8.07 Lecture Slides 6
September 23, 2019

ELECTROSTATICS

Announcements

Office hours for Yitian Sun

Current: Thurs, 5:30-7:30, Fri: 2:00-3:00

Proposal: Change to 1 hour on Thurs, 2 on Friday

Outcome [decided during lecture]:

Thurs 5:30-6:30, and Fri 1:30-3:30.

Blackboard Discussion of Work and Energy

After the slides, we had a blackboard discussion of work and energy in electrostatics, with the following key results:

- ☆ For a test charge q moving in the field of other, fixed charges, $qV(\vec{r}) =$ potential energy of charge q at position $\vec{r}$.
- ☆ The work needed to assemble a system of n point charges q_i at positions $\vec{r}_i$, starting with all the charges at infinity, can be written as

$$\boxed{W_{\rm tot} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{q_i q_j}{r_{ij}}} \tag{1}$$

or

$$\boxed{W_{\rm tot} = \frac{1}{2}\frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}} \; .} \tag{2}$$

☆ For a continuous charge density $\rho(\vec{\boldsymbol{r}})$, the work needed to assemble an arbitrary configuration, starting with all the charge at infinity, is given by

$$\boxed{W_{\rm tot} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x \int \mathrm{d}^3x' \, \frac{\rho(\vec{\boldsymbol{r}})\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \; .} \tag{3}$$

If $\rho(\vec{\boldsymbol{r}})$ is a well-defined **function**, there is no need to exclude $\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}}$, in analogy to excluding $j = i$ in Eqs. (1) and (2), because $\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}}$ is a set of measure zero, which does not affect the integral. But if $\rho(\vec{\boldsymbol{r}})$ is allowed to contain δ-functions, to describe point charges, then the above expression will diverge. When $\rho(\vec{\boldsymbol{r}})$ includes δ-functions, Eq. (3) describes the full energy, including the energy needed to create the point charges, which is infinite. Eqs. (1) and (2) only include the work needed to move the point charges into their positions from infinity.

☆ For a continuous charge density $\rho(\vec{r})$, the work needed to assemble an arbitrary configuration can also be written as

$$\boxed{W_{\rm tot} = \frac{1}{2}\int \mathrm{d}^3x\, \rho(\vec{r}\,)V(\vec{r}\,)} \tag{4}$$

or as

$$\boxed{W_{\rm tot} = \frac{1}{2}\epsilon_0 \int \left|\vec{E}\right|^2 \mathrm{d}^3x\ .} \tag{5}$$

☆ Aside: if the same exercise is carried out for Newtonian gravity, one finds

$$\boxed{W_{\rm tot} = -\frac{1}{8\pi G}\int |\vec{g}\,|^2\ \mathrm{d}^3x\ .} \tag{6}$$

The negative energy of gravitational fields allows for the possibility that the total energy of the universe is zero!

Self-Energy of a Point Charge is Infinite

Recall

$$W = \frac{1}{2}\epsilon_0 \int \left|\vec{\boldsymbol{E}}\right|^2 \, \mathrm{d}^3x \ .$$

For a point charge,

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \ ,$$

so

$$W = \frac{1}{2}\epsilon_0 \frac{q^2}{(4\pi\epsilon_0)^2}\int \frac{4\pi r^2 \, \mathrm{d}r}{r^4} = \infty \ .$$

Status of Self-Energy in Classical E&M

Can build a classical model of a point electron that works for some questions, but not all. Treat the electron as a uniformly charged ball of charge, with radius r_0. The physical mass of the electron is then

$$m_{e,\text{phys}} = m_0 + \frac{\frac{1}{2}\epsilon_0 \int \left|\vec{\boldsymbol{E}}\right|^2 \, \mathrm{d}^3x}{c^2} \, ,$$

where m_0 is the "bare mass," i.e., the mass before electrostatic interactions are considered.

Take the limit as $r_0 \to 0$ with $m_{e,\text{phys}}$ fixed (so $m_0 \to -\infty$!)

