

8.07 Lecture Slides 15 October 30, 2019

ELECTRIC FIELDS IN MATTER

Announcements

Quiz 2 will be given on Wednesday, November 13, two weeks from today. Problem Set 6 is due this Friday, 11/1/19, and Problem Set 7 will be due the next Friday, 11/8/19. The quiz will include material through Problem Set 7.

Review of Lecture 14

Electric Dipoles

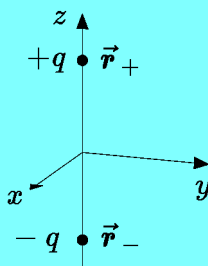
The dipole term of the multipole expansion looks like

$$V_{\text{dip}}(\vec{r}) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \quad \text{where} \quad \vec{p} = \int \rho(\vec{r}') \vec{r}' d^3x'.$$

A **physical dipole** is defined to be two charges, $+q$ and $-q$, at positions \vec{r}_+ and \vec{r}_- , respectively.

$$\vec{p} = q(\vec{r}_+ - \vec{r}_-).$$

An **ideal dipole** is the limit of a physical dipole as $|\vec{r}_+ - \vec{r}_-| \rightarrow 0$, $q \rightarrow \infty$, with \vec{p} fixed. An ideal dipole is a pure dipole, with no moments other than the dipole moment.



Review of Lecture 14

Properties of Electric Dipoles

Charge Density:

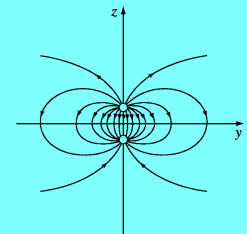
$$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla} \delta^3(\vec{r} - \vec{r}_0),$$

where \vec{r}_0 is the position of the dipole.

Electric Field:

$$\begin{aligned} \vec{E}_{\text{dip}}(\vec{r}) &= -\vec{\nabla} V_{\text{dip}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} \vec{p} \delta^3(\vec{r}). \end{aligned}$$

The delta function describes the contribution to $\int d^3x \vec{E}(\vec{r})$ of the strong \vec{E} field at the center of the dipole, as shown in the diagram.



Field of a "physical" dipole
FIGURE 3.37

Review of Lecture 14

Torque on a dipole:

$$\vec{\tau} = \vec{p} \times \vec{E} .$$

Force on a dipole:

$$\vec{F}_{\text{tot}} = (\vec{p} \cdot \vec{\nabla}) \vec{E} ,$$

i.e., $F_{\text{tot},i} = (p_j \partial_j) E_i$, or $F_{\text{tot},x} = \vec{p} \cdot \vec{\nabla} E_x$, etc.

Review of Lecture 14

Microscopic and Macroscopic Fields

In matter,

$$\rho_{\text{micro}}(\vec{r}) = \sum_i q_i \delta^3(\vec{r} - \vec{r}_i) ,$$

Define a macroscopic field by

$$\begin{aligned} \rho_{\text{macro}}(\vec{r}) &= \text{average of } \rho_{\text{micro}} \text{ in small region centered at } \vec{r} . \\ &= \langle \rho_{\text{micro}}(\vec{r}) \rangle . \end{aligned}$$

Choose size of region to be

- (a) large compared to the size of atoms.
- (b) small compared to macroscopic dimensions (i.e., the sizes of physical objects).

We will treat $\rho_{\text{macro}}(\vec{r})$ as a smooth function.

Review of Lecture 14

Similarly, in matter we distinguish between $\vec{E}_{\text{micro}}(\vec{r})$ and $\vec{E}_{\text{macro}}(\vec{r})$.

Convention: in matter, with no subscript, $\rho(\vec{r})$ and $\vec{E}(\vec{r})$ refer to the macro quantities.

Review of Lecture 14

Bound Charges

Matter can become “polarized,” meaning that it acquires a nonzero density of dipoles.

$$\vec{P}(\vec{r}) = \text{dipole moment per unit volume} .$$

$\vec{P}(\vec{r})$ is just a particular way of describing a distribution of charge. In principle, one can equivalently use $\rho(\vec{r})$

Given $\vec{P}(\vec{r})$, what is $\rho(\vec{r})$?

Answer:

$$\rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P}(\vec{r}) ,$$

and on the surface of a polarized material,

$$\sigma_b = \vec{P} \cdot \hat{n} ,$$

where \hat{n} is the outward unit normal.

Review of Lecture 14

Derivation of Bound Charge Density

We will explore **THREE** ways to derive this important relation.

Method 1 (from Griffiths, Sec. 4.2.1, pp. 173–174, done on blackboard in Lecture 14):

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}') \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3x'$$

$$= \frac{1}{4\pi\epsilon_0} \left\{ - \int_V \frac{\vec{\nabla} \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x' + \int_S \frac{\vec{P}(\vec{r}') \cdot \hat{n}}{|\vec{r} - \vec{r}'|} da' \right\}.$$

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Review of Lecture 14

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}') \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3x'$$

$$= \frac{1}{4\pi\epsilon_0} \left\{ - \int_V \frac{\vec{\nabla} \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x' + \int_S \frac{\vec{P}(\vec{r}') \cdot \hat{n}}{|\vec{r} - \vec{r}'|} da' \right\}.$$

For comparison, for an arbitrary charge density $\rho(\vec{r})$ and surface charge density $\sigma(\vec{r})$,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left\{ - \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x' + \int_S \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} da' \right\}.$$

Comparing the two expressions, we see that

$$\rho(\vec{r}) = -\vec{\nabla} \cdot \vec{P}(\vec{r}), \text{ and } \sigma(\vec{r}) = \vec{P}(\vec{r}) \cdot \hat{n}.$$

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Review of Lecture 14

The total charge on a polarized object is zero:

$$Q_{\text{tot}} = \int_V \rho_b d^3x + \int_S \sigma_b da = 0.$$

Pictures of $\rho_b = -\vec{\nabla} \cdot \vec{P}$ and $\sigma_b = \vec{P} \cdot \hat{n}$:

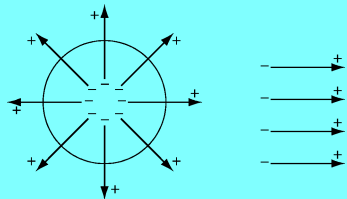


FIGURE 4.14

Shortcoming of Method 1: It makes it seem that Coulomb's law and $V(\vec{r})$ are relevant. They are not!

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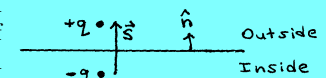
Derivation of Bound Charge Density Method 2: from Feynman Lectures

Calculate total charge in an arbitrary volume \mathcal{V} .

Dipoles completely within the volume do not contribute. Only dipoles cut by the surface contribute.

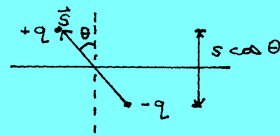
On surfaces, consider first dipoles perpendicular to surface:

Look at an infinitesimal patch of surface. Can treat as a plane. Layer of dipoles of thickness s contribute. Extra charge on surface is $\Delta q = -s da \mathcal{N}q$, where da = area of patch, \mathcal{N} = number density of dipoles.



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Now consider the case where \vec{P} is at an angle θ relative to the normal. Now a layer of thickness $s \cos \theta$ contributes. Then



$$\begin{aligned}\Delta q &= -s (da \cos \theta) \mathcal{N} q \\ &= -\mathcal{N}(q\vec{s}) \cdot d\vec{a} \\ &= -\vec{P} \cdot d\vec{a} .\end{aligned}$$

So, using the divergence theorem,

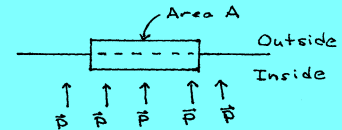
$$Q = - \int_S \vec{P} \cdot d\vec{a} = - \int_V \vec{\nabla} \cdot \vec{P} d^3x ,$$

$$\text{so } \rho_b = -\vec{\nabla} \cdot \vec{P} .$$

But what about σ_b ?

It is okay to use **only** $\rho_b = -\vec{\nabla} \cdot \vec{P}$, if one uses it over all space, including at the boundaries the polarized material:

Applying divergence theorem to a Gaussian pillbox that straddles the surface of the material. The only contribution is from the bottom surface:



$$\begin{aligned}Q_{\text{enc}} &= - \int_V \vec{\nabla} \cdot \vec{P} d^3x \\ &= - \int_S \vec{P} \cdot d\vec{a} = A \vec{P} \cdot \hat{n} ,\end{aligned}$$

so

$$\sigma_b = \frac{Q_{\text{enc}}}{A} = \vec{P} \cdot \hat{n} .$$

Derivation of Bound Charge Density Method 3: Use δ -functions

Reminder about microscopic and macroscopic charge density:

$$\begin{aligned}\rho_{\text{micro}}(\vec{r}) &= \sum_i q_i \delta^3(\vec{r} - \vec{r}_i) , \\ \rho_{\text{macro}}(\vec{r}) &= \left\langle \sum_i q_i \delta^3(\vec{r} - \vec{r}_i) \right\rangle .\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{P}_{\text{micro}}(\vec{r}) &= \sum_i \vec{p}_i \delta^3(\vec{r} - \vec{r}_i) , \\ \vec{P}_{\text{macro}}(\vec{r}) &= \left\langle \sum_i \vec{p}_i \delta^3(\vec{r} - \vec{r}_i) \right\rangle .\end{aligned}$$

For one dipole, $\rho = -\vec{p} \cdot \vec{\nabla} \delta^3(\vec{r} - \vec{r}_0)$. For ρ_b ,

$$\begin{aligned}\rho_{b,\text{micro}} &= - \sum_i \vec{p}^{(i)} \cdot \vec{\nabla} \delta^3(\vec{r} - \vec{r}_i) \\ &= \sum_i p_j^{(i)} \frac{\partial}{\partial x_j} \delta^3(\vec{r} - \vec{r}_i) \\ &= - \frac{\partial}{\partial x_j} \sum_i p_j^{(i)} \delta^3(\vec{r} - \vec{r}_i) .\end{aligned}$$

$$\begin{aligned}\rho_{b,\text{macro}} &= - \left\langle \frac{\partial}{\partial x_j} \sum_i p_j^{(i)} \delta^3(\vec{r} - \vec{r}_i) \right\rangle \\ &= - \frac{\partial}{\partial x_j} \left\langle \sum_i p_j^{(i)} \delta^3(\vec{r} - \vec{r}_i) \right\rangle \\ &= -\vec{\nabla} \cdot \vec{P}_{\text{macro}} .\end{aligned}$$