MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 9, 2019
Prof. Alan Guth

PROBLEM SET 7 SOLUTIONS

PROBLEM 1: FORCE ON A DIELECTRIC SLAB PART WAY INSIDE A CIRCULAR CAPACITOR *(15 points)*

We can consider the system as two capacitors: the part of the disks with only air between them, and the part with the dielectric between them. The two parts are attached in parallel—clearly the charge of the combined capacitor is the sum of the charges on the two capacitors, while the potential difference V applies to each of the capacitors, so the capacitance of the combined capacitor $C = Q/V$ is the sum of the two capacitances:

$$C = \frac{\epsilon_0 A_1}{d} + \frac{\epsilon A_2}{d}, \tag{1.1}$$

where A_1 and A_2 are the area of the empty region and dielectric region respectively. Due to fringe fields near the edges an attractive force occurs between the dielectric material and the conducting plates. This force can be found using the principle of virtual work, considering the thought experiment of pulling the dielectric outward by an infinitesimal distance. Although the capacitor is described in the problem as being held at a fixed potential difference V_0, we have a choice when we formulate our thought experiment. The actual force is determined by the distribution of charge, free and bound, when the capacitor is charged to potential V_0, and this distribution of charge will not change if the battery is disconnected after the capacitor is charged. The force could be measured statically, so it cannot depend on whether, if I were to later move the dielectric, I choose to move it at fixed Q or fixed V. The thought experiment is simpler if we imagine that the battery is disconnected for the thought experiment, pulling out the dielectric while keeping the charge Q on the capacitor fixed. We will therefore calculate it that way. [If the battery were connected, we would have to take into accou nt the work done by the battery, and it has been shown (in lecture, in the textbook on p. 196, and in Problem 4 of Problem Set 2) that one finds the same force, with a little extra effort. That will be shown again below.]

Let F_x^{applied} be the mechanical force that you need to apply in the x-direction to hold the dielectric slab in place. By applying an infinitesimally larger force you can move the dielectric to the right by a distance $\mathrm{d}x$, keeping the charge Q on the capacitor fixed at its original value, $Q_0 = C(x_0)V_0$. We can determine F_x^{applied} by using the fact that the change in the energy stored in the capacitor must equal the work done by F_x^{applied}. Thus,

$$F_x^{\text{applied}}\,\mathrm{d}x = \Delta\left(\frac{1}{2}\frac{Q_0^2}{C(x)}\right) \tag{1.2a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\frac{Q_0^2}{C(x)}\right)\mathrm{d}x \tag{1.2b}$$

$$= -\frac{1}{2}\frac{Q_0^2}{C(x_0)^2}\left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0}\mathrm{d}x \tag{1.2c}$$

$$= -\frac{1}{2}V_0^2\left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0}\mathrm{d}x \tag{1.2d}$$

$$= -\frac{V_0^2}{2d}\left.\left(\epsilon_0\frac{\mathrm{d}A_1}{\mathrm{d}x} + \epsilon\frac{\mathrm{d}A_2}{\mathrm{d}x}\right)\right|_{x=x_0}\mathrm{d}x\ . \tag{1.2e}$$

The left-hand edge of the square is at x, where initially $x = x_0$. When we move the square an infinitesimal distance $\mathrm{d}x$ to $x = x_0 + \mathrm{d}x$, the infinitesimal change of area for the empty region is $\mathrm{d}A_1 = \mathrm{d}x(2\sqrt{R^2 - x_0^2})$, and for the dielectric region is $\mathrm{d}A_2 = -\mathrm{d}x(2\sqrt{R^2 - x_0^2})$. Using these results with Eq. (1.2), we find

$$F_x^{\text{applied}} = \frac{V_0^2}{d}(\epsilon - \epsilon_0)\sqrt{R^2 - x_0^2}\ . \tag{1.3}$$

The force $\vec{\boldsymbol{F}}$ that the capacitor exerts on the dielectric is the negative of F_x^{applied}, so

$$\boxed{\vec{\boldsymbol{F}} = -\frac{V_0^2}{d}(\epsilon - \epsilon_0)\sqrt{R^2 - x_0^2}\,\hat{\boldsymbol{e}}_x\ .} \tag{1.4}$$

