Revised 9/28/19 to add summary of blackboard work on image charges

8.07 Lecture Slides 7
September 25, 2019

ELECTRIC POTENTIAL: LAPLACE'S EQUATION, METHOD OF IMAGES

Announcements

Office hours for Yitian Sun:

This week:

Thurs 9/26/19, 5:30–6:30 pm, Room 8-320

Next week:

Tues 10/1/19, 4:00–5:00 pm, Blackboard area outside Room 6-415

Thurs 10/3/19, 5:30–6:30 pm, Room 8-320

Fri 10/4/19, 1:30–3:30 pm, Room 8-320

Following weeks:

Thurs, 5:30–6:30 pm, Fri: 1:30–3:30 pm, all in Room 8-320

Review of Conductors from Lecture 6

- Inside a conductor $\vec{\boldsymbol{E}} = 0$.
- Inside a conductor, $\rho = 0$.
- Within a conductor, $V =$ constant.
- All charges on a conductor reside on the surface.
- Just outside a conductor,

$$\boxed{\vec{\boldsymbol{E}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}\ ,}$$

where $\sigma =$ surface charge density, $\hat{\boldsymbol{n}} =$ outward normal.

- Metal enclosures shield their interiors from electric fields. (Argument was not solid: it is obvious that $\vec{\boldsymbol{E}} = 0$, inside the enclosure, satisfies the static Maxwell eqs, but not yet obvious that this is the only possibility.)

Blackboard Discussion of Capacitance

Consider a system of isolated conductors, labeled 1, 2, 3, etc., each with charge Q_1, Q_2, Q_3, . . . , respectively.

Each conductor is an equipotential, where V_i denotes the potential of the i'th conductor.

Since the $\vec{\boldsymbol{E}}$ for many charges is just the sum of the field for each charge, and V is determined from $\vec{\boldsymbol{E}}$, the V_i should be a linear function of the Q_i's. So we can write

$$\boxed{V_i = \sum_j P_{ij} Q_j \ ,}$$

where the P_{ij}'s are the *potential coefficients*.

This matrix relation can be inverted, so we can write

$$\boxed{Q_i = \sum_j C_{ij} V_j \ ,}$$

where $C = P^{-1}$, and the C_{ij}'s are called coefficients of capacitance.

The total electrostatic energy of the system is given by

$$\boxed{W = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_{ij} Q_i P_{ij} Q_j = \frac{1}{2} \sum_{ij} V_i C_{ij} V_j \ .}$$

How Do We Know P is Invertible?

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Proof by contradiction. From linear algebra: if P is not invertible, then it has a zero eigenvector. That is, there exists a multiplet Q_i, where the Q_i are not all zero, and $\sum_j P_{ij}\bar{Q}_j = 0$.

But $\sum_j P_{ij}\bar{Q}_j = V_i$, where V_i is the potential on conductor i produced by the charges $\bar{Q}_j$

That is, it must be possible to put charges on the conductors in such a way that no potentials are produced.

But: $\bar{Q}_i \neq 0$ implies that there must be places where $\vec{\boldsymbol{E}} \neq 0$, by Gauss's law. So $W = \frac{1}{2}\epsilon_0 \int |\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3x > 0$.

But: $W = \frac{1}{2}\sum_i \bar{Q}_i V_i$ implies $W = 0$.

So we have a contradiction, so P must be invertible.

Proof that P Not Invertible $\Longrightarrow$ Zero Eigenvector

Step 1: If the functional relation $V_i = P_{ij} Q_j$ (where repeated indices are now summed) is invertible, then P is invertible, and the inverse function is therefore linear.

