOLUTIONS

PROBLEM 1: BAR SLIDING ON TWO RAILS IN A UNIFORM MAGNETIC FIELD *(20 points)*

This problem was carried over from Problem Set 8.

Griffiths Problem 7.7 (p. 310).

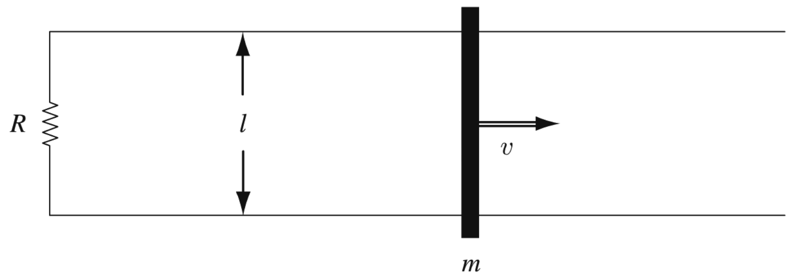


FIGURE 7.17

(a) As the bar slides along the rails, it increases the area of the circuit. To apply Faraday's law using the right-hand convention described by Griffiths on pp. 308-309, I will define the flux through the loop as positive when out of the page, which corresponds to defining the current as positive when counter-clockwise. Since the magnetic field B points into the page, the flux $\Phi = -B\ell x$, where x is the horizontal coordinate of the bar, with $x = 0$ corresponding to the x-coordinate of the resistor. Then

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = B\ell\frac{\mathrm{d}x}{\mathrm{d}t} = B\ell v \ .$$

The induced current is then given by

$$I = \mathcal{E}/R = \frac{v\ell B}{R} \ .$$

Since I is positive, it is counterclockwise and hence the current runs downward through the resistor.

(b) We complete the coordinate system by choosing the y axis to be vertical in the diagram, and the z axis to point out of the page. Note that to use the standard conventions, we must choose the coordinates so that $\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}$. Then $\vec{\boldsymbol{B}} = -B\hat{\boldsymbol{z}}$ and the current within the bar is $\vec{\boldsymbol{I}} = I\hat{\boldsymbol{y}}$, so the Lorentz force acting on the bar is

$$\vec{\boldsymbol{F}} = \ell\vec{\boldsymbol{I}} \times \vec{\boldsymbol{B}} = -\frac{v\ell^2 B^2}{R}\hat{\boldsymbol{x}} \ .$$

The force points leftwards, towards the interior of the loop and opposes the motion of the bar.

(c) Once set in motion, the bar induces a current in the circuit and then slows down by the Lorentz force acting on this current. The speed of the bar changes as

$$\begin{aligned}\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} &= \frac{\vec{\boldsymbol{F}}}{m} = -\frac{I\ell B}{m}\hat{\boldsymbol{x}} \\ &= -\frac{\ell^2 B^2}{mR} v\hat{\boldsymbol{x}}\end{aligned}$$

and so

$$\boxed{v(t) = v_0 \exp\left(-\frac{\ell^2 B^2}{mR}t\right).}$$

(d) Here we want to check that all of the kinetic energy of the bar is eventually converted to heat through Joule heating in the resistor. The instantaneous power of Joule heating is:

$$P = I^2 R = \frac{v^2 \ell^2 B^2}{R} ,$$

and the total amount of energy loss is

$$\begin{aligned}\Delta E &= \int_0^\infty P(t)\, \mathrm{d}t \\ &= \frac{\ell^2 B^2}{R} v_0^2 \int_0^\infty \exp\left(-2\frac{\ell^2 B^2}{mR}t\right) \mathrm{d}t \\ &= \frac{\ell^2 B^2}{R} v_0^2 \cdot \left(\frac{mR}{2\ell^2 B^2}\right) \\ &= \frac{m v_0^2}{2} .\end{aligned}$$

So, as expected, all of the original kinetic energy is converted to heat in the resistor.

PROBLEM 2: DONUT-SHAPED MAGNETS ON A VERTICAL ROD *(15 points)*

Griffiths Problem 6.2 (p. 293).

This problem was carried over from Problem Set 8.

(a) The magnetic field on upper magnet is,

$$\vec{B}_1 = \vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}}{z^3} = \frac{\mu_0 \vec{m}}{2\pi z^3}. \tag{2.1}$$

The magnetic force is,

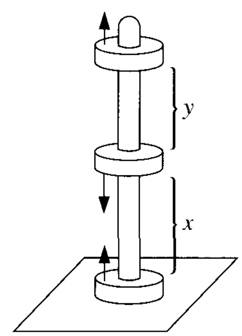
$$\vec{F} = \vec{\nabla}(\vec{m}_2 \cdot \vec{B}_1) = -\frac{\mu_0 m^2}{2\pi}\vec{\nabla}\frac{1}{z^3} = \frac{3\mu_0 m^2}{2\pi z^4}\hat{z}\ . \tag{2.2}$$

This force balances the gravitational force downward,

$$\frac{3\mu_0 m^2}{2\pi z^4} = m_d g \quad \Longrightarrow \quad \boxed{z = \left[\frac{3\mu_0 m^2}{2\pi m_d g}\right]^{1/4}\ ,} \tag{2.3}$$

in agreement with the given answer.

(b) Since we found the force exerted by two magnets in part (a), we can immediately say that the middle magnet is repelled upward by lower magnet and downward by upper magnet. Using x and y to denote the separations between the two pairs, as shown in the diagram from the problem reproduced at the right, the balancing of forces on the middle magnet gives



$$\frac{3\mu_0 m^2}{2\pi x^4} - \frac{3\mu_0 m^2}{2\pi y^4} - m_d g = 0\ . \tag{2.4}$$

The upper magnet is repelled upward by middle magnet, and attracted downward by lower magnet, so

$$\frac{3\mu_0 m^2}{2\pi y^4} - \frac{3\mu_0 m^2}{2\pi (x+y)^4} - m_d g = 0. \tag{2.5}$$

Substracting Eq. (2.5) from Eq. (2.4), one finds

$$\frac{1}{x^4} - \frac{1}{y^4} - \frac{1}{y^4} + \frac{1}{(x+y)^4} = 0\ .$$

Defining $\alpha \equiv x/y$, we have

$$2 = \frac{1}{\alpha^4} + \frac{1}{(\alpha+1)^4} \quad \Longrightarrow \quad \alpha = \boxed{x/y = 0.8501...\ ,} \tag{2.6}$$

which verifies the answer stated in the problem.

PROBLEM 3: PERFECT CONDUCTORS AND SUPERCONDUCTORS

(15 points plus 10 points extra credit)

Griffiths Problem 7.44 (p. 346). Parts (a)-(c) are required, for 15 points. Part (d) can optionally be done for 10 points of extra credit.

(a) Faraday's law reads as

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \ . \tag{3.1}$$

Since the electric field inside a conductor is zero ,

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0 \quad \Longrightarrow \quad \frac{\partial \vec{\boldsymbol{B}}}{\partial t} = 0 \ , \tag{3.2}$$

which implies that $\vec{\boldsymbol{B}}$ is independent of t.

(b) The integral form of Faraday's law says

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ . \tag{3.3}$$

For a loop inside a perfect conductor, $\vec{\boldsymbol{E}} = 0$, so Faraday's law implies that Φ through this loop is constant in time.

(c) The Ampere-Maxwell law can be written as

$$\mu_0 \vec{\boldsymbol{J}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} - \mu_0 \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t} = 0 \quad \Longrightarrow \quad \vec{\boldsymbol{J}} = 0 \tag{3.4}$$

inside the superconductor. Since the current must vanish in the interior, any current must be on the surface.

(d) When temperature is decreased below the critical temperature the flux is expelled. Recall, as discussed on pp. 245–247 of Griffiths, Example 5.11, that a rotating and uniformly charged spherical shell produces a uniform magnetic field inside the shell equal to

$$\vec{\boldsymbol{B}} = \frac{2}{3} \mu_0 \sigma \vec{\boldsymbol{\omega}} \ , \tag{3.5}$$

where σ is the surface charge density and $\vec{\boldsymbol{\omega}}$ is the angular velocity. That is, the surface current $\vec{\boldsymbol{K}} = \sigma \vec{\boldsymbol{v}} = \sigma \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} = \sigma \omega \sin\theta \hat{\boldsymbol{\phi}}$ produces a uniform magnetic field $\vec{\boldsymbol{B}} = \frac{2}{3} \mu_0 \sigma \vec{\boldsymbol{\omega}}$ inside the shell. Thus, to compensate $\vec{\boldsymbol{B}}_0$, we should have the surface current

$$\boxed{\vec{\boldsymbol{K}} = -\frac{3}{2\mu_0} B_0 \sin\theta \hat{\boldsymbol{\phi}} \ .} \tag{3.6}$$

PROBLEM 4: THE EFFECT OF A WEAK MAGNETIC FIELD ON THE RADIUS OF THE ORBIT OF AN ATOMIC ELECTRON *(10 points)*

Griffiths Problem 7.52 (p. 348).

Initially, the magnetic field is off and the electron circles in an orbit of radius r with the velocity $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\phi}}$. The centripetal force is provided by the Coulomb attraction (note that $q < 0$):

$$-\frac{mv^2}{r}\hat{\boldsymbol{r}} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{\boldsymbol{r}} \ . \tag{4.1}$$

After the magnetic field $\mathrm{d}\vec{\boldsymbol{B}} = \mathrm{d}B\hat{\boldsymbol{z}}$ is turned on, the electron circles in a new orbit, of radius $r_1 \equiv r + \mathrm{d}r$ and velocity $\vec{\boldsymbol{v}}_1 \equiv (v + \mathrm{d}v)\hat{\boldsymbol{\phi}}$. The Lorentz force from the magnetic field contributes to the centripetal force:

$$-\frac{mv_1^2}{r_1}\hat{\boldsymbol{r}} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1^2}\hat{\boldsymbol{r}} + qv_1 dB\hat{\boldsymbol{r}} \ . \tag{4.2}$$

The infinitesimal change in the kinetic energy is then

$$\mathrm{d}T = \frac{1}{2}mv_1^2 - \frac{1}{2}mv^2 = -\frac{1}{2}\frac{Qq}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r}\right) - \frac{1}{2}qvr_1\mathrm{d}B \ . \tag{4.3}$$

We have $r_1 = r + \mathrm{d}r$, so $r_1^{-1} \approx r^{-1}(1 - \frac{\mathrm{d}r}{r})$. To the lowest order in infinitesimals, Eq. (4.3) implies that the change in kinetic energy is given by

$$\mathrm{d}T = \frac{qQ}{8\pi\epsilon_0 r^2}\mathrm{d}r - \frac{qvr}{2}\mathrm{d}B \ . \tag{4.4}$$

We can also calculate $\mathrm{d}T$ directly, by calculating the magnitude of the electric field $\vec{\boldsymbol{E}}$ induced by the changing magnetic field $\vec{\boldsymbol{B}}$. Faraday's law (Griffiths, Eq. (7.19)) states that

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ ,$$

where Φ is the magnetic flux through the loop. The applied magnetic field $\vec{\boldsymbol{B}}$ is uniform and perpendicular to the electron's trajectory. We will assume for simplicity that the magnetic field is turned on in a circular region with the atom at its center, which is an important simplifying assumption that Griffiths somehow failed to include in the wording of the problem. In that case the symmetry of the situation implies that the induced electric field $\vec{\boldsymbol{E}}$ will be tangential to the trajectory (Griffiths, Example 7.7, p. 317):

$$E(2\pi r) = -(\pi r^2)\frac{\mathrm{d}B}{\mathrm{d}t}. \tag{4.5}$$

The electric force acting on the electron is then

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = qE = -\frac{qr}{2}\frac{\mathrm{d}B}{\mathrm{d}t} \quad \Longrightarrow \quad m\mathrm{d}v = -\frac{qr}{2}\mathrm{d}B\ . \tag{4.6}$$

The induced electric force changes the kinetic energy of the electron by amount

$$\mathrm{d}T = \mathrm{d}\left(\frac{1}{2}mv^2\right) = mv\mathrm{d}v = -\frac{qvr}{2}\mathrm{d}B\ . \tag{4.7}$$

Comparing the two equations for $\mathrm{d}T$, Eq. (4.4) and Eq. (4.7), we conclude that $\mathrm{d}r = 0$ and the electron continues revolving around the nucleus at the same radius r.

PROBLEM 5: CAPACITANCE AND INDUCTANCE FOR A SIMPLE TRANSMISSION LINE *(15 points)*

Griffiths Problem 7.62 (p. 352).

