8.07 Lecture Slides 8
September 30, 2019

ELECTRIC POTENTIAL: METHOD OF IMAGES, SEPARATION OF VARIABLES

Announcements

Quiz 1 is one week from today.

Quiz 1 will "cover" everything through Problem Set 3.

Practice problems for Quiz 1, with solutions, will be posted later today.

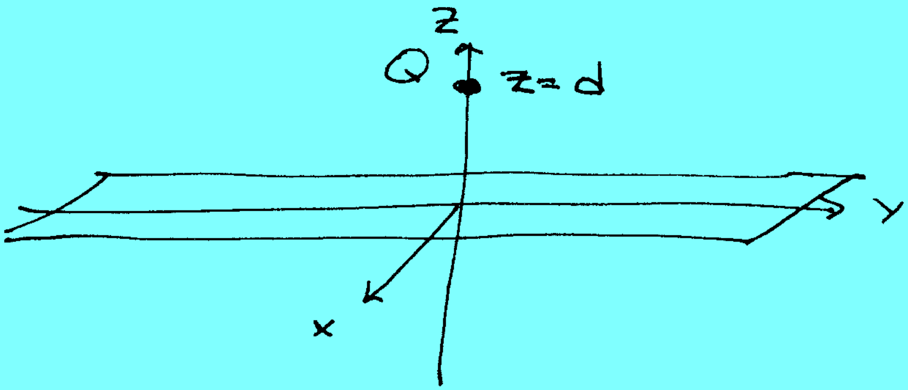
A formula sheet for Quiz 1 will also be posted later today, and will be included with the Quiz.

We need to schedule a review session, to be led by Yitian Sun.

Added after lecture: the review session will be on Sunday, October 6, from 3:00 to 5:00 pm, in Room 2-131.

Summary: Blackboard Discussion of Image Charges

Sample Problem: a conducting plane, in the x-y plane, with a charge Q at $(0, 0, d)$:



The problem is to find a solution to

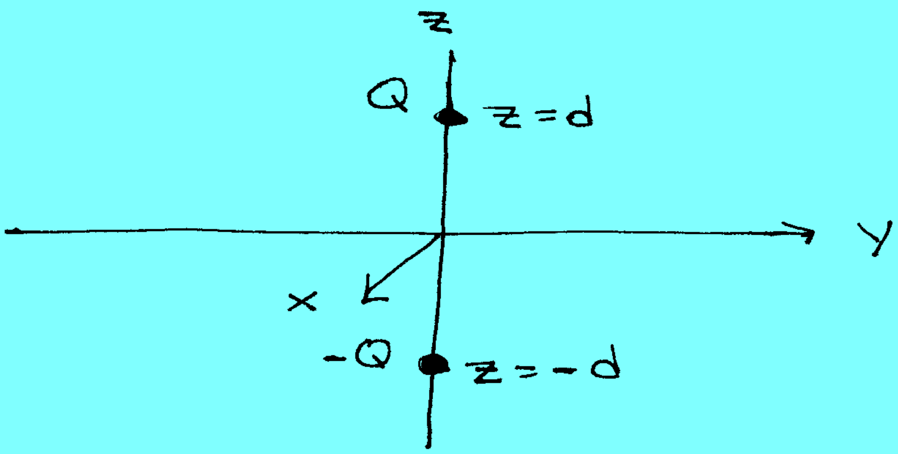
$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0} = -\frac{Q}{\epsilon_0}\delta^3(\vec{r} - \vec{r}_d)\ ,} \tag{1}$$

for $z > 0$, where $\vec{r}_d \equiv d\hat{z}$, with the boundary condition

$$\boxed{V(x, y, 0) = 0\ .}$$

Solution:

Imagine an *image charge* $-Q$ at the mirror image location, $(0, 0, -d)$. The total potential, due to the real and the image charge, will then vanish on the conducting plane and will satisfy Eq. (1) for $z > 0$. By the first uniqueness theorem, this is the unique solution.



Note that the image charge solution describes $V(\vec{r})$ only for $z \geq 0$. For $z < 0$, $V = 0$ is the unique solution. For $z \geq 0$

$$\boxed{V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} + \frac{-Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} .}$$

What is $\vec{\boldsymbol{E}}$? Ans: $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V$.

What is σ on conducting surface?

Ans: $\sigma = \epsilon_0 \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}$,

where $\hat{\boldsymbol{n}}$ = outward normal of conductor

What is total charge induced on the surface?

$$\text{Ans: } Q_{\text{induced}} \equiv \int \sigma(x, y)\, \mathrm{d}x\, \mathrm{d}y = -Q \ ,$$

where the answer is found from Gauss's law: the integral is the electric flux, and half the flux coming from Q and half the flux coming from $-Q$ intersect the x-y plane.

What is force on Q?

Ans: $\vec{\boldsymbol{F}} = Q\vec{\boldsymbol{E}}$, where $\vec{\boldsymbol{E}}$ is the electric field caused by all charges, real or image, except for Q itself.

What is the potential energy W?

