8.07 Lecture Slides 5
September 18, 2019

ELECTROSTATICS

Announcements

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Blackboard Washer!

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$10 each day (for 10 minutes work!)

An opportunity not to be passed up!

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See me after class!

Whoops! This slide was omitted from the last lecture!

Derivative of a Distribution

A derivative of a distribution is defined so that generalized integration is consistent with integration by parts. So

$$\underbrace{\int_{-\infty}^{\infty} \varphi(x)\, \delta'(x - x_0)\, \mathrm{d}x}_{\text{Defined by this equation}} \equiv - \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\, \delta(x - x_0)\, \mathrm{d}x = -\varphi'(x_0)\ .$$

where a prime ($'$) denotes a derivative with respect to x. For an arbitrary distribution,

$$F'[\varphi(x)] \equiv -F[\varphi'(x)]\ .$$

Any distribution is **infinitely** differentiable.

$\delta'(x - x_0)$ can be thought of intuitively as the limit of a sequence of derivatives of Gaussians $g'_\sigma(x)$, provided that the limit is delayed until after integration.

A Subtlety

If $\boxed{f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)}$ is an identity, then we would expect that we should find the same result if we differentiate both sides with respect to x.

But if we differentiate the left-hand side, we find

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\delta(x-x_0)\right] = f'(x)\delta(x-x_0) + f(x)\delta'(x-x_0)\ ,}$$

while if we differentiate the right-hand side, we find

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x_0)\delta(x-x_0)\right] = f(x_0)\delta'(x-x_0)\ .}$$

So, were we wrong to say that $f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$? What's happening?

I'll ask you to tell me on Wednesday.

Resolution

The two expressions are equal!

LHS: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\delta(x-x_0)\right] = f'(x)\delta(x-x_0) + f(x)\delta'(x-x_0) \quad :$$

$$\begin{aligned} F_{LHS}[\varphi(x)] &= \int_{-\infty}^{\infty} \varphi(x)\left[f'(x)\delta(x-x_0) + f(x)\delta'(x-x_0)\right] \mathrm{d}x \\ &= \varphi(x_0)f'(x_0) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi(x)f(x)\right]\big|_{x_0} \\ &= -\varphi'(x_0)f(x_0) \end{aligned}$$

RHS: $$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x_0)\delta(x-x_0)\right] = f(x_0)\delta'(x-x_0) \ .$$

$$F_{RHS}[\varphi(x)] = \int_{-\infty}^{\infty} \varphi(x)f(x_0)\delta'(x-x_0)\,\mathrm{d}x = -\varphi'(x_0)f(x_0) \ .$$

Responding to a question: not part of the course

Fourier Transforms and the Dirac δ-function

The Fourier transform of a function $\varphi(t)$ can be defined by

$$\tilde{\varphi}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t' \, e^{i\omega t'} \varphi(t') \; .$$

The inversion formula gives

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega t} \tilde{\varphi}(\omega) \; .$$

Putting the two together and rearranging a little,

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \left[\int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega(t-t')} \right] \varphi(t') \; .$$

Putting the two together and rearranging a little,

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t' \left[\int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega(t-t')} \right] \varphi(t') \ .$$

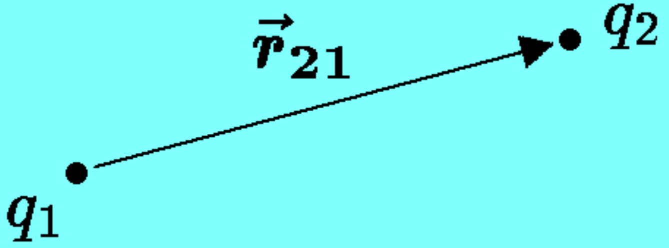
If $\varphi(t)$ is a Schwartz function, then $\tilde{\varphi}(\omega)$ exists and is also a Schwartz function.

From the above equation,

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega(t-t')} = \delta(t - t') \ .}$$

Coulomb's Law

$$\vec{F}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_{12}|^2} \hat{r}_{21}$$
$$= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_{12}|^3} (\vec{r}_2 - \vec{r}_1),$$



where

$$\vec{r}_{21} = \text{vector toward 2 from 1}$$
$$= \vec{r}_2 - \vec{r}_1$$

Units

In SI units, q is measured in coulombs.

