MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II December 8, 2019
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FORMULA SHEET FOR FINAL EXAM

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Table of Contents:

* In Sec. 29(e), the coefficient in the equation for $\vec{A}(\vec{r}, t)$ has been corrected. In Sec. 30(b), a minus sign in the equation for $\vec{E}$ has been deleted.

A few items below are marked with asterisks, ***. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

1. Index Notation:

Unit Vectors: $\hat{\boldsymbol{x}} \equiv \hat{\boldsymbol{\imath}} \equiv \hat{\boldsymbol{e}}_x \equiv \hat{\boldsymbol{e}}_1\,,\ \hat{\boldsymbol{y}} \equiv \hat{\boldsymbol{\jmath}} \equiv \hat{\boldsymbol{e}}_y \equiv \hat{\boldsymbol{e}}_2\,,\ \hat{\boldsymbol{z}} \equiv \hat{\boldsymbol{k}} \equiv \hat{\boldsymbol{e}}_z \equiv \hat{\boldsymbol{e}}_3\,,\ \vec{\boldsymbol{A}} \equiv A_i \hat{\boldsymbol{e}}_i$

General vector: $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} = x_i\,\hat{\boldsymbol{e}}_i$

Dot Product, Cross Product, and Determinant:

$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = A_i B_i\,, \qquad \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}_i = \epsilon_{ijk}A_jB_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2\cdots i_n} A_{1,i_1} A_{2,i_2}\cdots A_{n,i_n} \quad ***$$

Rotation of a Vector:

$$A'_i = R_{ij}A_j\,, \qquad \text{Orthogonality: } R_{ij}R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ: $$R_z(\phi)_{ij} = \begin{array}{c c} & \begin{array}{ccc} j{=}1 & j{=}2 & j{=}3 \end{array} \\ \begin{array}{c} i{=}1 \\ i{=}2 \\ i{=}3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{\boldsymbol{n}}$ by ϕ: ***

$$R(\hat{\boldsymbol{n}},\phi)_{ij} = \delta_{ij}\cos\phi + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j(1-\cos\phi) - \epsilon_{ijk}\hat{\boldsymbol{n}}_k\,\sin\phi\,.$$

2. Vector Calculus:

Gradient: $$(\vec{\boldsymbol{\nabla}}\varphi)_i = \partial_i\varphi = \frac{\partial\varphi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\varphi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{z}}\,, \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} \equiv \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl: $$(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})_i = \epsilon_{ijk}\partial_j A_k$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Laplacian: $$\nabla^2\varphi = \vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{\nabla}}\varphi) = \frac{\partial^2\varphi}{\partial x_i\partial x_i} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

3. Fundamental Theorems of Vector Calculus:

Gradient: $$\int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \vec{\boldsymbol{\nabla}}\varphi\cdot d\vec{\boldsymbol{\ell}} = \varphi(\vec{\boldsymbol{b}}) - \varphi(\vec{\boldsymbol{a}})$$

Divergence: $$\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}\,d^3x = \oint_S \vec{\boldsymbol{A}}\cdot d\vec{\boldsymbol{a}}$$

where S is the boundary of $\mathcal{V}$

Curl: $$\int_S (\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}})\cdot d\vec{\boldsymbol{a}} = \oint_P \vec{\boldsymbol{A}}\cdot d\vec{\boldsymbol{\ell}}$$

where P is the boundary of S

4. Vector Identities:

Triple Products:

$$\vec{\boldsymbol{A}}\cdot(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}}\cdot(\vec{\boldsymbol{C}}\times\vec{\boldsymbol{A}}) = \vec{\boldsymbol{C}}\cdot(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}})$$
$$\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}}) - \vec{\boldsymbol{C}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})$$

Product Rules:

$$\vec{\boldsymbol{\nabla}}(fg) = f\vec{\boldsymbol{\nabla}}g + g\vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}) + \vec{\boldsymbol{B}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}) + (\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}$$
$$\vec{\boldsymbol{\nabla}}\cdot(f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}})$$
$$\vec{\boldsymbol{\nabla}}\times(f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} - \vec{\boldsymbol{A}}\times\vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) = (\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}})$$

Second Derivatives:

$$\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}) = 0$$
$$\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}f) = 0$$
$$\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}) - \nabla^2\vec{\boldsymbol{A}}$$

5. Spherical Coordinates:

$x = r\sin\theta\cos\phi$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi$ $\qquad \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$

$z = r\cos\theta$ $\qquad \phi = \tan^{-1}(y/x)$

$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$ $\qquad \hat{\boldsymbol{x}} = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$

$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$ $\qquad \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$

$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$ $\qquad \hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$

Point separation: $d\vec{\boldsymbol{\ell}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$

Volume element: $d^3x \to r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}}$$
$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

6. Cylindrical Coordinates:

$x = s\cos\phi$	$s = \sqrt{x^2+y^2}$
$y = s\sin\phi$	$\phi = \tan^{-1}(y/x)$
$z = z$	$z = z$
$\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}} = \cos\phi\,\hat{\boldsymbol{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{y}} = \sin\phi\,\hat{\boldsymbol{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$

Point separation: $d\vec{\boldsymbol{\ell}} = ds\,\hat{\boldsymbol{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\boldsymbol{z}}$

Volume element: $d^3x \to s\,ds\,d\phi\,dz$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial s}\,\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial\varphi}{\partial z}\,\hat{\boldsymbol{z}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\boldsymbol{\phi}}$$
$$+\frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}$$

7. Delta Functions:

$$\int\varphi(x)\delta(x-x')\,dx = \varphi(x')\,, \qquad \int\varphi(\vec{\boldsymbol{r}})\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')\,d^3x = \varphi(\vec{\boldsymbol{r}}\,')$$

$$\int\varphi(x)\frac{d}{dx}\delta(x-x')\,dx = -\left.\frac{d\varphi}{dx}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i\frac{\delta(x-x_i)}{|g'(x_i)|}\,,\quad \text{where } i \text{ is summed over all points for which } g(x_i)=0$$

$$\nabla^2\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = -\vec{\boldsymbol{\nabla}}\cdot\left(\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')$$

$$\partial_i\partial_j\left(\frac{1}{r}\right) = -\partial_i\left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i\left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j-\delta_{ij}}{r^3} - \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{\boldsymbol{r}})$$

8. Electrostatics:

$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$, where

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i\frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')\,q_i}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int\frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\rho(\vec{\boldsymbol{r}}\,')\,d^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times10^{-12}\ \mathrm{C^2/(N{\cdot}m^2)}$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^9\ \mathrm{N{\cdot}m^2/C^2}$$

$$V(\vec{\boldsymbol{r}}) = V(\vec{\boldsymbol{r}}_0) - \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,')\cdot d\vec{\boldsymbol{\ell}}\,' = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d^3x'$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0\,, \qquad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V$$

$$\nabla^2V = -\frac{\rho}{\epsilon_0}\ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2V = 0\ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2V = 0$, then the average value of V on a spherical surface equals its value at the center.

