8.07 Lecture Slides 25
December 11, 2019

LAST LECTURE!
RADIATION

Announcements

Practice Problems for the Final Exam, and a Formula Sheet for the Final Exam, have been posted. The Formula Sheet is very thorough, and is intended as a tool for reviewing the course. If you understand all the formulas, you are in great shape for the Final Exam.

The Final Exam will be given on Thursday, December 19, from 1:30 pm - 4:30 pm, in this room (6-120). The final exam will include material from the entire course, but will emphasize material since the last quiz.

Two of the problems on the Final Exam will be taken verbatim, or at least almost verbatim, from Problem Sets 8 and 9, or the Practice Problems for the Final Exam. Extra credit problems on the homework will be possible choices.

Office hours will continue through Friday of this week, at the usual times and places. They are listed on the Staff tab of the website.

Yitian Sun will hold a review session for the Final Exam next weekend: Saturday, 1:00 pm., in Room 4-145. The ending time is flexible, but 2 hours is an estimate of how long it might last.

During final exam week, there will be no regular office hours or review sessions. But if you have questions, feel free to email Marin, Yitian, and me, and we will try to arrange a time to meet with you. Or you can just send your questions to us by email, and we will try to either answer you, or set up a meeting if the answer is hard to answer by email. In some cases we may decide it would be helpful to send the answer to the entire class. Please let us know, with your question, if it is okay for us to tell the class who asked the question, or if you would prefer to remain anonymous.

Last Friday I emailed to you a link to a survey about the video capture. Please respond!

You have also received a link to the end-of-term course evaluations, which are open until Monday December 16, at 9 am. Please respond! We very much value your feedback. Remember, it is your feedback that helps to keep the quality of teaching at MIT high.

POTENTIALS AND FIELDS

Maxwell's Equations with Sources:

$$
\begin{aligned}
&\text{(i)}\quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho &\qquad &\text{(iii)}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\ , \\
&\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0 & &\text{(iv)}\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\ ,
\end{aligned}
\tag{1}
$$

Question: If we are given the sources $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, can we find $\vec{E}$ and $\vec{B}$? If we accept the proposition that all integrals are in principle doable (at least numerically), then the answer is $\boxed{\text{YES}}$.

Electromagnetic Potentials

If $\vec{B}$ depends on time, then $\vec{\nabla} \times \vec{E} \neq \vec{0}$, so we cannot write $\vec{E} = -\vec{\nabla} V$. BUT: we can still write

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}\ .} \tag{2}$$

Then notice that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \quad \Longrightarrow \quad \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0\ . \tag{3}$$

so we can write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \quad \Longrightarrow \quad \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\ .} \tag{4}$$

Retarded Time Solutions

$$V(\vec{\boldsymbol{r}},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{\mathscr{r}}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{\mathscr{r}} \quad , \tag{5}$$

where

$$\vec{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,' \ , \quad \mathscr{r} = |\vec{\boldsymbol{\mathscr{r}}}| \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}{c} = t - \frac{\mathscr{r}}{c} \ . \tag{6}$$

The Liénard-Wiechert Potentials

Finally,

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}\left(1-\dfrac{\vec{v}_p}{c}\cdot\hat{\mathscr{r}}\right)}\ ,} \tag{7}$$

where $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}_p$, and $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at t_r. Similarly, starting with

$$\vec{J}(\vec{r},t) = q\vec{v}\delta^3\Big(\vec{r} - \vec{r}_p(t)\Big) \tag{8}$$

for a point particle, we find

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}_p}{\mathscr{r}\left(1-\dfrac{\vec{v}_p}{c}\cdot\hat{\mathscr{r}}\right)} = \frac{\vec{v}_p}{c^2}V(\vec{r},t)\ .} \tag{9}$$

The Fields of a Point Charge

Differentiating the Liénard-Wiechert potentials, after several pages, one finds

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{(\vec{u}\cdot\vec{\boldsymbol{\mathscr{r}}})^3}\left[(c^2 - v_p^2)\vec{u} + \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a}_p)\right] \ ,} \tag{10}$$

where

$$\boxed{\vec{u} = c\hat{\boldsymbol{\mathscr{r}}} - \vec{v}_p \ .} \tag{11}$$

And

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\vec{E}(\vec{r},t) \ .} \tag{12}$$

Here $\vec{r}_p$, $\vec{v}_p$, and $\vec{a}_p$ are the position, velocity, and acceleration, respectively, of the particle at the retarded time.

