MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 2, 2019
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PROBLEM SET 6 SOLUTIONS

PROBLEM 4: A SIMPLE ATOMIC MODEL OF POLARIZABILITY *(10 points)*

Assume that the electron cloud is rigid, and that in response to the applied electric field the proton moves relative to the cloud until electrostatic equilibrium is established. Let $\vec{E}(\vec{r})$ be the electric field produced by cloud and $\vec{d}$ be the distance that the proton will shift to reach the equilibrium where $\vec{E}(\vec{d}) + \vec{E}_{\rm ext} = 0$. We first find the electric field at position $\vec{r}$ where $|\vec{r}| \ll a$, using Gauss' law:

$$Q_{\rm enc}(r) = \int_0^r \rho\, d\tau = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2s/a} s^2\, ds. \tag{4.1}$$

Since $s < r$ and $r \ll a$, we can expand $e^{-2s/a} \approx 1 - \left(\frac{2s}{a}\right) + \ldots$. Then after integrating term by term we get

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{Q_{\rm enc}(r)}{4\pi\epsilon_0 r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[\frac{4}{3}\left(\frac{r}{a}\right)^3 + O(r/a)^4\right]\hat{r} \\ &= \frac{q}{3\pi\epsilon_0 a^3}\vec{r}\,[1 + O(r/a)]\ .\end{aligned} \tag{4.2}$$

The equilibrium condition $\vec{E}(\vec{d}) + \vec{E}_{\rm ext} = 0$ then gives

$$\vec{d} = -\frac{3\pi\epsilon_0 a^3}{q}\vec{E}_{\rm ext}\ . \tag{4.3}$$

The charge of the proton is $-q$, so the dipole moment is

$$\vec{p} = (-q)\vec{d} = 3\pi\epsilon_0 a^3 \vec{E}_{\rm ext}\ . \tag{4.4}$$

The atomic polarizability is defined by $\vec{p} = \alpha\vec{E}_{ext}$, so we find

$$\boxed{\alpha = 3\pi\epsilon_0 a^3\ .} \tag{4.5}$$

Alternatively, we could have integrated Eq. (4.1) exactly to find the electric field, and then we could have expanded it in powers of (d/a).

PROBLEM 2: A DELTA FUNCTION IDENTITY AND THE FIELD OF AN ELECTRIC DIPOLE *(20 points plus 10 points extra credit):*

This problem explores the identity

$$\boxed{\partial_i\partial_j\left(\frac{1}{r}\right) = -\partial_i\left(\frac{\hat{r}_j}{r^2}\right) = -\partial_i\left(\frac{x_j}{r^3}\right) = \frac{3\hat{r}_i\hat{r}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{r})\ .} \tag{2.1}$$

(a) [5 pts] In Cartesian coordinates

$$\nabla^2\left(\frac{1}{r}\right) = \partial_j\partial_j\left(\frac{1}{r}\right)\ , \tag{2.2}$$

so we can use the identity (2.1) to write

$$\nabla^2\left(\frac{1}{r}\right) = \frac{3\hat{r}_j\hat{r}_j - \delta_{jj}}{r^3} - \frac{4\pi}{3}\delta_{jj}\,\delta^3(\vec{r})\ . \tag{2.3}$$

Then we use $\delta_{jj} = 3$ and $\hat{r}_j\hat{r}_j = 1$, so

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})\ .} \tag{2.4}$$

(b) [5 pts] Starting from the potential of an electric dipole,

$$V_{\rm dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}\ , \tag{2.5}$$

we find using the index notation that

$$\begin{aligned}E_{\rm dip}, i &= -\partial_i V_{\rm dip} = -\frac{p_j}{4\pi\epsilon_0}\partial_i\left(\frac{\hat{r}_j}{r^2}\right) \\ &= -\frac{p_j}{4\pi\epsilon_0}\left[\frac{\delta_{ij} - 3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{r})\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{r})\hat{r}_i - p_i}{r^3} - \frac{1}{3\epsilon_0}p_i\delta^3(\vec{r})\ .\end{aligned} \tag{2.6}$$

Converting back to the vector notation, this becomes

$$\boxed{\vec{E}_{\rm dip}(\vec{r}) = -\vec{\nabla}V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0}\vec{p}\,\delta^3(\vec{r})\ .} \tag{2.7}$$

