MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 14, 2019
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PROBLEM SET 1 SOLUTIONS

Revised September 20, 2019*

PROBLEM 1: VECTOR IDENTITIES INVOLVING CROSS PRODUCTS

(20 points)

(a) [4 pts] Show that

$$\epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} \ . \tag{1.1}$$

Note that the free indices j, k, n, and m take on 3 values each, for a total of $3^4 = 81$ combinations. Thus, the proof can be carried out by simply working out all 81 cases. However, it can be shortened considerably by identifying groups of cases that can be discussed together.

Consider first the case when either $j = k$ or $n = m$ or both. Then the LHS (left-hand side) is zero, since $\epsilon_{ijk} = 0$ if any two indices are identical. The RHS (right-hand side) is also zero in this case, because the two terms cancel if either $j = k$ or $n = m$ or both. Thus, we need only consider further the cases where $j \neq k$ and $n \neq m$.

If $j \neq k$ there is a unique index p that is different from both j and k. Similarly, if $n \neq m$, there is a unique index q that is different from both n and m. But then ϵ_{ijk} is nonzero only when $i = p$, and ϵ_{inm} is nonzero only when $i = q$. If we consider the cases $p = q$ and $p \neq q$, then all cases will be covered.

If $p = q$, then the sum over i on the LHS gives a nonzero contribution only when $i = p = q$. The fact that $p = q$ implies that the pair of indices j, k matches the pair n, m, which in turn means that either $j = n$ and $k = m$, or $j = m$ and $k = n$. In the first case ($j = n, k = m$) the two ϵ factors on the LHS are identical, so the value of the LHS is $+1$. In the second case the ordering of the indices on the ϵ factors differs by an interchange of the 2nd two indices, and so one factor is the negative of the other, and the LHS is -1. The delta-functions on the RHS also give $+1$ in the first case and -1 in the second, so the equality holds.

If $p \neq q$, then the LHS is zero, because there is no value of i for which both factors are nonzero. The RHS is also zero, since $p \neq q$ implies that the pair of indices j, k does not match the pair n, m. Thus at least one of the indices j or k must be different from both n and m. If it is j, then δ_{jn} and δ_{jm} vanish, so the RHS vanishes. Similarly, if it is k that is different from both n and m, then $\delta_{km} = \delta_{kn} = 0$, so again the RHS vanishes. So, if $p \neq q$, then LHS = RHS = 0. The formula has now been shown to hold in all cases.

* The solution to Problem 3(c) has been rewritten, adding more explanation.

(b) [4 pts] $\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk} A_j B_k$, so for any vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$,

$$\left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right)_i = \epsilon_{ijk} A_j \left(\vec{B} \times \vec{C}\right)_k \ , \tag{1.2}$$

but

$$\left(\vec{B} \times \vec{C}\right)_k = \epsilon_{kmn} B_m C_n \ . \tag{1.3}$$

So

$$\left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right)_i = \epsilon_{ijk} A_j \epsilon_{kmn} B_m C_n = \epsilon_{ijk}\epsilon_{kmn} A_j B_m C_n \ . \tag{1.4}$$

But $\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$, so using Eq. (1.1),

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \ . \tag{1.5}$$

So, Eq. (1.4) can be rewritten as

$$\begin{aligned} \left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right)_i &= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j B_m C_n \\ &= \delta_{im}\delta_{jn} A_j B_m C_n - \delta_{in}\delta_{jm} A_j B_m C_n \\ &= A_n B_i C_n - A_j B_j C_i = (\vec{A}\cdot\vec{C})B_i - (\vec{A}\cdot\vec{B})C_i \ , \end{aligned} \tag{1.6}$$

Which is what we want, that is,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B}) \ . \tag{1.7}$$

(c) [4 pts] Using ∂_i to denote $\partial/\partial x_i$, we can write

$$\begin{aligned} \vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \partial_i(\epsilon_{ijk}A_jB_k) = \epsilon_{ijk}\partial_i\left(A_jB_k\right) = \epsilon_{ijk}A_j\left(\partial_iB_k\right) + \epsilon_{ijk}B_k\left(\partial_iA_j\right) \\ &= -\epsilon_{jik}A_j\left(\partial_iB_k\right) + \epsilon_{kij}B_k\left(\partial_iA_j\right) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \ . \end{aligned} \tag{1.8}$$

