

PROBLEM SET 4 SOLUTIONS

PROBLEM 1: TRACELESS SYMMETRIC PART OF $\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n$ (15 points)

The terms that can appear in the expansion of $\{\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n\}_{\text{TS}}$ can be classified by the number of Kronecker δ -functions that they contain. With 5 indices, there can be terms with no Kronecker δ -functions, terms with one Kronecker δ -function, and terms with two. Once the number of δ -functions is specified, the symmetry in the indices dictates the form of the terms up to an unknown coefficient for each classification. So we can begin by writing

$$\{\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n\}_{\text{TS}} = \hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n + c_1(\hat{n}_i\hat{n}_j\hat{n}_k\delta_{mn} + 9 \text{ other permutations}) + c_2(\hat{n}_i\delta_{jk}\delta_{mn} + 14 \text{ other permutations}). \quad (1.1)$$

There are 10 terms with one δ -function, since a term can be identified by specifying the two indices on the δ -function. The first of these indices is one choice out of 5 possibilities, while the second is a choice out of 4 possibilities. This would give $5 \cdot 4 = 20$ possible combinations, but the order of the two indices does not matter, so there are 10 possibilities. They can be listed in full:

$$\begin{aligned} \text{Terms with 1 } \delta\text{-function} = & \hat{n}_i\hat{n}_j\hat{n}_k\delta_{mn} + \hat{n}_j\hat{n}_k\hat{n}_m\delta_{in} + \hat{n}_i\hat{n}_k\hat{n}_m\delta_{jn} + \hat{n}_i\hat{n}_j\hat{n}_m\delta_{kn} \\ & + \hat{n}_j\hat{n}_k\hat{n}_n\delta_{im} + \hat{n}_i\hat{n}_k\hat{n}_n\delta_{jm} + \hat{n}_i\hat{n}_j\hat{n}_n\delta_{km} \\ & + \hat{n}_j\hat{n}_m\hat{n}_n\delta_{ik} + \hat{n}_i\hat{n}_m\hat{n}_n\delta_{jk} + \hat{n}_k\hat{n}_m\hat{n}_n\delta_{ij}. \end{aligned} \quad (1.2)$$

The terms with two δ -functions can be identified by first specifying the index on \hat{n}_i , which is one choice out of 5 possibilities. For each choice of the index on \hat{n}_i , there are 3 ways of arranging the other indices. For example, suppose that the index on \hat{n}_i is i , so that the remaining indices are j, k, m and n . The δ -functions can then pair j with either k, m , or n , and the term is completely specified once that choice is made. If we wish to write out all 15 terms, they are:

$$\begin{aligned} \text{Terms with 2 } \delta\text{-functions} = & \hat{n}_i\delta_{jk}\delta_{mn} + \hat{n}_i\delta_{jm}\delta_{kn} + \hat{n}_i\delta_{jn}\delta_{km} \\ & + \hat{n}_j\delta_{ik}\delta_{mn} + \hat{n}_j\delta_{im}\delta_{kn} + \hat{n}_j\delta_{in}\delta_{km} \\ & + \hat{n}_k\delta_{ij}\delta_{mn} + \hat{n}_k\delta_{im}\delta_{jn} + \hat{n}_k\delta_{in}\delta_{jm} \\ & + \hat{n}_m\delta_{ij}\delta_{kn} + \hat{n}_m\delta_{ik}\delta_{jn} + \hat{n}_m\delta_{in}\delta_{jk} \\ & + \hat{n}_n\delta_{ij}\delta_{km} + \hat{n}_n\delta_{ik}\delta_{jm} + \hat{n}_n\delta_{im}\delta_{jk}. \end{aligned} \quad (1.3)$$

