MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 27, 2019
Prof. Alan Guth

PROBLEM SET 5 SOLUTIONS

PROBLEM 1: INTEGRATION OF THE LEGENDRE POLYNOMIALS *(15 points)*

(a) [5 pts] Putting the equations together,

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\ , \tag{1.1}$$

so

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \frac{1}{2\pi}\left[\frac{(2\ell)!}{2^\ell(\ell!)^2}\right]\left[\frac{(2\ell')!}{2^{\ell'}(\ell'!)^2}\right] \times\int d\Omega\,\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\{\,\hat{\boldsymbol z}_{j_1}\ldots\hat{\boldsymbol z}_{j_{\ell'}}\,\}_{\rm TS}\,\hat{\boldsymbol n}_{j_1}\ldots\hat{\boldsymbol n}_{j_{\ell'}}\ . \tag{1.2}$$

Note that since the integrand does not depend on ϕ,

$$\int d\Omega = \int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi = 2\pi\int_0^\pi\sin\theta\,d\theta\ ,$$

so we could replace $\int_0^\pi \sin\theta\,d\theta$ by $\frac{1}{2\pi}\int d\Omega$. But the integral now has the form of Eq. (1.3) of the problem set, which includes the factor $\delta_{\ell,\ell'}$ which explicitly vanishes whenever ℓ and ℓ' are not equal.

Since the problem said to use Eq. (1.3), it is certainly a valid answer to point to the Kronecker δ-function in the equation. But if one wants to understand where this Kronecker δ-function comes from, one can go back to Eq. (3.28) in Lecture Notes 3,

$$\int d\Omega\,\left[C^{(\ell)}_{i_1\ldots i_\ell}\{\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\,\}_{\rm TS}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\,\hat{\boldsymbol n}_{j_1}\ldots\hat{\boldsymbol n}_{j_{\ell'}}\,\}_{\rm TS}\right] \propto C^{(\ell)}_{i_1\ldots i_\ell}C'^{(\ell')}_{j_1\ldots j_{\ell'}}\sum_{\text{all pairings}}\delta_{k_1,k_2}\ldots\delta_{k_{\ell+\ell'-1},k_{\ell+\ell'}}\ . \tag{LN 3.28}$$

The integral over $d\Omega$ of the product of $\hat{\boldsymbol n}$'s produces a sum of products of Kronecker δ-functions with all possible pairings. But $C^{(\ell)}$ and $C'^{(\ell)}$ are both traceless, so nonzero terms can only arise if every Kronecker δ-function contracts an index on $C^{(\ell)}$ with an index on $C'^{(\ell)}$, which can only happen if they each have the same number of indices.

(b) [10 pts] Using Eq. (1.3) of the problem set with Eq. (1.2) above, we find

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \frac{1}{2\pi}\left[\frac{(2\ell)!}{2^\ell(\ell!)^2}\right]^2\frac{4\pi\,2^\ell(\ell!)^2}{(2\ell+1)!}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\ . \tag{1.3}$$

As the problem set hinted, we can replace

$$\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}$$

by

$$\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\ ,$$

because the second factor of $\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}$ can be expanded as in Eq. (2.35) of Lecture Notes 2,

$$\{\,S_{i_1\ldots i_\ell}\,\}_{\rm TS} = S_{i_1\ldots i_\ell} + \mathop{\rm Sym}_{i_1\ldots i_\ell}\big[a_{1,\ell}\delta_{i_1i_2}\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1i_2}\delta_{i_3i_4}\delta^{j_1j_2}\delta^{j_3j_4}S_{j_1j_2j_3j_4i_5\ldots i_\ell}+\ldots\big]\ . \tag{LN 2.35}$$

All the terms beyond the first contain Kronecker δ-functions that will contract indices on the first factor of $\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}$. But since $\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}$ is traceless, these contributions will vanish. Thus, the only nonzero contribution comes from the first term of the expansion, which in this case is simply $\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}$.

Thus we need to evaluate $\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}$, which is exactly what we evaluated in Problem 3 of Problem Set 4. As was shown in that solution, this quantity can be evaluated by successively applying the relation (1.1) of the problem set:

$$\begin{aligned}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell} &= \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_{\ell-1}}\,\}_{\rm TS}\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{\ell-1}\\ &= \left(\frac{\ell}{2\ell-1}\right)\left(\frac{\ell-1}{2\ell-3}\right)\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_{\ell-2}}\,\}_{\rm TS}\\ &= \left(\frac{\ell}{2\ell-1}\right)\left(\frac{\ell-1}{2\ell-3}\right)\cdots\left(\frac{1}{1}\right)\{\,1\,\}_{\rm TS}\\ &= \frac{\ell!}{(2\ell-1)!!}\ .\end{aligned}$$

