

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II
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FORMULA SHEET FOR QUIZ 2

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A few items below are marked with asterisks, ***. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

1. Index Notation:

Unit Vectors: $\hat{x} \equiv \hat{i} \equiv \hat{e}_x \equiv \hat{e}_1$, $\hat{y} \equiv \hat{j} \equiv \hat{e}_y \equiv \hat{e}_2$, $\hat{z} \equiv \hat{k} \equiv \hat{e}_z \equiv \hat{e}_3$, $\vec{A} \equiv A_i \hat{e}_i$ General vector: $\vec{r} = x \hat{x} + y \hat{y} + z \hat{z} = x_i \hat{e}_i$

Dot Product, Cross Product, and Determinant:

$$\vec{A} \cdot \vec{B} = A_i B_i, \quad \vec{A} \times \vec{B}_i = \epsilon_{ijk} A_j B_k, \quad \epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \dots i_n} A_{1, i_1} A_{2, i_2} \dots A_{n, i_n} \quad ***$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j, \quad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \quad (R^T T = I)$$

$$\text{Rotation about } z\text{-axis by } \phi: R_z(\phi)_{ij} = \begin{matrix} & \begin{matrix} j=1 & j=2 & j=3 \end{matrix} \\ \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix} & \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Rotation about axis \hat{n} by ϕ : ***

$$R(\hat{n}, \phi)_{ij} = \delta_{ij} \cos \phi + \hat{n}_i \hat{n}_j (1 - \cos \phi) - \epsilon_{ijk} \hat{n}_k \sin \phi.$$

2. Vector Calculus:

$$\text{Gradient: } (\vec{\nabla} \varphi)_i = \partial_i \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y} + \frac{\partial \varphi}{\partial z} \hat{z}, \quad \partial_i \equiv \frac{\partial}{\partial x_i}$$

$$\text{Divergence: } \vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{Curl: } (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}$$

$$\text{Laplacian: } \nabla^2 \varphi = \vec{\nabla} \cdot (\vec{\nabla} \varphi) = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

3. Fundamental Theorems of Vector Calculus:

Gradient:
$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} \varphi \cdot d\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence:
$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{A} d^3x = \oint_S \vec{A} \cdot d\vec{a}$$

where S is the boundary of \mathcal{V}

Curl:
$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_P \vec{A} \cdot d\vec{\ell}$$

where P is the boundary of S

4. Vector Identities:

Triple Products:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Product Rules:

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$$

$$\vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla}f$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times (f\vec{A}) = f\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla}f$$

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

Second Derivatives:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

5. Spherical Coordinates:

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right)$$

$$\phi = \tan^{-1} (y/x)$$

$$\hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$$

$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

Point separation: $d\vec{\ell} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$

Volume element: $d^3x \rightarrow r^2 \sin \theta dr d\theta d\phi$

Gradient: $\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \phi} \hat{\phi}$

Divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$

Curl:
$$\begin{aligned} \vec{\nabla} \times \vec{A} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} \\ & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

Laplacian: $\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$

6. Cylindrical Coordinates:

$$x = s \cos \phi \qquad s = \sqrt{x^2 + y^2}$$

$$y = s \sin \phi \qquad \phi = \tan^{-1}(y/x)$$

$$z = z \qquad z = z$$

$$\hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \qquad \hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi}$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \qquad \hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi}$$

$$\hat{z} = \hat{z} \qquad \hat{z} = \hat{z}$$

Point separation: $d\vec{\ell} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$

Volume element: $d^3x \rightarrow s ds d\phi dz$

Gradient: $\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial \varphi}{\partial \phi} \hat{\phi} + \frac{\partial \varphi}{\partial z} \hat{z}$

Divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

Curl:
$$\begin{aligned} \vec{\nabla} \times \vec{A} = & \left[\frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right] \hat{\phi} \\ & + \frac{1}{s} \left[\frac{\partial}{\partial s} (s A_\phi) - \frac{\partial A_s}{\partial \phi} \right] \hat{z} \end{aligned}$$

Laplacian: $\nabla^2 \varphi = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial \varphi}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2}$

