*8.07 Lecture Slides 17*
*November 6, 2019*

# ELECTRIC FIELDS IN MATTER
# MAGNETOSTATICS

# Announcements

Quiz 2 will be given on Wednesday, November 13, one week from today. Problem Set 7 will be due this Friday, 11/8/19. The quiz will include material through Problem Set 7.

Office hour modifications:

Yitian Sun is away this week.

Wednesday (today), 5:00–6:00 pm: office hour by me, Room 6-322 (as usual).

Thursday, 5:30–6:30 pm: office hour by me, Room 8-320.

Friday, 1:30–2:30 pm: office hour by me, Room 8-320.

Friday, 2:30–3:30 pm: office hour by Marin, Room 8-320.

Tuesday (11/12/19): 3:00–4:00 pm: office hour by Marin, Room 6C-419.

Tuesday (11/12/19): 4:30–5:30 pm: office hour by me, Room 6-322.

Review Session, Tuesday evening, 11/12/19, 7:30 pm, Room 4-153, by Yitian.

A revised version of Problem Set 7 has been posted. Problem 4 has been reworded, correcting a miswritten symbol, clarifying the relevant approximations, and adding a hint.

# Energy Stored in $\vec{E}$ Fields in Matter

Starting point: If I change the free charge from $\rho_f(\vec{\boldsymbol{r}})$ to $\rho_f(\vec{\boldsymbol{r}}) + \Delta\rho_f(\vec{\boldsymbol{r}})$, the work that I have to do is

$$\boxed{\Delta W = \int \Delta\rho_f \, V \, \mathrm{d}^3x \ .}$$

Conclusion:

$$\boxed{\Delta W = \int \Delta\vec{\boldsymbol{D}} \cdot \vec{\boldsymbol{E}} \, \mathrm{d}^3x \ .}$$

If the material is a linear (not necessarily homogenous) dielectric, then $\vec{D} = \epsilon(\vec{r})\vec{E}$, and the total work needed to create the electric field is

$$\boxed{W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,\mathrm{d}^3x\ .}$$

# Intuitive Picture: "Spring" Energy

In the absence of matter, we showed earlier that the energy stored in an $\vec{\boldsymbol{E}}$ field is

$$W_{\text{vacuum}} = W_{\text{old}} = \frac{1}{2}\epsilon_0 \int |\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3x \ ,$$

while for linear dielectrics,

$$W_{\text{diel}} = \frac{1}{2} \int \epsilon(\vec{\boldsymbol{r}})|\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3x \ ,$$

But $\epsilon = (1+\chi_e)\epsilon_0$, with $\chi_e > 0$, so $\epsilon > \epsilon_0$. For the same $\vec{\boldsymbol{E}}$, $W_{\text{diel}} > W_{\text{vacuum}}$. Can write

$$W_{\text{diel}} = W_{\text{old}} + W_{\text{spring}} \ ,$$

where we can think of the dipoles as positive and negative charges connected by a spring, and then $W_{\text{spring}}$ can be interpretted as the energy necessary to stretch the springs.

# Capacitance of a Dielectric-Filled Capacitor

$$C_{\text{dielectric}} = \epsilon_r C_{\text{vacuum}} \ ,$$

where $C_{\text{dielectric}}$ is the capacitance of the capacitor when it it is filled with a linear dielectric material, $C_{\text{vacuum}}$ is the capacitance when the inside of the capacitor is empty, and $\epsilon_r$ is the dielectric constant of the material.

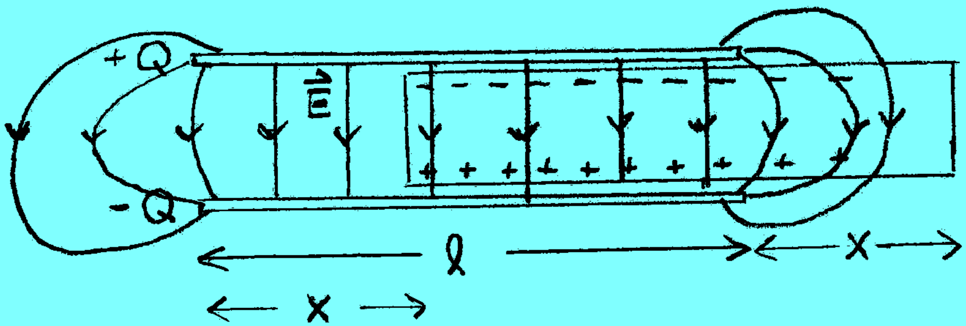
Since the energy stored in a capacitor is $W = \frac{1}{2}CV^2$, it follows that the stored energy when the capacitor is filled with a dielectric is given by