Ans: It is easiest to use

$$\boxed{W = \frac{1}{2}\int \rho(\vec{\boldsymbol{r}})V(\vec{\boldsymbol{r}})\,\mathrm{d}^3x\ ,}$$

where $\rho(\vec{\boldsymbol{r}})$ includes only physical charges, not image charges. The physical charges are the point charge Q, and the surface charge $\sigma(x,y)$ induced on the surface. But $\sigma(x,y)$ does not contribute, since $V(x,y,0)=0$. So

$$W = \frac{1}{2}\int Q\,\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_d)\,V(\vec{\boldsymbol{r}})\,\mathrm{d}^3x = \frac{1}{2}QV(\vec{\boldsymbol{r}}_d)\ .$$

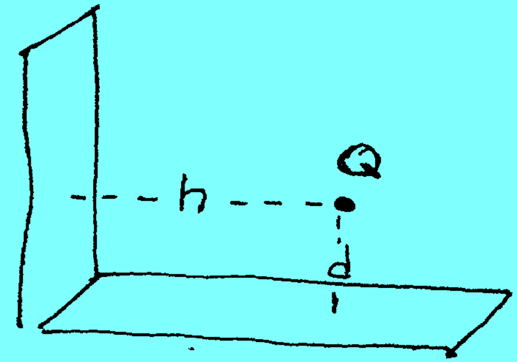
We exclude the infinite self-energy of Q by taking $V(\vec{\boldsymbol{r}}_d)$ to be the potential of all charges, real and image, excluding Q itself, so

$$\boxed{W = -\frac{Q^2}{8\pi\epsilon_0}\frac{1}{2d}\ .}$$

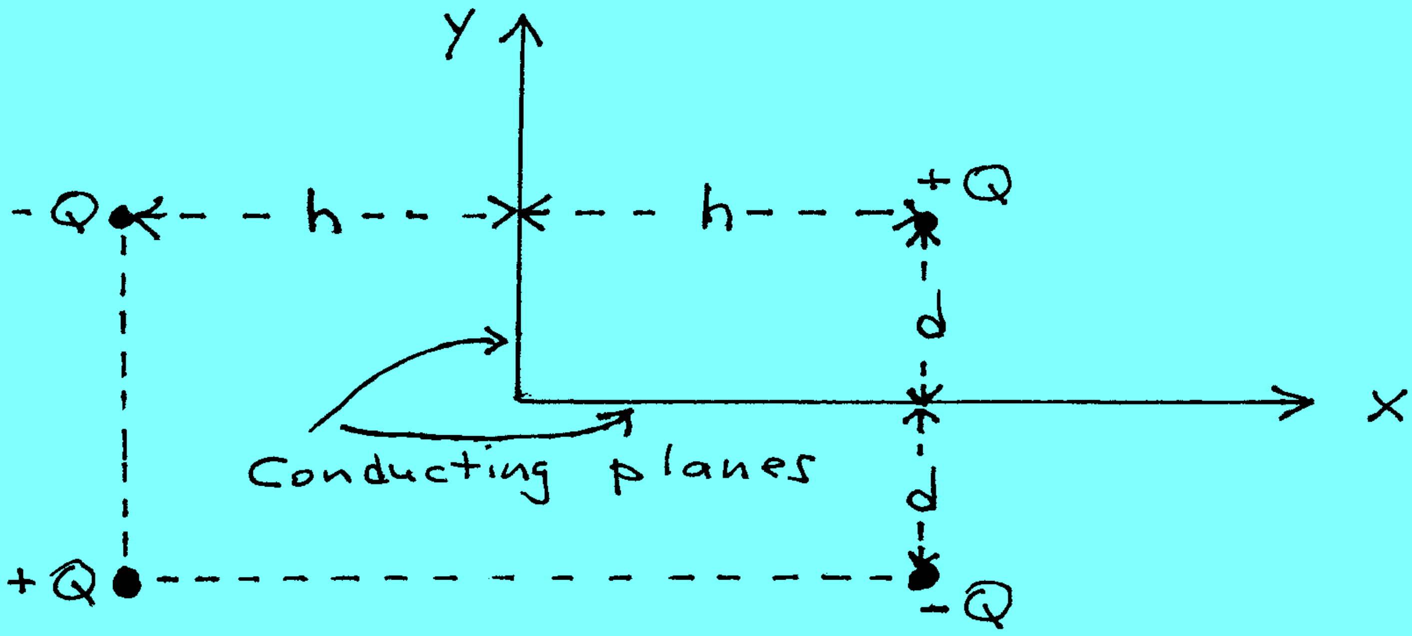
In this case the answer is half the potential energy we would have found if we treated the image charge as if it were real — but that is not always the case.

In general, the potential energy found by treating the image charges as real has no fixed relation to the actual potential energy of the system.

Sample Image Charge Problem #2: two conducting planes, at right angles.