(c) [10 pts] For $r \neq 0$, the calculation is straightforward. We first note that

$$\partial_i r = \partial_i (x_j x_j)^{1/2} = (x_k x_k)^{-1/2} x_j \partial_i x_j = \frac{x_j \delta_{ij}}{r} = \frac{x_i}{r} = \hat{\boldsymbol{r}}_i \;, \tag{2.15}$$

which should be familiar by now. Then

$$\partial_i \left(\frac{x_j}{r^3} \right) = \frac{\delta_{ij}}{r^3} - 3\frac{x_j}{r^4}\partial_i r = \boxed{\frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} \quad (\text{for } r \neq 0) \;.} \tag{2.16}$$

To find the δ-function, we integrate $\partial_i(\hat{\boldsymbol{r}}_j/r^2)$ over a small sphere $r < \epsilon$ about the origin. The contribution from Eq. (2.16) vanishes when integrated over all angles, but we will nonetheless find a contribution by integrating over the origin, which is encoded in the δ-function. In Griffiths' Problem 1.61(a), p. 56, we showed that

$$\int_{\mathcal{V}} (\vec{\boldsymbol{\nabla}} T) \mathrm{d}^3 x = \int_S T \mathrm{d}\vec{\boldsymbol{a}} \;, \tag{2.17}$$

where S is the surface bounding the volume $\mathcal{V}$. (We proved this identity in Problem 7(a) of Problem Set 1.) In index notation, this becomes

$$\int_{\mathcal{V}} \partial_i T \mathrm{d}^3 x = \int_S T \mathrm{d}a_i \;, \tag{2.18}$$

where $\mathrm{d}a_i$ is the i'th component of the surface area element vector $\mathrm{d}\vec{\boldsymbol{a}} = \mathrm{d}a\, \hat{\boldsymbol{n}}$, where $\mathrm{d}a$ is the infinitesimal area of the surface element and $\hat{\boldsymbol{n}}$ is the outward normal vector. For integration over a sphere, $\mathrm{d}\vec{\boldsymbol{a}} = r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \hat{\boldsymbol{r}}$. Now we let $T = (\hat{\boldsymbol{r}}_j/r^2)$, and we have

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3 x = \int_{r=\epsilon} \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \hat{\boldsymbol{r}}_i \, \mathrm{d}a = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi \;. \tag{2.19}$$

The last integral can be evaluated by brute force, using

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\, \hat{\boldsymbol{e}}_{\boldsymbol{x}} + \sin\theta\sin\phi\, \hat{\boldsymbol{e}}_{\boldsymbol{y}} + \cos\theta\, \hat{\boldsymbol{e}}_{\boldsymbol{z}} \;, \tag{2.20}$$

but it can be found easily if one recognizes that it must be rotationally invariant, and that the only rotationally invariant tensor is δ_{ij}. Thus it must have the form

$$\int \hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = A\, \delta_{ij} \;, \tag{2.21}$$

where we can determine A most easily by multiplying both sides by δ_{ij} and summing over the repeated indices. This gives $4\pi = 3A$, so $A = 4\pi/3$ and

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3 x = \int \hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = \frac{4\pi}{3}\delta_{ij} \;. \tag{2.22}$$

Since this result is obtained for arbitrarily small ϵ, we conclude that

$$\boxed{\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \quad \text{must contain a term} \quad \frac{4\pi}{3}\delta_{ij}\delta^3(\vec{\boldsymbol{r}}) \;.} \tag{2.23}$$

(d) [10 points extra credit] We are to start with the equation

$$LHS = -\lim_{\epsilon\to 0} \int_{r>\epsilon} \left\{ \partial_i \left[\varphi(\vec{\boldsymbol{r}}) \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \right] - \varphi(\vec{\boldsymbol{r}}) \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \right\} \mathrm{d}^3 x \;, \tag{2.24}$$

and show that it is equal to the RHS,

$$RHS = \lim_{\epsilon\to 0} \int_{r>\epsilon} \varphi(\vec{\boldsymbol{r}}) \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} \mathrm{d}^3 x + \frac{4\pi}{3}\delta_{ij}\, \varphi(\vec{0}) \;. \tag{2.25}$$

If we break up LHS into its two terms, $LHS = LHS_1 + LHS_2$, then the second term is easily evaluated using Eq. (2.16):

$$LHS_2 = \lim_{\epsilon\to 0} \int_{r>\epsilon} \varphi(\vec{\boldsymbol{r}}) \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3 x = \lim_{\epsilon\to 0} \int_{r>\epsilon} \varphi(\vec{\boldsymbol{r}}) \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} \mathrm{d}^3 x \;. \tag{2.26}$$

