MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 26, 2019
Prof. Alan Guth

PROBLEM SET 6

DUE DATE: Friday, November 1, 2019, at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

READING ASSIGNMENT: Griffiths Chapter 4: *Electric Fields in Matter.*

CREDIT: This problem set has 105 points of credit, plus the option of earning 10 points extra credit.

— PART A —

PROBLEM 1: A SIMPLE ATOMIC MODEL OF POLARIZABILITY *(10 points)*

Griffths, Problem 4.2 (p. 170). Assume that the electron cloud is rigid, and that in response to the applied electric field it moves relative to the point proton, until electrostatic equilibrium is established.

Text from Griffiths:

According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a} \ ,$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [Hint: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$. [Footnote from Griffiths: For a more sophisticated approach, see W. A. Bowers, *Am. J. Phys.* **54**, 347 (1986).]

PROBLEM 2: A DELTA FUNCTION IDENTITY AND THE FIELD OF AN ELECTRIC DIPOLE *(20 points plus 10 points extra credit):*

Essentially every E&M equation involving δ-functions can be derived from the identity

$$\boxed{\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\vec{\boldsymbol{r}}) \ .} \tag{2.1}$$

To make sure the definitions are clear, I remind you that $\vec{\boldsymbol{r}} \equiv x_i \hat{\boldsymbol{e}}_{\boldsymbol{i}} \equiv x\hat{\boldsymbol{e}}_{\boldsymbol{x}} + y\hat{\boldsymbol{e}}_{\boldsymbol{y}} + z\hat{\boldsymbol{e}}_{\boldsymbol{z}}$, $r \equiv |\vec{\boldsymbol{r}}|$, $\hat{\boldsymbol{r}} \equiv \vec{\boldsymbol{r}}/r$, $\hat{\boldsymbol{r}}_i \equiv x_i/r$, and $\partial_i \equiv \partial/\partial x_i$. [Despite the usefulness of this identity, it is not very well-known. A student in my 2014 class found it in a 1983 paper in the *American Journal of Physics* (vol. 51, p. 826) by Charles Frahm, and also in the textbook by Anupam Garg, **Classical Electrodynamics in a Nutshell** (Princeton University Press, 2012, p. 108). If you find any other references to it, please let me know.]

(a) [5 pts] Show that the famous identity,

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}}) \ , \tag{2.2}$$

follows immediately from Eq. (2.1).

(b) [5 pts] Using the identity (2.1), show that the potential for an electric dipole,

$$V_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}}}{r^2} \ , \tag{2.3}$$

can be differentiated straightforwardly to give the electric field of a dipole,

$$\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} - \frac{1}{3\epsilon_0}\vec{\boldsymbol{p}}\,\delta^3(\vec{\boldsymbol{r}}) \ . \tag{2.4}$$

(c) [10 pts] Demonstrate that the identity holds by first showing that it is valid for $r \neq 0$, and then verifying the δ-function by integrating over a small sphere of radius ϵ about the origin. Use the identity of Problem 8(a) of Problem Set 1 to convert

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) \, \mathrm{d}^3x \tag{2.5}$$

into a surface integral on a sphere of radius ϵ. (Here the integrand has an extra vector index j, but the identity holds for any function $T(\vec{\boldsymbol{r}})$; if the function happens to be the j'th component of a vector, the identity still applies.) Evaluate the surface integral to show that the δ-function term in Eq. (2.1) is correct.

(d) [10 points extra credit] Prove the identity (2.1), using the mathematical definitions of distribution theory. The calculation will not be much different from that of part (d), but the justification needs to be phrased more carefully. In particular, mathematicians would criticize the calculation in part (d) by saying that the integral in Eq. (2.5) was ill-defined from the beginning, since $(\hat{\boldsymbol{r}}_j/r^2)$ is not differentiable at $r = 0$.

