8.07 Lecture Slides 4
September 16, 2019

DIRAC DELTA FUNCTION

Electrostatics

Announcements

Planned Quiz dates:
Monday, October 7, 2019
Wednesday, November 13, 2019

These dates are now the final choices, unless there is a major emergency, such as Earth being attacked by aliens.

Dirac δ-function as a Distribution

Intuitively, $\delta(x - x_0)$ can be thought of as a limit of, for example, Gaussian functions

$$\boxed{g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} \ ,}$$

as $\sigma \to 0$. This function has integral $= 1$ for any σ. However, it is **crucial** that the limit is "delayed," in that integrals must be done **before** the limit is taken. Otherwise the integrals vanish.

Since a delayed limit is hard to define precisely, the mathematicians instead invented the **distribution**.

A **distribution,** or **generalized function,** is a linear mapping from "test" functions $\varphi(x)$ to numbers.

An ordinary integral is an example of a distribution, since it maps a function to a number, namely the integral of the function. But this is not the only way to define a distribution.

When we want to be formal, we denote a distribution by $F[\varphi(x)]$, where $\varphi(x)$ is the function and $F[\varphi(x)]$ is the number it is mapped to.

The distribution corresponding to $\delta(x - x_0)$ is defined by

$$\boxed{F_{\delta(x-x_0)}[\varphi(x)] \equiv \varphi(x_0) \ . \quad \text{(definition)}}$$

In standard notation it can be written to look like an integral, but it is defined as above:

$$\boxed{\int_{-\infty}^{\infty} \varphi(x)\delta(x-x_0)\mathrm{d}x \equiv F_{\delta(x-x_0)}[\varphi(x)] \equiv \varphi(x_0) \ .}$$

The Space of Test Functions

When mathematicians define a mapping, they like to be precise about the spaces being mapped from and to. So they are precise about the class of allowed test functions $\varphi(x)$.

Different choices are in use, but one common choice is the space of Schwartz functions, which are infinitely differentiable, and which have the property that the function and all its derivatives fall off faster than any power at large $|x|$. The distributions associated with this definition of smoothness are called tempered distributions. The Fourier transform of a Schwartz function always exists, and is also a Schwartz function.

These details will not be important to us, but we will call the test functions "smooth", and assume that they are as differentiable and as rapidly decreasing as needed for any application.

Promoting a Function to a Distribution

Any integrable function $f(x)$ can be used to define a distribution, by

$$F_{f(x)}[\varphi(x)] \equiv \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x \ .$$

Note that the more restrictive we are in the class of allowed $\varphi(x)$'s, the larger the class of allowed $f(x)$'s becomes.

"Equal in the Sense of Distributions"

When two expressions are said to be "equal in the sense of distributions," it means that when each expression is interpreted as a distribution, the two expressions represent the same distribution.

In our notation, we usually denote a distribution as an integral over a test function $\varphi(x)$, even when we are describing a δ-function distribution, which is not really defined as an integral. So, in this notation, two expressions $D_1(x)$ and $D_2(x)$ are equal in the sense of distributions if

$$\int_{-\infty}^{\infty} \varphi(x)D_1(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)D_2(x)\,\mathrm{d}x \ .$$

This matches Griffiths' Eq. (1.93) on p. 48.

This slide was mistakenly skipped in lecture

Derivative of a Distribution

A derivative of a distribution is defined so that generalized integration is consistent with integration by parts. So

$$\underbrace{\int_{-\infty}^{\infty} \varphi(x)\,\delta'(x-x_0)\,\mathrm{d}x}_{\text{Defined by this equation}} \equiv -\int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\delta(x-x_0)\,\mathrm{d}x = -\varphi'(x_0) \ .$$

where a prime ($'$) denotes a derivative with respect to x. For an arbitrary distribution,

$$F'[\varphi(x)] \equiv -F[\varphi'(x)] \ .$$

Any distribution is **infinitely** differentiable.

$\delta'(x-x_0)$ can be thought of intuitively as the limit of a sequence of derivatives of Gaussians $g'_\sigma(x)$, provided that the limit is delayed until after integration.

A Question

Are either of the following "identities" true?

$$\begin{aligned}&1)\ f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)\\&2)\ f(x)\delta'(x-x_0)=f(x_0)\delta'(x-x_0)\end{aligned}$$

where x_0 is a constant, and a prime denotes differentiation with respect to x.