Now we contract Eq. (1.1) on two indices. I will choose these indices as m and n , so I multiply the left- and right-hand sides by δ_{mn} , insisting that the trace vanish. The first term on the right-hand side gives

$$\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n\delta_{mn} = \hat{n}_i\hat{n}_j\hat{n}_k, \quad (1.4)$$

since $\hat{n}_m\hat{n}_n\delta_{mn} = \hat{n} \cdot \hat{n} = 1$. The second term (proportional to c_1) is more complicated. For the first term shown in Eq. (1.2), $\hat{n}_i\hat{n}_j\hat{n}_k\delta_{km}\delta_{mn} = \hat{n}_i\hat{n}_j\hat{n}_k\delta_{ik} = 3\hat{n}_i\hat{n}_j\hat{n}_k$. For the 6 terms for which either m or n appears on the δ -function, the result is $\hat{n}_i\hat{n}_j\hat{n}_k$ for each case. For the 3 terms for which both m and n appear on the \hat{n} 's, the result is $\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij}$. So finally,

$$c_1(\hat{n}_i\hat{n}_j\hat{n}_k\delta_{mn} + 9 \text{ other permutations})\delta_{mn} = c_1 \left[9\hat{n}_i\hat{n}_j\hat{n}_k + (\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij}) \right]. \quad (1.5)$$

For the 15 terms proportional to c_2 , 3 of them are proportional to δ_{mn} . The result of multiplying these three terms by δ_{mn} is then $3(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij})$. For 6 of the terms, m appears on one δ -function and n appears on the other. The result for these 6 terms is then $2(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij})$. Finally, for 6 of the terms either m or n appears as the index of \hat{n}_i , and the other appears as an index on one of the δ -functions. These 6 terms also give the result $2(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij})$, so

$$c_2(\hat{n}_i\delta_{jk}\delta_{mn} + 14 \text{ other permutations})\delta_{mn} = 7(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij}). \quad (1.6)$$

Collecting the results of Eqs. (1.4), (1.5), and (1.6), we have

$$\begin{aligned} \{\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n\}_{\text{TS}}\delta_{mn} &= 0 \\ &= \hat{n}_i\hat{n}_j\hat{n}_k + c_1 \left[9\hat{n}_i\hat{n}_j\hat{n}_k + (\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij}) \right] \\ &\quad + 7c_2(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij}). \end{aligned} \quad (1.7)$$

To insure that the coefficient of $\hat{n}_i\hat{n}_j\hat{n}_k$ equals zero, we find

$$c_1 = -\frac{1}{9}. \quad (1.8)$$

To insure that the coefficient of $(\hat{n}_i\delta_{jk} + \hat{n}_j\delta_{ik} + \hat{n}_k\delta_{ij})$ is zero, we find

$$c_2 = -\frac{1}{7}c_1 = \frac{1}{63}. \quad (1.9)$$

So finally,

$$\begin{aligned} \{\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n\}_{\text{TS}} &= \hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_m\hat{n}_n - \frac{1}{9}(\hat{n}_i\hat{n}_j\hat{n}_k\delta_{mn} + 9 \text{ other permutations}) \\ &\quad + \frac{1}{63}(\hat{n}_i\delta_{jk}\delta_{mn} + 14 \text{ other permutations}). \end{aligned} \quad (1.10)$$

As a check, we can rewrite Eq. (1.10) for $\hat{\mathbf{n}} = \hat{\mathbf{z}}$,

$$\begin{aligned} \{\hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \hat{z}_n\}_{\text{TS}} &= \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \hat{z}_n - \frac{1}{9} (\hat{z}_i \hat{z}_j \hat{z}_k \delta_{mn} + 9 \text{ other permutations}) \\ &\quad + \frac{1}{63} (\hat{z}_i \delta_{jk} \delta_{mn} + 14 \text{ other permutations}), \end{aligned} \quad (1.11)$$

as the coefficients do not depend on what unit vector is used in the construction. Then we can use Eq. (1.51) of Lecture Notes 1 to evaluate

$$\begin{aligned} P_3(\cos \theta) &= \frac{10!}{2^5 \cdot (5!)^2} \{\hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \hat{z}_n\}_{\text{TS}} \hat{n}_i \hat{n}_j \hat{n}_k \hat{n}_m \hat{n}_n \\ &= \frac{63}{8} \left[\cos^5 \theta - \frac{10}{9} \cos^3 \theta + \frac{15}{63} \cos \theta \right] \\ &= \frac{1}{8} [63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta], \end{aligned} \quad (1.12)$$

where the last line agrees with the standard expression for $P_3(x)$, as can be found, for example, in the Wikipedia. It can also be computed in Mathematica as `LegendreP[5,x]`.

