8.07 Lecture Slides 20
November 25, 2019

MAGNETIC DIPOLES, MAGNETIC FIELDS IN MATTER, and ELECTRODYNAMICS

Announcements

Problem Set 9 has been posted, and will be due Friday, December 6. It covers two weeks of work, and will be the last problem set to be handed in. There will also be a set of practice problems for the final exam, which will include solutions. Two of the problems on the final exam will be taken verbatim, or almost verbatim, from Problem Sets 8 and 9, and the practice problems for the final exam.

Magnetic Multipole Expansion

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \left\{ \hat{\boldsymbol{r}}_{i_1} \ldots \hat{\boldsymbol{r}}_{i_\ell} \right\}_{\mathrm{TS}} \left\{ \hat{\boldsymbol{r}}'_{i_1} \ldots \hat{\boldsymbol{r}}'_{i_\ell} \right\}_{\mathrm{TS}}$$

for $r' < r$. Here TS denotes the traceless symmetric part. Combine this with

$$A_j(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int \frac{J_j(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \, \mathrm{d}^3 x' \; .$$

Then

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \{\, \hat{\boldsymbol{r}}_{i_1} \ldots \hat{\boldsymbol{r}}_{i_\ell} \,\}_{\rm TS} \{\, \hat{\boldsymbol{r}}'_{i_1} \ldots \hat{\boldsymbol{r}}'_{i_\ell} \,\}_{\rm TS}$$

and

$$A_j(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int \frac{J_j(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \, \mathrm{d}^3x'$$

imply

$$A_j(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell} \frac{\{\, \hat{\boldsymbol{r}}_{i_1} \ldots \hat{\boldsymbol{r}}_{i_\ell} \,\}_{\rm TS}}{r^{\ell+1}} \;,$$

where

$$\mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \mathrm{d}^3x \, J_j(\vec{\boldsymbol{r}}) \{\, x_{i_1} \ldots x_{i_\ell} \,\}_{\rm TS} \;.$$

So the $\mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell}$ determine the coefficients of each term in the magnetic multipole expansion. That's good.

But we still need to explore what values of $\mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell}$ are allowed.

The $\mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell}$ are manifestly traceless and symmetric in $i_1 \ldots i_\ell$; but if that were the only restriction on their values, there would be $3\times(2\ell+1)$ independent components. (Reminder: a traceless symmetric tensor of rank ℓ [in 3 dimensions] has $2\ell+1$ independent components.) By contrast, we know that in the electric multipole expansion, there are only $2\ell+1$ independent components for each value of ℓ.

But there are further restrictions, coming from current conservation. For magnetostatics,

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = 0\ .}$$

Consequences of Current Conservation for $\ell = 0$ (monopole)

$$\mathcal{M}_j^{(0)} \equiv \int \mathrm{d}^3x\, J_j(\vec{\boldsymbol{r}}) \ .$$

But, by integration by parts

$$0 = \int \mathrm{d}^3x\, x_j\, \partial_i J_i = -\int \mathrm{d}^3x\, \delta_{ij} J_i = -\int \mathrm{d}^3x\, J_j \ ,$$

So,

$$\boxed{\mathcal{M}_j^{(0)} = 0 \ .}$$

Consequences of Current Conservation for $\ell = 1$ (dipole)

$$\mathcal{M}^{(1)}_{j;i} \equiv \int \mathrm{d}^3x \, J_j(\vec{r}) x_i \; .$$

Again, using integration by parts, we see

$$\begin{aligned} 0 &= \int \mathrm{d}^3x \, x_i x_j \, \partial_k J_k = -\int \mathrm{d}^3x \, J_k \left(\delta_{ik} x_j + \delta_{jk} x_i \right) \\ &= -\int \mathrm{d}^3x \, \left(J_i x_j + J_j x_i \right) \; . \end{aligned}$$

So,

$$\boxed{\mathcal{M}^{(1)}_{j;i} = -\mathcal{M}^{(1)}_{i;j} \; .}$$

So $\mathcal{M}^{(1)}_{j;i}$ is not an arbitrary 3×3 matrix, with 9 components, but is instead an antisymmetric 3×3 matrix, with 3 independent components.

The 3 independent components of an antisymmetric matrix can be written as a vector, using the Levi-Civita tensor ϵ_{ijk}. Define

$$\boxed{m_i = \frac{1}{2}\epsilon_{ijk}\mathcal{M}^{(1)}_{k;j} \ .}$$

Then, we can if we want recover $\mathcal{M}^{(1)}_{j;i}$ by

$$\boxed{\mathcal{M}^{(1)}_{j;i} = m_k\,\epsilon_{kij} \ .}$$

This agrees with the number of independent components for an electric dipole, and in fact that agreement persists for every $\ell \geq 1$.

Properties of Magnetic Dipoles

$$m_i = \frac{1}{2}\epsilon_{ijk}\mathcal{M}^{(1)}_{k;j} = \frac{1}{2}\epsilon_{ijk}\int \mathrm{d}^3x\, x_j J_k \ ,$$

or

$$\boxed{\vec{\boldsymbol{m}} = \frac{1}{2}\int \mathrm{d}^3x\, \vec{\boldsymbol{r}} \times \vec{\boldsymbol{J}} \ .}$$

For wires, $\mathrm{d}^3x\,\vec{\boldsymbol{J}} \to I\mathrm{d}\vec{\boldsymbol{\ell}}$, so

$$\boxed{\vec{\boldsymbol{m}} = \frac{1}{2}I\int \vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{\ell}} \ .}$$

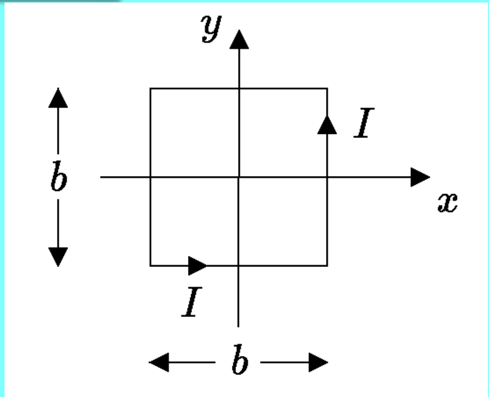
You may show on Problem Set 8 (Problem 9, extra credit), that one can also write

$$\boxed{\vec{\boldsymbol{m}} = I \int_S \mathrm{d}\vec{\boldsymbol{a}} \ ,}$$

where S is any surface that spans the wire loop.

An Ideal Magnetic Dipole

An ideal magnetic dipole is described as the limit of a square current loop, of side b and current I, in the limit as $b \to 0$ with $m = Ib^2$ held fixed.



Current density of an ideal magnetic dipole:

In Problem Set 9, there is a "Pedagogical Appendix" that describes this calculation.

Before taking the limit,

$$J_x(\vec{\boldsymbol{r}}) = \begin{cases} I\,\delta(z)\left[\delta\left(y + \frac{b}{2}\right) - \delta\left(y - \frac{b}{2}\right)\right] & \text{if } |x| < \frac{b}{2} \\ 0 & \text{otherwise} . \end{cases}$$

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Taking the limit,

$$J_x(\vec{\boldsymbol{r}}) = m\frac{\partial}{\partial y}\delta^3(\vec{\boldsymbol{r}}) \ .$$

Similarly,

$$J_y(\vec{\boldsymbol{r}}) \to -m\frac{\partial}{\partial x}\delta^3(\vec{\boldsymbol{r}}) \ ,$$

which can be put together into a vector equation

$$\boxed{\vec{\boldsymbol{J}}_{\rm dip} = -\vec{\boldsymbol{m}}\times\vec{\boldsymbol{\nabla}}\delta^3(\vec{\boldsymbol{r}}) \ .}$$

In Problem 11 of Problem Set 9, you will examine a more general picture of an ideal magnetic dipole, with an arbitrary $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$. The limit involves shrinking $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$ to a point, while scaling up $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$ to keep $\vec{\boldsymbol{m}}$ fixed. In the limit, the current density agrees with the formula above.

Vector potential of a magnetic dipole:

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1\ldots i_\ell} \frac{\{\hat{r}_{i_1} \ldots \hat{r}_{i_\ell}\}_{\text{TS}}}{r^{\ell+1}}$$

$$= \frac{\mu_0}{4\pi} \left[0 + m_k \epsilon_{kij} \frac{\hat{r}_i}{r^2} + \ldots \right] ,$$

so

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} + \ldots \;.}$$

Magnetic field of a magnetic dipole:

You showed on Problem Set 8 (Problem 2) that

$$\boxed{\vec{B}_{\text{dip}}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\,\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3} \vec{m}\, \delta^3(\vec{r}) \;.}$$

$$\boxed{\vec{B}_{\rm dip}(\vec{r}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3}\vec{m}\,\delta^3(\vec{r})\ .}$$

Note that for $\vec{r} \neq \vec{0}$, the form is the same as $\vec{E}$ for an electric dipole. The δ-function at the origin has a different coefficient. It must be. We expect

$$\boxed{\vec{\nabla}\cdot\vec{B}_{\rm dip} = 0\ , \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{\rm dip}\ ,}$$

while for an electric dipole we expect

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho_{\rm dip}\ , \quad \vec{\nabla}\times\vec{E} = 0\ .}$$

Using the identity

$$\boxed{\partial_i\partial_j\left(\frac{1}{r}\right) = -\partial_i\left(\frac{\hat{r}_j}{r^2}\right) = -\partial_i\left(\frac{x_j}{r^3}\right) = \frac{3\hat{r}_i\hat{r}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{r})\ ,}$$

these equations are straightforward to derive.

Forces, Torque, and Potential Energy for Magnetic Dipoles

Recall electric dipole formulas:

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad \text{or} \quad \vec{\nabla}(\vec{p}\cdot\vec{E}) \qquad \text{(force)}$$

$$\vec{\tau} = \vec{p}\times\vec{E} + \vec{r}\times\vec{F} \qquad \text{(torque)}$$

$$U = -\vec{p}\cdot\vec{E} \qquad \text{(potential energy).}$$

For magnetic dipoles,

$$\vec{F} = q\vec{v}\times\vec{B}\ ,$$

so for a current,

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell}\times\vec{B} = \vec{J}\times\vec{B}\,\mathrm{d}^3x\ .$$

For uniform $\vec{\boldsymbol{B}}$, there is no force, since

$$\int \mathrm{d}\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{B}} = \left[\int \mathrm{d}\vec{\boldsymbol{\ell}}\right] \times \vec{\boldsymbol{B}} = \vec{0} \times \vec{\boldsymbol{B}} = \vec{\mathbf{0}} \;,$$

or

$$\int \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}}\, \mathrm{d}^3x = \left[\int \vec{\boldsymbol{J}}\, \mathrm{d}^3x\right] \times \vec{\boldsymbol{B}} = \vec{\mathbf{0}} \times \vec{\boldsymbol{B}} = \vec{\mathbf{0}} \;.$$

For nonuniform $\vec{\boldsymbol{B}}$,

$$\boxed{\begin{aligned} \vec{\boldsymbol{F}} &= \int \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}}\, \mathrm{d}^3x = -\int \vec{\boldsymbol{B}} \times \vec{\boldsymbol{J}}\, \mathrm{d}^3x \\ &= -\int \vec{\boldsymbol{B}} \times \left[\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}}\delta^3(\vec{\boldsymbol{r}})\right] \mathrm{d}^3x \\ &= \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}}) \;. \end{aligned}}$$

Torque:

$$\vec{\boldsymbol{\tau}} = \int \vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{F}} \ , \qquad \mathrm{d}\vec{\boldsymbol{F}} = \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}} \, \mathrm{d}^3x \ ,$$

so

$$\begin{aligned} \vec{\boldsymbol{\tau}} &= -\int \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{J}}) \, \mathrm{d}^3x \\ &= \int \vec{\boldsymbol{r}} \times \left[\vec{\boldsymbol{B}} \times \left(\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}} \delta^3(\vec{\boldsymbol{r}}) \right) \right] \mathrm{d}^3x \\ &= \boxed{\vec{\boldsymbol{m}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \ ,} \end{aligned}$$

where the $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$ term contributes if the dipole is not at the origin.

Potential Energy:

Keep $\vec{\boldsymbol{m}}$ fixed, define $U = 0$ at ∞, and calculate how much work is needed to bring the dipole in from infinity:

$$U(\vec{\boldsymbol{r}}) = -\int_{\infty}^{\vec{r}} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = -\int_{\infty}^{\vec{r}} \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{\ell}} \ ,$$

So

$$\boxed{U(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) \ .}$$

$U(\vec{\boldsymbol{r}})$ also describes the changes in energy when the orientation of the dipole is changed. The formula for the torque can be derived from $U(\vec{\boldsymbol{r}})$.

Physical Relevance of δ-function in $\vec{B}_{\rm dip}$

$$\vec{\boldsymbol{B}}_{\rm dip}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi} \frac{3(\vec{\boldsymbol{m}} \cdot \hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{r^3} + \frac{2\mu_0}{3} \vec{\boldsymbol{m}}\, \delta^3(\vec{\boldsymbol{r}}) \ .$$

In the ground state of the hydrogen atom, one can think of the proton p as being stationary at the origin, with the electron e described by a spherically symmetric probability cloud. Both p and e have magnetic dipole moments, aligned with their spins. The magnetic energy can be written as

$$U_{\rm magnetic} = -\vec{\boldsymbol{m}}_{\rm electron} \cdot \vec{\boldsymbol{B}}_{\rm dip,proton} \ .$$

The first term in $B_{\rm dip}$ vanishes when averaged over the spherical cloud, so ONLY the δ-function term contributes. The transition from the spin aligned to the spin-aligned to the spin-anti-aligned state is responsible for the famous 21-centimeter line that is very important in astronomy.

Bound Currents

As we did for electric fields in matter, we can describe the magnetic properties of matter by specifying the density of magnetic dipoles, $\vec{\boldsymbol{M}}$. $\vec{\boldsymbol{M}}$ describes the magnetic moment per volume in the material.

$\vec{\boldsymbol{M}}$ is always viewed as a macroscopic quantity, defined by averaging over a region large compared to atoms, but small compared to the size of objects.

Since $\vec{\boldsymbol{M}}$ describes magnetic dipoles, it is really describing currents. The function $\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}})$ is equivalent to specifying the macroscopic current density of the dipoles, called the bound current, and given by

$$\boxed{\vec{\boldsymbol{J}}_{\text{bound}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}) \ .}$$

This is analogous to $\rho_{\text{bound}} = -\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{P}}$.

On the surface of a magnetized material, there can be a discontinuity in $\vec{M}$, giving a singular contribution to $\vec{J}$. This can be described as a surface current

$$\boxed{\vec{K}_{\text{bound}} = \vec{M} \times \hat{n} \ ,}$$

where $\hat{n}$ is the outward normal.

Sketch on blackboard.

The formula for $\vec{J}_{\rm bound}$ can be derived easily from the formula for the current density of a magnetic dipole,

$$\vec{J}_{\rm dip} = -\vec{m} \times \vec{\nabla} \delta^3(\vec{r}\,) = \vec{\nabla} \times \left(\vec{m}\, \delta^3(\vec{r}\,)\right) \ .$$

For a system of point dipoles $\vec{m}^{\,n}$ at positions $\vec{r}^{\,n}$, respectively, the microscopic current density is given by

$$\vec{J}_{\rm micro}(\vec{r}\,) = \vec{\nabla}_{\vec{r}} \times \sum_n \vec{m}^{\,n}\, \delta^3(\vec{r} - \vec{r}^{\,n}) \ .$$

The equation for $\vec{J}_{\rm bound} = \vec{J}_{\rm macro}$ is found simply by taking the macroscopic averages of both sides of the equation, recognizing that

$$\vec{M} = \left\langle \vec{M}_{\rm micro}(\vec{r}\,) \right\rangle = \left\langle \sum_n \vec{m}^{\,n}\, \delta^3(\vec{r} - \vec{r}^{\,n}) \right\rangle \ .$$

So,

$$\boxed{\vec{J}_{\rm bound}(\vec{r}\,) = \vec{\nabla} \times \vec{M}(\vec{r}\,) \ .}$$

Macroscopic Fields in Magnetic Materials

Faraday's Law for Moving Loops

Faraday's Law for Changing Magnetic Fields

The Displacement Current and the Full Maxwell Equations