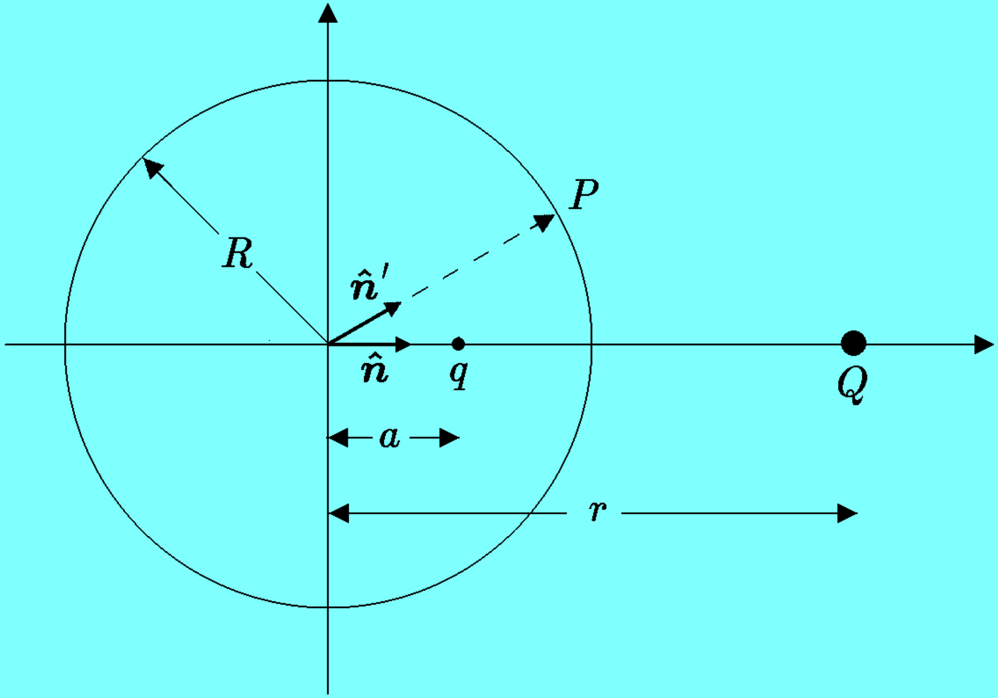


This problem can be solved by introducing 3 image charges:



Sample Image Charge Problem #3: A Grounded Sphere

Consider a grounded ($V = 0$) conducting sphere, of radius R, with a charge Q at a distance r from its center. ($V = 0$ means same potential as ∞.)

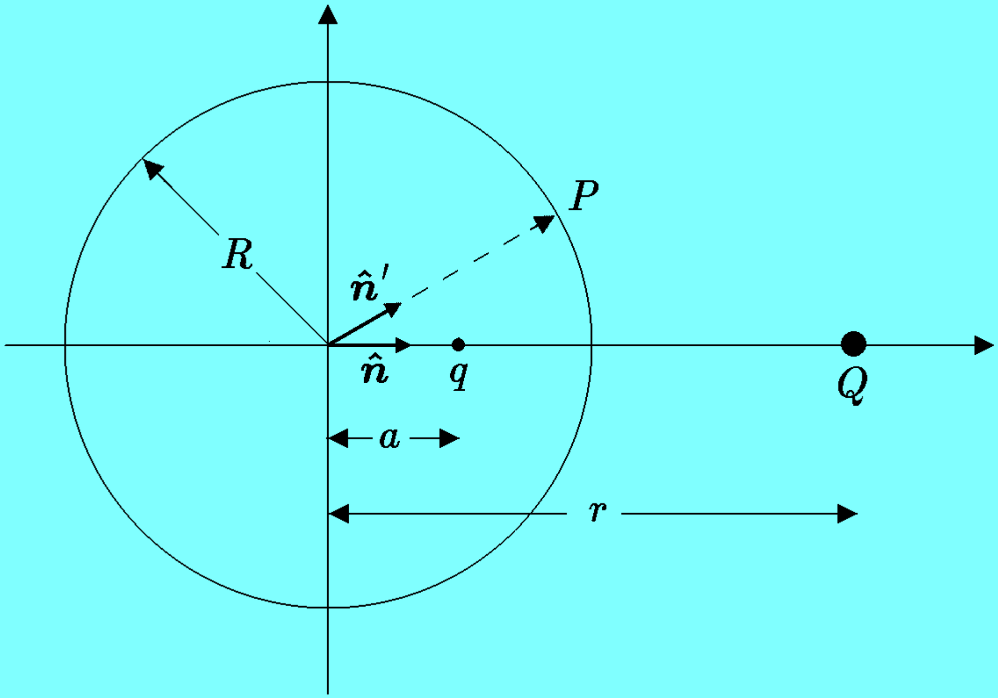


Is it possible to satisfy $V = 0$ on the sphere by adding only a single image charge?

Let's try. Put an image charge q along the line joining the center of the sphere to Q, at a distance a from the center. Calculate $V(P)$, where P is a point on the surface of the sphere, at a direction $\hat{\boldsymbol{n}}'$ from the center.

