8.07 Lecture Slides 13
October 23, 2019

ELECTRIC POTENTIAL: SPHERICAL HARMONICS VIA TRACELESS SYMMETRIC TENSORS

Announcements

Problem Set 5, Problem 5: the hint was worded incorrectly. You should calculate the **potential** along the z axis, not the **field**.

Review of Lecture 12

Detailed Form of the Trace Decomposition Theorem

For an symmetric tensor $S_{i_1\ldots i_\ell}$, the traceless symmetric part can be written as

$$\{\, S_{i_1\ldots i_\ell}\,\}_{\rm TS} = S_{i_1\ldots i_\ell} + \mathop{\rm Sym}_{i_1\ldots i_\ell}\Big[a_{1,\ell}\delta_{i_1 i_2} S_{j_1 j_1 i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4} S_{j_1 j_1 j_2 j_2 i_5\ldots i_\ell} + \ldots\Big] \;,$$

where

$$\mathop{\rm Sym}_{i_1\ldots i_\ell}\big[T_{i_1\ldots i_\ell}\big] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} T_{i_1\ldots i_\ell} \;,$$

and

$$a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell-2n)!}{n!(\ell-2n)!(\ell-n)!(2\ell)!} \;.$$

In the last lecture we derived $a_{1,\ell}$.

Review of Lecture 12

$$\{\, S_{i_1\ldots i_\ell}\,\}_{\rm TS} = S_{i_1\ldots i_\ell} + \mathop{\rm Sym}_{i_1\ldots i_\ell}\Big[a_{1,\ell}\delta_{i_1 i_2} S_{j_1 j_1 i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4} S_{j_1 j_1 j_2 j_2 i_5\ldots i_\ell} + \ldots\Big] \;,$$

We need to take the trace of the right-hand side (RHS), choosing the $a_{n\ell}$ so that it vanishes. Take the trace in i_1, i_2. Calculating through the $a_{1\ell}$ term,

$$\text{Tr}\,(\text{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2} \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \big[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\big] \;.$$

To evaluate the second term, we consider 4 cases for where the indices i_1 and i_2 can appear.

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$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

To evaluate the second term, we consider 4 cases for where the indices i_1 and i_2 can appear.

Case I: i_1 and i_2 can appear on the Kronecker δ-function in the square brackets.

Case I: Multiplicity $= 2(\ell-2)!$, Value $= 3S_{jji_3\ldots i_\ell}\dfrac{a_{1,\ell}}{\ell!}$.

Case II: i_1 can appear on the Kronecker δ-function in the square brackets, while i_2 appears as one of the indices of S.

Case II: Multiplicity $= 2(\ell-2)(\ell-2)!$, Value $= S_{jji_3\ldots i_\ell}\dfrac{a_{1,\ell}}{\ell!}$.

Review of Lecture 12

Case III: i_2 can appear on the Kronecker δ-function in the square brackets, while i_1 appears as one of the indices of S.

Case III: Multiplicity $= 2(\ell-2)(\ell-2)!$, Value $= S_{jji_3\ldots i_\ell}\dfrac{a_{1,\ell}}{\ell!}$.

Case IV: Both i_1 and i_2 can appear on S.

Case IV: Multiplicity $= (\ell-2)(\ell-3)(\ell-2)!$

Value $= \mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right]\dfrac{a_{1,\ell}}{\ell!}$.

Review of Lecture 12

$$\begin{aligned}\mathrm{Tr}\,(\mathrm{RHS}_1) = {} & S_{jji_3\ldots i_\ell}\left[1+\frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3+4(\ell-2)\right)\right] \\ & + \mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] a_{1,\ell}\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)} .\end{aligned}$$

$$\left[1+\frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3+4(\ell-2)\right)\right]=0 \quad\Longrightarrow\quad \boxed{a_{1,\ell} = -\frac{\ell(\ell-1)}{2(2\ell-1)} .}$$

Review of Lecture 12

Application to $\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}}$

$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}$ is traceless and symmetric and constructed from $\hat{\boldsymbol{n}}$, so it **must be proportional to** $\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\mathrm{TS}}$.