(d) [4 pts] For any vector field $\vec{A}$, show that $\vec{A}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{2}\vec{\nabla}A^2 - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{A}$.

$$\left(\vec{\nabla}\times\vec{A}\right)_i = \epsilon_{ijk}\partial_jA_k \qquad \text{where} \qquad \partial_j \equiv \frac{\partial}{\partial x_j} \ . \tag{1.9}$$

So

$$\begin{aligned} \left[\vec{A}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_i &= \epsilon_{ijk}A_j\left(\vec{\nabla}\times\vec{A}\right)_k = \epsilon_{ijk}A_j\left(\epsilon_{kmn}\partial_mA_n\right) \\ &= \epsilon_{ijk}\epsilon_{kmn}A_j\partial_mA_n \ . \end{aligned} \tag{1.10}$$

Following the same steps that led to Eq. (1.6), we have

$$\begin{aligned}\left[\vec{A}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_i &= \epsilon_{kij}\epsilon_{kmn}A_j\partial_m A_n = \left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)A_j\partial_m A_n \\ &= \left(A_n\partial_i A_n - A_m\partial_m A_i\right) = \frac{1}{2}\partial_i\left(A_n A_n\right) - \left(A_m\partial_m\right)A_i\ ,\end{aligned} \tag{1.11}$$

or, as desired,

$$\vec{A}\times\left(\vec{\nabla}\times\vec{A}\right) = \frac{1}{2}\vec{\nabla}A^2 - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{A}\ . \tag{1.12}$$

(e) [4 pts] Show that $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)$.

Using $\left(\vec{A}\times\vec{B}\right)_i = \epsilon_{ijk}A_jB_k$, for any vectors $\vec{A}$ and $\vec{B}$,

$$\begin{aligned}\left[\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)\right]_i &= \epsilon_{ijk}\partial_j\left(\vec{A}\times\vec{B}\right)_k = \epsilon_{ijk}\partial_j\left(\epsilon_{kmn}A_mB_n\right) \\ &= \left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\partial_j\left(A_mB_n\right) = \partial_n\left(A_iB_n\right) - \partial_m\left(A_mB_i\right) \\ &= A_i\left(\partial_nB_n\right) + \left(B_n\partial_n\right)A_i - B_i\left(\partial_mA_m\right) - \left(A_m\partial_m\right)B_i\ ,\end{aligned} \tag{1.13}$$

or

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} - \vec{B}\left(\vec{\nabla}\cdot\vec{A}\right) - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\ . \tag{1.14}$$

PROBLEM 2: TRIPLE CROSS PRODUCTS *(10 points)*

(a) [5 pts] Griffiths Problem 1.2 (p. 4): Simple counterexample: $(\hat{\boldsymbol{e}}_x\times\hat{\boldsymbol{e}}_y)\times\hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z\times\hat{\boldsymbol{e}}_y = -\hat{\boldsymbol{e}}_x$ while on the other hand $\hat{\boldsymbol{e}}_x\times(\hat{\boldsymbol{e}}_y\times\hat{\boldsymbol{e}}_y) = \vec{0}$.

(b) [5 pts] Griffiths Problem 1.6 (p. 8): We expand

$$\begin{aligned}&\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) \\ &= [\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})] + [\vec{C}(\vec{A}\cdot\vec{B}) - \vec{A}(\vec{B}\cdot\vec{C})] + [\vec{A}(\vec{B}\cdot\vec{C}) - \vec{B}(\vec{A}\cdot\vec{C})] = 0\end{aligned}$$

as one can see that the terms cancel in pairs. This type of identity is a *Jacobi identity* satisfied also by commutators.

In order to find out when $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\times\vec{C}$, we rewrite it as

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0\,.$$

Comparing with the Jacobi identity we just proved, we see that this requires

$$\vec{B}\times(\vec{C}\times\vec{A}) = 0\,.$$

A cross product of two vectors vanishes if they are parallel or either one is zero. With non-zero vectors $\vec{A},\vec{B}$ and $\vec{C}$ this requires that either (1) $\vec{B}$ is parallel to $\vec{C}\times\vec{A}$ (equivalently $\vec{B}$ is orthogonal to both $\vec{A}$ and $\vec{C}$) or, (2) that $\vec{C}\times\vec{A}$ vanish (equivalently, $\vec{A}$ is parallel to $\vec{C}$.).