Proof: Suppose there exist an inverse, so $Q_k = F_k(V_i)$, where $V_i = P_{ij} Q_j$.
By chain rule,

$$\frac{\partial Q_k}{\partial Q_j} = \frac{\partial F_k}{\partial V_i} \frac{\partial V_i}{\partial Q_j} = \frac{\partial F_k}{\partial V_i} P_{ij} \ , \tag{1}$$

Since $\partial Q_k / \partial Q_j = \delta_{kj}$, Eq. (1) implies that

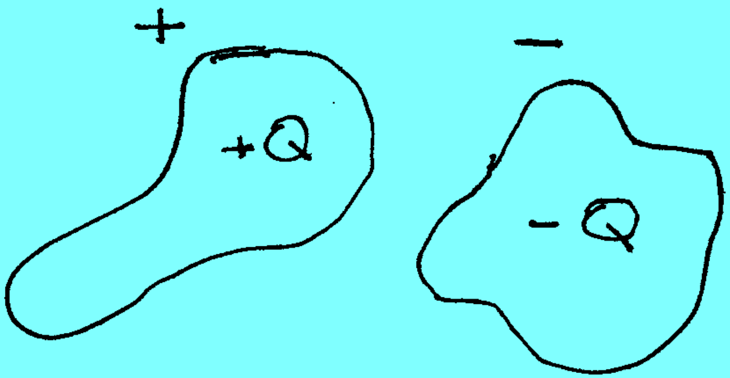
$$\frac{\partial F_k}{\partial V_i} = P^{-1}_{ki} \ .$$

Step 2: From Step 1, if P is not invertible, then the functional relation $V_i = P_{ij} Q_j$ is not invertible. Then there must be two different multiplets $Q_i^{(1)}$ and $Q_i^{(2)}$ that produce the same V_i:

$$V_i = \sum_j P_{ij} Q_j^{(1)} = \sum_j P_{ij} Q_j^{(2)} \quad \Longrightarrow \quad \sum_j P_{ij}\, \delta Q_j = 0 \ ,$$

where $\delta Q_j \equiv Q_j^{(2)} - Q_j^{(1)}$. So P not invertible implies P has a zero eigenvector.

One Last Slide About Capacitors: The Standard Capacitor



The standard capacitor consists of two conductors, labeled $+$ and $-$, with charges $+Q$ and $-Q$, respectively. The potential difference is called V, where $V = V_+ - V_-$. The **capacitance** C is defined by

$$\boxed{Q = CV.}$$

The units are coulomb per volt, which is also called a farad, abbreviated F. A farad is huge. Microfarads (10^{-6} F) or even picofarads (10^{-12} F) are much more frequently seen.

For a simple parallel plate capacitor,

$$C = \frac{\epsilon_0 A}{d} ,$$

where $\epsilon_0 = 8.85 \times 10^{-12} \, \frac{\mathrm{C}^2}{\mathrm{N\,m}^2}$.

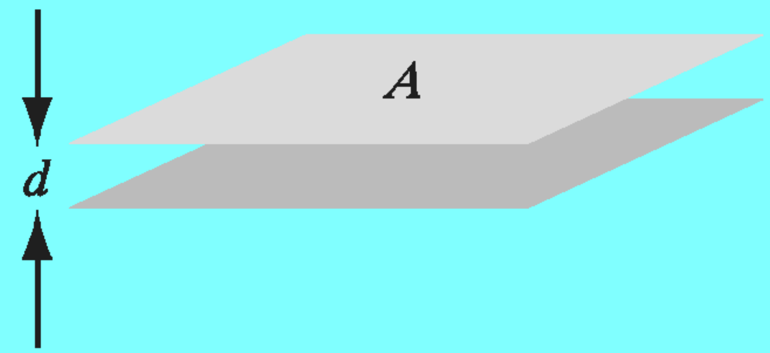


FIGURE 2.52

(Griffiths: *Electrodynamics*)

Electric Potential: Poisson and Laplace Equations

Basic equations of electrostatics:

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}\,\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ .}$$

or

$$\boxed{V(\vec{\boldsymbol{r}}) \equiv \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x' \ ,}$$

where

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V \ .}$$

Shortcoming: if we have conductors, charges move, so we may not know $\rho(\vec{\boldsymbol{r}})$.