What about $\vec{u} \cdot \vec{\mathscr{r}}$ in the denominator?

Can it vanish, leading to an infinite $\vec{\boldsymbol{E}}$? Answer, no:

$$\begin{aligned}\vec{\boldsymbol{\mathscr{r}}} \cdot \vec{\boldsymbol{u}} &= \vec{\boldsymbol{\mathscr{r}}} \cdot (c\hat{\boldsymbol{\mathscr{r}}} - \vec{\boldsymbol{v}}_p) \\ &= c|\vec{\boldsymbol{\mathscr{r}}}| - \vec{\boldsymbol{v}}_p \cdot \vec{\boldsymbol{\mathscr{r}}} \\ &= c\mathscr{r} - v_p \mathscr{r} \cos\Theta \\ &= c\mathscr{r}\left(1 - \frac{v_p}{c}\cos\Theta\right) > 0 \ ,\end{aligned}$$

where Θ is the angle between $\vec{\boldsymbol{v}}_p$ and $\vec{\boldsymbol{\mathscr{r}}}$. But one should not try to infer the angular dependence from this equation, since $\mathscr{r}$ also depends on angle.

If the particle is moving at constant velocity, then the acceleration term in Eq. (10) is absent, so the expression simplifies. The most informative way to express the electric field is to write it in terms of the velocity $\vec{v}$ and $\vec{R} \equiv \vec{r} - \Big(\vec{r}_p + \vec{v}_p(t - t_r)\Big)$, which is the vector from the *current* position of the particle to the point of observation $\vec{r}$.

With some algebra one finds that

$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2} \;.}$$

where θ is the angle beteen $\vec{R}$ and $\vec{v}$.

Thus, the electric field of a particle moving at constant velocity looks like:

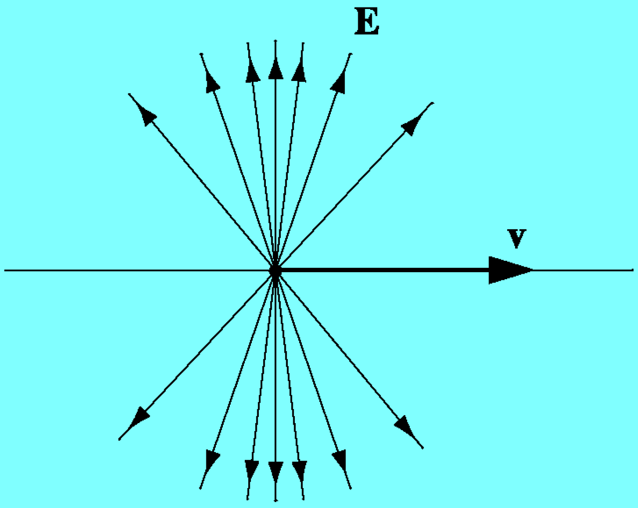


FIGURE 10.10

(Diagram taken from D.W. Griffiths, **Introduction to Electrodynamics,** *4th edition.)*

RADIATION!

Radiation: Electromagnetic fields that carry energy off to infinity.

At large distances, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fall off only as $1/r$, so the Poynting vector falls off as $1/r^2$. If the Poynting vector is then integrated over a large sphere, of area $4\pi r^2$, the contribution approaches a constant as $r \to \infty$.

Recall the Liénard-Wiechert potentials:

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}\left(1 - \dfrac{\vec{v}_p}{c}\cdot\hat{\mathscr{r}}\right)}} \;, \tag{13}$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}_p}{\mathscr{r}\left(1 - \dfrac{\vec{v}_p}{c}\cdot\hat{\mathscr{r}}\right)} = \frac{\vec{v}_p}{c^2}V(\vec{r},t)} \;, \tag{14}$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at t_r,