PROBLEM 2: QUADRUPOLE AND OCTOPOLE TERMS OF THE MULTIPOLE EXPANSION (20 points)

(a) [5 pts] The preamble said that this problem can be attacked in any one of three methods, so we give three different answers.

Method (i) (Using Griffiths' Eq. (3.95)): The starting equation is

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (\mathbf{r}')^n P_n(\cos \alpha) \rho(\vec{\mathbf{r}}') d^3x', \quad (\text{Griffiths 3.95})$$

where I am using d^3x' where Griffiths uses $d\tau'$. Using the expression for $P_2(x)$, the quadrupole term becomes

$$\begin{aligned} V_{\text{quad}}(\vec{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int r'^2 P_2(\cos \theta') \rho(\vec{\mathbf{r}}') d^3x' \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int \frac{1}{2} (3r'^2 \cos^2 \theta' - r'^2) \rho(\vec{\mathbf{r}}') d^3x'. \end{aligned} \quad (2.1)$$

We are asked to show that we can rewrite the quadrupole term in terms of the quadrupole moment:

$$V_{\text{quad}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_{i,j=1}^3 \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j Q_{ij}, \quad (2.2)$$

where

$$Q_{ij} = \frac{1}{2} \int [3r'_i r'_j - r'^2 \delta_{ij}] \rho(\vec{\mathbf{r}}') d^3x'. \quad (2.3)$$

If we write $\cos \theta' = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \hat{\mathbf{r}}_i \hat{\mathbf{r}}'_i$, and then $\cos^2 \theta' = \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j \hat{\mathbf{r}}'_i \hat{\mathbf{r}}'_j$, we see that

$$(3r'^2 \cos^2 \theta' - r'^2) = \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j (3r'_i r'_j - r'^2 \delta_{ij}), \quad (2.4)$$

and then Eq. (2.1) becomes

$$V_{\text{quad}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j \left[\frac{1}{2} \int (3r'_i r'_j - r'^2 \delta_{ij}) \rho(\vec{\mathbf{r}}') d^3x' \right], \quad (2.5)$$

which is the desired result.

Method (ii) (Using Griffiths' Eq. (2.29)): The starting equation in this case is

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d^3x, \quad (\text{Griffiths 2.29})$$

where I am using $|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$ where Griffiths uses \mathbf{r} . We now want to expand $f(\vec{\mathbf{r}}) \equiv 1/|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$ in a power series in \mathbf{r}'_i , the components of $\vec{\mathbf{r}}'$:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{\mathbf{r}}') \left[f(\vec{\mathbf{0}}) + \frac{\partial f}{\partial r'_i} \bigg|_{\vec{\mathbf{r}}'=\vec{\mathbf{0}}} r'_i + \frac{1}{2} \frac{\partial^2 f}{\partial r'_i \partial r'_j} \bigg|_{\vec{\mathbf{r}}'=\vec{\mathbf{0}}} r'_i r'_j + \dots \right] d^3x' \quad (2.6)$$

The quadrupole term is the term quadratic in \mathbf{r}'_i :

$$V_{\text{quad}}(\vec{\mathbf{r}}) = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int \rho(\vec{\mathbf{r}}') \frac{\partial^2 f}{\partial r'_i \partial r'_j} \bigg|_{\vec{\mathbf{r}}'=\vec{\mathbf{0}}} r'_i r'_j d^3x'. \quad (2.7)$$

To evaluate the partial derivatives of f , define $w \equiv |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2 = (r'_k - r_k)(r'_k - r_k)$, so $f(\vec{\mathbf{r}}) = 1/w^{1/2}$, and

$$\frac{\partial w}{\partial r'_i} = 2\delta_{ik}(r'_k - r_k) = 2(r'_i - r_i). \quad (2.8)$$

