8.07 Lecture Slides 13
October 23, 2019

ELECTRIC POTENTIAL: SPHERICAL HARMONICS VIA TRACELESS SYMMETRIC TENSORS

Announcements

Problem Set 5, Problem 5: the hint was worded incorrectly. You should calculate the **potential** along the z axis, not the **field**.

Detailed Form of the Trace Decomposition Theorem

For an symmetric tensor $S_{i_1 \ldots i_\ell}$, the traceless symmetric part can be written as

$$\boxed{\begin{aligned}\{\, S_{i_1 \ldots i_\ell} \,\}_{\rm TS} = S_{i_1 \ldots i_\ell} + \mathop{\rm Sym}_{i_1 \ldots i_\ell} \Big[& a_{1,\ell}\, \delta_{i_1 i_2} S_{j_1 j_1 i_3 \ldots i_\ell} \\ & + a_{2,\ell}\, \delta_{i_1 i_2} \delta_{i_3 i_4} S_{j_1 j_1 j_2 j_2 i_5 \ldots i_\ell} + \ldots \Big] \ ,\end{aligned}}$$

where

$$\mathop{\rm Sym}_{i_1 \ldots i_\ell} \left[T_{i_1 \ldots i_\ell} \right] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} T_{i_1 \ldots i_\ell} \ ,$$

and

$$\boxed{a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n! (\ell - 2n)! (\ell - n)! (2\ell)!} \ .}$$

In the last lecture we derived $a_{1,\ell}$.

$$\left\{ S_{i_1\ldots i_\ell} \right\}_{\rm TS} = S_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\rm Sym}\Big[a_{1,\ell}\delta_{i_1 i_2} S_{j_1 j_1 i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4} S_{j_1 j_1 j_2 j_2 i_5\ldots i_\ell} + \ldots \Big] ,$$

We need to take the trace of the right-hand side (RHS), choosing the $a_{n\ell}$ so that it vanishes. Take the trace in i_1, i_2. Calculating through the $a_{1\ell}$ term,

$$\mathrm{Tr}\left(\mathrm{RHS}_1\right) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \Big[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell} \Big] .$$

To evaluate the second term, we consider 4 cases for where the indices i_1 and i_2 can appear.

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

To evaluate the second term, we consider 4 cases for where the indices i_1 and i_2 can appear.

Case I: i_1 and i_2 can appear on the Kronecker δ-function in the square brackets.

$$\text{Case I:} \qquad \text{Multiplicity} = 2(\ell-2)!\ , \qquad \text{Value} = 3S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ .$$

Case II: i_1 can appear on the Kronecker δ-function in the square brackets, while i_2 appears as one of the indices of S.

$$\text{Case II:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ .$$

Case III: i_2 can appear on the Kronecker δ-function in the square brackets, while i_1 appears as one of the indices of S.

$$\text{Case III:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\,\frac{a_{1,\ell}}{\ell!}\ .$$

Case IV: Both i_1 and i_2 can appear on S.

$$\text{Case IV:} \qquad \text{Multiplicity} = (\ell-2)(\ell-3)(\ell-2)!$$

$$\text{Value} = \text{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right]\frac{a_{1,\ell}}{\ell!}\ .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3 + 4(\ell-2))\right] + \mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] a_{1,\ell}\,\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)}\ .$$

$$\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3 + 4(\ell-2))\right] = 0 \qquad \Longrightarrow \qquad \boxed{a_{1,\ell} = -\frac{\ell(\ell-1)}{2(2\ell-1)}\ .}$$

Application to $\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\rm TS}$

$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}$ is traceless and symmetric and constructed from $\hat{\boldsymbol{n}}$, so it **must be proportional to $\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}$.**

Then

$$\begin{aligned}\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS} &= \hat{\boldsymbol{n}}_{i_\ell}\left\{\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell} - \frac{\ell(\ell-1)}{2(2\ell-1)}\mathop{\rm Sym}_{i_1\ldots i_\ell}\left[\delta_{i_1 i_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right]+\ldots\right\}\\ &= \hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\left[1-\frac{\ell(\ell-1)}{2(2\ell-1)}\frac{2}{\ell}\right]+\ldots\\ &= \frac{\ell}{2\ell-1}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}+\ldots\ .\end{aligned}$$

The omitted terms cannot contribute to the term with no Kronecker delta functions, so

$$\boxed{\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}\ .}$$

Integration over Spherical Harmonics

Result:

$$\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell}\,\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\,\hat{\boldsymbol{n}}_{j_1}\ldots\hat{\boldsymbol{n}}_{j_{\ell'}}\,\}_{\rm TS}\right]$$

$$= 4\pi\frac{2^\ell \ell!^2}{(2\ell+1)!}\,C^{(\ell)}_{i_1\ldots i_\ell}\,C'^{(\ell)}_{i_1\ldots i_\ell} \quad \text{if } \ell'=\ell.$$

And it equals zero if $\ell' \neq \ell$.