Finally, for those who insist on keeping V fixed during the thought experiment, we show explicitly that we get the same answer. When the capacitance changes with fixed V, a charge $dQ = V_0\mathrm{d}C$ must flow to the capacitor, and must therefore flow through the battery. The work the battery does is then

$$dW_{\text{battery}} = V_0\,\mathrm{d}Q = V_0^2\,\mathrm{d}C = V_0^2\left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0}\mathrm{d}x\ . \tag{1.5}$$

We now describe the energy stored in the capacitor as

$$W_{\text{capacitor}} = \frac{1}{2}CV_0^2\ , \tag{1.6}$$

since $V = V_0$ is fixed. Eq. (1.2a) is then replaced by

$$F_x^{\text{applied}}\, \mathrm{d}x + V_0^2 \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0} \mathrm{d}x = \Delta\left(\frac{1}{2}C(x)V_0^2\right) , \tag{1.7}$$

which says that the work that I do, plus the work that the battery does, is equal to the change in the stored energy in the capacitor. Expanding the right-hand side, we find

$$F_x^{\text{applied}}\, \mathrm{d}x + V_0^2 \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0} \mathrm{d}x = \frac{1}{2}V_0^2 \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=x_0} \mathrm{d}x , \tag{1.8}$$

which is clearly equivalent to Eq. (1.2d).

PROBLEM 2: J.J. THOMPSON AND THE CHARGE TO MASS RATIO OF THE ELECTRON *(10 points)*

Griffiths Problem 5.3 (p. 216).

(a) The beam is not deflected by the uniform magnetic field $\vec{\boldsymbol{B}}$ and the electric field $\vec{\boldsymbol{E}}$ when the forces exerted by these fields cancel each other:

$$\vec{\boldsymbol{F}} = q\big[\vec{\boldsymbol{E}} + (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})\big] = 0 . \tag{2.1}$$

Since the fields $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{B}}$ and the velocity $\vec{\boldsymbol{v}}$ of the particles in the beam are perpendicular to each other, both fields will exert a force along the same direction. This allows us to write Eq. (2.1) in terms of the magnitudes of the fields E and B and the magnitude of the velocity v:

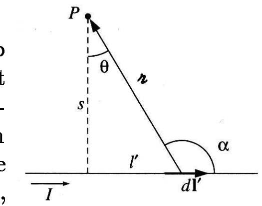
$$E = vB \quad \Longrightarrow \quad \boxed{v = \frac{E}{B}\ .} \tag{2.2}$$

(b) When the electric field is turned off, each particle will experience only the force from the magnetic field $\vec{\boldsymbol{B}}$. This force will always be orthogonal to its velocity and thus will cause it to move along a circular orbit with the radius R. In this case, the force exerted by the magnetic field must supply the centripetal force for the orbit of radius R, which gives us

$$qvB = m\frac{v^2}{R} \quad \Longrightarrow \quad \frac{q}{m} = \frac{v}{BR} = \boxed{\frac{E}{B^2R}\ .} \tag{2.3}$$

PROBLEM 3: EXAMPLES OF THE USE OF THE BIOT-SAVART LAW *(15 points)*

Griffiths, Problem 5.8 (p. 228).



Griffiths, Fig. 5.18

(a) We can find the magnetic field in the center of a loop by summing the contributions of each of the four straight sides of the square loop. The figure on the right illustrates one convenient way to parametrize the calculation for a single side: we consider a straight piece of wire located at a distance s from P, the center of the loop, and break it into infinitesimal segments of wire, each located in the angular sector $(\theta;\ \theta + \mathrm{d}\theta)$ with respect to P. Then the horizontal coordinate of each infinitesimal segment is $\ell' = s\tan\theta$ along the wire, and the length of the segment is $\mathrm{d}\ell' = s\,\mathrm{d}(\tan\theta) = s\,\mathrm{d}\theta/\cos^2\theta$. The distance from $\mathrm{d}\vec{\boldsymbol{\ell}}$ to P is $\mathscr{r} = s/\cos\theta$. The infinitesimal magnetic field at point P is pointed towards the reader and has magnitude

$$dB = \frac{\mu_0 I}{4\pi}\frac{|d\vec{\boldsymbol{\ell}}' \times \hat{\boldsymbol{\mathscr{r}}}|}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\left(\frac{s\,\mathrm{d}\theta}{\cos^2\theta}\right)\cos\theta\frac{\cos^2\theta}{s^2} = \frac{\mu_0 I}{4\pi s}d\theta\cos\theta. \tag{3.1}$$