Then

$$\frac{\partial f}{\partial r'_i} = -\frac{1}{2} \frac{1}{w^{3/2}} \frac{\partial w}{\partial r'_i} = -\frac{1}{w^{3/2}} (r'_i - r_i), \quad (2.9)$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial r'_j \partial r'_i} &= -\frac{\frac{\partial}{\partial r'_j} (r'_i - r_i)}{w^{3/2}} + \frac{3}{2} \frac{1}{w^{5/2}} (r'_i - r_i) \frac{\partial w}{\partial r'_j} \\ &= \frac{1}{w^{5/2}} [3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}]. \end{aligned} \quad (2.10)$$

Then

$$\begin{aligned} \frac{\partial^2 f}{\partial r_i' \partial r_i'} \bigg|_{\vec{r}'=0} r_i' r_j' &= \frac{1}{r^5} [3r_i r_j - r^2 \delta_{ij}] r_i' r_j' \\ &= \frac{1}{r^3} \hat{r}_i \hat{r}_j [3r_i' r_j' - r'^2 \delta_{ij}] , \end{aligned} \quad (2.11)$$

and finally, using Eq. (2.7), we find

$$V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \hat{r}_i \hat{r}_j \left[\frac{1}{2} \int [3r_i' r_j' - r'^2 \delta_{ij}] \rho(\vec{r}') d^3 x' \right] . \quad (2.12)$$

Method (iii) (Using traceless symmetric tensors): The starting point in this case is Eqs. (2.26) and (2.27) of Lecture Notes 2:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C_{i_1 \dots i_\ell}^{(\ell)} \hat{r}_{i_1} \hat{r}_{i_2} \dots \hat{r}_{i_\ell} , \quad (\text{Lecture Notes 2.26})$$

where

$$C_{i_1 \dots i_\ell}^{(\ell)} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}') \{ \vec{r}'_{i_1} \dots \vec{r}'_{i_\ell} \}_{\text{TS}} d^3 x' . \quad (\text{Lecture Notes 2.27})$$

Identifying Q_{ij} with $C_{ij}^{(2)}$, we see that $V(\vec{r})$ already has the desired form, so all we have to do is write out the quadrupole term, $\ell=2$, explicitly:

$$\begin{aligned} Q_{ij} &= \frac{3}{2} \int \rho(\vec{r}') \{ r_i' r_j' \}_{\text{TS}} d^3 x' \\ &= \frac{1}{2} \int \rho(\vec{r}') [3r_i' r_j' - r'^2 \delta_{ij}] d^3 x' . \end{aligned} \quad (2.13)$$

(b) [5 pts] The square in Griffiths Fig. 3.30 has side a and lies in the xy plane, centered at the origin. From the definition of Q_{ij} given in the problem, or equivalently in

Eq. (2.3),

$$\begin{aligned} Q_{xx} &= Q_{yy} = \frac{1}{2} [3(a/2)^2 - (\sqrt{2}a/2)^2] (q - q + q - q) = \boxed{0} \\ Q_{xz} &= Q_{zx} = Q_{yz} = Q_{zy} = \boxed{0} \quad \text{since } z = 0 \\ Q_{zz} &= -\frac{1}{2} (\sqrt{2}a/2)^2 (q - q + q - q) = \boxed{0} \\ Q_{xy} &= Q_{yx} = \frac{3}{2} \left[\left(\frac{a}{2} \right) \left(\frac{a}{2} \right) q + \left(\frac{a}{2} \right) \left(\frac{-a}{2} \right) (-q) + \left(\frac{-a}{2} \right) \left(\frac{a}{2} \right) q \right. \\ &\quad \left. + \left(\frac{-a}{2} \right) \left(\frac{a}{2} \right) (-q) \right] = \boxed{\frac{3}{2} a^2 q} . \end{aligned} \quad (2.14)$$

(c) [5 pts] For a given charge distribution the quadrupole moment is defined as

$$Q_{ij} = \frac{1}{2} \int [3r_i r_j - r^2 \delta_{ij}] \rho(\vec{r}) d^3 x . \quad (2.15)$$

