MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 23, 2019
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PROBLEM SET 8 SOLUTIONS

PROBLEM 1: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

Griffiths Problem 5.60 (p. 264).

(a) A uniformly charged solid sphere of radius R carries a total charge Q, hence it has charge density $\rho = Q/(\frac{4}{3}\pi R^3)$. To find the magnetic moment of sphere we can divide the sphere into infinitesimal charges. Using spherical polar coordinates, we can take $\mathrm{d}q = \rho\,\mathrm{d}\tau = \rho\, r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, with the contribution to the dipole moment given by $d\vec{\boldsymbol{m}} = \frac{1}{2}\vec{\boldsymbol{r}}\times\vec{\boldsymbol{J}}\,\mathrm{d}\tau$. One method would be to write down the volume integral directly, using $\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}} = \rho\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}$. We can, however, integrate over ϕ before we start, so we are breaking the sphere into rings, where a given ring is indicated by its coordinates r and θ, and its size $\mathrm{d}r$ and $\mathrm{d}\theta$. The volume of each ring is $\mathrm{d}\tau = 2\pi r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta$. The current $\mathrm{d}I$ in the ring is given by dq/T, where $T = 2\pi/\omega$ is the period, so

$$\mathrm{d}I = \frac{\mathrm{d}q}{T} = \frac{\omega\rho\,\mathrm{d}\tau}{2\pi} = \omega\rho r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\ . \tag{1.1}$$

The magnetic dipole moment of each ring is then given by

$$\mathrm{d}\vec{\boldsymbol{m}}_{\text{ring}} = \frac{1}{2}\int_{\text{ring}}\vec{\boldsymbol{r}}\times\vec{\boldsymbol{J}}\,\mathrm{d}\tau = \frac{1}{2}\mathrm{d}I\int_{\text{ring}}\vec{\boldsymbol{r}}\times d\vec{\boldsymbol{\ell}} = \mathrm{d}I(\pi r^2\sin^2\theta)\,\hat{\boldsymbol{z}}\ . \tag{1.2}$$

The total magnetic dipole moment is then

$$\begin{aligned}\vec{\boldsymbol{m}} &= \int\omega\rho r^2\sin\theta\,(\pi r^2\sin^2\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\hat{\boldsymbol{z}}\\ &= \pi\omega\rho\int_0^R r^4\,\mathrm{d}r\int_0^\pi(1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\hat{\boldsymbol{z}}\\ &= \pi\omega\frac{Q}{\frac{4}{3}\pi R^3}\frac{R^5}{5}\frac{4}{3} = \boxed{\frac{1}{5}Q\omega R^2\,\hat{\boldsymbol{z}}\ .}\end{aligned} \tag{1.3}$$

(b) Griffiths Eq. (5.93) gives the average field inside the sphere as,

$$\vec{\boldsymbol{B}}_{\text{avg}} = \frac{\mu_0}{4\pi}\frac{2\vec{\boldsymbol{m}}}{R^3} = \boxed{\frac{\mu_0}{4\pi}\frac{2}{5}\frac{Q\omega}{R}\,\hat{\boldsymbol{z}}\ .} \tag{1.4}$$

(c) The vector potential in dipole approximation is,

$$\vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{m}}\times\vec{\boldsymbol{r}}}{r^3} = \frac{\mu_0}{4\pi}\frac{|\vec{\boldsymbol{m}}|\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0}{4\pi}\frac{Q\omega R^2}{5}\frac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}\ .} \tag{1.4}$$

(d) To calculate the exact vector potential outside the sphere, we split the sphere into shells. Let r' be the integration variable and the radius of a shell, moreover let dr' denote the thickness of the shell. Then we can use the results of Example 5.11 (pp. 245-47) in Griffiths, if we replace σ by its value for this case. The value of σ is found equating charges

$$\sigma(4\pi r'^2) = \frac{Q}{\frac{4}{3}\pi R^3}(4\pi r'^2)dr' \tag{1.5}$$

and therefore we must replace

$$\sigma \to \frac{Q}{\frac{4}{3}\pi R^3}dr'\ .$$

Making this replacement in Griffiths' Eq. (5.69) we now have

$$\mathrm{d}A_\phi(r,\theta,\phi) = \frac{Q}{\frac{4}{3}\pi R^3}dr'\,\frac{\mu_0\omega}{3}\sin\theta\begin{cases} r'r & \text{if } r<r' \\ \dfrac{r'^4}{r^2} & \text{if } r>r'\ .\end{cases} \tag{1.6}$$

Note that the R of Griffiths has been replaced by r', which is the radius of the integration shell. For $r > R$ the quantity r is always bigger than the integration variable, so we use the second form in Eq. (1.6),

$$A_\phi(r,\theta,\phi) = \frac{Q}{\frac{4}{3}\pi R^3}\frac{\mu_0\omega}{3}\frac{\sin\theta}{r^2}\int_0^R r'^4 dr' = \frac{\mu_0}{4\pi}\frac{Q\omega R^2}{5}\frac{\sin\theta}{r^2} = \boxed{\frac{\mu_0}{4\pi}\frac{|\vec{\boldsymbol{m}}|\sin\theta}{r^2}\ .} \tag{1.7}$$

This shows that the vector potential outside the sphere is exactly that of a magnetic dipole with magnetic moment equal to that calculated before.

(e) We first calculate the vector potential inside the sphere at some radius $r < R$. This time the integration will require two pieces, a piece where $0 < r' < r$ and the other where $r < r' < R$, thus using the two options in Eq. (1.6):

$$A_\phi(r,\theta,\phi) = \frac{\mu_0}{4\pi}\frac{Q\omega}{R^3}\sin\theta\left[\int_0^r dr'\frac{r'^4}{r^2} + \int_r^R dr' rr'\right]\ . \tag{1.8}$$

Doing the integrals one finds

$$A_\phi(r,\theta,\phi) = \frac{\mu_0}{4\pi}\frac{Q\omega}{R^3}\sin\theta\Big[-\frac{3r^3}{10} + \frac{rR^2}{2}\Big]. \tag{1.9}$$

To find the magnetic field we use

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\,\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\boldsymbol{\theta}}\,. \tag{1.10}$$

The calculation, using Eqs. (1.9) and (1.10), gives

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{Q\omega}{R^3}\Big[-\frac{3}{5}r^2\cos\theta\,\hat{\boldsymbol{r}} + \frac{6}{5}r^2\sin\theta\,\hat{\boldsymbol{\theta}} + R^2\hat{\boldsymbol{z}}\Big]\,, \tag{1.11}$$

where we used the relation $\hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta = \hat{\boldsymbol{z}}$. This is the exact magnetic field inside the sphere, it is not constant! To find its average, we note, by symmetry, that the average is going to be in the z direction. We can therefore replace $\hat{\boldsymbol{r}} \to \cos\theta\,\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\theta}} \to -\sin\theta\,\hat{\boldsymbol{z}}$. Doing this in Eq. (1.11) we have that

$$\vec{\boldsymbol{B}}_{\rm avg} = \frac{\mu_0}{4\pi}\frac{Q\omega}{R^3}\Big[-\frac{3}{5}\langle r^2\cos^2\theta\rangle - \frac{6}{5}\langle r^2\sin^2\theta\rangle + R^2\Big]\,\hat{\boldsymbol{z}}\,, \tag{1.12}$$

where we have used $\langle\cdots\rangle$ to denote averages over a sphere.

$$\begin{aligned}\langle r^2\cos^2\theta\rangle &= \frac{1}{\frac{4\pi}{3}R^3}\int_{\rm sphere} r^2\cos^2\theta\, r^2\sin\theta\,{\rm d}\theta\,{\rm d}\phi\,{\rm d}r\\ &= \frac{1}{\frac{4\pi}{3}R^3}\,\frac{R^5}{5}\,2\pi\,\frac{2}{3} = \frac{R^2}{5}\,,\end{aligned}$$

and similarly $\langle r^2\sin^2\theta\rangle = \frac{2}{5}R^2$. (One might note that r and θ are uncorrelated, so $\langle r^2\cos^2\theta\rangle = \langle r^2\rangle\langle\cos^2\theta\rangle$, with $\langle r^2\rangle = \frac{3}{5}R^2$, $\langle\sin^2\theta\rangle = \frac{2}{3}$, and $\langle\cos^2\theta\rangle = \frac{1}{3}$.) Thus we get

$$\begin{aligned}\vec{\boldsymbol{B}}_{\rm avg} &= \frac{\mu_0}{4\pi}\frac{Q\omega}{R^3}\left[-\frac{3}{5}\left(\frac{3R^2}{5}\right)\left(\frac{1}{3}\right) - \frac{6}{5}\left(\frac{3R^2}{5}\right)\left(\frac{2}{3}\right) + R^2\right]\hat{\boldsymbol{z}}\\ &= \boxed{\frac{\mu_0}{4\pi}\frac{Q\omega}{R}\,\frac{2}{5}\hat{\boldsymbol{z}}\,,}\end{aligned} \tag{1.13}$$

which confirms the result in part (b).

PROBLEM 2: THE COMPLETE MAGNETIC FIELD OF A MAGNETIC DIPOLE: *(10 points)*

This problem explores the identity

$$\boxed{\partial_i\partial_j\left(\frac{1}{r}\right) = -\partial_i\left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i\left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{\boldsymbol{r}})\,.} \tag{2.1}$$

Starting from the vector potential of a magnetic dipole,

$$\vec{\boldsymbol{A}}_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{m}}\times\hat{\boldsymbol{r}}}{r^2}\,, \tag{2.2}$$

we find using index notation that

$$\begin{aligned}B_{{\rm dip},i} = \epsilon_{ijk}\partial_j A_k &= \frac{\mu_0}{4\pi}\epsilon_{ijk}\partial_j\left(\frac{\vec{\boldsymbol{m}}\times\hat{\boldsymbol{r}}}{r^2}\right)_k = \frac{\mu_0}{4\pi}\epsilon_{ijk}\partial_j\left(\frac{\epsilon_{k\ell m}m_\ell\hat{\boldsymbol{r}}_m}{r^2}\right)\\ &= \frac{\mu_0 m_\ell}{4\pi}\epsilon_{kij}\epsilon_{k\ell m}\partial_j\left(\frac{\hat{\boldsymbol{r}}_m}{r^2}\right)\,.\end{aligned} \tag{2.3}$$

Now we use the identity

$$\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \tag{2.4}$$

as well as the identity (2.1) to give

$$\begin{aligned}B_{{\rm dip},i} &= \frac{\mu_0 m_\ell}{4\pi}(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\left[\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} + \frac{4\pi}{3}\,\delta_{jm}\,\delta^3(\vec{\boldsymbol{r}})\right]\\ &= \frac{\mu_0}{4\pi}(m_i\delta_{jm} - m_j\delta_{im})\left[\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} + \frac{4\pi}{3}\,\delta_{jm}\,\delta^3(\vec{\boldsymbol{r}})\right]\,.\end{aligned} \tag{2.5}$$

Before we continue, note that

$$\delta_{jm}\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} = \frac{3-3}{r^3} = 0\,. \tag{2.6}$$

Thus,

$$\begin{aligned}B_{{\rm dip},i} &= \frac{\mu_0}{4\pi}\left[-\frac{m_i - 3(m_j\hat{\boldsymbol{r}}_j)\hat{\boldsymbol{r}}_i}{r^3} + \frac{4\pi}{3}\,(3m_i - m_i)\,\delta^3(\vec{\boldsymbol{r}})\right]\\ &= \frac{\mu_0}{4\pi}\frac{3(m_j\hat{\boldsymbol{r}}_j)\hat{\boldsymbol{r}}_i - m_i}{r^3} + \frac{2\mu_0}{3}m_i\delta^3(\vec{\boldsymbol{r}})\,.\end{aligned} \tag{2.7}$$

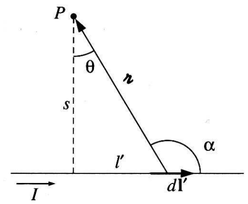
In vector notation, this becomes

$$\boxed{\vec{\boldsymbol{B}}_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{r^3} + \frac{2\mu_0}{3}\vec{\boldsymbol{m}}\,\delta^3(\vec{\boldsymbol{r}})\,.} \tag{2.8}$$

PROBLEM 3: SQUARE CURRENT LOOP ON AXIS: BIOT-SAVART AND THE MAGNETIC DIPOLE APPROXIMATION *(15 points)*

Griffiths Problem 5.36 (p. 255).

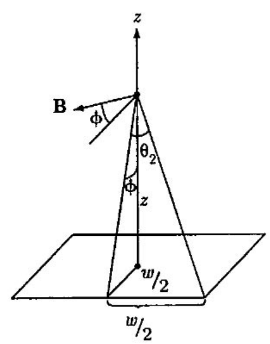
The magnetic field that a finite current carrying line segment produces at a point P at a distance s from the line was found in Problem 2 in of this problem set as



$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1) \;, \tag{3.1}$$

where θ_1 and θ_2 are the angles, as seen from P, of the endpoints of the line, measured from the line through P which intersects the segment at a right angle, as shown in the diagram at the right. The direction is perpendicular to the plane including the line segment and the point P.

For this case s is the distance from the point $(0,0,z)$ to the center of one side of the square, $s = \sqrt{z^2+(w/2)^2}$, and $\sin\theta_2 = -\sin\theta_1 = \frac{(w/2)}{\sqrt{z^2+(w/\sqrt{2})^2}}$, as shown in the diagram at the right. (Note that $w/\sqrt{2}$ is the distance from the center of the square to one corner.) The magnetic field from one side of the square then has magnitude



$$B = \frac{\mu_0 I}{4\pi}\frac{w}{\sqrt{\left(z^2+\frac{w^2}{2}\right)\left(z^2+\frac{w^2}{4}\right)}} \;. \tag{3.2}$$

For all four sides, we will pick the vertical components by multiplying by $\sin\phi = \frac{(w/2)}{\sqrt{z^2+(w/2)^2}}$. The contribution of all four sides is then

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi}\frac{w^2}{\left(z^2+\frac{w^2}{4}\right)\sqrt{z^2+\frac{w^2}{2}}}\,\hat{z} \;.} \tag{3.3}$$

(A useful check is to evaluate this answer for $z=0$, comparing with the answer to Problem 2(a) of this problem set: $B_{\rm tot} = \frac{\sqrt{2}\mu_0 I}{\pi R}$. Using $R = w/2$, we can see that they agree.) For $z \gg w$,

$$\vec{B} \approx \frac{\mu_0 I w^2}{2\pi z^3}\hat{z} = \frac{\mu_0 m}{2\pi z^3}\hat{z}, \tag{3.4}$$

where $\vec{m} = m\hat{z}$, and $m = Iw^2$ is the dipole moment of a square loop. The magnetic field of an equivalent dipole on the z-axis is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}}{z^3} = \frac{\mu_0\vec{m}}{2\pi z^3} \;, \tag{3.5}$$

so they agree.

PROBLEM 4: A BAR MAGNET IN THE SHAPE OF A RIGHT CIRCULAR CYLINDER *(20 points)*

(a) We can find the bound currents and then the magnetic fields. The permanent magnetization $\vec{M}_0$ results in $\vec{K}_b = \vec{M}_0 \times \hat{n} = M_0\hat{\phi}$ and $\vec{J}_b = \vec{\nabla}\times\vec{M}_0 = 0$. We can divide the cylinder into infinitesimal rings of $dI = M_0 dz$, and the contribution for each ring to the magnetic field on the axis at $z = z_0$ can be found using the Biot-Savart law, as given by Griffiths' Eq. (5.41) (p. 227):

$$dB_z = \frac{\mu_0 M_0 dz}{2}\frac{a^2}{[a^2+(z_0-z)^2]^{3/2}} \;. \tag{4.1}$$

The total magnetic field at $z = z_0$ is then

$$B_z = \frac{\mu_0 M_0}{2}\int_{-L/2}^{L/2}\frac{a^2}{[a^2+(z-z_0)^2]^{3/2}}dz \;. \tag{4.2}$$

The integral can be carried out by the change of variable $z - z_0 = a\tan\theta$, and $dz = a\dfrac{d\theta}{\cos^2\theta}$, giving

$$\int\frac{a^2}{[a^2+(z-z_0)^2]^{3/2}}dz = \int\frac{a^2\cos^3\theta}{a^3}\frac{a\,d\theta}{\cos^2\theta} = \int\cos\theta\, d\theta = \sin\theta \;. \tag{4.3}$$

Using $\sin\theta = \frac{z-z_0}{\sqrt{(z-z_0)^2+a^2}}$ and making use of the limits of integration,

$$\vec{B} = \frac{\mu_0 M_0}{2}\left[\frac{L/2-z_0}{\sqrt{(L/2-z_0)^2+a^2}} - \frac{-L/2-z_0}{\sqrt{(L/2+z_0)^2+a^2}}\right]\hat{z} \;. \tag{4.4}$$

Thus,

$$\vec{B}(z) = \boxed{-\frac{\mu_0 M_0}{2}\left[\frac{z-L/2}{\sqrt{(z-L/2)^2+a^2}} - \frac{z+L/2}{\sqrt{(z+L/2)^2+a^2}}\right]\hat{z} \;.} \tag{4.5}$$

To calculate $\vec{H}$, we use $\vec{H}(z) = \vec{B}(z)/\mu_0 - \vec{M}_0(z)$, where $\vec{M}_0(z)$ can be written in terms of the step function $\Theta(x)$, where $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ otherwise. Then $\vec{M}_0(z) = M_0\Theta(L/2 - |z|)\hat{z}$, and

$$\boxed{\vec{H}(z) = -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{(z-L/2)^2 + a^2}} - \frac{z + L/2}{\sqrt{(z+L/2)^2 + a^2}} + 2\Theta(L/2 - |z|)\right]\hat{z}\ .}$$

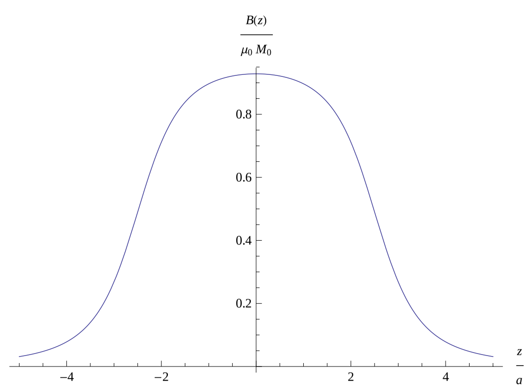
(b) Let $z/a \equiv x$ and $L/a \equiv 2b$, then $\vec{B}$ on the axis becomes

$$B_z(x) = -\frac{\mu_0 M_0}{2}\left[\frac{x-b}{\sqrt{(x-b)^2+1}} - \frac{x+b}{\sqrt{(x+b)^2+1}}\right]\ , \tag{4.5}$$

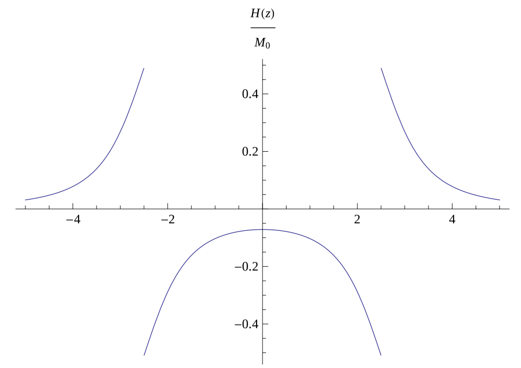
and $\vec{H}$ becomes

$$H_z(x) = -\frac{M_0}{2}\left[\frac{x-b}{\sqrt{(x-b)^2+1}} - \frac{x+b}{\sqrt{(x+b)^2+1}} + 2\Theta(b - |x|)\right], \tag{4.6}$$

For $L/a = 5$, the we can plot $B_z(z)/\mu_0 M_0$ as



and $H_z(z)/M_0$ as



(c) The bound currents were found in part (a): $\vec{K}_b = \vec{M}_0 \times \hat{n} = M_0\hat{\phi}$ and $\vec{J}_b = \vec{\nabla} \times \vec{M}_0 = 0$.

(d) The field far away from the magnet $x \gg b$ is approximately

$$\begin{aligned} B_z &= -\frac{\mu_0 M_0}{2}\left[\frac{1}{\sqrt{(1 + \frac{1}{(x-b)^2}}} - \frac{1}{\sqrt{1 + \frac{1}{(x+b)^2}}}\right] \\ &\approx -\frac{\mu_0 M_0}{2}\left[\left(1 - \frac{1}{2(x-b)^2}\right) - \left(1 - \frac{1}{2(x+b)^2}\right)\right] = \frac{\mu_0 M_0}{4}\left[\frac{1}{(x-b)^2} - \frac{1}{(x+b)^2}\right] \\ &\approx \frac{\mu_0 M_0 b}{x^3} = \frac{\mu_0 M_0 L a^2}{2z^3}\ . \end{aligned} \tag{4.7}$$

The magnetic field of a dipole moment $\vec{m} = m\hat{z}$ for points on the z-axis is

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{z})\hat{z} - \vec{m}}{z^3} = \frac{\mu_0\vec{m}}{2\pi z^3}. \tag{4.8}$$

Comparing Eq. (4.7) and Eq. (4.8) we can get $|\vec{m}| = M_0 L\pi a^2$. We could also find the same result by integrating $dm = M_0 dz(\pi a^2)$, or by multiplying $\vec{M} = M_0\hat{z}$ by the volume of the cylinder.

PROBLEM 5: CURRENT TRAVELING ON A LONG STRAIGHT WIRE MADE OF A MATERIAL WITH LINEAR MAGNETIZATION *(15 points)*

Griffiths Problem 6.17 (p. 287).

The current is uniformly distributed with $J_f = I/(\pi a^2)$. We choose a coordinate system so that the wire runs along the z-axis, and we take advantage of the cylindrical symmetry to apply Ampère's law to a circular loop of radius s about the z-axis:

$$\oint \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{\ell}} = H_\phi(2\pi s) = I_{f,\text{enc}} \ , \tag{5.1}$$

where $I_{f,\text{enc}} = I(s^2/a^2)$ for $s < a$, and $I_{f,\text{enc}} = I$ for $s > a$. Then we find

$$H_\phi = \begin{cases} \dfrac{Is}{2\pi a^2} & \text{if } s < a \ , \\ \dfrac{I}{2\pi s} & \text{if } s > a \ . \end{cases} \tag{5.2}$$

The magnetic field $\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}}$, where $\mu = \mu_0(1+\chi_m)$, so

$$B_\phi = \begin{cases} \dfrac{\mu_0(1+\chi_m)Is}{2\pi a^2} & \text{if } s < a \ , \\ \dfrac{\mu_0 I}{2\pi s} & \text{if } s > a \ . \end{cases} \tag{5.3}$$

The bound current density is then given by

$$\vec{\boldsymbol{J}}_b = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}} = \vec{\boldsymbol{\nabla}} \times (\chi_m \vec{\boldsymbol{H}}) = \chi_m \vec{\boldsymbol{J}}_f \quad \Longrightarrow \quad \boxed{\vec{\boldsymbol{J}}_b = \frac{\chi_m I}{\pi a^2}\,\hat{\boldsymbol{z}} \ .} \tag{5.4}$$

The bound surface current density is

$$\vec{\boldsymbol{K}}_b = \vec{\boldsymbol{M}} \times \hat{\boldsymbol{n}} = \chi_m \vec{\boldsymbol{H}} \times \hat{\boldsymbol{n}} \quad \Longrightarrow \quad \boxed{\vec{\boldsymbol{K}}_b = -\frac{\chi_m I}{2\pi a}\,\hat{\boldsymbol{z}} \ .} \tag{5.5}$$

The net bound current flowing is then

$$I_b = J_b(\pi a^2) + K_b(2\pi a) = \chi_m I - \chi_m I = 0 \ , \tag{5.6}$$

as it must be. The bound current is constructed from current loops, so it cannot give a net flow of current through the wire.

PROBLEM 6: BAR SLIDING ON TWO RAILS IN A UNIFORM MAGNETIC FIELD *(20 points)*

Griffiths Problem 7.7 (p. 310).
This problem is being held over to Problem Set 9, so it is no longer part of Problem Set 8.

PROBLEM 7: VECTOR POTENTIAL FOR A UNIFORM $\vec{B}$ FIELD *(10 points extra credit)*

Griffiths Problem 5.25 (p. 248).

Let's first check $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$:

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} &= -\frac{1}{2}\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}), \\ &= -\frac{1}{2}\left[\vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{r}}) - \vec{\boldsymbol{r}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}})\right] = 0 \ , \end{aligned} \tag{7.1}$$

where the first term vanishes because the curl of a radial vector field is zero, $(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{r}}) = 0$, and the second term vanishes because the magnetic field $\vec{\boldsymbol{B}}$ is uniform. Then

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} &= -\frac{1}{2}\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}}) \\ &= -\frac{1}{2}\left[(\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{r}} - (\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + \vec{\boldsymbol{r}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{r}})\right] \ . \end{aligned} \tag{7.2}$$

The terms involving the partial derivative of the magnetic field give zero, $(\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} = 0$, and $\vec{\boldsymbol{r}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}) = 0$. The first term in Eq. (7.2) is

$$\begin{aligned} (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{r}} &= \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)(x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z) \\ &= B_x\hat{\boldsymbol{e}}_x + B_y\hat{\boldsymbol{e}}_y + B_z\hat{\boldsymbol{e}}_z = \vec{\boldsymbol{B}} \ . \end{aligned} \tag{7.3}$$

The last term in Eq (7.2) is,

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{r}}) = \vec{\boldsymbol{B}}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\vec{\boldsymbol{B}} \ . \tag{7.4}$$

Then we get

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = -\frac{1}{2}(\vec{\boldsymbol{B}} - 3\vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}} \ . \tag{7.5}$$

Is this result unique? In localized situations, where we can impose the boundary condition that $|\vec{\boldsymbol{A}}| \to \vec{\boldsymbol{0}}$ as $|\vec{\boldsymbol{r}}| \to \infty$, the conditions $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}$ have a unique solution. In this case, however, with the $\vec{\boldsymbol{B}}$ field extending to infinity, no such boundary condition is possible, so $\vec{\boldsymbol{A}}$ is not unique. One possible change would be to add any constant vector. Bigger differences are possible, too. If $\vec{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}}$, then $\vec{\boldsymbol{A}} = B_0 x\hat{\boldsymbol{y}}$ or $\vec{\boldsymbol{A}} = -B_0 y\hat{\boldsymbol{x}}$ both work.

PROBLEM 8: DONUT-SHAPED MAGNETS ON A VERTICAL ROD *(15 points)*

Griffiths Problem 6.2 (p. 293).
This problem is being held over to Problem Set 9, so it is no longer part of Problem Set 8.

PROBLEM 9: THE MAGNETIC DIPOLE MOMENT OF A CURRENT LOOP *(10 points extra credit)*

We start with the definition of a magnetic dipole moment:

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell}\,.$$

The i^{th} component of the dipole moment $\vec{m}$ can be written as

$$m_i = \frac{1}{2} I \int_P \hat{e}_i \cdot (\vec{r} \times \mathrm{d}\vec{\ell})\,.$$

Using the property that the triple product is invariant under cyclic permutations, we change this to

$$m_i = \frac{1}{2} I \int_P \mathrm{d}\vec{\ell} \cdot (\hat{e}_i \times \vec{r})\,.$$

