8.07 Lecture Slides 22
December 2, 2019

ELECTRODYNAMICS, CONSERVATION LAWS, and ~~ELECTROMAGNETIC WAVES~~

(Next time)

Review of Lecture 21

The Complete Maxwell Equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

- ☆ $-\dfrac{\partial\vec{B}}{\partial t}$ added by Faraday. It restores Galilean relativity: if the magnetic flux through a loop changes, with this addition it does not matter whether $\vec{B}$ changes, or the loop moved.
- ☆ $\mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t}$ added by Maxwell to make RHS divergenceless.

Review of Lecture 21

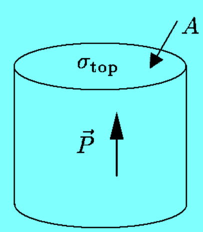
The Complete Maxwell Equations In Matter

There is one more bound charge effect: if $\vec{P}$ changes with time, currents flow.

$$\sigma_{\rm top} = \vec{P}\cdot\hat{n}\ ,$$

where $\hat{n}$ is the upward normal, the direction of $\vec{P}$. So $Q = PA$, where $P = |\vec{P}|$, so

$$I = \frac{dQ}{dt} = \frac{dP}{dt}A \quad\Longrightarrow\quad |\vec{J}| = \frac{dP}{dt}\ .$$



So

$$\boxed{\vec{J}_b = \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\ .}$$

Review of Lecture 21

If polarized material moves, there are other terms, but we'll assume that all our matter is static.

Starting with

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

and using

$$\rho=\rho_{\rm free}-\vec{\nabla}\cdot\vec{P}\ , \quad \vec{J}=\vec{J}_{\rm free}+\vec{\nabla}\times\vec{M}+\frac{\partial\vec{P}}{\partial t}\ ,$$

and the definitions

$$\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M}\ , \quad \vec{D}\equiv\epsilon_0\vec{E}+\vec{P}\ ,$$

one finds

$$\boxed{\begin{array}{ll}\vec{\nabla}\cdot\vec{D}=\rho_{\rm free} & \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\cdot\vec{B}=0 & \vec{\nabla}\times\vec{H}=\vec{J}_{\rm free}+\dfrac{\partial\vec{D}}{\partial t}\end{array}}$$

Conductivity: Ohm's Law

$$\boxed{\vec{J}=\sigma(\vec{E}+\vec{v}\times\vec{B})\ ,}$$

where $\vec{v}$ is the velocity of the medium, and

$$\boxed{\sigma\equiv\text{conductivity}\ ,}$$

$$\boxed{\rho\equiv\frac{1}{\sigma}=\text{resistivity}.}$$

Units:

$$\left[\vec{J}\right]=\frac{\rm C}{\rm s\cdot m^2}\ , \quad \left[\vec{E}\right]=\frac{\rm N}{\rm C} \quad\Longrightarrow\quad [\sigma]=\frac{\rm C}{\rm s\cdot m^2}\frac{\rm C}{\rm N}=\frac{\rm C^2}{\rm J\,m\,s}$$

$$1\ \text{Ohm}\ (\Omega)\equiv\frac{1\ \text{volt}}{\text{ampere}}=\frac{\rm V}{\rm A}=\frac{\rm J/C}{\rm C/s}=\frac{\rm J\,s}{\rm C^2} \quad\Longrightarrow\quad [\rho]=\Omega\cdot{\rm m}\ .$$

Drude Model of Resistivity

This is a very crude classical model. Assume:

1) Interatomic distance $=\lambda$.
2) Thermal velocity of electrons is given by
$$\frac{1}{2}m_e v_{\rm th}^2=\frac{3}{2}kT\ ,$$
where k = Boltzmann constant, T = temperature. For $T=300\,{\rm K}$, $v_{\rm th}=1.17\times10^7$ cm/s $=2.6\times10^5$ mile/hour.
3) $\Delta t=\lambda/v_{\rm th}$ = mean time between collisions,
4) $\langle\vec{v}\rangle=\frac{1}{2}\vec{a}\,\Delta t$, where $\vec{a}=q\vec{E}/m_e$.
5) Let f = number of free electrons per atom or molecule.
6) Let n = number density of atoms or molecules. Then

$$\boxed{\vec{J}=fnq\,\langle\vec{v}\rangle=\left(\frac{nf\lambda q^2}{2mv_{\rm th}}\right)\vec{E} \quad\Longrightarrow\quad \sigma=\frac{nf\lambda q^2}{2mv_{\rm th}}\ .}$$

Numerically, for copper

$$\begin{aligned} n&=\frac{\text{density}}{\text{atomic weight}}(\text{Avogadro's number})\\ &=\frac{(8.96\ {\rm g/cm^3})}{(63.5\ {\rm g/mole})}\left(6.022\times10^{23}\frac{\rm atoms}{\rm mole}\right)\\ &=8.50\times10^{22}\,/\,{\rm cm^3}\ .\end{aligned}$$

$$\lambda=\frac{1}{n^{1/3}}=2.27\times10^{-8}\ {\rm cm}.$$

Drude model gives

$$\rho_{\rm Drude}=4.3\times10^{-7}\ \Omega\text{-m}\ ,$$

while in reality

$$\rho_{\rm Copper}=1.68\times10^{-8}\ \Omega\text{-m}.$$

So copper conducts about 25 times better than the Drude model predicts.

Typical Values of Resistivity

Griffiths, Table 7.1, resistivities in Ω-m:

Conductors:

- Silver: 1.59×10^{-8}
- Iron: 9.61×10^{-8}
- Graphite: 1.6×10^{-5}

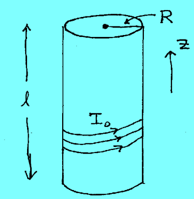
Semiconductors:

- Sea water: 0.2
- Germanium: 0.46
- Silicon: 2500

Insulators:

- Water (pure): 8.3×10^{3}
- Glass: $10^{9} - 10^{14}$
- Teflon: $10^{22} - 10^{24}$

Faraday's Law and Inductors



Consider a long thin circular solenoid, of n turns per unit length, length ℓ, radius R, with $\ell \gg R$.

Inside $\vec{B} \approx B_0 \hat{z}$, outside $\vec{B} \approx 0$, since the return flux is spread over an area that grows with ℓ.

Ampere's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 E_{\rm enc} \quad \Longrightarrow \quad \vec{B} = n\mu_0 I_0 \hat{z}$.

Magnetic flux $\Phi_B = \pi R^2 (n\mu_0 I_0)$ per turn, so

$$\Phi_B = \pi R^2 n^2 \mu_0 I_0 \ell = \mathcal{V} n^2 \mu_0 I_0 \ ,$$

where $\mathcal{V}$ = volume $= \pi R^2 \ell$.

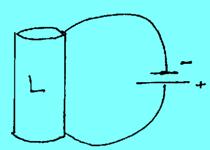
Inductance L is defined so that $\Phi_B = I_0 L$, and then

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt} = -L\frac{dI_0}{dt}}$$

For this system, $\boxed{L = \mu_0 n^2 \mathcal{V}}$

Inductors in Circuits

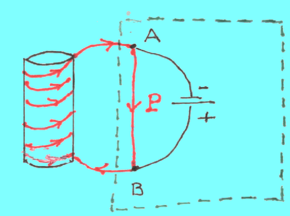
Suppose an inducting solenoid is connected to a battery:



Problem: Since $\vec{B}$ is changing, $\vec{\nabla} \times \vec{E} \neq 0$, so $V(\vec{r})$ is not well-defined.

Solution: If $\vec{B} \approx 0$ outside the inductor, there is a simple solution. ($\vec{B}$ outside is typically small because the return flux spreads out, especially if the length is much larger than the radius. Also, a large number of turns leads to signficant inductance, even if the field inside is not very strong.)

If $\vec{B}$ can be ignored outside the inductor, consider the path shown in red as P, and the region surrounded by the dashed line:



Inside the dashed line, $V(\vec{r})$ can be defined, since $\vec{\nabla} \times \vec{B} \approx 0$, and the region is simply connected (all loops are contractible).

The loop P runs along the center of the wire, except for segment between A and B. If the wire is a good conductor, $\vec{E}$ will vanish inside the wire, so

$$\varepsilon = \oint_P \vec{E} \cdot d\vec{\ell} = \int_A^B \vec{E} \cdot d\vec{\ell} = \frac{\partial \Phi_B}{\partial t} = L\frac{dI}{dt} \ .$$

So

$$\boxed{V(A) - V(B) = -L\frac{dI}{dt} \ .}$$

Boundary Conditions from Maxwell Equations

$$\vec{\nabla}\cdot\vec{D}=\rho_{\text{free}} \quad\Longrightarrow\quad D_1^{\perp}-D_2^{\perp}=\sigma_{\text{free}}\ ,$$
$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \quad\Longrightarrow\quad E_1^{\perp}-E_2^{\perp}=\frac{\sigma}{\epsilon_0}\ ,$$
$$\vec{\nabla}\cdot\vec{B}=0 \quad\Longrightarrow\quad B_1^{\perp}-B_2^{\perp}=0\ ,$$
$$\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M} \quad\Longrightarrow\quad H_1^{\perp}-H_2^{\perp}=M_2^{\perp}-M_1^{\perp}\ .$$

where 1 = outside material, 2 = inside material.

$$\vec{\nabla}\cdot\vec{D}=\rho_{\text{free}} \quad\Longrightarrow\quad D_1^{\perp}-D_2^{\perp}=\sigma_{\text{free}}\ ,$$
$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \quad\Longrightarrow\quad E_1^{\perp}-E_2^{\perp}=\frac{\sigma}{\epsilon_0}\ ,$$
$$\vec{\nabla}\cdot\vec{B}=0 \quad\Longrightarrow\quad B_1^{\perp}-B_2^{\perp}=0\ ,$$
$$\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M} \quad\Longrightarrow\quad H_1^{\perp}-H_2^{\perp}=M_2^{\perp}-M_1^{\perp}\ .$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\neq\infty \quad\Longrightarrow\quad \vec{E}_1^{\parallel}-\vec{E}_2^{\parallel}=0\ ,$$
$$\vec{D}=\epsilon_0\vec{E}+\vec{P} \quad\Longrightarrow\quad \vec{D}_1^{\parallel}-\vec{D}_2^{\parallel}=\vec{P}_1^{\parallel}-\vec{P}_2^{\parallel}\ ,$$
$$\vec{\nabla}\times\vec{H}=\vec{J}_{\text{free}}+\frac{\partial\vec{D}}{\partial t} \quad\Longrightarrow\quad \vec{H}_1^{\parallel}-\vec{H}_2^{\parallel}=-\hat{n}\times\vec{K}_{\text{free}}\ ,$$
$$\vec{\nabla}\times\vec{B}-\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad\Longrightarrow\quad \vec{B}_1^{\parallel}-\vec{B}_2^{\parallel}=-\mu_0\hat{n}\times\vec{K}\ ,$$

where $\hat{n}$ = unit outward vector, $\vec{K}$ = surface current density.

Conservation of Energy: Poynting's Theorem

Conjecture: the electromagnetic energy stored in a volume $\mathcal{V}$ is given by

$$\boxed{U_{\text{EM},\mathcal{V}}=\frac{1}{2}\int_{\mathcal{V}}\left[\epsilon_0|\vec{E}|^2+\frac{1}{\mu_0}|\vec{B}|^2\right]\mathrm{d}^3x,}$$

which is what we found by looking at the energy stored in capacitors and in solenoids. If so, then

$$\frac{\mathrm{d}U_{\text{EM}}}{\mathrm{d}t}=-P_{\text{particles}}-P_{\text{flux}}\ ,$$

where $P_{\text{particles}}$ is the power transferred to charged particles, and P_{flux} is the power transmitted electromagnetically through the boundary S of $\mathcal{V}$. We assume that no particles cross the boundary.

$$\begin{aligned}P_{\text{particles}}&=\sum_n\vec{F}_n\cdot\vec{v}_n\\&=\sum_n q_n(\vec{E}_n+\vec{v}_n\times\vec{B}_n)\cdot\vec{v}_n\\&=\sum_n q_n\vec{E}_n\cdot\vec{v}_n\ .\end{aligned}$$

For continuous matter, $q_n\to\rho\,\mathrm{d}^3x$, so

$$\boxed{P_{\text{particles}}=\int_{\mathcal{V}}\rho\vec{v}\cdot\vec{E}\,\mathrm{d}^3x=\int_{\mathcal{V}}\vec{J}\cdot\vec{E}\ ,\mathrm{d}^3x\ .}$$

If the original conjecture is true,

$$\begin{aligned} P_{\text{flux}} &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\left[\epsilon_0|\vec{\boldsymbol{E}}|^2+\frac{1}{\mu_0}|\vec{\boldsymbol{B}}|^2\right]\mathrm{d}^3x-\int_{\mathcal{V}}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{E}}\,\mathrm{d}^3x \\ &= -\int_{\mathcal{V}}\mathrm{d}^3x\left[\epsilon_0\vec{\boldsymbol{E}}\cdot\frac{\partial\vec{\boldsymbol{E}}}{\partial t}+\frac{1}{\mu_0}\vec{\boldsymbol{B}}\cdot\frac{\partial\vec{\boldsymbol{B}}}{\partial t}+\frac{1}{\mu_0}\vec{\boldsymbol{E}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}-\mu_0\epsilon_0\frac{\partial\vec{\boldsymbol{E}}}{\partial t}\right)\right] \\ &= -\int_{\mathcal{V}}\mathrm{d}^3x\left[-\frac{1}{\mu_0}\vec{\boldsymbol{B}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}})+\frac{1}{\mu_0}\vec{\boldsymbol{E}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}})\right] \\ &= \frac{1}{\mu_0}\int_{\mathcal{V}}\mathrm{d}^3x\,\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\,. \end{aligned}$$

$$P_{\text{flux}}=\frac{1}{\mu_0}\int_{\mathcal{V}}\mathrm{d}^3x\,\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\;,$$

so,

$$\boxed{P_{\text{flux}}=\oint_S\vec{\boldsymbol{S}}\cdot\mathrm{d}\vec{\boldsymbol{a}}\;,}$$

where

$$\boxed{\vec{\boldsymbol{S}}=\frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\;,}$$

where $\vec{\boldsymbol{S}}$ is called the Poynting vector. $\vec{\boldsymbol{S}}$ describes the flow of energy in space, with units of joules per meter^2 per second.

Forms of energy conservation:

$$\boxed{\frac{\mathrm{d}U_{\text{EM},\mathcal{V}}}{\mathrm{d}t}=-\int_{\mathcal{V}}\vec{\boldsymbol{J}}\cdot\vec{E}\,\mathrm{d}^3x-\oint_S\vec{\boldsymbol{S}}\cdot\mathrm{d}\vec{\boldsymbol{a}}\;,}$$

or in its differential form,

$$\boxed{\frac{\partial u_{\text{EM}}}{\partial t}=-\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{E}}-\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{S}}\;.}$$

The work-energy theorem implies that

$$\int_{\mathcal{V}}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{E}}\,d^3x=\frac{\mathrm{d}U_{\text{mech}}}{\mathrm{d}t}\;,$$

where U_{mech} is the mechanical energy of the particles, meaning the total kinetic energy plus any non-electromagnetic potential energy (e.g., gravitational energy, spring energy, etc.). Thus we can write

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}[U_{\text{EM},\mathcal{V}}+U_{\text{mech},\mathcal{V}}]=-\oint_S\vec{\boldsymbol{S}}\cdot\mathrm{d}\vec{\boldsymbol{a}}\;.}$$

From

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}[U_{\text{EM},\mathcal{V}}+U_{\text{mech},\mathcal{V}}]=-\oint_S\vec{\boldsymbol{S}}\cdot\mathrm{d}\vec{\boldsymbol{a}}\;,}$$

we can write the differential form

$$\boxed{\frac{\partial u}{\partial t}=-\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{S}}\;,}$$

where

$$u=u_{\text{EM}}+u_{\text{mech}}\;,$$

where u_{mech} is the density of mechanical energy.

Power Transmission

On the blackboard

Conservation of Momentum

On the blackboard