When we move the reference frame by \vec{d} , the quadrupole moment in the new reference frame becomes:

$$\begin{aligned} Q'_{ij} &= \frac{1}{2} \int [3(r_i - d_i)(r_j - d_j) - |\vec{r} - \vec{d}|^2 \delta_{ij}] \rho(\vec{r}) d^3 x \\ &= \frac{1}{2} \int [3r_i r_j - r^2 \delta_{ij}] \rho d^3 x - \frac{3}{2} d_i \int r_j \rho d^3 x - \frac{3}{2} d_j \int r_i \rho d^3 x + \frac{3}{2} d_i d_j \int \rho d^3 x \\ &\quad + \vec{d} \cdot \int \vec{r} \rho \delta_{ij} d^3 x - \frac{1}{2} d^2 \delta_{ij} \int \rho d^3 x \end{aligned}$$

$$= Q_{ij} - \frac{3}{2} (d_i p_j + d_j p_i) + \frac{3}{2} d_i d_j Q + \delta_{ij} \vec{d} \cdot \vec{p} - \frac{1}{2} d^2 \delta_{ij} Q , \quad (2.16)$$

where $Q = \int \rho d^3 x$ and $p_i = \int r_i \rho d^3 x$. So if $\vec{p} = 0$ and $Q = 0$ then $Q'_{ij} = Q_{ij}$.

Alternatively, using the notation of traceless symmetric tensors, we find

$$\begin{aligned} Q'_{ij} &= \frac{3}{2} \int \rho(\vec{r}') \{ (r_i - d_i)(r_j - d_j) \}_{\text{TS}} d^3 x \\ &= \frac{3}{2} \int \rho(\vec{r}') \{ r_i r_j - (r_i d_j + r_j d_i) + d_i d_j \}_{\text{TS}} d^3 x' \\ &= \frac{3}{2} [Q_{ij} - \{ p_i d_j + p_j d_i \}_{\text{TS}} + \{ d_i d_j \}_{\text{TS}} Q] , \end{aligned} \quad (2.17)$$

so again we see that $Q'_{ij} = Q_{ij}$ if $p_i = Q = 0$.

- (d) [5 pts] As in part (a), the preamble said that this problem can be attacked in any one of three methods:

Method (i) (Using Griffiths' Eq. (3.95)): Writing the $n = 3$ term of Griffiths' Eq. (3.95),

$$\begin{aligned} V_{\text{oct}}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \int (r')^n P_3(\cos\alpha) \rho(\vec{r}') d^3x' \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \int (r')^3 \left[\frac{1}{2} (5\cos^3\alpha - 3\cos\alpha) \right] \rho(\vec{r}') d^3x'. \end{aligned} \quad (2.18)$$

As in part (a), if we use $\cos\alpha = \hat{r}_i \hat{r}'_i$, the above equation can be rewritten as

$$\begin{aligned} V_{\text{oct}}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \hat{r}_i \hat{r}_j \hat{r}_k \int \left\{ \frac{1}{2} [5r'_i r'_j r'_k - r'^2 (\delta_{jk} r'_i + \delta_{ik} r'_j + \delta_{ij} r'_k)] \right\} \rho(\vec{r}') d^3x' \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \hat{r}_i \hat{r}_j \hat{r}_k Q_{ijk}, \end{aligned} \quad (2.19)$$

where

$$Q_{ijk} = \frac{1}{2} \int [5r'_i r'_j r'_k - r'^2 (\delta_{jk} r'_i + \delta_{ik} r'_j + \delta_{ij} r'_k)] \rho(\vec{r}') d^3x'.$$

Method (ii) (Using Griffiths' Eq. (2.29)): Continuing from part (a)(ii), the octopole term is the 3rd order term in the power expansion:

$$V_{\text{oct}}(\vec{r}) = \frac{1}{3!} \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}') \left. \frac{\partial^3 f}{\partial r'_i \partial r'_j \partial r'_k} \right|_{\vec{r}'=\vec{0}} r'_i r'_j r'_k d^3x'. \quad (2.21)$$

