8.07 Lecture Slides 11
October 16, 2019

ELECTRIC POTENTIAL: THE MULTIPOLE EXPANSION

Announcements

Quiz 1 results: Very good (mostly)

Average = 83.0%, Standard deviation = 14.9%

Top grades: 3 100%'s, 1 97%, 3 96%'s, ...

Average grade on Problem 2 (from homework) = 82.9%, slightly lower than the average grade on all problems. It should be 100%!

Low points:

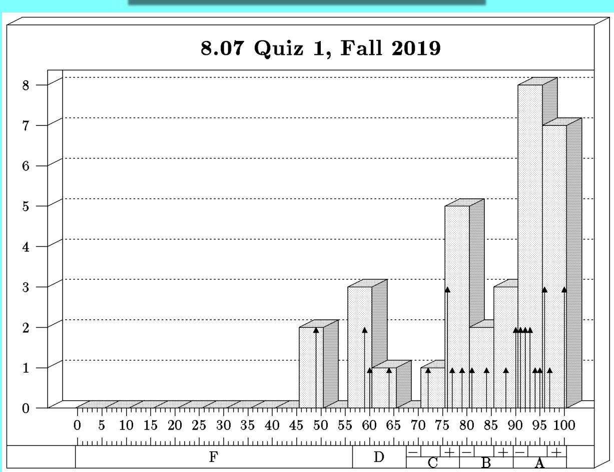
Part 1(c): 77.5% (δ-function of a function of x)

Part 2(d): 70% (total energy of spherical conductor and a conducting plane — it was on Problem Set 3!)

Part 3(c): 75.6% ($V(r)$ for spherical configuration, given $\vec{\boldsymbol{E}}$)

Part 3(d): 46.9% (total charge of a spherical configuration, given $\vec{\boldsymbol{E}}$ — just use Gauss's law)

Grade Histogram



Will be posted

Disclaimer

The grade cuts shown on the previous slide represent my estimate of what you have earned on this quiz, but they will not necessarily be the numerical cuts that will be used to evaluate your final grade. The final grade cuts will be adjusted to account for the level of difficulty of all the quizzes and problem sets.

Letter Grades

Slightly more than half the class scored in the A range.

If your grade was a B, C, D, or F — don't despair!

Quiz 1 is only 20% of the total grade. You have plenty of opportunity to raise your grades on Quiz 2, the problem sets, and the final exam.

As described on the Course Information Sheet, students whose final grade is slightly below a borderline might have their grades boosted. "Students whose grades have improved significantly during the term, and students whose average has been pushed down by single low grade, will be the ones most likely to be boosted." If your Quiz 1 grade is low, you have a great opportunity to put yourself into in one or both of these categories.

Review of Lecture 9 — Two weeks ago!!

Expansion of $F(\hat{n})$

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3 \ ,$$

so $F(\hat{\boldsymbol{n}})$ can also be written $F(\theta,\phi)$.

($\hat{\boldsymbol{n}}$ has the same definition as $\hat{\boldsymbol{r}}$ in spherical polar coordinates. I call it $\hat{\boldsymbol{n}}$ because we might not be using spherical polar coordinates.)

Can expand $F(\hat{\boldsymbol{n}})$ in a power series:

$$F(\hat{\boldsymbol{n}}) = C^{(0)} + C^{(1)}_i \hat{\boldsymbol{n}}_i + C^{(2)}_{ij}\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j + \ldots + C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell} + \ldots \ ,$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates), and each $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is **symmetric and traceless.**

Review of Lecture 9

Can expand $F(\hat{\boldsymbol{n}})$ in a power series:

$$F(\hat{\boldsymbol{n}}) = C^{(0)} + C^{(1)}_i \hat{\boldsymbol{n}}_i + C^{(2)}_{ij}\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j + \ldots + C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell} + \ldots \ ,$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates), and each $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is **symmetric and traceless.**

$C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ can be required to be **symmetric**, because $\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell}$ is.

$C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ can be required to be **traceless**, because pieces with nonzero trace contain Kronecker delta's ($\delta_{i_m i_n}$), which give a contribution with fewer $\hat{\boldsymbol{n}}$'s, which can then be absorbed into lower terms.