(a) Assume that the sheet has a length ℓ. Then the capacitance of the parallel plate capacitor is simply

$$C = \epsilon_0\frac{w\ell}{h} \quad \Longrightarrow \quad \mathcal{C} = \frac{C}{\ell} = \boxed{\epsilon_0\frac{w}{h}\ .} \tag{5.1}$$

(b) Consider the current I travelling down one strip and back along the other. We can find the total magnetic field by finding the magnetic field of each surface current, $K = I/w$, which flows down one strip and back on the other. In the region between the plates, each of them will give a contribution of magnitude $B = \mu_0 K/2$, in the plane of sheet and perpendicular to the current itself. The magnetic fields outside of the sheets cancel each other, so we have $B = \mu_0 K = \mu_0 I/w$ only in the region between the two parallel plates. It is then straightforward to calculate the inductance by calculating the energy U:

$$U = \frac{1}{2}LI^2 = \frac{B^2}{2\mu_0}(wh\ell) \quad \Longrightarrow \quad \mathcal{L} = \frac{L}{\ell} = \frac{\mu_0 h\ell/w}{\ell} = \boxed{\mu_0\frac{h}{w}\ .} \tag{5.2}$$

(c) The product $\mathcal{LC}$ is,

$$\mathcal{LC} = \mu_0\epsilon_0 = \frac{1}{c^2} = \boxed{1.112 \times 10^{-17}\ \mathrm{s}^2/\mathrm{m}^2.} \tag{5.3}$$

(d) When the dielectric is placed in between two plates, assume that we place charge $+\sigma$ and $-\sigma$ to both plates so that the displacement vector and the electric field becomes,

$$D = \sigma \quad \Longrightarrow \quad E = \sigma/\epsilon \quad \Longrightarrow \quad C = \frac{Q}{V} = \frac{\sigma w \ell}{\sigma h/\epsilon} = \epsilon \frac{w\ell}{h}, \tag{5.4}$$

and the capacitor per length becomes

$$\boxed{\mathcal{C} = \epsilon \frac{w}{h} \ .} \tag{5.5}$$

In order to find the inductance we assume there is a free current I flowing down one sheet and back on the other. As we did for magnetic field in part (b), the H field in between the two current carrying sheets is found by Ampere's law, where the free surface current is $K = I/w$. Ampere's law gives the H field as $H = K$, and then $B = \mu K$. Thus the B field is a factor μ/μ_0 times larger than in the case without the magnetic material, so the magnetic flux through any loop is increased by this factor. The inductance is therefore increased by this factor, so with Eq. (5.2) we conclude that

$$\boxed{\mathcal{L} = \mu \frac{h}{w} \ .} \tag{5.6}$$

Alternatively, we could again find the inductance from the total energy, as we did in part (b). However, to do this we need to know the magnetic energy density in a linear magnetic material.

To find the energy stored in electric and magnetic fields in a linear material, consider the free currents in a magnetic material. The rate the work done by the fields on a free charge q is given by

$$\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = q\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{v}}\,\mathrm{d}t \quad \Longrightarrow \quad \frac{dW}{\mathrm{d}t} = \int_{\mathcal{V}} (\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{J}}_f)\mathrm{d}\tau. \tag{5.7}$$

Using Maxwell's Equation for J_f in a media, we can express the quantity $(\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{J}}_f)$ as

$$\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{J}}_f = \vec{\boldsymbol{E}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}}) - \vec{\boldsymbol{E}} \cdot \frac{\partial \vec{\boldsymbol{D}}}{\partial t}. \tag{5.8}$$

Using the product rule,

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{H}}) = \vec{\boldsymbol{H}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}) - \vec{\boldsymbol{E}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}}), \tag{5.9}$$

and also using Faraday's law, ($\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$), the first term in Eq (5.8) becomes

$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{H}). \tag{5.10}$$

Putting this into Eq. (5.7) and applying the divergence theorem, we have

$$\begin{aligned}\frac{dW}{\mathrm{d}t} &= \int_{\mathcal{V}}\left[-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right]\mathrm{d}\tau \\ &= \int_{\mathcal{V}}\left[-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]\mathrm{d}\tau - \oint(\vec{E}\times\vec{H})\cdot\mathrm{d}\vec{a}\ .\end{aligned} \tag{5.11}$$

The above formula is valid for any medium, but we are interested in particular in a linear medium. (Note that ϵ and μ are defined only for linear media.) In that case $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, which means the first integral can be written as a time derivative:

$$\frac{dW}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}) - \oint(\vec{E}\times\vec{H})\cdot\mathrm{d}\vec{a}\ . \tag{5.12}$$

Here we can see that the energy stored in a linear magnetic media is given as $U = \frac{1}{2}\int\vec{B}\cdot\vec{H}\mathrm{d}\tau = \int\frac{B^2}{2\mu}\mathrm{d}\tau$. This energy equals to $U = \frac{1}{2}LI^2$, then we find the inductance per length as $\mathcal{L} = \mu\frac{h}{w}$. Then the product becomes

$$\boxed{\mathcal{L}\mathcal{C} = \mu\epsilon \quad\text{and}\quad v = 1/\sqrt{\mu\epsilon}\ .} \tag{5.13}$$

PROBLEM 6: ALFVEN'S THEOREM: FROZEN FLUX IN A PERFECTLY CONDUCTING FLUID *(15 points)*

Griffiths Problem 7.63 (p. 352).

(a) Griffiths' Eq. (7.2) says

$$\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})\ , \tag{6.1}$$

so if $\sigma = \infty$ and $\vec{J}$ is finite, then

$$\vec{E} + \vec{v}\times\vec{B} = 0\ . \tag{6.2}$$

Taking the curl of both sides and using Faraday's law,

$$-\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times(\vec{v}\times\vec{B}) = 0 \quad\Longrightarrow\quad \boxed{\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B})\ .} \tag{6.3}$$

(b) We have $\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = 0$ for any closed system. Applying this at time $(t+\mathrm{d}t)$ to the closed surface consisting of $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{R}$, as shown in the diagram at the right,

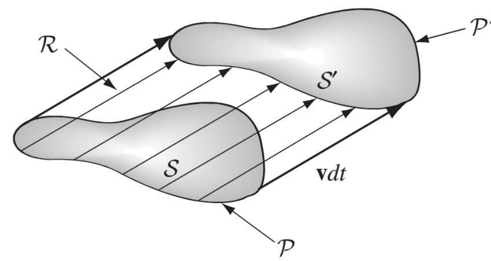


FIGURE 7.58

$$\int_{\mathcal{S}'} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} + \int_{\mathcal{R}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} = 0, \tag{6.4}$$

where the sign change for $\mathcal{S}$ is coming from a change in conventions: when computing $\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{a}}$ we need outward normals, while the direction of $\mathrm{d}\vec{\boldsymbol{a}}$ in $\int_{\mathcal{S}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}}$ is related to the direction of the path $\mathcal{P}$ by right-hand rule. The change in flux can then be expressed as

$$\begin{aligned}
\mathrm{d}\Phi &= \int_{\mathcal{S}'} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{a}} \\
&= \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{R}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{a}} \\
&= \int_{\mathcal{S}} \mathrm{d}t \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{R}} \vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} \; .
\end{aligned} \tag{6.5}$$

In the second integral in Eq. (6.5), the infinitesimal area vector for the ribbon can be written as $\mathrm{d}\vec{\boldsymbol{a}} = (\mathrm{d}\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{v}})\mathrm{d}t$. Then using the cyclic invariance of the triple vector product (i.e., $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}} \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}})$), we find $\vec{\boldsymbol{B}}(t+\mathrm{d}t) \cdot (\mathrm{d}\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{v}})\mathrm{d}t = \left[\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}(t+\mathrm{d}t)\right] \cdot \mathrm{d}\vec{\boldsymbol{\ell}}\,\mathrm{d}t$. We can replace $\vec{\boldsymbol{B}}(t+\mathrm{d}t)$ by $\vec{\boldsymbol{B}}(t)$ because it is already multiplied by $\mathrm{d}t$ and the distinction would be second order in $\mathrm{d}t$. Thus,

$$\mathrm{d}\Phi = \mathrm{d}t \int_{\mathcal{S}} \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \mathrm{d}\vec{\boldsymbol{a}} - \mathrm{d}t \int_{\mathcal{P}} (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{\ell}} \; . \tag{6.6}$$

Using Stoke's theorem in Eq. (6.6),

$$\mathrm{d}\Phi = \mathrm{d}t \left\{ \int_{\mathcal{S}} \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{S}} \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{a}} \right\} , \tag{6.7}$$

which implies that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int_{\mathcal{S}} \left[\frac{\partial \vec{\boldsymbol{B}}}{\partial t} - \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \right] \cdot \mathrm{d}\vec{\boldsymbol{a}} \; . \tag{6.8}$$

Then, with Eq. (6.3), we have Alfven's theorem,

$$\boxed{\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0\ .} \tag{6.9}$$

PROBLEM 7: A LONG, TIGHTLY WOUND SOLENOID, INDUCTANCE, AND THE ENERGY DENSITY OF A MAGNETIC FIELD *(20 points)*

(a) The magnetic field $\vec{\boldsymbol{B}}$, uniform and axial inside the solenoid and vanishing outside, is described mathematically as:

$$\vec{\boldsymbol{B}}(s) = \begin{cases} B\hat{\boldsymbol{z}}, & s < R, \\ 0, & s > R\,. \end{cases}$$

This expression can be written more concisely using the Heaviside step function, defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0\ . \end{cases}$$

With this definition,

$$\vec{\boldsymbol{B}}(s) = B_0\theta(R - s)\hat{\boldsymbol{z}} \tag{7.1}$$

Our goal is to show that the field described by the Eq. (7.1) satisfies Maxwell's equations for the magnetic field:

$$\nabla \cdot \vec{\boldsymbol{B}} = 0\,. \tag{7.2}$$

$$\nabla \times \vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{J}} + \frac{1}{c^2}\frac{\partial \vec{\boldsymbol{E}}}{\partial t}\,. \tag{7.3}$$

Assuming that there is no time-dependent electric field $\vec{\boldsymbol{E}}$, we set the second term in Eq. (7.3) to zero. The density of current in the solenoid can be written as

$$\vec{\boldsymbol{J}} = nI\delta(s - R)\hat{\boldsymbol{\phi}} \tag{7.4}$$

This allows us to rewrite the equations (7.2) and (7.3) as:

$$\nabla \cdot \vec{\boldsymbol{B}} = 0\,. \tag{7.5}$$

$$\nabla \times \vec{\boldsymbol{B}} = \mu_0 nI\delta(s - R)\hat{\boldsymbol{\phi}}\,. \tag{7.6}$$

Now we can show explicitly that the magnetic field described by Eq. (7.1) satisfies the equations (7.5) and (7.6). Since we operate in cylindrical coordinates, the mathematical expressions for divergence and curl are slightly different from those in Cartesian coordinates:

$$\nabla \cdot \vec{\boldsymbol{B}} = \frac{1}{s}\frac{\partial (sB_s)}{\partial s} + \frac{1}{s}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0 \tag{7.7}$$

$$\begin{aligned}\nabla \times \vec{\boldsymbol{B}} &= \left(\frac{1}{s}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\hat{\boldsymbol{s}} + \left(\frac{\partial B_s}{\partial z} - \frac{\partial B_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left(\frac{\partial (sB_\phi)}{\partial s} - \frac{\partial B_s}{\partial \phi}\right)\hat{\boldsymbol{z}} = \\ &= -\frac{\partial B_z}{\partial s}\hat{\boldsymbol{\phi}} = -\frac{\partial}{\partial s}\left(B_0\theta(R-s)\right)\hat{\boldsymbol{\phi}} = B_0\delta(s-R)\hat{\boldsymbol{\phi}}\,, \end{aligned} \tag{7.8}$$

where we used $\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x)$. Matching Eq. (7.7) with (7.2) and (7.8) with (7.3), we find that the field $\vec{\boldsymbol{B}}$ satisfies the Maxwell's equations and that its magnitude equals

$$\boxed{B_0 = \mu_0 n I} \tag{7.9}$$

(b) The (self) inductance L of the (finite-length) solenoid is defined as the coefficient of proportionality between the current I through the solenoid and the magnetic flux Φ through the solenoid. More precisely, Φ is the flux of the magnetic field produced by the current I through the loop that is formed if the wires at the two ends of the solenoid are connected. As an equation,

$$\Phi = LI\ .$$

In the case of a finite solenoid with a uniform axial field $\vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{B}}| = B_0$, the flux is given by

$$\Phi = B_0 A N,$$

where $A = \pi R^2$ is the area of a single loop of the solenoid and $N = n\ell$ is the total number of loops. Using the previously obtained relation (7.9) between the current I in the solenoid and the magnitude of the magnetic field $\vec{\boldsymbol{B}}$, we find that

$$\boxed{L = \mu_0 \pi n^2 R^2 \ell\ .} \tag{7.10}$$

(c) The total energy W of magnetic field inside a solenoid is given by

$$W = \frac{1}{2}LI^2\ ,$$

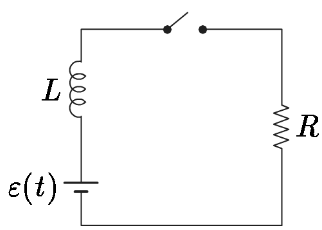
a formula which is found by asking how much energy is needed to increase the current in the inductor from zero to I. Using the previously obtained Eqs. (7.9) and (7.10), we can rearrange the factors to express the total energy W in terms of the magnitude of the magnetic field B_0:

$$W = \frac{1}{2}\mu_0 \pi n^2 R^2 \ell I^2 = \frac{1}{2\mu_0} B_0^2 \pi R^2 \ell = \frac{B_0^2}{2\mu_0}\mathcal{V},$$

where $\mathcal{V} = \pi R^2 \ell$ is the volume of the solenoid.

This allows us to find the energy density inside the solenoid:

$$\boxed{u = \frac{W}{\mathcal{V}} = \frac{B_0^2}{2\mu_0}\ .}$$



(d) Defining the current $I(t)$ in the RL-circuit as positive when clockwise, it obeys the differential equation

$$\mathcal{E}(t) - L\frac{\mathrm{d}I(t)}{\mathrm{d}t} - I(t)R = 0\ . \tag{7.11}$$

To see this, imagine walking around the circuit clockwise (in the direction that we have defined as positive I), starting at the lower left corner, keeping track of the changes in the electrostatic potential V. There is first an increase due to the battery, by $\mathcal{E}(t)$, followed by a decrease by $L\mathrm{d}I(t)/\mathrm{d}t$ and then a decrease by $I(t)R$. The sign of the $\mathcal{E}$ contribution is positive, by the convention that the longer cross bar in the diagram is the positive terminal. To understand the sign of the $L\mathrm{d}I(t)/\mathrm{d}t$ contribution, note that inductors always resist changes in $I(t)$. Thus, if $\mathrm{d}I/\mathrm{d}t$ is to be positive, there must be a higher V on the incoming side of the inductor to force the current to increase. Similarly, resisitors always resist the current itself, so if I is to be positive, there must again be a higher V on the incoming side to force the current through the resistor. The right-hand side of Eq. (7.11) is zero, because the sum of the changes in V around a closed circuit must vanish; that is, the walk around the loop returns us to where we started, and hence to the same value of V.

