8.07 Lecture Slides 15
October 30, 2019

ELECTRIC FIELDS IN MATTER

Announcements

Quiz 2 will be given on Wednesday, November 13, two weeks from today. Problem Set 6 is due this Friday, 11/1/19, and Problem Set 7 will be due the next Friday, 11/8/19. The quiz will include material through Problem Set 7.

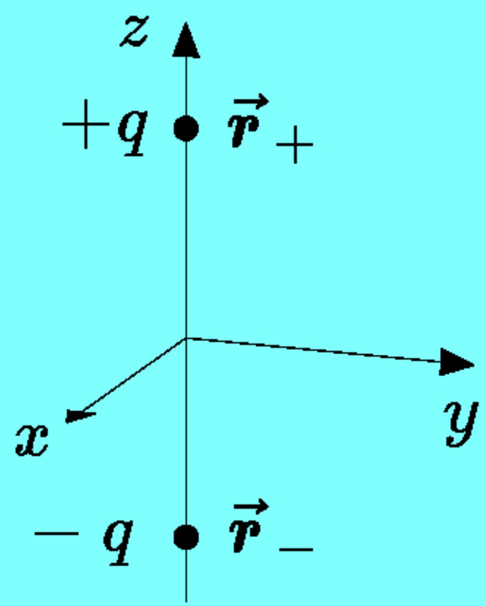
Electric Dipoles

The dipole term of the multipole expansion looks like

$$V_{\rm dip}(\vec{\boldsymbol r}) = \frac{\vec{\boldsymbol p}\cdot\hat{\boldsymbol r}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\ , \quad \text{where} \quad \vec{\boldsymbol p} = \int \rho(\vec{\boldsymbol r}\,')\vec{\boldsymbol r}\,'\,{\rm d}^3x'\ .$$

A **physical dipole** is defined to be two charges, $+q$ and $-q$, at positions $\vec{\boldsymbol r}_+$ and $\vec{\boldsymbol r}_-$, respectively.

$$\vec{\boldsymbol p} = q(\vec{\boldsymbol r}_+ - \vec{\boldsymbol r}_-)\ .$$



An **ideal dipole** is the limit of a physical dipole as $|\vec{\boldsymbol r}_+ - \vec{\boldsymbol r}_-| \to 0$, $q\to\infty$, with $\vec{\boldsymbol p}$ fixed. An ideal dipole is a pure dipole, with no moments other than the dipole moment.

Properties of Electric Dipoles

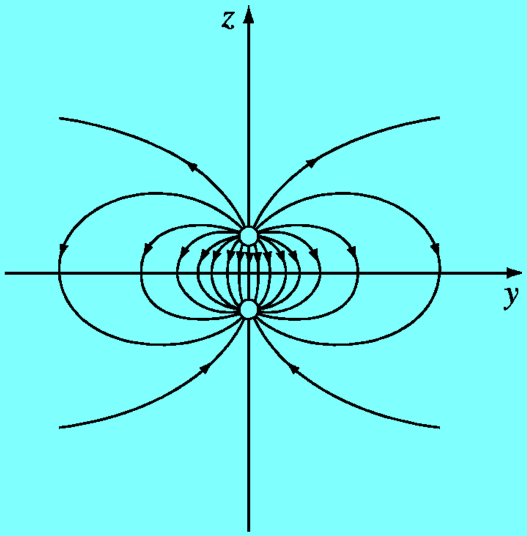
Charge Density:

$$\rho_{\text{dip}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{r}}\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) \ ,$$

where $\vec{\boldsymbol{r}}_0$ is the position of the dipole.

Electric Field:

$$\begin{aligned} \vec{\boldsymbol{E}}_{\text{dip}}(\vec{\boldsymbol{r}}) &= -\vec{\boldsymbol{\nabla}} V_{\text{dip}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{3(\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} - \frac{1}{3\epsilon_0} \vec{\boldsymbol{p}}\, \delta^3(\vec{\boldsymbol{r}}) \ . \end{aligned}$$



Field of a “physical” dipole

FIGURE 3.37

The delta function describes the contribution to $\int d^3x\, \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ of the strong $\vec{\boldsymbol{E}}$ field at the center of the dipole, as shown in the diagram.