Then

$$\{\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol z}_{i_1}\ldots\hat{\boldsymbol z}_{i_\ell} = \frac{\ell!}{(2\ell-1)!!} = \frac{2^\ell(\ell!)^2}{(2\ell)!}\ , \tag{1.4}$$

where we used the identity

$$(2\ell-1)!! = \frac{(2\ell)!}{2^\ell \ell!}\ , \tag{1.5}$$

which was derived in Lecture Notes 2 (Eq. (2.22)). Putting the pieces together,

$$\begin{aligned}\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta &= \frac{1}{2\pi}\left[\frac{(2\ell)!}{2^\ell(\ell!)^2}\right]^2\left[\frac{4\pi\,2^\ell(\ell!)^2}{(2\ell+1)!}\right]\left[\frac{2^\ell(\ell!)^2}{(2\ell)!}\right] \\ &= \boxed{\frac{2}{2\ell+1}\ .}\end{aligned} \tag{1.6}$$

PROBLEM 2: INTEGRATION OF THE SPHERICAL HARMONICS *(25 points)*

(a) [10 pts] Using the definition from Eq. (2.1) of the problem set and the integration formula from (1.3) of the problem set,

$$\begin{aligned}\int d\Omega\, Y^*_{\ell',m'}(\theta,\phi)\,Y_{\ell,m}(\theta,\phi) &= N(\ell',m')N(\ell,m) \\ &\quad\times\int d\Omega\,\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_{m'}}\,\hat{\boldsymbol z}_{i_{m'+1}}\ldots\hat{\boldsymbol z}_{i_{\ell'}}\,\}^*_{\rm TS}\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_{\ell'}} \\ &\quad\times\{\,\hat{\boldsymbol u}^+_{j_1}\ldots\hat{\boldsymbol u}^+_{j_m}\,\hat{\boldsymbol z}_{j_{m+1}}\ldots\hat{\boldsymbol z}_{j_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol n}_{j_1}\ldots\hat{\boldsymbol n}_{j_\ell} \\ &= \frac{4\pi\,2^\ell(\ell!)^2}{(2\ell+1)!}N(\ell',m')N(\ell,m) \\ &\quad\times\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_{m'}}\,\hat{\boldsymbol z}_{i_{m'+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}^*_{\rm TS}\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS}\,\delta_{\ell',\ell}\ .\end{aligned} \tag{2.1}$$

Thus, ℓ' must equal ℓ for the answer to be nonzero, as one can see from the Kronecker δ-function in Eq. (1.3) of the problem set. When the traceless symmetric tensors in the above expression are expanded, and the indices are contracted, every $\hat{\boldsymbol u}^{+*}_i$ must be contracted with something. Since $\hat{\boldsymbol u}^{+*}_i\hat{\boldsymbol u}^{+*}_i = \hat{\boldsymbol u}^{+*}_i\hat{\boldsymbol z}_i = 0$, the only nonzero terms will arise when every $\hat{\boldsymbol u}^{+*}_i$ is contracted with a $\hat{\boldsymbol u}^+_i$. Thus there must be at least as many $\hat{\boldsymbol u}^+_i$'s as there are $\hat{\boldsymbol u}^{+*}_i$'s, so $m \geq m'$. Similarly every $\hat{\boldsymbol u}^+_i$ must be contracted with a $\hat{\boldsymbol u}^{+*}_i$, so $m' \geq m$. Thus, a nonzero answer occurs only if $m' = m$.

(b) [15 pts] To evaluate the right-hand side of the above equation, we first recognize that we can drop the extraction of the traceless symmetric part, as in the previous

problem. Then we can iteratively use the identity of Eq. (2.5) from the problem set. Let us define

$$\begin{aligned}X &\equiv \{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}^*_{\rm TS}\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\rm TS} \\ &= \{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}^*_{\rm TS}\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell} \\ &= \left[\frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\right]\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_{\ell-1}}\,\}^*_{\rm TS}\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_{\ell-1}} \\ &= \left[\frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\right]\left[\frac{(\ell+m-1)(\ell-m-1)}{(\ell-1)(2\ell-3)}\right]\cdots\left[\frac{(2m+1)(1)}{(m+1)(2m+1)}\right] \\ &\qquad\times\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\}^*_{\rm TS}\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\ .\end{aligned} \tag{2.2}$$

Note that each factor in the final expression above has the same form as the first, but with ℓ replaced by $\ell-1$, then $\ell-2$, until finally ℓ is replaced by $m+1$, because $\hat{\boldsymbol z}_{i_{m+1}}$ is the last $\hat{\boldsymbol z}$ vector to be eliminated by the use of Eq. (2.5) of the problem set.

The last factor in X, $\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\}^*_{\rm TS}\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}$, is equal to 1. To see this, note that $\hat{\boldsymbol u}^+_i\hat{\boldsymbol u}^+_i = 0$, so $\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}$ is traceless without any subtractions. Thus,

$$\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\}^*_{\rm TS}\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m} = \hat{\boldsymbol u}^+{*_{i_1}}\ldots\hat{\boldsymbol u}^{+*}_{i_m}\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m} = 1\ .$$