To integrate LHS_1, we again use Eq. (2.18) to convert the volume integral to a surface integral. This time, however, the volume of integration is the region $r > \epsilon$, rather than $r < \epsilon$. But as a surface integral the calculation in part (d) and the current calculation connect, because here the integral over the surface at infinity vanishes, since the test function $\varphi(\vec{\boldsymbol{r}})$ is required to fall off rapidly at large r. The fact that $r = \epsilon$ is the inner surface causes $\mathrm{d}\vec{\boldsymbol{a}}$ to point inward, but the minus sign in Eq. (2.24) reverses it. So,

$$LHS_1 = \lim_{\epsilon\to 0} \int_{r=\epsilon} \varphi(\vec{\boldsymbol{r}}) \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \hat{\boldsymbol{r}}_i \, \mathrm{d}a \;. \tag{2.27}$$

The calculation differs from part (c) by the presence of $\varphi(\vec{\boldsymbol{r}})$, but $\varphi(\vec{\boldsymbol{r}}) \to \varphi(\vec{0})$ as $\epsilon \to 0$. Evaluating the integral as in part (d), we find

$$LHS_1 = \frac{4\pi}{3}\delta_{ij}\varphi(\vec{0}) \;, \tag{2.28}$$

which is exactly what we were trying to show, so that $LHS = RHS$.

PROBLEM 3: FORCES, TORQUES, AND ANGULAR MOMENTUM CONSERVATION WITH DIPOLE-DIPOLE INTERACTIONS *(15 points)*

(a) [10 pts] The force $\vec{\boldsymbol{F}}_2$ on dipole 2, caused by dipole 1, can be found from $\vec{\boldsymbol{F}}_2 = (\vec{\boldsymbol{p}}_2.\vec{\nabla})\vec{\boldsymbol{E}}_1$. The electric field produced by dipole 1 is given as

$$\vec{\boldsymbol{E}}_1 = \frac{1}{4\pi\epsilon_0}\frac{(3\vec{\boldsymbol{p}}_1\cdot\hat{r})\hat{r} - \vec{\boldsymbol{p}}_1}{r^3} = -\frac{p_1}{4\pi\epsilon_0 x^3}\hat{e}_z, \tag{3.1}$$

where we used $\vec{\boldsymbol{p}}_1 = p_1\hat{e}_z$ and $\hat{r} = \hat{e}_x$. While this expression is valid only for points along the x-axis, the force $\vec{\boldsymbol{F}}_2 = (\vec{\boldsymbol{p}}_2.\vec{\nabla})\vec{\boldsymbol{E}}_1$ depends only on the x derivative of $\vec{\boldsymbol{E}}$, since $\vec{\boldsymbol{p}}_2 = p_2\hat{\boldsymbol{e}}_{\boldsymbol{x}}$ points in the x direction. Thus this expression is sufficient, and the force $\vec{\boldsymbol{F}}_2$ follows as

$$\begin{aligned}\vec{\boldsymbol{F}}_2 &= (\vec{\boldsymbol{p}}_2\cdot\vec{\nabla})\vec{E}_1 = p_2\frac{\partial\vec{\boldsymbol{E}}_1}{\partial x} = -\frac{p_1p_2}{4\pi\epsilon_0}\left[\frac{d}{dx}\left(\frac{1}{x^3}\right)\right]\hat{e}_z \\ &= \boxed{\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\hat{e}_z\ .}\end{aligned} \tag{3.2}$$

To calculate the force $\vec{\boldsymbol{F}}_1$ on dipole 1 caused by dipole 2, we shift the origin to dipole 2 and use $\vec{\boldsymbol{F}}_1 = (\vec{\boldsymbol{p}}_1\cdot\vec{\nabla})\vec{\boldsymbol{E}}_2 = p_1\frac{\partial\vec{\boldsymbol{E}}_2}{\partial z}$, where $\vec{\boldsymbol{E}}_2$ is the electric field $\vec{\boldsymbol{p}}_2$ produces at the location of dipole 1.