Starting from Eq. (2.10),

$$\begin{aligned} \frac{\partial^3 f}{\partial r'_i \partial r'_j \partial r'_k} &= \frac{\partial}{\partial r'_k} \left\{ \frac{1}{w^{5/2}} [3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}] \right\} \\ &= -\frac{5}{2} \frac{1}{w^{7/2}} [3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}] 2(r'_k - r_k) \\ &\quad + \frac{1}{w^{5/2}} [3\delta_{ik}(r'_j - r_j) + 3\delta_{jk}(r'_i - r_i) - 2\delta_{ij}(r'_k - r_k)] \\ &= -\frac{15}{w^{7/2}} (r'_i - r_i)(r'_j - r_j)(r'_k - r_k) \\ &\quad + \frac{3}{w^{5/2}} [\delta_{ik}(r'_j - r_j) + \delta_{jk}(r'_i - r_i) + \delta_{ij}(r'_k - r_k)]. \end{aligned} \quad (2.22)$$

Then

$$\left. \frac{\partial^3 f}{\partial r'_i \partial r'_j \partial r'_k} \right|_{\vec{r}'=\vec{0}} = \frac{3}{r^4} [5\hat{r}_i \hat{r}_j \hat{r}_k - (\delta_{jk} \hat{r}_i + \delta_{ik} \hat{r}_j + \delta_{ij} \hat{r}_k)], \quad (2.23)$$

and from Eq. (2.21) we have

$$V_{\text{oct}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \hat{r}_i \hat{r}_j \hat{r}_k Q_{ijk}, \quad (2.24)$$

where

$$Q_{ijk} = \frac{1}{2} \int \rho(\vec{r}') [5r'_i r'_j r'_k - r'^2 (\delta_{jk} r'_i + \delta_{ik} r'_j + \delta_{ij} r'_k)] d^3x'. \quad (2.25)$$

Method (iii) (Using traceless symmetric tensors): Identifying Q_{ijk} with $C_{ijk}^{(3)}$, we see that $V(\vec{r})$ already has the desired form, so all we have to do is write out the octopole term, $\ell = 3$, of Lecture Notes 2 Eq. (2.27), quoted above in the solution to part (a).

$$\begin{aligned} Q_{ijk} &= \frac{5}{2} \int \rho(\vec{r}') \{ r'_i r'_j r'_k \}_{\text{TS}} d^3x' \\ &= \frac{1}{2} \int \rho(\vec{r}') [5r'_i r'_j r'_k - r'^2 (\delta_{jk} r'_i + \delta_{ik} r'_j + \delta_{ij} r'_k)] d^3x'. \end{aligned} \quad (2.26)$$

The expansion of $\{r'_i r'_j r'_k\}_{\text{TS}}$ in the above equation can be found directly from the definition, which says that it is equal to $r'_i r'_j r'_k$, combined with terms proportional to at least one Kronecker delta function to make it traceless. It is the same calculation that was done in Eq. (1.44) of Lecture Notes 1.

PROBLEM 3: NORMALIZING THE TRACELESS SYMMETRIC TENSOR REPRESENTATION OF THE LEGENDRE POLYNOMIALS (15 points)

From Eqs. (3.2) and (3.5) of the problem set, we have

$$\begin{aligned} N(\ell) &= \frac{1}{\{\hat{z}_{i_1} \dots \hat{z}_{i_\ell}\}_{\text{TS}} \hat{z}_{i_1} \dots \hat{z}_{i_\ell}}, \\ &\quad \{\hat{z}_{i_1} \dots \hat{z}_{i_\ell}\}_{\text{TS}} \hat{z}_{i_1} \dots \hat{z}_{i_\ell}. \end{aligned}$$

so our goal is to evaluate

But Eq. (3.6) of the problem set tells us that

$$\{\hat{z}_{i_1} \dots \hat{z}_{i_\ell}\}_{\text{TS}} \hat{z}_{i_\ell} = \frac{\ell}{2\ell-1} \{\hat{z}_{i_1} \dots \hat{z}_{i_{\ell-1}}\}_{\text{TS}} ,$$

where I have replaced \hat{n} by \hat{z} , since the equation was derived for any unit vector \hat{n} . Now we can proceed by iterating this relation:

$$\begin{aligned} \{\hat{z}_{i_1} \dots \hat{z}_{i_\ell}\}_{\text{TS}} \hat{z}_{i_1} \dots \hat{z}_{i_\ell} &= \frac{\ell}{2\ell-1} \{\hat{z}_{i_1} \dots \hat{z}_{i_{\ell-1}}\}_{\text{TS}} \hat{z}_{i_1} \dots \hat{z}_{i_{\ell-1}} \\ &= \left(\frac{\ell}{2\ell-1} \right) \left(\frac{\ell-1}{2\ell-3} \right) \{\hat{z}_{i_1} \dots \hat{z}_{i_{\ell-2}}\}_{\text{TS}} \\ &= \left(\frac{\ell}{2\ell-1} \right) \left(\frac{\ell-1}{2\ell-3} \right) \dots \left(\frac{1}{1} \right) \{1\}_{\text{TS}} \\ &= \frac{\ell!}{(2\ell-1)!!} . \end{aligned}$$