Then the magnetic field produced by a piece of wire at a distance s in the angular sector (θ_1, θ_2) with respect to P is equal

$$B = \int_{\theta_1}^{\theta_2} dB(\theta) = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1) , \tag{3.2}$$

where $s = R$, $\theta_2 = -\theta_1 = \frac{\pi}{4}$ for one side of the square. The total magnetic field is

$$\vec{\boldsymbol{B}}_{\text{tot}} = 4\frac{\mu_0 I}{4\pi R}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right]\hat{\boldsymbol{z}} = \boxed{\frac{\sqrt{2}\mu_0 I}{\pi R}\,\hat{\boldsymbol{z}}\ ,} \tag{3.3}$$

where the z axis is perpendicular to the square, upwards in Fig. 5.22 of the problem set.

(b) For an n-sided polygon, the angle between two lines drawn from the center to the two corners of one edge is $2\pi/n$. Then we get $\theta_2 = -\theta_1 = \frac{\pi}{n}$. The total magnetic field is

$$\vec{\boldsymbol{B}}_{\text{tot}} = n\frac{\mu_0 I}{4\pi R}\left[\sin\frac{\pi}{n} - \sin\left(-\frac{\pi}{n}\right)\right]\hat{\boldsymbol{z}} = \boxed{\frac{n\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)\hat{\boldsymbol{z}}\ .} \tag{3.4}$$

(c) When $n \to \infty$,

$$\vec{B}_{\text{tot}} = \lim_{n\to\infty} \frac{\mu_0 I}{2R} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \hat{z} = \frac{\mu_0 I}{2R} \lim_{x\to 0} \frac{\sin x}{x} = \boxed{\frac{\mu_0 I}{2R}\hat{z}\ .} \tag{3.5}$$

This result agrees with our previous result for the magnetic field at the center of a circular current carrying wire.

PROBLEM 4: MAGNETIC FIELD ON THE AXIS OF A TIGHTLY WOUND SOLENOID *(15 points)*

(a) The wire forms a helix, but we can approximate each turn as a circle in the x-y plane. Since the number of turns per unit length is n, which is large compared to $1/R$, the spacing between the turns is small compared to R, and hence neighboring turns give almost identical contributions to the magnetic field at the origin. We can therefore write a formula for the contribution of the turns between z and $z + \mathrm{d}z$, for which the current is $\mathrm{d}I = nI\mathrm{d}z$. Then the magnetic field at the origin produced by these turns is given by Griffiths' Eq. (5.41) as

$$\mathrm{d}B = \frac{\mu_0 nI\mathrm{d}z}{2} \frac{R^2}{(R^2+z^2)^{3/2}}\ , \tag{4.1}$$

where $d\vec{B} = dB\hat{z}$ is the contribution from the turns between z and $z+\mathrm{d}z$. The total magnetic field at the origin is then

$$\vec{B} = \frac{\mu_0 nI\hat{z}}{2} \int_{z_1}^{z_2} \frac{R^2}{(R^2+z^2)^{3/2}} dz\ . \tag{4.2}$$

Using the change of variable $z = R\tan\theta$, with $\mathrm{d}z = R\frac{\mathrm{d}\theta}{\cos^2\theta}$, we find

$$\int \frac{R^2}{(R^2+z^2)^{3/2}}\mathrm{d}z = \int \frac{R^2\cos^3\theta}{R^3}\frac{R\mathrm{d}\theta}{\cos^2\theta} = \int \cos\theta\mathrm{d}\theta = \sin\theta\ . \tag{4.3}$$

Using $\sin\theta = \frac{z}{\sqrt{z^2+R^2}}$ and restoring the limits of integration,

$$\boxed{\vec{B} = \frac{\mu_0 nI}{2}\left[\frac{z_2}{\sqrt{z_2^2+R^2}} - \frac{z_1}{\sqrt{z_1^2+R^2}}\right]\hat{z}\ .} \tag{4.4}$$

(b) Since $\lim_{z_2\to\infty} \frac{z_2}{\sqrt{z_2^2+R^2}} = 1$ and $\lim_{z_1\to-\infty} \frac{z_1}{\sqrt{z_1^2+R^2}} = -1$, for an infinite solenoid we find

$$\boxed{\vec{B}_{\text{solenoid}} = \mu_0 nI\hat{z}\ .} \tag{4.5}$$

PROBLEM 5: A CHARGED PARTICLE IN THE FIELD OF A MAGNETIC MONOPOLE *(20 points)*

Griffiths Problem 5.45 (p. 258).