PROBLEM 3: PROPERTIES OF THE ROTATION MATRIX R *(15 points)*

(a) [5 pts] Show that the elements (R_{ij}) of the three-dimensional rotation matrix must satisfy the constraint

$$R_{ij}R_{ik} = \delta_{jk} \tag{3.1}$$

in order to preserve the length of $\vec{A}$ for all $\vec{A}$. Griffiths equation (1.31), page 11, is $\bar{A}_i = R_{ij}A_j$, where A_i is an arbitrary vector and $\bar{A}_i$ is what it becomes after a rotation. The fact that rotations preserve the length of a vector can then be written

$$\bar{A}^2 = A^2 = A_kA_k = \delta_{jk}A_jA_k\ , \tag{3.2}$$

but we also have

$$\bar{A}^2 = \bar{A}_i\bar{A}_i = (R_{ij}A_j)(R_{ik}A_k) = (R_{ij}R_{ik})A_jA_k\ . \tag{3.3}$$

In order for Eqs. (3.2) and (3.3) to hold for all A_k, we must have

$$(R_{ij}R_{ik} - \delta_{jk})A_jA_k = 0 \quad \text{for all } A_j\ , \tag{3.4}$$

which implies that

$$R_{ij}R_{ik} = \delta_{jk}\ , \tag{3.5}$$

as desired.

[The implication of Eq. (3.5) from Eq. (3.4) may seem obvious, and full credit will be given to any student who treats it as obvious. Note, however, that if $R_{ij}R_{ik}$ were not manifestly symmetric in j and k, then the antisymmetric part of $R_{ij}R_{ik}$ would be unconstrained. If we wanted to actually prove Eq. (3.5), we could define $S_{jk} \equiv R_{ij}R_{ik} - \delta_{jk}$, so $S_{jk}A_jA_k = 0$ for all A_j. Then we can choose $A_j = \delta_{j1}$, which implies that $S_{11} = 0$. We can similarly show that $S_{22} = S_{33} = 0$. Then we can choose $A_j = \delta_{j1} + \delta_{j2}$. Given what we already know that the diagonal entries of S_{jk} vanish, we would find that $S_{12} + S_{21} = 0$. Using the fact that S_{jk} is symmetric, this shows that $S_{12} = 0$, and similarly we can show that any matrix element with two different indices must vanish.]

(b) [5 pts] Using Eq. (3.1) show that $A_i = R_{ji}\bar{A}_j$.

Take the equation $\bar{A}_i = R_{ij}A_j$, multiply by R_{ik}, and sum over i, giving

$$R_{ik}\bar{A}_i = R_{ik}R_{ij}A_j = \delta_{kj}A_j = A_k\ . \tag{3.6}$$

Renaming indices, Eq. (3.6) is

$$A_i = R_{ji}\bar{A}_j\ , \tag{3.7}$$

as desired.

(c) [5 pts] Show that $\vec{\nabla} f$ transforms as a vector.

The problem set tells us that

$$\bar{f}(\vec{\bar{r}}) = f(\vec{r}) \,, \tag{3.8}$$

where $\bar{x}_i = R_{ij}x_j$, and where I am using $\vec{r} \equiv (x_1, x_2, x_3)$ and $\vec{\bar{r}} \equiv (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ to avoid writing triplets explicitly. The relation between $\vec{r}$ and $\vec{\bar{r}}$ can be inverted by using the result of part (c): Eq. (3.3) follows from Eq. (3.1) for any vector $\vec{A}$, so we can let $\vec{A} = \vec{r}$, concluding that

$$x_i(\vec{\bar{r}}) = R_{ji}\bar{x}_j \,, \tag{3.9}$$

which is just the relation $R^{-1} = R^T$ written in component notation. We can rewrite the equation for $\bar{f}$ to show explicitly its dependence on $\vec{\bar{r}}$ by writing

$$\bar{f}(\vec{\bar{r}}) = f\Big(\vec{r}(\vec{\bar{r}})\Big) \,. \tag{3.10}$$

Applying the standard chain rule,

$$\frac{\partial \bar{f}(\vec{\bar{r}})}{\partial \bar{x}_i} = \frac{\partial f\Big(\vec{r}(\vec{\bar{r}})\Big)}{\partial x_j}\frac{\partial x_j}{\partial \bar{x}_i} \,. \tag{3.11}$$