To simplify the expression for X, note that

$$\begin{gathered}(\ell+m)(\ell+m-1)\cdots(2m+1) = \frac{(\ell+m)!}{(2m)!}\ , \\ (\ell-m)(\ell-m-1)\cdots(1) = (\ell-m)!\ , \\ \frac{1}{\ell}\frac{1}{(\ell-1)}\cdots\frac{1}{(m+1)} = \frac{m!}{\ell!}\ ,\end{gathered}$$

and

$$\frac{1}{(2\ell-1)}\frac{1}{(2\ell-3)}\cdots\frac{1}{(2m+1)} = \frac{(2m-1)!!}{(2\ell-1)!!} = \frac{(2m)!}{2^m m!}\frac{2^\ell\ell!}{(2\ell)!}\ ,$$

where I used the identity of Eq. (1.5) above,

$$(2\ell-1)!! = \frac{(2\ell)!}{2^\ell\ell!}\ .$$

Thus we can write

$$X = \frac{(\ell+m)!}{(2m)!}(\ell-m)!\frac{m!}{\ell!}\frac{(2m)!}{2^m m!}\frac{2^\ell\ell!}{(2\ell)!} = 2^{\ell-m}\frac{(\ell+m)!(\ell-m)!}{(2\ell)!}\ . \tag{2.3}$$

Finally, then,

$$\begin{aligned}\int d\Omega\, Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) &= \frac{4\pi\, 2^\ell(\ell!)^2}{(2\ell+1)!}N^2(\ell,m)X \\ &= \frac{4\pi\, 2^\ell(\ell!)^2}{(2\ell+1)!}2^{\ell-m}\frac{(\ell+m)!(\ell-m)!}{(2\ell)!}N^2(\ell,m) \\ &= \frac{4\pi\cdot 2^{2\ell-m}(\ell!)^2(\ell+m)!(\ell-m)!}{(2\ell+1)[(2\ell)!]^2}N^2(\ell,m)\end{aligned} \tag{2.4}$$

Requiring the integral to be equal to one, we get the result we wanted:

$$\boxed{N^2(\ell,m) = \frac{(2\ell+1)[(2\ell)!]^2}{4\pi\cdot 2^{2\ell-m}(\ell!)^2(\ell+m)!(\ell-m)!}\ .} \tag{2.5}$$

PROBLEM 3: EXPLICIT CALCULATION OF SPHERICAL HARMONICS *(15 points)*

(a) Using Eqs. (3.3) and (3.4) from the problem's description, we can write Y_{31} as

$$\begin{aligned}Y_{31}(\theta,\phi) &= C^{(3,1)}_{ijk}\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k \\ &= N(3,1)\{\,\hat{\boldsymbol u}^+_i\hat{\boldsymbol z}_j\hat{\boldsymbol z}_k\,\}_{\rm TS}\,\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k\ .\end{aligned}$$

Using the hint in the problem, we can rewrite this as

$$Y_{31}(\theta,\phi) = N(3,1)\hat{\boldsymbol u}^+_i\hat{\boldsymbol z}_j\hat{\boldsymbol z}_k\,\{\,\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k\,\}_{\rm TS}\ ,$$

which is a little easier to manipulate, since $\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k$ is already symmetric. Eq. (3.5) of the problem set gives $N(3,1) = -\frac{5}{4}\sqrt{\frac{21}{2\pi}}$. The traceless symmetric part of $\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k$ is given by

$$\{\,\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k\,\}_{\rm TS} = \hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k - \frac{1}{5}\big(\hat{\boldsymbol n}_i\delta_{jk} + \hat{\boldsymbol n}_j\delta_{ik} + \hat{\boldsymbol n}_k\delta_{ij}\big)\ .$$

Then using $\sum_{i=1}^3 \hat{\boldsymbol n}_i\hat{\boldsymbol u}^+_i = \hat{\boldsymbol n}\cdot\hat{\boldsymbol u}^+ = \frac{1}{\sqrt2}\sin\theta e^{i\phi}$, $\sum_{i=1}^3 \hat{\boldsymbol n}_i\hat{\boldsymbol z}_i = \hat{\boldsymbol n}\cdot\hat{\boldsymbol z} = \cos\theta$, and $\sum_{i=1}^3 \hat{\boldsymbol u}^+_i\hat{\boldsymbol z}_i = \hat{\boldsymbol u}^+\cdot\hat{\boldsymbol z} = 0$, we can evaluate the equation for Y_{31}:

$$\begin{aligned}Y_{31}(\theta,\phi) &= -\frac{5}{4}\sqrt{\frac{21}{2\pi}}\frac{1}{\sqrt2}\sin\theta e^{i\phi}\left(\cos^2\theta - \frac{1}{5}\right) \\ &= \boxed{-\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta e^{i\phi}(5\cos^2\theta - 1)\ .}\end{aligned}$$

For those who did not use the hint to move the curly brackets to $\{\,\hat{\boldsymbol n}_i\hat{\boldsymbol n}_j\hat{\boldsymbol n}_k\,\}_{\rm TS}$, you would need to extract the traceless symmetric part of $\hat{\boldsymbol u}^+_i\hat{\boldsymbol z}_j\hat{\boldsymbol z}_k$:

$$\{\,\hat{\boldsymbol u}^+_i\hat{\boldsymbol z}_j\hat{\boldsymbol z}_k\,\}_{\rm TS} = \frac{1}{3}\left[\hat{\boldsymbol u}^+_i\left(\hat{\boldsymbol z}_j\hat{\boldsymbol z}_k - \frac{1}{5}\delta_{jk}\right) + \hat{\boldsymbol u}^+_j\left(\hat{\boldsymbol z}_i\hat{\boldsymbol z}_k - \frac{1}{5}\delta_{ik}\right) + \hat{\boldsymbol u}^+_k\left(\hat{\boldsymbol z}_i\hat{\boldsymbol z}_j - \frac{1}{5}\delta_{ij}\right)\right]\ .$$

