MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 14, 2019
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PROBLEM SET 2

DUE DATE: Due to the student holiday on Friday, September 20, the due date for this problem set is postponed until Monday, September 23, 2019. You are STRONGLY encouraged, however, to finish the problem set by September 20, because Problem Set 3 will be posted on Saturday September 21, and will be due the following Friday. The problem set is due at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

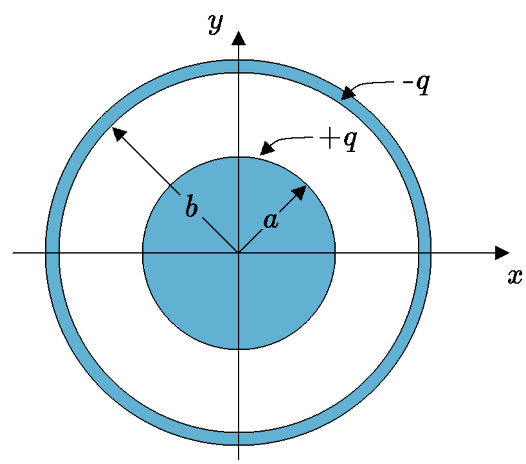
READING ASSIGNMENT: Chapter 2 of Griffiths: *Electrostatics.*

CREDIT: This problem set has 96 points of credit, plus the option of earning 20 points extra credit. (The Course Information description, on the General Info tab of the webpage, describes how extra credit grades will be handled.)

— PART A —

PROBLEM 1: CAPACITANCE OF A CYLINDRICAL CAPACITOR *(15 points)*

A very long conducting cylinder (length ℓ and radius a) carrying a total charge $+q$ is surrounded by a thin conducting cylindrical shell (length ℓ and radius b) with total charge -q, as shown in cross section in the sketch.



(a) [3 pts] Using Gauss's law, find an expression for the electric field $\vec{E}(\vec{r})$ at points $a < r < b$. Neglect end effects due to the finite length of the cylinder.

(b) [3 pts] Using your expression for $\vec{E}$ from part (a), find the potential difference ΔV between the outer shell and the inner cylinder.

(c) [3 pts] Derive an expression for the capacitance of this capacitor in terms of the quantities given. What is the capacitance per unit length?

(d) [3 pts] Compute the energy stored in the capacitor by integrating the energy density over the volume where $\vec{E}$ is nonzero, again ignoring any edge effects. Compare with the result you get using $W = \frac{1}{2}CV^2$. Do they agree? (Hint: they better agree!)

(e) [3 pts] Let the gap $d = b - a$ between the cylinders be small compared to the radii, a and b. Show that in this case your answer for part (c) reduces to that for a parallel plate capacitor (see Griffiths Eq. (2.54) on p. 106).

PROBLEM 2: THE ELECTRIC FIELD, POTENTIAL, AND ENERGY OF A UNIFORM SPHERE OF CHARGE *(16 points)*

(a) [4 pts] A uniformly charged sphere of charge has radius R and total charge Q. Using Gauss's law, calculate the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ everywhere.

(b) [4 pts] Using the electric field you calculated in part (a), find the electric potential $V(\vec{\boldsymbol{r}})$ everywhere.

(c) [4 pts] Using the expression

$$W = \frac{1}{2}\epsilon_0 \int_{\text{all space}} |\vec{\boldsymbol{E}}|^2 \, \mathrm{d}^3 x \ , \tag{2.1}$$

for the total work needed to assemble the charge configuration, calculate W using your expressions above.

(d) [4 pts] Using the expression

$$W = \frac{1}{2} \int_{\text{all space}} \rho V \, \mathrm{d}^3 x \ , \tag{2.2}$$

calculate W again using your expressions above.