But, from Eq. (3.9),

$$\frac{\partial x_j}{\partial \bar{x}_i} = \frac{\partial}{\partial \bar{x}_i}\left(R_{kj}\bar{x}_k\right) = R_{kj}\frac{\partial \bar{x}_k}{\partial \bar{x}_i} = R_{kj}\delta_{ki} = R_{ij} \,, \tag{3.12}$$

so we have the final result

$$\frac{\partial \bar{f}}{\partial \bar{x}_i}(\vec{\bar{r}}) = R_{ij}\frac{\partial f}{\partial x_j}\Big(\vec{r}(\vec{\bar{r}})\Big) \,. \tag{3.13}$$

Dropping the explicit arguments, we have

$$\boxed{\frac{\partial \bar{f}}{\partial \bar{x}_i} = R_{ij}\frac{\partial f}{\partial x_j} \,,} \tag{3.14}$$

which is exactly what we were asked to show.

Apparently some students were confused by the omission of arguments in Eq. (3.14) (which is identical to Eq. (3.5) in the Problem Set). Physicists often omit arguments when they *expect* that the arguments should be obvious to the reader. But the arguments are not ***always*** obvious. A few rules are usually followed: if the same function has previously been used with an explicit argument, the argument is the same as the previous use. That is the case in this problem, where $\bar{f}$ was first written with argument $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, and f was first written with argument (x_1, x_2, x_3). (Note that (x_1, x_2, x_3) is identical to $\vec{r}$, as stated in the problem, and that there is no significant difference here between $\vec{r}$ and $\vec{r}(\vec{\bar{r}})$, since the problem told us that $\vec{r}$ and $\vec{\bar{r}}$ are related by an invertible transformation, $\bar{x}_i = R_{ij}x_j$.) Another common situation in which arguments are omitted is when the equation describes functions of position and maybe also time, and when the quantities have not previously been written with an explicit argument. Then you can assume that the equation is valid at all positions (and times), as long as all terms are evaluated at the same position and time. For example, on the cover of Griffiths' book, he writes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \,,$$

which is meant to be interpreted as

$$\nabla^2 V(\vec{r}, t) = -\frac{1}{\epsilon_0}\rho(\vec{r}, t) \,.$$

PROBLEM 4: USE OF THE GRADIENT *(10 points)*

Griffiths Problem 1.12 (p. 15): The height function is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

a(i) [3 pts] The top of the hill must correspond to an extremal point of $h(x, y)$, in other words a point where its partial derivatives vanish, or briefly, the gradient vanishes. Thus we have the conditions

$$\frac{\partial h}{\partial x} = 10(2y - 6x - 18) = 0 \,; \quad \frac{\partial h}{\partial y} = 10(2x - 8y + 28) = 0 \,.$$

These have just one solution at $(x, y) = (-2, 3)$, so the hilltop is located $\boxed{\text{2 miles west and 3 miles north of S. Hadley.}}$

You were not asked to do so, but a thorough analysis should include a determination of whether this stationary point (i.e., point of vanishing gradient) is a local maximum, minimum, or saddle point. This is determined by the eigenvalues of the Hessian matrix, the 2×2 matrix of partial derivatives,

$$H_{ij} \equiv \frac{\partial^2 h}{\partial x_i\, \partial x_j} = \begin{pmatrix} -60 & 20 \\ 20 & -80 \end{pmatrix} \,.$$

The eigenvalues can be found by solving the characteristic equation

$$\mathrm{Det}\,(H - \lambda I) = 0 \,,$$

which expands to $\lambda^2+140\lambda+4400=0$. The roots are $\lambda=-70\pm10\sqrt{5}=-92.36$ and -47.74. Since all roots are negative, the stationary point is indeed a local maximum, as the wording of the problem implied.

a(ii) [2 pts] The height there is $\boxed{h(-2,3)=720 \text{ feet.}}$

a(iii) [2 pts] For this we just need to evaluate the gradient

$$\nabla h = 10(2y-6x-18\,,\,2x-8y+28)$$

at $(x,y)=(1,1)$. This gives $\nabla h=(-220,220)$, a vector pointing NW, and corresponds to the direction of steepest slope. Since $|\nabla h| = 220\sqrt{2} = 311.13$ the slope at that point is $\boxed{311.1 \text{ feet per mile.}}$

(b) [3 pts] Griffiths Problem 1.13 part (a) (p. 15): We have $\mathscr{r}^2=(x-x')^2+(y-y')^2+(z-z')^2$. Then $\vec{\nabla}(\mathscr{r}^2)=2(x-x')\hat{e}_x+2(y-y')\hat{e}_y+2(z-z')\hat{e}_z=2(\vec{r}-\vec{r}\,')=2\vec{\mathscr{r}}$.