Torque on a dipole:

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}} \ .}$$

Force on a dipole:

$$\boxed{\vec{\boldsymbol{F}}_{\rm tot} = (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{E}} \ ,}$$

i.e., $F_{{\rm tot},i} = (p_j \partial_j) E_i$, or $F_{{\rm tot},x} = \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}} E_x$, etc.

Microscopic and Macroscopic Fields

In matter,

$$\rho_{\text{micro}}(\vec{\boldsymbol{r}}) = \sum_i q_i \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \ ,$$

Define a macroscopic field by

$$\begin{aligned} \rho_{\text{macro}}(\vec{\boldsymbol{r}}) &= \text{average of } \rho_{\text{micro}} \text{ in small region centered at } \vec{\boldsymbol{r}}. \\ &= \langle \rho_{\text{micro}}(\vec{\boldsymbol{r}}) \rangle \ . \end{aligned}$$

Choose size of region to be

(a) large compared to the size of atoms.

(b) small compared to macroscopic dimensions (i.e., the sizes of physical objects).

We will treat $\rho_{\text{macro}}(\vec{\boldsymbol{r}})$ as a smooth function.

Similarly, in matter we distinguish between $\vec{\boldsymbol{E}}_{\rm micro}(\vec{\boldsymbol{r}})$ and $\vec{\boldsymbol{E}}_{\rm macro}(\vec{\boldsymbol{r}})$.

Convention: in matter, with no subscript, $\rho(\vec{\boldsymbol{r}})$ and $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ refer to the macro quantities.

Bound Charges

Matter can become "polarized," meaning that it acquires a nonzero density of dipoles.

$$\boxed{\vec{P}(\vec{\boldsymbol{r}}) = \text{dipole moment per unit volume.}}$$

$\vec{P}(\vec{\boldsymbol{r}})$ is just a particular way of describing a distribution of charge. In principle, one can equivalently use $\rho(\vec{\boldsymbol{r}})$

Given $\vec{P}(\vec{\boldsymbol{r}})$, what is $\rho(\vec{\boldsymbol{r}})$?

Answer:

$$\boxed{\rho_b(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} \cdot \vec{P}(\vec{\boldsymbol{r}}) \ ,}$$

and on the surface of a polarized material,

$$\boxed{\sigma_b = \vec{P} \cdot \hat{\boldsymbol{n}} \ ,}$$

where $\hat{\boldsymbol{n}}$ is the outward unit normal.

Derivation of Bound Charge Density

We will explore **THREE** ways to derive this important relation.

Method 1 (from Griffiths, Sec. 4.2.1, pp. 173–174, done on blackboard in Lecture 14):

$$\boxed{\begin{aligned} V(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \cdot \vec{P}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}\, \mathrm{d}^3x' \\ &= \frac{1}{4\pi\epsilon_0} \left\{ -\int_{\mathcal{V}} \frac{\vec{\boldsymbol{\nabla}} \cdot \vec{P}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, \mathrm{d}^3x' + \int_S \frac{\vec{P}(\vec{\boldsymbol{r}}\,') \cdot \hat{\boldsymbol{n}}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, \mathrm{d}a' \right\} . \end{aligned}}$$

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')\cdot\vec{P}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\,\mathrm{d}^3x'$$

$$= \frac{1}{4\pi\epsilon_0}\left\{-\int_{\mathcal{V}} \frac{\vec{\boldsymbol{\nabla}}\cdot\vec{P}(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x' + \int_S \frac{\vec{P}(\vec{\boldsymbol{r}}\,')\cdot\hat{\boldsymbol{n}}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}a'\right\} .$$

For comparison, for an arbitrary charge density $\rho(\vec{\boldsymbol{r}})$ and surface charge density $\sigma(\vec{\boldsymbol{r}})$,

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\left\{-\int_{\mathcal{V}} \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x' + \int_S \frac{\sigma(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}a'\right\} .$$