$$W_{\text{diel}} = \epsilon_r W_{\text{vacuum}} \ ,$$

which agrees with our general formulas for energy,

$$W_{\text{old}} = \frac{1}{2}\epsilon_0 \int |\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3x \ , \quad W = \frac{1}{2} \int \epsilon(\vec{\boldsymbol{r}}) |\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3x \ ,$$

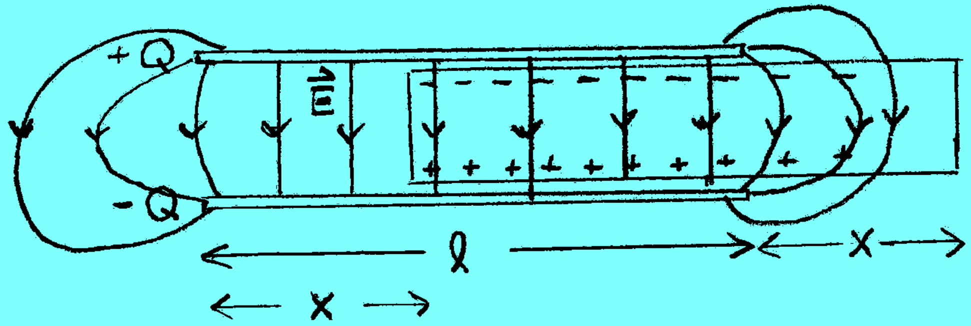
# Force on a Dielectric




Force is caused by fringing field, but can be calculated by considering only the total energy, and "virtual" work — just apply energy conservation to the process moving the dielectric by a distance $\delta x$.

Note (I didn't say this last time): there is no fringing fields at the left end of the dielectric. $V = \text{const}\, z$ satisfies all the boundary conditions.

Imagine moving dielectric at fixed $Q$.




$$C_{\text{vac}} = \frac{\epsilon_0 A}{d} \; , \qquad C_{\text{diel}} = \frac{\epsilon A}{d} \; ,$$

where $A$ is the area of the plates, and $d$ is the plate separation. Then

$$C(x) = \frac{x\left(\frac{\epsilon_0 A}{d}\right) + (\ell - x)\left(\frac{\epsilon A}{d}\right)}{\ell} \; .$$

Then

$$F_x = -\frac{\mathrm{d}W}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q^2}{2C(x)}\right) = \frac{Q^2}{2C(x)^2}\frac{\mathrm{d}C}{\mathrm{d}x} \; .$$

# Clausius-Mossotti Equation

In Griffiths, this is Problem 4.41, pp. 208.

Polarization can be induced by the polarization of atoms or molecules due to an applied $\vec{\boldsymbol{E}}$ field, or it can be due to the alignment of dipolar molecules. (See Griffiths, Sec. 4.1).

Here we consider the case of induced polarization. For an individual atom, the induced dipole moment is given by

$$\boxed{\vec{\boldsymbol{p}} = \alpha \vec{\boldsymbol{E}}_{\text{applied}} \ ,}$$

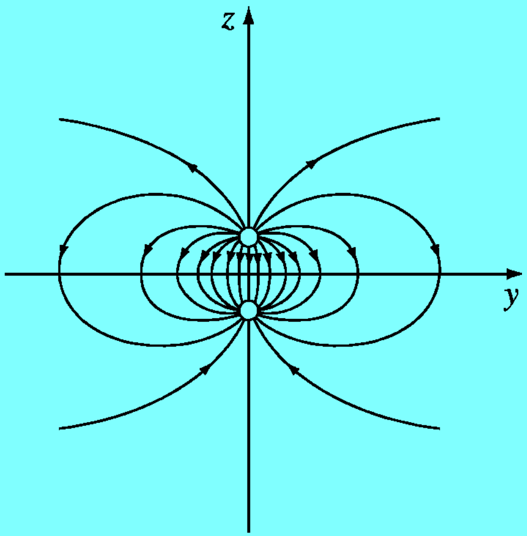
where $\alpha$ is the *atomic polarizability*, and $\vec{\boldsymbol{E}}$ is the applied electric field — i.e., the field in which the atom finds itself, not including its own field.