On Monday, I told you that 1 coulomb was defined as the charge that passes when 1 ampere of current flows for 1 second. The ampere, I told you, was the current which, if maintained on two infinite parallel wires 1 meter apart, results in a force/length of 2×10^{-7} N/m, newtons per meter.

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It's gotten simpler. As of May 20, 2019, the coulomb is defined so that the charge of the electron is exactly $1.602176634 \times 10^{-19}$ C. Forget about the two wires.

An ampere is now defined as 1 coulomb/second.

Values of Electromagnetic Constants

Under the new system, ϵ_0 and μ_0 no longer have defined values. Now they are measured.

Of course to 3 significant figures, nothing has changed:

$$\boxed{\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\,m}^2}\ , \\ k &= \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^{9}\ \frac{\mathrm{N\,m}}{\mathrm{C}^2}\ , \\ \mu_0 &= 4\pi \times 10^{-7}\,\mathrm{N/A}^2\ . \end{aligned}}$$

And

$$\boxed{\frac{1}{\mu_0\epsilon_0} \equiv c^2}$$

remains exactly true.

Electric Field $\vec{E}(\vec{r})$

A point charge q_1 at position $\vec{\boldsymbol{r}}_1$ produces an electric field

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_1|^3} (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_1) \; .$$

The force on a second charge q_2 at position $\vec{\boldsymbol{r}}_2$ is then

$$\vec{\boldsymbol{F}}_2 = q_2 \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}_2) \; .$$

In general, define

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \lim_{q \to 0} \frac{\vec{\boldsymbol{F}}_q}{q} \; ,$$

where F_q is the force on a test charge q at position $\vec{\boldsymbol{r}}$. (The limit $q \to 0$ is used to avoid having the other charges disturbed by the test charge.)

Multiple Charges — Superposition

For a set of point charges q_i at positions $\vec{\boldsymbol{r}}_i$,

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i|^3}\ .}$$

Notation reminder: $\vec{\boldsymbol{r}} = x_i\hat{\mathbf{e}}_i = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + x\hat{\mathbf{e}}_z$.
(Griffiths Eq. (1.19), p. 8)

Continuous Charge Densities

Let

$$\rho(\vec{\boldsymbol{r}}) = \text{charge per unit volume.}$$

Then

$$\mathrm{d}q = \rho(\vec{\boldsymbol{r}})\,\mathrm{d}^3x \ ,$$

and

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}\,\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ .$$

For surface charge densities,

$$\boxed{\sigma(\vec{\boldsymbol{r}}) = \text{charge per unit area.}}$$

$$\boxed{\mathrm{d}q = \sigma(\vec{\boldsymbol{r}})\,\mathrm{d}a\ ,}$$

where da is the infinitesimal area. For linear charge densities,

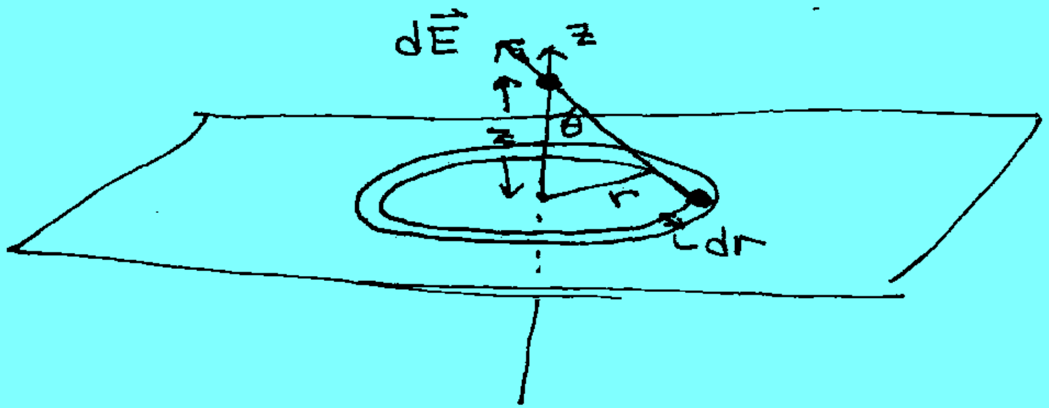
$$\boxed{\lambda(\vec{\boldsymbol{r}}) = \text{charge per unit length.}}$$

$$\boxed{\mathrm{d}q = \lambda(\vec{\boldsymbol{r}})\,\mathrm{d}\ell\ ,}$$

where dℓ is the infinitesimal length.

