8.07 Lecture Slides 12
October 21, 2019

ELECTRIC POTENTIAL: LEGENDRE POLYNOMIALS AND SPHERICAL HARMONICS VIA TRACELESS SYMMETRIC TENSORS

Expansion of $F(\hat{n})$

$$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3 \ ,$$

so $F(\hat{n})$ can also be written $F(\theta,\phi)$.

Can expand any square-integrable $F(\hat{n})$ in a power series:

$$F(\hat{n}) = C^{(0)} + C^{(1)}_i\hat{n}_i + C^{(2)}_{ij}\hat{n}_i\hat{n}_j + \ldots + C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} + \ldots \ ,$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates), and each $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is **symmetric and traceless.**

$$\nabla^2_{\text{ang}} F_\ell(\hat{n}) = -\ell(\ell+1)F_\ell(\hat{n}) \ ,$$

where

$$F_\ell(\hat{n}) = C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} \ .$$

General Solution to Laplace's Equation in Spherical Coordinates

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{n}$.

Trace Decomposition Theorem

Any symmetric matrix $S_{i_1\ldots i_\ell}$ can be uniquely written in the form

$$S_{i_1\ldots i_\ell} = S^{(\text{TS})}_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\text{Sym}}\left[M_{i_1\ldots i_{\ell-2}}\delta_{i_{\ell-1},i_\ell}\right] \ ,$$

where $S^{(\text{TS})}_{i_1\ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1\ldots i_{\ell-2}}$ is a symmetric tensor, and

$$\underset{i_1\ldots i_\ell}{\text{Sym}}[xxx]$$

means to symmetrize the expression xxx in the indices $i_1\ldots i_\ell$.

$$\{\,S_{i_1\ldots i_\ell}\,\}_{\text{TS}} \equiv S^{(\text{TS})}_{i_1\ldots i_\ell} = S_{i_1\ldots i_\ell} - \underset{i_1\ldots i_\ell}{\text{Sym}}\left[M_{i_1\ldots i_{\ell-2}}\delta_{i_{\ell-1},i_\ell}\right] .$$

Example: Azimuthal Symmetry

Azimuthal symmetry: symmetry under rotation about z axis. Construct traceless symmetric tensors from $\hat{\boldsymbol{z}}$:

$$\begin{aligned}
\{\,1\,\}_{\rm TS} &= 1\ ,\\
\{\,\hat{z}_i\,\}_{\rm TS} &= \hat{z}_i\ ,\\
\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS} &= \hat{z}_i\hat{z}_j - \tfrac{1}{3}\delta_{ij}\ ,\\
\{\,\hat{z}_i\hat{z}_j\hat{z}_k\,\}_{\rm TS} &= \hat{z}_i\hat{z}_j\hat{z}_k - \tfrac{1}{5}\left(\hat{z}_i\delta_{jk}+\hat{z}_j\delta_{ik}+\hat{z}_k\delta_{ij}\right)\ ,\\
\{\,\hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m\,\}_{\rm TS} &= \hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m - \frac{1}{7}\left(\hat{z}_i\hat{z}_j\delta_{km}+\hat{z}_i\hat{z}_k\delta_{mj}+\hat{z}_i\hat{z}_m\delta_{jk}+\hat{z}_j\hat{z}_k\delta_{im}\right.\\
&\qquad \left.+\hat{z}_j\hat{z}_m\delta_{ik}+\hat{z}_k\hat{z}_m\delta_{ij}\right)+\frac{1}{35}\left(\delta_{ij}\delta_{km}+\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right)\ .
\end{aligned}$$

Construction of $F(\theta)$

Any square-integrable function $F(\theta)$ can be expanded

$$\begin{aligned}
F(\theta) &= c_0 + c_1\{\,\hat{z}_i\,\}_{\rm TS}\hat{n}_i + c_2\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j + \ldots\\
&\quad + c_\ell\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}_{\rm TS}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell} + \ldots\ ,
\end{aligned}$$

where the c_ℓ's are constants.