PROBLEM 5: THE DIRAC DELTA FUNCTION AND $\nabla^2(1/4\pi r)$ *(20 points)*

The function $f_a(r)$ is given by

$$f_a(r) = -\frac{1}{4\pi}\frac{1}{\sqrt{r^2+a^2}}\,. \tag{5.1}$$

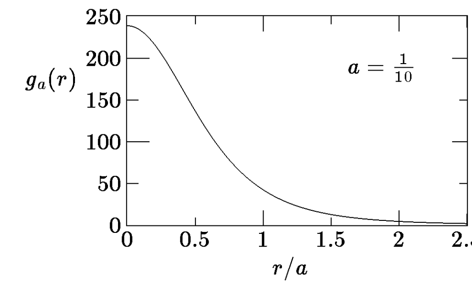
(a) [10 pts] To calculate $\nabla^2 f_a(r)$, use Eq. (1.73) (p. 42) of Griffiths for the Laplacian in spherical coordinates. For functions of only r, this gives

$$\begin{aligned}
g_a(r) = \nabla^2 f_a(r) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f_a(r)}{\partial r}\right)\\
&= -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+a^2}}\right) = \frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{(r^2+a^2)^{3/2}}\right)\\
&= \frac{1}{4\pi}\frac{1}{r^2}\left\{\left(\frac{3r^2}{(r^2+a^2)^{3/2}}\right) - \frac{3r^4}{(r^2+a^2)^{5/2}}\right\}\\
&= \frac{1}{4\pi}\frac{3}{(r^2+a^2)^{5/2}}\left\{(r^2+a^2)-r^2\right\}\\
&= \boxed{\frac{3}{4\pi}\frac{a^2}{(r^2+a^2)^{5/2}}\,.}
\end{aligned} \tag{5.2}$$

To sketch $g_a(r)$ as a function of r/a, rewrite it as

$$g_a(r) = \frac{3}{4\pi a^3}\frac{1}{\left(1+\frac{r^2}{a^2}\right)^{5/2}}\,. \tag{5.3}$$

Thus, the value of a affects only the multiplicative factor, and not the shape of the curve when plotted as a function of r/a. Choosing $a=\frac{1}{10}$, the graph looks like



(b) [7 pts] We must show that the integral of $g_a(r)$ over all space is 1. Denoting the integral by I,

$$I = 4\pi\int_0^\infty g_a(r)\,r^2\,dr = 4\pi\int_0^\infty \frac{3}{4\pi a^3}\frac{1}{\left(1+\frac{r^2}{a^2}\right)^{5/2}}\,r^2\,dr \tag{5.4}$$

$$= 3\int_0^\infty \frac{\eta^2\,d\eta}{(1+\eta^2)^{5/2}}\,, \tag{5.5}$$

where $\eta=r/a$. To carry out the integration use the substitution $\eta=\tan\theta$, so

$$d\eta = \frac{d\theta}{\cos^2\theta}$$

and

$$1+\eta^2 = \frac{1}{\cos^2\theta}\,.$$

Then

$$I = 3\int_0^{\pi/2}\tan^2\theta\,\cos^5\theta\,\frac{d\theta}{\cos^2\theta} = 3\int_0^{\pi/2}\sin^2\theta\,\cos\theta\,d\theta\,.$$

Now let $u=\sin\theta$, $du=\cos\theta\,d\theta$, so

$$I = 3\int_0^1 u^2\,du = 3\left[\frac{1}{3}u^3\right]\Bigg|_0^1 = 1\,, \tag{5.6}$$

as desired.

For those who like to do their computations using computer programs such as Mathematica, it would be acceptable here to set up the integral in the form of Eqs. (5.4) or (5.5), and then quote the answer from the computer program. If there are sometimes problems where I think that the detailed calculations are important to the understanding, I will make it clear in the wording of the problem that the calculations should be worked out by hand.