Can we find a charge q and a distance a such that

$$V(P) = \frac{q}{|R\hat{\boldsymbol{n}}' - a\hat{\boldsymbol{n}}|} + \frac{Q}{|R\hat{\boldsymbol{n}}' - r\hat{\boldsymbol{n}}|} = 0 \ \textbf{?}$$



If so, then

$$|R\hat{\boldsymbol{n}}' - a\hat{\boldsymbol{n}}| \underset{?}{=} -\frac{q}{Q}|R\hat{\boldsymbol{n}}' - r\hat{\boldsymbol{n}}| \ .$$

Squaring,

$$R^2 + a^2 - 2aR\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}} \underset{?}{=} \left(\frac{q}{Q}\right)^2 \left(R^2 + r^2 - 2rR\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}}\right) \ .$$

$$R^2 + a^2 - 2aR\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}} \underset{?}{=} \left(\frac{q}{Q}\right)^2 \left(R^2 + r^2 - 2rR\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}}\right) \ .$$

For this equation to hold for all $\hat{\boldsymbol{n}}'$, we need

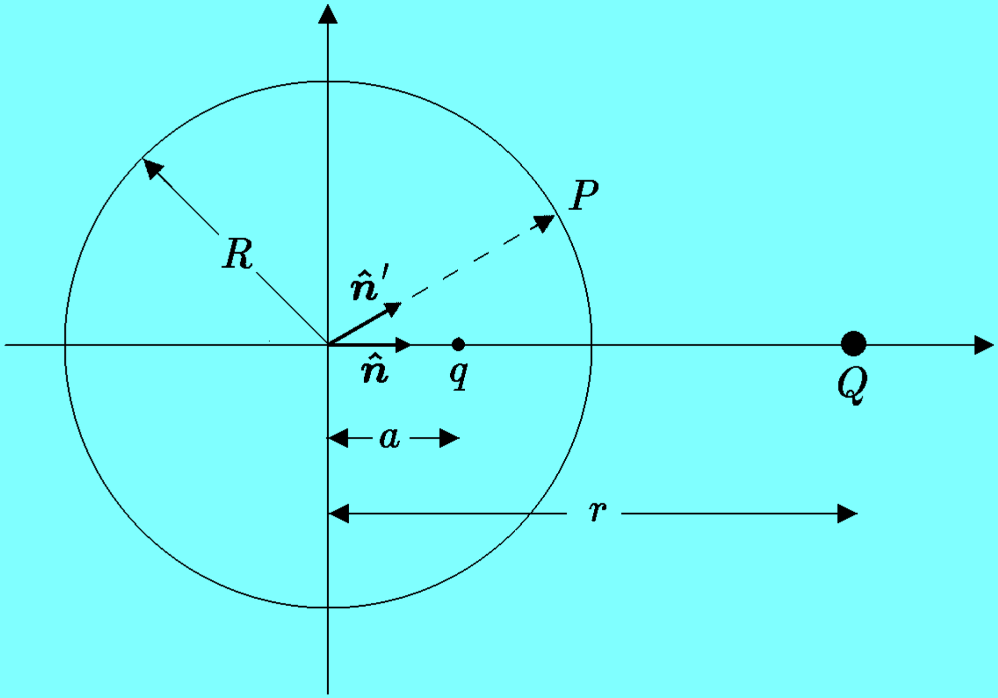
$$R^2 + a^2 = \left(\frac{q}{Q}\right)^2 (R^2 + r^2)$$

$$a = \left(\frac{q}{Q}\right)^2 r \ .$$

But these are two equations for two unknowns q and a, so we expect a solution. A little algebra shows it is

$$\boxed{q = -\frac{R}{r} Q \quad \text{and} \quad a = \frac{R^2}{r} \ .}$$

$$q = -\frac{R}{r} Q \quad \text{and} \quad a = \frac{R^2}{r} \ .$$



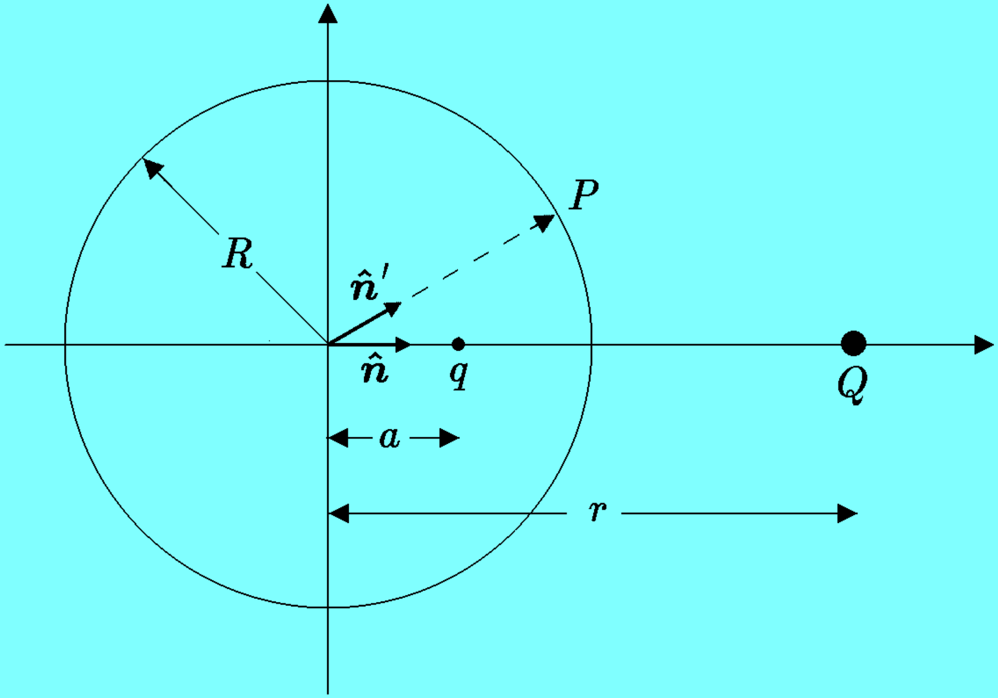
What is the force $\vec{F}$ on Q?