(1) is true:

$$\begin{aligned}LHS:&\quad \int_{-\infty}^{\infty}\varphi(x)f(x)\delta(x-x_0)\,\mathrm{d}x=\varphi(x_0)f(x_0)\\RHS:&\quad \int_{-\infty}^{\infty}\varphi(x)f(x_0)\delta(x-x_0)\,\mathrm{d}x=\varphi(x_0)f(x_0)\ ,\end{aligned}$$

so the two sides are equal in the sense of distributions.

A Question

Are either of the following "identities" true?

$$\begin{aligned}&1)\ f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)\\&2)\ f(x)\delta'(x-x_0)=f(x_0)\delta'(x-x_0)\end{aligned}$$

where x_0 is a constant, and a prime denotes differentiation with respect to x.

(2) is false:

$$\begin{aligned}LHS:&\quad \int_{-\infty}^{\infty}\varphi(x)f(x)\delta'(x-x_0)\,\mathrm{d}x=-\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\Big[\varphi(x)f(x)\Big]\delta(x-x_0)\\&\qquad=-\varphi'(x_0)f(x_0)-\varphi(x_0)f'(x_0)\\RHS:&\quad \int_{-\infty}^{\infty}\varphi(x)f(x_0)\delta'(x-x_0)\,\mathrm{d}x=-\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\Big[\varphi(x)f(x_0)\Big]\delta(x-x_0)\\&\qquad=-\varphi'(x_0)f(x_0)\ ,\end{aligned}$$

so the two sides are **NOT** equal in the sense of distributions.

A Subtlety

If $\boxed{f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)}$ is an identity, then we would expect that we should find the same result if we differentiate both sides with respect to x.

But if we differentiate the left-hand side, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\delta(x-x_0)\right]=f'(x)\delta(x-x_0)+f(x)\delta'(x-x_0)\ ,$$

while if we differentiate the right-hand side, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x_0)\delta(x-x_0)\right]=f(x_0)\delta'(x-x_0)\ .$$

So, were we wrong to say that $f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)$? What's happening?

I'll ask you to tell me on Wednesday.

A δ-function of a Function of x

How do we treat

$$\delta\big(f(x)\big)?$$

Answer: this is defined by changing variables of integration to $y=f(x)$, so the integral over y involves simply $\delta(y)$.

Since the integral over the δ-function is not truly an integral, the change of variable of integration is not truly justified. However, $\delta(f(x))$ has not yet been defined, so we are **defining** it in a way that is consistent with the manipulations of changing variables of integration.

We assume that $f(x)$ is smooth, and for simplicity we assume that it vanishes only at one point x_0, so the only contribution comes from integrating over x_0.

Choose numbers a and b such that $a < x_0 < b$, and so that $f(x)$ is monotonic in this range. Then, calling the desired integral I for future reference, the change of variables to $y = f(x)$ gives

$$\boxed{I = \int_a^b \varphi(x)\delta\big(f(x)\big)\,\mathrm{d}x = \int_{f(a)}^{f(b)} \varphi\big(x(y)\big)\frac{\delta(y)}{f'\big(x(y)\big)}\,\mathrm{d}y\ .}$$

$$\boxed{I = \int_a^b \varphi(x)\delta\big(f(x)\big)\,\mathrm{d}x = \int_{f(a)}^{f(b)} \varphi\big(x(y)\big)\frac{\delta(y)}{f'\big(x(y)\big)}\,\mathrm{d}y\ .}$$

Suppose that $f'(x_0) > 0$, which means that $f'(x) > 0$ for $a < x < b$, since the function is monotonic in this range. Then, using the properties of the δ-function,

$$\boxed{I = \frac{\varphi(x_0)}{f'(x_0)}\ ,}$$

which is equivalent to writing

$$\boxed{\delta\big(f(x)\big) = \frac{\delta(x - x_0)}{f'(x_0)} \quad (\text{if } f'(x_0) > 0)\ .}$$

Now suppose that $f'(x_0) < 0$, in which case $f(b) < f(a)$. This misordering of the limits of integration produces a minus sign in the integral, so

$$\boxed{\delta\big(f(x)\big) = -\frac{\delta(x - x_0)}{f'(x_0)} \quad (\text{if } f'(x_0) < 0)\ .}$$

The answer for both signs can be put together, to give a simple, final answer

$$\boxed{\delta\big(f(x)\big) = \frac{\delta(x - x_0)}{|f'(x_0)|}\ .}$$

If there is more than one place where $f(x) = 0$, each zero gives a contribution to the integral exactly like the one shown.