9. Electrostatic Energy:*

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int d^3x\,d^3x'\,\frac{\rho(\vec{\boldsymbol{r}})\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}$$

$$W = \frac{1}{2}\int d^3x\,\rho(\vec{\boldsymbol{r}})V(\vec{\boldsymbol{r}}) = \frac{1}{2}\epsilon_0\int\left|\vec{\boldsymbol{E}}\right|^2d^3x$$

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* See Sec. 14(c) for energy in the presence of dielectrics.

10. Conductors:

Just outside, $\vec{\boldsymbol{E}} = \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{\boldsymbol{E}}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j\,, \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j\,, \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

11. Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cosh\gamma z \\ \sinh\gamma z \end{Bmatrix} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

or, more generally,

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cos\gamma z \\ \sin\gamma z \end{Bmatrix}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$, each of α, β, and γ can be real or imaginary, with $\sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$. $\begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix}$ means any linear combination of $\cos\alpha x$ or $\sin\alpha x$, but usually one or the other suffices.

12. Separation of Variables for Laplace's Equation in Spherical Coordinates:

12(a) Traceless Symmetric Tensor expansion:

$$\nabla^2\varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\nabla^2_{\text{ang}}\,\varphi = 0\,,$$

where the angular part is given by

$$\nabla^2_{\text{ang}}\,\varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\nabla^2_{\text{ang}}\, C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell} = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3$ (unit vector in (θ,ϕ) direction)

12(a)(i) General solution to Laplace's equation:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,, \quad \text{where } \vec{\boldsymbol{r}} = r\hat{\boldsymbol{n}}$$

12(a)(ii)Azimuthal Symmetry:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\text{TS}}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}$$

where $\{\ldots\}_{\text{TS}}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{\,1\,\}_{\text{TS}} = 1$$

$$\{\,\hat{\boldsymbol{z}}_i\,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i$$

$$\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j - \tfrac{1}{3}\delta_{ij}$$

$$\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k - \tfrac{1}{5}\big(\hat{\boldsymbol{z}}_i\delta_{jk} + \hat{\boldsymbol{z}}_j\delta_{ik} + \hat{\boldsymbol{z}}_k\delta_{ij}\big)$$

$$\begin{aligned}\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m\,\}_{\text{TS}} = {} & \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m - \tfrac{1}{7}\big(\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\delta_{km} + \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_k\delta_{mj} + \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_m\delta_{jk} \\ & + \hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\delta_{im} + \hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_m\delta_{ik} + \hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m\delta_{ij}\big) + \tfrac{1}{35}\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big)\end{aligned}$$

12(b) Legendre Polynomial / Spherical Harmonic expansion:

12(b)(i) Azimuthal Symmetry:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\text{TS}}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}$$

12(b)(ii) General solution to Laplace's equation:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m}\, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right)Y_{\ell m}(\theta,\phi)$$

12(b)(iii) Orthonormality: $\displaystyle\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,Y^*_{\ell' m'}(\theta,\phi)\,Y_{\ell m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm}$

12(b)(iv) Spherical Harmonics in terms of Traceless Symmetric Tensors:

$$Y_{\ell,m}(\theta,\phi) = C^{(\ell,m)}_{i_1\ldots i_\ell}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ ,$$

where

$$C^{(\ell,m)}_{i_1\ldots i_\ell} = \begin{cases} N(\ell,m)\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\ge 0\ ,\\ N(\ell,m)\{\,\hat{\boldsymbol{u}}^-_{i_1}\ldots\hat{\boldsymbol{u}}^-_{i_{|m|}}\,\hat{\boldsymbol{z}}_{i_{|m|+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\le 0\ ,\end{cases}$$

$$\hat{\boldsymbol{u}}^+ \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} + i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right)\ ,\quad \hat{\boldsymbol{u}}^- \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} - i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right)$$

$$N(\ell,m) = \frac{(-1)^m(2\ell)!}{2^\ell\ell!}\sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}} \qquad (\text{for } m\ge 0).$$

Connection between m and $-m$: $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$, which holds for all m.

12(b)(v) More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P^m_\ell(\cos\theta)e^{im\phi}$$

where $P^m_\ell(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P^m_\ell(x) = \frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

13. Electric Multipole Expansion:

13(a) First several terms:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\left[\,\frac{Q}{r} + \frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}{r^2} + \frac{\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j}{r^3}\,Q_{ij} + \cdots\right]\ , \text{ where}$$

$$Q = \int \mathrm{d}^3x\,\rho(\vec{\boldsymbol{r}}),\ \ p_i = \int d^3x\,\rho(\vec{\boldsymbol{r}})\,x_i\ ,\ \ Q_{ij} = \frac{1}{2}\int d^3x\,\rho(\vec{\boldsymbol{r}})(3x_ix_j - \delta_{ij}|\vec{\boldsymbol{r}}|^2)$$

$$\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}}) = -\frac{1}{4\pi\epsilon_0}\vec{\boldsymbol{\nabla}}\left(\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} - \frac{1}{3\epsilon_0}p_i\delta^3(\vec{\boldsymbol{r}})$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}}) = 0\ ,\qquad \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{1}{\epsilon_0}\rho_{\rm dip}(\vec{\boldsymbol{r}}) = -\frac{1}{\epsilon_0}\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}\delta^3(\vec{\boldsymbol{r}})$$

13(b) Traceless Symmetric Tensor version:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\,C^{(\ell)}_{i_1\ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ ,$$

where

$$C^{(\ell)}_{i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!}\int\rho(\vec{\boldsymbol{r}})\,\{\,x_{i_1}\ldots x_{i_\ell}\,\}_{\rm TS}\,\mathrm{d}^3x \qquad (\vec{\boldsymbol{r}}\equiv r\hat{\boldsymbol{n}}\equiv x_i\hat{\boldsymbol{e}}_i)$$

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{r'^\ell}{r^{\ell+1}}\,\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}'_{i_1}\ldots\hat{\boldsymbol{n}}'_{i_\ell}\ ,\qquad \text{for } r'<r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 = \frac{(2\ell)!}{2^\ell\ell!}\ ,\ \text{with}\ \ (-1)!!\equiv 1\ .$$

Reminder: $\{\ldots\}_{\rm TS}$ denotes the traceless symmetric part of $\ldots$.