(c) [3 pts] Looking at Eq. (5.2), it is obvious that as $a \to 0$, the numerator vanishes and the denominator approaches r^5, so

$$\lim_{a\to 0} g_a(r) = 0 \qquad \text{for } r \neq 0. \tag{5.7}$$

PROBLEM 6: EXERCISES WITH δ-FUNCTIONS *(10 points)*

Recall that the Dirac delta is defined by what happens when it is integrated:

$$\int_a^b f(x)\delta(c-x)dx \equiv \begin{cases} f(c), & \text{if } c \in (a,b) \\ 0, & \text{if } c \notin (a,b). \end{cases} \tag{6.1}$$

The above "integral" is not really an integral, as it is not defined the area under any curve. Rather, it is an example of a distribution, which is defined simply as a linear mapping from a test function $f(x)$ to a number, given by the expression on the right-hand side. We write it as an integral, however, because we think of $\delta(c-x)$ intuitively as the limit of a sequence of functions that are more and more sharply peaked, such as

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-c)^2/(2\sigma^2)}$$

in the limit as $\sigma \to 0$, where the limit is valid as long as we remember to integrate first, and then take the limit.

(a) [2 pts] Evaluate $\int_0^5 \cos x\, \delta(x-\pi)\, dx$:

$$\int_0^5 \cos x\, \delta(x-\pi)\, dx = \cos\pi = -1 \ . \tag{6.2}$$

(b) [2 pts] Evaluate $\int_0^2 (x^3+3x+1)\delta(1-x)\, dx$:

$$\int_0^2 (x^3+3x+1)\delta(1-x)\, dx = 1^3+3+1 = 5 \ . \tag{6.3}$$

(c) [2 pts] Evaluate $\int_0^1 9x^2\, \delta(3x+1)\, dx$. This expression vanishes, since the argument of the δ-function vanishes at $x = -1/3$, which is outside the domain of integration.

(d) [4 pts] Integration over a δ-function of a function of x is defined to be exactly what one would find if one treated the δ-function as an ordinary function, and changed variables of integration to convert the expression into an expression of the form of Eq. (6.1). The problem tells us to consider an integral

$$I = \int \phi(x)\delta[f(x)]dx \ , \tag{6.4}$$

assuming that $\phi(x)$ has only one zero in the range of integration, at x_0, and that it is a simple zero. Let us first assume that $f'(x_0) > 0$, and then we will consider the alternative. We assume also that $f(x)$ is continuous, so there is a range (x_a, x_b) including x_0 over which $f'(x) > 0$. Since the δ-function gives a contribution only when its argument vanishes, we can restrict the region of integration to the interval (x_a, x_b). Within this range $y = f(x)$ is monotonic and hence invertible, so we call the inverse $x(y)$, where $x(0) = x_0$. Then

$$dy = f'(x)dx \quad \Longrightarrow \quad dx = \frac{dy}{f'(x)} \ , \tag{6.5}$$

so Eq. (6.4) becomes

$$I = \int_{f(x_a)}^{f(x_b)} \phi\big(x(y)\big)\,\delta(y) \frac{dy}{f'(x)|_{x(y)}} \ , \tag{6.6}$$

which can then be evaluated according to the rule defined by Eq. (6.1). This gives

$$I = \frac{\phi(x_0)}{f'(x_0)} \qquad (\text{for } f'(x_0) > 0) \ . \tag{6.7}$$

If $f'(x_0)$ were negative, then we would again obtain Eq. (6.6), but in this case $f(x_b) < f(x_a)$, so the upper limit of integration would then be smaller than the lower limit. The integral would then be evaluated as

$$I = -\int_{f(x_b)}^{f(x_a)} \phi\big(x(y)\big)\,\delta(y) \frac{dy}{f'(x)|_{x(y)}} = -\frac{\phi(x_0)}{f'(x_0)} \qquad (\text{for } f'(x_0) < 0) \ . \tag{6.8}$$

The two cases can then be combined by writing the result as

$$I = \frac{\phi(x_0)}{|f'(x_0)|} \ . \tag{6.9}$$

By comparing with the standard definition of Eq. (6.1), one sees that the result can be written

$$\boxed{\delta\big(f(x)\big) = \frac{\delta(x-x_0)}{|f'(x_0)|} \ .} \tag{6.10}$$

PROBLEM 7: EXERCISES WITH δ-FUNCTIONS *(10 points)*

Here I will write the answers and confirm the normalizations (of course, when doing this for the first time, one typically puts an undetermined constant which is fixed by the normalization condition).