In the initial time period $0 < t < \frac{2L}{R}$, the voltage source provides a positive and constant emf:

$$\mathcal{E}(t) = \mathcal{E}_0 \ \text{ for }\ 0 < t < \frac{2L}{R}\ ,$$

so

$$L\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} + I_1(t)R = \mathcal{E}_0\ .$$

This differential equation has a solution of the form

$$I_1(t) = \frac{\mathcal{E}_0}{R} + Ce^{-\frac{L}{R}t} \ .$$

The constant C is unknown, but can be found from the boundary condition $I(0) = 0$ (i.e., there was no current in the system before the switch was closed). The result is

$$I_1(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) \ .$$

In the later period of time, the voltage source provides no emf:

$$\mathcal{E}(t) = 0 \ \text{ for } \ t > \frac{2L}{R} \ ,$$

so

$$L\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + I_2(t)R = 0$$

The solution to this equation is

$$I_2(t) = C_2 e^{-\frac{R}{L}t},$$

where the constant C_2 can be found by joining the two solutions at $t = \frac{2L}{R}$:

$$I_2\left(\frac{2L}{R}\right) = I_1\left(\frac{2L}{R}\right) \ ,$$

so

$$C_2 e^{-2} = \frac{\mathcal{E}_0}{R}(1 - e^{-2}) \quad \Longrightarrow \quad C_2 = \frac{\mathcal{E}_0}{R}(e^2 - 1) \ .$$

Thus,

$$I_2(t) = \frac{\mathcal{E}_0}{R}(e^2 - 1)e^{-\frac{R}{L}t} \ .$$

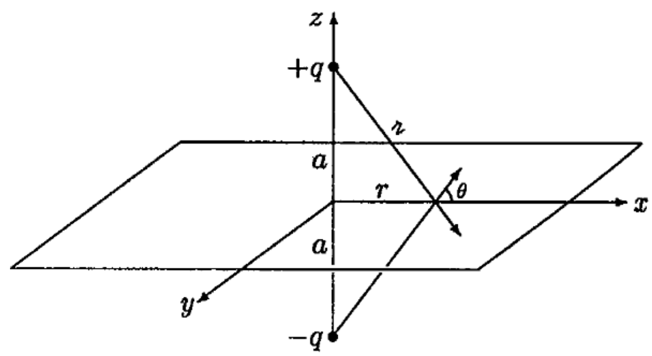
The final answer is then:

$$\boxed{I(t) = \begin{cases} 0 & \text{for } t < 0 \ , \\ \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) & \text{for } 0 < t < \frac{2L}{R} \\ \frac{\mathcal{E}_0}{R}(e^2 - 1)e^{-\frac{R}{L}t} & \text{for } t > \frac{2L}{R} \ . \end{cases} \ ,}$$

PROBLEM 8: CALCULATING THE FORCE BETWEEN TWO POINT CHARGES USING THE MAXWELL STRESS TENSOR *(15 points)*

Griffiths Problem 8.4 (p. 366).

In this problem we calculate the force between two point particles by integrating the stress tensor over a surface that surrounds one of them. It may seem like a lot of work just to find the force between two point charges, but the calculation illustrates an important principle: the total force acting on all the particles in any region of space can be calculated by integrating the stress tensor, which depends only on the values of the fields on the boundary. The total force does not depend on the details of how the charges are distributed inside the region.



In both cases, we will calculate the force on the bottom charge by integrating the stress tensor over the boundary of the region $z \leq 0$. This boundary will consist of the x-y plane, plus the $z < 0$ half of a spherical surface at radius $R = \infty$. Since the elecric fields fall off at least like $1/r^2$ at large distances, the stress tensor, being quadratic in the electric field, falls of like $1/r^4$. Since the area of the surface at $R = \infty$ grows like R^2, the surface integral of the stress tensor over the spherical surface will fall off like $1/R^2$ and will therefore not contribute. Therefore, in both cases we will only have to consider the surface integral over the plane $z = 0$, denoted as S_0. Since the volume is below this plane, the normal vector is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the force in both cases will be given by

$$F_i = \int_{S_0} T_{ij}\, n_j \mathrm{d}a = \int_{S_0} T_{i3} \mathrm{d}a = \epsilon_0 \int_{S_0} \Big(E_i E_z - \frac{1}{2}\delta_{i3}|E|^2 \Big) \mathrm{d}a\,. \tag{8.1}$$

(a) In the case that the charges have the same sign, E_z vanishes on S_0 and the electric field is tangential. Therefore Eq. (8.1) gives

$$F_i = -\frac{1}{2}\epsilon_0 \delta_{i3} \int_{S_0} |E|^2 \mathrm{d}a \quad \Longrightarrow \quad F = \Big(-\frac{1}{2}\epsilon_0 \int_{S_0} |E|^2 \mathrm{d}a \Big)\, \hat{\boldsymbol{z}}\,, \tag{8.2}$$

which correctly predicts that the force will be in the negative z-direction. The electric field is computed by adding the contributions of the two charges:

$$|\vec{\boldsymbol{E}}(r)| = \frac{1}{4\pi\epsilon_0} \frac{q}{(a^2 + r^2)} (2 \sin\theta) = \frac{1}{2\pi\epsilon_0} \frac{qr}{(a^2 + r^2)^{3/2}}\,,$$

and therefore

$$|\vec{\boldsymbol{E}}(r)|^2 = \frac{1}{4\pi^2\epsilon_0^2}\frac{q^2r^2}{(a^2+r^2)^3}\,. \tag{8.3}$$

Substituting into Eq. (8.2) we have

$$\begin{aligned} F_z &= -\frac{1}{2}\epsilon_0\,\frac{q^2}{4\pi^2\epsilon_0^2}\int_0^\infty \frac{r^2}{(a^2+r^2)^3}(2\pi r\,\mathrm{d}r) = -\frac{q^2}{4\pi\epsilon_0}\int_0^\infty \frac{r^2\frac{1}{2}\,\mathrm{d}(r^2)}{(a^2+r^2)^3} \\ &= -\frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\int_0^\infty \frac{xdx}{(a^2+x)^3} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\frac{1}{2a^2} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}\,. \end{aligned} \tag{8.4}$$

This is the correct value for the force.

(b) If the charges have opposite values the electric field at S_0 only has z-component, and therefore, from Eq. (8.1)

$$F = F_z\hat{\boldsymbol{z}} \quad \text{with} \quad F_z = \epsilon_0\int_{S_0}\Big(E_z^2 - \frac{1}{2}|E_z|^2\Big)\mathrm{d}a = \frac{\epsilon_0}{2}\int_{S_0}E_z^2\mathrm{d}a\,. \tag{8.5}$$

The electric field at S_0 is given by

$$|\vec{\boldsymbol{E}}_z(r)| = \frac{1}{4\pi\epsilon_0}\frac{q}{(a^2+r^2)}\,\frac{2a}{(a^2+r^2)^{1/2}} = \frac{1}{2\pi\epsilon_0}\frac{qa}{(a^2+r^2)^{3/2}}\,, \tag{8.6}$$

and substituting into Eq. (8.5) we have

$$\begin{aligned} F_z &= \frac{\epsilon_0}{2}\frac{q^2a^2}{4\pi^2\epsilon_0^2}\int_0^\infty \frac{2\pi r\,\mathrm{d}r}{(a^2+r^2)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{a^2}{2}\int_0^\infty \frac{\mathrm{d}r^2}{(a^2+r^2)^3} \\ &= \frac{q^2}{4\pi\epsilon_0}\frac{a^2}{2}\int_0^\infty \frac{dx}{(a^2+x)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{a^2}{2}\frac{1}{2a^4} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}\,. \end{aligned} \tag{8.7}$$

This is the expected answer.

PROBLEM 9: MOMENTUM STORAGE IN A PARALLEL-PLATE CAPACITOR IN A UNIFORM MAGNETIC FIELD *(15 points)*

Griffiths Problem 8.6 (p. 369).

(a) The momentum density stored in electromagnetic fields is

$$\vec{\wp}_{\mathrm{em}} = \epsilon_0(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}) = \epsilon_0 EB\,\hat{\boldsymbol{y}} \quad\Longrightarrow\quad \boxed{\vec{\boldsymbol{p}}_{\mathrm{em}} = \epsilon_0 EBAd\,\hat{\boldsymbol{y}}\ .} \tag{9.1}$$

(b) When we connect a wire between the plates, a magnetic force $\vec{\boldsymbol{F}} = I\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{B}} = IBd\,\hat{\boldsymbol{y}}$ will act on the wire. The total impulse $\Delta\vec{\boldsymbol{p}}$, while the capacitor is discharging, is given by

$$\Delta\vec{\boldsymbol{p}} = \int_0^\infty \vec{\boldsymbol{F}}\,\mathrm{d}t = Bd\,\hat{\boldsymbol{y}} \int_0^\infty I\,\mathrm{d}t = BdQ\,\hat{\boldsymbol{y}}, \tag{9.2}$$

where Q is the total charge that flows through the wire, which is the charge that is initially on the lower plate. Since the magnitude of the initial electric field is given as $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$, the total impulse can be written as

$$\boxed{\Delta\vec{\boldsymbol{p}} = \epsilon_0 EBAd\,\hat{\boldsymbol{y}}\ .} \tag{9.3}$$

The total momentum originally stored in the fields is delivered as an impulse to the wire.

PROBLEM 10: ANGULAR MOMENTUM AND A ROTATING SHELL OF CHARGE *(25 points)*

In this problem we can use example 5.11 noting that

$$\sigma = \frac{Q}{4\pi R^2}\,. \tag{10.1}$$

(a) From Eq. (5.68) the inside magnetic field is given as

$$B_{in} = \frac{2}{3}\mu_0\sigma R w\,\hat{\boldsymbol{z}} = \frac{\mu_0}{4\pi}\frac{2}{3}\frac{Q}{R}w\,\hat{\boldsymbol{z}}\,. \tag{10.2}$$

For convenience we will define

$$b_0 \equiv \frac{\mu_0}{4\pi}\frac{2}{3}\frac{Q}{R}\,, \quad \Longrightarrow \quad \vec{\boldsymbol{B}}_{in} = b_0\,\omega\,\hat{\boldsymbol{z}}\,. \tag{10.3}$$

For the magnetic field outside we have from Eq. (5.66) that it takes the form of a magnetic dipole $\vec{\boldsymbol{m}}$

$$\vec{\boldsymbol{A}} = \frac{\mu_0 R^4\sigma}{3r^3}\,\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{m}} \times \vec{\boldsymbol{r}}}{r^3}\,, \tag{10.4}$$

where we can identify

$$\vec{\boldsymbol{m}} = m\hat{\boldsymbol{z}} = \frac{4\pi}{3}R^4\sigma\omega\hat{\boldsymbol{z}} = \frac{1}{3}QR^2\omega\,\hat{\boldsymbol{z}}\,. \tag{10.5}$$

The magnetic field outside is thus

$$\vec{\boldsymbol{B}}_{out} = \frac{\mu_0 m}{4\pi r^3}\Big(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\Big) = \frac{b_0\omega R^3}{2r^3}\Big(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\Big)\,. \tag{10.6}$$

The static electric field is

$$\vec{\boldsymbol{E}} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\boldsymbol{r}} & r > R, \\ 0, & r < R\,. \end{cases} \tag{10.7}$$

(b) Consider a circle of constant θ on the surface of the sphere and apply Faraday's law. The electric field will be in the $\hat{\phi}$ direction, and since the circle has radius $R\sin\theta$ we have

$$2\pi R\sin\theta E_\phi(\theta) = -\frac{\partial}{\partial t}\Phi = -\frac{\partial}{\partial t}\pi R^2\sin^2\theta(b_0\omega)\,, \tag{10.8}$$

where we used the inside magnetic field in Eq. (10.3). Therefore

$$E_\phi(\theta) = -\frac{1}{2}R\sin\theta\, b_0\,\dot{\omega}\,, \qquad b_0 \equiv \frac{\mu_0}{4\pi}\frac{2}{3}\frac{Q}{R}\,. \tag{10.9}$$

To calculate the torque we use a thin element of area extending on the surface of the sphere from θ to $\theta + d\theta$. The z-component of the torque (around the origin) in that strip will be given by

$$|d\tau_z| = |E_\phi|(\sigma \mathrm{d}a)(R\sin\theta)\,. \tag{10.10}$$

Note that the $R\sin\theta$ arises because in the computation of $\vec{\boldsymbol{r}}\times\hat{\phi}$ needed for the torque we can split $\vec{\boldsymbol{r}} = R\cos\theta\hat{\boldsymbol{z}} + R\sin\theta\hat{\boldsymbol{s}}$. The first vector in this sum will not give torque in the z direction but the second will. Writing out what we have in Eq. (10.10) we get

$$|d\tau_z| = \frac{1}{2}R\sin\theta\, b_0\,\dot{\omega}\Big(\frac{Q}{2}\sin\theta d\theta\Big)(R\sin\theta) = \frac{1}{4}QR^2 b_0\dot{\omega}\sin^3\theta\,.$$

$$\Longrightarrow \quad |\tau_z| = \frac{1}{4}QR^2 b_0\dot{\omega}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}QR^2 b_0\dot{\omega}\frac{4}{3} = \frac{1}{3}QR^2 b_0\dot{\omega}\,. \tag{10.11}$$