Sample problem:

Consider a conducting plane, located in the x-y plane, and imagine that there is a set of point charges $i = 1 \ldots n$, of charge q_i at locations $\vec{\boldsymbol{r}}_i$, where we assume for now that $z_i > 0$ for all i. That is, all charges are on one side of the plane.

How can we find $\vec{\boldsymbol{E}}$?

The formulas on the previous slide cannot be used directly, because charges will move within the conductor until $\vec{\boldsymbol{E}} = 0$ inside it.

Problems with conductors are most easily approached by using the potential $V(\vec{\boldsymbol{r}})$.

We know that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \;, \quad \text{and} \quad \vec{E} = -\vec{\nabla} V \;,$$

from which it follows that

$$\vec{\nabla} \cdot (\vec{\nabla} V) \equiv \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0} \;.}$$

The boxed equation is called the **Poisson equation**.

In a region where $\rho(\vec{r}\,)$ happens to be zero,

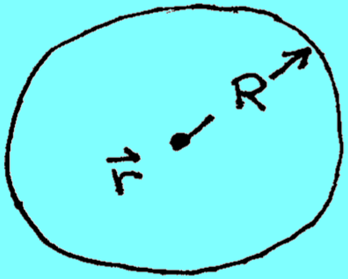
$$\boxed{\nabla^2 V = 0 \;,}$$

which is called **Laplace's equation**. A function V satisfying Laplace's equation is called a **harmonic function**.

A Very Important Theorem

The theorem has no name that I have seen, so I'll name it the Laplacian Mean Value Theorem.

Thm: If $\nabla^2 V = 0$, then the value of V at any point $\vec{\boldsymbol{r}}$ is equal to the **average** value of V on a sphere centered at $\vec{\boldsymbol{r}}$ of any radius R. That is,



$$\boxed{V(\vec{\boldsymbol{r}}) = \bar{V}(\vec{\boldsymbol{r}}, R)\ ,}$$

where

$$\boxed{\bar{V}(\vec{\boldsymbol{r}}, R) \equiv \frac{1}{4\pi R^2} \oint_{\substack{\text{sphere of} \\ \text{radius } R}} V(\vec{\boldsymbol{r}}\,')\,\mathrm{d}a'\ .}$$

Proof: It is a corollary of the "Lumpiness" Theorem for the Laplacian,

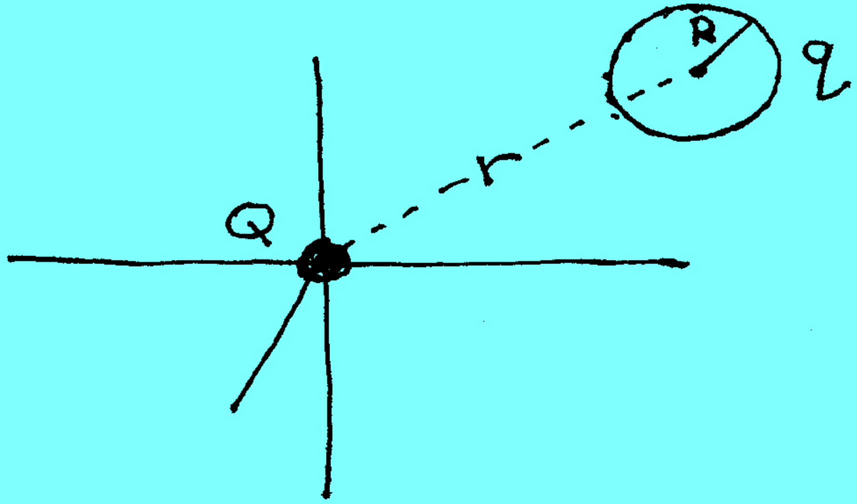
$$\boxed{V(\vec{\boldsymbol{r}}) = \bar{V}(\vec{\boldsymbol{r}}, R) - \frac{1}{4\pi}\int_{r<R} d^3x\,\nabla^2 V\left[\frac{1}{r} - \frac{1}{R}\right] \ ,}$$

which was stated in Lecture 3, and you will prove on your next problem set. If $\nabla^2 V = 0$, the right-hand side vanishes, and we have $V(\vec{\boldsymbol{r}}) = \bar{V}(\vec{\boldsymbol{r}}, R)$.