Then

$$\begin{aligned}\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}} &= \hat{\boldsymbol{n}}_{i_\ell}\left\{\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell} - \frac{\ell(\ell-1)}{2(2\ell-1)}\mathop{\mathrm{Sym}}_{i_1\ldots i_\ell}\left[\delta_{i_1 i_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right]+\ldots\right\} \\ &= \hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\left[1-\frac{\ell(\ell-1)}{2(2\ell-1)}\frac{2}{\ell}\right]+\ldots \\ &= \frac{\ell}{2\ell-1}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}+\ldots .\end{aligned}$$

The omitted terms cannot contribute to the term with no Kronecker delta functions, so

$$\boxed{\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\mathrm{TS}} .}$$

Integration over Spherical Harmonics

Result:

$$\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell}\{\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\}_{\rm TS}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\hat{\boldsymbol n}_{j_1}\ldots\hat{\boldsymbol n}_{j_{\ell'}}\}_{\rm TS}\right]$$
$$= 4\pi\frac{2^\ell \ell!^2}{(2\ell+1)!}\,C^{(\ell)}_{i_1\ldots i_\ell}\,C'^{(\ell)}_{i_1\ldots i_\ell}\quad \text{if } \ell'=\ell.$$

And it equals zero if $\ell'\neq\ell$.

We started this derivation on Monday.

Useful Integral: $\int_0^\infty r^{2\ell+2}\,e^{-r^2/2}\,\mathrm{d}r$

$$\boxed{\int_0^\infty r^{2\ell+2}\,e^{-r^2/2}\,\mathrm{d}r=\sqrt{\frac{\pi}{2}}\,\frac{(2\ell+1)!}{2^\ell \ell!}\ .}$$

Sketch of proof:

Define $\boxed{I_2(\lambda)\equiv\int_0^\infty dr\,e^{-\lambda r^2}\ .}$

Then

$$I_2^2(\lambda)=\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\,e^{-\lambda(x^2+y^2)}=\int_0^{\pi/2}d\phi\int_0^\infty r\,dr\,e^{-\lambda r^2}=\frac{\pi}{4\lambda}\ ,$$

so $I_2(\lambda)=\dfrac{1}{2}\sqrt{\dfrac{\pi}{\lambda}}$, and $\displaystyle\int_0^\infty r^{2\ell+2}\,e^{-r^2/2}\,\mathrm{d}r=(-1)^{\ell+1}\frac{\mathrm{d}^{\ell+1}}{\mathrm{d}\lambda^{\ell+1}}I_2(\lambda)\bigg|_{\lambda=\frac{1}{2}}$,

which leads to the boxed result above.

Another Useful Integral: $\int \mathrm{d}^3x\,e^{-\frac{|\vec r|^2}{2}+\vec J\cdot\vec r}$

Define $\boxed{I_1(\vec J)\equiv\int \mathrm{d}^3x\,e^{-\frac{|\vec r|^2}{2}+\vec J\cdot\vec r}\ .}$ Then

$$I_1(\vec J)=e^{\vec J^2/2}\int \mathrm{d}^3x\,e^{-\frac12(\vec r-\vec J)^2}=e^{\vec J^2/2}\int \mathrm{d}^3x'\,e^{-\frac12\vec r\,'^2}$$
$$=4\pi e^{\vec J^2/2}\int_0^\infty r^2\,\mathrm{d}r\,e^{-r^2/2}=\boxed{(2\pi)^{3/2}e^{\vec J^2/2}\ .}$$

The Most Useful Integral: $\int \mathrm{d}\Omega\,\hat n_{i_1}\ldots\hat n_{i_{2\ell}}$

where $\mathrm{d}\Omega\equiv\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=$ area element on sphere of radius 1.

Define $I_{i_1\ldots i_{2\ell}}\equiv\int \mathrm{d}\Omega\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_{2\ell}}$. Can find from $I_1(\vec J)\equiv\int \mathrm{d}^3x\,e^{-\frac{|\vec r|^2}{2}+\vec J\cdot\vec r}$, since

$$\frac{\partial^{2\ell}}{\partial J_{i_1}\ldots\partial J_{i_{2\ell}}}I_1(\vec J)=\int \mathrm{d}^3x\,x_{i_1}\ldots x_{i_{2\ell}}\,e^{-\frac{|\vec r|^2}{2}+\vec J\cdot\vec r}\ ,$$

so

$$\frac{\partial^{2\ell}}{\partial J_{i_1}\ldots J_{i_{2\ell}}}I_1(\vec J)\bigg|_{\vec J=0}=\int \mathrm{d}^3x\,x_{i_1}\ldots x_{i_{2\ell}}\,e^{-\frac{|\vec r|^2}{2}}$$
$$=\int_0^\infty r^2\,\mathrm{d}r\int \mathrm{d}\Omega\,x_{i_1}\ldots x_{i_{2\ell}}\,e^{-\frac{|\vec r|^2}{2}}=\int_0^\infty r^{2\ell+2}\,e^{-r^2/2}\,\mathrm{d}r\int \mathrm{d}\Omega\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_{2\ell}}$$
$$=I_{i_1\ldots i_{2\ell}}\int_0^\infty r^{2\ell+2}\,e^{-r^2/2}\,\mathrm{d}r\ ,$$

So we need to evaluate

$$\begin{aligned}\frac{\partial^{2\ell}}{\partial J_{i_1}\dots J_{i_{2\ell}}}\, e^{\vec{\boldsymbol J}^2/2} &= \frac{\partial^{2\ell-1}}{\partial J_{i_1}\dots\partial J_{i_{2\ell-1}}}\left[J_{i_{2\ell}}e^{\vec{\boldsymbol J}^2/2}\right]\\ &= \frac{\partial^{2\ell-2}}{\partial J_{i_1}\dots\partial J_{i_{2\ell-2}}}\left[\left(\delta_{i_{2\ell-1},i_{2\ell}} + J_{i_{2\ell-1}}J_{i_{2\ell}}\right)e^{\vec{\boldsymbol J}^2/2}\right]\\ &= \frac{\partial^{2\ell-3}}{\partial J_{i_1}\dots\partial J_{i_{2\ell-3}}}\Big[\big(J_{i_{2\ell-2}}\delta_{i_{2\ell-1},i_{2\ell}} + J_{i_{2\ell-2}}J_{i_{2\ell-1}}J_{i_{2\ell}}\\ &\qquad + \delta_{i_{2\ell-2},i_{2\ell-1}}J_{i_{2\ell}} + \delta_{i_{2\ell-2},i_{2\ell}}J_{i_{2\ell-1}}\big)\,e^{\vec{\boldsymbol J}^2/2}\Big]\end{aligned}$$

Want $\left.\dfrac{\partial^{2\ell}}{\partial J_{i_1}\dots J_{i_{2\ell}}}\, e^{\vec{\boldsymbol J}^2/2}\right|_{\vec{\boldsymbol J}=0}$. To have a nonzero term after setting $\vec{\boldsymbol J} = 0$, need for half of the derivatives to act on $e^{\vec{\boldsymbol J}^2/2}$, generating factors of J_i, and for the other half to differentiate each of these J_i's, producing Kronecker delta functions.

$$\boxed{\left.\frac{\partial^{2\ell}}{\partial J_{i_1}\dots J_{i_{2\ell}}}\, e^{\vec{\boldsymbol J}^2/2}\right|_{\vec{\boldsymbol J}=0} = \sum_{\text{all pairings}} \delta_{i_1,i_2}\,\delta_{i_3,i_4}\dots\delta_{i_{2\ell-1},i_{2\ell}}\ .}$$

Finally,

$$\boxed{I_{i_1\dots i_{2\ell}} = 4\pi\frac{2^\ell\ell!}{(2\ell+1)!}\sum_{\text{all pairings}} \delta_{i_1,i_2}\,\delta_{i_3,i_4}\dots\delta_{i_{2\ell-1},i_{2\ell}}\ .}$$

A pairing does not depend on the order in which the δ-function factors are written, and it does not depend on the ordering of the two indices on each δ. For example, for $\ell = 2$,

$$\sum_{\text{all pairings}} \delta_{i_1,i_2}\delta_{i_3,i_4} = \delta_{i_1,i_2}\delta_{i_3,i_4} + \delta_{i_1,i_3}\delta_{i_2,i_4} + \delta_{i_1,i_4}\delta_{i_1,i_3}\ .$$

Back to the original problem:

$$\begin{aligned}&\int \mathrm{d}\Omega\,\left[C^{(\ell)}_{i_1\dots i_\ell}\,\{\,\hat{\boldsymbol n}_{i_1}\dots\hat{\boldsymbol n}_{i_\ell}\,\}_{\rm TS}\right]\,\left[C'^{(\ell')}_{j_1\dots j_{\ell'}},\{\,\hat{\boldsymbol n}_{j_1}\dots\hat{\boldsymbol n}_{j_{\ell'}}\,\}_{\rm TS}\right]\\ &\qquad\propto C^{(\ell)}_{i_1\dots i_\ell}C'^{(\ell')}_{j_1\dots j_{\ell'}}\sum_{\text{all pairings}}\delta_{k_1,k_2}\dots\delta_{k_{\ell+\ell'-1},k_{\ell+\ell'}}\ ,\end{aligned}$$

where the sum is over all pairings of the full set of indices $\{i_1,\dots,i_\ell,j_1,\dots,j_{\ell'}\}$. If two i_m indices are paired, or if two j_m indices are paired, then the term vanishes, since $C^{(\ell)}$ and $C'^{\ell')}$ are traceless. The only nonzero contribution arises when every pair involves one i and one j, so the integral vanishes unless $\ell = \ell'$.

So

$$\begin{aligned}&\int d\Omega\,\left[C^{(\ell)}_{i_1\dots i_\ell}\,\{\,\hat{\boldsymbol n}_{i_1}\dots\hat{\boldsymbol n}_{i_\ell}\,\}_{\rm TS}\right]\,\left[C'^{(\ell')}_{j_1\dots j_{\ell'}},\{\,\hat{\boldsymbol n}_{j_1}\dots\hat{\boldsymbol n}_{j_{\ell'}}\,\}_{\rm TS}\right]\\ &\qquad= 4\pi\frac{2^\ell\ell!}{(2\ell+1)!}\,C^{(\ell)}_{i_1\dots i_\ell}\,C'^{(\ell)}_{j_1\dots j_\ell}\sum_{\text{all pairings}}\delta_{i_1,j_1}\dots\delta_{i_\ell,j_\ell}\ .\end{aligned}$$

All pairing produce an identical result, since the C's are symmetric. How many pairings are there? i_1 can be paired with any of ℓ j's, i_2 can be paired with any of $\ell-1$ j's, etc., so there are $\ell!$ pairings. Finally,

$$\boxed{\begin{aligned}&\int d\Omega\,\left[C^{(\ell)}_{i_1\dots i_\ell}\,\{\,\hat{\boldsymbol n}_{i_1}\dots\hat{\boldsymbol n}_{i_\ell}\,\}_{\rm TS}\right]\,\left[C'^{(\ell')}_{j_1\dots j_{\ell'}},\{\,\hat{\boldsymbol n}_{j_1}\dots\hat{\boldsymbol n}_{j_{\ell'}}\,\}_{\rm TS}\right]\\ &\qquad= 4\pi\frac{2^\ell\ell!^2}{(2\ell+1)!}\,C^{(\ell)}_{i_1\dots i_\ell}\,C'^{(\ell)}_{i_1\dots i_\ell}\quad\text{if } \ell' = \ell.\end{aligned}}$$