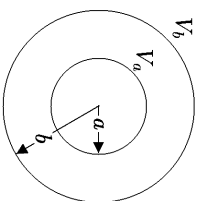
So

$$N(\ell) = \frac{(2\ell-1)!!}{\ell!} = \frac{(2\ell)!}{2^\ell(\ell!)^2} .$$

PROBLEM 4: CONCENTRIC SPHERICAL SHELLS OF CHARGE (25 points)

- (a) [5 pts] The situation is spherically symmetric, which means that only the $\ell = 0$ solution of the multipole expansion is allowed. Thus, $V(r)$ in any region has at most two terms, a constant and a term proportional to $1/r$. For the potential to vanish at infinity, the potential for $r > b$ can have only a $1/r$ term. Matching the boundary condition at $r = b$ then gives

$$V(r) = \frac{bV_b}{r} \quad (\text{for } r \geq b) . \quad (4.1)$$



Similarly, the potential for $r \leq a$ can only have the constant term, since the term proportional to $1/r$ would diverge. The constant must then match the potential at $r = a$, so

$$V(r) = V_a \quad (\text{for } r \leq a) . \quad (4.2)$$

For $a \leq r \leq b$, both terms are allowed, but the potential must match V_a on the inside and V_b on the outside. Writing

$$V(r) = A + \frac{B}{r}$$

for this region, the two boundary conditions give

$$A + \frac{B}{a} = V_a , \quad A + \frac{B}{b} = V_b .$$

The problem is then solved by solving these two equations for A and B . For example, we can subtract the second equation from the first to find

$$B \left(\frac{1}{a} - \frac{1}{b} \right) = V_a - V_b ,$$

which implies that

$$B = \frac{ab}{b-a} (V_a - V_b) .$$

Then from the first equation,

$$A = V_a - \frac{B}{a} = V_a - \frac{b}{b-a} (V_a - V_b) = \frac{bV_b - aV_a}{b-a} .$$

Finally, then,

$$V(r) = \frac{1}{b-a} \left[bV_b - aV_a + \frac{ab}{r} (V_a - V_b) \right] \quad (\text{for } a \leq r \leq b) . \quad (4.3)$$

- (b) [10 pts] In terms of the charge Q_a on the inner shell, the potential in the region $a \leq r \leq b$ can be written as

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q_a}{r} + \text{const} ,$$

which can be compared with Eq. (4.3) found in part (a) to see that

$$Q_a = 4\pi\epsilon_0 \left(\frac{ab}{b-a} \right) (V_a - V_b) . \quad (4.4)$$

Similarly, the potential outside of both shells can be expressed in terms of the total charge on both shells as

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{Q_a + Q_b}{r},$$

where in this case we do not need to include a constant term, since we know that we have chosen $V = 0$ at $r = \infty$. Comparing with Eq. (4.1), we see that

$$Q_a + Q_b = 4\pi\epsilon_0 b V_b.$$

Using Eq. (4.4), we find

$$\begin{aligned} Q_b &= 4\pi\epsilon_0 \left[bV_b - \left(\frac{ab}{b-a} \right) (V_a - V_b) \right] \\ &= 4\pi\epsilon_0 \left[\frac{b(b-a)V_b - ab(V_a - V_b)}{b-a} \right] \\ &= 4\pi\epsilon_0 \left(\frac{b}{b-a} \right) (bV_b - aV_a). \end{aligned} \quad (4.5)$$