Alternative Proof of Laplacian Mean Value Theorem

Consider a fixed charge Q at the origin, which creates a potential $V_Q(\vec{\boldsymbol{r}})$. A spherical shell of charge q, uniformly distributed on the shell, is brought in from infinity to a distance r.

How much work is done?



Method 1: The force between Q and the shell is the same as it would be if the shell were replaced by a point charge q at the center, so the work is that same as it would be for the point charge:

$$\boxed{W_{\text{Method 1}} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r} \ .}$$

Method 2: We can calculate the work needed to bring in each surface element $\mathrm{d}a$, and integrate. The charge of the surface element is $\sigma\,\mathrm{d}a$, where $\sigma = q/4\pi R^2$ is the surface charge density. So

$$\boxed{W_{\text{Method 2}} = \int V_Q(\vec{\boldsymbol{r}}\,')\frac{q}{4\pi R^2}\,\mathrm{d}a = q\bar{V}_Q(\vec{\boldsymbol{r}}, R) \ .}$$

Equating the two, we find

$$\boxed{\bar{V}_Q(\vec{\boldsymbol{r}}, R) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = V_Q(\vec{\boldsymbol{r}}) \ .}$$

This proves the result for the potential $V_Q(\vec{\boldsymbol{r}})$ caused by a single point charge Q. For many charges, use superposition: the $\vec{\boldsymbol{E}}$ field for many charges is just the vector sum of $\vec{\boldsymbol{E}}$ for each of them.

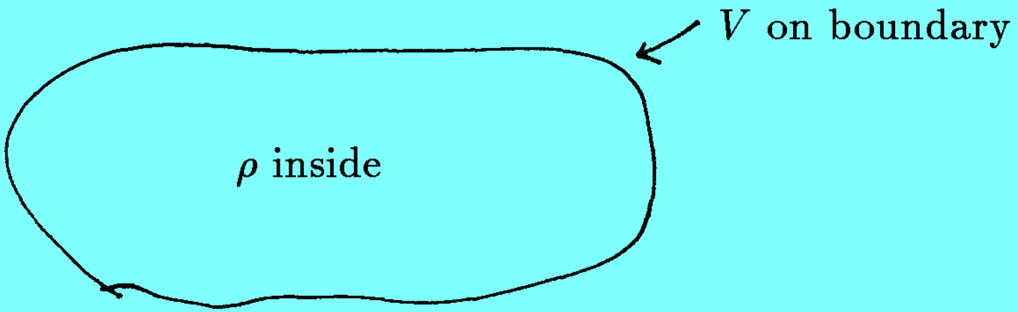
Corollary of Laplacian Mean Value Thm:

In a region where $\nabla^2 V = 0$, V cannot have any local minima or maxima.

Any minima or maxima must occur on the boundary of the region in which $\nabla^2 V = 0$.

A Uniqueness Theorem

Consider a region in which V is specified everywhere on the boundary, and ρ is specified inside the volume. The boundary could be at infinity. Then V is uniquely determined in the volume.