7. Delta Functions:

$$\int \varphi(x) \delta(x - x') dx = \varphi(x'), \quad \int \varphi(\vec{r}) \delta^3(\vec{r} - \vec{r}') d^3x = \varphi(\vec{r}')$$

$$\int \varphi(x) \frac{d}{dx} \delta(x - x') dx = - \left. \frac{d\varphi}{dx} \right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \quad \text{where } i \text{ is summed over all points for which } g(x_i) = 0$$

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -\vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = -4\pi \delta^3(\vec{r} - \vec{r}')$$

$$\partial_i \partial_j \left(\frac{1}{r} \right) = -\partial_i \left(\frac{\hat{r}_j}{r^2} \right) = -\partial_i \left(\frac{x_j}{r^3} \right) = \frac{3\hat{r}_i \hat{r}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3} \delta_{ij} \delta^3(\vec{r})$$

8. Electrostatics:

$$\vec{F} = q\vec{E}, \text{ where}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{(\vec{r} - \vec{r}') q_i}{|\vec{r} - \vec{r}'|^3} = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d^3x'$$

$$\epsilon_0 = \text{permittivity of free space} = 8.854 \times 10^{-12} \text{ C}^2/(\text{N}\cdot\text{m}^2)$$

$$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}') \cdot d\vec{\ell}' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x'$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = 0, \quad \vec{E} = -\vec{\nabla} V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \text{ (Poisson's Eq.)}, \quad \rho = 0 \implies \nabla^2 V = 0 \text{ (Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

9. Electrostatic Energy:*

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{\substack{ij \\ i \neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int d^3x d^3x' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$W = \frac{1}{2} \int d^3x \rho(\vec{r}) V(\vec{r}) = \frac{1}{2} \epsilon_0 \int |\vec{E}|^2 d^3x$$

* See 14(c) for energy in the presence of dielectrics.

10. Conductors:

Just outside, $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j, \quad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j, \quad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a : Image of Q at R is $q = -\frac{a}{R}Q$, $r = \frac{a^2}{R}$

11. Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix} \begin{Bmatrix} \cos \beta y \\ \sin \beta y \end{Bmatrix} \begin{Bmatrix} \cosh \gamma z \\ \sinh \gamma z \end{Bmatrix} \quad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

or, more generally,

$$V = \begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix} \begin{Bmatrix} \cos \beta y \\ \sin \beta y \end{Bmatrix} \begin{Bmatrix} \cos \gamma z \\ \sin \gamma z \end{Bmatrix}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$, each of α , β , and γ can be real or imaginary, with $\sin(i\theta) = i \sinh \theta$ and $\cos(i\theta) = \cosh \theta$. $\begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix}$ means any linear combination of $\cos \alpha x$ or $\sin \alpha x$, but usually one or the other suffices.

12. Separation of Variables for Laplace's Equation in Spherical Coordinates:**12(a) Traceless Symmetric Tensor expansion:**

$$\nabla^2 \varphi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \nabla_{\text{ang}}^2 \varphi = 0,$$

where the angular part is given by

$$\nabla_{\text{ang}}^2 \varphi \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$$

$$\nabla_{\text{ang}}^2 C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \hat{n}_{i_2} \dots \hat{n}_{i_\ell} = -\ell(\ell+1) C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \hat{n}_{i_2} \dots \hat{n}_{i_\ell},$$

where $C_{i_1 i_2 \dots i_\ell}^{(\ell)}$ is a symmetric traceless tensor and

$$\hat{\mathbf{n}} = \sin \theta \cos \phi \hat{\mathbf{e}}_1 + \sin \theta \sin \phi \hat{\mathbf{e}}_2 + \cos \theta \hat{\mathbf{e}}_3 \quad (\text{unit vector in } (\theta, \phi) \text{ direction})$$

12(a)(i) General solution to Laplace's equation:

$$V(\vec{\mathbf{r}}) = \sum_{\ell=0}^{\infty} \left(C_{i_1 i_2 \dots i_\ell}^{(\ell)} r^\ell + \frac{C'_{i_1 i_2 \dots i_\ell}^{(\ell)}}{r^{\ell+1}} \right) \hat{\mathbf{n}}_{i_1} \hat{\mathbf{n}}_{i_2} \dots \hat{\mathbf{n}}_{i_\ell}, \quad \text{where } \vec{\mathbf{r}} = r \hat{\mathbf{n}}$$

12(a)(ii) Azimuthal Symmetry:

$$V(\vec{\mathbf{r}}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{ \hat{\mathbf{z}}_{i_1} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell}$$

where $\{ \dots \}_{\text{TS}}$ denotes the traceless symmetric part of \dots

Special cases:

$$\{ 1 \}_{\text{TS}} = 1$$

$$\{ \hat{\mathbf{z}}_i \}_{\text{TS}} = \hat{\mathbf{z}}_i$$

$$\{ \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \}_{\text{TS}} = \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j - \frac{1}{3} \delta_{ij}$$

$$\{ \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \hat{\mathbf{z}}_k \}_{\text{TS}} = \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \hat{\mathbf{z}}_k - \frac{1}{5} (\hat{\mathbf{z}}_i \delta_{jk} + \hat{\mathbf{z}}_j \delta_{ik} + \hat{\mathbf{z}}_k \delta_{ij})$$

$$\begin{aligned} \{ \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \hat{\mathbf{z}}_k \hat{\mathbf{z}}_m \}_{\text{TS}} = & \hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \hat{\mathbf{z}}_k \hat{\mathbf{z}}_m - \frac{1}{7} (\hat{\mathbf{z}}_i \hat{\mathbf{z}}_j \delta_{km} + \hat{\mathbf{z}}_i \hat{\mathbf{z}}_k \delta_{mj} + \hat{\mathbf{z}}_i \hat{\mathbf{z}}_m \delta_{jk} \\ & + \hat{\mathbf{z}}_j \hat{\mathbf{z}}_k \delta_{im} + \hat{\mathbf{z}}_j \hat{\mathbf{z}}_m \delta_{ik} + \hat{\mathbf{z}}_k \hat{\mathbf{z}}_m \delta_{ij}) + \frac{1}{35} (\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) \end{aligned}$$

12(b) Legendre Polynomial / Spherical Harmonic expansion:

12(b)(i) Azimuthal Symmetry:

$$V(\vec{\mathbf{r}}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

$$P_\ell(\cos \theta) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \{ \hat{\mathbf{z}}_{i_1} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell}$$

12(b)(ii) General solution to Laplace's equation:

$$V(\vec{\mathbf{r}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

12(b)(iii) Orthonormality: $\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$

12(b)(iv) Spherical Harmonics in terms of Traceless Symmetric Tensors:

$$Y_{\ell,m}(\theta, \phi) = C_{i_1 \dots i_\ell}^{(\ell,m)} \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell} ,$$

where

$$C_{i_1 \dots i_\ell}^{(\ell,m)} = \begin{cases} N(\ell, m) \{ \hat{\mathbf{u}}_{i_1}^+ \dots \hat{\mathbf{u}}_{i_m}^+ \hat{\mathbf{z}}_{i_{m+1}} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} & \text{for } m \geq 0 , \\ N(\ell, m) \{ \hat{\mathbf{u}}_{i_1}^- \dots \hat{\mathbf{u}}_{i_{|m|}}^- \hat{\mathbf{z}}_{i_{|m|+1}} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} & \text{for } m \leq 0 , \end{cases}$$

$$\hat{\mathbf{u}}^+ \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y) , \quad \hat{\mathbf{u}}^- \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y)$$

$$N(\ell, m) = \frac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\frac{2^{|m|} (2\ell+1)}{4\pi (\ell+m)! (\ell-m)!}} \quad (\text{for } m \geq 0).$$

Connection between m and $-m$: $Y_{\ell,-m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$, which holds for all m .

12(b)(v) More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$$

where $P_\ell^m(\cos \theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

13. Electric Multipole Expansion:

13(a) First several terms:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2} \frac{\hat{\mathbf{r}}_i \hat{\mathbf{r}}_j}{r^3} Q_{ij} + \dots \right] , \text{ where}$$

$$Q = \int d^3x \rho(\vec{\mathbf{r}}) , \quad p_i = \int d^3x \rho(\vec{\mathbf{r}}) x_i \quad Q_{ij} = \int d^3x \rho(\vec{\mathbf{r}}) (3x_i x_j - \delta_{ij} |\vec{\mathbf{r}}|^2) ,$$

$$\vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \vec{\mathbf{p}}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{\mathbf{r}})$$

$$\vec{\nabla} \times \vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = 0 , \quad \vec{\nabla} \cdot \vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = \frac{1}{\epsilon_0} \rho_{\text{dip}}(\vec{\mathbf{r}}) = -\frac{1}{\epsilon_0} \vec{\mathbf{p}} \cdot \vec{\nabla} \delta^3(\vec{\mathbf{r}})$$

13(b) Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C_{i_1 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \dots \hat{n}_{i_\ell} ,$$

where

$$C_{i_1 \dots i_\ell}^{(\ell)} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \{x_{i_1} \dots x_{i_\ell}\}_{\text{TS}} d^3x \quad (\vec{r} \equiv r\hat{n} \equiv x_i \hat{e}_i)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{ \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \}_{\text{TS}} \hat{n}'_{i_1} \dots \hat{n}'_{i_\ell} , \quad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5) \dots 1 = \frac{(2\ell)!}{2^\ell \ell!} , \quad \text{with } (-1)!! \equiv 1 .$$

Reminder: $\{\dots\}_{\text{TS}}$ denotes the traceless symmetric part of \dots .

13(c) Griffiths version (azimuthal symmetry only):

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^\ell \rho(\vec{r}') P_\ell(\cos \theta') d^3x$$

where θ' = angle between \vec{r} and \vec{r}' .

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} P_\ell(\cos \theta') , \quad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell (x^2 - 1)^\ell , \quad (\text{Rodrigues' formula})$$

$$P_\ell(1) = 1 \quad P_\ell(-x) = (-1)^\ell P_\ell(x) \quad \int_{-1}^1 dx P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

13(d) Spherical Harmonic version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

$$\text{where } q_{\ell m} = \int Y_{\ell m}^* r'^\ell \rho(\vec{r}') d^3x'$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^\ell}{r^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) , \quad \text{for } r' < r$$

13(e) Properties of Traceless Symmetric Tensors:

13(e)(i) Trace Decomposition: Any symmetric tensor $S_{i_1 \dots i_\ell}$ can be written uniquely as

$$S_{i_1 \dots i_\ell} = S_{i_1 \dots i_\ell}^{(\text{TS})} + \text{Sym}_{i_1 \dots i_\ell} [M_{i_1 \dots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell}] ,$$

where $S_{i_1 \dots i_\ell}^{(\text{TS})}$ is a traceless symmetric tensor, $M_{i_1 \dots i_{\ell-2}}$ is a symmetric tensor, and $\text{Sym}_{i_1 \dots i_\ell} [xxx]$ symmetrizes xxx in the indices $i_1 \dots i_\ell$. $S_{i_1 \dots i_\ell}^{(\text{TS})}$ is called the *traceless symmetric part* of $S_{i_1 \dots i_\ell}$.

13(e)(ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1 \dots i_\ell}$:

$$\begin{aligned} \{S_{i_1 \dots i_\ell}\}_{\text{TS}} = S_{i_1 \dots i_\ell} + \text{Sym}_{i_1 \dots i_\ell} \Big[& a_{1,\ell} \delta_{i_1 i_2} \delta^{j_1 j_2} S_{j_1 j_2 i_3 \dots i_\ell} \\ & + a_{2,\ell} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta^{j_1 j_2} \delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5 \dots i_\ell} + \dots \Big] \end{aligned}$$

where

$$a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n! (\ell - 2n)! (\ell - n)! (2\ell)!}$$

The series terminates when the number of i indices on S is zero or one.

13(e)(iii) Integration:

$$\int d\Omega \hat{n}_{i_1} \dots \hat{n}_{i_{2\ell}} = 4\pi \frac{2^\ell \ell!}{(2\ell + 1)!} \sum_{\text{all pairings}} \delta_{i_1, i_2} \delta_{i_3, i_4} \dots \delta_{i_{2\ell-1}, i_{2\ell}}$$

$$\text{where } \int d\Omega \equiv \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi$$

The integral vanishes if the number of \hat{n} factors is odd.