Consider a material with a number density $n$ of such atoms. Question: what is the electric susceptibility $\chi_e$, defined by

$$\boxed{\vec{\boldsymbol{P}} = \epsilon_0 \chi_e \vec{\boldsymbol{E}}_{\text{macro}} \ .}$$

Electric field of a dipole at the origin:

$$\boxed{\begin{aligned}\vec{E}_{\rm dip}(\vec{r}\,) &= -\vec{\nabla} V_{\rm dip} \\ &= \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{r}\,)\,\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0}\vec{p}\,\delta^3(\vec{r}\,)\ .\end{aligned}}$$



Field of a "physical" dipole

**FIGURE 3.37**

The delta function describes the contribution to $\int d^3x\, \vec{E}(\vec{r}\,)$ of the strong $\vec{E}$ field at the core of the dipole, as shown in the diagram.

Why doesn't the contribution of the core vanish as the separation $a \to 0$?

Consider how the relevant quantities scale as $a \to 0$:

Volume of core $\sim a^3$.

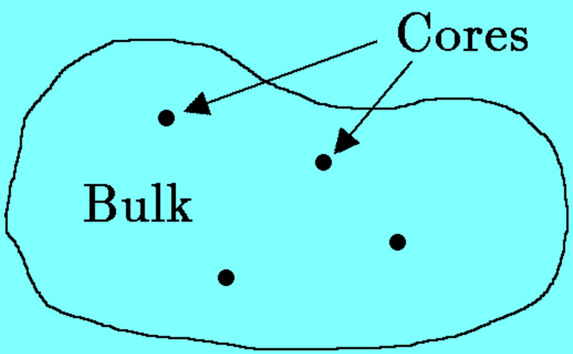
Typical field strength in the core $\sim \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{a^2}$.

The charge $q = |\vec{p}|/a$. So the typical field strength $\sim 1/a^3$, the volume $\sim a^3$, so $\int d^3x\, \vec{E}$ approaches a constant as $a \to 0$.

Given that $\vec{\boldsymbol{p}} = \alpha \vec{\boldsymbol{E}}_{\text{applied}}$ and $\vec{\boldsymbol{P}} = \epsilon_0 \chi_e \vec{\boldsymbol{E}}_{\text{macro}}$, it is tempting to think that $\chi_e = n\alpha/\epsilon_0$, but this is not quite right.

The subtlety is that $\vec{\boldsymbol{E}}_{\text{applied}} \neq \vec{\boldsymbol{E}}_{\text{macro}}$.

Picture of dielectric: The cores represent the $\delta$-function contributions to $\vec{\boldsymbol{E}}$ from each dipole, and the region outside the cores is called "bulk".

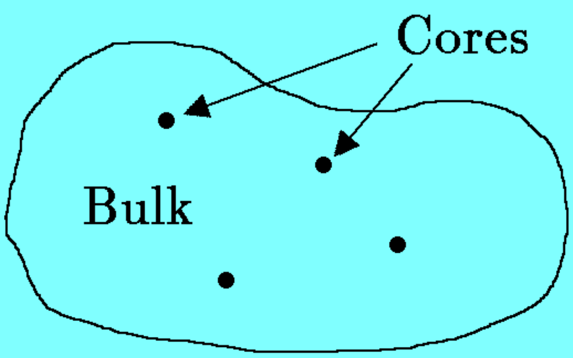



$$\boxed{\vec{\boldsymbol{E}}_{\text{macro}} = \vec{\boldsymbol{E}}_{\text{core}} + \vec{\boldsymbol{E}}_{\text{bulk}} \ ,}$$

where $\vec{\boldsymbol{E}}_{\text{macro}}$ is the average field, $\vec{\boldsymbol{E}}_{\text{core}}$ is the contribution to the average field from the cores, and $\vec{\boldsymbol{E}}_{\text{bulk}}$ is the average field outside of the cores.