Proof: Suppose there exists two solutions, V_1 and V_2. Let

$$\delta V \equiv V_2 - V_1 \ .$$

Then

$$\nabla^2(\delta V) = \nabla^2 V_2 - \nabla^2 V_1 = 0 \ ,$$

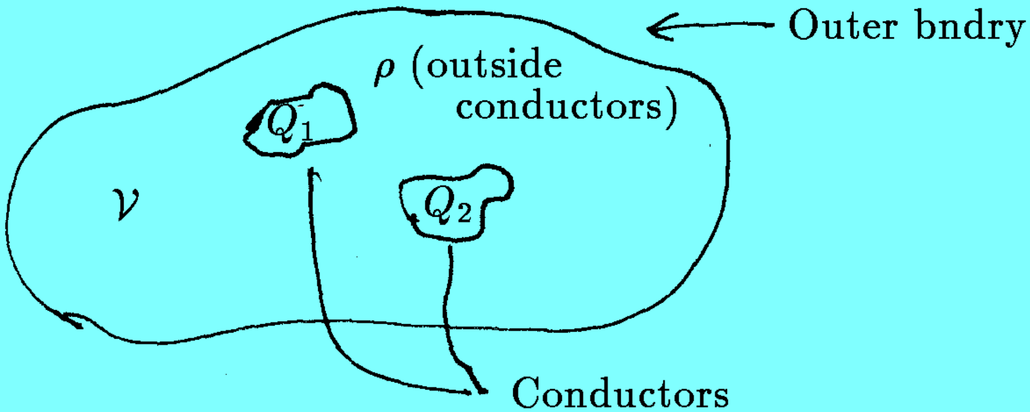
and $\delta V = 0$ on the boundary.

By the corollary of the Laplacian Mean Value Thm, δV cannot have a minimum or maximum in the volume. Therefore it must be zero everywhere, and $V_1 = V_2$.

This proves what we said earlier about conducting enclosures. The conducting outer shell is an equipotential at some potential V_0, so one solution to $\nabla^2 V = 0$ in the interior is clearly $V = V_0$, which means $\vec{\boldsymbol{E}} = 0$. The uniqueness theorem guarantees that this is the **ONLY** solution.

Second Uniqueness Theorem

Consider a volume $\mathcal{V}$ which is bounded by conductors, each with a specified charge Q_i, and possibly a conducting enclosure, with ρ specified everywhere in $\mathcal{V}$. Then $\vec{\boldsymbol{E}}$ is uniquely determined in $\mathcal{V}$. V is determined up to a constant.



Proof: Suppose there exists two solutions, $\vec{\boldsymbol{E}}_1$ and $\vec{\boldsymbol{E}}_2$. Let

$$\delta\vec{\boldsymbol{E}} \equiv \vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{E}}_1 \quad \text{and} \quad \delta V \equiv V_2 - V_1 \ .$$

Within $\mathcal{V}$,

$$\vec{\boldsymbol{\nabla}} \cdot \delta\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}_1 = 0 \ . \tag{1}$$

On the conducting surfaces,

$$\int_{S_i} \delta\vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \int_{S_i} \vec{\boldsymbol{E}}_2 \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{S_i} \vec{\boldsymbol{E}}_1 \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{Q_i}{\epsilon_0} - \frac{Q_i}{\epsilon_0} = 0 \ . \tag{2}$$

$\delta\vec{\boldsymbol{E}}$ and δV are related by $\delta\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}(\delta V)$.

Consider the quantity

$$\vec{\boldsymbol{\nabla}} \cdot (\delta V\, \delta\vec{\boldsymbol{E}}) = \delta V\, \vec{\boldsymbol{\nabla}} \cdot \delta\vec{\boldsymbol{E}} + \vec{\boldsymbol{\nabla}}(\delta V) \cdot \delta\vec{\boldsymbol{E}} \ . \tag{3}$$

The right-hand side simplifies to

$$0 - |\delta\vec{\boldsymbol{E}}|^2 \ . \tag{4}$$

If we integrate the equation over $\mathcal{V}$, we can apply the divergence theorem:

$$\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}} \cdot (\delta V\, \delta\vec{\boldsymbol{E}})\, \mathrm{d}^3x = \sum_i \int_{S_i} \delta V\, \delta\vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} + \int_{S_{\rm outer}} \delta V\, \delta\vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} \ . \tag{5}$$

We can simplify the first term on the right above by noticing that V_1 and V_2 must be constants on S_i, the boundary of the i'th conductor, so δV_i must be a constant. Taking it outside the integral, we are left with $\delta V \int_{S_i} \delta\vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}}$, which vanishes by Eq. (2).