(a) The force $\vec{F}$ on q_e is,

$$\vec{F} = q_e(\vec{v}\times\vec{B}) = \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^2}(\vec{v}\times\hat{r})\ . \tag{5.1}$$

Let the accelaration of q_e be $\vec{a}$, then

$$\vec{F} = m\vec{a} \quad\Longrightarrow\quad \boxed{\vec{a} = \frac{\mu_0}{4\pi}\frac{q_e q_m}{mr^3}(\vec{v}\times\vec{r})\ .} \tag{5.2}$$

(b) Since the force on particle q_e is found to be perpendicular to its velocity $\vec{v}$, this force will not perform work on the particle. Therefore the kinetic energy of particle does not change and the speed $v = |\vec{v}|$ is a constant of the motion.

$$\frac{dE_{\text{kin}}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = m\vec{a}\cdot\vec{v} = 0\ . \tag{5.3}$$

(c) It is claimed that the vector quantity

$$\vec{Q} \equiv m\,\vec{r}\times\vec{v} - \frac{\mu_0\, q_e\, q_m}{4\pi}\,\hat{r} \tag{5.4}$$

is conserved. To verify this, we differentiate $\vec{Q}$ with respect to time, finding

$$\begin{aligned}\frac{d\vec{Q}}{dt} &= m(\vec{v}\times\vec{v}) + m(\vec{r}\times\vec{a}) - \frac{\mu_0 q_e q_m}{4\pi}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)\ ,\\ &= 0 + \frac{\mu_0 q_e q_m}{4\pi r^3}\left[\vec{r}\times(\vec{v}\times\vec{r})\right] - \frac{\mu_0 q_e q_m}{4\pi}\left(\frac{\vec{v}}{r} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right)\ .\end{aligned} \tag{5.5}$$

The time derivative or r, $\frac{dr}{dt}$ is the velocity component along $\hat{r}$, $\frac{dr}{dt} = \vec{v}\cdot\hat{r} = \frac{1}{r}\vec{v}\cdot\vec{r}$. The triple product rule gives $\vec{r}\times(\vec{v}\times\vec{r}) = r^2\vec{v} - \vec{r}(\vec{r}\cdot\vec{v})$. Then putting these relations into Eq. (5.5), we get

$$\frac{d\vec{Q}}{dt} = \frac{\mu_0 q_e q_m}{4\pi}\left[\frac{1}{r^3}\left(r^2\vec{v} - \vec{r}(\vec{r}\cdot\vec{v})\right) - \frac{\vec{v}}{r} + \frac{\vec{r}}{r^2}\frac{(\vec{v}\cdot\vec{r})}{r}\right] = 0\ . \tag{5.6}$$

(d) In the new coordinate system $\vec{\boldsymbol{Q}}$ points along z-axis, so $\vec{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}} = Q\,\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = Q\cos\theta$. Then, from Eq. (5.4), we find

$$\vec{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}} = m(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) \cdot \hat{\boldsymbol{r}} - \frac{\mu_0 q_e q_m}{4\pi}(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}) = \frac{\mu_0 q_e q_m}{4\pi} \ , \tag{5.7}$$

so

$$Q\cos\theta = -\frac{\mu_0 q_e q_m}{4\pi} \ . \tag{5.8}$$

In part (c) we found that $\vec{\boldsymbol{Q}}$ is a constant of the motion. Then Eq. (5.8) implies that the motion is confined to a fixed value of the polar angle θ, hence to a cone. The polar angle θ is

$$\boxed{\cos\theta = -\frac{\mu_0 q_e q_m}{4\pi Q} \ .} \tag{5.9}$$

PROBLEM 6: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

This problem was held over to Problem Set 8.

PROBLEM 7: THE COMPLETE MAGNETIC FIELD OF A MAGNETIC DIPOLE: *(10 points)*

This problem was held over to Problem Set 8.

PROBLEM 8: SQUARE CURRENT LOOP ON AXIS: BIOT-SAVART AND THE MAGNETIC DIPOLE APPROXIMATION *(15 points)*

This problem was held over to Problem Set 8.

PROBLEM 9: VECTOR POTENTIAL FOR A UNIFORM $\vec{\boldsymbol{B}}$ FIELD *(10 points extra credit)*

This problem was held over to Problem Set 8.