Ans: $\vec{F} = Q\vec{E}_{\text{other}}(r\hat{n})$, where $\vec{E}_{\text{other}}$ means the electric field of all charges other than Q.

In this case, $\vec{E}_{\text{other}}$ is the field due to the image charge q. So

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{(r-a)^2} \hat{n} = \frac{1}{4\pi\epsilon_0} \frac{\left(-\frac{R}{r}Q\right) Q}{\left(r - \frac{R^2}{r}\right)^2} \hat{n} = -\frac{1}{4\pi\epsilon_0} \frac{RrQ^2}{(r^2 - R^2)^2} \hat{n} \ .$$

$$\boxed{q = -\frac{R}{r} Q \quad \text{and} \quad a = \frac{R^2}{r} \; .}$$



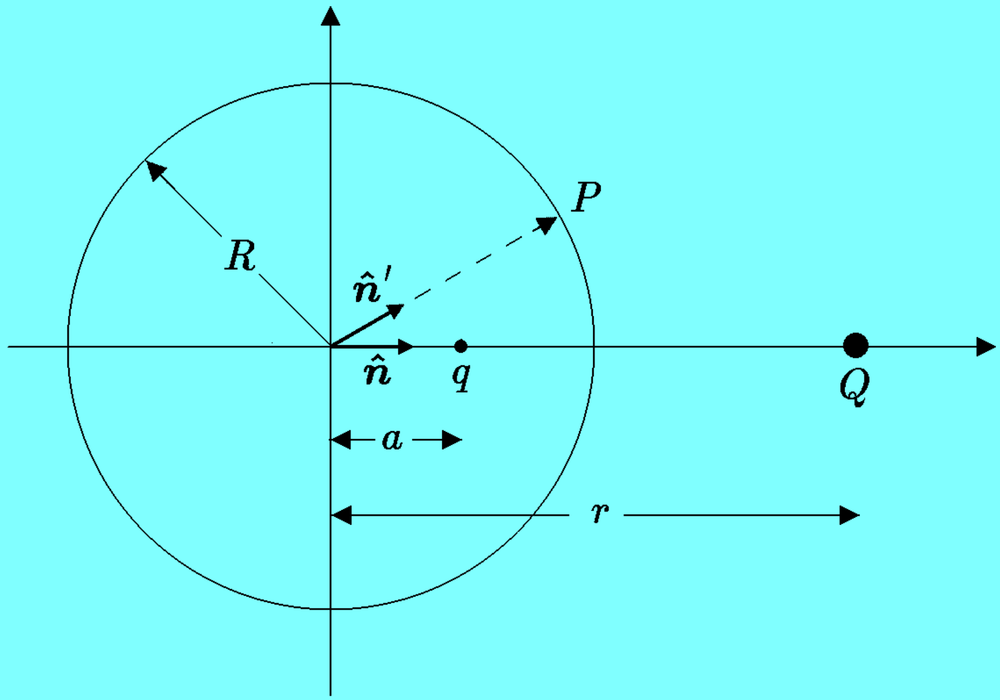
What is the total potential energy W of the system?

Ans: $\boxed{W = \frac{1}{2} \int \rho V_{\text{other}} \, \mathrm{d}^3 x,}$ where

$\boldsymbol{\rho}$ **includes only real charges,** which in this case includes the point charge Q and the surface charge q distributed on the surface of the sphere.