$$t_r = t - \frac{\mathscr{r}}{c} \;, \tag{15}$$

and

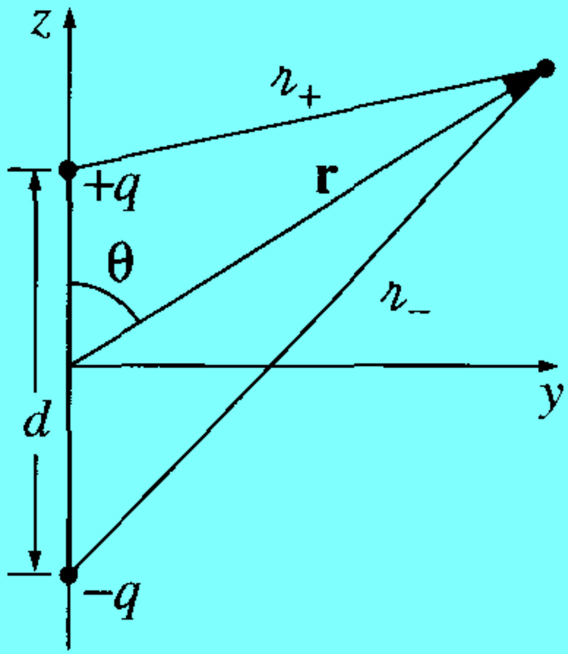
$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{r}_p}{|\vec{r} - \vec{r}_p|} \;, \qquad \mathscr{r} = |\vec{r} - \vec{r}_p| \;. \tag{16}$$

Electric Dipole Radiation

Simplest dipole: two tiny metal spheres separated by a distance d along the z-axis, connected by a wire, with charges

$$q(t) = q_0 \cos(\omega t) \tag{17}$$

on the top sphere, and $q(t) = -q_0 \cos(\omega t)$ on the bottom sphere.



Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0 \cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0 \cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} . \tag{18}$$

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Summary of approximations: $d \ll \lambda \ll r$.

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)] \ . \tag{19}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\boldsymbol{z}} \ . \tag{20}$$

Differentiating, and keeping terms that fall off as $1/r$, while dropping terms that fall off as $1/r^2$,

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}\ .} \tag{21}$$

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t)\ .} \tag{22}$$

Recall that θ is the angle between $\vec{\boldsymbol{p}}$ and $\hat{\boldsymbol{r}}$.

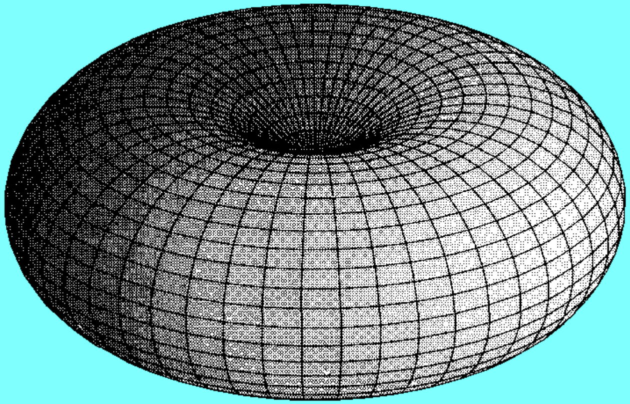
Poynting Vector:

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{r} \ . \tag{23}$$

Intensity:

Average the Poynting vector over a complete cycle: $\langle \cos^2 \rangle = 1/2$.

$$\left\langle \vec{S} \right\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2}\,\hat{r} \ . \tag{24}$$



All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Total Power:

Integrate over a sphere at large r.

$$\langle P \rangle = \int \left\langle \vec{\boldsymbol{S}} \right\rangle \cdot \mathrm{d}\vec{\boldsymbol{a}} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \int \frac{\sin^2\theta}{r^2}\, r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= \boxed{\frac{\mu_0 p_0^2 \omega^4}{12\pi c} \ .} \tag{25}$$