We will see later that this treatment gives the Abraham-Lorentz formula for radiation reaction:

$$\vec{\boldsymbol{F}} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{\boldsymbol{a}}} \ ,$$

describing the back-reaction of the radiation that a point charge emits if it accelerates.

Formula is believable under many circumstances, BUT it also has runaway solutions, for which the particle accelerates due its own radiation, with no applied forces.

In summary, we don't have a completely acceptable **classical** theory of point charges.

Status of Self-Energy in Quantum Electrodynamics (QED)

The situation is much better under control.

Again, write

$$m_{e,\text{phys}} = m_0 + \Delta m \ ,$$

where Δm is the electromagnetic correction, which, like the classical case, is divergent.

Like the classical model above, "regularize" it to make it finite. Classically, replace point by ball. In QED, can use short distance cutoff, high-k cutoff in Fourier space, or, rather shockingly, an analytic continuation in the number of spatial dimensions.

Then take limit as regulator disappears (like $r_0 \to 0$), with $m_{e,\text{phys}}$ fixed, so again $m_0 \to -\infty$.

Must do the same for the charge: $e_{\text{phys}} = e_0 + \Delta e$, with e_{phys} fixed.

Result: at least in perturbation theory (expansion in powers of $e_{\rm phys}$), results are finite to all orders.

But does the perturbation converge?

We don't think so. (We think it is an asymptotic series, but not convergent.)

So where are we?

Prevailing wisdom: QED by itself is not completely well-defined. But it can (presumably) be saved by adding other interactions that become significant at very short distances.

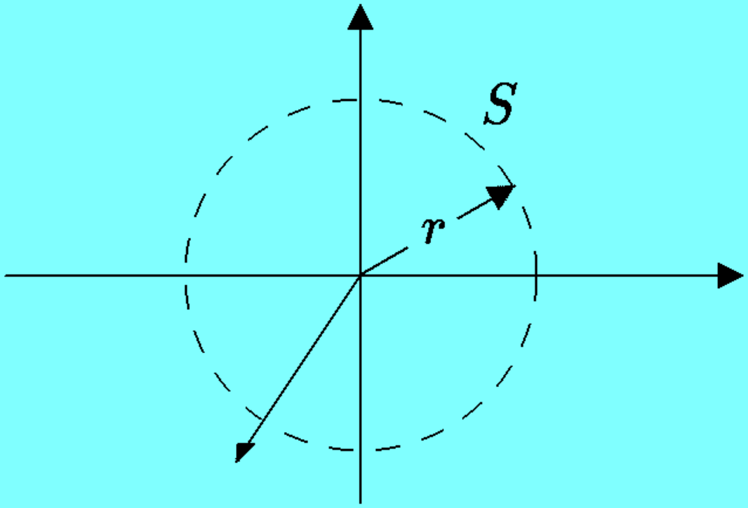
Use of Gauss's Law

Gauss's law is always true, but $\vec{\boldsymbol{E}}$ can be found from Gauss's law only in symmetric cases. There are 3 types of cases: spherical, cylindrical, and planar symmetries.

Always,

$$\boxed{\oint_S \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{1}{\epsilon_0} Q_{\rm enc}\ .}$$

Case I: Spherical Symmetry



By symmetry, we infer that

$$\boxed{\vec{\boldsymbol{E}} = E(r)\hat{\boldsymbol{r}} \ .}$$

S = spherical Gaussian surface, centered on spherical charge distribution, at radius r.