Since $\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}\cdot(\sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1+\sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2+\cos\theta\,\hat{\boldsymbol{e}}_3) = \cos\theta$,

$$\begin{aligned}
\{\,1\,\}_{\rm TS} &= 1\\
\{\,\hat{z}_i\,\}_{\rm TS}\,\hat{n}_i &= \cos\theta\\
\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j &= \cos^2\theta - \frac{1}{3}\\
\{\,\hat{z}_i\hat{z}_j\hat{z}_k\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j\hat{n}_k &= \cos^3\theta - \frac{3}{5}\cos\theta
\end{aligned}$$

Connection to Legendre Polynomials

Up to normalization, this is the standard expansion in Legendre polynomials.

The Legendre polynomials are normalized so that $P_\ell(\cos\theta=1)=1$, and you showed on Problem Set 4 that

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}_{\rm TS}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\ .$$

The Multipole Expansion

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\Phi(\vec{r}\,) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1i_2\ldots i_\ell}r^\ell + \frac{C'^{(\ell)}_{i_1i_2\ldots i_\ell}}{r^{\ell+1}}\right)\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\ ,$$

where $C^{(\ell)}_{i_1i_2\ldots i_\ell}$ and $C'^{(\ell)}_{i_1i_2\ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{\boldsymbol{n}}$. For localized charge distributions, $C^{(\ell)}_{i_1i_2\ldots i_\ell}=0$. So

$$\Phi(\vec{r}\,) = \sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\,C'^{(\ell)}_{i_1\ldots i_\ell}\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\ .$$

Since each term falls off faster, the first few terms are usually sufficient at large r.

How to Find Multipole Moments Method 1 (as in Griffiths)

Combine

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3x'\ ,}$$

with

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta')$$

to obtain

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^{\ell}\rho(\vec{r}\,')P_\ell(\cos\theta')\,\mathrm{d}^3x'\ .}$$

Drawback of Method 1

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^{\ell}\rho(\vec{r}\,')P_\ell(\cos\theta')\,\mathrm{d}^3x'\ .}$$

Since θ' is the angle between $\vec{r}$ and $\vec{r}\,'$, to use this equation directly one has to redo the integral for every direction of $\vec{r}$.

How to Find Multipole Moments Method 2: with Traceless Symmetric Tensors

Expand

$$f(\vec{r}\,') \equiv \frac{1}{|\vec{r}-\vec{r}\,'|}\ .$$

in a power series in the components of $\vec{r}\,' = x'_i\hat{e}_i = r'\hat{n}'_i$:

$$\boxed{\begin{aligned} f(\vec{r}\,') &= f(\vec{0}) + \left.\frac{\partial f}{\partial x'_i}\right|_{\vec{r}\,'=\vec{0}} x'_i + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x'_i\partial x'_j}\right|_{\vec{r}\,'=\vec{0}} x'_i x'_j + \ldots\ , \\ &= f(\vec{0}) + r'\left.\frac{\partial f}{\partial x'_i}\right|_{\vec{r}\,'=\vec{0}}\hat{n}'_i + \frac{r'^2}{2!}\left.\frac{\partial^2 f}{\partial x'_i\partial x'_j}\right|_{\vec{r}\,'=\vec{0}}\hat{n}'_i\hat{n}'_j + \ldots\ . \end{aligned}}$$

Since f is a function of $\vec{r}-\vec{r}\,'$, $\dfrac{\partial f}{\partial x'_i} = -\dfrac{\partial f}{\partial x_i}$, and

$$\left.\frac{\partial^\ell f}{\partial x'_{i_1}\ldots\partial x'_{i_\ell}}\right|_{\vec{r}\,'=\vec{0}} = (-1)^\ell\frac{\partial^\ell}{\partial x_{i_1}\ldots\partial x_{i_\ell}}\frac{1}{|\vec{r}|}\ .$$

This quantity is traceless, since

$$\left(\frac{\partial^\ell}{\partial x_j\partial x_j\partial x_3\ldots\partial x_{i_\ell}}\frac{1}{|\vec{r}|}\right) = \frac{\partial^\ell}{\partial x_3\ldots\partial x_{i_\ell}}\nabla^2\frac{1}{|\vec{r}|} = 0\ ,$$

because $\nabla^2(1/|\vec{r}|) = 0$ except at $\vec{r} = 0$.