Using the value of b_0 we get

$$|\tau_z| = \frac{\mu_0}{4\pi}\frac{2}{9}Q^2 R\,\dot{\omega}\,.$$

This torque is actually in the negative z- direction, it opposes the increase in angular velocity. Thus we can write

$$\boxed{\tau_z = -\frac{\mu_0}{4\pi}\frac{2}{9}Q^2 R\,\dot{\omega}\,.} \tag{10.12}$$

(c) If we have an external torque G_z along the positive z-axis the equation of motion would give

$$\tau_{\text{net},z} = G_z + \tau_z = I_{mech}\dot{\omega}\,.$$

Using Eq. (10.12) this gives

$$G_z = -\tau_z + I_{mech}\dot{\omega} = \Big(I_{mech} + \frac{\mu_0}{4\pi}\,\frac{2}{9}\,Q^2\,R\Big)\dot{\omega}\,. \tag{10.13}$$

We therefore identify the "magnetic" moment of inertia as

$$\boxed{I_{mag} = \frac{\mu_0}{4\pi}\,\frac{2}{9}\,Q^2\,R\,.} \tag{10.14}$$

(d) The calculation for the inside is simple, since the field is uniform:

$$U_{inside} = \frac{1}{2\mu_0}\,b_0^2\,\omega^2\,\frac{4}{3}\pi R^3\,. \tag{10.15}$$

Using equation (10.6), we find that the square of the field on the outside is given by

$$|\vec{\boldsymbol{B}}_{out}|^2 = \frac{b_0^2\omega^2 R^6}{4r^6}(3\cos^2\theta + 1)\,. \tag{10.16}$$

Therefore

$$\begin{aligned}
U_{out} &= \frac{1}{2\mu_0}\int d\tau\,\frac{b_0^2\omega^2 R^6}{4r^6}(3\cos^2\theta+1) = \frac{1}{2\mu_0}\frac{b_0^2R^6\omega^2}{4}\int d\Omega\,r^2\,\mathrm{d}r\frac{1}{r^6}(3\cos^2\theta+1)\\
&= \frac{1}{2\mu_0}\frac{b_0^2R^6\omega^2}{4}\int_R^\infty\frac{\mathrm{d}r}{r^4}\int d\Omega(3\cos^2\theta+1) = \frac{1}{2\mu_0}\frac{b_0^2R^6\omega^2}{4}\,\frac{1}{3R^3}\,(4\pi)(3\cdot\frac{1}{3}+1)\\
&= \frac{1}{2\mu_0}b_0^2\omega^2\,\frac{1}{2}\,\frac{4}{3}\pi R^3\,.
\end{aligned} \tag{10.17}$$

Comparing with Eq. (10.15) we indeed see that two thirds of the energy is inside the sphere and one-third is outside. The total energy is

$$\begin{aligned}
U_{tot} &= \frac{1}{2\mu_0}b_0^2\omega^2\,\frac{3}{2}\,\frac{4}{3}\pi R^3 = \frac{1}{2\mu_0}b_0^2\,2\pi R^3\omega^2\\
&= \frac{1}{2\mu_0}\Big(\frac{\mu_0}{4\pi}\frac{2}{3}\frac{Q}{R}\Big)^2(2\pi R^3)\omega^2 = \frac{1}{2}\frac{\mu_0}{4\pi}\frac{2}{9}Q^2R\omega^2 = \frac{1}{2}\,I_{mag}\,\omega^2\,,
\end{aligned} \tag{10.18}$$

where we used Eq. (10.14).

(e) The angular momentum volume density $\vec{\boldsymbol{g}}$ is given by

$$\begin{aligned}\vec{\boldsymbol{g}} &= \epsilon_0 \vec{\boldsymbol{r}} \times (E \times B) = \epsilon_0 r\, \hat{\boldsymbol{r}} \times \left(\frac{Q}{4\pi\epsilon_0 r^2} \hat{\boldsymbol{r}} \times \frac{b_0 \omega R^3}{2r^3} \sin\theta\, \hat{\boldsymbol{\theta}} \right) \\ &= \frac{Q b_0 \omega R^3}{8\pi} \frac{1}{r^4} \sin\theta\, \hat{\boldsymbol{r}} \times \hat{\phi} = \frac{Q b_0 \omega R^3}{8\pi} \frac{1}{r^4} \sin\theta\, (-\hat{\theta})\,. \end{aligned} \tag{10.19}$$

Once we integrate over ϕ there will only remain a z component. Since $\hat{\boldsymbol{\theta}} = -\sin\theta\, \hat{\boldsymbol{z}} + \cdots$, it suffices to replace the above angular momentum density by its z component

$$(\vec{\boldsymbol{g}})_z = \frac{Q b_0 \omega R^3}{8\pi} \frac{\sin^2\theta}{r^4}\,. \tag{10.20}$$

The total angular momentum is obtained by integrating this outside the sphere (inside the electric field vanishes for constant ω). We find

$$\begin{aligned} L_z &= \int d(vol) (\vec{\boldsymbol{g}})_z = \frac{Q b_0 \omega R^3}{8\pi} \int_R^\infty \frac{\mathrm{d}r}{r^2} (2\pi) \int_0^\pi \sin^3\theta d\theta \\ L_z &= \frac{1}{3} Q b_0 \omega R^2 = \frac{1}{3}\, Q\, \frac{\mu_0}{4\pi} \frac{2}{3} \frac{Q}{R} \omega R^2 = \frac{\mu_0}{4\pi} \frac{2}{9} Q^2 R\, \omega = I_{mag}\, \omega\,. \end{aligned} \tag{10.21}$$

This is what we wanted to show.

PROBLEM 11: THE CURRENT DENSITY FOR AN IDEAL MAGNETIC DIPOLE *(20 points extra credit)*

(a) [5 pts] Changing the variables of integration in the definition of $\vec{\boldsymbol{m}}$ from $\vec{\boldsymbol{r}}$ to $\vec{\boldsymbol{u}} = \lambda \vec{\boldsymbol{r}}$,

$$\begin{aligned} \vec{\boldsymbol{m}}(\lambda) &= \frac{1}{2} \int d^3x\, \vec{\boldsymbol{r}} \times \vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda) \\ &= \frac{1}{2} \int d^3x\, \vec{\boldsymbol{r}} \times \lambda^n \vec{\boldsymbol{J}}(\lambda \vec{\boldsymbol{r}}) \\ &= \frac{\lambda^n}{2} \int \frac{d^3u}{\lambda^3} \frac{1}{\lambda} \vec{\boldsymbol{u}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{u}}) \\ &= \frac{\lambda^{n-4}}{2} \int d^3u\, \vec{\boldsymbol{u}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{u}})\,, \end{aligned} \tag{11.1}$$

so $\vec{\boldsymbol{m}}$ is independent of λ if $\boxed{n = 4.}$

(b) [5 pts] We are given the hint that the solution involves expanding $\vec{\varphi}(\vec{r})$ in a Taylor series, so we begin by doing that. Since Eq. (11.9) of the problem set depends on $\vec{\varphi}(\vec{r})$ only through the terms $\vec{\varphi}(\vec{0})$ and $\left.\dfrac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}}$, it seems pretty clear that we should expand about the origin. Using index notation,

$$\varphi_i(\vec{r}) = \varphi_i(\vec{0}) + \left.\frac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}} x_j + \frac{1}{2}\left.\frac{\partial^2 \varphi_i(\vec{r})}{\partial x_j \partial x_k}\right|_{\vec{r}=\vec{0}} x_j\, x_k + \dots\ . \tag{11.2}$$

In the following integral we could expand $\vec{\varphi}(\vec{r})$ either before or after changing variables of integration, but here we choose to expand first. So

$$\begin{aligned}
\lim_{\lambda\to\infty} \int d^3x\, \vec{\varphi}(\vec{r}) \cdot \vec{J}_r(\vec{r},\lambda) &= \lim_{\lambda\to\infty} \lambda^4 \int d^3x \left[\varphi_i(\vec{0}) + \left.\frac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}} x_j + \mathcal{O}(x^2)\right] J_i(\lambda\vec{r}) \\
&= \lim_{\lambda\to\infty} \lambda^4 \int \frac{d^3u}{\lambda^3} \left[\varphi_i(\vec{0}) + \left.\frac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}} \frac{u_j}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] J_i(\vec{u}) \\
&= \lim_{\lambda\to\infty} \lambda\varphi_i(\vec{0}) \int d^3u\, J_i(\vec{u}) + \left.\frac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}} \int d^3u\, u_j\, J_i(\vec{u})\ .
\end{aligned} \tag{11.3}$$

The higher order terms are $\mathcal{O}(1/\lambda)$, so they disappear completely in the limit $\lambda \to \infty$.

(c) [5 pts] Starting with

$$\int d^3u\, u_j \partial_i J_i = 0\ , \tag{11.4}$$

where $\partial_i \equiv (\partial/\partial u_i)$, we can integrate by parts to find

$$0 = -\int d^3u\, (\partial_i u_j)\, J_i = -\int d^3u\, \delta_{ij}\, J_i = -\int d^3u\, J_j\ ,$$

so

$$\int d^3u\, \vec{J}(\vec{u}) = 0\ . \tag{11.5}$$

Then, starting from

$$\int d^3u\, u_j u_k\, \partial_i J_i = 0\ , \tag{11.6}$$

integration by parts gives

$$0 = -\int d^3u\, \left[\partial_i(u_j u_k)\right] J_i = -\int d^3u\, \left(\delta_{ij} u_k + u_j \delta_{ik}\right) J_i = -\int d^3u\, \left(J_j u_k + u_j J_k\right)\ , \tag{11.7}$$

so

$$\int d^3u\, u_j\, J_i(\vec{\boldsymbol{u}}) = -\int d^3u\, u_i\, J_j(\vec{\boldsymbol{u}})\ . \tag{11.8}$$

(d) [5 pts] Writing the definition of $\vec{\boldsymbol{m}}$ in index notation, starting with Eq. (11.2) of the problem set, we have

$$m_i = \frac{1}{2}\epsilon_{ijk}\int d^3x\, x_j J_k(\vec{\boldsymbol{r}})\ . \tag{11.9}$$

Multiplying by $\epsilon_{i\ell m}$, and using the identity

$$\epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}\ , \tag{11.10}$$

we find

$$\begin{aligned} m_i\,\epsilon_{i\ell m} &= \frac{1}{2}\epsilon_{ijk}\,\epsilon_{k\ell m}\int d^3x\, x_j J_k(\vec{\boldsymbol{r}}) \\ &= \frac{1}{2}\left[\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}\right]\int d^3x\, x_j J_k(\vec{\boldsymbol{r}}) \\ &= \frac{1}{2}\int d^3x\,\left[x_\ell J_m(\vec{\boldsymbol{r}}) - x_m J_\ell(\vec{\boldsymbol{r}})\right] \\ &= \int d^3x\, x_\ell J_m(\vec{\boldsymbol{r}})\ , \end{aligned} \tag{11.11}$$

where in the last line we used Eq. (11.8).

Combining the results of parts (b) and (c), we have shown that

$$\lim_{\lambda\to\infty}\int d^3x\,\vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}},\lambda) = \left.\frac{\partial\varphi_i(\vec{\boldsymbol{r}})}{\partial x_j}\right|_{\vec{\boldsymbol{r}}=\vec{\boldsymbol{0}}}\int d^3r\, x_j\, J_i(\vec{\boldsymbol{r}})\ , \tag{11.12}$$

where we have changed the name of the variable of integration in Eq. (11.3) from $\vec{\boldsymbol{u}}$ back to $\vec{\boldsymbol{r}}$. (You might be confused by the fact that we originally used $\vec{\boldsymbol{u}} = \lambda\vec{\boldsymbol{r}}$, but keep in mind that an integration variable is a "dummy" variable, meaning that one can always change its name to anything one likes. As long as one makes the corresponding changes in the way the integrand is written, the value of the integral is not changed.) Then, using Eq. (11.11),

$$\lim_{\lambda\to\infty}\int d^3x\,\vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\cdot\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}},\lambda) = \left.\frac{\partial\varphi_i(\vec{\boldsymbol{r}})}{\partial x_j}\right|_{\vec{\boldsymbol{r}}=\vec{\boldsymbol{0}}} m_k\,\epsilon_{kji} = \vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{\nabla}}\times\left.\vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\right|_{\vec{\boldsymbol{r}}=\vec{\boldsymbol{0}}}\ . \tag{11.13}$$

Thus, we have shown Eq. (11.8) of the problem set. As discussed in the problem statement, this equation is equivalent in the sense of distributions to showing that

$$\vec{\boldsymbol{J}}_{\rm dip}(\vec{\boldsymbol{r}}) \equiv \lim_{\lambda\to\infty}\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}},\lambda) = -\vec{\boldsymbol{m}}\times\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}}\,\delta^3(\vec{\boldsymbol{r}})\ , \tag{11.14}$$

describing the current density of a magnetic dipole $\vec{\boldsymbol{m}}$ at the origin.

PROBLEM 12: POWER TRANSMISSION IN A COAXIAL CABLE *(20 points)*

(a) [4 pts] As the current configuration is symmetric under translation along the z axis and rotation around the z axis, we use a cylindrical coordinate system (s, ϕ, z). The magnetic field components B_s, B_ϕ, and B_z can then depend only on s. Then, from Ampere's law, we can immediately determine $B_\phi(s)$:

$$B_\phi(s) = \begin{cases} 0 & 0 < s < a \\ \dfrac{\mu_0 I}{2\pi s} & a < s < b \\ 0 & b < s < \infty \end{cases} \tag{12.1}$$

Addendum:

We did not expect students to justify the statement that $B_z = B_s = 0$, but for completeness we will give such an argument here.

The conclusion that $B_z = B_s = 0$ can be justified in (at least) two different ways. One way is to use Maxwell's equations directly.

The fact that B_s vanishes can be seen from considering a cylindrical Gaussian surface whose radius s is in the range $a < s < b$. Let the range of z extend from $z = z_1$ to $z = z_2$. One has $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{a}} = 0$, as always, but because of translational invariance the contributions from B_z at the $z = z_1$ and $z = z_2$ end caps must cancel each other. So, the integral of $\vec{\boldsymbol{B}}$ along the side surface must vanish. Since B_s depends only on s, this integral equals $2\pi s(z_2 - z_1)B_s(s)$, so the vanishing of this integral implies that $B_s = 0$.

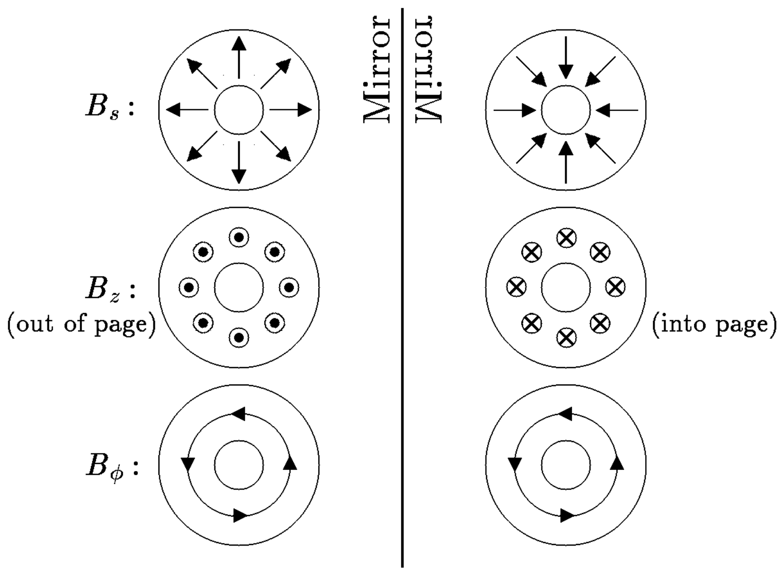
The vanishing of B_z can be seen easily in the differential form of the $\hat{\boldsymbol{\phi}}$ component of Ampere's law,

$$(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}})_\phi = \frac{\partial B_s}{\partial z} - \frac{\partial B_z}{\partial s} = \mu_0 J_\phi + \frac{1}{c^2}\frac{\partial E_\phi}{\partial t} \ . \tag{12.2}$$

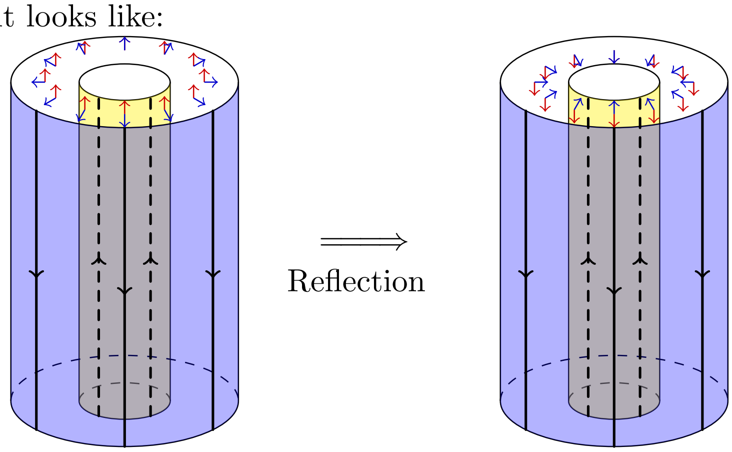
In this situation $J_\phi = E_\phi = 0$, and B_s does not depend on z, so $\partial B_z/\partial s = 0$. This means that the only possible B_z allowed by Maxwell's equations would be a uniform field throughout space, which we can exclude by imposing the boundary condition that $\vec{\boldsymbol{B}} \to \vec{\boldsymbol{0}}$ as $s \to \infty$.

Another way to see that B_z and B_s must vanish is to use a symmetry argument. It's a little subtle, however, because rotational symmetry and translational symmetry are not enough. One must consider reflection symmetry, taking into account the fact that $\vec{\boldsymbol{B}}$ is a pseudovector. A pseudovector (which involves a cross product or something equivalent in its definition) behaves the same as a vector under rotations, but under reflections it is exactly the opposite. To understand what the reflection of a pseudovector is, first visualize what a reflected vector would be—they reflect in exactly the same way as a physical arrow would reflect—and then take the negative of that vector.

The following diagram shows how the three possible components of $\vec{\boldsymbol{B}}$ behave under a mirror reflection, shown as a cross section of the cable:



The images on the left show the actual coaxial cable as seen in cross section, and the images on the right show the mirror images. In all cases the current flows out of the page on the inner cylinder, and flows into the page on the outer cylinder. In perspective, it looks like:



The point is that the laws of electromagnetism are invariant under reflections, so if any hypothetical system is consistent with the laws of E&M, then so is its mirror reflection. The diagrams above show that for this system, $B_s \geq 0$ is possible only if $B_s \leq 0$ is a possible consequence of the same currents, so the only consistent solution is $B_s = 0$. And the same is true for B_z.

(b) [3 pts] From the azimuthal symmetry of the problem, the potential V can be only a function of s. The Laplace equation in cylindrical coordinates becomes:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0 \ . \tag{12.3}$$

This equation can be solved easily as:

$$V = A\ln s + B \ . \tag{12.4}$$

Here, A and B are constants. For the region $0 < s < a$, we must have $A = 0$, since $\ln s$ would blow up at $s = 0$. The constant B is undetermined, but we are told that the potential of the inner cylinder is higher than that of the outer cylinder by V_0. I'll choose to set $V(a) = V_0$, and then $V(b) = 0$. For the region $a < s < b$, we can determine the constants A and B by requiring that $V(s)$ has the right values at $s = a$ and $s = b$:

$$\begin{aligned} V(a) &= A\ln a + B = V_0 \ , \\ V(b) &= \ln b + B = 0 \ . \end{aligned} \tag{12.5}$$

Solving the two equations simultaneously gives

$$A = \frac{V_0}{\ln(a/b)} \ , \quad B = -\frac{V_0}{\ln(a/b)}\ln b \ , \tag{12.6}$$

and then

$$V(s) = \frac{V_0}{\ln(a/b)}\ln s - \frac{V_0}{\ln(a/b)}\ln b = \frac{V_0}{\ln(a/b)}\ln(s/b) \ . \tag{12.7}$$

For $s > b$, any $A \neq 0$ would correspond to a radial (i.e., $\hat{\boldsymbol{s}}$-directed) electric field, which in turn would correspond to a net charge per length on the cable. Since the coaxial cable is presumably not charged (since nothing in the problem suggests that it is), we must have $A = 0$ for $s > b$. The constant potential B is then determined by continuity at $s = b$, which for the conventions we adopted means $B = 0$, and hence $V = 0$ for $s > b$. (Regardless of conventions, $V(s) =$ constant for $s > b$.) To summarize,

$$V(s) = \begin{cases} V_0 & \text{if } 0 \leq s \leq a \\ \dfrac{V_0}{\ln(a/b)}\ln(s/b) & \text{if } a \leq s \leq b \\ 0 & \text{if } b \leq s < \infty \ . \end{cases} \tag{12.8}$$

Then we find E_s from

$$\boxed{E_s(s) = -\frac{\partial V(s)}{\partial s} = \begin{cases} 0 & \text{if } 0 < s < a \\ \dfrac{1}{s}\dfrac{V_0}{\ln(b/a)} & \text{if } a < s < b \\ 0 & \text{if } b < s < \infty \ . \end{cases}} \tag{12.9}$$

(c) [4 pts] Electromagnetic energy in a length ℓ of the cable is given as:

$$\begin{aligned} W(\ell) &= \frac{\ell}{2}\int 2\pi s ds\left(\epsilon_0 E_s^2(s) + \frac{1}{\mu_0}B_\phi^2(s)\right) \\ &= \pi\ell\left(\frac{\epsilon_0 V_0^2}{\ln\left(\frac{b}{a}\right)} + \frac{\mu_0 I^2}{4\pi^2}\ln\left(\frac{b}{a}\right)\right) . \end{aligned} \tag{12.10}$$

(d) [4 pts] From Gauss' Law, we find that the charge of a length ℓ of the (inner) cable is given by

$$Q(\ell) = 2\pi\epsilon_0\ell\frac{V_0}{\ln(\frac{b}{a})} . \tag{12.11}$$

Then, the capacitance of the length ℓ of the cable is given as

$$C(\ell) = \frac{Q(\ell)}{V_0} = \frac{2\pi\epsilon_0\ell}{\ln(\frac{b}{a})} . \tag{12.12}$$

To calculate the self inductance we are going to use the relation:

$$W_{\text{mag}} = \frac{1}{2}LI^2 . \tag{12.13}$$

The magnetic energy stored in a length ℓ of the cable is

$$\begin{aligned} W_{\text{mag}}(\ell) &= \frac{\ell}{2\mu_0}\int 2\pi s ds B_\phi^2(s) \\ &= \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) . \end{aligned} \tag{12.14}$$

From Eqs. (12.13) and (12.14), we obtain

$$L = \frac{\mu_0\ell}{2\pi}\ln\left(\frac{b}{a}\right) . \tag{12.15}$$

(e) [5 pts] The Poynting vector $\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ is given as:

$$\vec{\boldsymbol{S}} = \begin{cases} 0 & \text{if } 0 < s < a \\ \dfrac{1}{s^2}\dfrac{IV_0}{2\pi\ln(\frac{b}{a})}\hat{\boldsymbol{z}} & \text{if } a < s < b \\ 0 & \text{if } b < s < \infty \end{cases} . \tag{12.16}$$

The power being transported is:

$$\begin{aligned} P &= \int_{z=const} s\, ds\, d\phi\, \vec{\boldsymbol{S}} \cdot \hat{\boldsymbol{z}} \\ &= IV_0 . \end{aligned} \tag{12.17}$$

PROBLEM 13: MANIPULATING AMPLITUDES AND PHASES *(10 points)*

Let $\tilde{A}_1 = A_1 e^{i\delta_1}$, $\tilde{A}_2 = A_2 e^{i\delta_2}$, $\tilde{A}_3 = A_3 e^{i\delta_3}$. To find A_3, consider $\left|\tilde{A}_3\right|^2$:

$$\begin{aligned} A_3^2 = \left|\tilde{A}_3\right|^2 &= \left(\tilde{A}_1^* + \tilde{A}_2^*\right)\left(\tilde{A}_1 + \tilde{A}_2\right) \\ &= \left|\tilde{A}_1\right|^2 + \left|\tilde{A}_2\right|^2 + \tilde{A}_1^*\tilde{A}_2 + \tilde{A}_2^*\tilde{A}_1 \\ &= A_1^2 + A_2^2 + A_1 A_2\left(e^{i(\delta_2-\delta_1)} + e^{i(\delta_1-\delta_2)}\right) \\ &= A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2) \ . \end{aligned} \tag{13.1}$$

Therefore:

$$\boxed{A_3 = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1-\delta_2)} \ .} \tag{13.2}$$

To find δ_3, consider $e^{i\delta_3} = \tilde{A}_3/A_3$:

$$\begin{aligned} e^{i\delta_3} &= \frac{\tilde{A}_3}{A_3} = \frac{A_1 e^{i\delta_1} + A_2 e^{i\delta_2}}{A_3} \\ &= \frac{A_1\cos\delta_1 + A_2\cos\delta_2 + i\left(A_1\sin\delta_1 + A_2\sin\delta_2\right)}{A_3} \ . \end{aligned} \tag{13.3}$$

Take ratios of the imaginary part to the real part for both sides of Eq. (13.3) to obtain:

$$\tan\delta_3 = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2} \ . \tag{13.4}$$

Therefore

$$\boxed{\delta_3 = \arctan\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right) \ .} \tag{13.5}$$

Addendum: Instead of computing $\tan\delta_3$, we could equally well have computed $\sin\delta_3$ or $\cos\delta_3$, obtaining

$$\delta_3 = \arcsin\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_3}\right) \tag{13.6}$$

or

$$\delta_3 = \arccos\left(\frac{A_1\cos\delta_1 + A_2\cos\delta_2}{A_3}\right) \ , \tag{13.7}$$

where A_3 is given by Eq. (13.2). All of these are acceptable answers as far as the problem set is concerned, but none of them are completely satisfactory, because the sin, cos, and tan functions are not really invertible: for example, $\arctan(1) = \frac{\pi}{4}$ by the usual definitions, but it is also true that $\tan\left(\frac{5\pi}{4}\right) = 1$, so $\arctan(\tan(\theta))$ is not necessarily equal to θ. This problem can be avoided by introducing a two-argument version of the inverse tangent function, $\arctan2(y, x)$, which is defined as the value of $\arctan(y/x)$ in the same quadrant as the point (x, y). Most computer implementations of arctan2 return values in the range $-\pi < \theta \leq \pi$, but if we consider θ to be an angle that is defined mod 2π (i.e., θ and $\theta + 2\pi$ are equivalent), then $\theta = \arctan2(y, x)$ determines θ uniquely. See the Wikipedia article on "atan2" for more details.

PROBLEM 14: THE MAXWELL STRESS TENSOR FOR A MONOCHROMATIC LINEARLY POLARIZED PLANE WAVE *(15 points)*

Griffiths Problem 9.13 (p. 400)

I will do all computations here with time averaged quantities. We write for our monochromatic, x-polarized plane wave

$$\begin{aligned} \vec{\boldsymbol{E}} &= \Re\left\{\vec{\boldsymbol{\mathcal{E}}}\exp\big(i(kz-\omega t)\big)\right\} \ , \\ \vec{\boldsymbol{B}} &= \Re\left\{\vec{\boldsymbol{\mathcal{B}}}\exp\big(i(kz-\omega t)\big)\right\} \ , \end{aligned} \tag{14.1}$$

and we have

$$\vec{\boldsymbol{\mathcal{E}}} = E_0 e^{i\delta}\hat{\boldsymbol{x}} \ , \qquad \vec{\boldsymbol{\mathcal{B}}} = \frac{E_0}{c}e^{i\delta}\hat{\boldsymbol{y}} \ . \tag{14.2}$$

Since $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ have the same phase, the time-averaging is particularly simple: for any two sinusoidally varying quantities $X(t)$ and $Y(t)$, if $X(t) = \Re\bar{X}e^{i(\omega t+\delta)}$ and $Y(t) = \Re\bar{Y}e^{i(\omega t+\delta)}$, where $\bar{X}$ and $\bar{Y}$ are real, the time average $\langle X(t)Y(t)\rangle$ is given by

$$\langle X(t)Y(t)\rangle = \bar{X}\bar{Y}\left\langle\cos^2(\omega t+\delta)\right\rangle = \frac{1}{2}\bar{X}\bar{Y} \ , \tag{14.3}$$

where we used the fact that

$$\left\langle\cos^2(\omega t+\delta)\right\rangle = \frac{1}{2} \ . \tag{14.4}$$

(To see this, note that

$$\left\langle\cos^2(\omega t+\delta)+\sin^2(\omega t+\delta)\right\rangle = 1 \ , \tag{14.5}$$

and since $\cos x = \sin\left(x+\frac{\pi}{2}\right)$, the two terms in Eq. (14.5) must be equal when averaged over a cycle.) The Maxwell stress tensor is given in general by

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \ , \tag{14.6}$$

so its time average in this case is given by

$$\langle T_{ij}\rangle = \frac{\epsilon_0}{2}\left(\mathcal{E}_i^*\mathcal{E}_j - \frac{1}{2}\left|\vec{\boldsymbol{\mathcal{E}}}\right|^2\delta_{ij}\right) + \frac{1}{2\mu_0}\left(\mathcal{B}_i^*\mathcal{B}_j - \frac{1}{2}\left|\vec{\boldsymbol{\mathcal{B}}}\right|^2\delta_{ij}\right) \ . \tag{14.7}$$

Since both $\mathcal{E}$ and $\mathcal{B}$ only have components in one direction ($\mathcal{E}_1$ and $\mathcal{B}_2$), $T_{ij}=0$ for $i\neq j$. Using the values in Eq. (14.2) and $c^2\epsilon_0\mu_0=1$, we simplify the above to

$$\langle T_{ij}\rangle = \frac{\epsilon_0}{2}\,\mathcal{E}_i^*\mathcal{E}_j + \frac{1}{2\mu_0}\mathcal{B}_i^*\mathcal{B}_j - \frac{\epsilon_0}{2}\,E_0^2\,\delta_{ij} \ . \tag{14.8}$$

We now find

$$\langle T_{11}\rangle = \frac{\epsilon_0}{2}\,\mathcal{E}_1^*\mathcal{E}_1 - \frac{\epsilon_0}{2}\,E_0^2 = 0\,, \quad \langle T_{22}\rangle = \frac{1}{2\mu_0}\,\mathcal{B}_1^*\mathcal{B}_1 - \frac{\epsilon_0}{2}\,E_0^2 = 0\,, \quad \langle T_{33}\rangle = -\frac{\epsilon_0}{2}\,E_0^2\,.$$

All in all

$$\boxed{\langle T_{ij}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon_0}{2}\,E_0^2 \end{pmatrix}\,.} \tag{14.9}$$

This makes sense, there is only flow of the z component of momentum and only in the z direction—thus only T_{33}. The sign is also correct: T_{ij} represents the negative of the flow of i-momentum in the j direction, so $T_{33} < 0$ means that there is a positive flow of z-momentum in the z direction. This is what we expect if the wave carries z-momentum and moves in the positive z direction.

The time-averaged energy density is given by (see Eq. (8.5) in the textbook)

$$\begin{aligned} \langle u\rangle &= \frac{1}{2}\left(\epsilon_0\left\langle|\vec{\boldsymbol{E}}|^2\right\rangle + \frac{1}{\mu_0}\left\langle|\vec{\boldsymbol{B}}|^2\right\rangle\right) \\ &= \frac{1}{2}\left(\epsilon_0 E_0^2\left\langle\cos^2(kz-\omega t+\delta)\right\rangle + \frac{E_0^2}{\mu_0 c^2}\left\langle\cos^2(kz-\omega t+\delta)\right\rangle\right) \\ &= \frac{1}{2}\epsilon_0 E_0^2 = -T_{33}\ . \end{aligned} \tag{14.10}$$

Identifying $-T_{33}$ with the momentum flux density (the flow of z-momentum in the z direction), we then have

$$\boxed{\text{momentum flux density} = \text{energy density.}} \tag{14.11}$$

[You were not asked to verify the units, but it is worth showing that these quantities have the same units, even though it is not obvious. Momentum flux density is momentum per area per time, so its units are

$$\big[T_{ij}\big] = \frac{\text{kg}\cdot\text{m}\cdot\text{s}^{-1}}{\text{m}^2\cdot\text{s}} = \frac{\text{kg}}{\text{m}\cdot\text{s}^2}\ . \tag{14.12}$$

The units of energy density are

$$\big[u\big] = \frac{\text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}}{\text{m}^3} = \frac{\text{kg}}{\text{m}\cdot\text{s}^2}\ , \tag{14.13}$$

so they agree. The units of pressure, by the way, are the same, and in fact a gas under pressure can be described by a stress tensor $T_{ij} = -P\delta_{ij}$, where P is the pressure.]

[You were also not asked about the momentum density, but it is also related to T_{33} in an interesting way. The momentum density $\vec{\boldsymbol{g}}$, from Eq. (8.29) in the textbook, is given by $\vec{\boldsymbol{g}} = \frac{1}{c^2}\vec{\boldsymbol{S}}$. $\vec{\boldsymbol{S}}$ is the energy flux, which for plane waves is the energy density u times the velocity c, so $\left|\langle\vec{\boldsymbol{S}}\rangle\right| = \langle u\rangle\, c$. One can assign a relativistic mass density ρ_m to an electromagnetic wave which is equal to its energy density u divided by c^2, and then $\left|\langle\vec{\boldsymbol{g}}\rangle\right| = \langle\rho_m\rangle\, c$, as one would expect.]

PROBLEM 15: REFLECTION AND TRANSMISSION OF A PLANE WAVE AT NORMAL INCIDENCE *(15 points)*

Griffiths Problem 9.14 (p. 405).

Using Eq. (9.82), we can write the reflection coefficient in Eq. (9.86) as,

$$R = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 , \tag{15.1}$$

where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$. Also the exact transmission coefficient in Eq. (9.87) becomes,

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 . \tag{15.2}$$

Since we have $\epsilon_1 = \frac{1}{\mu_1 v_1^2}$ and $\epsilon_2 = \frac{1}{\mu_2 v_2^2}$, then

$$\frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1 v_1^2}{\mu_2 v_2^2}\frac{v_2}{v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} = \beta. \tag{15.3}$$

Putting this to Eq. (15.2), and using again Griffiths' Eq. (9.82), we see

$$T = \beta\left(\frac{2}{1+\beta}\right)^2 . \tag{15.4}$$

Thus,

$$R + T = \frac{1}{(1+\beta)^2}\left[4\beta + (1-\beta)^2\right] = \frac{1}{(1+\beta)^2}(1 + 2\beta + \beta^2) = 1. \tag{15.5}$$

PROBLEM 16: TRANSMISSION OF LIGHT THROUGH THREE MEDIA
(25 points)

Griffiths Problem 9.36 (p. 433).

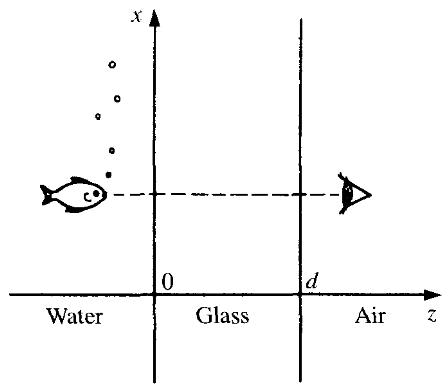


Figure 9.27

We can choose the polarization of the radiation, so here we choose to have the electric field point along the x axis. We can begin by writing the form of the fields in each of the three regions, which we label as Region 1 (water, $z < 0$), Region 2 (glass, $0 < z < d$), and Region 3 (air, $z > d$). The angular frequency will be ω in all regions, since each of the waves involved here is a consequence of the incident wave, and so has the same frequency. Each region is assumed to have $\mu = \mu_0$, which means that for this problem, $\vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{H}}$. The general rule is that $\vec{\boldsymbol{H}}^{\parallel}$ is continuous at the interfaces, but if the μ's are equal on both sides of a boundary, then $\vec{\boldsymbol{B}}^{\parallel}$ is also continuous. The three materials have indices of refraction n_1, n_2, and n_3, with wave speeds $v_i = c/n_i$, and wave numbers $k_i = \omega/v_i = \omega n_i/c$. In the following equations, $\bar{E}_I$, $\bar{E}_R$, $\bar{E}_r$, $\bar{E}_\ell$, and $\bar{E}_T$ are complex constants.

Region 1 ($\boldsymbol{z < 0}$): There is an incident wave (I) moving to the right,

$$\vec{\boldsymbol{E}}_I(z,t) = \bar{E}_I e^{i(k_1 z - \omega t)}\,\hat{\boldsymbol{x}}\ ,\quad \vec{\boldsymbol{B}}_I(z,t) = \frac{n_1}{c}\bar{E}_I e^{i(k_1 z - \omega t)}\,\hat{\boldsymbol{y}}\ . \tag{16.1}$$

The reflected wave propagates in the opposite direction, so $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ must point in the opposite direction:

$$\vec{\boldsymbol{E}}_R(z,t) = \bar{E}_R e^{i(-k_1 z - \omega t)}\,\hat{\boldsymbol{x}}\quad \vec{\boldsymbol{B}}_R(z,t) = -\frac{n_1}{c}\bar{E}_R e^{i(-k_1 z - \omega t)}\ . \tag{16.2}$$

Region 2 ($\boldsymbol{0 < z < d}$): There is a right-moving wave (r) and a left-moving wave (ℓ):

$$\vec{\boldsymbol{E}}_r(z,t) = \bar{E}_r e^{i(k_2 z - \omega t)}\,\hat{\boldsymbol{x}}\ ,\quad \vec{\boldsymbol{B}}_r(z,t) = \frac{n_2}{c}\bar{E}_r e^{i(k_2 z - \omega t)}\,\hat{\boldsymbol{y}}\ . \tag{16.3}$$

$$\vec{\boldsymbol{E}}_\ell(z,t) = \bar{E}_\ell e^{i(-k_2 z-\omega t)}\,\hat{\boldsymbol{x}}\ , \quad \vec{\boldsymbol{B}}_\ell(z,t) = -\frac{n_2}{c}\bar{E}_\ell e^{i(-k_2 z-\omega t)}\,\hat{\boldsymbol{y}}\ . \tag{16.4}$$

Region 3 ($z > d$): In this region there is only a transmitted wave (T) moving to the right:

$$\vec{\boldsymbol{E}}_T(z,t) = \bar{E}_T e^{i(k_3 z-\omega t)}\,\hat{\boldsymbol{x}}\ , \quad \vec{\boldsymbol{B}}_T(z,t) = \frac{n_3}{c}\bar{E}_T e^{i(k_3 z-\omega t)}\,\hat{\boldsymbol{y}}\ . \tag{16.5}$$

Boundary conditions:

At $z = 0$,

$$\bar{E}_I + \bar{E}_R = \bar{E}_r + \bar{E}_\ell\ , \tag{16.6}$$

$$\frac{n_1}{c}(\bar{E}_I - \bar{E}_R) = \frac{n_2}{c}(\bar{E}_r - \bar{E}_\ell) \quad \Longrightarrow \quad \bar{E}_I - \bar{E}_R = \frac{n_2}{n_1}(\bar{E}_r - \bar{E}_\ell)\ . \tag{16.7}$$

At $z = d$,

$$\bar{E}_r e^{ik_2 d} + \bar{E}_\ell e^{-ik_2 d} = \bar{E}_T e^{ik_3 d}\ , \tag{16.8}$$

$$\frac{n_2}{c}(\bar{E}_r e^{ik_2 d} - \bar{E}_\ell e^{-ik_2 d}) = \frac{n_3}{c}\bar{E}_T e^{ik_3 d} \quad \Longrightarrow \quad \bar{E}_r e^{ik_2 d} - \bar{E}_\ell e^{-ik_2 d} = \frac{n_3}{n_2}\bar{E}_T e^{ik_3 d}\ . \tag{16.9}$$

Solving the equations:

Now we have four equations (Eqs. (16.6)–(16.9)) for the four unknowns $\bar{E}_R$, $\bar{E}_r$, $\bar{E}_\ell$, and $\bar{E}_T$, where we think of the (complex) incident wave amplitude $\bar{E}_I$ as being given. Fortunately they can be simplified fairly easily. By adding Eqs. (16.6) and (16.7), $\bar{E}_R$ can be eliminated:

$$2\bar{E}_I = \left(1 + \frac{n_2}{n_1}\right)\bar{E}_r + \left(1 - \frac{n_2}{n_1}\right)\bar{E}_\ell\ . \tag{16.10}$$

Similarly Eqs. (16.8) and (16.9) can be simplified by considering their sums and differences:

$$2\bar{E}_r e^{ik_2 d} = \left(1 + \frac{n_3}{n_2}\right)\bar{E}_T e^{ik_3 d} \quad \Longrightarrow \quad \bar{E}_r = \frac{1}{2}e^{-ik_2 d}\left(1 + \frac{n_3}{n_2}\right)e^{ik_3 d}\bar{E}_T\ , \tag{16.11}$$

$$2\bar{E}_\ell e^{-ik_2 d} = \left(1 - \frac{n_3}{n_2}\right)\bar{E}_T e^{ik_3 d} \quad \Longrightarrow \quad \bar{E}_\ell = \frac{1}{2}e^{ik_2 d}\left(1 - \frac{n_3}{n_2}\right)e^{ik_3 d}\bar{E}_T\ . \tag{16.12}$$

Now Eqs. (16.11) and (16.12) can be used to replace $\bar{E}_r$ and $\bar{E}_\ell$ in Eq. (16.10), giving us the desired relation between $\bar{E}_T$ and $\bar{E}_I$:

$$\bar{E}_I = \frac{1}{4n_1 n_2}\left[e^{-ik_2 d}(n_2 + n_3)(n_1 + n_2) + e^{ik_2 d}(n_2 - n_3)(n_1 - n_2)\right]e^{ik_3 d}\bar{E}_T\ . \tag{16.13}$$

The intensity is given by Griffiths' Eq. (9.73) on p. 402,

$$I = \frac{1}{2}\epsilon v E_0^2\ , \tag{16.14}$$

which can be rewritten by using $v = c/n = 1/\sqrt{\epsilon\mu}$ to give

$$I = \frac{1}{2}\frac{n}{\mu c}E_0^2 \ . \tag{16.15}$$

In this problem $\mu = \mu_0$ is the same for all three media, so we can compare intensities by using $I \propto nE_0^2$. Thus, the transmission coefficient T that we are seeking is given by

$$T = \frac{n_3|\bar{E}_T|^2}{n_1|\bar{E}_I|^2} \ , \tag{16.16}$$

which implies that

$$T^{-1} = \frac{n_1}{n_3}\frac{1}{16n_1^2n_2^2}\left|e^{-ik_2d}(n_2+n_3)(n_1+n_2) + e^{ik_2d}(n_2-n_3)(n_1-n_2)\right|^2 \ . \tag{16.17}$$

To simplify the above expression, I will start by simplifying the square of the magnitude ($|\dots|^2$) by writing it as

$$F \equiv |\alpha e^{-i\delta} + \beta e^{i\delta}|^2 \ , \tag{16.18}$$

where

$$\delta \equiv k_2 d \ , \quad \alpha \equiv (n_2+n_3)(n_1+n_2) \ , \quad \beta \equiv (n_2-n_3)(n_1-n_2) \ . \tag{16.19}$$

Then

$$\begin{aligned} F &= |(\beta+\alpha)\cos\delta + i(\beta-\alpha)\sin\delta|^2 \\ &= (\beta+\alpha)^2\cos^2\delta + (\beta-\alpha)^2\sin^2\delta \\ &= (\beta+\alpha)^2(1-\sin^2\delta) + (\beta-\alpha)^2\sin^2\delta \\ &= (\beta+\alpha)^2 - 4\alpha\beta\sin^2\delta \ , \end{aligned} \tag{16.20}$$

where

$$\beta+\alpha = 2n_2(n_1+n_3) \ , \quad \alpha\beta = (n_2^2-n_3^2)(n_1^2-n_2^2) \ . \tag{16.21}$$

Then finally,

$$\begin{aligned} T^{-1} &= \frac{n_1}{n_3}\frac{1}{16n_1^2n_2^2}\left\{[2n_2(n_1+n_3)]^2 - 4(n_2^2-n_3^2)(n_1^2-n_2^2)\sin^2(k_2d)\right\} \\ &= \boxed{\frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]} \ . \end{aligned} \tag{16.22}$$

Addendum: Alternative solution

Since the problem involves the transmission of the wave from water to glass and then to air, one wonders if maybe one could solve the problem one step at a time. Maybe one could make use of the results already derived for the transmission from one medium to a second medium, and then calculate the transmission to the third medium by using the same formulas.

An approach of this type does work, but it is more complicated, because one has to take into account multiple reflections. There is a part of the beam that travels directly from water to glass to air, and this piece of the beam can be described simply by combining the two-medium equations. But then there is also a piece of the beam that travels from water to glass, but is then reflected at the glass-air interface and bounces backward, and is then reflected at the glass-water interface and bounces forward, finally escaping into the air after this extra loop. This piece can also be described by combining two-medium equations, but one has to include the extra round-trip detour through the glass. But then there are also pieces of the beam that make two such detours, or three such detours, or any number of such detours. If this infinite sum is carried out, one can obtain the same final answer as above.

To carry out this calculation, we can begin with the two-medium results that Griffiths presents as Eqs. (9.83), on p. 404. Assuming that the two media can both be treated as satisfying $\mu = \mu_0$, the complex amplitudes of the reflected and transmitted beams through a normal interface of the two media are given by

$$\tilde{E}_{0_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0_I} \ , \quad \tilde{E}_{0_T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0_I} \ . \tag{16.23}$$

Using $v_i = c/n_i$, these formulas can be rewritten in terms of the indices of refraction n_i:

$$\tilde{E}_{0_R} = \left(\frac{n_1 - n_2}{n_2 + n_1}\right)\tilde{E}_{0_I} \ , \quad \tilde{E}_{0_T} = \left(\frac{2n_1}{n_2 + n_1}\right)\tilde{E}_{0_I} \ . \tag{16.24}$$

The specific factors that are relevant for this problem can then be tabulated:

$$\begin{aligned}
\text{Transmission from water to glass: } \alpha &= \frac{2n_1}{n_1 + n_2} \\
\text{Transmission from glass to air: } \beta &= \frac{2n_2}{n_2 + n_3} \\
\text{Reflection in glass at the water interface: } \alpha' &= \frac{n_2 - n_1}{n_1 + n_2} \\
\text{Reflection in glass at the air interface: } \beta' &= \frac{n_2 - n_3}{n_2 + n_3} \\
\text{Change of phase through glass: } \delta &= e^{ik_2 d} \ .
\end{aligned} \tag{16.25}$$

Assume that the complex amplitude of the incident electric field is E_I. We want to find E_T, the complex amplitude of the electric field in the air. E_T is the sum of infinitely many terms. The first term corresponds to the light wave that was never reflected from the air. One has

$$E_{T,0} = \beta\delta\alpha E_I \ . \tag{16.26}$$

The second term corresponds to the light wave that was reflected from the air exactly once:

$$E_{T,1} = \beta\delta\alpha'\delta\beta'\delta\alpha E_I \ . \tag{16.27}$$

The factors, from right to left, correspond to the history of light: It is transmitted from water to glass (the factor α), then it propagates through the glass (the factor δ), then it is reflected from the interface with the air (the factor β'), then it again propagates through the glass (the factor δ), then it is reflected from the interface with the water (the factor α'), then it propagates once more through the glass (the factor δ) and finally it is transmitted from the glass to the air (the factor β). In a similar fashion, the wave that is reflected from the interface with the air n times is given by

$$E_{T,n} = \beta\delta\alpha(\alpha'\delta^2\beta')^n E_I \ . \tag{16.28}$$

The total transmitted amplitude is then given by

$$\begin{aligned} E_T &= \sum_{n=0}^{\infty} E_{T,n} \\ &= \beta\delta\alpha E_I \sum_{n=0}^{\infty} (\alpha'\delta^2\beta')^n \\ &= \frac{\beta\delta\alpha}{1-\alpha'\beta'\delta^2} E_I \ . \end{aligned} \tag{16.29}$$

We are interested in $\left|\frac{E_T}{E_I}\right|^2$:

$$\begin{aligned}
\left|\frac{E_T}{E_I}\right|^2 &= \left|\frac{\beta\delta\alpha}{1-\alpha'\beta'\delta^2}\right|^2 \\
&= \frac{\alpha^2\beta^2}{1-2\alpha'\beta'\cos(2k_2d)+\alpha'^2\beta'^2} \\
&= \frac{\alpha^2\beta^2}{1-2\alpha'\beta'(1-2\sin^2(k_2d))+\alpha'^2\beta'^2} \\
&= \frac{\alpha^2\beta^2}{1-2\alpha'\beta'(1-2\sin^2(k_2d))+\alpha'^2\beta'^2} \\
&= \frac{\alpha^2\beta^2}{(1-\alpha'\beta')^2+4\alpha'\beta'\sin^2(k_2d)} \\
&= \frac{\frac{16n_1^2n_2^2}{(n_1+n_2)^2(n_2+n_3)^2}}{\frac{4n_2^2(n_1+n_3)^2}{(n_1+n_2)^2(n_2+n_3)^2}+\frac{4(n_1-n_2)(n_3-n_2)}{(n_1+n_2)(n_2+n_3)}\sin^2(k_2d)} \\
&= \frac{4n_1^2}{(n_1+n_3)^2+\frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2(k_2d)}\ .
\end{aligned} \tag{16.30}$$

The inverse of the transmission coefficient is given by

$$\begin{aligned}
T^{-1} &= \frac{n_1}{n_3}\left|\frac{E_T}{E_I}\right|^{-2} \\
&= \frac{n_1}{n_3}\,\frac{(n_1+n_3)^2+\frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2(k_2d)}{4n_1^2} \\
&= \frac{1}{4n_1n_3}\left[(n_1+n_3)^2+\frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2(k_2d)\right]\ ,
\end{aligned} \tag{16.31}$$

which agrees with the previous answer.

PROBLEM 17: ENERGY CONSERVATION IN THE PRESENCE OF MAGNETIC MONOPOLES *(15 points extra credit)*

The total electromagnetic force on the charges in an infinitesimal volume d^3x is given by

$$d\vec{F} = \rho_e\big(\vec{E} + \vec{v} \times \vec{B}\big)d^3x + \rho_m\big(\vec{B} - \frac{1}{c^2}\vec{v} \times \vec{E}\big)d^3x \; . \tag{17.1}$$

The rate at which work is done by electromagnetic fields is then given as

$$\frac{dW}{dt} = \int_{\mathcal{V}} \big[\rho_e \vec{E} \cdot \vec{v}_e + \rho_m \vec{B} \cdot \vec{v}_m\big] d^3x = \int_{\mathcal{V}} \big[\vec{E} \cdot \vec{J}_e + \vec{B} \cdot \vec{J}_m\big] d^3x \; . \tag{17.2}$$

Using Maxwell's equations to express $\vec{J}_e$ and $\vec{J}_m$ in terms of the fields and their time derivatives,

$$\begin{aligned} \vec{E} \cdot \vec{J}_e &= \frac{1}{\mu_0}\vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}, \\ \vec{B} \cdot \vec{J}_m &= -\frac{1}{\mu_0}\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \frac{1}{\mu_0}\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \; . \end{aligned} \tag{17.3}$$

Using the product rule,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \; , \tag{17.4}$$

we can put Eqs. (17.3) into Eq. (17.2) as,

$$\begin{aligned} \frac{dW}{dt} &= \int_{\mathcal{V}} \left[-\frac{1}{\mu_0}\vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)\right] d^3x, \\ &= -\frac{1}{\mu_0}\oint (\vec{E} \times \vec{B}) \cdot d\vec{a} - \int_{\mathcal{V}} \frac{1}{2}\frac{d}{dt}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) d^3x \; . \end{aligned} \tag{17.5}$$

As it is seen in Eq. (17.5) there is no need to modify the expressions for the Poynting vector or for the energy density of an electromagnetic field.

If the force on a magnetic monopole were given by $q\big(\vec{B} + \frac{1}{c^2}\vec{v} \times \vec{E}\big)$, energy conservation would still hold — the $\vec{v} \times \vec{E}$ term disappeared from the calculation above at Eq. (17.2), because it does not contribute to the work done by the fields. (If we had looked at momentum conservation, however, we would have found that it requires the negative sign in front of the $\vec{v} \times \vec{E}$ term in the force law.)

PROBLEM 18: MORE FUN WITH δ-FUNCTIONS *(20 points extra credit)*

(a) [3 pts] By the definition of the derivative of a distribution,

$$\int_{-\infty}^{\infty} \varphi(x) \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\,\mathrm{d}x \equiv -\int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\theta(x)\,\mathrm{d}x\ . \tag{18.1}$$

By the definition of the θ-function, this can be rewritten as

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(x) \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\,\mathrm{d}x &= -\int_{0}^{\infty} \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x = \varphi(0) - \varphi(\infty) \\ &= \varphi(0)\ ,\end{aligned} \tag{18.2}$$

where in the last line we used the fact that test functions $\varphi(x)$ are required to fall off rapidly at large $|x|$. Thus

$$\int_{-\infty}^{\infty} \varphi(x) \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\,\delta(x)\,\mathrm{d}x\ , \tag{18.3}$$

from which it follows that

$$\boxed{\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x)\ ,} \tag{18.4}$$

since distributions are defined by the result of multiplying by a test function and integrating.