Mathematicians give a rigorous meaning to the identity (2.1) by defining both sides as distributions, which means that they are defined by the result of multiplying them by an arbitrary test function $\varphi(\vec{\boldsymbol{r}})$ (which is assumed to be smooth and to fall off rapidly as $r \to \infty$) and then integrating both sides over all space. (The distribution is thus a mapping from the test function $\varphi(\vec{\boldsymbol{r}})$ to a number, the value of the integral.) If the integrals obtained on each side of the equation are equal, then the identity (2.1) is valid — i.e., the two sides of Eq. (2.1) are said to be equal in the sense of distributions.* For the right-hand side (RHS),

$$\begin{aligned} RHS &= \int \varphi(\vec{\boldsymbol{r}}) \left[\frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} + \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\vec{\boldsymbol{r}}) \right] \mathrm{d}^3x \\ &= \int \varphi(\vec{\boldsymbol{r}}) \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3}\, \mathrm{d}^3x + \frac{4\pi}{3} \delta_{ij}\, \varphi(\vec{\boldsymbol{0}}) \ . \end{aligned} \tag{2.6}$$

The integral above is singular, but may be defined precisely by

$$\int \varphi(\vec{\boldsymbol{r}}) \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3}\, \mathrm{d}^3x \equiv \lim_{\epsilon \to 0} \int_{r > \epsilon} \varphi(\vec{\boldsymbol{r}}) \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3}\, \mathrm{d}^3x \ . \tag{2.7}$$

In this form the integral is well-defined, since the measure $\mathrm{d}^3x = r^2 \mathrm{d}r \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ cancels two factors of r in the denominator, and the remaining factor of r is harmless because $\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j$ vanishes when integrated over angles.

To evaluate the left-hand side (LHS), integrate by parts:

$$LHS = \int \varphi(\vec{\boldsymbol{r}}) \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3x \tag{2.8a}$$

$$= - \int \partial_i \varphi(\vec{\boldsymbol{r}}) \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3x \ . \tag{2.8b}$$

Note that Eq. (2.8a) involves the same ill-defined derivative as Eq. (2.5), but the mathematicians give it unambiguous meaning by defining the derivative of a distribution by integration by parts: all the derivatives are applied to the test function $\varphi(\vec{\boldsymbol{r}})$. Thus the LHS is defined by Eq. (2.8b).

To manipulate Eq. (2.8b), we first write it as

$$LHS = - \lim_{\epsilon \to 0} \int_{r > \epsilon} \partial_i \varphi(\vec{\boldsymbol{r}}) \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3x \ . \tag{2.9}$$

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* Note, however, that "equal in the sense of distributions" might not correspond completely to your intuitive sense of "equal". For example, a function of x that is equal to 1 at $x = 0$ and zero everywhere else is equal in the sense of distributions to the function that is zero everwhere.

The restriction of the integral to $r > \epsilon$ makes no difference in the limit $\epsilon \to 0$, since the integrand behaves smoothly (recall that $\mathrm{d}^3x = r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$). By writing it this way, however, we can separate it into pieces which individually will not behave smoothly as $r \to 0$. Specifically, we can integrate by parts:

$$LHS = -\lim_{\epsilon\to 0}\int_{r>\epsilon}\left\{\partial_i\left[\varphi(\vec{\boldsymbol r})\left(\frac{\hat{\boldsymbol r}_j}{r^2}\right)\right] - \varphi(\vec{\boldsymbol r})\partial_i\left(\frac{\hat{\boldsymbol r}_j}{r^2}\right)\right\}\,\mathrm{d}^3x\ . \tag{2.10}$$

Here's where you take over. Show that the second term in the integrand reproduces Eq. (2.7), and that the first term can be converted to a surface integral that evaluates to $(4\pi/3)\delta_{ij}\,\varphi(\vec{\boldsymbol r})$.