(b) Since $\hat{\boldsymbol u}^+\cdot\hat{\boldsymbol u}^+ = 0$, the constructions such as $\hat{\boldsymbol u}^+_i\hat{\boldsymbol u}^+_j$ are traceless as well as symmetric. Therefore we can simply derive a general expression for $Y_{\ell\ell}$,

$$\begin{aligned}Y_{\ell\ell} &= N(\ell,\ell)\{\hat{\boldsymbol u}^+_{i_1}\hat{\boldsymbol u}^+_{i_2}...\hat{\boldsymbol u}^+_{i_\ell}\}\hat{\boldsymbol n}_{i_1}\hat{\boldsymbol n}_{i_2}\ldots\hat{\boldsymbol n}_{i_\ell} \\ &= N(\ell,\ell)(\hat{\boldsymbol u}^+_{i_1}\hat{\boldsymbol u}^+_{i_2}\ldots\hat{\boldsymbol u}^+_{i_\ell})\hat{\boldsymbol n}_{i_1}\hat{\boldsymbol n}_{i_2}\ldots\hat{\boldsymbol n}_{i_\ell} \\ &= \frac{N(\ell,\ell)}{2^{\ell/2}}\sin^\ell\theta e^{i\ell\phi} \\ &= \boxed{\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)(2\ell)!}{4\pi}}\sin^\ell\theta e^{i\ell\phi}\ .}\end{aligned}$$

PROBLEM 4: A SPHERE WITH OPPOSITELY CHARGED HEMISPHERES *(15 points)*

Griffiths Problem 3.23 (p. 150).

Given that the potential of the spherical shell should remain finite at both $r = 0$ and as $r \to \infty$, the general expansion of the solution to Laplace's equation must be restricted to

$$V(r,\theta) = \begin{cases}\displaystyle\sum_{\ell=0}^\infty A_\ell r^\ell P_\ell(\cos\theta)\,, & r \le R \\ \displaystyle\sum_{\ell=0}^\infty \frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta)\,, & r \ge R\ .\end{cases} \tag{4.1}$$

The potential must be continuous at $r = R$ (since the electric field is finite), which means that the coefficients in front of corresponding $P_\ell(\cos\theta)$ should be equal at $r = R$:

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad\Longrightarrow\quad B_\ell = A_\ell R^{2\ell+1}\ . \tag{4.2}$$

We can use the known distribution of the surface charge density $\sigma(\theta)$ by writing the boundary condition for the normal component of electric field on the surface of the sphere at $r = R$:

$$E_\perp|_{R^+} - E_\perp|_{R^-} = -\frac{\partial V(r,\theta)}{\partial r}\bigg|_{r=R^+} + \frac{\partial V(r,\theta)}{\partial r}\bigg|_{r=R^-} = \frac{\sigma(\theta)}{\epsilon_0}\,, \tag{4.3}$$

where

$$\sigma(\theta) = \begin{cases} \sigma_0, & \text{for } 0 < \theta < \frac{\pi}{2}, \\ -\sigma_0, & \text{for } \frac{\pi}{2} < \theta < \pi. \end{cases} \tag{4.4}$$

The discontinuity condition in Eq. (4.3) can then be rewritten using Eqs. (4.1) and (4.2), giving

$$\sum_\ell (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0} . \tag{4.5}$$

Now we can use the orthogonality of Legendre polynomials to find the values of A_ℓ:

$$\int_{-1}^{1} P_{\ell'}(x) P_\ell(x)\, \mathrm{d}x = \frac{2}{2\ell+1} \delta_{\ell'\ell} . \tag{4.6}$$

We multiply both sides of Eq. (4.5) by $P_{\ell'}(\cos\theta)\sin\theta$ and integrate over θ in $(0, \pi)$:

$$\sum_\ell (2\ell+1) A_\ell R^{\ell-1} \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta = \frac{1}{\epsilon_0} \int_0^\pi \sigma(\theta) P_{\ell'}(\cos\theta) \sin\theta d\theta . \tag{4.7}$$

which, thanks to the orthogonality of Legendre polynomials shown in Eq. (4.6), becomes:

$$A_\ell = \frac{1}{2\epsilon_0 R^{\ell-1}} \int_0^\pi \sigma(\theta) P_\ell(\cos\theta) \sin\theta d\theta . \tag{4.8}$$

We can simplify this expression by rewriting $\sigma(\theta)$ in terms of the sign function $\mathrm{sgn}(\cos\theta)$:

$$\sigma(\theta) = \sigma_0 \ \mathrm{sgn}(\cos\theta) , \quad \text{with} \quad \mathrm{sgn}(x) = \begin{cases} 1, & \text{for} \quad x > 0, \\ 0, & \text{for} \quad x = 0, \\ -1, & \text{for} \quad x < 0, \end{cases} \tag{4.9}$$

and then

$$A_\ell = \frac{\sigma_0}{2\epsilon_0 R^{\ell-1}} s_\ell , \tag{4.10}$$

where

$$s_\ell = \int_0^\pi \mathrm{sgn}(\cos\theta) P_\ell(\cos\theta) \sin\theta d\theta = \int_{-1}^{1} \mathrm{sgn}(x) P_\ell(x) dx . \tag{4.11}$$

Since the sign function $\mathrm{sgn}(x)$ is odd, the integral of $\mathrm{sgn}(x)P_\ell(x)$ over the symmetric region $[-1; 1]$ equals zero for even P_ℓ's:

$$s_0 = s_2 = s_4 = s_6 = 0 . \tag{4.12}$$

The odd s_ℓ's can be calculated directly:

$$\begin{aligned} s_1 &= \int_{-1}^{1} \mathrm{sgn}(x) P_1(x) dx = 2 \int_0^1 x\, dx = 1, \\ s_3 &= \int_{-1}^{1} \mathrm{sgn}(x) P_3(x) dx = 2 \int_0^1 \frac{1}{2}(5x^3 - 3x)\, dx = -\frac{1}{4}, \\ s_5 &= \int_{-1}^{1} \mathrm{sgn}(x) P_5(x) dx = 2 \int_0^1 \frac{1}{8}(63x^5 - 70x^3 + 15x)\, dx = \frac{1}{8}, \end{aligned} \tag{4.13}$$

which finally gives us the values of A_ℓ's:

$$A_1 = \frac{\sigma_0}{\epsilon_0}\left(\frac{1}{2}\right); \quad A_3 = \frac{\sigma_0}{\epsilon_0 R^2}\left(-\frac{1}{8}\right); \quad A_5 = \frac{\sigma_0}{\epsilon_0 R^4}\left(\frac{1}{16}\right) \tag{4.14}$$

and B_ℓ's (via Eq. (4.2)):

$$B_1 = \frac{\sigma_0}{\epsilon_0} R^3 \left(\frac{1}{2}\right); \quad B_3 = \frac{\sigma_0}{\epsilon_0} R^5 \left(-\frac{1}{8}\right); \quad B_5 = \frac{\sigma_0}{\epsilon_0} R^7 \left(\frac{1}{16}\right) . \tag{4.15}$$

Thus, the potential created by the spherical shell is:

$$\boxed{V(r,\theta) = \begin{cases} \dfrac{\sigma_0 r}{2\epsilon_0}\left[P_1(\cos\theta) - \dfrac{1}{4}\left(\dfrac{r}{R}\right)^2 P_3(\cos\theta) + \dfrac{1}{8}\left(\dfrac{r}{R}\right)^4 P_5(\cos\theta)\cdots\right], & r \le R \\ \dfrac{\sigma_0 R^3}{2\epsilon_0 r^2}\left[P_1(\cos\theta) - \dfrac{1}{4}\left(\dfrac{R}{r}\right)^2 P_3(\cos\theta) + \dfrac{1}{8}\left(\dfrac{R}{r}\right)^4 P_5(\cos\theta)\cdots\right], & r \ge R . \end{cases}} \tag{4.16}$$

Alternatively, the coefficients s_ℓ could be obtained from the following property of Legendre polynomials:

$$(2\ell+1) P_\ell = \frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} . \tag{4.17}$$

Integrating both sides, we get:

$$s_\ell = 2\int_0^1 P_\ell(x) dx = \frac{2}{2\ell+1}(P_{\ell+1} - P_{\ell-1})\Big|_0^1 = \frac{2}{2\ell+1}[P_{\ell-1}(0) - P_{\ell+1}(0)] , \tag{4.18}$$

where we used the fact that $P_{\ell+1}(1) = P_{\ell-1}(1) = 1$ for odd ℓ. The computation of the s_ℓ's requires the value of Legendre polynomials at $x = 0$. This could be most easily obtained from the generating function for Legendre polynomials:

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x) \quad \Longrightarrow \quad \frac{1}{\sqrt{1+h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(0) . \tag{4.19}$$

The Taylor expansion of the square root gives us:

$$\sum_{\ell=0}^{\infty} h^\ell P_\ell(0) = 1 - \frac{1}{2}h^2 + \frac{3}{8}h^4 - \frac{5}{16}h^6 + \frac{35}{128}h^8 + \cdots , \tag{4.20}$$

so

$$P_0(0) = 1, \quad P_2(0) = -\frac{1}{2}, \quad P_4(0) = \frac{3}{8}, \quad P_6(0) = -\frac{5}{16}. \tag{4.21}$$

Using these values of $P_\ell(0)$, we find s_ℓ's:

$$s_1 = 1\,, \quad s_3 = -\frac{1}{4}, \quad s_5 = \frac{1}{8}\,, \tag{4.22}$$

which agree with Eq. (4.13).