Sample Problem: Plane of Charge

Find $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ due to a plane of constant surface charge density σ.



$E_x = E_y = 0$ due to symmetry: there is no preferred direction in the x-y plane.

$$\mathrm{d}q = (2\pi r\,\mathrm{d}r)\sigma$$

$$\mathrm{d}E_z = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}q}{(z^2+r^2)}\cos\theta$$

$$\cos\theta = \frac{z}{\sqrt{z^2+r^2}}$$

Putting these together and integrating,

$$E_z = \frac{\sigma z}{2\epsilon_0} \int_0^\infty \frac{r\,\mathrm{d}r}{(z^2 + r^2)^{3/2}}$$

$$\boxed{\vec{\boldsymbol{E}} = \frac{\sigma}{2\epsilon_0} \hat{\boldsymbol{e}}_z \ .}$$

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Electric field does not fall off with distance.

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Can be seen by dimensional analysis. From the formula for $\vec{\boldsymbol{E}}$ of a point charge, we know that $E = \frac{1}{4\pi\epsilon_0} \times \ \mathrm{Q}/\mathrm{L}^2$, i.e., (charge/length^2). Since σ has units of Q/L^2, there cannot be any powers of the distance.

Electric Potential $V(\vec{r})$

Recall that

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}\, \rho(\vec{\boldsymbol{r}}\,')\, \mathrm{d}^3x'$$

$$= \frac{1}{4\pi\epsilon_0} \int \left(\vec{\boldsymbol{\nabla}}_{\vec{r}}\, \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \right) \rho(\vec{\boldsymbol{r}}\,')\, \mathrm{d}^3x'$$

$$= -\vec{\boldsymbol{\nabla}}_{\vec{r}}\, \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, \mathrm{d}^3x' \ .$$

So, define

$$\boxed{V(\vec{\boldsymbol{r}}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, \mathrm{d}^3x' \ ,}$$

and then

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V \ .}$$

So, define

$$\boxed{V(\vec{\boldsymbol{r}}) \equiv \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x'\ ,}$$

and then

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V\ .}$$

The potential $V(\vec{\boldsymbol{r}})$ is a scalar quantity, making it a simpler quantity than $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$, which is a vector.

Note that $V(\vec{\boldsymbol{r}})$ for a point charge falls off as $1/r$, which, when differentiated, gives an electric field that falls off like $1/r^2$.

Maxwell's Equations of Electostatics

What are the equations for the divergence and curl of $\vec{\boldsymbol{E}}$?

Curl:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} V = 0 \ ,$$

since the curl of a gradient is always zero.

Divergence:

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{\nabla} V) = -\nabla^2 V$$

$$= -\frac{1}{4\pi\epsilon_0} \nabla^2_{\vec{r}} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, \mathrm{d}^3x'$$

$$= -\frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}\,') \underbrace{\left(\nabla^2_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}\,'|} \right)}_{-4\pi\delta^3(\vec{r} - \vec{r}\,')} \mathrm{d}^3x'$$

$$= \frac{\rho(\vec{r})}{\epsilon_0} \; .$$

Putting these together,

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \; , \qquad \vec{\nabla} \times \vec{E} = 0 \; .}$$

Comparison with Full Maxwell Equations

Static Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} , \quad \vec{\nabla} \times \vec{E} = 0 .$$

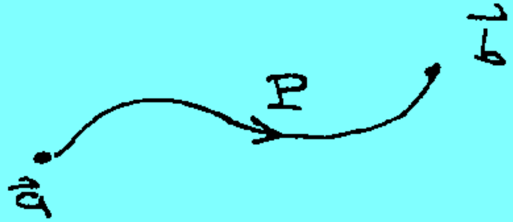
Full Maxwell equations for time-dependent $\vec{E}$:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} , \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} .$$

No need to think about time-dependent equations now. But it is worth noting that the static equation for the divergence is golden, while the static equation for the curl is only bronze.