PROBLEM 3: POTENTIAL OF A HEMISPHERICAL BOWL *(10 points)*

Griffiths Problem 2.48 (p. 108): An inverted hemispherical bowl of radius R carries a uniform surface charge density σ. Find the potential difference between the "north pole" and the center. [Answer: $(R\sigma/2\epsilon_0)(\sqrt{2}-1)$]

PROBLEM 4: CALCULATING FORCES USING VIRTUAL WORK *(10 points)*

Use "virtual work" to calculate the attractive force between conductors in the parallel plate capacitor (area A, separation d). That is, use conservation of energy to determine how much work must be done to move one plate by an infinitesimal amount, and then use the value of the work to determine the force. Do your virtual work computations in two ways:

(a) [5 pts] keeping fixed the charges on the plates, and,

(b) [5 pts] keeping fixed the voltage between the plates.

— PART B (To be handed in separately from Part A) —

PROBLEM 5: MUTUAL CAPACITANCE *(15 points)*

In lecture we discussed relations of the form

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j , \quad i, j = 1, 2, \ldots , n \ . \tag{5.1}$$

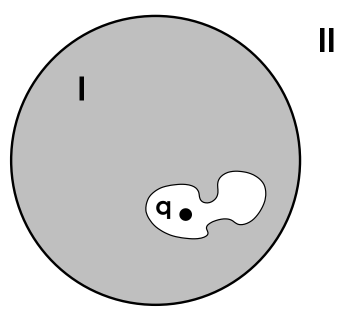
governing the potentials and charges of n conductors (with the potential taken to be zero at spatial infinity).

(a) [5 pts] Prove that $C_{ij} = C_{ji}$. [*Hint:* Consider how much energy is needed to start with the system uncharged, then add charge Q_i to conductor i, and then add charge Q_j to conductor j. Then consider starting again with the system uncharged, and performing these operations in the opposite order. That is, add charge Q_j to conductor j, and then Q_i to conductor i. Then think about how to use your answers to prove the desired result.]

(b) [5 pts] Consider a two-conductor configuration. Calculate the conventional capacitance C in terms of C_{11}, C_{12}, C_{21}, and C_{22}.

(c) [5 pts] Consider two concentric spherical conducting shells of radii a and b with $a < b$. Call the inner shell conductor 1, and the outer shell conductor 2. Calculate the matrix of capacitances C_{ij} and use your result from part (b) to infer the conventional capacitance C. Compare your answer with Example 2.12 in Griffiths, p. 106.

PROBLEM 6: A CAVITY IN A CONDUCTING SPHERE *(8 points)*



An irregular region is carved out of a conducting sphere. A charge q is introduced and held at a fixed position in that irregular region. Take the origin of coordinates to be at the center of the sphere.

(a) [2 pts] What is the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ in region I?

(b) [2 pts] What is the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ in region II?

(c) [2 pts] Will there be a layer of charge on the surface of the cavity? If so, what will be the total charge of this layer? Will it be uniformly distributed?

(d) [2 pts] Will there be a layer of charge on the outer surface of the sphere? If so, what will be the total charge of this layer? Will it be uniformly distributed?

PROBLEM 7: SPACE CHARGE, VACUUM DIODES, AND THE CHILD-LANGMUIR LAW *(22 points)*

Griffiths Problem 2.53 (p. 109). Challenging! For part (e), you can solve the differential equation either by guessing a solution and showing that it works, or by finding a first integral by the same method that is used in mechanics to go from Newton's 2nd order equation of motion to the first order equation for the conservation of mechanical energy. In that case, you may recall, one multiplies the second order equation of motion $m\ddot{x} = -\frac{dV}{dx}$ *by* $\dot{x}$*, and then rewrites* $\dot{x}\ddot{x}$ *as* $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}^2)$ *and* $\dot{x}\frac{dV}{dx}$ *by* $\frac{dV}{dt}$*. Here is the text of the problem:*

In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

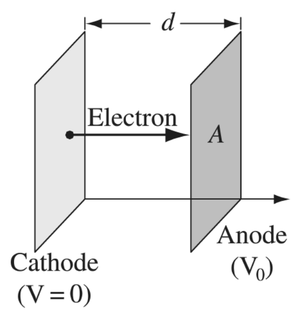


FIGURE 2.55

(a) [3 pts] Write Poisson's equation for the region between the plates.