$\boldsymbol{V_{\text{other}}}$ **is equal to** $\boldsymbol{V(\vec{r}\,)}$**,** the full potential (which is caused by all charges, real or image), **except when** $\boldsymbol{\rho(\vec{r}\,)}$ **contains** $\boldsymbol{\delta}$**-functions (point charges).** In that case, when $V_{\text{other}}(\vec{r}\,)$ is evaluated at the location of a point charge q_i, infinite self-energies are avoided by evaluating V_{other} as the potential due to all charges (real or image) **other than** the point charge q_i.

$$q = -\frac{R}{r}Q \quad \text{and} \quad a = \frac{R^2}{r} \; .$$



The charge q on the surface of the sphere gives no contribution, since $V = 0$. So $W = \frac{1}{2}\int \rho V_{\text{other}}\, \mathrm{d}^3x = \frac{1}{2} Q V_{\text{other}}(r\hat{\boldsymbol{n}})$, which gives

$$W = \frac{1}{2}Q\left[\frac{1}{4\pi\epsilon_0}\frac{q}{(r-a)}\right] = \frac{1}{8\pi\epsilon_0}\frac{\left(-\frac{R}{r}Q\right)Q}{\left(r-\frac{R^2}{r}\right)} = -\frac{1}{8\pi\epsilon_0}\frac{RQ^2}{(r^2-R^2)} \; .$$

About the Total Energy of a Real + Image Charge System?

IF we pretended that the image charge in the previous problem was real, and evaluated the potential energy W_{pretend} of the system, we would have found

$$\boxed{W_{\text{pretend}} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{(r-a)} \ ,}$$

which is exactly twice the answer that we found. For the problem of a point charge and a conducting plane, we would find the same relation.

Is it always true that the total energy W of a system solved with image charges is equal to $\frac{1}{2} W_{\text{pretend}}$, i.e. half the total energy that would be found if the image charges were real?

Answer:

About the Total Energy of a Real + Image Charge System?

IF we pretended that the image charge in the previous problem was real, and evaluated the potential energy W_{pretend} of the system, we would have found

$$\boxed{W_{\text{pretend}} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{(r-a)} \ ,}$$

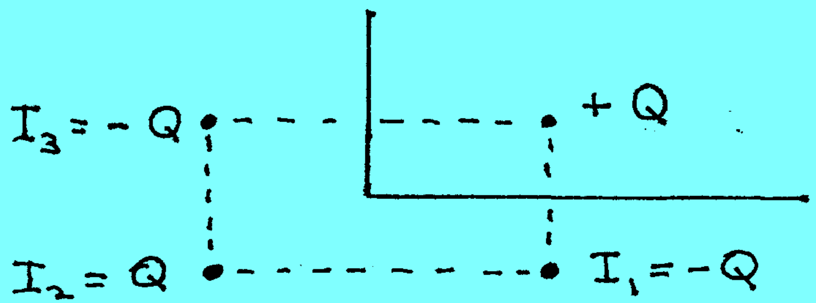
which is exactly twice the answer that we found. For the problem of a point charge and a conducting plane, we would find the same relation.

Is it always true that the total energy W of a system solved with image charges is equal to $\frac{1}{2}W_{\text{pretend}}$, i.e. half the total energy that would be found if the image charges were real?

Answer: **NO!**

Exceptions:

1) The problem with two conducting planes at right angles. The right answer is



$$\boxed{W = \frac{1}{2} Q \left[V_1 + V_2 + V_3 \right] \ ,}$$

where $V_i = \frac{1}{4\pi\epsilon_0}(Q_{I_i}/r_i)$, where r_i is the distance from I_i to the real charge $+Q$. But in W_{pretend}, where the image charges are treated as real, there would also be terms for each pair of image charges.

2) An *uncharged* conducting sphere and a point charge Q.

An image charge $q = -(R/r)Q$ at radius $a = R^2/r$ will lead to $V = 0$ on the surface of the sphere. For this problem we need $V =$ constant on the sphere, but with no net charge on the sphere.

Solution: add an image charge $-q$ at the center of the sphere.

Then $W = \frac{1}{2}Q\left[V_q + V_{-q}\right]$, where V_q and V_{-q} are the potentials of the image charges at the location of Q.

But in calculating W_{pretend}, which treats the image charges as if they are real, there would also be a contribution to W from the interaction of the two image charges.

So when is $W = \frac{1}{2} W_{\text{pretend}}$?

There are two cases:

1) When there is exactly one real charge and one image charge.

2) When the only conductor is an infinite plane. Then, no matter how complicated the charge distribution, $W = \frac{1}{2} W_{\text{pretend}}$.

Separation of Variables in Cartesian Coordinates

How to solve $\nabla^2 V = 0$?

In Cartesian coordinates,

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \ .}$$

Try a solution of the form

$$\boxed{V(x, y, z) = X(x)Y(y)Z(z) \ .}$$

(Even if this is not general enough, sums of solutions of this form might work.)

Laplace's equation implies

$$\boxed{YZ\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + XZ\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + XY\frac{d^2Z}{\mathrm{d}z^2} = 0\ .}$$

Now divide by $V = XYZ$:

$$\boxed{\underbrace{\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}_{C_1} + \underbrace{\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}}_{C_2} + \underbrace{\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2}}_{C_3} = 0\ .}$$

Since the 1st term depends only on x, the 2nd depends only on y, and the 3rd depends only on z, each term must be a constant. So

$$\boxed{C_1 + C_2 + C_3 = 0\ .}$$

General Form for $X(x)$

We look first at $X(x)$. The other two functions $Y(y)$ and $Z(z)$ are analogous. We have

$$\boxed{\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = C_1 X \ .}$$

There are three possibilities:

1) If $C_1 > 0$, write $C_1 \equiv \alpha^2$. Then the general solution can be written as $\boxed{X(x) = Ae^{\alpha x} + Be^{-\alpha x}}$, or $\boxed{X(x) = A\sinh(\alpha x) + B\cosh(\alpha x)}$, where A and B are constants.