A derivation using traceless symmetric tensors would be essentially identical, using Eq. (1.1) to relate the Legendre polynomials to the azimuthally symmetric functions constructed from traceless symmetric tensors.

PROBLEM 5: A CIRCULAR DISK AT A FIXED POTENTIAL *(20 points extra credit)*

(a) [12 pts extra] We are given that

$$\sigma(s) = \frac{A}{\sqrt{R^2 - s^2}}\,, \tag{5.1}$$

where A is a constant of proportionality we must determine, given the information that the disk is at potential V_0. We now calculate the potential $V(z)$ on the z-axis by integration

$$V(z) = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{2\pi s\sigma(s)\,\mathrm{d}s}{\sqrt{s^2+z^2}} = \frac{A}{2\epsilon_0}\int_0^R \frac{s ds}{\sqrt{(z^2+s^2)(R^2-s^2)}}\,. \tag{5.2}$$

We can determine the constant A by evaluating this integral for $z = 0$, setting $V(0) = V_0$. The integral can be carried out by substituting $s = R\sin\theta$, so

$$V(0) = V_0 = \frac{A}{2\epsilon_0}\int_0^R \frac{\mathrm{d}s}{\sqrt{R^2-s^2}} = \frac{A}{2\epsilon_0}\int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi A}{4\epsilon_0}\,, \tag{5.3}$$

so

$$A = \frac{4\epsilon_0 V_0}{\pi} \tag{5.4}$$

and

$$V(z) = \frac{2V_0}{\pi}\int_0^R \frac{s ds}{\sqrt{(z^2+s^2)(R^2-s^2)}}\,. \tag{5.5}$$

There are two ways to proceed at this point: we can first expand in a power series in $1/z$, and then integrate, or we can first integrate and then expand in a power series. We will do it both ways.

Method 1: Expand and then Integrate:

To expand in powers of $1/z$, we write

$$\frac{1}{\sqrt{z^2+s^2}} = \frac{1}{z}\frac{1}{\sqrt{1+\frac{s^2}{z^2}}}\,. \tag{5.6}$$

We can expand $1/\sqrt{1+x}$ by writing

$$f(x) = \frac{1}{\sqrt{1+x}}\,, \tag{5.7}$$

which can be differentiated n times to give

$$f^{(n)}(x) = \frac{(-1)^n(2n-1)!!}{2^n}(1+x)^{-(2n+1)/2}\,, \tag{5.8}$$

where

$$(2n-1)!! \equiv (2n-1)(2n-3)(2n-5)\ldots 1 = \frac{(2n)!}{2^n n!}\,, \text{ with } (-1)!! \equiv 1\,. \tag{5.9}$$

Eq. (5.8) can easily be verified by induction. Thus, the Taylor series for $f(x)$ is

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}x^n\,. \tag{5.10}$$

Using the expansion (5.10) with Eq. (5.6) to replace $1/\sqrt{z^2+s^2}$ in Eq. (5.5), we find

$$V(z) = \frac{2V_0}{\pi z}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n\, n!\, z^{2n}}I_n(R)\,, \tag{5.11}$$

where

$$I_n(R) \equiv \int_0^R \frac{s^{2n+1}\,\mathrm{d}s}{\sqrt{R^2-s^2}}\,. \tag{5.12}$$

To carry out the integral, use the substitution

$$x = \frac{\sqrt{R^2 - s^2}}{R} , \tag{5.13}$$

which gives

$$I_n(R) = R^{2n+1}\tilde{I}_n \ , \ \text{where } \tilde{I}_n = \int_0^1 (1-x^2)^n \, \mathrm{d}x \ . \tag{5.14}$$

To evaluate $\tilde{I}_n$, one can integrate by parts to find that, for $n \geq 1$,

$$\tilde{I}_n = 2n \int_0^1 (1-x^2)^{n-1} x^2 \, \mathrm{d}x = -2n\tilde{I}_n + 2n\tilde{I}_{n-1} \quad \Longrightarrow \quad \tilde{I}_n = \frac{2n}{2n+1}\tilde{I}_{n-1} \ . \tag{5.15}$$

Combined with $\tilde{I}_0 = 1$, the recursion relation (5.15) leads to the result

$$\tilde{I}_n = \frac{2^n n!}{(2n+1)!!} , \tag{5.16}$$

which can be combined with Eqs. (5.11) and (5.14) to give

$$V(z) = \frac{2V_0 R}{\pi z} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{R^2}{z^2}\right)^n \ . \tag{5.17}$$

This solution is valid on the z-axis, and only for $z > R$, where the power series used in Eq. (5.10) converges. For $r > R$ the potential $V(\vec{r})$ obeys Laplace's equation, and we know that the most general possible solution with azimuthal symmetry can be written as

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta) \ . \tag{5.18}$$

For $r > R$ we must have $A_\ell = 0$, so that the potential is finite as $r \to \infty$. On the z-axis we can equate z and r, and we see from Eq. (5.18) that the Legendre polynomial that multiplies the n'th term, which is proportional to $1/z^{2n+1} = 1/r^{2n+1}$, is $P_{2n}(\cos\theta)$. That is, the index should be one less than the power of r in the denominator. Changing the summation variable to ℓ, which is more traditional in this context, we have

$$\boxed{V(\vec{r}) = \frac{2V_0 R}{\pi r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \left(\frac{R}{r}\right)^{2\ell} P_{2\ell}(\cos\theta) \ .} \tag{5.19}$$