$$\vec{\boldsymbol{E}}_2 = \frac{1}{4\pi\epsilon_0}\frac{(3\vec{\boldsymbol{p}}_2\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}} - \vec{\boldsymbol{p}}_2|\vec{\boldsymbol{r}}|^2}{|\vec{\boldsymbol{r}}|^5}, \tag{3.3}$$

where $\vec{\boldsymbol{r}} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ is the position vector from dipole 2 to dipole 1. Then we will set $(x,y,z) = (-r,0,0)$ for dipole 1 when we evaluate the force. Hence,

$$\begin{aligned}\vec{\boldsymbol{E}}_2 &= \frac{p_2}{4\pi\epsilon_0}\left[\frac{3x(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z) - \hat{e}_x(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right], \\ \frac{\partial\vec{\boldsymbol{E}}_2}{\partial z} &= \frac{p_2}{4\pi\epsilon_0}\left[-\frac{5}{2}\frac{2z}{|\vec{r}|^7}\left[3x(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z) - \hat{e}_x(x^2+y^2+z^2)\right] + \frac{1}{|\vec{\boldsymbol{r}}|^5}(3x\hat{e}_z - 2z\hat{e}_x)\right], \\ \vec{\boldsymbol{F}}_1 &= p_1\frac{\partial\vec{\boldsymbol{E}}_2}{\partial z}\bigg|_{(-r,0,0)} = \frac{p_1p_2}{4\pi\epsilon_0}\frac{1}{r^5}3(-r)\hat{e}_z = \boxed{-\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\hat{e}_z\ .}\end{aligned} \tag{3.4}$$

These forces are equal in magnitude and opposite in direction. However the forces do not point along the line joining two dipoles.

(b) [5 pts] Considering dipole 1 at the origin, we can write positions as $\vec{\boldsymbol{r}}_1 = 0$ and $\vec{\boldsymbol{r}}_2 = r\hat{e}_x$. The torque about the origin follows as

$$\begin{aligned}\vec{\tau} &= \vec{\boldsymbol{r}}_1\times\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{r}}_2\times\vec{\boldsymbol{F}}_2, \\ &= r\hat{e}_x\times\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\hat{e}_z = -\frac{3p_1p_2}{4\pi\epsilon_0 r^3}\hat{e}_y.\end{aligned} \tag{3.5}$$

Including the torques that each dipole experiences due to the field of the other,

$$\begin{aligned}\vec{\tau}_{total} &= \vec{\boldsymbol{r}}_1\times\vec{\boldsymbol{F}}_1 + \vec{r}_2\times\vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{p}}_1\times\vec{\boldsymbol{E}}_2 + \vec{\boldsymbol{p}}_2\times\vec{\boldsymbol{E}}_1, \\ &= -\frac{3p_1p_2}{4\pi\epsilon_0 r^3}\hat{e}_y + p_1\hat{e}_z\times\frac{p_2}{4\pi\epsilon_0 r^3}2\hat{e}_x + p_2\hat{e}_x\times\left(-\frac{p_1}{4\pi\epsilon_0 r^3}\hat{e}_z\right), \\ &= -\frac{3p_1p_2}{4\pi\epsilon_0 r^3}\hat{e}_y + \frac{2p_1p_2}{4\pi\epsilon_0 r^3}\hat{e}_y + \frac{p_1p_2}{4\pi\epsilon_0 r^3}\hat{e}_y, \\ &= 0.\end{aligned}$$

PROBLEM 4: A POINT CHARGE AT THE CENTER OF A SPHERICAL DIELECTRIC *(10 points)*

The displacement can be found using Gauss' law.

$$\oint\vec{\boldsymbol{D}}\cdot d\vec{\boldsymbol{a}} = Q_{fenc} \quad\Longrightarrow\quad \vec{\boldsymbol{D}} = \frac{q}{4\pi r^2}\hat{r}. \tag{4.1}$$

For a linear media the electric field is,

$$\vec{\boldsymbol{E}} = \frac{1}{\epsilon}\vec{\boldsymbol{D}} = \frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{r}}{r^2}, \tag{4.2}$$

and the polarization vector is,

$$\vec{\boldsymbol{P}} = \epsilon_0\chi_e\vec{\boldsymbol{E}} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{r}}{r^2}. \tag{4.3}$$

The bound charge density ρ_b within the sphere and the surface charge density σ_b on its outer edge are given by,

$$\begin{aligned}\rho_b &= -\vec{\nabla}\cdot\vec{\boldsymbol{P}} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) = -\frac{q\chi_e}{(1+\chi_e)}\delta^3(\vec{\boldsymbol{r}}), \\ \sigma_b &= \vec{\boldsymbol{P}}\cdot\hat{r} = \frac{q\chi_e}{4\pi(1+\chi_e)R^2}.\end{aligned} \tag{4.4}$$

The total bound charge on the surface is,

$$Q_{surface} = \sigma_b(4\pi R^2) = q\frac{\chi_e}{1+\chi_e}. \tag{4.5}$$

The compensating negative charge is at the center:

$$\int\rho_b d\tau = -\frac{q\chi_e}{(1+\chi_e)}\int\delta^3(\vec{\boldsymbol{r}})d\tau = -q\frac{\chi_e}{1+\chi_e}. \tag{4.6}$$

PROBLEM 5: SPHERE OF UNIFORM DIELECTRIC MATERIAL IN AN OTHERWISE UNIFORM ELECTRIC FIELD *(5 points)*

Griffiths, Example 4.7 (p. 193).