2) If $C_1 < 0$, write $C_1 = -\alpha^2$. The general solution can then be written as $\boxed{X(x) = Ae^{i\alpha x} + Be^{-i\alpha x}}$, or $\boxed{X(x) = A\sin(\alpha x) + B\cos(\alpha x)}$, where A and B are constants.

3) If $C_1 = 0$, then the general solution can be written as $\boxed{X(x) = A + Bx}$.

Separation of Variables Example: Infinitely Long Rectangular Pipe

This is Griffiths's Example 3.5, pp. 134–136.

An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

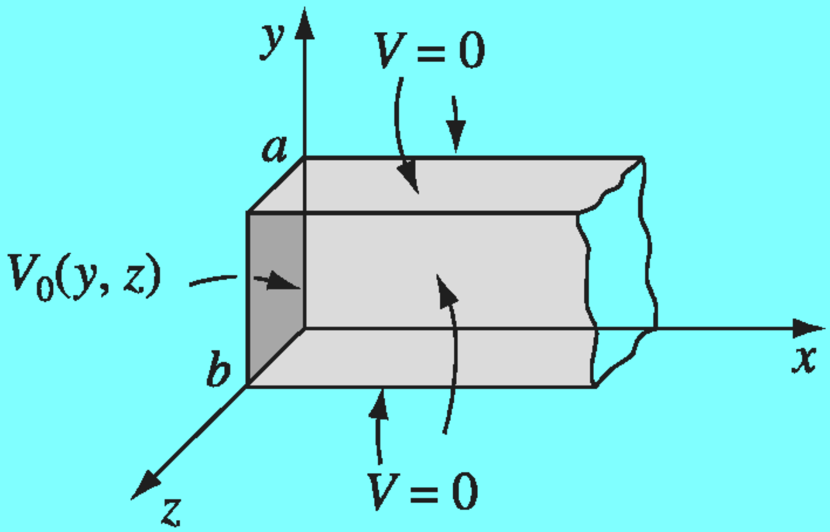


FIGURE 3.22

Suppose $C_2 > 0$. Then

$$Y(y) = A\sinh(\alpha y) + B\cosh(\alpha y) \ .$$

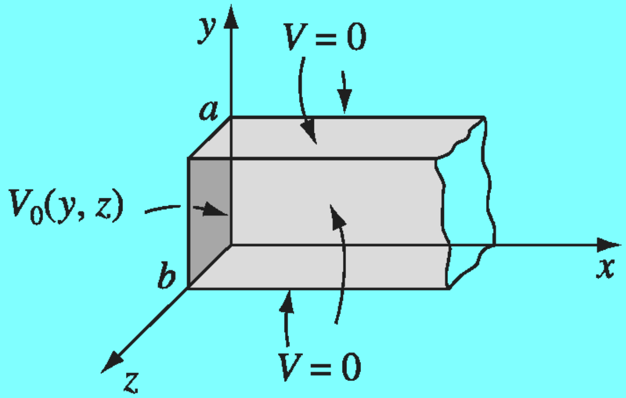


FIGURE 3.22

But look at boundary conditions:
$Y(0) = 0 \quad \Longrightarrow \quad B = 0,$
so $Y = A\sinh(\alpha y)$. But this cannot vanish at $y = a$.

$C_2 = 0$ is also ruled out — a straight line that vanishes as $y = 0$ and $y = a$ must vanish everywhere.

Conclusion: $C_2 < 0$. We must have $C_3 < 0$ for the same reason. Then $C_1 = -(C_2 + C_3) > 0$.

So let $C_2 = -\alpha^2$, so

$$Y(y) = A\sin(\alpha y) + B\cos(\alpha y)\ .$$

$$Y(0) = 0 \quad \Longrightarrow \quad B = 0.\ Y(a) = 0 \quad \Longrightarrow$$

$$\boxed{\alpha = \frac{n\pi}{a}\ ,}$$

where n is a positive integer. Why positive? $n = 0$ gives only $Y = 0$, and $n = -1$ is the same solution as $n = +1$, with A replaced by $-A$.

