MASSACHUSETTS INSTITUTE OF TECHNOLOGY Physics Department

Prof. Alan Guth Physics 8.07: Electromagnetism II

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FORMULA SHEET FOR FINAL EXAM

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Sec. 30(b), a minus sign in the equation for \vec{E} has been deleted. In Sec. 29(e), the coefficient in the equation for $A(\vec{r},t)$ has been corrected. In

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A few items below are marked with asterisks, ***. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

1. Index Notation:

Unit Vectors:
$$\hat{x} \equiv \hat{i} \equiv \hat{e}_x \equiv \hat{e}_1$$
, $\hat{y} \equiv \hat{j} \equiv \hat{e}_y \equiv \hat{e}_2$, $\hat{z} \equiv \hat{k} \equiv \hat{e}_z \equiv \hat{e}_3$, $\vec{A} \equiv A_i \hat{e}_i$
General vector: $\vec{r} = x \hat{x} + y \hat{y} + z \hat{z} = x_i \hat{e}_i$

Dot Product, Cross Product, and Determinant:

$$ec{m{A}}\cdotec{m{B}}=A_iB_i\,, \qquad ec{m{A}} imesec{m{B}}_i=\epsilon_{ijk}A_jB_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}$$

 $\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1, i_1} A_{2, i_2} \cdots A_{n, i_n} **$

Rotation of a Vector:

$$A_i' = R_{ij} A_j$$
, Orthogonality: $R_{ij} R_{ik} = \delta_{jk}$ $(R^T T = I)$

Rotation about z-axis by
$$\phi$$
: $R_z(\phi)_{ij} = i=1 \atop i=$

Rotation about axis \hat{n} by ϕ : ***

$$R(\hat{m{n}},\phi)_{ij}=\delta_{ij}\cos\phi+\hat{m{n}}_i\hat{m{n}}_j(1-\cos\phi)-\epsilon_{ijk}\hat{m{n}}_k\,\sin\phi$$

2. Vector Calculus:

Gradient:
$$(\vec{\nabla}\varphi)_i = \partial_i\varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} + \frac{\partial \varphi}{\partial z}\hat{z}$$
, $\partial_i \equiv \frac{\partial}{\partial x_i}$
Divergence: $\vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$$(ec{m{
abla}} imesec{m{A}})_i=\epsilon_{ijk}\partial_jA_k$$

Curl:

$$egin{align*} ec{m{\nabla}} imes ec{m{A}} &= \left(rac{\partial A_y}{\partial y} - rac{\partial A_y}{\partial z}
ight) \hat{m{x}} + \left(rac{\partial A_x}{\partial z} - rac{\partial A_z}{\partial x}
ight) \hat{m{y}} + \left(rac{\partial A_y}{\partial x} - rac{\partial A_x}{\partial y}
ight) \hat{m{z}} \
abla^2 arphi &= rac{\partial^2 arphi}{\partial x_i \partial x_i} = rac{\partial^2 arphi}{\partial x^2} + rac{\partial^2 arphi}{\partial y^2} + rac{\partial^2 arphi}{\partial z^2}
onumber \end{aligned}$$

Laplacian:

3. Fundamental Theorems of Vector Calculus:

Gradient:
$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} \varphi \cdot d\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence:
$$\int_{\mathcal{V}} \vec{\mathbf{\nabla}} \cdot \vec{A} \, \mathrm{d}^3 x = \oint_{S} \vec{A} \cdot \mathrm{d} \vec{a}$$

where S is the boundary of $\mathcal V$

url:
$$\int_S (ec{m{ au}} imes ec{m{A}}) \cdot \mathrm{d} m{a} = \oint_P ec{m{A}} \cdot \mathrm{d} m{l}$$
 where P is the boundary of S

4. Vector Identities:

Triple Products:

$$ec{m{A}}\cdot(ec{m{B}} imesec{m{C}}) = ec{m{B}}\cdot(ec{m{C}} imesec{m{A}}) = ec{m{C}}\cdot(ec{m{A}} imesec{m{B}}) \ ec{m{A}}\cdot(ec{m{B}} imesec{m{C}}) = ec{m{B}}(ec{m{A}}\cdotec{m{C}}) - ec{m{C}}(ec{m{A}}\cdotec{m{B}})$$

Product Rules:

$$ec{m{
abla}}\cdot(fec{m{A}})=fec{m{
abla}}\cdotec{m{A}}+ec{m{A}}\cdotec{m{
abla}}f$$

$$\vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla} f$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$ec{m{
abla}} imes(fec{m{A}})=fec{m{
abla}} imesec{m{A}}-ec{m{A}} imesec{m{
abla}}f$$

$$ec{m{\nabla}} imes(ec{m{A}} imesec{m{B}})=(ec{m{B}}\cdotec{m{\nabla}})ec{m{A}}-(ec{m{A}}\cdotec{m{\nabla}})ec{m{B}}+ec{m{A}}(ec{m{\nabla}}\cdotec{m{B}})-ec{m{B}}(ec{m{\nabla}}\cdotec{m{A}})$$

Second Derivatives:

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= 0 \\ \vec{\nabla} \times (\vec{\nabla} f) &= 0 \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \end{aligned}$$

ġ Spherical Coordinates

 $x = r \sin \theta \cos \phi$

$$y = r \sin \theta \sin \phi$$
 $z = r \cos \theta$
 $\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$
 $\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$

 $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$

 $\hat{z} = \cos\theta \,\hat{r} - \sin\theta \,\hat{\theta}$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$$

$$\phi = \tan^{-1}\left(y/x\right)$$

$$\hat{x} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$$

$$\hat{y} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}$$

Point separation: $d\vec{\ell} = dr \, \hat{r} + r \, d\theta \, \hat{\theta} + r \sin \theta \, d\phi \, \hat{\phi}$

Volume element: $d^3x \rightarrow r^2 \sin\theta dr d\theta d$

 $ec{m{\nabla}}arphi = rac{\partial arphi}{\partial r}\,\hat{m{r}} + rac{1}{r}rac{\partial arphi}{\partial heta}\,\hat{m{ heta}} + rac{1}{r\sin heta}rac{\partial arphi}{\partial \phi}\,\hat{m{\phi}}$

Divergence: $ec{m{\nabla}}\cdotec{m{A}}=rac{1}{r^2}rac{\partial}{\partial r}\left(r^2A_r
ight)+rac{1}{r\sin heta}rac{\partial}{\partial heta}(\sin heta A_ heta)+rac{1}{r\sin heta}rac{\partial A_ heta}{\partial \phi}$

$$\begin{split} & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_{\phi}) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_{\theta}) - \frac{\partial A_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \\ & \nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2} \end{split}$$
 $ec{m{
abla}} imesec{m{A}}=rac{1}{r\sin heta}\left[rac{\partial}{\partial heta}(\sin heta A_\phi)-rac{\partial A_ heta}{\partial\phi}
ight]\mathbf{\hat{r}}$

Cylindrical Coordinates

 $s=\sqrt{x^2+y^2} \ \phi= an^{-1}(y/x) \ z=z$

 $\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$

 $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$ $\hat{\boldsymbol{x}} = \cos\phi \,\hat{\boldsymbol{s}} - \sin\phi \,\hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{y}} = \sin\phi \,\hat{\boldsymbol{s}} + \cos\phi \,\hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$

Point separation: $d\vec{\ell} = ds \, \hat{s} + s \, d\phi \, \hat{\phi} + dz \, \hat{z}$ Volume element: $d^3x \rightarrow s \, ds \, d\phi \, dz$

 $ec{m{
abla}}arphi = rac{\partial arphi}{\partial s}\,\hat{m{s}} + rac{1}{s}rac{\partial arphi}{\partial \phi}\,\hat{m{\phi}} + rac{\partial arphi}{\partial z}\,\hat{m{z}}$

Gradient:

Divergence:

 $egin{align*} ec{m{\nabla}} \cdot ec{m{A}} &= rac{1}{s} rac{\partial}{\partial s} (sA_s) + rac{1}{s} rac{\partial A_\phi}{\partial \phi} + rac{\partial A_z}{\partial z} \ ec{m{\nabla}} imes ec{m{A}} &= \left[rac{1}{s} rac{\partial A_z}{\partial \phi} - rac{\partial A_\phi}{\partial z}
ight] \hat{m{s}} + \left[rac{\partial A_s}{\partial z} - rac{\partial A_z}{\partial z}
ight] \hat{m{s}} + \left[rac{\partial A_s}{\partial z} - rac{\partial A_z}{\partial z}
ight] \hat{m{s}} + \left[rac{\partial A_s}{\partial z} - rac{\partial A_s}{\partial z}
ight] \hat{m{z}} + rac{1}{s} \left[rac{\partial}{\partial s} \left(sA_\phi \right) - rac{\partial A_s}{\partial \phi}
ight] \hat{m{z}} + rac{\partial^2 \varphi}{\partial z^2} +$

Laplacian:

 $\delta(g(x)) = \sum_i rac{\delta(x-x_i)}{|g'(x_i)|}$, where i is summed over all points for which $g(x_i) = 0$ $\int arphi(x) rac{\mathrm{d}}{\mathrm{d}x} \delta(x-x') \, \mathrm{d}x = - \left. rac{\mathrm{d}arphi}{\mathrm{d}x}
ight|_{x=x'}$ $\int arphi(x) \delta(x-x') \, \mathrm{d}x = arphi(x') \,, \qquad \int arphi(ec{m{r}}) \delta^3(ec{m{r}}-ec{m{r}}') \, \mathrm{d}^3x = arphi(ec{m{r}}')$

 $\begin{aligned} & \nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -\vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = -4\pi \delta^3 (\vec{r} - \vec{r}') \\ & \partial_i \partial_j \left(\frac{1}{r} \right) = -\partial_i \left(\frac{\hat{r}_j}{r^2} \right) = -\partial_i \left(\frac{x_j}{r^3} \right) = \frac{3 \hat{r}_i \hat{r}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3} \, \delta_{ij} \, \delta^3 (\vec{r}) \end{aligned}$

8. Electrostatics: $\vec{F} = q\vec{E} \text{ , where } \\ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \frac{q_i}{4\pi\epsilon_0} = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') \, \mathrm{d}^3 x'$ $\epsilon_0 = \text{permittivity of free space} = 8.854 \times 10^{-12} \, \mathrm{C}^2/(\mathrm{N \cdot m}^2)$ $\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \, \mathrm{N \cdot m}^2/\mathrm{C}^2$

$$\begin{split} V(\vec{\boldsymbol{r}}) &= V(\vec{\boldsymbol{r}}_0) - \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}') \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}')}{\left|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'\right|} \mathrm{d}^3x' \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} &= \frac{\rho}{\epsilon_0} \,, \qquad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0 \,, \qquad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V \\ \nabla^2 V &= -\frac{\rho}{\epsilon_0} \,\, \text{(Poisson's Eq.)} \,, \qquad \rho = 0 \quad \Longrightarrow \quad \nabla^2 V = 0 \,\,\, \text{(Laplace's Eq.)} \end{split}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V=0$, then the average value of V on a spherical surface equals its value at the center.

9. Electrostatic Energy:*

$$W = rac{1}{2}rac{1}{4\pi\epsilon_0}\sum_{ij}rac{q_iq_j}{r_{ij}} = rac{1}{2}rac{1}{4\pi\epsilon_0}\int\mathrm{d}^3x\,\mathrm{d}^3x'\,rac{
ho(ec{m{r}}')
ho(ec{m{r}}')}{\left|ec{m{r}}-ec{m{r}}'
ight|} \ W = rac{1}{2}\int\mathrm{d}^3x
ho(ec{m{r}})V(ec{m{r}}) = rac{1}{2}\epsilon_0\int\!\left|ec{m{E}}
ight|^2\mathrm{d}^3x$$

See Sec. 14(c) for energy in the presence of dielectrics

10. Conductors:

Just outside, $m{ec{E}} = rac{\sigma}{\epsilon_0} \hat{m{n}}$

Pressure on surface: $\frac{1}{2}\sigma |\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and -Q: $Q = CV, W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j$$
 , $P_{ij} = ext{elastance matrix}, ext{ or reciprocal capacitance matrix}$

 $Q_i = \sum_j C_{ij} V_j$, $C_{ij} = ext{capacitance matrix}$

11. Separation of Variables for Laplace's Equation in Cartesian Coordinates: Image charge in sphere of radius a: Image of Q at R is $q = -\frac{a}{R}Q$, $r = \frac{a^2}{R}$

$$V = \left\{ egin{array}{ll} \cos lpha x \ \sin lpha x \end{array}
ight\} \left\{ egin{array}{ll} \cos eta y \ \sin eta y \end{array}
ight\} \left\{ egin{array}{ll} \cosh \gamma z \ \sinh \gamma z \end{array}
ight\} \quad ext{where } \gamma^2 = lpha^2 + eta^2$$

or, more generally,

$$V = \begin{cases} \cos \alpha x \\ \sin \alpha x \end{cases} \begin{cases} \cos \beta y \\ \sin \beta y \end{cases} \begin{cases} \cos \gamma z \\ \sin \gamma z \end{cases}$$
 where $\alpha^2 + \beta^2 + \gamma^2 = 0$, each of α , β , and γ can be real or imaginary, with $\sin(i\theta) = i \sinh \theta$ and $\cos(i\theta) = \cosh \theta$. $\begin{cases} \cos \alpha x \\ \sin \alpha x \end{cases}$ means any linear combination of $\cos \alpha x$ or $\sin \alpha x$, but usually one or the other suffices.

12. Separation of Variables for Laplace's Equation in Spherical Coordinates:

12(a) Traceless Symmetric Tensor expansion:

 $abla^2 arphi(r, heta,\phi) = rac{1}{r^2}rac{\partial}{\partial r}\left(r^2rac{\partial arphi}{\partial r}
ight) + rac{1}{r^2}
abla^2_{
m ang}\,arphi = 0 \; ,$

where the angular part is given by
$$\nabla_{\rm ang}^2 \varphi \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$$

$$abla_{\mathrm{ang}} \varphi \equiv \frac{1}{\sin \theta} \frac{1}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi^2}$$

 $abla_{ ext{ang}}^2 C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{m{n}}_{i_1} \hat{m{n}}_{i_2} \dots \hat{m{n}}_{i_\ell} = -\ell(\ell+1) C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{m{n}}_{i_1} \hat{m{n}}_{i_2} \dots \hat{m{n}}_{i_\ell} \; ,$ where $C_{i_1i_2...i_\ell}^{(\ell)}$ is a symmetric traceless tensor and

 $\hat{\boldsymbol{n}} = \sin \theta \cos \phi \, \hat{\boldsymbol{e}}_1 + \sin \theta \sin \phi \, \hat{\boldsymbol{e}}_2 + \cos \theta \, \hat{\boldsymbol{e}}_3$ (unit vector in (θ, ϕ) direction)

12(a)(i) General solution to Laplace's equation:

$$V(ec{m{r}}) = \sum_{\ell=0}^{\infty} \left(C_{i_1 i_2 \dots i_\ell}^{(\ell)} r^\ell + rac{C_{i_1 i_2 \dots i_\ell}^{(\ell)}}{r^{\ell+1}}
ight) \hat{m{n}}_{i_1} \hat{m{n}}_{i_2} \dots \hat{m{n}}_{i_\ell} \,, \quad ext{where } ec{m{r}} = r \hat{m{n}}$$

12(a)(ii)Azimuthal Symmetry:

$$V(ec{m{r}}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + rac{B_\ell}{r^{\ell+1}}
ight) \{ \, \hat{m{z}}_{i_1} \ldots \hat{m{z}}_{i_\ell} \, \}_{ ext{TS}} \, \hat{m{n}}_{i_1} \ldots \hat{m{n}}_{i_\ell} \ \, ext{where} \, \{ \ldots \}_{ ext{TS}} \, ext{denotes the traceless symmetric part of} \, \ldots \, .$$