(b) [3 pts] Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

(c) [4 pts] In the steady state, I is independent of x. What, then, is the relation between ρ and v?

(d) [3 pts] Use these three results to obtain a differential equation for V, by eliminating ρ and v.

(e) [6 pts] Solve this equation for V as a function of x, V_0, and d. Plot $V(x)$, and compare it to the potential *without* space-charge. Also, find ρ and v as functions of x.

(f) [3 pts] Show that

$$I = KV_0^{3/2} \ , \tag{2.56}$$

and find the constant K . (Equation 2.56 is called the **Child-Langmuir law**. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is *nonlinear*—it does not obey Ohm's law.)

PROBLEM 8: $\nabla^2(1/r)$ IN THE LANGUAGE OF DISTRIBUTIONS *(20 points extra credit)*

This problem will have a longwinded pedagogical introduction, since it concerns an approach which was discussed in lecture, but is not discussed in the textbook. Think of this introduction as a set of lecture notes.

In Problem 5 of Problem Set 1, you evaluated $-\nabla^2(1/4\pi r)$ by replacing $1/r$ by $1/\sqrt{r^2+a^2}$. After calculating $g_a(r) \equiv -\nabla^2(1/4\pi\sqrt{r^2+a^2})$, you showed that its integral over all space is 1, and that for any $r \neq 0$ it approaches 0 as $a \to 0$. This exercise conveys useful intuition about δ-functions, and about the relation

$$-\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{\boldsymbol{r}}) \ . \tag{8.1}$$

However, from the standpoint of a mathematically rigorous treatment, there is a shortcoming to this and all similar treatments of the δ-function as a limit of a sequence of functions. While the sequence of functions leads to reliable intuition, the precise mathematical picture is complicated by the ordering of limits. That is, you showed in your problem set solutions that $\int_{\text{all space}} g_a(r)\, \mathrm{d}^3x = 1$ for any $a > 0$, and hence

$$\lim_{a\to 0} \int_{\text{all space}} g_a(r)\, \mathrm{d}^3x = 1 \ . \tag{8.2}$$

However, you also showed on your problem set solutions that

$$g_{\,\text{lim}}(r) \equiv \lim_{a\to 0} g_a(r) = \begin{cases} 0 & \text{if } r \neq 0 \\ \infty & \text{if } r = 0 \end{cases} \ , \tag{8.3}$$

and we showed in lecture that this implies that

$$\int_{\text{all space}} \lim_{a\to 0} g_a(r)\, \mathrm{d}^3x = 0 \ , \tag{8.4}$$

where the integral is defined as usual as the area under the curve. By comparing Eqs. (8.2) and (8.4), we see that the ordering of the limit and the integration matters! So we cannot quite say that $g_{\,\text{lim}}(r)$ behaves as a δ-function. Instead, we have to keep in mind the slightly more complicated picture in which $g_a(r)$ acts almost like a δ-function when a is very very small, and behaves exactly as a δ-function if we take the limit $a \to 0$ *after* any integrations have been carried out.

Since the integral of $g_{\lim}(r)$ vanishes, there is no ordinary function that, when speaking precisely, behaves as a Dirac δ-function. Thus the δ-function is technically not a function, but rather what the mathematicians call a *generalized function*, or a *distribution*. It is really the concept of integration that is being generalized. Starting with functions of one variable, we can consider an arbitrary function $\varphi(x)$. Its integral,

$$\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x \ , \tag{8.5}$$

maps the function $\varphi(x)$ into a single real number, the value of its integral. It is a linear map, in the sense that

$$\int_{-\infty}^{\infty} [\varphi_1(x) + \lambda\varphi_2(x)]\ \mathrm{d}x = \int_{-\infty}^{\infty} \varphi_1(x)\,\mathrm{d}x + \lambda \int_{-\infty}^{\infty} \varphi_2(x)\,\mathrm{d}x \ , \tag{8.6}$$

where λ is a constant. A distribution defines a generalized integral, which is an *arbitrary* linear map from the space of smooth "test" functions $\varphi(x)$ to real numbers. These test functions are required not only to be smooth, but also to fall off rapidly at large values of $|x|$.* The distribution that corresponds to $\delta(x - x_0)$, where x_0 is a constant, is the map which takes the function $\varphi(x)$ to the number $\varphi(x_0)$, its value at the particular point x_0.

In equations, we can define a distribution

$$F_{\delta(x-x_0)}[\varphi(x)] \equiv \varphi(x_0) \ . \tag{8.7}$$

Here, $F_{\delta(x-x_0)}$ is the distribution corresponding to $\delta(x - x_0)$, and the equation says that this distribution maps the function $\varphi(x)$ to its value at x_0, namely $\phi_(x_0)$. Note that the pair of square brackets used for the argument of F is a common notation for a functional, i.e., a function of a function.

While there is no function that behaves as a Dirac δ-function, it is perfectly clear that Eq. (8.7) is well-defined. Thinking of this map as a generalization of integration, we can write it as

$$\int_{-\infty}^{\infty} \varphi(x)\,\delta(x - x_0)\,\mathrm{d}x \equiv \varphi(x_0) \ . \tag{8.8}$$

* Various choices can be made for the precise restrictions on the space of test functions. A frequently used choice is the space of Schwartz functions, which are infinitely differentiable, and which have the property that the function and all its derivatives fall off faster than any power at large $|x|$. The distributions associated with this definition of smoothness are called tempered distributions.

But remember that the integral sign here does not describe the area under a curve. Instead the left-hand side of Eq. (8.8) is defined to be simply an alternative notation for $F_{\delta(x-x_0)}[\varphi(x)]$, which in turn is defined by Eq. (8.7). Mathematically, Eq. (8.8) defines the δ-function, which is defined solely as a prescription for a generalized type of integration.

The derivative of a distribution is defined so that generalized integration is consistent with the usual procedure of integration by parts, which for two functions $\phi(x)$ and $f(x)$ can be written as

$$\int_{-\infty}^{\infty} \varphi(x) \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x = \varphi(x) f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} f(x)\,\mathrm{d}x \; . \tag{8.9}$$

In analogy, we define the derivative of a δ-function by

$$\int_{-\infty}^{\infty} \varphi(x)\, \frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x_0)\,\mathrm{d}x \equiv -\int \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\,\delta(x-x_0)\,\mathrm{d}x \equiv -\varphi'(x_0) \; , \tag{8.10}$$

where $\varphi'(x) \equiv \mathrm{d}\varphi(x)/\mathrm{d}x$. Note that the boundary term, the first term on the right of Eq. (8.9), is not included in the definition of $\int \varphi(x)\, \frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x_0)\,\mathrm{d}x$. This choice is motivated by the intuition that $\varphi(x)\delta(x-x_0)$ is zero for $x \neq x_0$, so the intuitive value of the boundary term is zero. For arbitrary distributions $F[\varphi(x)]$, the definition of the derivative in Eq. (8.10) can be generalized to

$$F'[\varphi(x)] \equiv -F[\varphi'(x)] \; . \tag{8.11}$$

Since the allowed test functions $\varphi(x)$ are required to be infinitely differentiable, distributions are always infinitely differentiable, too.