PROBLEM 3: FORCES, TORQUES, AND ANGULAR MOMENTUM CONSERVATION WITH DIPOLE-DIPOLE INTERACTIONS *(15 points)*

Consider a system of two ideal electric dipoles. The first is located at the origin, and points along the z-axis, so

$$\vec{\boldsymbol p}_1 = p_1\,\hat{\boldsymbol e}_{\boldsymbol z}\ . \tag{3.1}$$

The second dipole lies along the positive x-axis at

$$\vec{\boldsymbol r} = r\,\hat{\boldsymbol e}_{\boldsymbol x}\ . \tag{3.2}$$

and is oriented along the x-axis,

$$\vec{\boldsymbol p}_2 = p_2\,\hat{\boldsymbol e}_{\boldsymbol x}\ . \tag{3.3}$$

(a) [10 pts] Calculate the force $\vec{\boldsymbol F}_2$ on dipole 2 caused by the field of dipole 1, and the force $\vec{\boldsymbol F}_1$ on dipole 1 due to the field of dipole 2. Are these forces equal in magnitude and opposite in direction, as described by Newton's third law of motion? Do the forces point along the line joining the two dipoles?

(b) [5 pts] In part (a) you should have found that the forces are equal and opposite, but that they do not point along the line joining the particles. But this raises a question about the conservation of angular momentum, since the standard proof of the conservation of angular momentum from Newton's laws relies on the "strong form" of Newton's third law, which requires the force to point along the line of centers. Under that assumption, one argues that for an isolated two-particle system, $\vec{\boldsymbol F}_2 = -\vec{\boldsymbol F}_1$ and the total torque is

$$\begin{aligned}\vec{\boldsymbol\tau} &= \vec{\boldsymbol r}_1\times\vec{\boldsymbol F}_1 + \vec{\boldsymbol r}_2\times\vec{\boldsymbol F}_2\\ &= (\vec{\boldsymbol r}_1 - \vec{\boldsymbol r}_2)\times\vec{\boldsymbol F}_1 = \vec{\boldsymbol 0}\ ,\end{aligned} \tag{3.4}$$

since the cross product between parallel vectors is zero. Show, however, that the conservation of angular momentum is nonetheless valid in this case, if one includes the torques that each dipole experiences due to the field of the other, $\vec{\boldsymbol\tau} = \vec{\boldsymbol p}\times\vec{\boldsymbol E}$. Calculate the total torque on the system, for example about the origin, and show that it vanishes.

PROBLEM 4: A POINT CHARGE AT THE CENTER OF A SPHERICAL DIELECTRIC *(10 points)*

Griffiths, Problem 4.35 (p. 206).

A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

— PART B (To be handed in separately from Part A) —

PROBLEM 5: SPHERE OF UNIFORM DIELECTRIC MATERIAL IN AN OTHERWISE UNIFORM ELECTRIC FIELD *(10 points)*

In lecture I started to discuss Example 4.7 (p. 193) from Griffiths' book, but I left the end of the derivation for you to do here.

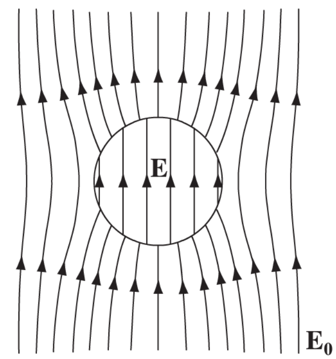


FIGURE 4.27

I'll remind you what we did in lecture. The problem consists of considering a uniform dielectric sphere of radius R and dielectric constant ϵ_r, placed at the origin of the coordinate system, in an initially uniform electric field $\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 = E_0\hat{\boldsymbol{z}}$. The form of the electric field very far from the sphere will be unperturbed, so

$$\vec{\boldsymbol{E}} \sim \vec{\boldsymbol{E}}_0 \quad \text{as } r \to \infty \ , \tag{5.1}$$

which in turn implies that

$$V \sim -E_0 z = -E_0 r\cos\theta \quad \text{as } r \to \infty \ , \tag{5.2}$$

where I chose $V = 0$ at $z = 0$. Outside the dielectic we clearly have $\nabla^2 V = 0$, since it is empty space. Inside we also have $\nabla^2 V = 0$, since $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{D}} = \rho_{\text{free}} = 0$, and $\vec{\boldsymbol{D}} = \epsilon\vec{\boldsymbol{E}}$, so $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = 0$. We also have the boundary conditions

$$\begin{array}{lll} \text{(i)} & V_{\text{in}} = V_{\text{out}} \ , & \text{at } r = R \ , \\[2ex] \text{(ii)} & \epsilon\dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r} \ , & \text{at } r = R \ , \\[2ex] \text{(iii)} & V_{\text{out}} \to -E_0 r\cos\theta \ , & \text{for } r \gg R \ . \end{array} \tag{5.3}$$