We started this derivation on Monday.

Useful Integral: $\int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\, \mathrm{d}r$

$$\boxed{\int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\, \mathrm{d}r = \sqrt{\frac{\pi}{2}}\,\frac{(2\ell+1)!}{2^\ell \ell!}\ .}$$

Sketch of proof:

Define $\boxed{I_2(\lambda) \equiv \int_0^\infty dr\, e^{-\lambda r^2}\ .}$

Then

$$I_2^2(\lambda) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, e^{-\lambda(x^2+y^2)} = \int_0^{\pi/2} d\phi \int_0^\infty r\, dr\, e^{-\lambda r^2} = \frac{\pi}{4\lambda}\ ,$$

so $I_2(\lambda) = \dfrac{1}{2}\sqrt{\dfrac{\pi}{\lambda}}$, and $\displaystyle\int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\, \mathrm{d}r = (-1)^{\ell+1}\, \frac{\mathrm{d}^{\ell+1}}{\mathrm{d}\lambda^{\ell+1}} I_2(\lambda)\bigg|_{\lambda=\frac{1}{2}}$,

which leads to the boxed result above.

Another Useful Integral: $\int d^3x \, e^{-\frac{|\vec{r}\,|^2}{2} + \vec{J}\cdot\vec{r}}$

Define $\boxed{I_1(\vec{J}\,) \equiv \int d^3x \, e^{-\frac{|\vec{r}\,|^2}{2} + \vec{J}\cdot\vec{r}} \ .}$ Then

$$I_1(\vec{J}\,) = e^{\vec{J}^{\,2}/2} \int d^3x \, e^{-\frac{1}{2}(\vec{r} - \vec{J}\,)^2} = e^{\vec{J}^{\,2}/2} \int d^3x' \, e^{-\frac{1}{2}\vec{r}\,'^2}$$

$$= 4\pi e^{\vec{J}^{\,2}/2} \int_0^\infty r^2 \, dr \, e^{-r^2/2} = \boxed{(2\pi)^{3/2} e^{\vec{J}^{\,2}/2} \ .}$$

The Most Useful Integral: $\int \mathrm{d}\Omega\, \hat{n}_{i_1} \ldots \hat{n}_{i_{2\ell}}$

where $\mathrm{d}\Omega \equiv \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ = area element on sphere of radius 1.

Define $I_{i_1 \ldots i_{2\ell}} \equiv \int \mathrm{d}\Omega\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{2\ell}}$. Can find from $I_1(\vec{\boldsymbol{J}}) \equiv \int \mathrm{d}^3 x\, e^{-\frac{|\vec{\boldsymbol{r}}|^2}{2} + \vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{r}}}$, since

$$\frac{\partial^{2\ell}}{\partial J_{i_1} \ldots \partial J_{i_{2\ell}}} I_1(\vec{\boldsymbol{J}}) = \int \mathrm{d}^3 x\, x_{i_1} \ldots x_{i_{2\ell}}\, e^{-\frac{|\vec{\boldsymbol{r}}|^2}{2} + \vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{r}}} \;,$$

so

$$\frac{\partial^{2\ell}}{\partial J_{i_1} \ldots J_{i_{2\ell}}} I_1(\vec{\boldsymbol{J}})\bigg|_{\vec{\boldsymbol{J}}=0} = \int \mathrm{d}^3 x\, x_{i_1} \ldots x_{i_{2\ell}}\, e^{-\frac{|\vec{\boldsymbol{r}}|^2}{2}}$$

$$= \int_0^\infty r^2\,\mathrm{d}r \int \mathrm{d}\Omega\, x_{i_1} \ldots x_{i_{2\ell}}\, e^{-\frac{|\vec{\boldsymbol{r}}|^2}{2}} = \int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\,\mathrm{d}r \int \mathrm{d}\Omega\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{2\ell}}$$

$$= I_{i_1 \ldots i_{2\ell}} \int_0^\infty r^{2\ell+2}\, e^{-r^2/2}\,\mathrm{d}r \;,$$