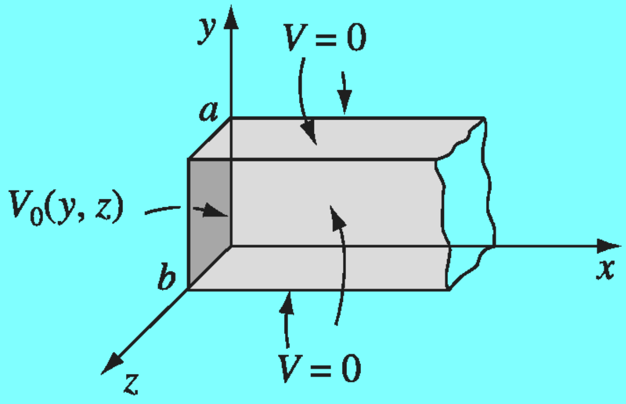


FIGURE 3.22

Similarly, let $C_3 = -\beta^2$, so

$$Z(z) = A\sin(\beta z) + B\cos(\beta z) \ .$$

The boundary conditions imply that B = 0, and

$$\boxed{\beta = \frac{m\pi}{b} \ ,}$$

where m is a positive integer.

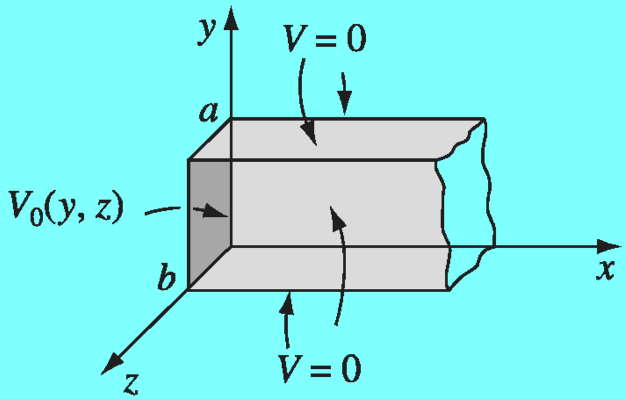


FIGURE 3.22

$C_1 = \gamma^2 > 0$, so

$$X(x) = Ae^{-\gamma x} + Be^{\gamma x} \ .$$

Both terms satisfy Laplace's equation, but we assume the boundary condition $X(x) \to 0$ as $x \to \infty$, because otherwise the total energy is infinite. So $B = 0$.

Full solution:

$$\boxed{V(x,y,z) = \sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}e^{-\gamma x}\ .}$$

Try a sum of solutions of this form, with as yet unspecified coefficients:

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,x\right]\ .}$$

This slide was added after lecture.

Summary: Blackboard Discussion Continuing the Calculation

For any choice of the V_{nm}'s, the solution

1) satisfies $\nabla^2 V = 0$;
2) satisfies boundary conditions on the sides of the pipe, at $y = 0, a$ and $z = 0, b$;
3) satisfies the boundary condition $V \to 0$ as $x \to \infty$.

We must still arrange to satisfy the boundary condition at $x = 0$,

$$\boxed{V(0, y, z) = V_0(y, z)\ .}$$

So we need to choose the V_{nm}'s so that

$$\boxed{V(0, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} = V_0(y, z)\ .}$$

So we need to choose the V_{nm}'s so that

$$\boxed{V(0,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} = V_0(y,z)\ .}$$

Can we do this? **YES!**

This is a Fourier series. Fourier's theorem (often called Dirichlet's theorem in its most rigorous formulation) says that this series is complete. Any piecewise continuous bounded function, satisfying the boundary conditions that it vanishes when $y = 0$ or $y = a$ or $z = 0$ or $z = b$, can be expanded this way.

This slide was added after lecture.

We will not prove completeness, which is not an easy thing to prove — so we will leave it to math classes.

But once one knows that $V_0(y,z)$ can be written as such a series, it is easy to find the V_{nm}'s that do it. We use the fact that

$$\boxed{\int_0^a \sin\frac{n'\pi y}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}y = \frac{1}{2}a\,\delta_{n'n}\ .}$$

For $n' = n$, the integral is evaluated by recognizing that the average value of $\sin^2 w$, when averaged over any number of half periods of $\sin w$, is equal to $1/2$. For $n' \neq n$, trig identities can be used to convert the integral into integrations of $\cos w$ over integral numbers of periods, which vanish.

This slide was added after lecture.

So we integrate both sides,

$$\boxed{\begin{aligned}&\int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\\ &\qquad = \int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\, V_0(y,z)\ .\end{aligned}}$$

Using

$$\int_0^a \sin\frac{n'\pi y}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}y = \frac{1}{2}a\,\delta_{n'n}$$

and the analogous formula for z, we find

$$\frac{1}{4}\,ab\,V_{n'm'} = \int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\, V_0(y,z)\ .$$

$$\frac{1}{4}\, ab\, V_{n'm'} = \int_0^a \sin\frac{n'\pi y}{a} \int_0^b \sin\frac{m'\pi z}{b}\, V_0(y,z) \ .$$

So we can write the final solution as

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} \exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, x\right] \ ,}$$

where

$$\boxed{V_{nm} = \frac{4}{ab}\int_0^a \sin\frac{n\pi y}{a} \int_0^b \sin\frac{m\pi z}{b}\, V_0(y,z) \ .}$$

We have dropped the primes in the formula above, which we can do because the equation no longer contains any instances of the original indices m and n.