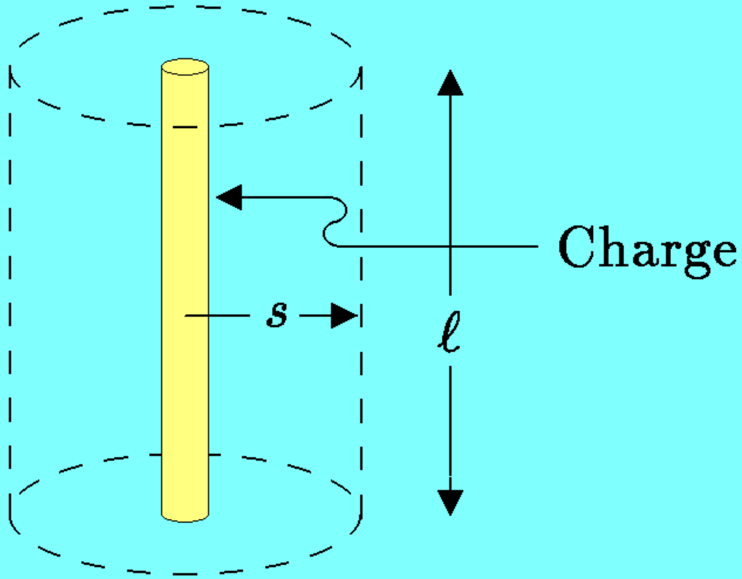
Then

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = 4\pi r^2 E(r) \ ,$$

so

$$\boxed{E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enclosed}}(r)}{r^2} \ .}$$

Case II: Cylindrical Symmetry



By symmetry, we infer that

$$\boxed{\vec{\boldsymbol{E}} = E(s)\hat{\boldsymbol{s}} \ .}$$

$S =$ cylindrical Gaussian surface, centered on cylindrically symmetric charge distribution, at radius s, with arbitrary length ℓ.

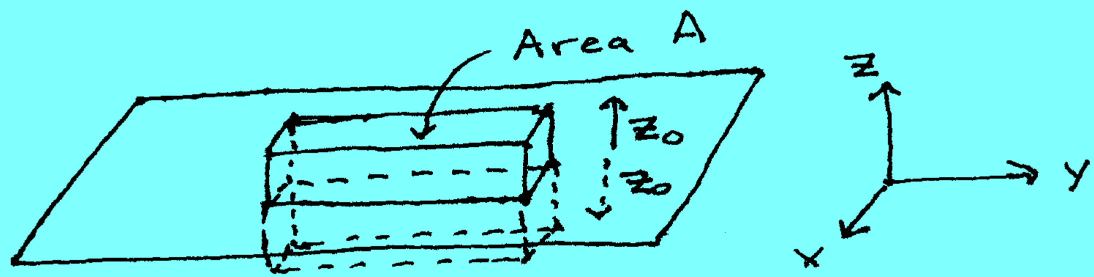
Then

$$\frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{\lambda_{\text{enclosed}}(s)\ell}{\epsilon_0} \ , \quad \text{and} \quad \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = 2\pi s \ell E(s) \ ,$$

so

$$\boxed{E(s) = \frac{1}{2\pi\epsilon_0} \frac{\lambda_{\text{enclosed}}(s)}{s} \ .}$$

Case III: Planar Symmetry



By symmetry, we infer that

$$\boxed{\vec{\boldsymbol{E}} = \begin{cases} E(z_0)\hat{\boldsymbol{z}} & \text{if } z > 0 \\ -E(z_0)\hat{\boldsymbol{z}} & \text{if } z < 0 \end{cases} .}$$

For a Gaussian surface, take a "pillbox," a rectangular box that extends a distance z_0 both above and below the plane of symmetry, with an area A in the symmetry plane.

Then

$$\frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{\sigma_{\text{enclosed}}(z_0)A}{\epsilon_0} , \quad \text{and} \quad \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = 2E(z_0)A ,$$

so

$$\boxed{E(z_0) = \frac{\sigma_{\text{enclosed}}(z_0)}{2\epsilon_0} .}$$

Conductors

What are conductors?

Electrons are free to move. Any nonzero $\vec{E}$ causes a current to flow, and very quickly the electrons move to a configuration that makes $\vec{E} = 0$ inside conductor. (Statics for now: no batteries or generators.)