Comparing the two expressions, we see that

$$\rho(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}\cdot\vec{P}(\vec{\boldsymbol{r}}) \ , \ \text{and} \ \ \sigma(\vec{\boldsymbol{r}}) = \vec{P}(\vec{\boldsymbol{r}})\cdot\hat{\boldsymbol{n}} \ .$$

The total charge on a polarized object is zero:

$$Q_{\text{tot}} = \int_{\mathcal{V}} \rho_b \, \mathrm{d}^3x + \int_S \sigma_b \, \mathrm{d}a = 0 \; .$$

Pictures of $\rho_b = -\vec{\boldsymbol{\nabla}} \cdot \vec{P}$ and $\sigma_b = \vec{P} \cdot \hat{\boldsymbol{n}}$:

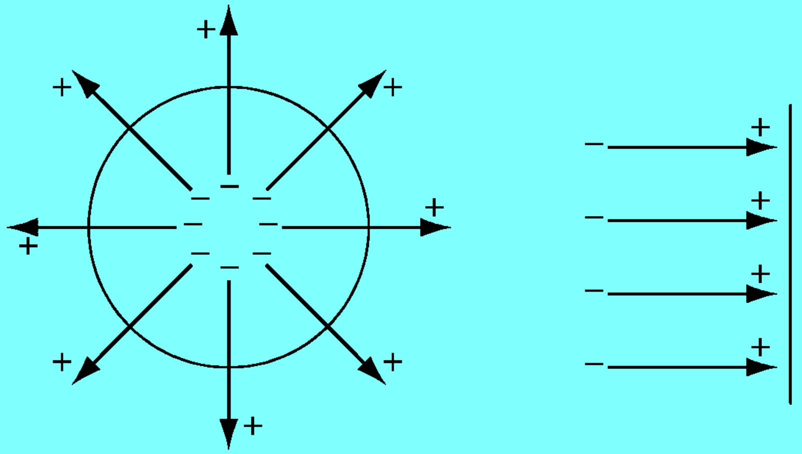


FIGURE 4.14

Shortcoming of Method 1: It makes it seem that Coulomb's law and $V(\vec{\boldsymbol{r}})$ are relevant. They are not!

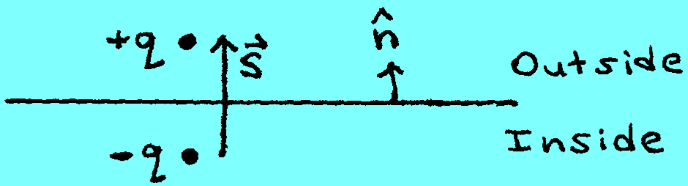
Derivation of Bound Charge Density Method 2: from Feynman Lectures

Calculate total charge in an arbitrary volume $\mathcal{V}$.

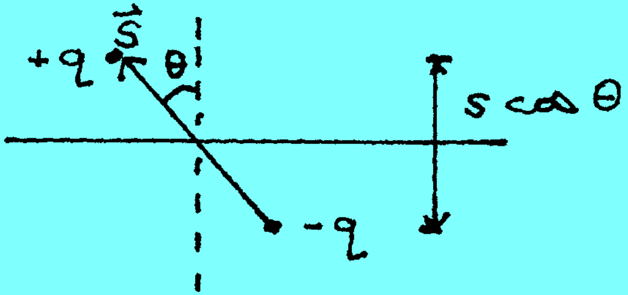
Dipoles completely within the volume do not contribute. Only dipoles cut by the surface contribute.

On surfaces, consider first dipoles perpendicular to surface:

Look at an infinitesimal patch of surface. Can treat as a plane. Layer of dipoles of thickness s contribute. Extra charge on surface is $\Delta q = -s\,\mathrm{d}a\,\mathcal{N}q$, where $\mathrm{d}a$ = area of patch, $\mathcal{N}$ = number density of dipoles.