We are now ready to evaluate $\nabla^2(1/r)$ in the language of distributions. Note that in the language of functions $\nabla^2(1/r)$ is ill-defined, because $1/r$ is not differentiable at $r = 0$. But we can promote $1/r$ to a distribution by defining it as a mapping from test functions $\varphi(\vec{\boldsymbol{r}})$ to numbers, where the mapping is given by the (ordinary) integral

$$\int \varphi(\vec{\boldsymbol{r}})\, \frac{1}{r}\,\mathrm{d}^3x \; . \tag{8.12}$$

Note that even though $1/r$ is singular at $r = 0$, this integral is perfectly well defined, since in spherical polar coordinates we have

$$\frac{1}{r}\,\mathrm{d}^3x = r\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \; . \tag{8.13}$$

By defining the derivative of a distribution by integration by parts, as in Eq. (8.11), we can write the distribution corresponding to $\nabla^2(1/r)$, which I will call $F_A[\varphi(\vec{\boldsymbol{r}})]$ for future reference:

$$F_A[\varphi(\vec{\boldsymbol{r}})] \equiv \int \varphi(\vec{\boldsymbol{r}})\,\nabla^2\left(\frac{1}{r}\right)\,\mathrm{d}^3x = \int \varphi(\vec{\boldsymbol{r}})\,\partial_i\partial_i\left(\frac{1}{r}\right)\,\mathrm{d}^3x \tag{8.14a}$$

$$= -\int \partial_i\varphi(\vec{\boldsymbol{r}})\,\partial_i\left(\frac{1}{r}\right)\,\mathrm{d}^3x \tag{8.14b}$$

$$= \int \nabla^2\varphi(\vec{\boldsymbol{r}})\,\left(\frac{1}{r}\right)\,\mathrm{d}^3x\ , \tag{8.14c}$$

so

$$F_A[\varphi(\vec{\boldsymbol{r}})] \equiv \int \nabla^2\varphi(\vec{\boldsymbol{r}})\,\left(\frac{1}{r}\right)\,\mathrm{d}^3x\ . \tag{8.15}$$

Thus $F_A[\varphi(x)]$ maps the function $\varphi(\vec{\boldsymbol{r}})$ to the number given by the (ordinary) integral on the right-hand side of Eq. (8.15).

AT LAST: THE HOMEWORK PROBLEM:

(a) [10 pts] Evaluate $F_A[\varphi(\vec{\boldsymbol{r}})]$ (as defined by Eq. (8.15)) for an arbitrary smooth test function $\varphi(\vec{\boldsymbol{r}})$ which falls off rapidly for large $|\vec{\boldsymbol{r}}|$. Specifically, show that

$$F_A[\varphi(\vec{\boldsymbol{r}})] \equiv \int \nabla^2\varphi(\vec{\boldsymbol{r}})\,\left(\frac{1}{r}\right)\,\mathrm{d}^3x = -4\pi\varphi(\vec{\boldsymbol{0}})\ . \tag{8.16}$$

Since

$$\int \varphi(\vec{\boldsymbol{r}})\,\delta^3(\vec{\boldsymbol{r}}) = \varphi(\vec{\boldsymbol{0}})\ , \tag{8.17}$$

Eq. (8.16) is equivalent to writing

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}}) \tag{8.18}$$

in the sense of distributions, which is the result we seek. (Two expressions, say $D_1(\vec{\boldsymbol{r}})$ and $D_2(\vec{\boldsymbol{r}})$, are equal in the sense of distributions if

$$\int_{\text{all space}} \varphi(\vec{\boldsymbol{r}})D_1(\vec{\boldsymbol{r}})\,\mathrm{d}^3x = \int_{\text{all space}} \varphi(\vec{\boldsymbol{r}})D_2(\vec{\boldsymbol{r}})\,\mathrm{d}^3x$$

for all smooth test functions $\varphi(x)$, which is closely analogous to Griffiths' Eq. (1.93) on p. 48.) [*Hint:* Although Eq. (8.15) is the defining equation, there is nothing that prevents you from integrating by parts once to retrieve the integral in the form of Eq. (8.14b). Write this in spherical polar coordinates, and then try to evaluate it.]

(b) [10 pts] Use the language of distributions to evaluate $\nabla^2 \ln r$ in two dimensions. (See Problem 7(d) of Problem Set 1.)