Magnetic Dipole Radiation

Consider a wire loop of radius b, with alternating current

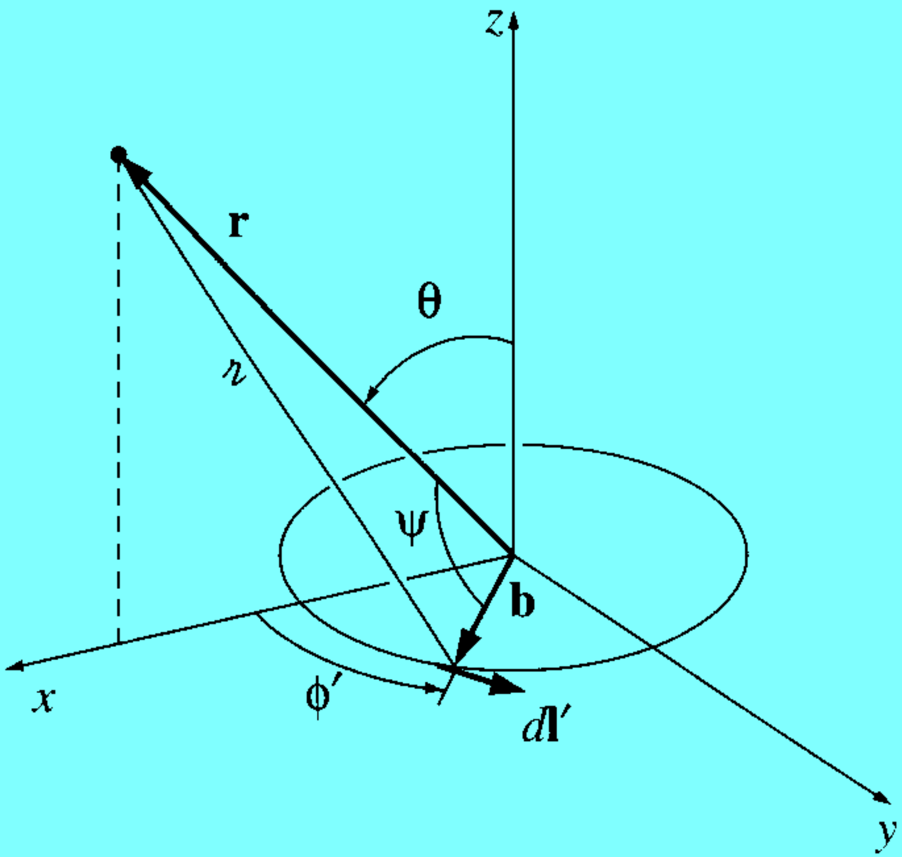
$$I(t) = I_0 \cos(\omega t) \ , \qquad (26)$$

with magnetic dipole moment

$$\begin{aligned} \vec{\boldsymbol{m}}(t) &= \pi b^2 I(t)\,\hat{\boldsymbol{z}} \\ &= m_0 \cos(\omega t)\,\hat{\boldsymbol{z}} \ , \end{aligned} \qquad (27)$$

where

$$m_0 \equiv \pi b^2 I_0 \ . \qquad (28)$$



All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Make the same approximations as before, with b replacing d as the size of the system: $b \ll \lambda \ll r$. Find

$$\vec{\boldsymbol{E}} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}} \; . \tag{29}$$

Compared to the electric dipole radiation,

$$p_0 \to \frac{m_0}{c} \; , \qquad -\hat{\boldsymbol{\theta}} \to \hat{\boldsymbol{\phi}} \; . \tag{30}$$

As always for radiation,

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) \; . \tag{31}$$

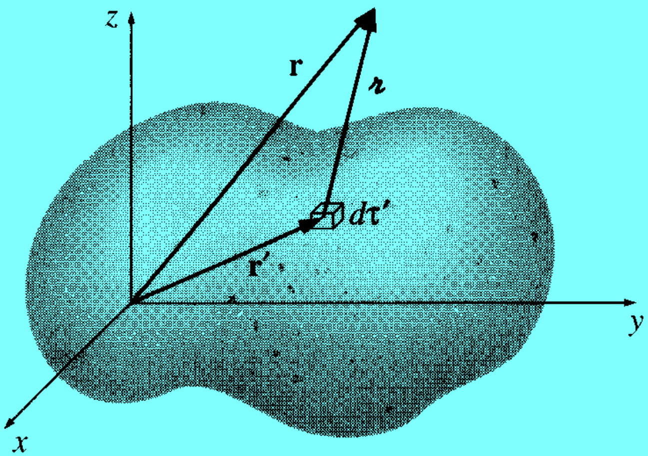
$$\vec{\boldsymbol{S}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{c} \left\{ \frac{m_0 \omega^2}{4\pi c} \left(\frac{\sin\theta}{r} \right) \cos\left[\omega(t - r/c)\right] \right\}^2 \hat{\boldsymbol{r}} \ . \tag{32}$$

$$\left\langle \vec{\boldsymbol{S}} \right\rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \right) \frac{\sin^2\theta}{r^2} \hat{\boldsymbol{r}} \ . \tag{33}$$

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} \ . \tag{34}$$

(Electric Dipole) Radiation From an Arbitrary Source

Consider an arbitrary time-dependent charge distribution $\rho(\vec{r}, t)$. Then



$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} \mathrm{d}\tau' \ . \qquad (35)$$

Expand $1/|\vec{r} - \vec{r}\,'|$ and t_r in powers of $\vec{r}\,'$, using similar approximations as before.