13(c) Griffiths version (azimuthal symmetry only):

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^\ell\rho(\vec{\boldsymbol{r}}\,')P_\ell(\cos\theta')\,\mathrm{d}^3x$$

where θ' = angle between $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}\,'$.

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\frac{r^\ell_<}{r^{\ell+1}_>}P_\ell(\cos\theta')\ ,\qquad \frac{1}{\sqrt{1-2\lambda x+\lambda^2}} = \sum_{\ell=0}^{\infty}\lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell\ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell(x^2-1)^\ell\ , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1}\mathrm{d}x\,P_{\ell'}(x)P_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell'\ell}$$

13(d) Spherical Harmonic version:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{q_{\ell m}}{r^{\ell+1}}\,Y_{\ell m}(\theta,\phi)$$

where $q_{\ell m} = \int Y^*_{\ell m}r'^\ell\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x'$

$$\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)\ ,\qquad \text{for } r'<r$$

13(e) Properties of Traceless Symmetric Tensors:

13(e)(i) Trace Decomposition: Any symmetric tensor $S_{i_1\ldots i_\ell}$ can be written uniquely as

$$S_{i_1\ldots i_\ell} = S^{(\mathrm{TS})}_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\mathrm{Sym}}\left[M_{i_1\ldots i_{\ell-2}}\delta_{i_{\ell-1},i_\ell}\right] ,$$

where $S^{(\mathrm{TS})}_{i_1\ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1\ldots i_{\ell-2}}$ is a symmetric tensor tensor, and $\underset{i_1\ldots i_\ell}{\mathrm{Sym}}[\,xxx\,]$ symmetrizes xxx in the indices $i_1 \ldots i_\ell$. $S^{(\mathrm{TS})}_{i_1\ldots i_\ell}$ is called the ***traceless symmetric part*** of $S_{i_1\ldots i_\ell}$.

13(e)(ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1\ldots i_\ell}$:

$$\{\,S_{i_1\ldots i_\ell}\,\}_{\mathrm{TS}} = S_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\mathrm{Sym}}\Big[a_{1,\ell}\,\delta_{i_1 i_2}\,\delta^{j_1 j_2}\,S_{j_1 j_2 i_3\ldots i_\ell} + a_{2,\ell}\,\delta_{i_1 i_2}\,\delta_{i_3 i_4}\,\delta^{j_1 j_2}\,\delta^{j_3 j_4}\,S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big]$$

where

$$a_{n,\ell} = (-1)^n \frac{\ell!^2(2\ell-2n)!}{n!(\ell-2n)!(\ell-n)!(2\ell)!}$$

The series terminates when the number of i indices on S is zero or one.

13(e)(iii) Integration:

$$\int d\Omega\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_{2\ell}} = 4\pi\frac{2^\ell \ell!}{(2\ell+1)!}\sum_{\text{all pairings}}\delta_{i_1,i_2}\,\delta_{i_3,i_4}\ldots\delta_{i_{2\ell-1},i_{2\ell}}$$

where $\displaystyle\int d\Omega \equiv \int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi$

The integral vanishes if the number of $\hat{\boldsymbol n}$ factors is odd.

$$\int d\Omega\,\left[C^{(\ell)}_{i_1\ldots i_\ell}\{\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\,\}_{\mathrm{TS}}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\,\hat{\boldsymbol n}_{j_1}\ldots\hat{\boldsymbol n}_{j_{\ell'}}\,\}_{\mathrm{TS}}\right]$$

$$= \begin{cases} 4\pi\dfrac{2^\ell \ell!^2}{(2\ell+1)!}C^{(\ell)}_{i_1\ldots i_\ell}\,C'^{(\ell)}_{i_1\ldots i_\ell} & \text{if } \ell'=\ell \\ 0 & \text{otherwise}\end{cases}$$

13(e)(iv) Other identities:

$$\hat{\boldsymbol n}_{i_\ell}\{\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol n}_{i_1}\ldots\hat{\boldsymbol n}_{i_{\ell-1}}\,\}_{\mathrm{TS}} \qquad (\hat{\boldsymbol n} = \text{any unit vector})$$

$$\hat{\boldsymbol z}_{i_\ell}\,\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\{\,\hat{\boldsymbol u}^+_{i_1}\ldots\hat{\boldsymbol u}^+_{i_m}\,\hat{\boldsymbol z}_{i_{m+1}}\ldots\hat{\boldsymbol z}_{i_{\ell-1}}\,\}_{\mathrm{TS}}$$

where $\displaystyle\hat{\boldsymbol u}^+ \equiv \frac{1}{\sqrt2}\left(\hat{\boldsymbol e}_x + i\hat{\boldsymbol e}_y\right)$

*** For any symmetric traceless tensor $S_{i_1\ldots i_\ell}$,

$$\delta_{i_{\ell+2n-1},i_{\ell+2n}}\left\{\underset{i_1\ldots i_{\ell+2n}}{\mathrm{Sym}}\left[S_{i_1\ldots i_\ell}\,\underbrace{\delta_{i_{\ell+1},i_{\ell+2}}\ldots\delta_{i_{\ell+2n-1},i_{\ell+2n}}}_{n\text{ Kronecker }\delta\text{-functions}}\right]\right\}$$

$$= F(n,\ell)\underset{i_1\ldots i_{\ell+2n-2}}{\mathrm{Sym}}\left[S_{i_1\ldots i_\ell}\,\underbrace{\delta_{i_{\ell+1},i_{\ell+2}}\ldots\delta_{i_{\ell+2n-3},i_{\ell+2n-2}}}_{n-1\text{ Kronecker }\delta\text{-functions}}\right]$$

where

$$F(n,\ell) = \frac{2n(2\ell+2n+1)}{(\ell+2n)(\ell+2n-1)}$$

14. Electric Fields in Matter:

14(a) Electric Dipoles:

$$\vec{\boldsymbol p} = \int d^3x\,\rho(\vec{\boldsymbol r})\,\vec{\boldsymbol r}$$

$\rho_{\mathrm{dip}}(\vec{\boldsymbol r}) = -\vec{\boldsymbol p}\cdot\vec{\boldsymbol\nabla}_{\vec{\boldsymbol r}}\,\delta^3(\vec{\boldsymbol r}-\vec{\boldsymbol r}_d)$, where $\vec{\boldsymbol r}_d$ = position of dipole

$\vec{\boldsymbol F} = (\vec{\boldsymbol p}\cdot\vec{\boldsymbol\nabla})\vec{\boldsymbol E} = \vec{\boldsymbol\nabla}(\vec{\boldsymbol p}\cdot\vec{\boldsymbol E})$ (force on a dipole)

$\vec{\boldsymbol\tau} = \vec{\boldsymbol p}\times\vec{\boldsymbol E}$ (torque on a dipole)

$U = -\vec{\boldsymbol p}\cdot\vec{\boldsymbol E}$ (potential energy)

14(b) Electrically Polarizable Materials:

$\vec{\boldsymbol P}(\vec{\boldsymbol r})$ = polarization = electric dipole moment per unit volume

$\rho_{\mathrm{bound}} = -\nabla\cdot\vec{\boldsymbol P}$, $\sigma_{\mathrm{bound}} = \vec{\boldsymbol P}\cdot\hat{\boldsymbol n}$

$\vec{\boldsymbol D} \equiv \epsilon_0\vec{\boldsymbol E} + \vec{\boldsymbol P}$, $\vec{\boldsymbol\nabla}\cdot\vec{\boldsymbol D} = \rho_{\mathrm{free}}$, $\vec{\boldsymbol\nabla}\times\vec{\boldsymbol E} = 0$

Boundary conditions:

$$E^\perp_{\mathrm{above}} - E^\perp_{\mathrm{below}} = \frac{\sigma}{\epsilon_0} \qquad D^\perp_{\mathrm{above}} - D^\perp_{\mathrm{below}} = \sigma_{\mathrm{free}}$$

$$\vec{\boldsymbol E}^\parallel_{\mathrm{above}} - \vec{\boldsymbol E}^\parallel_{\mathrm{below}} = 0 \qquad \vec{\boldsymbol D}^\parallel_{\mathrm{above}} - \vec{\boldsymbol D}^\parallel_{\mathrm{below}} = \vec{\boldsymbol P}^\parallel_{\mathrm{above}} - \vec{\boldsymbol P}^\parallel_{\mathrm{below}}$$

14(c) Linear Dielectrics:

$\vec{\boldsymbol P} = \epsilon_0\chi_e\vec{\boldsymbol E}$, χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1+\chi_e)$ = permittivity, $\vec{\boldsymbol D} = \epsilon\vec{\boldsymbol E}$

$\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{n\alpha/\epsilon_0}{1-\frac{n\alpha}{3\epsilon_0}}$, where n = number density of atoms or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{\boldsymbol{p}} = \alpha\vec{\boldsymbol{E}}$)

Energy: $W = \dfrac{1}{2}\displaystyle\int \vec{\boldsymbol{D}}\cdot\vec{\boldsymbol{E}}\,\mathrm{d}^3x$ (linear materials only)

Force on a dielectric: $\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}}W$, where W is the potential energy stored in the system. Even if one or more potential differences are held fixed, the force can be found by computing the gradient of W with the total charge on each conductor fixed.

15. Magnetostatics:

15(a) Lorentz Force Law:

$$\vec{\boldsymbol{F}} \equiv \frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}) \ , \qquad \text{where } \vec{\boldsymbol{p}} = \gamma m_0\vec{\boldsymbol{v}} \ , \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\vec{\boldsymbol{F}} = \int I\,\mathrm{d}\vec{\boldsymbol{\ell}}\times\vec{\boldsymbol{B}} = \int \vec{\boldsymbol{J}}\times\vec{\boldsymbol{B}}\,\mathrm{d}^3x$$

15(b) Current Density:

Current through a surface S: $I_S = \displaystyle\int_S \vec{\boldsymbol{J}}\cdot\mathrm{d}\vec{\boldsymbol{a}}$

Charge conservation: $\dfrac{\partial\rho}{\partial t} = -\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}}$

Moving density of charge: $\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}}$

15(c) Biot-Savart Law:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}I\int\frac{\mathrm{d}\vec{\boldsymbol{\ell}}'\times(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^3}$$

where μ_0 = permeability of free space $\simeq 4\pi\times10^{-7}$ N/A^2 (to about 8-figure accuracy)

Examples:

Infinitely long straight wire: $\vec{\boldsymbol{B}} = \dfrac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}$

Infintely long tightly wound solenoid: $\vec{\boldsymbol{B}} = \mu_0 n I_0\,\hat{\boldsymbol{z}}$, where n = turns per unit length

Loop of current on axis: $\vec{\boldsymbol{B}}(0,0,z) = \dfrac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}\,\hat{\boldsymbol{z}}$

15(d) Vector Potential:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})_{\rm coul} = \frac{\mu_0}{4\pi}\int\frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}^3x' \ , \qquad \vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} \ , \qquad \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}}_{\rm coul} = 0$$

$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = 0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{\boldsymbol{A}}'(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) + \vec{\boldsymbol{\nabla}}\Lambda(\vec{\boldsymbol{r}})$ for any $\Lambda(\vec{\boldsymbol{r}})$. $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}$ is unchanged.

15(e) Ampère's Law:

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{J}} \ , \quad \text{or equivalently} \quad \int_P \vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{\ell}} = \mu_0 I_{\rm enc}$$

16. Magnetic Multipole Expansion:

16(a) Traceless Symmetric Tensor version:

$$A_j(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\mathcal{M}^{(\ell)}_{j;i_1i_2\ldots i_\ell}\frac{\{\,\hat{\boldsymbol{r}}_{i_1}\ldots\hat{\boldsymbol{r}}_{i_\ell}\,\}_{\rm TS}}{r^{\ell+1}}$$

where $\mathcal{M}^{(\ell)}_{j;i_1i_2\ldots i_\ell} = \dfrac{(2\ell-1)!!}{\ell!}\displaystyle\int \mathrm{d}^3x J_j(\vec{\boldsymbol{r}})\{\,x_{i_1}\ldots x_{i_\ell}\,\}_{\rm TS}$

Current conservation restriction: $\displaystyle\int \mathrm{d}^3x\,\underset{i_1\ldots i_\ell}{\mathrm{Sym}}(x_{i_1}\ldots x_{i_{\ell-1}}J_{i_\ell}) = 0$

where $\underset{i_1\ldots i_\ell}{\mathrm{Sym}}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1\ldots i_\ell$

Special cases:

$\ell = 1$: $\displaystyle\int \mathrm{d}^3x\,J_i = 0$

$\ell = 2$: $\displaystyle\int \mathrm{d}^3x\,(J_i x_j + J_j x_i) = 0$

Leading term (dipole): $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \dfrac{\mu_0}{4\pi}\dfrac{\vec{\boldsymbol{m}}\times\hat{\boldsymbol{r}}}{r^2}$,

where

$$m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}^{(1)}_{j;k}$$

$$\vec{\boldsymbol{m}} = \frac{1}{2}I\int_P \vec{\boldsymbol{r}}\times\mathrm{d}\vec{\boldsymbol{\ell}} = \frac{1}{2}\int \mathrm{d}^3x\,\vec{\boldsymbol{r}}\times\vec{\boldsymbol{J}} = I\vec{\boldsymbol{a}} \ ,$$

where $\vec{\boldsymbol{a}} = \displaystyle\int_S \mathrm{d}\vec{\boldsymbol{a}}$ for any surface S spanning P

$$\vec{B}_{\rm dip}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r}-\vec{m}}{r^3} + \frac{2\mu_0}{3}\,\vec{m}\,\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\vec{B}_{\rm dip}(\vec{r}) = 0\ , \qquad \vec{\nabla}\times\vec{B}_{\rm dip}(\vec{r}) = \mu_0\vec{J}_{\rm dip}(\vec{r}) = -\mu_0\vec{m}\times\vec{\nabla}\delta^3(\vec{r})$$

16(b) Griffiths version (azimuthal symmetry only, current in wires):

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\oint (r')^\ell P_\ell(\cos\theta')\,d\vec{\ell}\,, \text{ where } \theta' = \text{angle between } \vec{r} \text{ and } \vec{r}\,'.$$

17. Magnetic Fields in Matter:

17(a) Magnetic Dipoles:

$$\vec{m} = \frac{1}{2}I\int_P \vec{r}\times d\vec{\ell} = \frac{1}{2}\int d^3x\,\vec{r}\times\vec{J} = I\vec{a}\ , \text{ where } \vec{a} = \textstyle\int_S d\vec{a} \text{ for any } S \text{ spanning } P$$

$\vec{J}_{\rm dip}(\vec{r}) = -\vec{m}\times\vec{\nabla}_{\vec{r}}\,\delta^3(\vec{r}-\vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$ (force on a dipole)

$\vec{\tau} = \vec{m}\times\vec{B}$ (torque on a dipole)

$U = -\vec{m}\cdot\vec{B}$ (potential energy)

17(b) Magnetically Polarizable Materials:

$\vec{M}(\vec{r})$ = magnetization = magnetic dipole moment per unit volume

$\vec{J}_{\rm bound} = \vec{\nabla}\times\vec{M}\ , \qquad \vec{K}_{\rm bound} = \vec{M}\times\hat{n}$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}\ , \qquad \vec{\nabla}\times\vec{H} = \vec{J}_{\rm free} + \frac{\partial\vec{D}}{\partial t}\ , \qquad \vec{\nabla}\cdot\vec{B} = 0$$

Boundary conditions:

$B^\perp_{\rm above} - B^\perp_{\rm below} = 0 \qquad H^\perp_{\rm above} - H^\perp_{\rm below} = -(M^\perp_{\rm above} - M^\perp_{\rm below})$

$\vec{B}^\parallel_{\rm above} - \vec{B}^\parallel_{\rm below} = \mu_0(\vec{K}\times\hat{n}) \qquad \vec{H}^\parallel_{\rm above} - \vec{H}^\parallel_{\rm below} = \vec{K}_{\rm free}\times\hat{n}$

17(c) Linear Magnetic Materials:

$\vec{M} = \chi_m\vec{H}, \qquad \chi_m$ = magnetic susceptibility

$\mu = \mu_0(1+\chi_m)$ = permeability, $\qquad \vec{B} = \mu\vec{H}$

18. Magnetic Monopoles:

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\dfrac{q_m}{r^2}\hat{r}\ ;$ Force on a static monopole: $\vec{F} = q_m\vec{B}$

Angular momentum of monopole/charge system: $\vec{L} = \dfrac{\mu_0 q_e q_m}{4\pi}\hat{r}$, where $\hat{r}$ points from q_e to q_m

Dirac quantization condition: $\dfrac{\mu_0 q_e q_m}{4\pi} = \dfrac{1}{2}\hbar \times$ integer

19. Maxwell's Equations:

(i) $\vec{\nabla}\cdot\vec{E} = \dfrac{1}{\epsilon_0}\rho$ (iii) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$,

(ii) $\vec{\nabla}\cdot\vec{B} = 0$ (iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$

where $\mu_0\epsilon_0 = \dfrac{1}{c^2}$

Lorentz force law: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

Charge conservation: $\dfrac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$

20. Maxwell's Equations in Matter:

Polarization $\vec{P}$ and magnetization $\vec{M}$:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}\ , \quad \vec{J}_P = \frac{\partial\vec{P}}{\partial t}\ , \quad \vec{J}_b = \vec{\nabla}\times\vec{M}\ , \quad \rho = \rho_f + \rho_b\ , \quad \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P$$

Polarization current if the matter is in motion, with velocity $\vec{v}(\vec{r},t)$: ***

$$\vec{J}_P = \frac{\partial\vec{P}}{\partial t} - \vec{\nabla}\times(\vec{v}\times\vec{P})$$

Auxiliary Fields: $\vec{H} \equiv \dfrac{\vec{B}}{\mu_0} - \vec{M}\ , \qquad \vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$

Maxwell's Equations:

(i) $\vec{\nabla}\cdot\vec{D} = \rho_f$ (iii) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$,

(ii) $\vec{\nabla}\cdot\vec{B} = 0$ (iv) $\vec{\nabla}\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t}$

For linear media:

$\vec{D} = \epsilon\vec{E}\ , \qquad \vec{H} = \dfrac{1}{\mu}\vec{B}$

where ϵ = dielectric constant, μ = relative permeability

$\vec{J}_d \equiv \dfrac{\partial\vec{D}}{\partial t}$ = displacement current

21. Boundary Conditions:

$D_1^\perp - D_2^\perp = \sigma_f \qquad \vec{E}_1^\parallel - \vec{E}_2^\parallel = 0$

$E_1^\perp - E_2^\perp = \dfrac{1}{\epsilon_0}\sigma \qquad \vec{D}_1^\parallel - \vec{D}_2^\parallel = \vec{P}_1^\parallel - \vec{P}_2^\parallel$

$B_1^\perp - B_2^\perp = 0 \qquad \vec{H}_1^\parallel - \vec{H}_2^\parallel = -\hat{n}\times\vec{K}_f$

$H_1^\perp - H_2^\perp = M_2^\perp - M_1^\perp \qquad \vec{B}_1^\parallel - \vec{B}_2^\parallel = -\mu_0\hat{n}\times\vec{K}$

22. Maxwell's Equations with Magnetic (Monopole) Charge:

(i) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho_e$ (iii) $\vec{\nabla}\times\vec{E} = -\mu_0\vec{J}_m - \frac{\partial\vec{B}}{\partial t}$,

(ii) $\vec{\nabla}\cdot\vec{B} = \mu_0\rho_m$ (iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_e + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

Magnetic Lorentz force law: $\vec{F} = q_m\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right)$

23. Current, Resistance, and Ohm's Law:

$\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$, where σ = conductivity. $\rho = 1/\sigma$ = resistivity

Resistors: $V = IR$, $P = IV = I^2R = V^2/R$

Resistance in a wire: $R = \frac{\ell}{A}\rho$, where ℓ = length, A = cross-sectional area, and ρ = resistivity

Charging an RC circuit: $I = \frac{V_0}{R}e^{-t/RC}$, $Q = CV_0\left[1 - e^{-t/RC}\right]$

Emf (Electromotive force): $\mathcal{E} \equiv \oint(\vec{f}_s + \vec{E} + \vec{v}\times\vec{B})\cdot \mathrm{d}\vec{\ell}$. Here $\vec{f}_s$ is the force per unit charge of any external power source, such as a battery. $\vec{v}$ can be either the velocity of the wire, or the velocity of the charge carriers (since the velocity of the charge carriers relative to the wire points along the wire, and gives no contribution to $(\vec{v}\times\vec{B})\cdot \mathrm{d}\vec{\ell}$).

24. Inductance:

Universal flux rule: Whenever the flux through a loop changes, whether due to a changing $\vec{B}$ or motion of the loop, $\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$, where Φ_B is the magnetic flux through the loop. Here $\mathcal{E}$ is the emf induced by the magnetic field, and does not include possible contributions from sources $\vec{f}_s$.

Mutual inductance: $\Phi_2 = M_{21}I_1$, M_{21} = mutual inductance

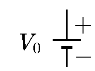
(Franz) Neumann's formula: $M_{21} = M_{12} = \frac{\mu_0}{4\pi}\oint_{P_1}\oint_{P_2}\frac{\mathrm{d}\vec{\ell}_1\cdot \mathrm{d}\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$

Self inductance: $\Phi = LI$, $\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$; L = inductance

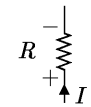
Self inductance of a solenoid: $L = n^2\mu_0\mathcal{V}$, where n = number of turns per length, $\mathcal{V}$ = volume

Rising current in an RL circuit: $I = \frac{V_0}{R}\left[1 - e^{\frac{R}{L}t}\right]$

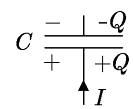
25. Circuit Elements:

Voltage source: 

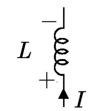
$V_+ - V_- \equiv V_0$

Resistor: 

$V_+ - V_- \equiv V_R = IR$

Capacitor: 

$V_+ - V_- \equiv V_C$, $Q = CV_C$, $\frac{dQ}{dt} = I.$

Inductor: 

$V_+ - V_- \equiv V_L = L\frac{dI}{dt}$

26. Conservation Laws:

Energy density: $u_{\mathrm{EM}} = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right]$

Poynting vector (flow of energy): $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

Conservation of energy:

Integral form: $\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{\mathrm{EM}} + U_{\mathrm{mech}}\right] = -\int\vec{S}\cdot \mathrm{d}\vec{a}$

Differential form: $\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$, where $u = u_{\mathrm{EM}} + u_{\mathrm{mech}}$

Momentum density: $\vec{\wp}_{\mathrm{EM}} = \frac{1}{c^2}\vec{S}$; $\frac{1}{c^2}S_i$ is the density of momentum in the i'th direction

Maxwell stress tensor: $T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2\right)$

where $-T_{ij} = -T_{ji}$ = flow in j'th direction of momentum in the i'th direction

Conservation of momentum:

Integral form: $\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\mathrm{mech},i} + \frac{1}{c^2}\int_{\mathcal{V}}S_i\,\mathrm{d}^3x\right) = \oint_S T_{ij}\,\mathrm{d}a_j$, for a volume $\mathcal{V}$ bounded by a surface S

Differential form: $\frac{\partial}{\partial t}\left(\wp_{\mathrm{mech},i} + \wp_{\mathrm{EM},i}\right) = \partial_j T_{ji}$

Angular momentum:

Angular momentum density (about the origin):

$$\vec{\boldsymbol{\ell}}_{\rm EM} = \vec{\boldsymbol{r}} \times \vec{\wp}_{\rm EM} = \epsilon_0[\vec{\boldsymbol{r}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}})]$$

27. Wave Equation in 1 Dimension:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \text{ , where } v \text{ is the wave velocity}$$

Sinusoidal waves:

$$f(z,t) = A\cos\left[k(z-vt)+\delta\right] = A\cos\left[kz-\omega t+\delta\right]$$

where

ω = angular frequency = $2\pi\nu$	ν = frequency
$v = \dfrac{\omega}{k}$ = phase velocity	δ = phase (or phase constant)
k = wave number	$\lambda = 2\pi/k$ = wavelength
$T = 2\pi/\omega$ = period	A = amplitude

Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$

Complex notation: $f(z,t) = \Re[\tilde{A}e^{i(kz-\omega t)}]$, where $\tilde{A} = Ae^{i\delta}$; "$\Re$" is usually dropped.

Wave velocities: $v = \dfrac{\omega}{k}$ = phase velocity; $v_{\rm group} = \dfrac{{\rm d}\omega}{{\rm d}k}$ = group velocity

28. Electromagnetic Waves:

Wave Equations: $\nabla^2\vec{\boldsymbol{E}} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{\boldsymbol{E}}}{\partial t^2} = 0$, $\quad \nabla^2\vec{\boldsymbol{B}} - \dfrac{1}{c^2}\dfrac{\partial^2\vec{\boldsymbol{B}}}{\partial t^2} = 0$

28(a) Linearly Polarized Plane Waves:

$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t) = \tilde{E}_0 e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)}\,\hat{\boldsymbol{n}}$, where $\tilde{E}_0$ is a complex amplitude, $\hat{\boldsymbol{n}}$ is a unit vector, and $\omega/|\vec{\boldsymbol{k}}| = v_{\rm phase} = c$.

$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{k}} = 0$ (transverse wave)

$$\vec{\boldsymbol{B}} = \frac{1}{c}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{E}}$$

Energy and Momentum:

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz-\omega t+\delta)}_{\text{averages to } 1/2} , \quad (\vec{\boldsymbol{k}} = k\,\hat{\boldsymbol{z}})$$

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = uc\,\hat{\boldsymbol{z}} , \qquad I \text{ (intensity)} = \left\langle |\vec{\boldsymbol{S}}| \right\rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\vec{\wp}_{\rm EM} = \frac{1}{c^2}\vec{\boldsymbol{S}} = \frac{u}{c}\,\hat{\boldsymbol{z}}$$

Electromagnetic Waves in Matter:

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{index of refraction}$$

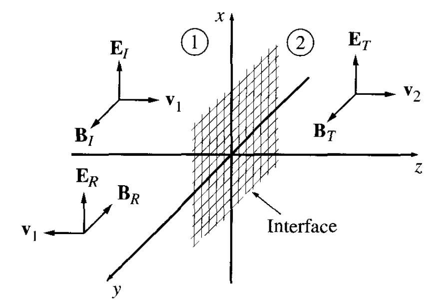
$$v = \text{phase velocity} = \frac{c}{n}$$

$$u = \frac{1}{2}\left[\epsilon|\vec{\boldsymbol{E}}|^2 + \frac{1}{\mu}|\vec{\boldsymbol{B}}|^2\right]$$

$$\vec{\boldsymbol{B}} = \frac{n}{c}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{E}}$$

$$\vec{\boldsymbol{S}} = \frac{1}{\mu}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \frac{uc}{n}\,\hat{\boldsymbol{z}}$$

28(b) Reflection and Transmission at Normal Incidence:



Boundary conditions:

$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \qquad \vec{\boldsymbol{E}}_1^\parallel = \vec{\boldsymbol{E}}_2^\parallel ,$$

$$B_1^\perp = B_2^\perp \qquad \frac{1}{\mu_1}\vec{\boldsymbol{B}}_1^\parallel = \frac{1}{\mu_2}\vec{\boldsymbol{B}}_2^\parallel .$$

Incident wave ($z < 0$):

$$\vec{\boldsymbol{E}}_I(z,t) = \tilde{E}_{0,I}\,e^{i(k_1 z-\omega t)}\,\hat{\boldsymbol{x}}$$

$$\vec{\boldsymbol{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}\,e^{i(k_1 z-\omega t)}\,\hat{\boldsymbol{y}} .$$

Transmitted wave ($z > 0$):

$$\vec{\boldsymbol{E}}_T(z,t) = \tilde{E}_{0,T}\,e^{i(k_2 z-\omega t)}\,\hat{\boldsymbol{x}}$$

$$\vec{\boldsymbol{B}}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0,T}\,e^{i(k_2 z-\omega t)}\,\hat{\boldsymbol{y}} .$$

Reflected wave ($z < 0$):

$$\vec{\boldsymbol{E}}_R(z,t) = \tilde{E}_{0,R}\,e^{i(-k_1 z-\omega t)}\,\hat{\boldsymbol{x}}$$

$$\vec{\boldsymbol{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0,R}\,e^{i(-k_1 z-\omega t)}\,\hat{\boldsymbol{y}} .$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1} , \qquad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2}$$

Applying boundary conditions and solving, approximating $\mu_1 = \mu_2 = \mu_0$,

$$\tilde{E}_{0,R} = \frac{n_1 - n_2}{n_1 + n_2}\,\tilde{E}_{0,I} \qquad E_{0,T} = \left(\frac{2n_1}{n_1+n_2}\right)\tilde{E}_{0,I}$$

29. Electromagnetic Potentials:

29(a) The fields: $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\ , \quad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \dfrac{\partial \vec{\boldsymbol{A}}}{\partial t}$

29(b) Gauge transformations: $\vec{\boldsymbol{A}}' = \vec{\boldsymbol{A}} + \vec{\boldsymbol{\nabla}}\Lambda\ , \quad V' = V - \dfrac{\partial \Lambda}{\partial t}$

29(c) Coulomb gauge: $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0 \quad \Longrightarrow \quad \nabla^2 V = -\dfrac{1}{\epsilon_0}\rho$ (but $\vec{\boldsymbol{A}}$ is complicated)

29(d) Lorenz gauge: $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = -\dfrac{1}{c^2}\dfrac{\partial V}{\partial t} \quad \Longrightarrow$

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho\ , \qquad \Box^2 \vec{\boldsymbol{A}} = -\mu_0 \vec{\boldsymbol{J}}\ , \qquad \text{where } \Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$\Box^2 \equiv$ D'Alembertian

29(e) Retarded time solutions (Lorenz gauge):

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\, \frac{\rho(\vec{\boldsymbol{r}}', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\ , \qquad \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}$$

where

$$t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}{c}$$

29(f) Liénard-Wiechert Potentials (potentials of a point charge):

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\, \frac{q}{\mathscr{r}\left(1 - \frac{1}{c}\vec{\boldsymbol{v}}_p \cdot \hat{\boldsymbol{\mathscr{r}}}\right)}$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi}\, \frac{q\vec{\boldsymbol{v}}_p}{\mathscr{r}\left(1 - \frac{1}{c}\vec{\boldsymbol{v}}_p \cdot \hat{\boldsymbol{\mathscr{r}}}\right)} = \frac{\vec{\boldsymbol{v}}_p}{c^2}\, V(\vec{\boldsymbol{r}}, t)$$

where $\vec{\boldsymbol{r}}_p$ and $\vec{\boldsymbol{v}}_p$ are the position and velocity of the particle at the retarded time t_r, and

$$\vec{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p\ , \qquad \mathscr{r} = |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p|\ , \qquad \hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p|}$$

29(g) Fields of a point charge (from the Liénard-Wiechert potentials):

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\mathscr{r}}})^3}\left[(c^2 - v_p^2)\vec{\boldsymbol{u}} + \vec{\boldsymbol{\mathscr{r}}} \times (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{a}}_p)\right]$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{\mathscr{r}}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t)$$

where $\vec{\boldsymbol{u}} = c\,\hat{\boldsymbol{\mathscr{r}}} - \vec{\boldsymbol{v}}_p$

30. Radiation:

30(a) Radiation from an oscillating electric dipole along the z axis:

$p(t) = p_0 \cos(\omega t)\ , \quad p_0 = q_0 d$

Approximations: $d \ll \lambda \ll r$,

$$V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]$$

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\boldsymbol{z}}$$

$$\vec{\boldsymbol{E}} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}\ , \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t)$$

Poynting vector: $\vec{\boldsymbol{S}} = \dfrac{1}{\mu_0}(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = \dfrac{\mu_0}{c}\left\{\dfrac{p_0\omega^2}{4\pi}\left(\dfrac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\boldsymbol{r}}$

Intensity: $\vec{\boldsymbol{I}} = \left\langle \vec{\boldsymbol{S}} \right\rangle = \left(\dfrac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\dfrac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}\ , \qquad$ using $\left\langle \cos^2 \right\rangle = \dfrac{1}{2}$

Total power: $\langle P \rangle = \displaystyle\int \left\langle \vec{\boldsymbol{S}} \right\rangle \cdot \mathrm{d}\vec{\boldsymbol{a}} = \dfrac{\mu_0 p_0^2 \omega^4}{12\pi c}$

30(b) Magnetic Dipole Radiation:

Dipole moment: $\vec{\boldsymbol{m}}(t) = m_0 \cos(\omega t)\,\hat{\boldsymbol{z}}$, at the origin

$$\vec{\boldsymbol{E}} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}\ , \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t)$$

Compared to the electric dipole radiation, $p_0 \to \dfrac{m_0}{c}\ , \qquad -\hat{\boldsymbol{\theta}} \to \hat{\boldsymbol{\phi}}$

30(c) General Electric Dipole Radiation:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi r}[(\hat{\boldsymbol{r}} \cdot \ddot{\vec{\boldsymbol{p}}})\hat{\boldsymbol{r}} - \ddot{\vec{\boldsymbol{p}}}\,]\ , \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = -\frac{\mu_0}{4\pi r c}[\hat{\boldsymbol{r}} \times \ddot{\vec{\boldsymbol{p}}}]$$

30(d) Multipole Expansion for Radiation:

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, … radiation, and also magnetic dipole, quadrupole, … radiation.

30(e) Radiation from a Point Particle:

When the particle is at rest at the retarded time,

$$\vec{\boldsymbol{E}}_{\mathrm{rad}} = \frac{q}{4\pi\epsilon_0 c^2 |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}[\,\hat{\boldsymbol{\mathscr{r}}} \times (\,\hat{\boldsymbol{\mathscr{r}}} \times \vec{\boldsymbol{a}}_p)]$$

Poynting vector: $\vec{\boldsymbol{S}}_{\mathrm{rad}} = \dfrac{1}{\mu_0 c}|\vec{\boldsymbol{E}}_{\mathrm{rad}}|^2\,\hat{\boldsymbol{\mathscr{r}}} = \dfrac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\dfrac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}}$

where θ is the angle between $\vec{\boldsymbol{a}}_p$ and $\hat{\boldsymbol{\mathscr{r}}}$.

Total power (Larmor formula): $P = \dfrac{\mu_0 q^2 a^2}{6\pi c}$

(valid for $\vec{v}_p = 0$ or $|\vec{v}_p| \ll c$)

Liénard's Generalization if $\vec{v}_p \neq 0$: ***

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}}_{\text{For relativists only}}$$

31. Radiation Reaction:***

31(a) Abraham-Lorentz formula:

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}$$

The Abraham-Lorentz formula is guaranteed to give the correct average energy loss for periodic or nearly periodic motion, but one would like a formula that works under general circumstances. The Abraham-Lorentz formula leads to runaway solutions which are clearly unphysical. The problem of radiation reaction for point particles in classical electrodynamics apparently remains unsolved.

32. Table of Legendre Polynomials $P_\ell(x)$:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \dfrac{1}{2}(3x^2 - 1)$

$P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$

$P_4(x) = \dfrac{1}{8}(35x^4 - 30x^2 + 3)$

33. Table of Spherical Harmonics $Y_{\ell m}(\theta,\phi)$:

$\ell = 0 \qquad Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$$\ell = 1 \quad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\,\sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}}\,\cos\theta \end{cases}$$

$$\ell = 2 \quad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\,\sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\,\sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \dfrac{1}{2}\sqrt{\dfrac{5}{4\pi}}\,(3\cos^2\theta - 1) \end{cases}$$

$$\ell = 3 \quad \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}}\,\sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}}\,\sin^2\theta\cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}}\,\sin\theta(5\cos^2\theta - 1)e^{i\phi} \\ Y_{30} = \dfrac{1}{2}\sqrt{\dfrac{7}{4\pi}}\,(5\cos^3\theta - 3\cos\theta) \end{cases}$$

$$\ell = 4 \quad \begin{cases} Y_{44} = \dfrac{3}{16}\sqrt{\dfrac{35}{2\pi}}\,\sin^4\theta e^{4i\phi} \\ Y_{43} = -\dfrac{3}{2}\sqrt{\dfrac{35}{2\pi}}\,\sin^3\theta\cos\theta e^{3i\phi} \\ Y_{42} = \dfrac{3}{8}\sqrt{\dfrac{5}{2\pi}}\,\sin^2\theta(7\cos^2\theta - 1)e^{2i\phi} \\ Y_{41} = -\dfrac{3}{2}\sqrt{\dfrac{5}{2\pi}}\,\sin\theta(7\cos^3\theta - 3\cos\theta)e^{i\phi} \\ Y_{40} = \dfrac{3}{\sqrt{4\pi}}(35\cos^4\theta - 30\cos^2\theta + 3) \end{cases}$$

For $m < 0$, use $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$, which is valid for all m.