We found that

$$\frac{\partial^\ell}{\partial x_{i_1} \dots \partial x_{i_\ell}} \frac{1}{|\vec{r}|} = \frac{(-1)^\ell (2\ell-1)!!}{r^{\ell+1}} \{ \hat{n}_{i_i} \dots \hat{n}_{i_\ell} \}_{\rm TS} ,$$

where

$$\begin{aligned}(2\ell-1)!! &\equiv (2\ell-1)(2\ell-3)(2\ell-5)\dots 1 , \text{ with } (-1)!! \equiv 1 . \\ &= \frac{(2\ell)!}{2^\ell \ell!}\end{aligned}$$

The power series then becomes

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{ \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \}_{\rm TS} \{ \hat{n}'_{i_1} \dots \hat{n}'_{i_\ell} \}_{\rm TS} , \ (\text{for } r' < r)}$$

where either one (but not both) of the TS's can be dropped, since the difference is proportional to a Kronecker delta function, which leads to taking a trace of the other TS expression, which vanishes.

The analogous equation for the standard spherical harmonics is

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) , \qquad (\text{for } r' < r).$$

Inserting the above boxed equation into

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, \mathrm{d}^3x' ,$$

we find the final result

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \dots i_\ell} \hat{n}_{i_1} \dots \hat{n}_{i_\ell} ,}$$

where the multipole moments $C^{(\ell)}_{i_1 \dots i_\ell}$ are given by

$$\boxed{C^{(\ell)}_{i_1 \dots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}\,') \{ \vec{r}\,'_{i_1} \dots \vec{r}\,'_{i_\ell} \}_{\rm TS} \, \mathrm{d}^3x' .}$$

The analogous formulas for the standard spherical harmonic treatment are

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} ,}$$

where the multipole moments $q_{\ell m}$ are given by

$$\boxed{q_{\ell m} = \int \rho(\vec{r}\,') \, r'^\ell \, Y^*_{\ell m}(\theta', \phi') \, \mathrm{d}^3x' .}$$

Once we calculate the $C^{(\ell)}_{i_1 \dots i_\ell}$ or the $q_{\ell m}$ for a given charge distribution $\rho(\vec{r}\,')$, from

$$\boxed{\begin{aligned} C^{(\ell)}_{i_1 \dots i_\ell} &= \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}\,') \{ \vec{r}\,'_{i_1} \dots \vec{r}\,'_{i_\ell} \}_{\rm TS} \, \mathrm{d}^3x' , \\ q_{\ell m} &= \int \rho(\vec{r}\,') \, r'^\ell \, Y^*_{\ell m}(\theta', \phi') \, \mathrm{d}^3x' , \end{aligned}}$$

then we can calculate $V(\vec{r})$ for any $\vec{r}$:

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \dots i_\ell} \hat{n}_{i_1} \dots \hat{n}_{i_\ell} , \\ V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} , \end{aligned}}$$

avoiding the drawback of "Method 1".

Detailed Form of the Trace Decomposition Theorem

For an symmetric tensor $S_{i_1\ldots i_\ell}$, the traceless symmetric part can be written as

$$\{ S_{i_1\ldots i_\ell} \}_{\rm TS} = S_{i_1\ldots i_\ell} + \mathop{\rm Sym}_{i_1\ldots i_\ell}\Big[a_{1,\ell}\delta_{i_1i_2}\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1i_2}\delta_{i_3i_4}\delta^{j_1j_2}\delta^{j_3j_4}S_{j_1j_2j_3j_4i_5\ldots i_\ell} + \ldots\Big] ,$$

where

$$\mathop{\rm Sym}_{i_1\ldots i_\ell}\big[T_{i_1\ldots i_\ell}\big] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} T_{i_1\ldots i_\ell} ,$$

and

$$a_{n,\ell} = (-1)^n \frac{\ell!^2(2\ell-2n)!}{n!(\ell-2n)!(\ell-n)!(2\ell)!} .$$

Today we will derive $a_{1,\ell}$.

$$\{ S_{i_1\ldots i_\ell} \}_{\rm TS} = S_{i_1\ldots i_\ell} + \mathop{\rm Sym}_{i_1\ldots i_\ell}\Big[a_{1,\ell}\delta_{i_1i_2}\delta^{j_1j_2}S_{j_1j_2i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1i_2}\delta_{i_3i_4}\delta^{j_1j_2}\delta^{j_3j_4}S_{j_1j_2j_3j_4i_5\ldots i_\ell} + \ldots\Big] ,$$

The coefficients $a_{n,\ell}$ are determined by the requirement that the trace of $\{ S_{i_1\ldots i_\ell} \}_{\rm TS}$ equals zero. We can trace on any two indicies, since they are symmetric. Choose i_1 and i_2. The trace of the RHS, through the $a_{1,\ell}$ term, is

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1i_2}\frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \big[\delta_{i_1i_2}S_{jji_3\ldots i_\ell}\big] .$$

Evaluation of this trace is tricky, since the right-hand piece is summed over all $\ell!$ orderings of the indices. So the indices are not fixed at the locations shown. For example, for $\ell=3$,

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3} + a_{1,3}\delta_{i_1i_2}\frac{1}{6}\big[\delta_{i_1i_2}S_{jji_3} + \delta_{i_1i_3}S_{jji_2} + \delta_{i_2i_3}S_{jji_1} + \delta_{i_2i_1}S_{jji_3} + \delta_{i_3i_1}S_{jji_2} + \delta_{i_3i_2}S_{jji_1}\big] .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1i_2}\frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \big[\delta_{i_1i_2}S_{jji_3\ldots i_\ell}\big] .$$

To carry out the sum, we have to consider 4 possible cases of where the indices i_1 and i_2 can be.

Case I: i_1 and i_2 can appear on the Kronecker δ-function in the square brackets. There are two possible orderings of i_1 and i_2. There are $(\ell-2)!$ orderings of the other indices, $i_3,\ldots,i_\ell$. The value includes a factor of $\delta_{i_1i_2}\delta_{i_1i_2}=3$. Summary:

$$\text{Case I:} \qquad \text{Multiplicity} = 2(\ell-2)!\ , \qquad \text{Value} = 3S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!} .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1i_2}\frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \big[\delta_{i_1i_2}S_{jji_3\ldots i_\ell}\big] .$$

Case II: i_1 can appear on the Kronecker δ-function in the square brackets, while i_2 appears as one of the indices of S. There are two possible positions for i_1, and $\ell-2$ positions for i_2. Again, there are $(\ell-2)!$ orderings for the other indices. The value can be seen by looking at a sample term:

$$a_{1,\ell}\delta_{i_1i_2}\frac{1}{\ell!}\big[\delta_{i_1i_3}S_{jji_2i_4\ldots i_\ell}\big] = \frac{a_{1,\ell}}{\ell!}S_{jji_3\ldots i_\ell} .$$

Summarizing,

$$\text{Case II:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!} .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2}S_{jji_3\ldots i_\ell}\right] .$$

Case III: i_2 can appear on the Kronecker δ-function in the square brackets, while i_1 appears as one of the indices of S. This case is equivalent to Case II:

Case III: Multiplicity $= 2(\ell-2)(\ell-2)!$, Value $= S_{jji_3\ldots i_\ell}\dfrac{a_{1,\ell}}{\ell!}$.

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2}S_{jji_3\ldots i_\ell}\right] .$$

Case IV: Both i_1 and i_2 can appear on S. i_1 has $\ell-2$ possible places, and i_2 has $\ell-3$ places. The other indices again have $(\ell-2)!$ orderings. The value can again be seen by looking at a representative term:

$$a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}[\delta_{i_3 i_4}S_{jji_1 i_2 i_5\ldots i_\ell}] = \frac{a_{1,\ell}}{\ell!}\,\mathrm{Sym}\left[\delta_{i_3 i_4}S_{jjkki_5\ldots i_\ell}\right] .$$

Summary:

Case IV: Multiplicity $= (\ell-2)(\ell-3)(\ell-2)!$

Value $= \mathrm{Sym}\left[\delta_{i_3 i_4}S_{jjkki_5\ldots i_\ell}\right]\dfrac{a_{1,\ell}}{\ell!}$.

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2}S_{jji_3\ldots i_\ell}\right] .$$

Check on multiplicities. Remember, multiplicity = number of terms in above sum for each case, so they should sum to $\ell!$. Check:

$$(\ell-2)!\,[2+2(\ell-2)+2(\ell-2)+(\ell-2)(\ell-3)] = \ell! \,.$$

TERRIFIC!

Then

$$a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index}\\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2}S_{jji_3\ldots i_\ell}\right] =$$

$$\sum_{\text{Case}}\text{Multiplicity}(\text{Case})\times\text{Value}(\text{Case})$$

Finally,

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}\left[1+\frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3+4(\ell-2)\right)\right] + \mathrm{Sym}\left[\delta_{i_3 i_4}S_{jjkki_5\ldots i_\ell}\right]a_{1,\ell}\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)} .$$

Recall,

$$\{\,S_{i_1\ldots i_\ell}\,\}_{\mathrm{TS}} = S_{i_1\ldots i_\ell} + \underset{i_1\ldots i_\ell}{\mathrm{Sym}}\Big[a_{1,\ell}\delta_{i_1 i_2}\delta^{j_1 j_2}S_{j_1 j_2 i_3\ldots i_\ell} + a_{2,\ell}\delta_{i_1 i_2}\delta_{i_3 i_4}\delta^{j_1 j_2}\delta^{j_3 j_4}S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big] .$$

We must require that the trace of the full RHS vanish. However, note that all terms not yet included have 2 or more Kronecker delta functions, so when traced they will have 1 or more. So, they cannot contribute to the terms with no Kronecker delta functions. So the term in square brackets in the top equation must vanish.

$$\operatorname{Tr}(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3 + 4(\ell-2)\right)\right]$$
$$+\operatorname{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] a_{1,\ell}\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)} .$$

$$\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3 + 4(\ell-2)\right)\right] = 0 \quad\Longrightarrow\quad \boxed{a_{1,\ell} = -\frac{\ell(\ell-1)}{2(2\ell-1)} .}$$

Application to $\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}}$

$\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}}$ is traceless and symmetric and constructed from $\hat{n}$, so it must be proportional to $\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}}\,\}_{\mathrm{TS}}$. Then

$$\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}} = \hat{n}_{i_\ell}\left\{\hat{n}_{i_1}\ldots\hat{n}_{i_\ell} - \frac{\ell(\ell-1)}{2(2\ell-1)}\operatorname*{Sym}_{i_1\ldots i_\ell}\left[\delta_{i_1 i_2}\hat{n}_{i_3}\ldots\hat{n}_{i_\ell}\right] + \ldots\right\} .$$

Clearly

$$\hat{n}_{i_\ell}\left(\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\right) = \hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}} .$$

Also

$$\hat{n}_{i_\ell}\operatorname*{Sym}_{i_1\ldots i_\ell}\left[\delta_{i_1 i_2}\hat{n}_{i_3}\ldots\hat{n}_{i_\ell}\right] = \frac{2}{\ell}\hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}} .$$

Then

$$\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}} = \hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}}\left[1 - \frac{\ell(\ell-1)}{2(2\ell-1)}\frac{2}{\ell}\right] + \ldots$$
$$= \frac{\ell}{2\ell-1}\hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}} + \ldots .$$

$$\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}} = \hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}}\left[1 - \frac{\ell(\ell-1)}{2(2\ell-1)}\frac{2}{\ell}\right] + \ldots$$
$$= \frac{\ell}{2\ell-1}\hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}} + \ldots .$$

The omitted terms cannot contribute to the term with no Kronecker delta functions, so

$$\boxed{\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{\ell}{2\ell-1}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_{\ell-1}}\,\}_{\mathrm{TS}} .}$$

Integration over Legendre Polynomials

Result:

$$\boxed{\begin{aligned}&\int d\Omega\,\left[C^{(\ell)}_{i_1\ldots i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\mathrm{TS}}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}\{\,\hat{n}_{j_1}\ldots\hat{n}_{j_{\ell'}}\,\}_{\mathrm{TS}}\right]\\ &\quad = 4\pi\frac{2^\ell \ell!^2}{(2\ell+1)!}\,C^{(\ell)}_{i_1\ldots i_\ell}C'^{(\ell)}_{i_1\ldots i_\ell}\quad \text{if } \ell' = \ell.\end{aligned}}$$

And it equals zero if $\ell' \neq \ell$.

Note added after class: we started this derivation on the blackboard, and will finish it on Wednesday.