We can similarly simplify the 2nd term on the right of Eq. (5). If it is a conducting enclosure, then the same argument applies. If there is no enclosure, then the outer boundary is at infinity. (This is similar to our calculation of the total energy as an integral over space.) We expect δV falls off at least as $1/R$, and $\delta\vec{\boldsymbol{E}}$ at least as fast as $1/R^2$. Since $\int |\mathrm{d}\vec{\boldsymbol{a}}| \propto R^2$, the term approaches zero at $R \to \infty$. Thus, the RHS of Eq. (5) is zero.

Using Eqs. (3) and (4), with the fact that the RHS of Eq. (5) vanishes, we find

$$\int_{\mathcal{V}} |\delta\vec{\boldsymbol{E}}|^2 = 0 \ ,$$

which implies that

$$\boxed{\delta\vec{\boldsymbol{E}} = 0 \quad \text{in } \mathcal{V} \ .}$$

and therefore

$$\boxed{\delta V = \text{constant} \qquad \text{in } \mathcal{V} \ .}$$

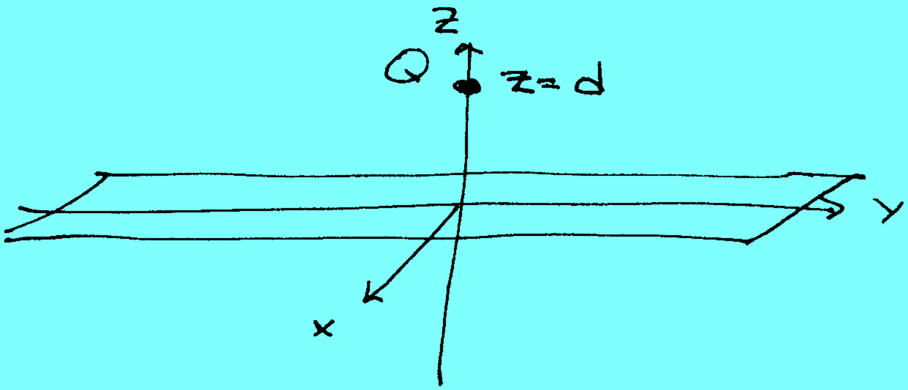
MORAL:

If you find a solution to a problem of this type, no matter how you found it, you are done. There cannot be another solution.

This slide was added after lecture.

Summary: Blackboard Discussion of Image Charges

Sample Problem: a conducting plane, in the x-y plane, with a charge Q at $(0, 0, d)$:



The problem is to find a solution to

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0} = -\frac{Q}{\epsilon_0}\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d)\ ,} \tag{1}$$

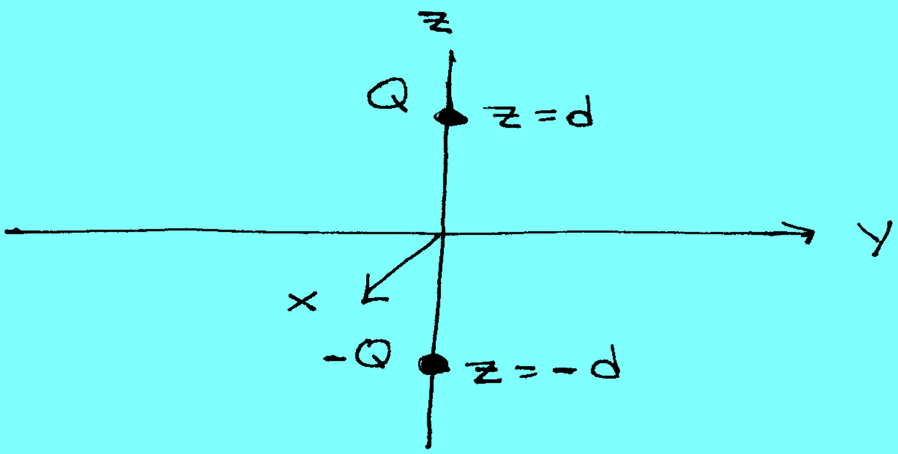
for $z > 0$, where $\vec{\boldsymbol{r}}_d \equiv d\hat{\boldsymbol{z}}$, with the boundary condition