Electric Potential as a Line Integral

We know that $\vec{E} = -\vec{\nabla} V$. Consider the line integral of $\vec{E}$ along a path $\mathcal{P}$ from point $\vec{a}$ to point $\vec{b}$. Using the fundamental theorem of the gradient,



$$\int_{\vec{a},\mathcal{P}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} = -\int_{\vec{a},\mathcal{P}}^{\vec{b}} \vec{\nabla} V \cdot d\vec{\ell} = V(\vec{a}) - V(\vec{b}) \; .$$

This allows an alternative definition of $V(\vec{r}\,)$:

$$\boxed{V(\vec{r}\,) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,') \cdot d\vec{\ell}\,' \; ,}$$

where $\vec{r}_0$ is an arbitrary reference point. By this definition, only potential differences are meaningful.

$$V(\vec{\boldsymbol{r}}) = V(\vec{\boldsymbol{r}}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,') \cdot \mathrm{d}\vec{\boldsymbol{\ell}}\,' \ ,$$

where $\vec{\boldsymbol{r}}_0$ is an arbitrary reference point. By this definition, only potential differences are meaningful.

By the previous definition,

$$V(\vec{\boldsymbol{r}}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \mathrm{d}^3x' \ ,$$

$V(\vec{\boldsymbol{r}}) \to 0$ as $|\vec{\boldsymbol{r}}| \to \infty$, at least for finite distributions of charge. So the previous definition corresponds to choosing $|\vec{\boldsymbol{r}}_0| \to \infty$, with $V(\vec{\boldsymbol{r}}_0) = 0$.

How to Remember the Formula

$$V(\vec{\boldsymbol{r}}) = V(\vec{\boldsymbol{r}}_0) - \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}') \cdot \mathrm{d}\vec{\boldsymbol{\ell}}' \ ,$$

How to Remember the Formula

$$V(\vec{\boldsymbol{r}}) = V(\vec{\boldsymbol{r}}_0) - \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,') \cdot \mathrm{d}\vec{\boldsymbol{\ell}}\,' \ ,$$

This slide was added after lecture.

Blackboard Discussion of Work and Energy

After the slides, we had a blackboard discussion of work and energy in electrostatics, with the following key results:

- ☆ For a test charge q moving in the field of other, fixed charges, $qV(\vec{r}) =$ potential energy of charge q at position $\vec{r}$.
- ☆ The work needed to assemble a system of n point charges q_i at positions $\vec{r}_i$, starting with all the charges at infinity, can be written as

$$\boxed{W_{\rm tot} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{q_i q_j}{r_{ij}}} \qquad (1)$$

or

$$\boxed{W_{\rm tot} = \frac{1}{2}\frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}} \ .} \qquad (2)$$

This slide was added after lecture.

☆ For a continuous charge density $\rho(\vec{\boldsymbol{r}})$, the work needed to assemble an arbitrary configuration, starting with all the charge at infinity, is given by

$$\boxed{W_{\text{tot}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x \int \mathrm{d}^3x'\, \frac{\rho(\vec{\boldsymbol{r}})\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\ .} \tag{3}$$

If $\rho(\vec{\boldsymbol{r}})$ is a well-defined **function**, there is no need to exclude $\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}}$, in analogy to excluding $j = i$ in Eqs. (1) and (2), because $\vec{\boldsymbol{r}}\,' = \vec{\boldsymbol{r}}$ is a set of measure zero, which does not affect the integral. But if $\rho(\vec{\boldsymbol{r}})$ is allowed to contain δ-functions, to describe point charges, then the above expression will diverge. When $\rho(\vec{\boldsymbol{r}})$ includes δ-functions, Eq. (3) describes the full energy, including the energy needed to create the point charges, which is infinite. Eqs. (1) and (2) only include the work needed to move the point charges into their positions from infinity.

☆ For a continuous charge density $\rho(\vec{r})$, the work needed to assemble an arbitrary configuration can also be written as

$$\boxed{W_{\rm tot} = \frac{1}{2}\int \mathrm{d}^3x\, \rho(\vec{r})V(\vec{r})} \tag{4}$$

or as

$$\boxed{W_{\rm tot} = \frac{1}{2}\epsilon_0 \int \left|\vec{E}\right|^2 \mathrm{d}^3x\ .} \tag{5}$$

☆ Aside: if the same exercise is carried out for Newtonian gravity, one finds

$$\boxed{W_{\rm tot} = -\frac{1}{8\pi G}\int |\vec{g}\,|^2\ \mathrm{d}^3x\ .} \tag{6}$$

The negative energy of gravitational fields allows for the possibility that the total energy of the universe is zero!