☆ $\rho = 0$ inside conductors:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \ , \quad \text{so} \quad \vec{E} = 0 \quad \Longrightarrow \quad \boxed{\rho = 0 \ .}$$

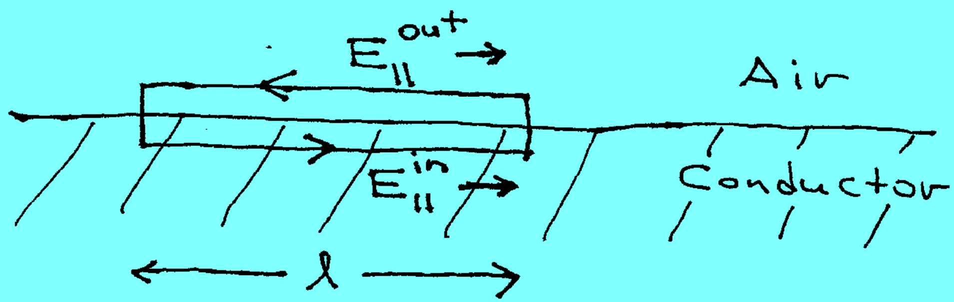
☆ Conductors are equipotential, since inside the conductor,

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot \mathrm{d}\vec{\ell} = 0 \ .$$

☆ Boundary conditions for $\vec{\boldsymbol{E}}$:

Just outside the surface, $\vec{\boldsymbol{E}}$ is $\perp$ to surface.

Proof:



Consider a loop that straddles the surface, with an upper leg in the air, and a lower leg inside the conductor. Loop is small, so take the field in each region to be uniform. The

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \left(E_{||}^{\text{in}} - E_{||}^{\text{out}} \right) \ell = 0 \ ,$$

so

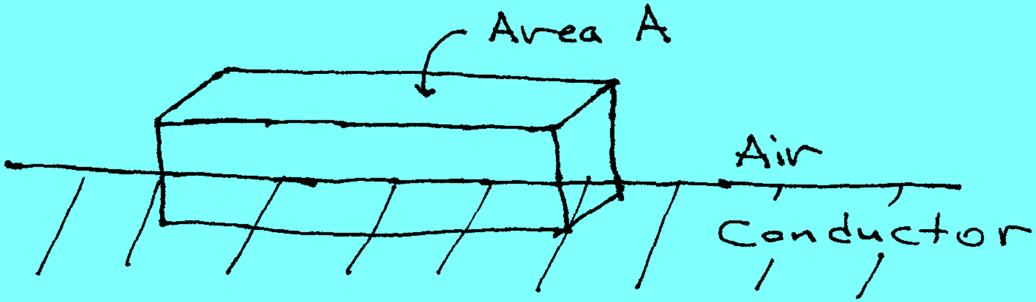
$$\boxed{E_{||}^{\text{out}} = E_{||}^{\text{in}} = 0 \ .}$$

Just outside the surface,

$$\boxed{\vec{\boldsymbol{E}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \ ,}$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal from the conductor.

Proof:



$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{Q_{\rm enc}}{\epsilon_0} \ ,$$