Now consider the case where $\vec{P}$ is at an angle θ relative to the normal. Now a layer of thickness $s\cos\theta$ contributes. Then



$$\begin{aligned}\Delta q &= -s\,(\mathrm{d}a\cos\theta)\mathcal{N}q \\ &= -\mathcal{N}(q\vec{\boldsymbol{s}})\cdot\mathrm{d}a \\ &= -\vec{P}\cdot\mathrm{d}\vec{\boldsymbol{a}}\ .\end{aligned}$$

So, using the divergence theorem,

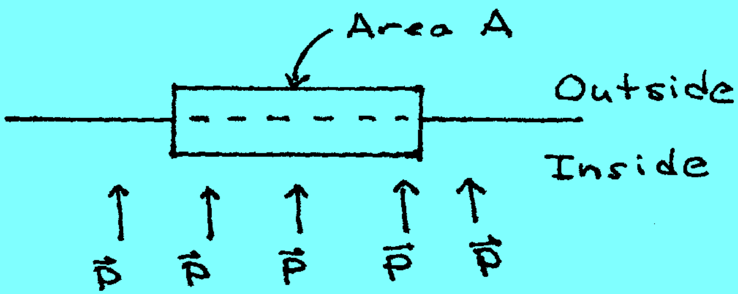
$$\boxed{Q = -\int_S \vec{P}\cdot\mathrm{d}\vec{\boldsymbol{a}} = -\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}}\cdot\vec{P}\,\mathrm{d}^3x\ ,}$$

so $\rho_b = -\vec{\boldsymbol{\nabla}}\cdot\vec{P}$.

But what about σ_b?

It is okay to use **only** $\rho_b = -\vec{\boldsymbol{\nabla}} \cdot \vec{P}$, if one uses it over all space, including at the boundaries the polarized material:

Applying divergence theorem to a Gaussian pillbox that straddles the surface of the material. The only contribution is from the bottom surface:



$$
\begin{aligned}
Q_{\rm enc} &= -\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}} \cdot \vec{P}\, \mathrm{d}^3x \\
&= -\int_S \vec{P} \cdot \mathrm{d}\vec{\boldsymbol{a}} = A\vec{P} \cdot \hat{\boldsymbol{n}}\ ,
\end{aligned}
$$

so

$$
\sigma_b = \frac{Q_{\rm enc}}{A} = \vec{P} \cdot \hat{\boldsymbol{n}}\ .
$$

Derivation of Bound Charge Density
Method 3: Use δ-functions

Reminder about microscopic and macroscopic charge density:

$$\rho_{\rm micro}(\vec{\boldsymbol r}) = \sum_i q_i \delta^3(\vec{\boldsymbol r} - \vec{\boldsymbol r}_i) \ ,$$

$$\rho_{\rm macro}(\vec{\boldsymbol r}) = \left\langle \sum_i q_i \delta^3(\vec{\boldsymbol r} - \vec{\boldsymbol r}_i) \right\rangle \ .$$

Similarly,

$$\vec{P}_{\rm micro}(\vec{\boldsymbol r}) = \sum_i \vec{\boldsymbol p}_i \delta^3(\vec{\boldsymbol r} - \vec{\boldsymbol r}_i) \ ,$$

$$\vec{P}_{\rm macro}(\vec{\boldsymbol r}) = \left\langle \sum_i \vec{\boldsymbol p}_i \delta^3(\vec{\boldsymbol r} - \vec{\boldsymbol r}_i) \right\rangle \ .$$

For one dipole, $\rho = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)$. For ρ_b,

$$\begin{aligned}
\rho_{b,\text{micro}} &= -\sum_i \vec{\boldsymbol{p}}^{(i)} \cdot \vec{\boldsymbol{\nabla}} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \\
&= \sum_i p_j^{(i)} \frac{\partial}{\partial x_j} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \\
&= -\frac{\partial}{\partial x_j} \sum_i p_j^{(i)} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \ .
\end{aligned}$$

$$\begin{aligned}
\rho_{b,\text{macro}} &= -\left\langle \frac{\partial}{\partial x_j} \sum_i p_j^{(i)} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \right\rangle \\
&= -\frac{\partial}{\partial x_j} \left\langle \sum_i p_j^{(i)} \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \right\rangle \\
&= -\vec{\boldsymbol{\nabla}} \cdot \vec{P}_{\text{macro}} \ .
\end{aligned}$$