Since the problem has azimuthal symmetry, we can expand the solution in Legendre polynomials,

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \ . \tag{5.4}$$

We use separate expansions on the inside and on the outside, because $\nabla^2 V \neq 0$ at $r = R$, where there is a bound surface charge. For the inside, all the B_ℓ's must vanish, because otherwise V would be singular at $r = 0$. For the outside, all the A_ℓ's must vanish to avoid having V blow up at infinity, except that A_1 must equal $-E_0$ to comply with boundary condition (iii). Thus,

$$V_{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \ , \tag{5.5}$$

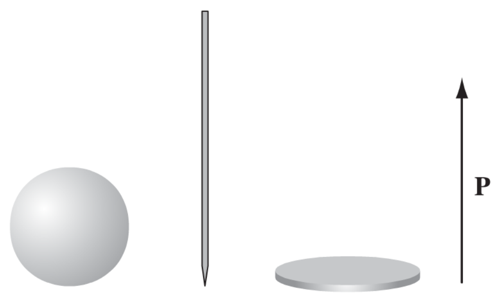
and

$$V_{\rm out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \ . \tag{5.6}$$

Now it is your job to finish the calculation by imposing the boundary conditions (5.3), finding $V(r,\theta)$ everywhere. You may look at the solution in the book if you want, but it is recommended that you first try to solve the problem on your own.

PROBLEM 6: CAVITIES IN DIELECTRIC MEDIA *(15 points)*

Griffiths, Problem 4.16 (p. 184).



(a) Sphere (b) Needle (c) Wafer

FIGURE 4.19

Suppose the field inside a large piece of dielectric is $\vec{\boldsymbol{E}}_0$, so that the electric displacement is $\vec{\boldsymbol{D}}_0 = \epsilon_0\vec{\boldsymbol{E}}_0 + \vec{\boldsymbol{P}}$.

(a) [5 pts] Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\vec{\boldsymbol{E}}_0$ and $\vec{\boldsymbol{P}}$. Also find the displacement at the center of the cavity in terms of $\vec{\boldsymbol{D}}_0$ and $\vec{\boldsymbol{P}}$. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

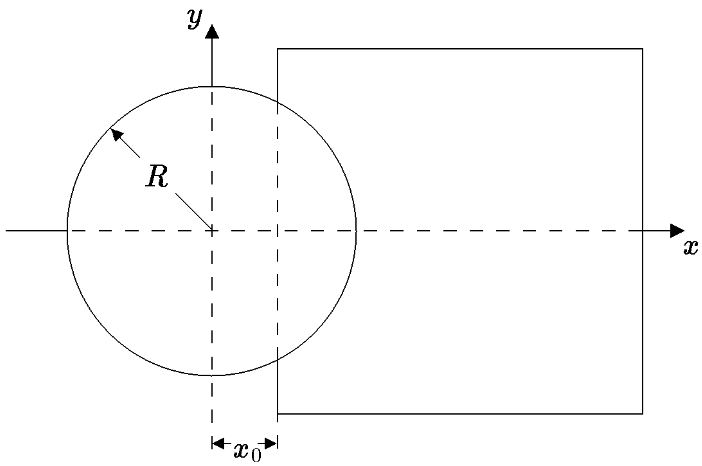
(b) [5 pts] Do the same for a long needle-shaped cavity running parallel to $\vec{\boldsymbol{P}}$ (Fig. 4.19b).

(c) [5 pts] Do the same for a thin wafer-shaped cavity perpendicular to $\vec{\boldsymbol{P}}$ (Fig. 4.19c).