We want to solve the Laplace's equation for the electric potentials $V_{\rm in}(r,\theta)$ inside the sphere ($r < R$), and $V_{\rm out}(r,\theta)$ outside the sphere ($r \geq R$) with the following boundary conditions:

$$\begin{array}{lll} \text{(i)} & V_{\rm in} = V_{\rm out}\ , & \text{at } r = R\ , \\ \text{(ii)} & \epsilon \dfrac{\partial V_{\rm in}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\rm out}}{\partial r}\ , & \text{at } r = R\ , \\ \text{(iii)} & V_{\rm out} \to -E_0 r\cos\theta\ , & \text{for } r \gg R\ . \end{array}$$

Since our system has azimuthal symmetry, the electric potentials can be decomposed into a series of Legendre polynomials:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\ .$$

We use separate expansions on the inside and on the outside, because $\nabla^2 V \neq 0$ at $r = R$, where there is a bound surface charge. For the inside, all the B_ℓ's must vanish, because otherwise V would be singular at $r = 0$. For the outside, all the A_ℓ's must vanish to avoid having V blow up at infinity, except that A_1 must equal $-E_0$ to comply with boundary condition (iii). Thus,

$$V_{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{\rm out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta)\ .$$

Since we explicitly set $V = 0$ at $z = 0$, $V_{\rm in}(r,\theta)$ must be equal zero at $r = 0$ and:

$$A_0 = 0\ .$$

Boundary condition (i) requires that

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{R^{\ell+1}}P_\ell(\cos\theta)\ .$$

As we remember, Legendre polynomials are orthogonal to each other. Thus, if we multiply both sides of this equation by $P_\ell(\cos\theta)\sin\theta$ and integrate over θ from zero to π, we obtain a relation between A_ℓ and B_ℓ:

$$A_1 R = -E_0 R + \frac{B_1}{R^2}, \tag{5.1a}$$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}, \quad \text{for} \quad \ell \neq 1\ , \tag{5.1b}$$

which also allows us to find the value of B_0:

$$B_0 = \frac{A_0}{R} = 0\ .$$

Boundary condition (ii) gives us

$$\epsilon_r \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -E_0\cos\theta - \sum_{\ell=0}^{\infty}\frac{(\ell+1)B_\ell}{R^{\ell+2}}P_\ell(\cos\theta)\ ,$$

which, using the orthogonality of Legende polynomials, gives us

$$\epsilon_r A_1 = -E_0 - \frac{2B_1}{R^3}\ , \tag{5.2a}$$

$$\epsilon_r \ell A_\ell R^{\ell-1} = -\frac{(\ell+1)B_\ell}{R^{\ell+2}}, \quad \text{for} \quad \ell > 1\ . \tag{5.2b}$$

Then Eqs. (5.1a) and (5.2a) can be solved for A_1 and B_1, and Eqs. (5.1b) and (5.2b) can be solved A_ℓ and B_ℓ for $\ell > 1$, giving

$$A_1 = -\frac{3}{\epsilon_r+2}E_0, \qquad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2}R^3 E_0\ ,$$

$$A_\ell = B_\ell = 0, \quad \text{for} \quad \ell > 1\ .$$

These coefficients define the electric potentials inside and outside of the sphere:

$$\boxed{\begin{aligned} V_{\rm in}(r,\theta) &= -\frac{3E_0}{\epsilon_r+2}r\cos\theta = -\frac{3}{\epsilon_r+2}E_0 z\ , \\ V_{\rm out}(r,\theta) &= \left[\frac{\epsilon_r-1}{\epsilon_r+2}\left(\frac{R}{r}\right)^3 - 1\right]E_0 z\ . \end{aligned}}$$

Interestingly, the field inside the sphere is uniform:

$$\vec{\boldsymbol{E}} = \frac{3}{\epsilon_r+2}\vec{\boldsymbol{E}}_0\ .$$

PROBLEM 6: CAVITIES IN DIELECTRIC MEDIA *(15 points)*

Griffiths, Problem 4.16 (p. 184).