Method 2: Integrate and then Expand:

You were of course expected to only do this problem one way, but both ways are worth looking at. Starting with Eq. (5.5), we can substitute $u = s^2$ to rewrite it as

$$V(z) = \frac{V_0}{\pi} \int_0^{R^2} \frac{\mathrm{d}u}{\sqrt{(z^2+u)(R^2-u)}} , \tag{5.20}$$

The integral can be carried out, probably with the help of a computer algebra program, to give

$$V(z) = \frac{2V_0}{\pi} \tan^{-1}\left(\sqrt{\frac{z^2+u}{R^2-u}}\right)\Bigg|_0^{R^2} \tag{5.21a}$$

$$= \frac{2V_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{z}{R}\right)\right] \tag{5.21b}$$

$$= \frac{2V_0}{\pi} \tan^{-1}\left(\frac{R}{z}\right) \ . \tag{5.21c}$$

where you can always check the expression by differentiating the indefinite integral shown in Eq. (5.21a). To expand $\tan^{-1} x$ in a power series, one can use a computer algebra program to find

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots , \tag{5.22}$$

where the pattern is obvious. If one nonetheless prefers a derivation to pattern recognition, one can begin by writing

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2} , \tag{5.23}$$

but $1/(1+x^2)$ has a well-known and easily verified power series,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \ . \tag{5.24}$$

We can then integrate term by term, fixing the constant of integration by $\tan^{-1}(0) = 0$, so

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \ . \tag{5.25}$$

Thus we have

$$V(z) = \frac{2V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{R}{z}\right)^{2n+1} , \tag{5.26}$$

in agreement with Eq. (5.17). The argument to justify the angular dependence shown in Eq. (5.19) is the same as above.

(b) [8 pts extra] The total charge on the disk is calculated as

$$Q = \int_0^R 2\pi s \sigma(s) \mathrm{d}s = 2\pi A \int_0^R \frac{s ds}{\sqrt{R^2 - s^2}} . \tag{5.27}$$

The integral can be carried out with the same substitution $s = R \sin\theta$ that was used in Eq. (5.3), giving

$$Q = 2\pi AR \int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta = 2\pi AR = 8\epsilon_0 RV_0 , \tag{5.28}$$

where we used Eq. (5.4) for A. Thus, $Q = CV$ implies that

$$C = 8\epsilon_0 R . \tag{5.29}$$

Further Discussion: The Potential for $r < R$:

You were not asked to discuss the fields for $r < R$, but I want to include a discussion here for completeness. The connection between the fields for $r < R$ and the fields for $r > R$ is subtle, and when I taught this course in 2012 there was a web posting from another university which claimed that Jackson's answer is wrong. When I contacted that professor, however, he agreed that Jackson is right, and his posting has since been taken down.*

* Although correct, Jackson's argument is certainly not complete, since Eq. (5.1) for the charge density was assumed and not justified. A very clever argument justifying this charge density has been posted by Kirk T. McDonald at http://www.physics.princeton.edu/~mcdonald/examples/ellipsoid.pdf. The argument was first published by William Thomson Kelvin (Lord Kelvin) in **Reprint of Papers on Electrostatics and Magnetism,** 1884, pp. 7 and 178–179, in an article internally dated as "Jan. 1869, not hitherto published". McDonald's article includes a Princeton-only link to Kelvin's book, but it can also be accessed through Google books, with a web interface at https://books.google.com/books?hl=en&lr=&id=rTu4AAAAIAAJ&oi=fnd&pg=PA1&dq=Papers+on+Electricity+and+Magnetism+Kelvin&ots=0L_Q8Wnglr&sig=fMUEoJ726WszKMcu3y6PVpUxyms#v=onepage&q=Papers%20on%20Electricity%20and%20Magnetism%20Kelvin&f=false and a PDF link at https://books.google.com/books/download/Reprint_of_Papers_on_Electrostatics_and.pdf?id=rTu4AAAAIAAJ&output=pdf&sig=ACfU3U1hUcAZFcHfytWujcM1sSQx9-19lw. Note that McDonald's eq. (26) has a sign error.

The complication for $r < R$ is the fact that Laplace's equation does not hold throughout the region, but is violated at $z = 0$, on the conducting disk, where there is a nonzero surface charge density. This is handled by considering separately the two disconnected regions, $z > 0$ and $z < 0$, within which Laplace's equation holds. These regions must be treated separately, although of course they are related by a $z \to -z$ symmetry. But since they are treated separately, we are dealing (for $z > 0$) with values of $\cos\theta$ that extend from 0 to 1, rather than the usual -1 to 1. We will still use Legendre polynomials, but we need to remember that these polynomials obey an orthogonality relation when integrated over the full range -1 to 1, but not when integrated over the restricted range used here. This means that the expansion in Legendre polynomials will not be uniquely determined, as it is in more standard cases.