Notational subtlety: $\hat{\boldsymbol{n}}_i$ denotes the i'th component of the vector $\hat{\boldsymbol{n}}$. A component of a vector is just a number, so Marin pointed out to me that $\hat{\boldsymbol{n}}_i$ should not be set in boldface. Griffiths, for example, uses ordinary italics for the components of vectors. However, since I have already used boldface in a number of documents, for consistency I will continue to do so. But be aware that $\hat{\boldsymbol{n}}_i$, although set in boldface, is **not** a vector.

Not Quite a Review of Lecture 9

This expansion is equivalent to the standard spherical harmonic expansion, which is known to converge for square-integrable functions

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi |f(\theta,\phi)|^2 < \infty \ ,$$

in the sense that

$$\lim_{\ell_{\max}\to\infty} \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \left| f(\theta,\phi) - \sum_{\ell=0}^{\ell_{\max}} C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell} \right|^2 = 0 \ .$$

Review of Lecture 9

General Solution to Laplace's Equation in Spherical Coordinates

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{n}$.

Review of Lecture 9

Connection to Standard $Y_{\ell m}(\theta, \phi)$'s

Later we will be explicit, but for now we state that the ℓ'th term, $F_\ell(\hat{n}) \equiv C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}$ is equivalent to the sum over all $Y_{\ell m}$'s for a given ℓ.

Since m runs from $-\ell$ to ℓ, so there are $2\ell + 1$ possible values, it must take $2\ell+1$ independent parameters to specify the general traceless symmetric tensor $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$. We checked that.

Review of Lecture 9

Trace Decomposition Theorem

Any symmetric matrix $S_{i_1 \ldots i_\ell}$ can be uniquely written in the form

$$S_{i_1 \ldots i_\ell} = S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} + \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] \ ,$$

where $S^{(\mathrm{TS})}_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1 \ldots i_{\ell-2}}$ is a symmetric tensor, and

$$\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} [xxx]$$

means to symmetrize the expression xxx in the indices $i_1 \ldots i_\ell$. I.e.,

$$\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} [M_{i_1 \ldots i_\ell}] = \frac{1}{\ell!} \sum_{\substack{\text{permutations} \\ \{j_1 \ldots j_\ell\} \text{ of } \{i_1 \ldots i_\ell\}}} M_{j_1 \ldots j_\ell} \ .$$

Traceless Symmetric Part of a Tensor

I will use the symbol $\{\ \}_{\mathrm{TS}}$ to denote the traceless symmetric part of any symmetric tensor; i.e., given that

$$S_{i_1 \ldots i_\ell} = S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} + \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] \ ,$$

define $\{ S_{i_1 \ldots i_\ell} \}_{\mathrm{TS}}$ by

$$\{ S_{i_1 \ldots i_\ell} \}_{\mathrm{TS}} \equiv S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} = S_{i_1 \ldots i_\ell} - \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] .$$

By the trace decomposition theorem, this is uniquely defined.

Example: Azimuthal Symmetry

Azimuthal symmetry: symmetry under rotation about z axis.

$\hat{\boldsymbol{z}}$ is invariant under rotations about the z axis, and higher rank tensors can be constructed from $\hat{\boldsymbol{z}}$.

Tensors of rank 0 and 1 (i.e., scalars and vectors) are by definition traceless. Thus,

$$\{\,1\,\}_{\rm TS} = 1\ ,$$
$$\{\,\hat{z}_i\,\}_{\rm TS} = \hat{z}_i\ .$$

But for rank 2, the trace of $\hat{z}_i\hat{z}_j$ equals $\hat{z}_i\hat{z}_i = \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = 1$. But we can subtract a constant times δ_{ij} so that the result is traceless:

$$\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS} = \hat{z}_i\hat{z}_j - \tfrac{1}{3}\delta_{ij}\ .$$

The coefficient is $1/3$, because the trace of δ_{ij} is $\delta_{ii} = 3$.

For rank 3, $\hat{z}_i\hat{z}_j\hat{z}_k$ has trace $\hat{z}_i\hat{z}_i\hat{z}_k = \hat{z}_k$, but we can make it traceless with a subtraction

$$\{\,\hat{z}_i\hat{z}_j\hat{z}_k\,\}_{\rm TS} = \hat{z}_i\hat{z}_j\hat{z}_k - \tfrac{1}{5}\left(\hat{z}_i\delta_{jk} + \hat{z}_j\delta_{ik} + \hat{z}_k\delta_{ij}\right)\ .$$

The subtraction must be symmetric, and each term must contain at least one Kronecker delta function. We verify tracelessness on the blackboard.

For rank 4, there are two possible subtractions:

$$\begin{aligned}\{\,\hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m\,\}_{\rm TS} = \hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m &+ c_1\big(\hat{z}_i\hat{z}_j\delta_{km} + \hat{z}_i\hat{z}_k\delta_{mj} + \hat{z}_i\hat{z}_m\delta_{jk} + \hat{z}_j\hat{z}_k\delta_{im}\\ &+ \hat{z}_j\hat{z}_m\delta_{ik} + \hat{z}_k\hat{z}_m\delta_{ij}\big) + c_2\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big)\ .\end{aligned}$$

We verify on the blackboard that $c_1 = -1/7$ and $c_2 = 1/35$.

Construction of $F(\theta)$

We have one term for each rank, so

$$\begin{aligned}F(\theta) = c_0 + c_1\{\,\hat{z}_i\,\}_{\rm TS}\hat{n}_i + c_2\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j + \ldots\\ + c_\ell\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}_{\rm TS}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell} + \ldots\ ,\end{aligned}$$

where the c_ℓ's are constants. Since $\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}} = \cos\theta$,

$$\{\,1\,\}_{\rm TS} = 1$$
$$\{\,\hat{z}_i\,\}_{\rm TS}\,\hat{n}_i = \cos\theta$$
$$\{\,\hat{z}_i\hat{z}_j\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j = \cos^2\theta - \frac{1}{3}$$
$$\{\,\hat{z}_i\hat{z}_j\hat{z}_k\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j\hat{n}_k = \cos^3\theta - \frac{3}{5}\cos\theta$$
$$\{\,\hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m\,\}_{\rm TS}\,\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m = \cos^4\theta - \frac{6}{7}\cos^2\theta + \frac{3}{35}\ .$$

Connection to Legendre Polynomials

Up to normalization, this is the standard expansion in Legendre polynomials.

The Legendre polynomials are normalized so that $P_\ell(\cos\theta = 1) = 1$, and you will see on Problem Set 4 that

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}_{\rm TS}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\ .$$

The Multipole Expansion

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{n}$. Consider the potential due to a charge distribution entirely contained within a radius R of the origin. Then $\Phi(\vec{r})$ must be valid, and hence finite, for all $r > R$, which implies $C^{(\ell)}_{i_1 i_2 \ldots i_\ell} = 0$. So

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C'^{(\ell)}_{i_1 \ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ .}$$

Since each term falls off faster, the first few terms are usually sufficient at large r.

How to Find Multipole Moments Method 1 (as in Griffiths)

Start with

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3x' \ ,}$$

and then expand

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{\sqrt{|\vec{r}|^2 + |\vec{r}\,'|^2 - 2\vec{r} \cdot \vec{r}\,'}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta'}} \ ,$$

in powers of r', where $r = |\vec{r}|$, $r' = |\vec{r}\,'|$, and θ' is the angle between $\vec{r}$ and $\vec{r}\,'$.

The function to be expanded,

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta'}} = \frac{1}{r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'}}$$

has the form

$$g(x, \lambda) = \frac{1}{\sqrt{1 + \lambda^2 - 2\lambda x}} \ ,$$

and we can define the ℓ'th term in the power expansion to be $P_\ell(x)$, the Legendre polynomial:

$$\boxed{g(x, \lambda) = \frac{1}{\sqrt{1 + \lambda^2 - 2\lambda x}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x) \ .}$$

Then

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'}} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta') \ .$$

Then

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3x'$$

and

$$\frac{1}{|\vec{r} - \vec{r}\,'|} == \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta')$$

imply

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^\ell \rho(\vec{r}\,') P_\ell(\cos\theta') \, d^3x' \ .}$$

Drawback of Method 1

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta')\, \mathrm{d}^3x' \ .$$

Since θ' is the angle between $\vec{r}$ and $\vec{r}\,'$, to use this equation directly one has to redo the integral for every direction of $\vec{r}$.

How to Find Multipole Moments Method 2: with Traceless Symmetric Tensors

Expand

$$f(\vec{r}\,') \equiv \frac{1}{|\vec{r} - \vec{r}\,'|} \ .$$

in a power series in the components of $\vec{r}\,' = x'_i \hat{e}_i = r' \hat{n}'_i$:

$$\begin{aligned} f(\vec{r}\,') &= f(\vec{0}) + \left.\frac{\partial f}{\partial x'_i}\right|_{\vec{r}\,'=\vec{0}} x'_i + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x'_i \partial x'_j}\right|_{\vec{r}\,'=\vec{0}} x'_i x'_j + \ldots \ , \\ &= f(\vec{0}) + r' \left.\frac{\partial f}{\partial x'_i}\right|_{\vec{r}\,'=\vec{0}} \hat{n}'_i + \frac{r'^2}{2!} \left.\frac{\partial^2 f}{\partial x'_i \partial x'_j}\right|_{\vec{r}\,'=\vec{0}} \hat{n}'_i \hat{n}'_j + \ldots \ . \end{aligned}$$

To be completed on the blackboard.

This slide was added after lecture.

Summary: Blackboard Discussion Continuing the Calculation

Since f is a function of $\vec{r} - \vec{r}\,'$, $\dfrac{\partial f}{\partial x'_i} = -\dfrac{\partial f}{\partial x_i}$, and

$$\left.\frac{\partial^\ell f}{\partial x'_{i_1} \ldots \partial x'_{i_\ell}}\right|_{\vec{r}\,'=\vec{0}} = (-1)^\ell \frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{r}|} \ .$$

This quantity is traceless, since

$$\left(\frac{\partial^\ell}{\partial x_j \partial x_j \partial x_3 \ldots \partial x_{i_\ell}} \frac{1}{|\vec{r}|} \right) = \frac{\partial^\ell}{\partial x_3 \ldots \partial x_{i_\ell}} \nabla^2 \frac{1}{|\vec{r}|} = 0 \ ,$$

because $\nabla^2(1/|\vec{r}|) = 0$ except at $\vec{r} = 0$.

This slide was added after lecture.

We found that

$$\frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{r}|} = \frac{(-1)^\ell (2\ell-1)!!}{r^{\ell+1}} \{ \hat{n}_{i_i} \ldots \hat{n}_{i_\ell} \}_{\mathrm{TS}} \ ,$$

where

$$\begin{aligned} (2\ell-1)!! &\equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 \ , \text{ with } \ (-1)!! \equiv 1 \ . \\ &= \frac{(2\ell)!}{2^\ell \ell!} \end{aligned}$$

The power series then becomes

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{ \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \}_{\mathrm{TS}} \{ \hat{n}'_{i_1} \ldots \hat{n}'_{i_\ell} \}_{\mathrm{TS}} \ , \ (\text{for } r' < r)$$

where either one (but not both) of the TS's can be dropped, since the difference is proportional to a Kronecker delta function, which leads to taking a trace of the other TS expression, which vanishes.

This slide was added after lecture.

The analogous equation for the standard spherical harmonics is

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \ , \qquad (\text{for } r' < r).$$

Inserting the above boxed equation into

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, \mathrm{d}^3x' \ ,$$

we find the final result

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1\ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ ,}$$

where the multipole moments $C^{(\ell)}_{i_1\ldots i_\ell}$ are given by

$$\boxed{C^{(\ell)}_{i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}\,') \, \{ \, \vec{r}\,'_{i_1} \ldots \vec{r}\,'_{i_\ell} \, \}_{\mathrm{TS}} \, \mathrm{d}^3x' \ .}$$

This slide was added after lecture.

The analogous formulas for the standard spherical harmonic treatment are

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \ ,}$$

where the multipole moments $q_{\ell m}$ are given by

$$\boxed{q_{\ell m} = \int \rho(\vec{r}\,') \, r'^{\ell} \, Y^*_{\ell m}(\theta',\phi') \, \mathrm{d}^3x' \ .}$$

This slide was added after lecture.

Once we calculate the $C^{(\ell)}_{i_1\ldots i_\ell}$ or the $q_{\ell m}$ for a given charge distribution $\rho(\vec{r}\,')$, from

$$\boxed{\begin{aligned} C^{(\ell)}_{i_1\ldots i_\ell} &= \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}\,') \, \{ \, \vec{r}\,'_{i_1} \ldots \vec{r}\,'_{i_\ell} \, \}_{\mathrm{TS}} \, \mathrm{d}^3x' \ , \\ q_{\ell m} &= \int \rho(\vec{r}\,') \, r'^{\ell} \, Y^*_{\ell m}(\theta',\phi') \, \mathrm{d}^3x' \ , \end{aligned}}$$

then we can calculate $V(\vec{r})$ for any $\vec{r}$:

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1\ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \\ V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \ , \end{aligned}}$$

avoiding the drawback of “Method 1”.