(b) [3 pts] For any $x \neq 0$, it is clear that $\theta^n(x) = \theta(x)$. Since $x = 0$ is a set of measure zero, it follows that

$$\int_{-\infty}^{\infty} \varphi(x)\theta^n(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\theta(x)\,\mathrm{d}x\ . \tag{18.5}$$

(If $\varphi(x)$ were allowed to be $\delta(x)$, this would not be true, but test functions are required to be smooth functions. $\delta(x)$, by contrast, is not even a function.) Thus,

$$\boxed{\theta^n(x) = \theta(x)\ .} \tag{18.6}$$

(c) [4 pts] Suppose that we could believe that

$$n\theta^{n-1}(x)\delta(x) \overset{?}{=} \delta(x)\ . \tag{18.6}$$

If so, the cases $n = 2$ and $n = 3$ would give

$$2\theta(x)\delta(x) \stackrel{?}{=} \delta(x) \tag{18.7a}$$

$$3\theta^2(x)\delta(x) \stackrel{?}{=} \delta(x) \tag{18.7b}$$

But from Eq. (18.6), $\theta^2(x) = \theta(x)$, so Eq. (18.7b) implies that $3\theta(x)\delta(x) = \delta(x)$, which clearly contradicts Eq. (18.7a).

What then is $\mathrm{d}\theta^n(x)/\mathrm{d}x$? Using Eqs. (18.6) and (18.4),

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\theta^n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \delta(x)\ .} \tag{18.8}$$

(d) [3 pts] To verify the stated equality, we need to verify that

$$\int_{-\infty}^{\infty} \varphi(x) f(x)\delta(x - x_0)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x) f(x_0)\delta(x - x_0)\,\mathrm{d}x\ . \tag{18.9}$$

To evaluate the LHS, note that if $f(x)$ is a smooth function, then $\varphi(x)f(x)$ is a suitable test function, so by the definition of a δ-function, the LHS equals $\varphi(x_0)f(x_0)$. The RHS can be evaluated by factoring out the constant $f(x_0)$, and the remaining integral equals $\varphi(x_0)$, again by the definition of the δ-function. So the two sides are equal, and therefore

$$\boxed{f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)} \tag{18.10}$$

in the sense of distributions.

(e) [4 pts] The RHS is simpler, so I will first evaluate the result of integrating the RHS with a test function:

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(x) f(x_0) \frac{\mathrm{d}\delta(x - x_0)}{\mathrm{d}x}\,\mathrm{d}x &= f(x_0)\int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}\delta(x - x_0)}{\mathrm{d}x}\,\mathrm{d}x \\
&= -f(x_0)\int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\delta(x - x_0)\,\mathrm{d}x \\
&= -f(x_0)\varphi'(x_0)\ ,
\end{aligned} \tag{18.11}$$

where a prime denotes a derivative with respect to x. Evaluating the LHS,

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(x) \left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\delta(x-x_0) + f(x)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x}\right] \mathrm{d}x \\ &= \varphi(x_0)f'(x_0) + \int_{-\infty}^{\infty} \varphi(x)f(x)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x}\,\mathrm{d}x \\ &= \varphi(x_0)f'(x_0) - \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\big[\varphi(x)f(x)\big]\delta(x-x_0)\,\mathrm{d}x \\ &= \varphi(x_0)f'(x_0) - \int_{-\infty}^{\infty} \big[\varphi'(x)f(x) + \varphi(x)f'(x)\big]\delta(x-x_0)\,\mathrm{d}x \\ &= \varphi(x_0)f'(x_0) - \big[\varphi'(x_0)f(x_0) + \varphi(x_0)f'(x_0)\big] \\ &= -f(x_0)\varphi'(x_0)\ .\end{aligned} \tag{18.12}$$

So the two sides are equal, and therefore

$$\boxed{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\delta(x-x_0) + f(x)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x} = f(x_0)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x}\ .} \tag{18.13}$$

(f) [3 pts] To evaluate the δ-function of a function of x, we use the following formula (from the formula sheet for Quiz 2):

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\ , \quad g(x_i) = 0\ . \tag{18.14}$$

The function $g(x) = x^2 - a^2$ vanishes at $x = a$ and $x = -a$, with derivatives $g'(x) = 2x$. Thus,

$$\boxed{\int_{-\infty}^{\infty} \varphi(x)\delta(x^2-a^2)\,\mathrm{d}x = \frac{1}{2a}\big[\varphi(a) + \varphi(-a)\big]\ .} \tag{18.15}$$

PROBLEM 19: SPHERICAL WAVES AND THE POYNTING VECTOR

Read Only Problem: the solution was given in the problem set, so you were only asked to read and understand it. (It would have been worth 30 points if you had been asked to solve it.)

Griffiths Problem 9.35 (p. 432).

For this problem it will be useful to keep in mind the formulas for the divergence and curl in spherical coordinates. For an arbitrary vector field $\vec{\boldsymbol{A}}(r,\theta,\phi)$,

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{19.1a}$$

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\ .\end{aligned} \tag{191.b}$$

(a) [16 pts] First consider the equation $\nabla\cdot E = 0$. In spherical coordinates, this equation becomes (since $E_r, E_\theta = 0$):

$$\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0\ . \tag{19.2}$$

We have

$$E_\phi = \frac{A\sin\theta}{r}\left(\cos\left(kr-\omega t\right) - \frac{1}{kr}\sin\left(kr-\omega t\right)\right)\ . \tag{19.3}$$

Since E_ϕ is independent of ϕ, we have

$$\boxed{\nabla\cdot E = 0\ .} \tag{19.4}$$

Next consider $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}$, which in spherical coordinates becomes:

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right)\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\boldsymbol{\theta}} \\ &= \frac{2}{r}\cot\theta E_\phi\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\boldsymbol{\theta}} \\ &= \frac{2A\cos\theta}{r^2}\left(\cos\left(kr-\omega t\right) - \frac{1}{kr}\sin\left(kr-\omega t\right)\right)\hat{\boldsymbol{r}} \\ &\quad + \frac{Ak\sin\theta}{r}\left(\left(1-\frac{1}{k^2r^2}\right)\sin\left(kr-\omega t\right) + \frac{1}{kr}\cos\left(kr-\omega t\right)\right)\hat{\boldsymbol{\theta}}\ .\end{aligned} \tag{19.5}$$

From $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t}$ and Eq. (19.5), we obtain:

$$\begin{aligned}\vec{\boldsymbol{B}}(r,\theta,\phi,t) &= \vec{\boldsymbol{B}}_0(r,\theta,\phi) + \frac{2A\cos\theta}{ckr^2}\left(\frac{1}{kr}\cos\left(kr-\omega t\right) + \sin\left(kr-\omega t\right)\right)\hat{\boldsymbol{r}} \\ &\quad + \frac{A\sin\theta}{cr}\left[\left(\frac{1}{k^2r^2}-1\right)\cos\left(kr-\omega t\right) + \frac{1}{kr}\sin\left(kr-\omega t\right)\right]\hat{\boldsymbol{\theta}}\ .\end{aligned} \tag{19.6}$$

Here $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ is an integration "constant" which needs to be determined later. (It is a constant in the sense that it is independent of t, but it can vary with position.) Students who ignored $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ should get full credit, because in the end we will ignore it, but we will nonetheless discuss it for the sake of thoroughness.

Next we consider $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$. We could directly calculate $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$, but we don't need to. From $\frac{\partial}{\partial t}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}\right) = -\vec{\boldsymbol{\nabla}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}\right) = 0$, we see that $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$ is time independent. If we imagine taking the divergence of both sides of Eq. (19.6), it is clear that it will have the form

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0(r,\theta,\phi) + f_1(r,\theta)\cos(kr-\omega t) + f_2(r,\theta)\sin(kr-\omega t)\ , \tag{19.7}$$

where $f_1(r,\theta)$ and $f_2(r,\theta)$ must vanish by the previous argument, so we don't need to calculate them. Thus we have $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0$, and therefore

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0(r,\theta,\phi) = 0\ . \tag{19.8}$$

Now, let us consider $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}$:

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} &= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\phi}\right)\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial B_r}{\partial\theta}\hat{\boldsymbol{\phi}} - \frac{1}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi}\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\theta)\right)\hat{\boldsymbol{\phi}}\\ &= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{Ak\sin\theta}{cr}\left(\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right)\hat{\boldsymbol{\phi}}\\ &= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{c^2}\frac{\partial E_\phi}{\partial t}\hat{\boldsymbol{\phi}}\\ &= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{c^2}\frac{\partial\vec{\boldsymbol{E}}}{\partial t}\end{aligned} \tag{19.9}$$

Therefore we must have

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0(r,\theta,\phi) = 0\ . \tag{19.10}$$

Eqs. (19.8) and (19.10) imply that $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ must be a solution to the source-free equations of magnetostatics, which have an identical form to the source-free equations of electrostatics. Eq. (5.9) guarantees that we can write

$$\vec{\boldsymbol{B}}_0(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}V_B(\vec{\boldsymbol{r}})\ , \tag{19.11}$$

where V_B is a magnetic scalar potential which is applicable only when $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = 0$. Since $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0 = 0$, it follows that $\nabla^2 V_B = 0$. There is actually a large class of solutions to these equations, corresponding to the multipole expansion with positive powers of r:

$$V_B(\vec{\boldsymbol{r}}) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} C^{(\ell)}_{i_1\ldots i_\ell} x_{i_1}\ldots x_{i_\ell} \\ \text{or} \\ \displaystyle\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m} r^\ell Y_{\ell,m}(\theta,\phi) \ , \end{cases} \tag{19.12}$$

where the $C^{(\ell)}_{i_1\ldots i_\ell}$ are arbitrary traceless symmetric tensors, and the $a_{\ell,m}$ are arbitrary coefficients. Either of the two expressions on the right-hand side describe a complete set of solutions.

Maxwell's equations allow the existence of a static $\vec{\boldsymbol{B}}_0(\vec{\boldsymbol{r}})$ in this situation, but we are not interested in it. We can eliminate this possibility by imposing the (realistic) boundary condition that $|\vec{\boldsymbol{B}}| \to 0$ at infinity, which implies that $\vec{\boldsymbol{B}}_0 = 0$. (To see that this is the only solution in this case, remember that $\nabla^2 V_B = 0$ implies that V_B cannot have any local minima or maxima. So if it is required to approach a constant at infinity, it must be a constant everywhere.) For the rest of the problem, we will take $\vec{\boldsymbol{B}}_0 = 0$.

Therefore $\vec{\boldsymbol{B}}$ is given by:

$$\boxed{\begin{aligned}\vec{\boldsymbol{B}}(r,\theta,\phi,t) = {}& \frac{2A\cos\theta}{ckr^2}\left(\frac{1}{kr}\cos(kr-\omega t) + \sin(kr-\omega t)\right)\hat{\boldsymbol{r}} \\ &+ \frac{A\sin\theta}{cr}\left[\left(\frac{1}{k^2r^2}-1\right)\cos(kr-\omega t) + \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\boldsymbol{\theta}} \ .\end{aligned}} \tag{19.13}$$

(b) [8 pts]

$$\boxed{\begin{aligned}\vec{\boldsymbol{S}} &= \frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} \\ &= \frac{A^2\sin\theta}{\mu_0 ckr^2}\Bigg\{\sin\theta\left[k\cos^2(kr-\omega t) + \left(\frac{1}{2k^2r^3}-\frac{1}{r}\right)\sin\big(2(kr-\omega t)\big)\right. \\ &\qquad\qquad \left. - \frac{1}{kr^2}\cos\big(2(kr-\omega t)\big)\right]\hat{\boldsymbol{r}} \\ &\qquad + \frac{2\cos\theta}{r}\left[\frac{1}{2}\left(1-\frac{1}{k^2r^2}\right)\sin\big(2(kr-\omega t)\big) + \frac{1}{kr}\cos\big(2(kr-\omega t)\big)\right]\hat{\boldsymbol{\theta}}\Bigg\}\end{aligned}} \tag{19.14}$$

Here we have used the trigonometric identities $\sin(2x) = 2\sin x \cos x$ and $\cos(2x) = \cos^2 x - \sin^2 x$. The average of $\sin(2x)$ and $\cos(2x)$ is 0 over a period, while the averages of $\sin^2(x)$ and $\cos^2(x)$ are $\frac{1}{2}$. Therefore, for intensity we obtain:

$$\boxed{\vec{\boldsymbol{I}} = \frac{A^2 \sin^2\theta}{2\mu_0 c r^2}\hat{\boldsymbol{r}}\ .} \tag{19.15}$$

From Eq. (19.15) we see that intensity vector points our $\hat{\boldsymbol{r}}$, as expected. It also falls of as $\frac{1}{r^2}$ as it should.

(c) [6 pts]

$$\begin{aligned}
P &= \int \vec{\boldsymbol{I}} \cdot d\vec{\boldsymbol{a}} \\
&= \int\limits_{\theta=0}^{\pi} \int\limits_{\phi=0}^{2\pi} \sin\theta d\theta d\phi\, I(\theta, r) r^2 \\
&= \frac{A^2\pi}{\mu_0 c}\int_{\theta=0}^{\pi} \sin^3\theta d\theta \\
&= -\frac{A^2\pi}{\mu_0 c}\int_{\theta=0}^{\pi} \sin^2\theta d\left(\cos\theta\right) \\
&= -\frac{A^2\pi}{\mu_0 c}\int_{\theta=0}^{\pi} \left(1-\cos^2\theta\right) d\left(\cos\theta\right) \qquad (\text{let } u = \cos\theta) \\
&= \frac{A^2\pi}{\mu_0 c}\int_{u=0}^{1} (1-u^2) du \\
&= \boxed{\frac{4}{3}\frac{A^2\pi}{\mu_0 c}.}
\end{aligned} \tag{19.16}$$