One way of finding an expression for $V(\vec{r})$ for $r < R$ is to proceed the same way that we did for $r > R$, starting with a power series in z that is valid on the z-axis. This time we will be interested in small z rather than large z, so we will want a power series in z instead of a power series in $1/z$. There is no simple way to do this by Method 1 above, since the integral over s starts at $s = 0$. Hence there will always be parts of the integration for which $z \gg s$, so we cannot treat z/s as a small quantity. But the integral we found in Method 2 is still valid, so we can use Eqs. (5.21b) and (5.25) to obtain a small z expansion:

$$V(z) = V_0 - \frac{2V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z}{R}\right)^{2n+1} . \tag{5.30}$$

We can extend this expression to a function of both θ and r by using the general solution of Eq. (5.18), where this time we insist that $B_\ell = 0$, so that $V(\vec{r})$ remains finite at the origin. We find that for $\cos\theta > 0$,

$$V(\vec{r}) = V_0 - \frac{2V_0}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \left(\frac{r}{R}\right)^{2\ell+1} P_{2\ell+1}(\cos\theta) \qquad (\text{if } \cos\theta > 0) . \tag{5.31}$$

Thus, even though we used only even Legendre polynomials for the solution for $r > R$, we are using only odd Legendre polynomials for $r < R$. One might think that this means that this solution cannot be continuous at $r = R$, but here is where it is important that Eq. (5.31) holds only the range $0 < \cos\theta < 1$, which is too small a range for orthogonality relations to apply. If we had claimed that Eq. (5.19) and Eq. (5.31) agreed at $r = R$ for all θ, we would have a contradiction, because the two would be manifestly orthogonal. Eq. (5.19) holds for all θ, but for $r < R$ and $\cos\theta < 0$ we must repeat the calculation. The $z \to -z$ symmetry immediately tells us what we will find: the potential must be even in θ, and therefore for $\cos\theta < 0$,

$$V(\vec{r}) = V_0 + \frac{2V_0}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \left(\frac{r}{R}\right)^{2\ell+1} P_{2\ell+1}(\cos\theta) \qquad (\text{if } \cos\theta < 0) . \tag{5.32}$$

We will come back to the issue of matching at $r = R$, but let us first check that Eqs. (5.31) and (5.32) give the correct electric field just above and below the $z = 0$ plane. For $z = 0+$, Eq. (5.31) applies, and E_z is given by

$$E_z(r, z{=}0+) = -\frac{\partial V}{\partial z} = -\frac{1}{r}\frac{\partial V}{\partial \cos\theta} = \frac{2V_0}{\pi R}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{r}{R}\right)^{2\ell} P'_{2\ell+1}(0) \ . \tag{5.33}$$

To find a formula for $P'_{2\ell+1}(0)$, we can use the generating function for the Legendre polynomials:

$$\frac{1}{\sqrt{1+\lambda^2-2\lambda x}} = \sum_{\ell=0}^{\infty}\lambda^\ell P_\ell(x) \ . \tag{5.34}$$

By differentiating both sides with respect to x and then setting $x = 0$, we find

$$\frac{\lambda}{(1+\lambda^2)^{3/2}} = \sum_{\ell=0}^{\infty}\lambda^\ell P'_\ell(0) \ . \tag{5.35}$$

Expanding the left-hand side of the above equation in a power series, we learn that for odd Legendre polynomials,

$$P'_{2\ell+1}(0) = \frac{(-1)^\ell(2\ell+1)!!}{2^\ell \ell!} \ , \tag{5.36}$$

and therefore

$$E_z(r, z{=}0+) = \frac{2V_0}{\pi R}\sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{2^\ell \ell!}\left(\frac{r}{R}\right)^{2\ell} \ . \tag{5.37}$$

It is easily checked that this is exactly the power series expansion of

$$E_z(r, z{=}0+) = \frac{2V_0}{\pi}\frac{1}{\sqrt{R^2-r^2}} = \frac{\sigma(r)}{2\epsilon_0} \ , \tag{5.38}$$

which is exactly what we expect for the surface charge density given by Eqs. (5.1) and (5.4).

Now we return to the question of checking the matching at $r = R$. If the expression in Eq. (5.19) for $r > R$ is to match the expression in Eq. (5.31) for $r < R$, we require

$$\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left[P_{2\ell}(x) + P_{2\ell+1}(x)\right] = \frac{\pi}{2} \qquad (\text{for } 0 < x < 1) \ . \tag{5.39}$$

I have not been able to prove this conjectured identity, but I have written a computer program to test it numerically. Evaluating the Legendre polynomials by a recursive procedure from **Numerical Recipes,** 3rd Edition, by W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, and summing up to $\ell = 1{,}000{,}000$ in extended precision (10-byte real numbers), I found that on a grid of x from 0.01 to 0.99, in steps of 0.01, the identity consistently holds to better than 8 significant figures. At $x = 0$ on the boundary of the expected range of convergence, I found only 3 significant figures of agreement; at $x = 1$ I found 6 significant figures of agreement. So, I'm convinced that Eq. (5.39) is valid, so the solution we have constructed satisfies all the relevant tests.