Minor difference: here we have no ω. Previously we assumed that $d \ll \lambda$ or equivalently $\omega \ll c/d$. Here we need to assume that $|\ddot{\rho}/\dot{\rho}| \ll c/d$, with similar bounds on higher time derivatives.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

We find

$$V(\vec{\boldsymbol{r}},t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}}(t_0)}{r^2} + \frac{\hat{\boldsymbol{r}}\cdot\dot{\vec{\boldsymbol{p}}}(t_0)}{rc}\right] , \tag{36}$$

where t_0 is the retarded time at the origin. And

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) \simeq \frac{\mu_0}{4\pi}\frac{\dot{\vec{\boldsymbol{p}}}(t_0)}{r} . \tag{37}$$

Final result:

$$\boxed{\begin{aligned} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) &\simeq \frac{\mu_0}{4\pi r}[(\hat{\boldsymbol{r}}\cdot\ddot{\vec{\boldsymbol{p}}})\hat{\boldsymbol{r}} - \ddot{\vec{\boldsymbol{p}}}\,] \\ \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}},t) &\simeq -\frac{\mu_0}{4\pi rc}[\hat{\boldsymbol{r}}\times\ddot{\vec{\boldsymbol{p}}}\,] . \end{aligned}} \tag{38}$$

Although this looks different, it is really the same as what we had for the simple electric dipole, changing to vector notation and replacing $-\omega^2\vec{\boldsymbol{p}}_0$ by $\ddot{\vec{\boldsymbol{p}}}$. Note that $\vec{\boldsymbol{E}}$ is proportional to $\ddot{\vec{\boldsymbol{p}}} - (\hat{\boldsymbol{r}}\cdot\ddot{\vec{\boldsymbol{p}}})\hat{\boldsymbol{r}}$, which is the projection of $\ddot{\vec{\boldsymbol{p}}}$ into the plane perpendicular to the line of sight.

The Poynting vector is given by

$$\boxed{\vec{\boldsymbol{S}}(\vec{\boldsymbol{r}}\,, t) \simeq \frac{1}{\mu_0}(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2 \left(\frac{\sin^2\theta}{r^2}\right) \hat{\boldsymbol{r}} \ ,} \tag{39}$$

where θ is the angle between $\ddot{\vec{\boldsymbol{p}}}(t_0)$ and $\hat{\boldsymbol{r}}$. The total power radiated is given by

$$\boxed{P_{\mathrm{rad}}(t_0) \simeq \frac{\mu_0}{6\pi c}[\ddot{p}(t_0)]^2 \ .} \tag{40}$$

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, … radiation, and also magnetic dipole, quadrupole, … radiation.

Radiation by Point Charges

Recall the fields (found by differentiating the Liénard-Wiechert potentials):

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{u}\cdot\vec{\mathscr{r}})^3}\left[(c^2 - v_p^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u}\times\vec{a}_p)\right] \ , \qquad (41)$$

where

$$\vec{u} = c\hat{\mathscr{r}} - \vec{v}_p \ , \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}_p \ . \qquad (42)$$

If $\vec{v}_p = 0$ (at the retarded time), then the radiation part of the electric field is given by

$$\vec{E}_{\rm rad}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}\left[\hat{\mathscr{r}} \times (\hat{\mathscr{r}}\times\vec{a}_p)\right] \ . \qquad (43)$$

Poynting Vector (particle at rest):

$$\vec{S}_{\rm rad} = \frac{1}{\mu_0 c}|\vec{E}_{\rm rad}|^2\,\hat{\boldsymbol{r}} = \boxed{\frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\boldsymbol{r}}\;,} \tag{44}$$

where θ is the angle between $\hat{\boldsymbol{r}}$ and $\vec{a}$.

Total Power (Larmor formula):

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}\;.} \tag{45}$$

Total Power (Larmor formula):