So we need to evaluate

$$\frac{\partial^{2\ell}}{\partial J_{i_1} \ldots J_{i_{2\ell}}}\, e^{\vec{J}^2/2} = \frac{\partial^{2\ell-1}}{\partial J_{i_1} \ldots \partial J_{i_{2\ell-1}}} \left[J_{i_{2\ell}} e^{\vec{J}^2/2} \right]$$

$$= \frac{\partial^{2\ell-2}}{\partial J_{i_1} \ldots \partial J_{i_{2\ell-2}}} \left[\left(\delta_{i_{2\ell-1},i_{2\ell}} + J_{i_{2\ell-1}} J_{i_{2\ell}} \right) e^{\vec{J}^2/2} \right]$$

$$= \frac{\partial^{2\ell-3}}{\partial J_{i_1} \ldots \partial J_{i_{2\ell-3}}} \left[\left(J_{i_{2\ell-2}} \delta_{i_{2\ell-1},i_{2\ell}} + J_{i_{2\ell-2}} J_{i_{2\ell-1}} J_{i_{2\ell}} \right.\right.$$

$$\left.\left. + \; \delta_{i_{2\ell-2},i_{2\ell-1}} J_{i_{2\ell}} + \delta_{i_{2\ell-2},i_{2\ell}} J_{i_{2\ell-1}} \right) e^{\vec{J}^2/2} \right]$$

Want $\left. \dfrac{\partial^{2\ell}}{\partial J_{i_1} \ldots J_{i_{2\ell}}}\, e^{\vec{J}^2/2} \right|_{\vec{J}=0}$. To have a nonzero term after setting $\vec{J} = 0$, need for half of the derivatives to act on $e^{\vec{J}^2/2}$, generating factors of J_i, and for the other half to differentiate each of these J_i's, producing Kronecker delta functions.

$$\boxed{\left. \frac{\partial^{2\ell}}{\partial J_{i_1} \ldots J_{i_{2\ell}}}\, e^{\vec{J}^2/2} \right|_{\vec{J}=0} = \sum_{\text{all pairings}} \delta_{i_1,i_2}\, \delta_{i_3,i_4} \ldots \delta_{i_{2\ell-1},i_{2\ell}} \; .}$$

Finally,

$$\boxed{I_{i_1 \ldots i_{2\ell}} = 4\pi \frac{2^\ell \ell!}{(2\ell+1)!} \sum_{\text{all pairings}} \delta_{i_1,i_2}\, \delta_{i_3,i_4} \ldots \delta_{i_{2\ell-1},i_{2\ell}} \ .}$$

A pairing does not depend on the order in which the δ-function factors are written, and it does not depend on the ordering of the two indices on each δ. For example, for $\ell = 2$,

$$\sum_{\text{all pairings}} \delta_{i_1,i_2}\delta_{i_3,i_4} = \delta_{i_1,i_2}\delta_{i_3,i_4} + \delta_{i_1,i_3}\delta_{i_2,i_4} + \delta_{i_1,i_4}\delta_{i_1,i_3} \ .$$

Back to the original problem:

$$\int \mathrm{d}\Omega \left[C^{(\ell)}_{i_1 \ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\mathrm{TS}} \right] \left[C'^{(\ell')}_{j_1 \ldots j_{\ell'}} \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\mathrm{TS}} \right]$$

$$\propto C^{(\ell)}_{i_1 \ldots i_\ell} C'^{(\ell')}_{j_1 \ldots j_{\ell'}} \sum_{\text{all pairings}} \delta_{k_1, k_2} \ldots \delta_{k_{\ell+\ell'-1}, k_{\ell+\ell'}} \ ,$$

where the sum is over all pairings of the full set of indices $\{i_1, \ldots, i_\ell, j_1, \ldots, j_{\ell'}\}$. If two i_m indices are paired, or if two j_m indices are paired, then the term vanishes, since $C^{(\ell)}$ and $C'^{\ell')}$ are traceless. The only nonzero contribution arises when every pair involves one i and one j, so the integral vanishes unless $\ell = \ell'$.

So

$$\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell} \{\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\rm TS}\right] \left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}, \{\, \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \,\}_{\rm TS}\right]$$

$$= 4\pi \frac{2^\ell \ell!}{(2\ell+1)!} C^{(\ell)}_{i_1\ldots i_\ell} C'^{(\ell)}_{j_1\ldots j_\ell} \sum_{\text{all pairings}} \delta_{i_1,j_1} \ldots \delta_{i_\ell,j_\ell} \ .$$

All pairing produce an identical result, since the C's are symmetric. How many pairings are there? i_1 can be paired with any of ℓ j's, i_2 can be paired with any of $\ell-1$ j's, etc., so there are $\ell!$ pairings. Finally,

$$\boxed{\begin{aligned} &\int d\Omega \left[C^{(\ell)}_{i_1\ldots i_\ell} \{\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\rm TS}\right] \left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}, \{\, \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \,\}_{\rm TS}\right] \\ &\qquad = 4\pi \frac{2^\ell \ell!^2}{(2\ell+1)!} C^{(\ell)}_{i_1\ldots i_\ell} C'^{(\ell)}_{i_1\ldots i_\ell} \quad \text{if } \ell' = \ell. \end{aligned}}$$