$$\boxed{V(x, y, 0) = 0\ .}$$

This slide was added after lecture.

Solution:

Imagine an *image charge* $-Q$ at the mirror image location, $(0, 0, -d)$. The total potential, due to the real and the image charge, will then vanish on the conducting plane and will satisfy Eq. (1) for $z > 0$. By the first uniqueness theorem, this is the unique solution.



Note that the image charge solution describes $V(\vec{r})$ only for $z \geq 0$. For $z < 0$, $V = 0$ is the unique solution. For $z \geq 0$

$$\boxed{V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} .}$$

This slide was added after lecture.

What is $\vec{\boldsymbol{E}}$? Ans: $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V$.

What is σ on conducting surface?

Ans: $\sigma = \epsilon_0 \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}$,

where $\hat{\boldsymbol{n}}$ = outward normal of conductor

What is total charge induced on the surface?

Ans: $Q_{\text{induced}} \equiv \displaystyle\int \sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y = -Q$,

where the answer is found from Gauss's law: the integral is the electric flux, and half the flux coming from Q and half the flux coming from $-Q$ intersect the x-y plane.

What is force on Q?

Ans: $\vec{\boldsymbol{F}} = Q\vec{\boldsymbol{E}}$, where $\vec{\boldsymbol{E}}$ is the electric field caused by all charges, real or image, except for Q itself.

This slide was added after lecture.

What is the potential energy W?

Ans: It is easiest to use

$$\boxed{W = \frac{1}{2}\int \rho(\vec{\boldsymbol{r}})V(\vec{\boldsymbol{r}})\,\mathrm{d}^3x\ ,}$$

where $\rho(\vec{\boldsymbol{r}})$ includes only physical charges, not image charges. The physical charges are the point charge Q, and the surface charge $\sigma(x,y)$ induced on the surface. But $\sigma(x,y)$ does not contribute, since $V(x,y,0)=0$. So

$$W = \frac{1}{2}\int Q\,\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_d)\,V(\vec{\boldsymbol{r}})\,\mathrm{d}^3x = \frac{1}{2}QV(\vec{\boldsymbol{r}}_d)\ .$$

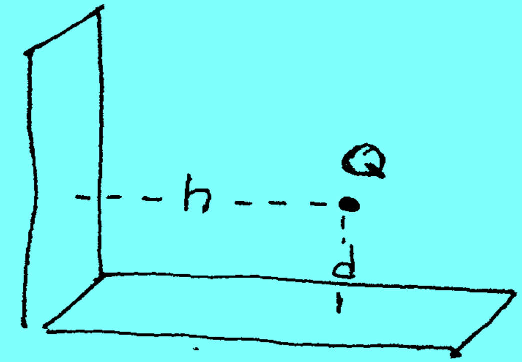
We exclude the infinite self-energy of Q by taking $V(\vec{\boldsymbol{r}}_d)$ to be the potential of all charges, real and image, excluding Q itself, so

$$\boxed{W = -\frac{Q^2}{8\pi\epsilon_0}\frac{1}{2d}\ .}$$

In this case the answer is half the potential energy we would have found if we treated the image charge as if it were real — but that is not always the case.

In general, the potential energy found by treating the image charges as real has no fixed relation to the actual potential energy of the system.

2nd Sample Image Charge Problem: two conducting planes, at right angles.



This problem can be solved by introducing 3 image charges: