MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE NOTES 2

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TRACELESS SYMMETRIC TENSOR APPROACH TO LEGENDRE POLYNOMIALS AND SPHERICAL HARMONICS, CONTINUED

7. THE MULTIPOLE EXPANSION:

The most general solution to Laplace's equation, in spherical coordinates, was given as Eq. (1.32). We now wish to apply that result to a common situation: suppose we have a charged object, and we wish to describe the potential outside of the object. Let's say for definiteness that the charge of the object is entirely contained within a sphere of radius R, centered at the origin. In that case Laplace's equation will hold for all $r > R$, so there should be a solution of the form of Eq. (1.32) that is valid throughout this region. At infinity the potential of a localized charge distribution will always approach a constant, which so we can take to be zero, we can see that the $C^{(\ell)}_{i_1 \ldots i_\ell}$ coefficients that appear in Eq. (1.32) must all vanish. Thus we can write

$$\boxed{\Phi(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \, C'^{(\ell)}_{i_1 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ .} \tag{2.1}$$

Since the $C^{(\ell)}_{i_1 \ldots i_\ell}$ coefficients in Eq. (1.32) all vanish, we will simplify the notation by dropping the prime on $C'^{(\ell)}_{i_1 \ldots i_\ell}$ in Eq. (2.1). Since each successive term comes with an extra factor of $1/r$, at large distances the sum is dominated by the first term or maybe the first few terms. All the information about the charge distribution of the object is contained in the $C^{(\ell)}_{i_1 \ldots i_\ell}$, so knowledge of the first few $C^{(\ell)}_{i_1 \ldots i_\ell}$ is enough to describe the field at large distances, no matter how complicated the object.

The first few terms of this series have special names: the $\ell = 0$ term is the monopole term, the $\ell = 1$ term is the dipole, the $\ell = 2$ term is the quadrupole, and the $\ell = 3$ term is the octupole.

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* Errors in Eqs. (2.2) and (2.7) were fixed by adding primes, $\mathrm{d}^3x \to \mathrm{d}^3x'$. Eqs. (2.29) and (2.30) were added, showing the multipole expansion formulas for the standard spherical harmonic formulas. Just before Eq. (2.23), "this result" was changed to Eq. (2.20). Some references to equation numbers of Lecture Notes 1 were revised to use the numbering of the 10/17/19 revision of Lecture Notes 1.

If we want to calculate the $C^{(\ell)}_{i_1\ldots i_\ell}$ in terms of the charge distribution, we can start with the general equation for the potential of an arbitrary charge distribution:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'\right|}\,\mathrm{d}^3x' \ . \tag{2.2}$$

The multipole expansion can then be derived by expanding $1/\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'\right|$ in a power series in $\vec{\boldsymbol{r}}\,'$.

I'll begin by doing it as Griffiths does, which gives the simplest — but not the most useful — form of the multipole expansion. Griffiths rewrote the denominator as

$$\frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'\right|} = \frac{1}{\sqrt{\left|\vec{\boldsymbol{r}}\right|^2 + \left|\vec{\boldsymbol{r}}\,'\right|^2 - 2\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}\,'}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} \ , \tag{2.3}$$

where r and r' are the lengths of the vectors $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}\,'$, respectively, and θ' is the angle between these vectors. Next he used the fact that the Legendre polynomials can be defined by the generating function

$$g(x,\lambda) = \frac{1}{\sqrt{1+\lambda^2 - 2\lambda x}} \ , \tag{2.4}$$

which means that the Legendre polynomials $P_\ell(x)$ can be obtained by expanding $g(x,\lambda)$ in a power series in λ:

$$\boxed{g(x,\lambda) = \frac{1}{\sqrt{1+\lambda^2 - 2\lambda x}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x) \ .} \tag{2.5}$$

Eq. (2.5) is sometimes taken as the definition of the Legendre polynomials, and sometimes it is derived from some other definition. We will see shortly that Eq. (2.5) is equivalent to the definition in terms of traceless symmetric tensors that we gave earlier. In any case, if we accept Eq. (2.5) as valid, then

$$\frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'\right|} = \frac{1}{r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta') \ . \tag{2.6}$$

Inserting this relation into Eq. (2.2), we find

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^{\ell}\rho(\vec{\boldsymbol{r}}\,')P_\ell(\cos\theta')\,\mathrm{d}^3x' \ .} \tag{2.7}$$

This is the easiest way that I know to show that there is an expansion of $V(\vec{\boldsymbol{r}})$ in powers of $1/r$, but the complication is that $\cos\theta'$ appears inside the integral. If we could implement Eq. (2.1), we would be able to calculate (or maybe measure) a small number of the quantities $C^{(\ell)}_{i_1 \ldots i_\ell}$, and then we would be able to evaluate $V(\vec{\boldsymbol{r}})$ at large distances in any direction. To use Eq. (2.7) directly, however, one would have to redo the integration for every direction of $\vec{\boldsymbol{r}}$. Griffiths works around this problem by massaging the formula to extract the monopole and dipole terms, and in Problem 3.52 (p. 165) Griffith guides the reader in carrying out this procedure for the quadrupole and octupole terms. This problem will be assigned on the next problem set.

The standard method of "improving" Eq. (2.7) is to construct a general multipole expansion in terms of spherical harmonics, but here I will derive the equivalent relations using the traceless symmetric tensor approach. Later we will convert our result to the standard expression in terms of spherical harmonics, so you will be equipped to work with the multipole expansion in both formalisms.

Instead of expanding $1/\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|$ in powers of r', we will think of it as a function of three variables — the components x'_i of $\vec{\boldsymbol{r}}' \equiv x'_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$, and we will expand it as a Taylor series in these 3 variables. To make the formalism clear, I will define the function

$$f(\vec{\boldsymbol{r}}') \equiv \frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|} \ . \tag{2.8}$$

The function can then be expanded in a power series using the standard multi-variable Taylor expansion:

$$f(\vec{\boldsymbol{r}}') = f(\vec{0}) + \left.\frac{\partial f}{\partial x'_i}\right|_{\vec{\boldsymbol{r}}'=\vec{0}} x'_i + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x'_i \partial x'_j}\right|_{\vec{\boldsymbol{r}}'=\vec{0}} x'_i x'_j + \ldots \ , \tag{2.9}$$

where the repeated indices are summed. To separate the angular behavior, we write

$$x'_i = r' \hat{\boldsymbol{n}}'_i \ , \tag{2.10}$$

so Eq. (2.9) becomes

$$f(\vec{\boldsymbol{r}}') = f(\vec{0}) + r' \left.\frac{\partial f}{\partial x'_i}\right|_{\vec{\boldsymbol{r}}'=\vec{0}} \hat{\boldsymbol{n}}'_i + \frac{r'^2}{2!} \left.\frac{\partial^2 f}{\partial x'_i \partial x'_j}\right|_{\vec{\boldsymbol{r}}'=\vec{0}} \hat{\boldsymbol{n}}'_i \hat{\boldsymbol{n}}'_j + \ldots \ . \tag{2.11}$$

The expansion will be valid for small r', and it will turn out that the series will converge for $r' < r$. The notation can now be simplified by noting that since f is a function of $\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'$, the derivatives with respect to x'_i can be replaced by derivatives with respect to x_i with a change of sign:

$$\frac{\partial f}{\partial x'_i} = \frac{\partial}{\partial x'_i} \left.\left(\frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|}\right)\right|_{\vec{\boldsymbol{r}}'=\vec{0}} = -\frac{\partial}{\partial x_i} \left.\left(\frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|}\right)\right|_{\vec{\boldsymbol{r}}'=\vec{0}} = -\frac{\partial}{\partial x_i}\left(\frac{1}{\left|\vec{\boldsymbol{r}}\right|}\right) \ . \tag{2.12}$$

This allows us to write the derivatives in the expansion (2.11) much more simply. The ℓ'th derivative is found by repeating the above operation ℓ times:

$$\left.\frac{\partial^\ell f}{\partial x'_{i_1} \ldots \partial x'_{i_\ell}}\right|_{\vec{\boldsymbol{r}}'=\vec{0}} = (-1)^\ell \frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} \ . \tag{2.13}$$

Combining Eqs. (2.13) with (2.11), we can write

$$\frac{1}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell r'^\ell}{\ell!} \left(\frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} \right) \hat{\boldsymbol{n}}'_{i_1} \ldots \hat{\boldsymbol{n}}'_{i_\ell} \ . \tag{2.14}$$

Note that the quantity in parentheses in the equation above is traceless, because

$$\begin{aligned} \left(\frac{\partial^\ell}{\partial x_j \partial x_j \partial x_3 \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} \right) &= \nabla^2 \frac{\partial^\ell}{\partial x_3 \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} \\ &= \frac{\partial^\ell}{\partial x_3 \ldots \partial x_{i_\ell}} \nabla^2 \frac{1}{|\vec{\boldsymbol{r}}|} = 0 \ , \end{aligned} \tag{2.15}$$

because $\nabla^2(1/|\vec{\boldsymbol{r}}|) = 0$ except at $\vec{\boldsymbol{r}} = 0$. So we can see the traceless symmetric tensor formalism emerging.

To evaluate this quantity, we will work out the first several terms until we recognize the pattern. We write

$$\vec{\boldsymbol{r}} \equiv r\hat{\boldsymbol{n}} \ , \tag{2.16}$$

and adopt the abbreviation

$$\partial_i \equiv \frac{\partial}{\partial x_i} \ . \tag{2.17}$$

It is useful to start by evaluating the derivatives of the basic quantities r and $\hat{\boldsymbol{n}}_i$:

$$\begin{aligned} \partial_i r &= \partial_i (x_j x_j)^{1/2} = \frac{1}{2}(x_k x_k)^{-1/2} \partial_i (x_j x_j) = \frac{1}{2r} 2 x_j \delta_{ij} = \frac{x_i}{r} = \hat{\boldsymbol{n}}_i \ , \\ \partial_i \hat{\boldsymbol{n}}_j &= \partial_i \left(\frac{x_j}{r} \right) = \frac{\delta_{ij}}{r} - \frac{1}{r^2} x_j \partial_i r = \frac{1}{r} \left(\delta_{ij} - \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \right) \ . \end{aligned} \tag{2.18}$$

It is then straightforward to show that

$$\begin{aligned} \partial_i \left(\frac{1}{r} \right) &= -\frac{1}{r^2} \hat{\boldsymbol{n}}_i \ , \\ \partial_i \partial_j \left(\frac{1}{r} \right) &= \frac{3}{r^3} \{ \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \}_{\mathrm{TS}} \ , \\ \partial_i \partial_j \partial_k \left(\frac{1}{r} \right) &= -\frac{5 \cdot 3}{r^4} \{ \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \hat{\boldsymbol{n}}_k \}_{\mathrm{TS}} \ , \end{aligned} \tag{2.19}$$

where $\{\ \}_{\rm TS}$ denotes the traceless symmetric part, and the expressions for low values of ℓ are shown explicitly in Eqs. (1.41) – (1.44), and also Eq. (1.48). It becomes clear that the general formula, which can be proven by induction, is

$$\frac{\partial^\ell}{\partial x_{i_1}\dots\partial x_{i_\ell}}\frac{1}{|\vec{\boldsymbol{r}}|} = \frac{(-1)^\ell(2\ell-1)!!}{r^{\ell+1}}\{\,\hat{\boldsymbol{n}}_{i_i}\dots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\ , \tag{2.20}$$

where

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\dots 1\ ,\ \text{with}\ \ (-1)!! \equiv 1\ . \tag{2.21}$$

It is sometimes useful to rewrite the double factorial in terms of single factorials, using

$$\begin{aligned}(2\ell-1)!! &= \frac{(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)(2\ell-4)\dots 1}{(2\ell)(2\ell-2)(2\ell-4)\dots 1}\\ &= \frac{(2\ell)!}{2^\ell\,[(\ell)(\ell-1)\dots 1]}\\ &= \frac{(2\ell)!}{2^\ell \ell!}\ .\end{aligned} \tag{2.22}$$

Inserting Eq. (2.20) into Eq. (2.14), we find (for $r' < r$) that

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{r'^\ell}{r^{\ell+1}}\{\,\hat{\boldsymbol{n}}_{i_1}\dots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}'_{i_1}\dots\hat{\boldsymbol{n}}'_{i_\ell}\ . \tag{2.23}$$

One can write this more symmetrically by writing

$$\boxed{\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{r'^\ell}{r^{\ell+1}}\{\,\hat{\boldsymbol{n}}_{i_1}\dots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\{\,\hat{\boldsymbol{n}}'_{i_1}\dots\hat{\boldsymbol{n}}'_{i_\ell}\,\}_{\rm TS}\ ,\ (\text{for } r' < r)} \tag{2.24}$$

since $\{\,\hat{\boldsymbol{n}}'_{i_1}\dots\hat{\boldsymbol{n}}'_{i_\ell}\,\}_{\rm TS}$ differs from $\hat{\boldsymbol{n}}'_{i_1}\dots\hat{\boldsymbol{n}}'_{i_\ell}$ by terms proportional to Kronecker δ-functions, which vanish when summed with the traceless tensor $\{\,\hat{\boldsymbol{n}}_{i_1}\dots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}$. Starting with Eq. (2.24), one can if one wishes drop the curly brackets around either factor (but not both!).

For comparison, the analogous equation in terms of spherical harmonics is

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)\ ,\qquad (\text{for } r' < r) \tag{2.25}$$

Inserting this expression for $1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|$ into Eq. (2.2), we have the final result

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}}\, C^{(\ell)}_{i_1 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell}\ ,} \tag{2.26}$$

where

$$\boxed{C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{\boldsymbol{r}}\,')\, \{\, \vec{\boldsymbol{r}}\,'_{i_1} \ldots \vec{\boldsymbol{r}}\,'_{i_\ell}\, \}_{\mathrm{TS}}\, \mathrm{d}^3x'\ .} \tag{2.27}$$

Note that we can use Eq. (2.22) to rewrite the coefficient in the above expression as

$$\frac{(2\ell-1)!!}{\ell!} = \frac{(2\ell)!}{2^\ell (\ell!)^2}\ . \tag{2.28}$$

The analogous formulas for the standard spherical harmonic treatment are

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}\ ,} \tag{2.29}$$

where the multipole moments $q_{\ell m}$ are given by

$$\boxed{q_{\ell m} = \int \rho(\vec{\boldsymbol{r}}\,')\, r'^{\ell}\, Y^*_{\ell m}(\theta',\phi')\, \mathrm{d}^3x'\ .} \tag{2.30}$$

For purposes of illustration, I will write out the first two terms — the monopole and dipole terms — in a bit more detail. The monopole term can be written as

$$V_{\text{mono}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}\ , \tag{2.31}$$

where

$$Q = C^{(0)} = \int \rho(\vec{\boldsymbol{r}}\,')\, \mathrm{d}^3x'\ . \tag{2.32}$$

The dipole term is

$$V_{\text{dip}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{n}}}{r^2}\ , \tag{2.33}$$

where

$$p_i = C^{(1)}_i = \int \rho(\vec{\boldsymbol{r}}\,')\, x_i\, \mathrm{d}^3x\ . \tag{2.34}$$

8. ALGEBRAIC MANIPULATION OF TRACELESS SYMMETRIC TENSORS:

It is possible to write an explicit expression for the extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1\ldots i_\ell}$:

$$\begin{aligned}\{\, S_{i_1\ldots i_\ell}\,\}_{\text{TS}} = S_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\text{Sym}}\Big[& a_{1,\ell}\delta_{i_1 i_2}\delta^{j_1 j_2} S_{j_1 j_2 i_3\ldots i_\ell} \\ & + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4}\delta^{j_1 j_2}\delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big] \ ,\end{aligned} \tag{2.35}$$

where

$$\underset{i_1\ldots i_\ell}{\text{Sym}}\big[T_{i_1\ldots i_\ell}\big] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} T_{i_1\ldots i_\ell} \ , \tag{2.36}$$

and

$$a_{n,\ell} = (-1)^n \frac{\ell!^2(2\ell-2n)!}{n!(\ell-2n)!(\ell-n)!(2\ell)!} \ . \tag{2.37}$$

I will at some point show you a full proof of this relation, but it turns out that it is very useful to know even the first nontrivial term, $a_{1,\ell}$. So here I will derive only this term.

We begin by writing the right-hand side through the $a_{1,\ell}$ term, and taking its trace:

$$\text{Tr}\,(\text{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \big[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\big] \ . \tag{2.38}$$

Note that after summing over repeated indices, each term will be a tensor with indices $i_3 \ldots i_\ell$. Since Eq. (2.35) is an equation for the traceless part of $S_{i_1\ldots i_\ell}$, our goal is to find coefficients $a_{n,\ell}$ which cause the full right-hand side to be traceless. When we simplify the right-hand side of the above equation, we will find two terms, one involving no Kronecker δ-functions, and one involving one Kronecker δ-function. The $a_{2,\ell}$ and higher terms in Eq. (2.35) involve two or more Kronecker δ-functions in the indices $i_1 \ldots i_\ell$, so after taking a trace they will still involve at least one Kronecker δ-function. Thus, the contribution to $\text{Tr}\,(\text{RHS}_1)$ with no Kronecker δ-functions cannot be canceled by any later terms, and therefore it must vanish on its own. We will see that this requirement allows us to find $a_{1,\ell}$.

We need to carry out the trace operation shown in the second term of Eq. (2.38), by which I mean carrying out the sum over the repeated indices i_1 and i_2. This is tricky, because inside the square brackets [] the indices are not actually at the locations shown, as we are summing over all $\ell!$ orderings of the indices $i_1 \ldots i_\ell$. To evaluate the sum, we have to consider all possible cases of where the indices i_1 and i_2 can appear:

Case I: i_1 and i_2 can appear on the Kronecker δ-function in the square brackets. There are two such ways the indices i_1 and i_2 can be assigned, since they could occur in either order: $\delta_{i_1 i_2}$ or $\delta_{i_2 i_1}$. Since we are summing over all $\ell!$ orderings, these two cases count separately. Furthermore, once we specify the locations of i_1 and i_2, the remaining indices $i_3 \ldots i_\ell$ can still be ordered in all possible ways. Since there are $\ell - 2$ of these indices, there are $(\ell - 2)!$ ways that they can be ordered. So, the total number of orderings for Case I is $2(\ell - 2)!$. The value of the term for this case is $(a_{1,\ell}/\ell!) \times 3S_{jji_3\ldots i_\ell}$, since $\delta_{i_1 i_2}\delta_{i_1 i_2} = 3$. Thus, we can summarize the results for Case I by

$$\text{Case I:} \qquad \text{Multiplicity} = 2(\ell-2)!\ , \qquad \text{Value} = 3S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ . \tag{2.39}$$

Case II: i_1 can appear on the Kronecker δ-function in the square brackets, while i_2 appears as one of the indices of S. For this case there are 2 possibilities for the position of i_1 (first or second index of the Kronecker δ-function), and $\ell - 2$ positions for i_2. Then there are again $(\ell - 2)!$ orderings for all the other indices, so the multiplicity is $2(\ell - 2)(\ell - 2)!$. The value can be seen by looking at a representative term:

$$(a_{1,\ell}/\ell!)\delta_{i_1 i_2}\delta_{i_1 i_3}S_{jji_2 i_4\ldots i_\ell} = (a_{1,\ell}/\ell!)S_{jji_3\ldots i_\ell}\ . \tag{2.40}$$

Summarizing,

$$\text{Case II:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ . \tag{2.41}$$

Case III: i_2 can appear on the Kronecker δ-function in the square brackets, while i_1 appears as one of the indices of S. This case is equivalent to Case II:

$$\text{Case III:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ . \tag{2.42}$$

Case IV: Both i_1 and i_2 can appear on S. Since there are $\ell - 2$ indices i on S, i_1 has $\ell - 2$ possible places, and i_2 has $\ell - 3$ places. Then again the other indices have $(\ell - 2)!$ orderings. The value can again be seen by looking at a representative term:

$$(a_{1,\ell}/\ell!)\delta_{i_1 i_2}\delta_{i_3 i_4}S_{jji_1 i_2 i_5\ldots i_\ell} = (a_{1,\ell}/\ell!)\,\mathrm{Sym}\big[\delta_{i_3 i_4}S_{jjkki_5\ldots i_\ell}\big]\ . \tag{2.43}$$

The summary is then

$$\begin{aligned}\text{Case IV:} \qquad &\text{Multiplicity} = (\ell-2)(\ell-3)(\ell-2)! \\ &\text{Value} = \mathrm{Sym}\big[\delta_{i_3 i_4}S_{jjkki_5\ldots i_\ell}\big]\frac{a_{1,\ell}}{\ell!}\ .\end{aligned} \tag{2.44}$$

A useful check on this kind of accounting is the sum of the multiplicities, which should equal the total number of orderings, $\ell!$. One can check that

$$(\ell-2)!\,[2+2(\ell-2)+2(\ell-2)+(\ell-2)(\ell-3)] = \ell!\ , \tag{2.45}$$

so our result is consistent.

The final result is obtained by assembling each of the contributions, taking into account the multiplicity and the value of each:

$$\begin{aligned} \mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}&\left[1+\frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3+4(\ell-2))\right] \\ &+\mathrm{Sym}\big[\delta_{i_3i_4}S_{jjkki_5\ldots i_\ell}\big]\,a_{1,\ell}\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)}\ . \end{aligned} \tag{2.46}$$

As described above, the trace of the full right-hand side must vanish, but we are only looking at a piece of the right-hand side. Nonetheless, we know that the omitted terms will not produce any terms of the form $S_{jji_3\ldots i_\ell}$, with no Kronecker δ-functions, so the coefficient of this term must vanish by itself. So

$$\left[1+\frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3+4(\ell-2))\right]=0 \quad\Longrightarrow\quad \boxed{\;a_{1,\ell}=-\frac{\ell(\ell-1)}{2(2\ell-1)}\;}\ , \tag{2.47}$$

which agrees with Eq. (2.37).

From Eq. (2.47), we can derive several important relations that will be essential for some of the problems on Problem Set 4. First, we can apply our new-found knowledge to the particular traceless symmetric tensor $\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}$, where $\hat{\boldsymbol{n}}$ is any unit vector. From Eqs. (2.35) and (2.47), we have

$$\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}=\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}-\frac{\ell(\ell-1)}{2(2\ell-1)}\mathop{\mathrm{Sym}}_{i_1\ldots i_\ell}\left[\delta_{i_1i_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right]+\ldots\ , \tag{2.48}$$

where the omitted terms represented by $\ldots$ all include more than one Kronecker δ-function.

We will be interested in computing $\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}$, which is clearly symmetric and traceless in $i_1\ldots i_{\ell-1}$, and so must be proportional to $\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\mathrm{TS}}$. The equation above is enough to find the constant of proportionality. Multiplying the equation by $\hat{\boldsymbol{n}}_{i_\ell}$ and summing over the repeated index i_ℓ, the first term clearly gives $\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}(\hat{\boldsymbol{n}}_{i_\ell}\hat{\boldsymbol{n}}_{i_\ell})=\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}$. The second term will also give a term proportional to this tensor, but again we have to be careful with the symmetrization. The indices inside the square brackets [] will not always have the locations shown, since we are summing over all permutations. This is a little simpler than the previous calculation, however, since there is only one "special" index to be concerned with, i_ℓ. Thus there are only two cases: i_ℓ can be on the δ-function, or it can be on the tensor $\hat{\boldsymbol{n}}_{i_3} \ldots \hat{\boldsymbol{n}}_{i_\ell}$. If the index i_ℓ is on the δ-function, then the value will be proportional to $\hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}}$. If the index i_ℓ is in the tensor $\hat{\boldsymbol{n}}_{i_3} \ldots \hat{\boldsymbol{n}}_{i_\ell}$, however, then two $\hat{\boldsymbol{n}}$ vectors will be dotted together, and the Kronecker δ-function will remain. This will contribute to the later terms in the series, but not to the leading term. Thus, the index i_ℓ has ℓ possible positions, and we have found that two of them contribute to the leading term, resulting in a factor of $2/\ell$ in the result. Thus, the leading term (i.e., the term with no Kronecker δ-functions) is given by

$$\begin{aligned} \hat{\boldsymbol{n}}_{i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\mathrm{TS}} &= \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}} \left[1 - \frac{\ell(\ell-1)}{2(2\ell-1)} \frac{2}{\ell} \right] + \ldots \\ &= \frac{\ell}{2\ell-1} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}} + \ldots \; . \end{aligned} \tag{2.49}$$

Since $\hat{\boldsymbol{n}}_{i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\mathrm{TS}}$ must be proportional to $\{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}} \}_{\mathrm{TS}}$, for which the leading term is simply $\hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}}$, the result must be

$$\boxed{ \hat{\boldsymbol{n}}_{i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\mathrm{TS}} = \frac{\ell}{2\ell-1} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}} \}_{\mathrm{TS}} \; . } \tag{2.50}$$