The total electrostatic energy W is then given by

$$W = \frac{1}{2} (Q_a V_a + Q_b V_b). \quad (4.6)$$

Substituting Q_a and Q_b from Eqs. (4.4) and (4.5), one finds

$$W = \frac{2\pi\epsilon_0 b}{b-a} [aV_a^2 - bV_b^2 - 2aV_a V_b] \quad (4.7a)$$

or, equivalently,

$$W = \frac{2\pi\epsilon_0 b}{b-a} [a(V_a - V_b)^2 + (b-a)V_b^2]. \quad (4.7b)$$

(c) [10 pts] The expression for $V(r=b, \theta, \phi)$ can be rewritten as

$$V(r=b, \theta, \phi) = V_0 \hat{n}_x \hat{n}_y,$$

or $V_0 \hat{r}_x \hat{r}_y$, depending on notation. This can be rewritten as

$$V(r=b, \theta, \phi) = C_{ij} \hat{n}_i \hat{n}_j,$$

where

$$C_{ij} = \frac{1}{2} V_0 (\hat{x}_i \hat{y}_j + \hat{y}_i \hat{x}_j).$$

One sees that $\delta_{ij} C_{ij} = \hat{x} \cdot \hat{y} = 0$, so C_{ij} is traceless and symmetric. Thus, the angular dependence is described entirely by the $\ell = 2$ term of the general expansion (on the formula sheet(s)),

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C_{i_1 i_2 \dots i_\ell}^{(\ell)} r^\ell + \frac{C_{i_1 i_2 \dots i_\ell}^{(\ell)}}{r^{\ell+1}} \right) \hat{r}_{i_1} \hat{r}_{i_2} \dots \hat{r}_{i_\ell}, \quad \text{where } \vec{r} = r \hat{r}.$$

For those who prefer standard spherical harmonics, one can see from the tables in the formula sheet that

$$V(r=b, \theta, \phi) = \frac{1}{2} V_0 \left(4 \sqrt{\frac{2\pi}{15}} \right) \frac{Y_{2,2} - Y_{2,-2}}{2i},$$

so again it is pure $\ell = 2$. Thus, the allowed radial dependence, for solutions to Laplace's equation, is either r^2 or $1/r^3$.

For $r \leq a$, we see that the boundary condition of $V = 0$ at $r = a$ is trivially matched by

$$V(r) = 0 \quad (\text{for } r \leq a). \quad (4.8)$$

For $r \geq b$, a term proportional to r^2 would not be allowed, since it blows up at infinity, so V must be proportional to $1/r^3$. It is then seen that the boundary condition at $r = b$ is easily matched by writing

$$V(r) = \left(\frac{b}{r} \right)^3 V_0 \sin^2 \theta \sin \phi \cos \phi \quad (\text{for } r \geq b). \quad (4.9)$$

Finally, the region $a \leq r \leq b$ is the most complicated, since it can include both terms. We write

$$V(r) = V_0 \sin^2 \theta \sin \phi \cos \phi \left[A r^2 + \frac{B}{r^3} \right], \quad (4.10)$$

where the boundary conditions at $r = a$ and $r = b$ require that

$$Aa^2 + \frac{B}{a^3} = 0, \quad Ab^2 + \frac{B}{b^3} = 1. \quad (4.11)$$

Multiplying the first equation above by b^2 , and the second equation above by a^2 , and then subtracting the second equation from the first, one finds

$$B \left(\frac{b^2}{a^3} - \frac{a^2}{b^3} \right) = -a^2,$$

from which one finds

$$B = -\frac{a^5 b^3}{b^5 - a^5}.$$

Then, from the first of Eqs. (4.11),

$$A = -\frac{B}{a^5} = \frac{b^3}{b^5 - a^5}.$$

Substituting these values into Eq. (4.10), one finds

$$V(r) = \frac{V_0 \sin^2 \theta \sin \phi \cos \phi}{b^5 - a^5} \left[b^3 r^2 - \frac{a^5 b^3}{r^3} \right], \quad (4.12)$$

which could equivalently be written

$$V(r) = V_0 \sin^2 \theta \sin \phi \cos \phi \left(\frac{b^3}{b^5 - a^5} \right) \left(r^2 - \frac{a^5}{r^3} \right). \quad (4.13)$$