Then we use Stokes' theorem to convert the integral over the loop P to an integral over an arbitrary surface S enclosed by the loop P:

$$m_i = \frac{1}{2} I \int_S \mathrm{d}\vec{a} \cdot \vec{\nabla} \times (\hat{e}_i \times \vec{r})\,.$$

The expression under the integral can be expanded using the rule for the curl of a product of vector fields:

$$\vec{\nabla} \times (\hat{e}_i \times \vec{r}) = \hat{e}_i (\vec{\nabla} \cdot \vec{r}) - (\hat{e}_i \cdot \vec{\nabla})\vec{r}\ ,$$

where

$$\vec{\nabla} \cdot \vec{r} = 3$$

$$(\hat{e}_i \cdot \vec{\nabla})\vec{r} = \partial_i \vec{r} = \partial_i (r_j \hat{e}_j) = \delta_{ij} \hat{e}_j = \hat{e}_i\ .$$

Thus,

$$\vec{\nabla} \times (\hat{e}_i \times \vec{r}) = 2\hat{e}_i\ ,$$

and the resulting expression for m_i and $\vec{m}$ is

$$m_i = I \int_S \mathrm{d}\vec{a} \cdot \hat{e}_i\ ,$$

which is equalivalent to

$$\boxed{\vec{m} = I \int_S \mathrm{d}\vec{a}\,.}$$

PROBLEM 10: CURRENT LOOPS AND NEWTON'S THIRD LAW *(10 points)*

Griffiths Problem 5.50 (p. 259).

Using the Biot-Savart law, we find the field of the first loop as

$$\vec{B}(\vec{r}_2) = \frac{\mu_0 I_1}{4\pi} \oint_1 \frac{\mathrm{d}\vec{\ell}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\ , \tag{10.1}$$

where $\vec{\boldsymbol{\mathscr{r}}} \equiv \vec{r}_2 - \vec{r}_1$. The force on the second loop is then

$$\vec{F}_2 = I_2 \oint_2 \mathrm{d}\vec{\ell}_2 \times \vec{B}(\vec{r}_2) = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{\mathrm{d}\vec{\ell}_2 \times (\mathrm{d}\vec{\ell}_1 \times \hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2}\,. \tag{10.2}$$

Using the triple product rule, $\mathrm{d}\vec{\ell}_2 \times (\mathrm{d}\vec{\ell}_1 \times \hat{\boldsymbol{\mathscr{r}}}) = \mathrm{d}\vec{\ell}_1 (\mathrm{d}\vec{\ell}_2 \cdot \hat{\boldsymbol{\mathscr{r}}}) - \hat{\boldsymbol{\mathscr{r}}}(\mathrm{d}\vec{\ell}_1 \cdot \mathrm{d}\vec{\ell}_2)$, we find the force as

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{\hat{\boldsymbol{\mathscr{r}}}(\mathrm{d}\vec{\ell}_1 \cdot \mathrm{d}\vec{\ell}_2)}{\mathscr{r}^2} + \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \mathrm{d}\vec{\ell}_1 \oint_2 \frac{\mathrm{d}\vec{\ell}_2 \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,. \tag{10.3}$$

The second integral in Eq. (10.3) can be shown to vanish:

$$\oint_2 \mathrm{d}\vec{\ell}_2 \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \oint_2 \mathrm{d}\vec{\ell}_2 \cdot \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} = \oint_2 \mathrm{d}\vec{\ell}_2 \cdot \vec{\nabla}_{\vec{r}_2} \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right) = 0\ , \tag{10.4}$$

since the line integral of a gradient field over a closed path gives zero. Then only the first term remains,

$$\boxed{\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{\hat{\boldsymbol{\mathscr{r}}}(\mathrm{d}\vec{\ell}_1 \cdot \mathrm{d}\vec{\ell}_2)}{\mathscr{r}^2}\,.} \tag{10.5}$$

In this form it is clear that the forces that each loop exerts on the other obey $\vec{F}_1 = -\vec{F}_2$, since $\vec{\boldsymbol{\mathscr{r}}}$ changes direction when the roles of $\vec{r}_1$ and $\vec{r}_2$ are interchanged.