(a) We are given that the field inside a large piece of dielectric is $\vec{E}_0$, so that the electric displacement is $\vec{D}_0 = \epsilon_0\vec{E}_0 + \vec{P}$. Now a small spherical cavity is hollowed out of the material. We can think of this system as the superposition of a uniform dielectric with a sphere of uniform polarization $\vec{P}_{\rm sphere} = -\vec{P}$. Griffiths' wording was not very clear, but the hint makes it clear that we are to think of the dielectric as an electret, with a fixed polarization $\vec{P}$, rather than a linear dielectric that responds to the local electric field. Eq. (4.14) in Griffiths tells us that the electric field everywhere inside a uniform sphere of polarization $\vec{P}_{\rm sphere}$ is given by $\vec{E}_{in} = -\vec{P}_{\rm sphere}/3\epsilon_0$. For $P_{\rm sphere} = -\vec{P}$, the field caused by the sphere will be $\vec{E}_{in} = \vec{P}/3\epsilon_0$. The total electric field inside the cavity is then

$$\boxed{\vec{E}_{\rm cav} = \vec{E}_0 + \frac{1}{3\epsilon_0}\vec{P}\ ,} \tag{6.1}$$

and the displacement field inside the cavity is found by multiplying by ϵ_0:

$$\vec{D}_{\rm cav} = \epsilon_0\vec{E} = \epsilon_0\vec{E}_0 + \frac{1}{3}\vec{P} = (\vec{D}_0 - \vec{P}) + \frac{1}{3}\vec{P} = \boxed{\vec{D}_0 - \frac{2}{3}\vec{P}.} \tag{6.2}$$

(b) Using the same procedure as in part (a), we treat the needle-shaped cavity as a superposition of a needle-shaped rod with polarization $-\vec{P}$ and a large piece of uniform dielectric. At the two ends of the needle there will be (bound) surface charges, but these are small and far away from most of the needle, so the electric field is nearly constant throughout the region. The total electric field and the displacement in the cavity are therefore

$$\boxed{\vec{E}_{\rm cav} = \vec{E}_0} \quad \text{and} \quad \boxed{\vec{D}_{\rm cav} = \epsilon_0\vec{E} = \vec{D}_0 - \vec{P}\ .} \tag{6.3}$$

We should note that the electric field and displacement vectors satisfy the standard boundary conditions. Along the sides of the needle the parallel component of the electric field is continuous, $\vec{E}_{\rm cav} = \vec{E}_0$, and the normal component of $\vec{D}$ is also continuous, as it must be in the absence of free surface charges — it vanishes on both sides. Note, however, that the simple solution of Eq. (6.3) does not satisfy the right boundary conditions at the top and bottom of the needle; according to Eq. (6.3) the normal component of $\vec{D}$ undergoes a jump at the top and bottom of

the needle, even though there are no free surface charges. Thus, our solution is a good approximation for needles that are much thinner than they are high, but the actual fields would differ from this approximation by fringing effects at the ends of the needle.

(c) Using the same procedure in part (a), we treat the wafer-shaped cavity as the superposition of a large uniform dielectric and a wafer-shaped dielectric with polarization $\vec{P}_{\rm wafer} = -\vec{P}$. To find the (bound) surface charge density of the wafer we use $\sigma_b = \vec{P}_{\rm wafer}\cdot\hat{n}$ where $\hat{n}$ is the unit normal vector pointing outward from the surface of the wafer. These surface charges act just like a parallel plate capacitor. If $\vec{P}$ points upward, then $\vec{P}_{\rm wafer}$ points downward, and then σ_b on the upper plate is $-|\vec{P}|$, and σ_b on the lower plate is $|\vec{P}|$. The electric field produced by the plates has magnitude $|\vec{P}|/\epsilon_0$, directed upward, and therefore is equal vectorially to $\vec{P}/\epsilon_0$. The total electric field and displacement, inside the wafer-shaped cavity, are therefore

$$\boxed{\vec{E}_{\rm cav} = \vec{E}_0 + \frac{1}{\epsilon_0}\vec{P}} \quad \text{and} \quad \boxed{\vec{D} = \epsilon_0\vec{E}_0 + \vec{P} = \vec{D}_0\ .} \tag{6.4}$$

Note that this solution obeys the standard boundary conditions along the top and bottom surfaces of the wafer: the parallel component of $\vec{E}$ is continuous (zero on both sides), and the normal component of $\vec{D}$ is continuous, as it should be when there are no free charges. However, once again the simple solution that we write is not exact, as it violates the correct boundary conditions at the edges of the wafer. At the edges the solution shows a jump in the parallel component of $\vec{E}$ by $\vec{P}/\epsilon$, so the exact solution much show fringing fields that are not included in our approximation.

PROBLEM 7: FORCE ON A DIELECTRIC SLAB PART WAY INSIDE A CIRCULAR CAPACITOR *(15 points)*

Problem held over to Problem Set 7.

PROBLEM 8: A DIPOLE ON A CIRCULAR TRACK *(10 points)*

Griffiths, Problem 4.31, p. 206.

The force on an ideal dipole is given by Griffiths' Eq. (4.5),

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}\ .$$

In component notation, this is

$$F_i = p_j\frac{\partial}{\partial x_j}E_i\ ,$$

where in this case

$$E_i = \frac{Q}{4\pi\epsilon_0}\frac{x_i}{(x^2)^{3/2}} .$$

Then, using $r \equiv (x^2)^{1/2}$ when convenient,

$$\begin{aligned} F_i &= \frac{Q}{4\pi\epsilon_0} p_j \frac{\partial}{\partial x_j}\left[\frac{x_i}{(x^2)^{3/2}}\right] = \frac{Q}{4\pi\epsilon_0} p_j \left[\frac{\delta_{ij}}{(x^2)^{3/2}} - \frac{3}{2}\frac{x_i}{(x^2)^{5/2}}\frac{\partial(x^2)}{\partial x_j}\right] \\ &= \frac{Q}{4\pi\epsilon_0} p_j \left[\frac{\delta_{ij}}{(x^2)^{3/2}} - \frac{3}{2}\frac{x_i}{(x^2)^{5/2}} 2x_j\right] = \frac{Q}{4\pi\epsilon_0}\frac{p_j}{r^3}\left[\delta_{ij} - 3\frac{x_i x_j}{r^2}\right] \\ &= \frac{Q}{4\pi\epsilon_0 r^3}\left[p_i - 3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\hat{r}_i\right] . \end{aligned}$$

In this case $\vec{\boldsymbol{p}}$ is perpendicular to $\hat{\boldsymbol{r}}$, since $\vec{\boldsymbol{p}}$ is tangent to the track, so $\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}} = 0$ and the formula reduces to

$$\boxed{\vec{\boldsymbol{F}} = \frac{Q\vec{\boldsymbol{p}}}{4\pi\epsilon_0 r^3}} , \tag{8.1}$$

as we were told to show. That is the straightforward part of the problem.

We were also asked to explain why this is not a perpetual motion machine, which is the subtle part. I (Alan Guth) would begin by saying that the question has some ambiguity, because it does not tell us how the dipole is constrained to point tangent to the circle. The simplest answer to this question is that each charge runs along the track, so they are each constrained to be at radius R from the center (where R is the radius of the track), as shown in Fig. 8.1:

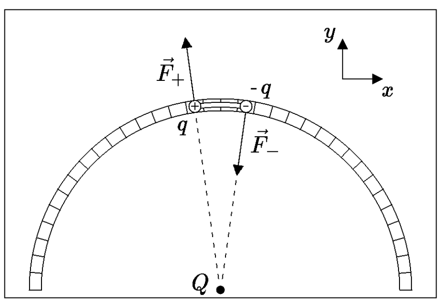

Figure 8.1: A dipole with orientation constrained by the track.

The diagram clearly shows how the force described by Eq. (8.1) arises: while the force $\vec{\boldsymbol{F}}_+$ on the positive charge and the force $\vec{\boldsymbol{F}}_-$ on the negative charge are both radial, they

are radial at different locations. Thus their x-components point in the same direction, giving a net electrostatic force in the $-x$ direction.

However, if the track is to constrain each charge to stay at radius R, then the track must exert radial forces on each charge. If we start by considering the dipole to be at rest, then the complete force diagram for the dipole looks like Fig. 8.2, where the constraint forces are labeled $\vec{\boldsymbol{F}}_{c,+}$ and $\vec{\boldsymbol{F}}_{c,-}$:

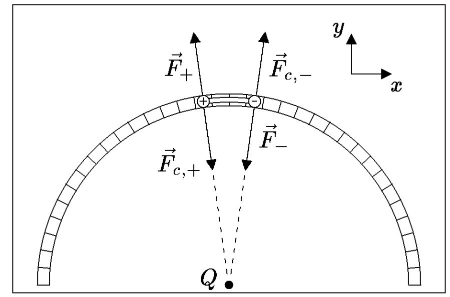

Figure 8.2: The complete force diagram for the dipole with orientation constrained by the track.

Each constraint force is shown as being radial, since any component of the constraint force tangential to the interface would be called friction, and we are told that the track is frictionless. As long as each force is radial, then to keep the charges at radius R the constraint forces must each be equal to the negative of the corresponding electrostatic force. Then one can see from the diagram that the constraint forces also give a net horizontal force, in this case in the $+x$ direction. The horizontal force from the constraints cancels the electrostatic force on the dipole, resulting in no tangential force and hence no tangential acceleration whatever. The cancelation can be seen much more simply by noting that the total force on each charge is exactly zero.

If the dipole is moving the situation is a little more complicated, because now the constraint forces are no longer equal in magnitude to the electrostatic forces. In this case, if we treat the connecting rod as massless, there must be an additional force $\vec{\boldsymbol{F}}_{q,\text{add}}$ acting on each charge q to supply the acceleration associated with the circular motion:

$$\vec{\boldsymbol{F}}_{q,\text{add}} = -\frac{m_q v^2}{R}\hat{\boldsymbol{r}} , \tag{8.2}$$

where m_q is the mass of the charge q and v is the speed with which both charges are moving. As long as $m_+ = m_-$, then these additional forces are equal in magnitude, and the left-right symmetry of Fig. 8.2 guarantees the the x components of these additional forces will cancel. Again we conclude that there will be no tangential acceleration, which means that the dipole can circle the track indefinitely, as long as the zero-friction idealization is valid.

But what if the two masses are not equal? In that case the two additional constraint forces $\vec{F}_{q,\text{add}}$ will again be radial, but they will have different magnitudes. From Fig. 8.2, we can see that the x-components of the two additional forces will no longer cancel, so it looks like we are again faced with a paradox of a seemingly nonzero tangential force. How do we reason our way out of the problem this time? The answer is that we have to remember that the total force on the dipole, which is a rigid body, is equal to the mass of the dipole times the acceleration of its center of mass. In this case, the center of mass is not at the center of the dipole. A short calculation shows that the sum of the forces in Eq. (8.2) always produces a force in the same direction as a vector from the center of the track to the center of mass of the dipole. Thus, the net force on the center of mass is radial, so it will undergo no tangential acceleration.

I said at the beginning that there is more than one method by which the orientation of the dipole might be constrained. Another choice would be to use an external "torque motor" to control the orientation. In this case one would imagine attaching the dipole to the track in a way that prevents the track from exerting any torque on the dipole. For example, the dipole can have a frictionless pivot at its center, which is attached to the track, as in Fig. 8.3:

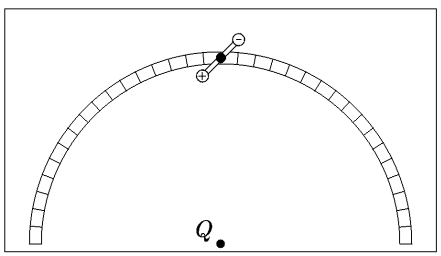

Figure 8.3: A dipole attached to the track via a frictionless pivot.

The "torque motor" would be attached rigidly to the ground, and would be connected to the dipole by means of a rod, perpendicular to the page in Figure 8.3. The orientation of the dipole would then be constrained to be tangential, with special care taken to arrange for the connecting rod to exert the necessary torque, while exerting no net force on the dipole. In this case the constraint force acts only on the pivot, and therefore must be radial, so it cannot cancel the tangential electrostatic force. Thus there will be a net tangential force, and the dipole will accelerate around the track without limit. However, in this case we can easily identify the source of the energy. There is no perpetual motion machine, but instead the energy is supplied by the torque motor. The dipole will continue to accelerate as long as the torque motor can keep up.

Finally, I'd like to explain where I think Griffiths went wrong in his solution. Griffiths wrote: "Indeed, if the dipole has the orientation indicated in the figure, and is moving in the $\hat{\phi}$ direction, the torque exerted by Q is clockwise, whereas the rotation is counterclockwise, so these constraint forces must actually be *larger* than the forces exerted by Q ...." Somehow it looks like Griffiths has overlooked the basic ideas of Newtonian physics. The constraint torque need not be larger than the electrostatic torque. If the two are equal in magnitude (as is the case in the analysis above for the case where the orientation constraint is imposed by the track), then the dipole can rotate at any fixed angular velocity, of either sign.

P.S.: The new edition of Griffiths' instructor's solutions manual describes the solution given here, including all three figures.