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}\ .} \qquad (45)$$

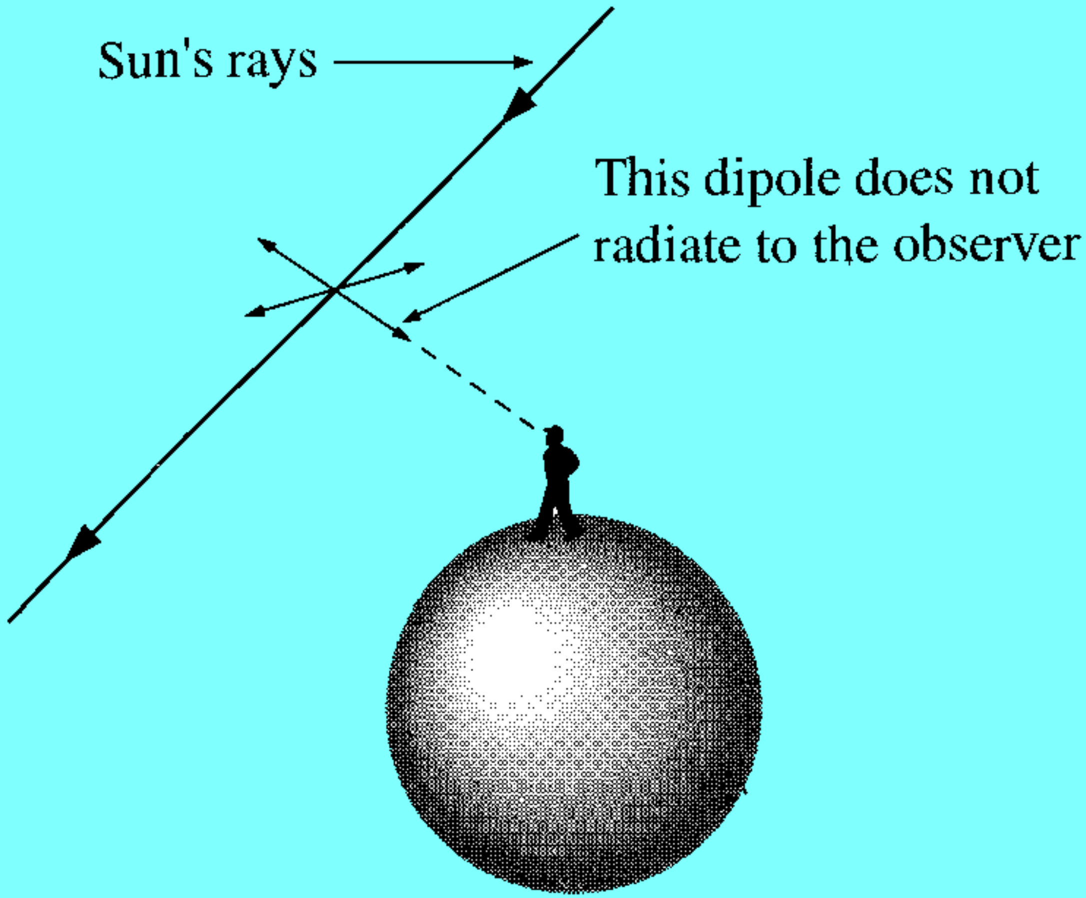
Liénard's Generalization if $\vec{v}_p \neq 0$:

$$\boxed{P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{\boldsymbol{v}} \times \vec{\boldsymbol{a}}}{c}\right|^2\right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}}_{\text{For relativists only}}\ .} \qquad (46)$$

P = rate at which energy that is destined to become radiation is leaving the particle.

Polarized Blue Skies

- Blue is the highest frequency in the visible spectrum. The sky appears blue largely because the ω^4 factor in dipole radiation implies that blue light is scattered more strongly than other frequencies.
- Sunsets are red because the blue light has been scattered out of the path of the sunlight.



★ When the line of sight is perpendicular to the Sun's rays, the light is polarized horizontally.

All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Radiation Reaction

If an accelerating particle radiates energy, it must lose energy — obviously! But it is not clear exactly how. Point charges and conservation laws cannot be combined rigorously.

For a nonrelativistic particle, radiation power is given by the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \ .$$

Maybe then,

$$\vec{\boldsymbol{F}}_{\rm rad} \cdot \vec{\boldsymbol{v}} \stackrel{?}{=} -\frac{\mu_0 q^2 a^2}{6\pi c} \ . \tag{47}$$

But not necessarily, since $P_{\rm Larmor}$ represents the power destined to become radiation. There can be other energy exchanges with the near fields. Since the RHS of Eq. (47) does not depend on $\vec{\boldsymbol{v}}$, there is no way to match the two sides.

