MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — December 7, 2019
Prof. Alan Guth

PRACTICE PROBLEMS FOR THE FINAL EXAM, PART B

DUE DATE: None. But the final exam will include two problems that will be taken verbatim, or at least almost verbatim, from Problem Sets 8 or 9, or from the practice problems for the final exam. This is the first part of the practice problems for the final exam.

READING ASSIGNMENT: Griffiths: Chapter 10 (*Potentials and Fields).* The final exam will include questions relevant to Chapter 10, and to selected topics in Chapter 11 (*Radiation*), but for both chapters you will only be responsible for those topics that are discussed in lecture, or which appear in the problem sets or practice problems.

PROBLEM 1: ORTHOGONALITY OF AN ANGULAR INTEGRAL

One of my (Alan Guth's) research groups is studying the possible production of primordial black holes in the context of a particular type of inflationary model, called hybrid inflation. For part of the calculation we found it useful to expand the density field in spherical harmonics, and then we wanted to know whether the different terms in our expansion were correlated with each other. The answer to this depended on whether the following integral,

$$I_{\ell' m',\ell m} \equiv \int d\Omega' \int d\Omega \, Y^*_{\ell' m'}(\theta' \phi') \, Y_{\ell m}(\theta, \phi) \, f(\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}}) \ , \tag{1.1}$$

vanishes if $\ell' \neq \ell$ or $m' \neq m$. Here $f(\hat{\boldsymbol{n}}' \cdot \hat{\boldsymbol{n}})$ is allowed to be any smooth function, and

$$\hat{\boldsymbol{n}} = \sin\theta \, \cos\phi \, \hat{\boldsymbol{x}} + \sin\theta \, \sin\phi \, \hat{\boldsymbol{y}} + \cos\theta \, \hat{\boldsymbol{z}} \ , \tag{1.2}$$

and similarly for $\hat{\boldsymbol{n}}'$:

$$\hat{\boldsymbol{n}}' = \sin\theta' \, \cos\phi' \, \hat{\boldsymbol{x}} + \sin\theta' \, \sin\phi' \, \hat{\boldsymbol{y}} + \cos\theta' \, \hat{\boldsymbol{z}} \ , \tag{1.3}$$

and

$$d\Omega = \sin\theta \, d\theta \, d\phi \ , \quad d\Omega' = \sin\theta' \, d\theta' \, d\phi' \ . \tag{1.4}$$

Others in the group did not see how to approach this question, but, using the traceless symmetric tensor description of spherical harmonics, I found the answer obvious. I used the fact that is well-known, although maybe not so easy to really prove, that a rotationally invariant tensor must be built out of Kronecker delta functions δ_{ij} or perhaps Levi-Civita tensors, ϵ_{ijk}. Using this assumption, prove that $I_{\ell' m',\ell m}$ vanishes if $\ell' \neq \ell$ or $m' \neq m$.

PROBLEM 2: A SPHERICAL SHELL WITH SURFACE CHARGE DENSITY $\sigma = \sigma_0 \cos^2\theta$

An anonymous student recommended on Piazza that we solve the problem of a spherical shell with surface charge density

$$\sigma = \sigma_0 \cos^2\theta \ . \tag{2.1}$$

Why not? Here θ is the usual polar angle, and we let R denote the radius of the thin spherical shell. The problem is to find the electric potential $V(\vec{\boldsymbol{r}})$ everywhere in space for this charge configuration.

PROBLEM 3: ELECTRIC DIPOLE RADIATION

In lecture, we derived the solutions

$$\begin{aligned} V(\vec{\boldsymbol{r}},t) &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d^3x' \\ \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) &= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}d^3x' \end{aligned} \tag{3.1}$$

for the Lorenz-gauge potentials arising from a time-varying source of charge and current. Here, $t_r = t - |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|/c$ is the retarded time, i.e., time accounting for how long it takes information to propagate from $\vec{\boldsymbol{r}}\,'$ to $\vec{\boldsymbol{r}}$. We further showed (see Lecture Slides 27) that the vector potential of an oscillating dipole $\vec{\boldsymbol{p}}(t) = p_0\cos(\omega t)\,\hat{\boldsymbol{z}}$, in the radiation approximation, is given by

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{z}} \ , \tag{3.2}$$

which can be generalized to a dipole with an arbitrary direction and time dependence by writing

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi r}\left(\frac{d\vec{\boldsymbol{p}}}{dt}\right)_{t=t_r} \ , \tag{3.3}$$

where $\vec{\boldsymbol{p}} \equiv \vec{\boldsymbol{p}}(t)$ is the dipole moment of the charge/current distribution, and where the subscripted "$t = t_r$" means that the quantity in parentheses is to be evaluated at the retarded time $t - r/c$. Note that this version of retarded time does *not* depend on $\vec{\boldsymbol{r}}\,'$.

(a) At several points in the calculations that follow, you will need to compute the divergence or curl of a vector function that depends on retarded time. Working in spherical coordinates, prove the following identities:

$$\begin{aligned} \vec{\nabla}\cdot\vec{\boldsymbol{F}}(t_r) &= -\hat{\boldsymbol{r}}\cdot\frac{1}{c}\left(\frac{\partial\vec{\boldsymbol{F}}}{\partial t}\right)_{t=t_r} \\ \vec{\nabla}\times\vec{\boldsymbol{F}}(t_r) &= -\hat{\boldsymbol{r}}\times\frac{1}{c}\left(\frac{\partial\vec{\boldsymbol{F}}}{\partial t}\right)_{t=t_r} \end{aligned} \tag{3.4}$$

These will make your life easier as you work through the rest of the calculations in this problem.

(b) Show that the Lorenz gauge condition $\vec{\nabla} \cdot \vec{\boldsymbol{A}} = -\mu_0\epsilon_0 \partial V/\partial t$ leads to the scalar potential

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\boldsymbol{r}} \cdot (d\vec{\boldsymbol{p}}/dt)}{cr} + \frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}}}{r^2}\right]_{t=t_r} . \tag{3.5}$$

There could be a time-independent contribution that arises as a constant of integration, but we are interested only in the time-dependent piece shown in Eq. (3.5). Don't forget that $c = 1/\sqrt{\mu_0\epsilon_0}$.

(c) Show that the electric and magnetic fields corresponding to these potentials are given by

$$\begin{aligned}\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) &= \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} + \frac{3[(d\vec{\boldsymbol{p}}/dt) \cdot \hat{\boldsymbol{r}}]\hat{\boldsymbol{r}} - (d\vec{\boldsymbol{p}}/dt)}{cr^2}\right. \\ &\qquad\qquad \left. + \frac{[(d^2\vec{\boldsymbol{p}}/dt^2) \cdot \hat{\boldsymbol{r}}]\hat{\boldsymbol{r}} - (d^2\vec{\boldsymbol{p}}/dt^2)}{c^2 r}\right]_{t=t_r} \\ \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) &= -\frac{\mu_0}{4\pi}\,\hat{\boldsymbol{r}} \times \left[\frac{d\vec{\boldsymbol{p}}/dt}{r^2} + \frac{d^2\vec{\boldsymbol{p}}/dt^2}{cr}\right]_{t=t_r} .\end{aligned} \tag{3.6}$$

In what follows, take the dipole to be aligned with the z axis, so that $\vec{\boldsymbol{p}} \propto \hat{\boldsymbol{z}}$. (This is not necessary for the calculations we ask you to do, but it makes life a little easier.)

(d) Show that, far from the source, the Poynting vector is given by

$$\vec{\boldsymbol{S}} = \frac{1}{16\pi^2\epsilon_0}\frac{\hat{\boldsymbol{r}}}{c^3 r^2}\left(1 - \cos^2\theta\right)\left(\frac{d^2 p}{dt^2}\right)^2_{t=t_r} . \tag{3.7}$$

Note: This form is only correct for $\vec{\boldsymbol{p}} = p\hat{\boldsymbol{z}}$. It is not too difficult to get a general form, but this is not necessary.

(e) Show that the total power radiated by this oscillating dipole is given by

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{1}{c^3}\left(\frac{d^2 p}{dt^2}\right)^2_{t=t_r} . \tag{3.8}$$

This is often called the *dipole formula*.

PROBLEM 4: POLARIZATION OF ELECTRIC DIPOLE RADIATION

Consider the $1/r$ piece of the electric field you worked out in the previous problem:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{c^2}\frac{1}{r}\left[\left(\hat{\boldsymbol{r}} \cdot \frac{d^2\vec{\boldsymbol{p}}}{dt^2}\right)\hat{\boldsymbol{r}} - \frac{d^2\vec{\boldsymbol{p}}}{dt^2}\right]_{t=t_r} . \tag{4.1}$$

This piece dominates far away. As discussed in lecture, it is considered the "radiation zone" solution, and is often denoted $\vec{\boldsymbol{E}}_{\text{rad}}$.

(a) Show that an equivalent way to write this is

$$\vec{\boldsymbol{E}}_{\text{rad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{c^2}\frac{\hat{\boldsymbol{r}}}{r} \times \left(\hat{\boldsymbol{r}} \times \frac{d^2\vec{\boldsymbol{p}}}{dt^2}\right)_{t=t_r} . \tag{4.2}$$

(b) Compute $\vec{\boldsymbol{E}}$ for $\vec{\boldsymbol{p}} = p_0 \cos\omega t\, \hat{\boldsymbol{z}}$. Show that $\vec{\boldsymbol{E}}$ is *linearly polarized*, meaning that $\vec{\boldsymbol{E}}$ at a fixed position points either parallel or anti-parallel to some constant direction.

(c) Compute $\vec{\boldsymbol{E}}$ for $\vec{\boldsymbol{p}} = p_0 \cos\omega t\, \hat{\boldsymbol{x}} + p_0 \sin\omega t\, \hat{\boldsymbol{y}}$. Show that $\vec{\boldsymbol{E}}$ is linearly polarized at $\theta = \pi/2$, and that it is *circularly polarized* at $\theta = 0$, meaning that the direction of $\vec{\boldsymbol{E}}$ rotates in a circle over a full oscillation cycle. (θ is the spherical coordinate. A rephrasing of this question: Show that $\vec{\boldsymbol{E}}$ is linearly polarized when measured by an observer in the x-y plane, and circular polarized when measured by someone on the z axis.)

PROBLEM 5: CONSISTENCY OF CHARGE AND CURRENT

Griffiths, Problem 10.7, p. 442. *Hint:* When we studied electrostatics, we learned that some kinds of problems with sufficient symmetry could be solved using Gauss's law alone. Think about whether these same arguments apply to part (c) of this problem, even though the fields are time-dependent.

A time-dependent charge $q(t)$ sits at the origin; we write the charge density associated with it as $\rho(\vec{\boldsymbol{r}}, t) = q(t)\delta^{(3)}(\vec{\boldsymbol{r}})$. It is fed by a current $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}, t) = -(1/4\pi)(\dot{q}/r^2)\hat{\boldsymbol{r}}$, where $\dot{q} = dq/dt$.

(a) Check that charge is conserved by confirming that the continuity equation is obeyed.

(b) Find the scalar and vector potentials in the Coulomb gauge. If you get stuck, try working on (c) first.

(c) Find the fields, and check that they satisfy all of Maxwell's equations.

PROBLEM 6: ENERGY RADIATED BY BRAKING ELECTRIC CHARGE

Griffiths, Problem 11.13, p. 487.

A positive charge q is fired head on at a distant positive charge Q (which is held stationary) with initial velocity v_0. It comes in, decelerates to $v = 0$, and returns to out to infinity. (The motion is one dimensional.) What fraction of its initial energy, $\frac{1}{2}mv_0^2$, is radiated away? Assume $v_0 \ll c$, and that you can safely ignore the backreaction of radiative losses on the motion of the particle. [*Answer:* fraction radiated $= (16/45)(q/Q)(v_0/c)^3$.]

PROBLEM 7: OSCILLATING POINT CHARGE

Griffiths, Problem 11.22, p. 496.

A particle of mass m and charge q is attached to a spring with force constant k, hanging from the ceiling (Fig. 11.18). Its equilibrium position is a distance h above the floor. It is pulled down a distance d below equilibrium and released, at time $t = 0$.

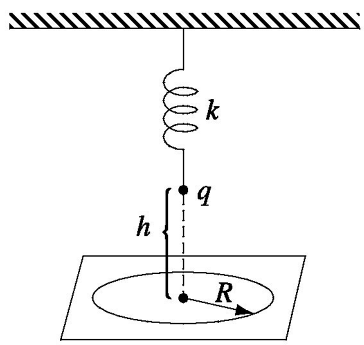

FIGURE 11.18

(a) Under the usual assumptions $(d \ll \lambda \ll h)$, calculate the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below q. [Note: The intensity here is the average power per unit area of floor.] At what R is the radiation most intense? Neglect the radiative damping of the oscillator. [Answer: $\mu_0 q^2 d^2 \omega^4 R^2 h / 32\pi^2 c(R^2 + h^2)^{5/2}$.]

(b) As a check on your formula, assume the floor is of infinite extent, and calculate the average energy per unit time striking the entire floor. Is it what you'd expect?

(c) Because it is losing energy in the form of radiation, the amplitude of the oscillation will gradually decrease. After what time r has the amplitude been reduced to d/e? (Assume the fraction of the total energy lost in one cycle is very small.)

PROBLEM 8: CONSISTENCY OF CHARGE AND CURRENT IN LIÉNARD-WIECHART DERIVATION

When deriving the Liénard-Wiechart potentials, we used the charge and current densities

$$\rho = q\delta^{(3)}[\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p(t)] \ , \quad \vec{\boldsymbol{J}} = q\vec{\boldsymbol{v}}_p\delta^{(3)}[\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p(t)], \tag{8.1}$$

where $\vec{\boldsymbol{r}}_p(t)$ is the instantaneous position of the charge, and where $\vec{\boldsymbol{v}}_p = d\vec{\boldsymbol{r}}_p/dt$.

Show that these densities are consistent with the continuity equation,

$$\vec{\nabla} \cdot \vec{\boldsymbol{J}} + \frac{\partial \rho}{\partial t} = 0 \ . \tag{8.2}$$

PROBLEM 9: THE ANGULAR MOMENTUM OF A CHARGE AND A MAGNETIC MONOPOLE

Griffiths, Problem 8.19, p. 380. This problem is motivated by Problem Set 7, Problem 5, *A Charged Particle in the Field of a Magnetic Monopole*. In that problem you showed that when a particle of mass m and electric charge q_e moves in the field of a stationary magnetic monopole q_m at the origin, then the quantity

$$\vec{\boldsymbol{Q}} = m(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) - \frac{\mu_0 q_e q_m}{4\pi}\hat{\boldsymbol{r}} \tag{9.1}$$

is conserved. Note that the conserved vector $\vec{\boldsymbol{Q}}$ is equal to the standard angular momentum vector, $\vec{\boldsymbol{r}} \times (m\vec{\boldsymbol{v}})$, plus a vector of fixed magnitude, pointing radially inward, $-[\mu_0 q_e q_m/(4\pi)]\hat{\boldsymbol{r}}$. "Radially inward" means that it points from the electric charge to the magnetic monopole (when q_e and q_m have the same sign). The fact that $\vec{\boldsymbol{Q}}$ is conserved means that the ordinary angular momentum vector is not. But when this new radially inward vector is added to the ordinary angular momentum, the sum is conserved. The radially inward vector must therefore be interpreted as a new contribution to the angular momentum. In this problem you will verify something that was mentioned in lecture: if one looks at the field configuration of a system containing one point electric charge and one point magnetic monopole, the momentum density of the electric and magnetic fields leads to an angular momentum of magnitude $\mu_0 q_e q_m/(4\pi)$, pointing from the electric charge to the magnetic monopole. It is a little surprising that this angular momentum does not depend on how far apart the two particles are, but one can see how this happens if one thinks carefully about how each of the relevant quantities depends on the distance. As Griffiths describes in Section 8.2.4, on p. 370, the angular momentum density of the field configuration is equal to $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{g}}$, where $\vec{\boldsymbol{g}}$ is the momentum density.

Griffiths' Problem 8.19:

Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \tag{9.2}$$

(of course), and the field of the magnetic monopole is

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\ . \tag{9.3}$$

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [Answer: $(\mu_0/4\pi)q_e\,q_m$.]

PROBLEM 10: ONE POINT CHARGE OSCILLATOR / TWO POINT CHARGE OSCILLATORS

Griffiths, Problem 11.18, p. 491.

A point charge q, of mass m, is attached to a spring of constant k. At time $t = 0$ it is given a kick, so its initial energy is $U_0 = \frac{1}{2}mv_0^2$. Now it oscillates, gradually radiating away this energy.

(a) Confirm that the total energy radiated is equal to U_0. Assume the radiation damping is small, so you can write the equation of motion as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\ , \tag{10.1}$$

and the solution as

$$x(t) = \frac{v_0}{\omega_0}e^{-\gamma t/2}\,\sin(\omega_0 t)\ , \tag{10.2}$$

with $\omega_0 \equiv \sqrt{k/m}$, $\gamma = \omega_0^2\tau$, where

$$\tau = \frac{\mu_0 q^2}{6\pi mc}\ , \tag{10.3}$$

and $\gamma \ll \omega_0$ (drop γ^2 in comparison to ω_0^2, and when you average over a complete cycle, ignore the change in $e^{-\gamma t}$).

(b) Suppose now we have *two* such oscillators, and we start them off with identical kicks. Regardless of their relative positions and orientations, the total energy radiated must be $2U_0$. But what if they are right on top of each other, so it's equivalent to a *single* oscillator with twice the charge; the Larmor formula says that the power radiated is *four* times as great, suggesting that the total will be $4U_0$. Find the error in this reasoning, and show that the total is actually $2U_0$, as it should be.