$$\vec{\boldsymbol{p}} = \alpha \vec{\boldsymbol{E}}_{\text{applied}} \ , \qquad \vec{\boldsymbol{P}} = \epsilon_0 \chi_e \vec{\boldsymbol{E}}_{\text{macro}}$$

$$\vec{\boldsymbol{E}}_{\text{macro}} = \vec{\boldsymbol{E}}_{\text{core}} + \vec{\boldsymbol{E}}_{\text{bulk}} \ ,$$




Then

$$\vec{\boldsymbol{E}}_{\text{core}} = \frac{1}{\mathcal{V}} \int \mathrm{d}^3 x' \left[ -\frac{1}{3\epsilon_0} \sum_i \vec{\boldsymbol{p}}_i \, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \right]$$
$$= \frac{N_{\text{dipole}}}{\mathcal{V}} \left( -\frac{1}{3\epsilon_0} \vec{\boldsymbol{p}} \right) = -\frac{1}{3\epsilon_0} n \vec{\boldsymbol{p}} \ .$$

Key point:

$$\vec{\boldsymbol{E}}_{\text{applied}} = \vec{\boldsymbol{E}}_{\text{bulk}} \ .$$

Recall

$$\vec{p} = \alpha \vec{E}_{\rm bulk} \ , \qquad \vec{P} = \epsilon_0 \chi_e \vec{E}_{\rm macro}$$

$$\vec{E}_{\rm macro} = \vec{E}_{\rm core} + \vec{E}_{\rm bulk} \ ,$$

$$\vec{E}_{\rm core} = -\frac{1}{3\epsilon_0} n \vec{p} \ .$$

Then

$$\vec{E}_{\rm macro} = -\frac{n\alpha}{3\epsilon_0}\vec{E}_{\rm bulk} + \vec{E}_{\rm bulk} = \left(1 - \frac{n\alpha}{3\epsilon_0}\right)\vec{E}_{\rm bulk} \ ,$$

so

$$\vec{P} = n\vec{p} = n\alpha\vec{E}_{\rm bulk} = \frac{n\alpha}{1 - \frac{n\alpha}{3\epsilon_0}}\vec{E}_{\rm macro} \ .$$

Finally, the Clausius-Mossotti equation:

$$\boxed{\chi_e = \frac{n\alpha/\epsilon_0}{1 - \frac{n\alpha}{3\epsilon_0}} \ .}$$

# Magnetostatics

Until now, all charges were assumed to be static. To discuss magnetoSTATICS, we allow motion of charge, but still insist that $\rho(\vec{\boldsymbol{r}})$ does not change with time. There can be steady currents, like a current flowing around a wire loop, but the current must be independent of time. We will learn how to calculate the force on a test charge, which can have arbitrary motion.

**Magnetic fields and the Lorentz force law:** Much experimentation has shown that the force on a moving test charge, with charge $q$ and velocity $\vec{\boldsymbol{v}}$, can be described in terms of an electric field $\vec{\boldsymbol{E}}$ and a magnetic field $B$,

$$\boxed{\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \ .}$$

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \ .$$

If we had such a test charge, how would we find $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$?

1) Measure $\vec{\boldsymbol{F}}$ for $\vec{\boldsymbol{v}} = 0$. Then $\vec{\boldsymbol{E}} = \vec{\boldsymbol{F}}/q$.

2) Try different velocities. The force law implies that we will find some direction such that a motion in that direction does not affect $\vec{\boldsymbol{F}}$. Call this the $z$-axis. $\vec{\boldsymbol{B}}$ will then have only a $z$-component.

3) Give the test particle a velocity in the $x$-$y$ plane. The force law implies that $|\vec{\boldsymbol{F}} - q\vec{\boldsymbol{E}}| = q|\vec{\boldsymbol{v}}||\vec{\boldsymbol{B}}|$, so $|\vec{\boldsymbol{B}}|$ is determined.

4) To decide whether $\vec{\boldsymbol{B}} = |\vec{\boldsymbol{B}}|\hat{\boldsymbol{z}}$ or $\vec{\boldsymbol{B}} = -|\vec{\boldsymbol{B}}|\hat{\boldsymbol{z}}$, use right-hand rule. Use the one for which $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ is in the same direction as $\vec{\boldsymbol{F}} - q\vec{\boldsymbol{E}}$.

# Comments

1) $\vec{\boldsymbol{E}}$ and $c\vec{\boldsymbol{B}}$ have the same units.