For cyclic motion, the total energy loss over one cycle should match the energy loss described by the Larmor formula. If a cycle extends from t_1 to t_2 then

$$\int_{t_1}^{t_2} a^2\,\mathrm{d}t = \int \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}\,\mathrm{d}t = \underbrace{\left(\vec{\boldsymbol{v}}\cdot\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t}\right)\Bigg|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2}\frac{\mathrm{d}^2\vec{\boldsymbol{v}}}{\mathrm{d}t^2}\cdot\vec{\boldsymbol{v}}\,\mathrm{d}t\ . \qquad (48)$$

So, energy conservation holds if

$$\int_{t_1}^{t_2}\left(\vec{\boldsymbol{F}}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{\boldsymbol{a}}}\right)\cdot\vec{\boldsymbol{v}}\,\mathrm{d}t = 0\ , \qquad (49)$$

which will hold if

$$\boxed{\vec{\boldsymbol{F}}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{\boldsymbol{a}}}\ .} \qquad (50)$$

$$\boxed{\vec{F}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}} \tag{50}$$

is the Abraham-Lorentz formula for the radiation reaction force. We have not proven it, but we have shown that it provides a force that is consistent with conservation of energy, at least for periodic motion.

The Abraham-Lorentz formula can be "derived" by modeling the point particle as an extended object, so that one can calculate the force that one part of the object exerts on the other parts. One then takes a limit in which the size of the extended object approaches zero. But the result depends on the details of the model, so the result remains inconclusive.

The Abraham-Lorentz formula gives reasonable predictions for the radiation reaction force under many circumstances, but it also has an important pathology: runaway solutions. Note that

$$\vec{\boldsymbol{F}}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{\boldsymbol{a}}} = m\vec{\boldsymbol{a}} \tag{51}$$

has the solution

$$a(t) = a_0\, e^{t/\tau} \ , \tag{52}$$

where

$$\tau = \frac{\mu_0 q^2}{6\pi m c} \approx 6 \times 10^{-24}\ \mathrm{s} \tag{53}$$

for an electron.

In quantum electrodynamics (QED) the situation is at least under much better control. Infinities arise, but they can be removed from the theory by a technique called renormalization. This process can be proven to succeed order by order in perturbation theory (an expansion in powers of the $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$), but the perturbation expansion is not believed to converge. Nothing can be proven rigorously, but there are reasons to believe that the theory is not completely well-defined, but that it nonetheless gives a consistent description up to extraordinarily high energies, well beyond the Planck energy, $E_{\text{Planck}} \equiv \sqrt{\frac{\hbar c}{G}}\, c^2$, the energy at which quantum gravity is believed to become essential. It is further believed/hoped that QED can be embedded into a more complete theory (perhaps string theory) that would be consistent at all energies.

A FINALE
ABOUT THE ARROW OF TIME

Retarded Time Solutions

$$
\begin{aligned}
V(\vec{\boldsymbol{r}}, t) &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{\mathscr{r}} \\
\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) &= \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,', t_r)}{\mathscr{r}} \quad ,
\end{aligned}
\tag{1}
$$

where

$$
\vec{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,' \; , \quad \mathscr{r} = |\vec{\boldsymbol{\mathscr{r}}}| \; , \quad t_r = t - \frac{\mathscr{r}}{c} \; . \tag{2}
$$

Advanced Time Solutions??

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}\,',t_a)}{\eta} \qquad (3)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{r}\,',t_a)}{\eta} \; ,$$

where

$$t_a = t + \frac{\eta}{c} \; . \qquad (4)$$

Maxwell's equations, and the laws of physics as we know them, make no distinction between the future and the past. The advanced solutions also satisfy Maxwell's equations.

Mystery of the Arrow of Time

Real Events:

Mystery of the Arrow of Time

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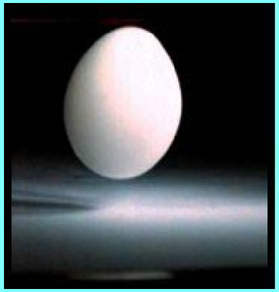
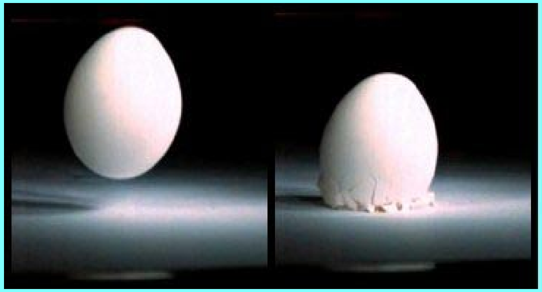


Mystery of the Arrow of Time

Real Events:



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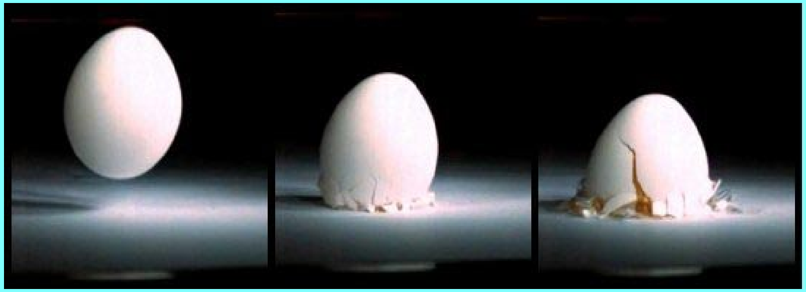

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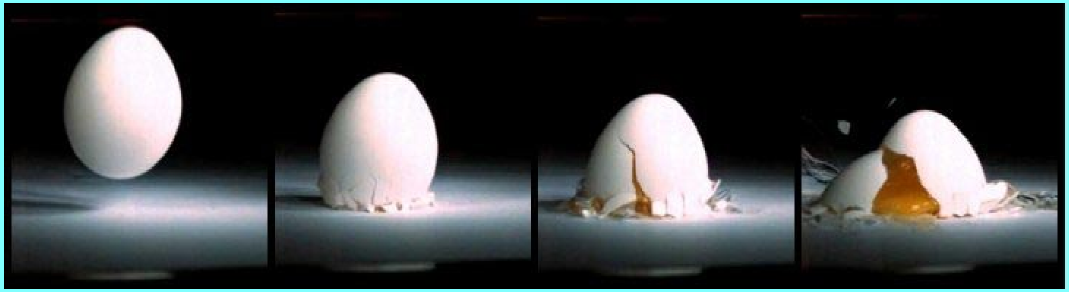

Mystery of the Arrow of Time

Real Events:

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Real Events:

Arrow of time

Mystery of the Arrow of Time

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Arrow of time

Laws of Physics:

Mystery of the Arrow of Time

Real Events:

Arrow of time

Laws of Physics:

Time symmetric

Alan Guth
Massachusetts Institute of Technology

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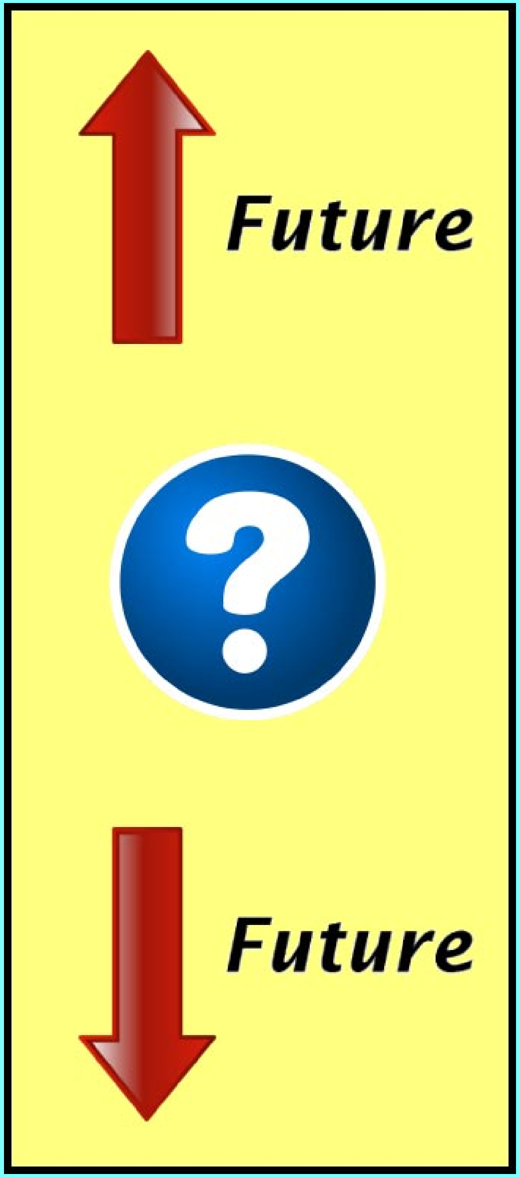
Why did the universe start in a low entropy state?

Who knows?

My preferred explanation:

It is possible that there is no upper limit to the entropy of the universe, so any state in which it may have started is low-entropy compared to what it can be. Sean Carroll (Caltech) and I (Alan Guth) are planning to write a paper about this.


Future
Future

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