Special cases:

$$\begin{split} &\{1\}_{\mathrm{TS}} = 1 \\ &\{\hat{x}_i\}_{\mathrm{TS}} = \hat{z}_i \\ &\{\hat{x}_i\hat{z}_j\}_{\mathrm{TS}} = \hat{z}_i\hat{z}_j - \frac{1}{3}\delta_{ij} \\ &\{\hat{z}_i\hat{z}_j\}_{\mathrm{TS}} = \hat{z}_i\hat{z}_j\hat{z}_k - \frac{1}{5}(\hat{z}_i\delta_{jk} + \hat{z}_j\delta_{ik} + \hat{z}_k\delta_{ij}) \\ &\{\hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m\}_{\mathrm{TS}} = \hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m - \frac{1}{7}(\hat{z}_i\hat{z}_j\delta_{km} + \hat{z}_i\hat{z}_k\delta_{mj} + \hat{z}_i\hat{z}_m\delta_{jk} \\ &+ \hat{z}_j\hat{z}_k\delta_{im} + \hat{z}_j\hat{z}_m\delta_{ik} + \hat{z}_k\hat{z}_m\delta_{ij}) + \frac{1}{35}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) \end{split}$$

12(b) Legendre Polynomial / Spherical Harmonic expansion:

12(b)(i) Azimuthal Symmetry:

$$V(ec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + rac{B_\ell}{r^{\ell+1}}
ight) P_\ell(\cos heta)$$

$$P_\ell(\cos heta) = rac{(2\ell)!}{2^\ell(\ell!)^2} \{\,\hat{m{z}}_{i_1}\dots\hat{m{z}}_{i_\ell}\,\}_{ ext{TS}}\,\hat{m{n}}_{i_1}\dots\hat{m{n}}_{i_\ell}$$

l2(b)(ii) General solution to Laplace's equation:

$$V(ec{m{r}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + rac{B_{\ell m}}{r^{\ell+1}}
ight) Y_{\ell m}(heta, \phi)$$

12(b)(iii) Orthonormality:
$$\int_0^{2\pi} \,\mathrm{d}\phi \int_0^\pi \,\sin\theta \,\mathrm{d}\theta \,Y^*_{\ell'm'}(\theta,\phi) \,Y_{\ell m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm}$$

12(b)(iv) Spherical Harmonics in terms of Traceless Symmetric Tensors:

$$Y_{\ell,m}(heta,\phi) = C_{i_1...i_\ell}^{(\ell,m)}\,\hat{m{n}}_{i_1}\dots\hat{m{n}}_{i_\ell}\;,$$

$$C_{i_{1}...i_{\ell}}^{(\ell,m)} = egin{cases} N(\ell,m) \{ \hat{m{u}}_{i_{1}}^{+} \ldots \hat{m{u}}_{i_{m}}^{+} \hat{m{z}}_{i_{m+1}} \ldots \hat{m{z}}_{i_{\ell}} \}_{\mathrm{TS}} & ext{for } m \geq 0 \;, \ \hat{m{u}}^{+} \equiv rac{1}{\sqrt{2}} \left(\hat{m{e}}_{m{x}} + i \hat{m{e}}_{m{y}}
ight) \; , & \hat{m{u}}^{-} \equiv rac{1}{\sqrt{2}} \left(\hat{m{e}}_{m{x}} - i \hat{m{e}}_{m{y}}
ight) \ N(\ell,m) = rac{(-1)^{m}(2\ell)!}{2^{\ell}\ell!} \sqrt{rac{2^{|m|}(2\ell+1)}{4\pi(\ell+m)!(\ell-m)!}} & ext{(for } m \geq 0). \end{cases}$$

Connection between m and -m: $Y_{\ell,-m}(\theta,\phi)=(-1)^mY_{\ell m}^*(\theta,\phi)$, which holds for all m.

12(b)(v) More Information about Spherical Harmonics:***

$$Y_{\ell m}(heta,\phi) = \sqrt{rac{2\ell+1}{4\pi}rac{(\ell-m)!}{(\ell+m)!}}P_{\ell}^m(\cos heta)e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = rac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} rac{\mathrm{d}^{\ell+m}}{\mathrm{d} x^{\ell+m}} (x^2-1)^\ell$$

13. Electric Multipole Expansion:

13(a) First several terms:

$$\begin{split} V(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}}}{r^2} + \frac{\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} Q_{ij} + \cdots \right] \text{, where} \\ Q &= \int \mathrm{d}^3 x \, \rho(\vec{\boldsymbol{r}}) \,, \ p_i = \int d^3 x \, \rho(\vec{\boldsymbol{r}}) \, x_i \,, \ Q_{ij} = \frac{1}{2} \int d^3 x \, \rho(\vec{\boldsymbol{r}}) (3x_i x_j - \delta_{ij} |\vec{\boldsymbol{r}}|^2) \\ \vec{\boldsymbol{E}}_{\mathrm{dip}}(\vec{\boldsymbol{r}}) &= -\frac{1}{4\pi\epsilon_0} \vec{\boldsymbol{\nabla}} \left(\frac{\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{\boldsymbol{r}}) \\ \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}_{\mathrm{dip}}(\vec{\boldsymbol{r}}) &= 0 \,, \qquad \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}_{\mathrm{dip}}(\vec{\boldsymbol{r}}) = \frac{1}{\epsilon_0} \rho_{\mathrm{dip}}(\vec{\boldsymbol{r}}) = -\frac{1}{\epsilon_0} \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}} \delta^3(\vec{\boldsymbol{r}}) \end{split}$$

13(b) Traceless Symmetric Tensor version:

$$V(ec{m{r}}) = rac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}rac{1}{r^{\ell+1}}\,C_{i_1\ldots i_\ell}^{(\ell)}\hat{m{n}}_{i_1}\ldots\hat{m{n}}_{i_{\ell}} \;,$$

$$\begin{aligned} &4\pi\epsilon_0 \underset{\ell=0}{\longleftarrow} r^{\ell+1} \quad ^{i_1...i_\ell - i_r} \\ & \quad C_{i_1...i_\ell}^{(\ell)} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{\boldsymbol{r}}) \left\{ x_{i_1} \dots x_{i_\ell} \right\}_{\mathrm{TS}} \mathrm{d}^3 x \qquad (\vec{\boldsymbol{r}} \equiv r \hat{\boldsymbol{n}} \equiv x_i \hat{\boldsymbol{e}}_i) \\ & \quad \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \left\{ \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_\ell} \right\}_{\mathrm{TS}} \hat{\boldsymbol{n}}_{i_1}' \dots \hat{\boldsymbol{n}}_{i_\ell}' , \qquad \text{for } r' < r \\ & \quad (2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\dots 1 = \frac{(2\ell)!}{2^{\ell}\ell!} , \text{ with } (-1)!! \equiv 1 . \end{aligned}$$

Reminder: $\{\dots\}_{TS}$ denotes the traceless symmetric part of \dots

13(c) Griffiths version (azimuthal symmetry only):

$$V(ec{m{r}}) = rac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} rac{1}{r^{\ell+1}} \int r'^{\ell}
ho(ec{m{r}}') P_{\ell}(\cos heta') \, \mathrm{d}^3x$$

where heta'= angle between $ec{m{r}}$ and $ec{m{r}}'$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta'), \qquad \frac{1}{\sqrt{1-2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_{\ell}(x) = rac{1}{2^{\ell}\ell!} \left(rac{\mathrm{d}}{\mathrm{d}x}
ight)^{\ell} (x^2-1)^{\ell}, \qquad ext{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^1 \mathrm{d}x \, P_{\ell'}(x) P_\ell(x) = rac{2}{2\ell+1} \delta_{\ell'\ell}$$

13(d) Spherical Harmonic version:

$$egin{align} V(ec{m{r}}) &= rac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} rac{4\pi}{2\ell+1} rac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(heta,\phi) \ & ext{where } q_{\ell m} &= \int Y_{\ell m}^* r'^\ell
ho(ec{m{r}}') \, \mathrm{d}^3 x' \ \end{aligned}$$

$$rac{1}{|ec{m{r}}-ec{m{r}'}|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} rac{4\pi}{2\ell+1} rac{r'^\ell}{r^{\ell+1}} Y_{\ell m}^*(heta',\phi') Y_{\ell m}(heta,\phi) \;\;, \qquad ext{for } r' < r$$

13(e) Properties of Traceless Symmetric Tensors:

13(e)(i) Trace Decomposition: Any symmetric tensor $S_{i_1...i_\ell}$ can be written

$$S_{i_1...i_\ell} = S_{i_1...i_\ell}^{(\mathrm{TS})} + \underset{i_1...i_\ell}{\operatorname{Sym}} \left[M_{i_1...i_{\ell-2}} \delta_{i_{\ell-1},i_\ell} \right] \; ,$$

where $S_{i_1...i_{\ell-2}}^{(\mathrm{TS})}$ is a traceless symmetric tensor, $M_{i_1...i_{\ell-2}}$ is a symmetric tensor tensor, and Sym[xxx] symmetrizes xxx in the indices $i_1 \dots i_\ell$. $S^{(\mathrm{TS})}_{i_1\dots i_\ell}$

is called the traceless symmetric part of $S_{i_1...i_\ell}$.

13(e)(ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1...i_\ell}$:

$$egin{align*} \{S_{i_1...i_\ell}\}_{ ext{TS}} &= S_{i_1...i_\ell} + \operatorname{Sym}_{i_1...i_\ell} \left[a_{1,\ell}\,\delta_{i_1\,i_2}\,\delta^{j_1j_2}\,S_{j_1j_2\,i_3...i_\ell}
ight. \ &+ a_{2,\ell}\,\delta_{i_1i_2}\,\delta_{i_3i_4}\,\delta^{j_1j_2}\,\delta^{j_3j_4}\,S_{j_1j_2j_3j_4i_5...i_\ell} + \ldots
ight] \end{split}$$

$$a_{n,\ell} = (-1)^n rac{\ell!^2 (2\ell-2n)!}{n! (\ell-2n)! (\ell-n)! (2\ell)!}$$

The series terminates when the number of i indices on S is zero or one. 13(e)(iii) Integration:

$$\int d\Omega \, \hat{m{n}}_{i_1} \dots \hat{m{n}}_{i_{2\ell}} = 4\pi rac{2^\ell \ell!}{(2\ell+1)!} \sum_{ ext{all pairings}} \delta_{i_1,i_2} \, \delta_{i_3,i_4} \dots \delta_{i_{2\ell-1},i_{2\ell}}$$

where
$$\int d\Omega \equiv \int_0^\pi \sin\theta \ d\theta \int_0^{2\pi} d\phi$$

The integral vanishes if the number of $\hat{\bm{n}}$ factors is odd.

$$egin{aligned} \int d\Omega \left[C_{i_1...i_\ell}^{(\ell)} \left\{ \hat{m{n}}_{i_1} \ldots \hat{m{n}}_{i_\ell}
ight.
igh$$

otherwise
$$13(e)(iv) \text{ Other identities:}$$

$$\hat{\boldsymbol{n}}_{i_{\ell}} \{ \hat{\boldsymbol{n}}_{i_{1}} \dots \hat{\boldsymbol{n}}_{i_{\ell}} \}_{\mathrm{TS}} = \frac{\ell}{2\ell-1} \{ \hat{\boldsymbol{n}}_{i_{1}} \dots \hat{\boldsymbol{n}}_{i_{\ell-1}} \}_{\mathrm{TS}} \qquad (\hat{\boldsymbol{n}} = \text{any unit vector})$$

$$\hat{\boldsymbol{n}}_{i_{\ell}} \{ \hat{\boldsymbol{n}}_{i_{1}} \dots \hat{\boldsymbol{n}}_{i_{\ell}} \}_{\mathrm{TS}} = \frac{\ell}{2\ell-1} \{ \hat{\boldsymbol{n}}_{i_{1}} \dots \hat{\boldsymbol{n}}_{i_{\ell-1}} \}_{\mathrm{TS}} \qquad (\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} + \hat{\boldsymbol{n}$$

$$\hat{oldsymbol{z}}_{i_\ell} \left\{ \hat{oldsymbol{u}}_{i_1}^+ \dots \hat{oldsymbol{u}}_{i_m}^+ \hat{oldsymbol{z}}_{i_{m+1}} \dots \hat{oldsymbol{z}}_{i_\ell}
ight\}_{\mathrm{TS}} = rac{(\ell+m)(\ell-m)}{\ell(2\ell-1)} \left\{ \hat{oldsymbol{u}}_{i_1}^+ \dots \hat{oldsymbol{u}}_{i_m}^+ \hat{oldsymbol{z}}_{i_{m+1}} \dots \hat{oldsymbol{z}}_{i_{\ell-1}}
ight\}_{\mathrm{TS}}$$

$$\text{where } \hat{oldsymbol{u}}^+ \equiv rac{1}{\sqrt{2}} \left(\hat{oldsymbol{e}}_{oldsymbol{x}} + i \hat{oldsymbol{e}}_{oldsymbol{y}} \right)$$

*** For any symmetric traceless tensor $S_{i_1...i_\ell}$.

$$\begin{split} \delta_{i\iota_{+2n-1},i\iota_{+2n}} \left\{ & \underset{i_{1}\dots i_{\ell+2n}}{\operatorname{Sym}} \left[S_{i_{1}\dots i_{\ell}} \underbrace{\delta_{i\iota_{+1},i\iota_{+2}} \cdots \delta_{i\iota_{+2n-1},i\iota_{+2n}}}_{n \text{ Kronecker } \delta-\text{functions}} \right] \right\} \\ &= F(n,\ell) \underset{i_{1}\dots i_{\ell+2n-2}}{\operatorname{Sym}} \left[S_{i_{1}\dots i_{\ell}} \underbrace{\delta_{i\iota_{+1},i\iota_{+2}} \cdots \delta_{i\iota_{+2n-3},i\iota_{+2n-2}}}_{n-1 \text{ Kronecker } \delta-\text{functions}} \right] \end{split}$$

$$F(n,\ell)=rac{2n(2\ell+2n+1)}{(\ell+2n)(\ell+2n-1)}$$

14. Electric Fields in Matter:

14(a) Electric Dipoles:

$$\begin{split} \vec{\boldsymbol{p}} &= \int d^3x \, \rho(\vec{\boldsymbol{r}}) \, \vec{\boldsymbol{r}} \\ \rho_{\mathrm{dip}}(\vec{\boldsymbol{r}}) &= -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}} \, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \text{, where } \vec{\boldsymbol{r}}_d = \text{position of dipole} \\ \vec{\boldsymbol{F}} &= (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}) \vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}} (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}) \quad \text{(force on a dipole)} \\ \vec{\boldsymbol{\tau}} &= \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}} \quad \text{(torque on a dipole)} \\ U &= -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}} \quad \text{(potential energy)} \end{split}$$

14(b) Electrically Polarizable Materials:

 $ec{m{P}}(ec{m{r}})= ext{polarization}= ext{electric dipole moment per unit volume}$

$$ho_{
m bound} = -
abla \cdot ec{m{P}} \;, \quad \sigma_{
m bound} = ec{m{P}} \cdot \hat{m{n}} \ ec{m{D}} \equiv \epsilon_0 \, ec{m{E}} + ec{m{P}} \;, \quad ec{m{
abla}} \cdot ec{m{D}} =
ho_{
m free} \;, \quad ec{m{
abla}} imes ec{m{E}} = 0$$

Boundary conditions:

$$egin{align*} E_{
m above}^\perp - E_{
m below}^\perp &= rac{\sigma}{\epsilon_0} & D_{
m above}^\perp - D_{
m below}^\perp &= \sigma_{
m free} \ & ec{m{E}}_{
m above}^\parallel - ec{m{E}}_{
m below}^\parallel &= 0 & ec{m{D}}_{
m above}^\parallel - ec{m{D}}_{
m below}^\parallel &= ec{m{P}}_{
m above}^\parallel - ec{m{P}}_{
m below}^\parallel - ec{m{P}_{
m below}^\parallel - ec{m{P}}_{
m be$$

14(c) Linear Dielectrics:

$$ec{m{P}} = \epsilon_0 \chi_e ec{m{E}}, \qquad \chi_e = ext{electric susceptibility}$$
 $\epsilon \equiv \epsilon_0 (1 + \chi_e) = ext{permittivity}, \qquad ec{m{D}} = \epsilon ec{m{E}}$
 $\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e = ext{relative permittivity}, \text{ or dielectric constant}$

Clausius-Mossotti equation: $\chi_e=rac{nlpha/\epsilon_0}{1-rac{nlpha}{3\epsilon_0}}$, where n= number density of atoms or (nonpolar) molecules, $\alpha={\rm atomic/molecular}$ polarizability ($\vec{\pmb p}=\alpha\vec{\pmb E})$

Energy:
$$W = \frac{1}{2} \int \vec{\boldsymbol{D}} \cdot \vec{\boldsymbol{E}} \, \mathrm{d}^3 x$$
 (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla}W$, where W is the potential energy stored in the system. Even if one or more potential differences are held fixed, the force can be found by computing the gradient of W with the total charge on each conductor fixed.

15. Magnetostatics:

15(a) Lorentz Force Law:
$$\vec{F} \equiv \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q(\vec{E} + \vec{v} \times \vec{B})$$
, where $\vec{p} = \gamma m_0 \vec{v}$, $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$m{ar{F}} = \int I \mathrm{d}m{ar{\ell}} imes m{ar{B}} = \int m{ar{J}} imes m{ar{B}} \, \mathrm{d}^3 x$$
15(b) Current Density:

Current through a surface $S\colon I_S = \int_S \vec{\boldsymbol{J}} \cdot \mathrm{d} \vec{\boldsymbol{a}}$

Charge conservation: $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Moving density of charge: $\vec{\boldsymbol{J}}=\rho\vec{\boldsymbol{v}}$

15(c) Biot-Savart Law:

$$ec{oldsymbol{B}}(ec{oldsymbol{r}}) = rac{\mu_0}{4\pi} I \int rac{\mathrm{d}ec{oldsymbol{\ell}} imes (ec{oldsymbol{r}} - ec{oldsymbol{r}'})}{|ec{oldsymbol{r}} - ec{oldsymbol{r}'}|^3}$$

where $\mu_0=$ permeability of free space $\simeq 4\pi \times 10^{-7}$ N/A² (to about 8-figure accuracy)

Infinitely long straight wire: $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$

Infintely long tightly wound solenoid: $\vec{\pmb{B}}=\mu_0 n I_0\,\hat{\pmb{z}}$, where n= turns per unit length

Loop of current on axis: $\vec{B}(0,0,z) = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{z}$

15(d) Vector Potential:

$$ec{m{A}}(ec{m{r}})_{
m coul} = rac{\mu_0}{4\pi} \int rac{ec{m{J}}(ec{m{r}}')}{|ec{m{r}}-ec{m{r}}'|}\, \mathrm{d}^3x' \;, \qquad ec{m{B}} = ec{m{
abla}} imes ec{m{A}} \;, \qquad ec{m{
abla}} \cdot ec{m{A}}_{
m coul} = 0$$

 $ec{m{\nabla}}\cdotec{m{B}}=0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla} \Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

$$ec{m{
abla}} imesec{m{B}}=\mu_0ec{m{J}}$$
 , or equivalently $\int_Pec{m{B}}\cdot\mathrm{d}ec{m{\ell}}=\mu_0I_{\mathrm{enc}}$

16. Magnetic Multipole Expansion:

16(a) Traceless Symmetric Tensor version:

$$A_j(ec{m{r}}) = rac{\mu_0}{4\pi} \sum_{\ell=0}^\infty \mathcal{M}_{j;i_1i_2...i_\ell}^{(\ell)} rac{\{\hat{m{r}}_{i_1}\dots\hat{m{r}}_{i_\ell}\}_{\mathrm{TS}}}{r^{\ell+1}}$$

where
$$\mathcal{M}_{j;i_1i_2...i_\ell}^{(\ell)} = rac{(2\ell-1)!!}{\ell!} \int \mathrm{d}^3x J_j(ec{m{r}}) \set{x_{i_1}\dots x_{i_\ell}}$$
Ts

Current conservation restriction: $\int \mathrm{d}^3 x \operatorname{Sym}_{i_1 \dots i_\ell} (x_{i_1} \dots x_{i_{\ell-1}} J_{i_\ell}) = 0$

where $\sup_{i_1...i_\ell}^{\mathrm{Sym}}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1 \dots i_\ell$

Special cases:
$$\ell=1$$
: $\int \mathrm{d}^3x\,J_i=0$ $\ell=2$: $\int \mathrm{d}^3x\,(J_ix_j+J_jx_i)=0$

Leading term (dipole): $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$,

$$egin{aligned} m{m}_i &= -rac{1}{2}\epsilon_{ijk}\mathcal{M}_{j;k}^{(1)} \ m{ec{m}} &= rac{1}{2}I\int_Pm{ec{r}} imes \mathrm{d}m{ec{\ell}} = rac{1}{2}\int\mathrm{d}^3x\,m{ec{r}} imesm{ec{J}} = Im{a}\;, \end{aligned}$$

where $\vec{\boldsymbol{a}} = \int_S \mathrm{d} \vec{\boldsymbol{a}}$ for any surface S spanning P

$$egin{align*} ec{m{B}}_{
m dip}(ec{m{r}}) = rac{\mu_0}{4\pi} ec{m{\nabla}} imes rac{ec{m{m}} imes \dot{m{r}}}{r^2} = rac{\mu_0}{4\pi} rac{3(ec{m{m}} \cdot \hat{m{r}}) \dot{m{r}} - ec{m{m}}}{r^3} + rac{2\mu_0}{3} ec{m{m}} \; m{\delta}^3(ec{m{r}}) \ ec{m{\nabla}} \cdot ec{m{B}}_{
m dip}(ec{m{r}}) = 0 \;, \qquad ec{m{\nabla}} imes ec{m{B}}_{
m dip}(ec{m{r}}) = \mu_0 ec{m{J}}_{
m dip}(ec{m{r}}) = -\mu_0 ec{m{m}} imes ec{m{\nabla}} m{\delta}^3(ec{m{r}}) \end{split}$$

16(b) Griffiths version (azimuthal symmetry only, current in wires):

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \oint (r')^{\ell} P_{\ell}(\cos \theta') \mathrm{d}\vec{\boldsymbol{\ell}}, \text{ where } \theta' = \text{angle between } \vec{\boldsymbol{r}} \text{ and } \vec{\boldsymbol{r}}'.$$

17. Magnetic Fields in Matter:

17(a) Magnetic Dipoles:

$$ec{m{m}}=rac{1}{2}I\int_Pec{m{r}} imes \mathrm{d}ec{m{\ell}}=rac{1}{2}\int\mathrm{d}^3x\,ec{m{r}} imesec{m{J}}=Im{a}$$
 , where $m{d}=\int_S\mathrm{d}m{d}$ for any S spanning P

 $ec{m{J}}_{
m dip}(ec{m{r}}) = -ec{m{m}} imes ec{m{\nabla}}_{ec{m{r}}} \delta^3 (ec{m{r}} - ec{m{r}}_d), ext{ where } ec{m{r}}_d = ext{position of dipole} \ ec{m{F}} = ec{m{\nabla}} (ec{m{m}} \cdot ec{m{B}}) \qquad ext{(force on a dipole)}$

$$\vec{r} = \vec{v} (\vec{m} \cdot \vec{B})$$
 (torque on a dipole)

$$I = -\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}}$$
 (potential energy)

17(b) Magnetically Polarizable Materials:

 $ec{M}(ec{r}) = ext{magnetization} = ext{magnetic dipole moment per unit volume} \ ec{J}_{ ext{bound}} = ec{m{\nabla}} imes ec{M} \ , \qquad ec{K}_{ ext{bound}} = ec{M} imes \hat{n}$

$$ec{m{f}}_{ ext{bound}} = ec{m{
abla}} imes ec{m{M}} \; , \qquad ec{m{K}}_{ ext{bound}} = ec{m{M}} imes \hat{m{n}} \; .$$

$$ec{m{H}}\equivrac{1}{\mu_0}ec{m{B}}-ec{m{M}}\;, \qquad ec{m{
abla}} imesec{m{H}}=ec{m{J}}_{
m free}+rac{\partial ec{m{D}}}{\partial t}\;, \qquad ec{m{
abla}}\cdotec{m{B}}=0$$

$$egin{align*} egin{align*} egin{align*}$$

17(c) Linear Magnetic Materials:

$$\vec{M} = \chi_m \vec{H}, \qquad \chi_m = \text{magnetic susceptibility}$$

 $\mu = \mu_0 (1 + \chi_m) = \text{permeability}, \qquad \vec{B} = \mu \vec{H}$

Magnetic Monopoles:

$$ec{m{B}}(ec{m{r}}) = rac{\mu_0}{4\pi} rac{q_m}{r^2} \hat{m{r}} \; ; \qquad ext{Force on a static monopole: } ec{m{F}} = q_m ec{m{B}} \;$$

Angular momentum of monopole/charge system: $\vec{L} = \frac{\mu_0 q_e q_m}{4\pi} \hat{r}$, where \hat{r} points from q_e to q_m

Dirac quantization condition: $\frac{\mu_0 q_e q_m}{4\pi} = \frac{1}{2}\hbar \times \text{integer}$

19. Maxwell's Equations:

(i)
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$
 (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,
(ii) $\vec{\nabla} \cdot \vec{B} = 0$ (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{\epsilon_0} \vec{J} = 0$

$$ec{m{\phi}}\cdotm{m{B}}=0 \qquad ext{(iv)} \;\; m{m{\nabla}} imesm{m{B}}=\mu_0m{m{J}}+rac{1}{c^2}rac{\partialm{m{E}}}{\partial t}$$

where
$$\mu_0\epsilon_0=rac{1}{c^2}$$

Lorentz force law: $\vec{m{F}} = q(\vec{m{E}} + \vec{m{v}} \times \vec{m{B}})$

Charge conservation: $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{\boldsymbol{J}}$

20. Maxwell's Equations in Matter:

Polarization \vec{P} and magnetization \vec{M}

$$ho_b = - ec{m{ au}} \cdot ec{m{P}} \; , \quad ec{m{J}}_P = rac{\partial ec{m{P}}}{\partial t} \; , \quad ec{m{J}}_b = ec{m{
abla}} imes ec{m{M}} \; , \quad
ho =
ho_f +
ho_b \; , \quad ec{m{J}} = ec{m{J}}_f + ec{m{J}}_b + ec{m{J}}_P$$

Polarization current if the matter is in motion, with velocity $\vec{v}(\vec{r},t)$: ***

$$ec{m{J}}_{P}=rac{\partial ec{m{P}}}{\partial t}-ec{m{
abla}} imes (ec{m{v}} imes ec{m{P}})$$

Auxiliary Fields: $ec{m{H}}\equivrac{ec{m{B}}}{\mu_0}-ec{m{M}}\;, \qquad ec{m{D}}\equiv\epsilon_0ec{m{E}}+ec{m{P}}$

Maxwell's Equations:

(i)
$$\vec{\nabla} \cdot \vec{D} = \rho_f$$
 (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

(ii)
$$\vec{\nabla} \cdot \vec{B} = 0$$
 (iv) $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

For linear media:

$$ec{m{m{m{eta}}}} = \epsilon ec{m{m{E}}} \;, \qquad ec{m{H}} = rac{1}{...} ec{m{B}} \;.$$

where $\epsilon =$ dielectric constant, $\mu =$ relative permeability

$$ec{m{J}}_d \equiv rac{\partial ec{m{D}}}{\partial t} = ext{displacement current}$$

21. Boundary Conditions:

$$egin{align} D_1^\perp - D_2^\perp &= \sigma_f & m{ec E}_1^\parallel - m{ec E}_2^\parallel &= 0 \ E_1^\perp - E_2^\perp &= rac{1}{\epsilon_0} & m{ec D}_1^\parallel - m{ec D}_2^\parallel &= m{ec P}_1^\parallel - m{ec P}_2^\parallel \ \end{pmatrix}$$

$$egin{align} B_1^\perp - B_2^\perp &= 0 & \vec{m{H}}_1^\parallel - \vec{m{H}}_2^\parallel &= -\hat{m{n}} imes \vec{m{K}}_f \ B_1^\perp - H_2^\perp &= M_2^\perp - M_1^\perp & \vec{m{B}}_1^\parallel - \vec{m{B}}_2^\parallel &= -\mu_0 \hat{m{n}} imes \vec{m{K}} \end{array}$$

22. Maxwell's Equations with Magnetic (Monopole) Charge

(i)
$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho_e$$
 (ii) $\vec{\nabla} \times \vec{E} = -\mu_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t}$, (ii) $\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m$ (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_e + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$

(ii)
$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = \mu_0 \rho_m$$
 (iv) $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}_e + \frac{1}{c'}$

Magnetic Lorentz force law:
$$ec{m{F}} = q_m \left(ec{m{B}} - rac{1}{c^2} ec{m{v}} imes ec{m{E}}
ight)$$

23. Current, Resistance, and Ohm's Law:

$$m{ec J} = \sigma(m{ec E} + m{ec v} imes m{ec B})$$
 , where $\sigma = ext{conductivity}.$ $ho = 1/\sigma = ext{resistivity}$

Resistors:
$$V = IR$$
, $P = IV = I^2R = V^2/R$

Resistance in a wire: $R=rac{\iota}{A}\,
ho$, where $\ell={
m length}, A={
m cross-sectional}$ area, and ho=

Charging an RC circuit:
$$I = \frac{V_0}{R} e^{-t/RC}$$
 , $Q = CV_0 \left[1 - e^{-t/RC}\right]$

Emf (Electromotive force): $\mathcal{E} \equiv \oint (\vec{f}_s + \vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell}$. Here \vec{f}_s is the force per unit charge of any external power source, such as a battery. \vec{v} can be either the velocity of the wire, or the velocity of the charge carriers (since the velocity of the charge carriers relative to the wire points along the wire, and gives no contribution to $(\vec{v} \times \vec{B}) \cdot d\vec{\ell}$).

Universal flux rule: Whenever the flux through a loop changes, whether due to a changing \vec{B} or motion of the loop, $\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$, where Φ_B is the magnetic flux through the loop. Here \mathcal{E} is the emf induced by the magnetic field, and does not include possible contributions from sources \vec{f}_s .

Mutual inductance: $\Phi_2=M_{21}I_1$, $M_{21}=$ mutual inductance

(Franz) Neumann's formula:
$$M_{21}=M_{12}=rac{\mu_0}{4\pi}\oint_{P_1}\oint_{P_2}rac{\mathrm{d} \vec{m{\ell}}_1\cdot\mathrm{d} \vec{m{\ell}}_2}{|\vec{m{r}}_1-\vec{m{r}}_2|}$$

Self inductance:
$$\Phi=LI$$
 , $\mathcal{E}=-L\frac{\mathrm{d}I}{\mathrm{d}t}$; $L=\mathrm{inductance}$

Self inductance of a solenoid: $L=n^2\mu_0\mathcal{V}$, where n= number of turns per length, $\mathcal{V}=$ volume

Rising current in an RL circuit: $I = \frac{V_0}{R} \left[1 - e^{\frac{R}{L}t} \right]$

25. Circuit Elements

Voltage source:
$$V_0 \stackrel{\downarrow+}{\mathsf{T}_-} V_+ - V_- \equiv V_0$$

Resistor:
$$R \stackrel{-}{\underset{+}{\gtrless}} V_{+} - V_{-} \equiv V_{R} = IR$$

Capacitor:
$$C = \frac{-1 - Q}{+1}$$

$$V_{+} - V_{-} \equiv V_{C}, \quad Q = CV_{C},$$
Inductor:
$$L = \begin{cases} -\frac{1}{4} & V_{+} - V_{-} \equiv V_{L} = L \frac{dI}{dt} \end{cases}$$

Energy density:
$$u_{\mathrm{EM}} = \frac{1}{2} \left[\epsilon_0 | \vec{m{E}}|^2 + \frac{1}{\mu_0} | \vec{m{B}}|^2
ight]$$

Poynting vector (flow of energy): $\vec{\boldsymbol{S}} = \frac{1}{\mu_0} \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$

$$\text{Integral form: } \frac{\mathrm{d}}{\mathrm{d}t} \left[U_{\mathrm{EM}} + U_{\mathrm{mech}} \right] = - \int \vec{\boldsymbol{S}} \cdot \, \mathrm{d}\vec{\boldsymbol{a}}$$

Differential form: $\frac{\partial u}{\partial t} = - \vec{m{\nabla}} \cdot \vec{m{S}}$, where $u = u_{\rm EM} + u_{
m mech}$

Momentum density: $\vec{\wp}_{\rm EM}=\frac{1}{c^2}\vec{\bf S}$; $\frac{1}{c^2}S_i$ is the density of momentum in the i'th direction

Maxwell stress tensor:
$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} |\vec{E}|^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} |\vec{B}|^2 \right)$$
 where $-T_{ij} = -T_{ji} =$ flow in j 'th direction of momentum in the i 'th direction

Integral form:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\mathrm{mech},i}+\frac{1}{c^2}\int_{\mathcal{V}}S_i\,\mathrm{d}^3x\right)=\oint_{S}T_{ij}\,\mathrm{d}a_j$$
, for a volume \mathcal{V} bounded by a surface S Differential form: $\frac{\partial}{\partial t}\left(\wp_{\mathrm{mech},i}+\wp_{\mathrm{EM},i}\right)=\partial_jT_{ji}$

Angular momentum:

Angular momentum density (about the origin):

$$m{ar{l}}_{ ext{EM}} = m{ec{r}} imes m{ec{arphi}}_{ ext{EM}} = \epsilon_0 [m{ec{r}} imes (m{ar{E}} imes m{ar{B}})]$$

27. Wave Equation in 1 Dimension:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$
 , where v is the wave velocity

Sinusoidal waves:

$$f(z,t) = A\cos\left[k(z-vt) + \delta\right] = A\cos\left[kz - \omega t + \delta\right]$$
 where

$$\omega = \mathrm{angular} \; \mathrm{frequency} = 2\pi \nu$$
 $v = \frac{\omega}{k} = \mathrm{phase} \; \mathrm{velocity}$
 $k = \mathrm{wave} \; \mathrm{number}$
 $T = 2\pi/\omega = \mathrm{period}$

$$\delta = {
m phase} \ ({
m or} \ {
m phase} \ {
m constant})$$
 $\lambda = 2\pi/k = {
m wavelength}$ $A = {
m amplitude}$

 $\nu = \text{frequency}$

Euler identity: $e^{i\theta} = \cos \theta + i \sin \theta$

Complex notation: $f(z,t)=\Re[\tilde{A}e^{i(kz-\omega t)}]$, where $\tilde{A}=Ae^{i\delta};$ " \Re " is usually dropped.

Wave velocities: $v=rac{\omega}{k}= ext{phase}$ velocity; $v_{ ext{group}}=rac{ ext{d}\omega}{ ext{d}k}= ext{group}$ velocity

28. Electromagnetic Waves:

Wave Equations: $\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$, $\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$

28(a) Linearly Polarized Plane Waves:

 $ec{m{E}}(ec{m{r}},t)= ilde{E}_0e^{i(ec{m{k}}\cdotec{m{r}}-\omega t)}\,\hat{m{n}}$, where $ilde{E}_0$ is a complex amplitude, $\hat{m{n}}$ is a unit vector, and $\omega/|ec{m{k}}| = v_{
m phase} = c.$

$$\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{k}} = 0$$
 (transverse wave)

$$ec{m{B}} = rac{1}{c} \hat{m{k}} imes ec{m{E}}$$

Energy and Momentum:

$$u=\epsilon_0 E_0^2 \underbrace{\cos^2(kz-\omega t+\delta)}_{t=1.0}, \quad (ec{m{k}}=k\,m{\hat{z}})$$

$$ec{\wp}_{\mathrm{EM}} = rac{1}{c^2}\,ec{oldsymbol{S}} = rac{u}{c}\,oldsymbol{\hat{z}}$$

$$u=\epsilon_0 E_0^2 \frac{\cos^2(kz-\omega t+\delta)}{\text{averages to } 1/2}, \quad (ec{m{k}}=k\,m{\hat{z}})$$

$$m{ec{S}} = rac{1}{\mu_0} m{ec{E}} imes m{ec{B}} = uc \; m{\hat{z}} \; , \qquad I \; (ext{intensity}) = \left\langle \left| m{ec{S}}
ight|
ight
angle = rac{1}{2} \epsilon_0 E_0^2$$

Electromagnetic Waves in Matter:

$$n \equiv \sqrt{rac{\mu \epsilon}{\mu_0 \epsilon_0}} = ext{index of refraction}$$

$$v= ext{phase velocity}=rac{c}{n} \ u=rac{1}{2}\left[\epsilon|oldsymbol{ec{E}}|^2+rac{1}{\mu}|oldsymbol{ec{B}}|^2
ight]$$

$$ec{m{B}} = rac{n}{c} \hat{m{k}} imes ec{m{E}}$$

$$ec{m{S}} = rac{1}{\mu} ec{m{E}} imes ec{m{B}} = rac{uc}{n} \; \hat{m{z}}$$

28(b) Reflection and Transmission at Normal Incidence:

Boundary conditions:

$$egin{align} egin{align} \epsilon_1 E_1^ot &= \epsilon_2 E_2^ot \ B_1^ot &= oldsymbol{ar{E}}_1^ot &= oldsymbol{ar{E}}_2^ot \ B_1^ot &= B_2^ot & rac{1}{\mu_1} oldsymbol{ar{B}}_1^ot &= rac{1}{\mu_2} oldsymbol{ar{B}}_2^ot \ . \end{align}$$

Incident wave (z < 0):

$$m{ec{E}}_I(z,t) = ilde{E}_{0,I} e^{i(k_1z-\omega t)} \, m{\hat{x}}$$

$$m{ec{B}}_I(z,t) = rac{1}{v_1} ilde{E}_{0,I} e^{i(k_1z-\omega t)} \, \hat{m{y}} \; .$$

Transmitted wave (z > 0):

$$ec{m{E}}_T(z,t) = ilde{E}_{0,T} \, e^{i(k_2 z - \omega t)} \, \hat{m{x}}$$

$$ec{m{B}}_T(z,t) = rac{1}{v_2} ilde{E}_{0,T} e^{i(k_2 z - \omega t)} \, \hat{m{y}} \; .$$

Reflected wave (z < 0):

$$ec{oldsymbol{E}}_{R}(z,t)= ilde{E}_{0,R}\,e^{i(-k_{1}z-\omega t)}\,\hat{oldsymbol{x}}$$

$$ec{m{B}}_R(z,t) = -rac{1}{v_1} ilde{E}_{0,R} \, e^{i(-k_1z-\omega t)} \, \hat{m{y}}$$
 ω must be the same on both sides, so

$$rac{\omega}{k_1}=v_1=rac{c}{n_1}\;, \qquad rac{\omega}{k_2}=v_2=rac{c}{n_2}$$

Applying boundary conditions and solving, approximating $\mu_1=\mu_2=\mu_0$,

$$ilde{E}_{0,R} = rac{n_1 - n_2}{n_1 + n_2} \, ilde{E}_{0,I} \qquad E_{0,T} = \left(rac{2n_1}{n_1 + n_2}
ight) \, ilde{E}_{0,I}$$

29. Electromagnetic Potentials:

29(a) The fields:
$$\vec{\pmb{B}} = \vec{\pmb{\nabla}} \times \vec{\pmb{A}}$$
 , $\vec{\pmb{E}} = -\vec{\pmb{\nabla}} V - \frac{\partial \pmb{A}}{\partial t}$

29(b) Gauge transformations:
$$ec{m{A}}'=ec{m{A}}+ec{m{\nabla}}\Lambda$$
 , $V'=V-rac{\partial\Lambda}{\partial t}$

29(b) Gauge transformations:
$$\vec{A}' = \vec{A} + \vec{\nabla} \Lambda$$
, $V' = V - \frac{\partial \Lambda}{\partial t}$
29(c) Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0 \implies \nabla^2 V = -\frac{1}{\epsilon_0} \rho$ (but \vec{A} is complicated)

29(d) Lorenz gauge:
$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \implies$$

$$m{\Box}^2 V = -rac{1}{\epsilon_0}
ho \;, \qquad m{\Box}^2 ec{m{A}} = -\mu_0 ec{m{J}} \;\;, \qquad ext{where} \;\; m{\Box}^2 \equiv
abla^2 - rac{1}{c^2} rac{\partial^2}{\partial t^2}$$

29(e) Retarded time solutions (Lorenz gauge):
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \,, \qquad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}$$

$$t_r = t - rac{|ec{oldsymbol{r}} - ec{oldsymbol{r}'}|}{c}$$

29(f) Liénard-Wiechert Potentials (potentials of a point charge):
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\hbar \left(1 - \frac{1}{c}\vec{v}_p \cdot \hat{\boldsymbol{x}}\right)}$$

$$m{ec{A}}(m{ec{r}},t) = rac{\mu_0}{4\pi} \; rac{qm{ec{v}}_p}{2\left(1-rac{1}{c}m{ec{v}}_p\cdot\hat{m{\lambda}}
ight)} = rac{m{ec{v}}_p}{c^2} V(m{ec{r}},t)$$

where \vec{r}_p and \vec{v}_p are the position and velocity of the particle at the retarded time t_r , and

$$ec{m{ ilde{ heta}}} = ec{m{r}} - ec{m{r}}_p \; , \qquad m{\hbar} = |ec{m{r}} - ec{m{r}}_p| \; , \qquad \hat{m{\hbar}} = rac{ec{m{r}} - ec{m{r}}_p}{|ec{m{r}} - ec{m{r}}_p|}$$

29(g) Fields of a point charge (from the Liénard-Wiechert potentials):

$$m{ec{E}}(m{ec{r}},t) = rac{q}{4\pi\epsilon_0}rac{\hbar}{\left(m{ec{u}}\cdotm{ec{z}}
ight)^3}\left[\left(c^2-v_p^2
ight)m{ec{u}} + m{ec{z}} imes\left(m{ec{u}} imesm{ec{a}}_p
ight)
ight]$$

$$ec{m{B}}(ec{m{r}},t)=rac{1}{c}\,\hat{m{\lambda}} imesec{m{E}}(ec{m{r}},t)$$

where $\vec{\boldsymbol{u}} = c \; \hat{\boldsymbol{z}} - \vec{\boldsymbol{v}}_p$

30. Radiation:

30(a) Radiation from an oscillating electric dipole along the z axis:

$$p_0(t)=p_0\cos(\omega t)\;,\quad p_0=q_0\,dt$$

Approximations: $d \ll \lambda \ll r$,

$$V(r, heta,t) = -rac{p_0\omega}{4\pi\epsilon_0\,c}\left(rac{\cos heta}{r}
ight)\sin[\omega(t-r/c)]$$

$$ec{m{A}}(ec{m{r}},t) = -rac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t-r/c)]~m{\hat{z}}$$

$$ec{m{E}} = -rac{\mu_0 p_0 \omega^2}{4\pi} \left(rac{\sin heta}{r}
ight) \cos[\omega(t-r/c)]\,\hat{m{ heta}} \;, \qquad ec{m{E}}(ec{m{r}},t) = rac{1}{c}\,\hat{m{r}} imesec{m{E}}(ec{m{r}},t)$$

Poynting vector:
$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0} (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = \frac{\mu_0}{c} \left\{ \frac{p_0 \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \right\}^2 \hat{\boldsymbol{r}}$$

Intensity: $\vec{\boldsymbol{I}} = \left\langle \vec{\boldsymbol{S}} \right\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2 \theta}{r^2} \hat{\boldsymbol{r}}$, using $\langle \cos^2 \rangle = \frac{1}{2}$

Total power:
$$\langle P \rangle = \int \left\langle \vec{S} \right\rangle \cdot d\vec{a} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$
30(b) Magnetic Dipole Radiation:

Dipole moment: $\vec{\boldsymbol{m}}(t) = m_0 \cos(\omega t) \, \hat{\boldsymbol{z}}$, at the origin

$$ec{m{E}} = rac{\mu_0 m_0 \omega^2}{4\pi c} \left(rac{\sin heta}{r}
ight) \cos[\omega(t-r/c)]\,\hat{m{\phi}} \;, \qquad ec{m{B}}(ec{m{r}},t) = rac{1}{c}\,\hat{m{r}} imesec{m{E}}(ec{m{r}},t)$$

Compared to the electric dipole radiation, $p_0 o rac{m_0}{c} \;, \qquad -\hat{m{ heta}} o \hat{m{\phi}}$

30(c) General Electric Dipole Radiation:

$$ec{m{E}}(ec{m{r}},t)=rac{\mu_0}{4\pi r}[(\hat{m{r}}\cdot\ddot{m{p}})\hat{m{r}}-\ddot{m{p}}~]~,~~ec{m{B}}(ec{m{r}},t)=rac{1}{c}\hat{m{r}} imesec{m{E}}(ec{m{r}})$$

30(d) Multipole Expansion for Radiation:

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, ... radiation, and also magnetic dipole, quadrupole, quadrupole, ...

 $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,,t) = \frac{\mu_0}{4\pi r} [(\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}})\hat{\boldsymbol{r}}-\ddot{\boldsymbol{p}}\,]\;, \qquad \vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}\,,t) = \frac{1}{c}\,\hat{\boldsymbol{r}}\times\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,,t) = -\frac{\mu_0}{4\pi rc}[\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}]$

30(e) Radiation from a Point Particle:

When the particle is at rest at the retarded time

$$ec{m{E}}_{\mathrm{rad}} = rac{q}{4\pi\epsilon_0 \, c^2 |ec{m{r}} - ec{m{r}}'|} [\, m{\hat{m{\lambda}}} imes (\, m{\hat{m{\lambda}}} imes m{ar{a}}_p)]$$

Poynting vector:
$$\vec{\boldsymbol{S}}_{\mathrm{rad}} = \frac{1}{\mu_0 c} |\vec{\boldsymbol{E}}_{\mathrm{rad}}|^2 \,\, \hat{\boldsymbol{\lambda}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left(\frac{\sin^2 \theta}{\mathcal{P}^2}\right) \,\hat{\boldsymbol{\lambda}}$$

where θ is the angle between \vec{a}_p and \hat{z} .

Total power (Larmor formula):
$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

alid for
$$\vec{\boldsymbol{v}}_p = 0$$
 or $|\vec{\boldsymbol{v}}_p| \ll c$

(valid for $\vec{\pmb{v}}_p=0$ or $|\vec{\pmb{v}}_p|\ll c$) Liénard's Generalization if $\vec{\pmb{v}}_p\neq 0$: ***

$$P = rac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| rac{ec{m{v}} imes ec{m{a}}}{c}
ight|^2
ight) = rac{\mu_0 q^2}{6\pi m_0^2 c} rac{\mathrm{d} p_\mu}{\mathrm{d} au} rac{\mathrm{d} p^\mu}{\mathrm{d} au}$$

For relativists only

31. Radiation Reaction:***

31(a) Abraham-Lorentz formula:

$$ec{m{F}}_{
m rad} = rac{\mu_0 q^2}{6\pi c} m{\dot{a}}$$

The Abraham-Lorentz formula is guaranteed to give the correct average energy remains unsolved. radiation reaction for point particles in classical electrodynamics apparently leads to runaway solutions which are clearly unphysical. The problem of that works under general circumstances. The Abraham-Lorentz formula loss for periodic or nearly periodic motion, but one would like a formula

32. Table of Legendre Polynomials $P_{\ell}(x)$:

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x)=rac{1}{8}(35x^4-30x^2+3)$$

33. Table of Spherical Harmonics $Y_{\ell m}(\theta, \phi)$:

$$\ell = 0 Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\ell = 1 \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \end{cases}$$

$$V_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$V_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} (3\cos^{2}\theta - 1)$$

$$V_{33} = -\frac{1}{4}\sqrt{\frac{35}{2\pi}} \sin^{3}\theta e^{3i\phi}$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^{2}\theta \cos \theta e^{2i\phi}$$

$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{2\pi}} \sin \theta (5\cos^{2}\theta - 1) e^{i\phi}$$

$$Y_{30} = \frac{1}{2}\sqrt{\frac{7}{4\pi}} (5\cos^{3}\theta - 3\cos\theta)$$

$$V_{44} = \frac{3}{16}\sqrt{\frac{35}{2\pi}} \sin^{4}\theta e^{4i\phi}$$

$$Y_{43} = -\frac{3}{2}\sqrt{\frac{35}{2\pi}} \sin^{3}\theta \cos \theta e^{3i\phi}$$

$$V_{42} = \frac{3}{8}\sqrt{\frac{5}{2\pi}} \sin^{2}\theta (7\cos^{2}\theta - 1) e^{2i\phi}$$

$$Y_{41} = -\frac{3}{2}\sqrt{\frac{5}{2\pi}} \sin \theta (7\cos^{2}\theta - 3\cos\theta) e^{i\phi}$$

$$Y_{40} = \frac{3}{\sqrt{4\pi}} (35\cos^{4}\theta - 30\cos^{2}\theta + 3)$$

For m<0, use $Y_{\ell,-m}(\theta,\phi)=(-1)^mY_{\ell m}^*(\theta,\phi)$, which is valid for all m.