2) Relativistically

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \ ,}$$

where

$$\boxed{\vec{\boldsymbol{p}} \equiv \gamma m_0 \vec{\boldsymbol{v}} \ , \quad \vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} \ , \quad \text{and} \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ ,}$$

where $m_0$ is the particle's rest mass.

3) Since $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{v}}$ are vectors, and $\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$, $\vec{\boldsymbol{B}}$ is a *pseudovector*. It behaves oppositely to normal vectors under mirror reflections or parity transformations. (A parity trasformation is $x \to -x$, $y \to -y$, **and** $z \to -z$.)

Sketch on blackboard.

4) Magnetic forces deflect moving particles, but do **NO** work:

$$\mathrm{d}W = \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \ ,$$

$$\text{Power} = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{\boldsymbol{F}} \cdot \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} \ .$$

Since $\vec{\boldsymbol{F}} \perp \vec{\boldsymbol{v}}$, Power $= 0$. So, for particles influenced only by $\vec{\boldsymbol{B}}$ fields, $|\vec{\boldsymbol{v}}| =$ constant.

5) Particle moving in a uniform $\vec{\boldsymbol{B}}$ field.

To show on blackboard:

The particle can move in a circle, with angular velocity

$$\boxed{\vec{\boldsymbol{\omega}} = -\frac{q\vec{\boldsymbol{B}}}{\gamma m_0}\ ,}$$

with radius

$$\boxed{r = \frac{p}{qB}\ .}$$

This slide was added after lecture.

# Blackboard Discussion: Law of Biot & Savart

For steady currents in a wire,

$$\boxed{\mathrm{d}\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\boldsymbol{\ell}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}}$$

where, to 8 significant figure accuracy,

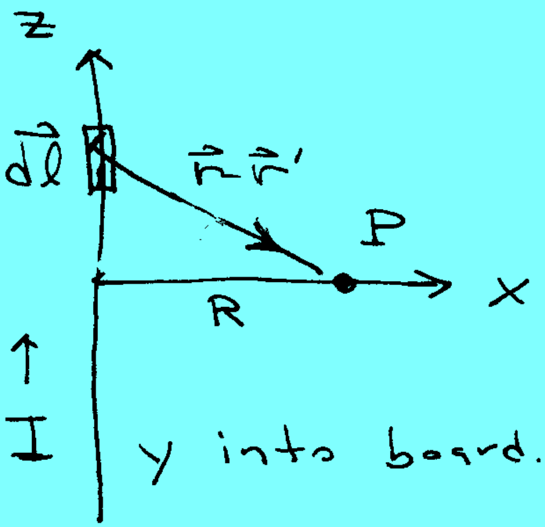
$$\boxed{\frac{\mu_0}{4\pi} = 10^{-7}\ \mathrm{N/A}^2\ ,}$$

where N = newton and A = ampere.

This slide was added after lecture.

# Example: Infinite Straight Wire

Alan's advice: I think it's best to attack these problems by writing the basic vectors explicitly, in terms of unit vectors is some coordinate system. From there, it is just a matter of mechanical manipulations.




$$\vec{r} - \vec{r}' = R\hat{e}_x - z\hat{e}_z$$

$$d\vec{\ell} = dz\,\hat{e}_z$$

$$d\vec{\ell} \times (\vec{r} - \vec{r}') = R\,dz\,\hat{e}_y \ .$$

$$\vec{B}(P) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{R\,dz}{(R^2 + z^2)^{3/2}}\,\hat{e}_y \ .$$

This slide was added after lecture.

$$\vec{\boldsymbol{B}}(P) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{R\,\mathrm{d}z}{(R^2 + z^2)^{3/2}}\,\hat{\boldsymbol{e}}_y \ .$$

Let $z \equiv Ru$, so

$$\vec{\boldsymbol{B}}(P) = \frac{\mu_0 I}{4\pi R}\,\hat{\boldsymbol{e}}_y \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1 + u^2)^{3/2}} \ .$$

With the further substitution $u = \tan\theta$, the integral can be evaluated,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(1 + u^2)^{3/2}} = 2 \ ,$$

so finally

$$\vec{\boldsymbol{B}}(P) = \frac{\mu_0 I}{2\pi R}\,\hat{\boldsymbol{e}}_y \ .$$