(a) [2 pts] For the sphere

$$\boxed{\rho = \frac{Q}{4\pi R^2}\delta(r-R)\ .} \tag{7.1}$$

This clearly has the right units Q/L^3 (an L^2 from the R's and the other from the units of the delta function). Moreover with spherical symmetry $d^3x = 4\pi r^2 dr$ and thus

$$\int d^3x \rho(x) = \int 4\pi r^2 dr\, \frac{Q}{4\pi R^2}\delta(r-R) = Q\ , \tag{7.2}$$

as should be the case.

For the charge distributed over the circle, the angle $\theta = \pi/2$, and we write

$$\boxed{\rho = \frac{Q}{2\pi R^2}\delta(r-R)\delta\left(\theta - \frac{\pi}{2}\right) = \frac{Q}{2\pi R^2}\delta(r-R)\delta\left(\cos\theta\right)\ .} \tag{7.3}$$

Make sure you understand why $\delta\left(\cos\theta\right) = \delta\left(\theta - \frac{\pi}{2}\right)$. To confirm the normalization we integrate again:

$$\begin{aligned}\int_0^\infty (r^2\,dr)\int_{-1}^1 d(\cos\theta)\int_0^{2\pi} d\phi\left[\frac{Q}{2\pi R^2}\delta(r-R)\delta\left(\cos\theta\right)\right]\\ = R^2\cdot 1\cdot 2\pi\,\frac{Q}{2\pi R^2} = Q\ .\end{aligned} \tag{7.4}$$

Similarly, we could write the charge distribution in cylindirical coordinates as

$$\rho = \frac{Q}{2\pi R}\delta(r-R)\delta(z-z')\ . \tag{7.5}$$

The normalization is assured by the integration with volume element $d^3x = dr(rd\phi)dz$.

Note that in general the product of two distributions is undefined, so $\delta(x)^2$ does not make any sense. But there is an exception when the two distributions depend on

different variables, as in Eq. (7.3) above. It is defined by the integral of an arbitrary smooth test function $\varphi(r,\theta)$,

$$\int_0^\pi\int_0^\infty \varphi(r,\theta)\delta(r-R)\delta(\theta-\theta_0) \equiv \varphi(R,\theta_0)\ . \tag{7.6}$$

(b) [2 pts] We write

$$\boxed{\rho = \frac{\lambda}{2\pi b}\delta(s-b)\ ,} \tag{7.8}$$

which we confirm by integrating over a finite cylinder $\mathcal{C}_L$ bounded by the planes $z = \pm L$:

$$\int_{\mathcal{C}_L} d^3x\,\rho = \int_0^\infty sds\int_0^{2\pi} d\phi\int_{-L}^L dz\,\frac{\lambda}{2\pi b}\delta(s-b) = 2L\lambda\ , \tag{7.9}$$

which is the expected result.

(c) [2 pts] The volume element here is $d^3x = ds(sd\phi)dz$ and we thus guess

$$\boxed{\delta(\vec{r}-\vec{r}') = \frac{1}{s'}\delta(s-s')\delta(\phi-\phi')\delta(z-z')\ ,} \tag{7.7}$$

which is confirmed by computing

$$\int d^3x\,\delta(\vec{r}-\vec{r}') = \int_0^\infty sds\int_0^{2\pi} d\phi\int_{-\infty}^\infty dz\,\frac{1}{s'}\delta(s-s')\delta(\phi-\phi')\delta(z-z') = 1\ .$$

(d) [4 pts] Away from the origin,

$$\nabla^2\ln r = \vec{\nabla}\cdot(\vec{\nabla}\ln r) = \vec{\nabla}\cdot\left(\frac{\hat{\boldsymbol{r}}}{r}\right) = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right) = \vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r} + \frac{1}{r^2}\vec{\nabla}\cdot\vec{r}\ .$$

In two dimensions, $\vec{\nabla}\cdot\vec{r} = 2$ and thus we get

$$\nabla^2\ln r = -\frac{2\hat{\boldsymbol{r}}}{r^2}\cdot\vec{r} + \frac{2}{r^2} = 0\ . \tag{7.10}$$