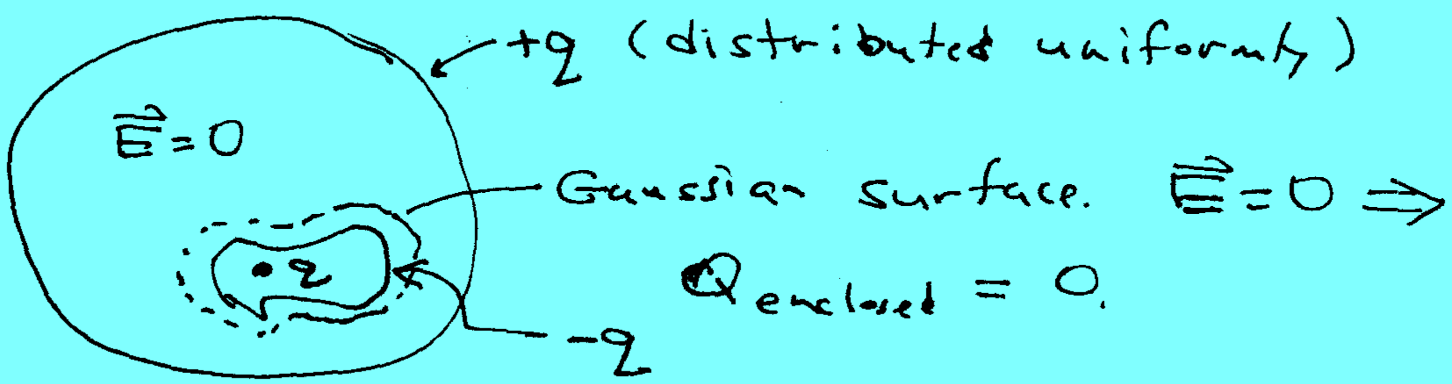
which implies

$$E_{\hat{n}} A = \frac{\sigma A}{\epsilon_0} \quad \Longrightarrow \quad \boxed{\vec{\boldsymbol{E}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \ .}$$

★ All charge resides on the surfaces

This has to be the case, since $\rho = 0$ inside the conductor.

Example



Consider a spherical ball with a cavity cut out, and a charge q in the cavity.

The Gaussian surface shown is in the conductor, so $\vec{\boldsymbol{E}} = 0$, so $Q_{\rm enc} = 0$. So there must be an **induced** charge on the surface of the cavity of $-q$. By conservation of charge, there must be an induced charge on the outer surface of $+q$. It must be distributed uniformly, because otherwise the $\vec{\boldsymbol{E}}$ produced would not be radial.

☆ Metal enclosures shield their interiors from electric fields.

On a metal enclosure, all charge flows to the outer surface of the metal. Any charge on the inner surface of the metal will produce an electric field in the metal, so there is no charge on the inner surface. Clearly, IF we assume that $\vec{\boldsymbol{E}} = 0$ in the enclosure, it will be consistent with the conditions $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0$ and $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = 0$, in the region inside the enclosure and on its boundary. We will learn shortly that once we have one such solution, it is always the only such solution. So $\vec{\boldsymbol{E}} = 0$ inside the enclosure.

A wire mesh does not completely shield $\vec{\boldsymbol{E}}$, but it can do a surprisingly good job. A wire mesh cage is known as a Faraday cage.

Blackboard Discussion of Virtual Work

All charges on conductors reside on the surface. Since $\vec{\boldsymbol{E}}$ is discontinuous, with $\vec{\boldsymbol{E}} = (\sigma/\epsilon_0)\hat{\boldsymbol{n}}$ outside and zero inside, it is not clear what the force on the surface will be.

We considered a spherical surface, of radius R and charge Q uniformly spread.

The total electrostatic energy is $\frac{1}{2}QV(R)$.

By calculating the change in total energy when R is changed to $R + \mathrm{d}R$, we found that the force per area on the surface, also called the pressure, is outward, with magnitude

$$\boxed{p = \frac{\sigma^2}{2\epsilon_0} = \frac{1}{2}\sigma \left|\vec{\boldsymbol{E}}\right|_{\text{outside}} \; .}$$

I mentioned without proof that the same result can be found by allowing the surface charge to have a small thickness, so $\vec{\boldsymbol{E}}$ varies continuously over the thickness. Although we derived the result only for a sphere, it is true for any shape.

Blackboard Discussion of Capacitance

Consider a system of isolated conductors, labeled 1, 2, 3, etc., each with charge Q_1, Q_2, Q_3, ..., respectively.

Each conductor is an equipotential, where V_i denotes the potential of the i'th conductor.

Since the $\vec{\boldsymbol{E}}$ for many charges is just the sum of the field for each charge, and V is determined from $\vec{\boldsymbol{E}}$, the V_i should be a linear function of the Q_i's. So we can write

$$\boxed{V_i = \sum_j P_{ij} Q_j \ ,}$$

where the P_{ij}'s are the *potential coefficients*.

This slide was added after lecture.

This matrix relation can be inverted, so we can write

$$\boxed{Q_i = \sum_j Q_{ij} V_j \ ,}$$

where $C = P^{-1}$, and the C_{ij}'s are called coefficients of capacitance.

The total electrostatic energy of the system is given by

$$\boxed{W = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_{ij} Q_i P_{ij} Q_j = \frac{1}{2} \sum_{ij} V_i C_{ij} V_j \ .}$$