Assume the cavities are small enough that $\vec{\boldsymbol{P}}$, $\vec{\boldsymbol{E}}_0$, and $\vec{\boldsymbol{D}}_0$ are essentially uniform. [Hint: Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

[Further comment by Alan Guth: When Griffiths says to assume that the cavities are small, he means to assume that the needle in (b) is thin, in that its width is small compared to its height, and that the wafer in (c) is thin, in that its height is small compared to its width. You should not worry if your solutions are inaccuate in regions whose size is of the order of the small dimensions.]

PROBLEM 7: FORCE ON A DIELECTRIC SLAB PART WAY INSIDE A CIRCULAR CAPACITOR *(15 points)*



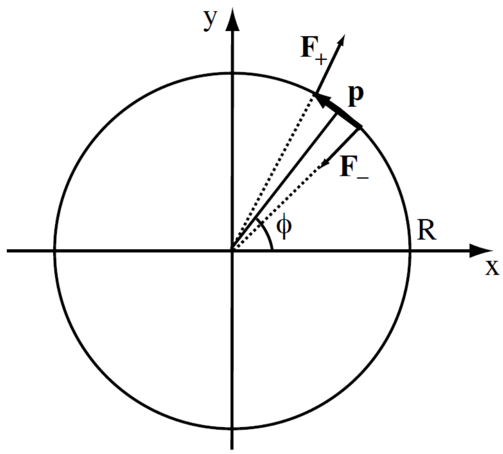
Consider a circular capacitor, consisting of two thin circular disks, of radius R, parallel to the x-y plane. Both are centered on the z-axis, one at $z = +\frac{1}{2}d$ and the other at $z = -\frac{1}{2}d$. The two disks are held at a potential difference V_0. A square sheet of dielectric, with thickness just a shade smaller than d and with side larger than $2R$, is placed part way between the two disks, as shown in the diagram. The left-hand edge of the square is at $x = x_0$. Calculate the force on the slab as a function of x_0.

PROBLEM 8: A DIPOLE ON A CIRCULAR TRACK *(10 points)*

Griffiths, Problem 4.31, p. 206. (Griffiths thanks K. Brownstein for the "charming paradox"):

> [5 pts] A point charge Q is "nailed down" on a table. Around it, at radius R, is a frictionless circular track on which a dipole $\vec{\boldsymbol{p}}$ rides, constrained always to point tangent to the circle. Use Eq. 4.5 to show that the electric force on the dipole is
>
> $$\vec{\boldsymbol{F}} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}}{R^3}\ .$$
>
> Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?

This is a subtle problem, and a good one to discuss. The solutions manual offers the following answer to the perpetual motion issue, which is certainly wrong:



> To keep the dipole going in a circle, there must be a centripetal force exerted by the track (we may as well take it to act at the center of the dipole, and it is irrelevant to the problem), and to keep it aiming in the tangential direction there must be a torque (which we could model by radial forces of equal magnitude acting at the two ends). Indeed, if the dipole has the orientation indicated in the figure, and is moving in the $\hat{\phi}$ direction, the torque exerted by Q is clockwise, whereas the rotation is counterclockwise, so these constraint forces must actually be *larger* than the forces exerted by Q, and the *net* force will be in the "backward" direction—tending to slow the dipole down. [If the motion is in the $-\hat{\phi}$ direction, then the *electrical forces* will dominate, and the net force will be in the direction of $\vec{p}$, but this again will tend to slow it down.]

While Griffiths is trying to explain why the dipole doesn't accelerate perpertually around the track in obvious violation of the conservation of energy, he is suggesting that instead the dipole slows down. But that also violates the conservation of energy! (In an email conversation in 2014, Griffiths agreed wholeheartedly that his solution was incorrect.)

[5 pts] Talk this over with your friends, and see if you can figure out how this system really behaves, and how it is consistent with the conservation of energy. *Hint:* The track exerts forces on the dipole which keep it on the track, and keep it oriented along the track. It is easiest to understand this by thinking of a "physical dipole," with some small separation between the two charges, and with each charge constrained to move along the frictionless track.