Since we could be missing a delta function at the origin we use the analog of the divergence theorem, which in two dimensions relates the area integral of the divergence to the flux across the boundary line:

$$\int_{r<R} da\,\vec{\nabla}\cdot(\vec{\nabla}\ln r) = \int_0^{2\pi} Rd\phi\ \hat{\boldsymbol{r}}\cdot(\vec{\nabla}\ln r)\Big|_{r=R} = \int_0^{2\pi} Rd\phi\ \hat{\boldsymbol{r}}\cdot(\hat{\boldsymbol{r}}/r)|_{r=R} = 2\pi\ . \tag{7.11}$$

We thus conclude that

$$\boxed{\nabla^2\ln r = 2\pi\delta^2(\vec{r})\ .} \tag{7.12}$$

PROBLEM 8: COROLLARIES OF THE FUNDAMENTAL INTEGRAL THEOREMS *(15 points extra credit)*

(a) [4 pts] Starting with the divergence theorem,

$$\int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}\, d^3x = \int_S \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{a}}\ ,$$

consider the case $\vec{\boldsymbol{F}} = \psi \vec{\boldsymbol{c}}$ for a constant vector $\vec{\boldsymbol{c}}$. Then

$$\vec{\boldsymbol{c}} \cdot \int_V \vec{\boldsymbol{\nabla}} \psi d^3x = \vec{\boldsymbol{c}} \cdot \int_S \psi d\vec{\boldsymbol{a}}\ .$$

Since $\vec{\boldsymbol{c}}$ was arbitrary this means

$$\int_V \vec{\boldsymbol{\nabla}} \psi d^3x = \int_S \psi d\vec{\boldsymbol{a}}\ . \tag{8.1}$$

(b) [4 pts] This time choose $\vec{\boldsymbol{F}} = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{c}}$, so $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} = \vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$. Then

$$\vec{\boldsymbol{c}} \cdot \int_V \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\right) d^3x = \int_V \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{c}}\right) d^3x = \int_S \left(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{c}}\right) \cdot d\vec{\boldsymbol{a}} = -\vec{\boldsymbol{c}} \cdot \int_S \vec{\boldsymbol{A}} \times d\vec{\boldsymbol{a}}\ . \tag{8.2}$$

(c) [4 pts] Apply Stokes theorem to $\psi \vec{\boldsymbol{b}}$, where $\vec{\boldsymbol{b}}$ is a constant vector:

$$\int_S \vec{\boldsymbol{\nabla}} \times (\psi\, \vec{\boldsymbol{b}}) \cdot d\vec{\boldsymbol{a}} = \oint_\Gamma \psi \vec{\boldsymbol{b}} \cdot d\vec{\boldsymbol{\ell}}\ .$$

Using the identity $\vec{\boldsymbol{\nabla}} \times (\psi\, \vec{\boldsymbol{b}}) = \vec{\boldsymbol{\nabla}} \psi \times \vec{\boldsymbol{b}}$,

$$\int_S (\vec{\boldsymbol{\nabla}} \psi \times \vec{\boldsymbol{b}}) \cdot d\vec{\boldsymbol{a}} = \vec{\boldsymbol{b}} \cdot \int_S (d\vec{\boldsymbol{a}} \times \vec{\boldsymbol{\nabla}} \psi) = \vec{\boldsymbol{b}} \cdot \oint_\Gamma \psi\, d\vec{\boldsymbol{\ell}}\ .$$

Since this equation holds for all $\vec{\boldsymbol{b}}$, we can write

$$\int_S \vec{\boldsymbol{\nabla}} \psi \times d\vec{\boldsymbol{a}} = -\oint_\Gamma \psi\, d\vec{\boldsymbol{\ell}}\ , \tag{8.3}$$

where we have reordered the cross product to match the identity in the problem statement.

(d) [3 pts] The boundary ∂S of a closed surface S is zero, therefore by Stokes' theorem we have $\oint_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} = \int_{\partial S} \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{\ell}} = 0$. Alternatively, we can assume that the closed surface S is non-self-intersecting, so that Klein bottle surfaces are not allowed. Then S will necessarily be the boundary of some volume V. In that case we can use the divergence theorem and the fact that $\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = 0$ for all $\vec{\boldsymbol{A}}$ to show

$$\oint_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot d\vec{\boldsymbol{a}} = \int_V \vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})\, d^3x = 0\ . \tag{8.4}$$