$$\begin{aligned} \int d\Omega \left[C_{i_1 \dots i_\ell}^{(\ell)} \{ \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \}_{\text{TS}} \right] \left[C_{j_1 \dots j_{\ell'}}^{(\ell')} \{ \hat{n}_{j_1} \dots \hat{n}_{j_{\ell'}} \}_{\text{TS}} \right] \\ = \begin{cases} 4\pi \frac{2^\ell \ell!^2}{(2\ell + 1)!} C_{i_1 \dots i_\ell}^{(\ell)} C_{i_1 \dots i_\ell}^{(\ell)} & \text{if } \ell' = \ell \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

13(e)(iv) Other identities:

$$\hat{n}_{i_\ell} \{ \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \}_{\text{TS}} = \frac{\ell}{2\ell - 1} \{ \hat{n}_{i_1} \dots \hat{n}_{i_{\ell-1}} \}_{\text{TS}} \quad (\hat{n} = \text{any unit vector})$$

$$\hat{z}_{i_\ell} \{ \hat{u}_{i_1}^+ \dots \hat{u}_{i_m}^+ \hat{z}_{i_{m+1}} \dots \hat{z}_{i_\ell} \}_{\text{TS}} = \frac{(\ell + m)(\ell - m)}{\ell(2\ell - 1)} \{ \hat{u}_{i_1}^+ \dots \hat{u}_{i_m}^+ \hat{z}_{i_{m+1}} \dots \hat{z}_{i_{\ell-1}} \}_{\text{TS}}$$

$$\text{where } \hat{\mathbf{u}}^+ \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y)$$

*** For any symmetric traceless tensor $S_{i_1 \dots i_\ell}$,

$$\begin{aligned} \delta_{i_{\ell+2n-1}, i_{\ell+2n}} & \left\{ \text{Sym}_{i_1 \dots i_{\ell+2n}} \left[S_{i_1 \dots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \dots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}}_{n \text{ times}} \right] \right\} \\ & = F(n, \ell) \text{Sym}_{i_1 \dots i_{\ell+2n-2}} \left[S_{i_1 \dots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \dots \delta_{i_{\ell+2n-3}, i_{\ell+2n-2}}}_{n-1 \text{ times}} \right] \end{aligned}$$

where

$$F(n, \ell) = \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)}$$

14. Electric Fields in Matter:

14(a) Electric Dipoles:

$$\vec{\mathbf{p}} = \int d^3x \rho(\vec{\mathbf{r}}) \vec{\mathbf{r}}$$

$$\rho_{\text{dip}}(\vec{\mathbf{r}}) = -\vec{\mathbf{p}} \cdot \vec{\nabla}_{\vec{\mathbf{r}}} \delta^3(\vec{\mathbf{r}} - \vec{\mathbf{r}}_d), \text{ where } \vec{\mathbf{r}}_d = \text{position of dipole}$$

$$\vec{\mathbf{F}} = (\vec{\mathbf{p}} \cdot \vec{\nabla}) \vec{\mathbf{E}} = \vec{\nabla}(\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}) \quad (\text{force on a dipole})$$

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} \quad (\text{torque on a dipole})$$

$$U = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}} \quad (\text{potential energy})$$

14(b) Electrically Polarizable Materials:

$$\vec{\mathbf{P}}(\vec{\mathbf{r}}) = \text{polarization} = \text{electric dipole moment per unit volume}$$

$$\rho_{\text{bound}} = -\nabla \cdot \vec{\mathbf{P}}, \quad \sigma_{\text{bound}} = \vec{\mathbf{P}} \cdot \hat{\mathbf{n}}$$

$$\vec{\mathbf{D}} \equiv \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}, \quad \vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_{\text{free}}, \quad \vec{\nabla} \times \vec{\mathbf{E}} = 0$$

Boundary conditions:

$$E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0} \quad D_{\text{above}}^\perp - D_{\text{below}}^\perp = \sigma_{\text{free}}$$

$$\vec{\mathbf{E}}_{\text{above}}^\parallel - \vec{\mathbf{E}}_{\text{below}}^\parallel = 0 \quad \vec{\mathbf{D}}_{\text{above}}^\parallel - \vec{\mathbf{D}}_{\text{below}}^\parallel = \vec{\mathbf{P}}_{\text{above}}^\parallel - \vec{\mathbf{P}}_{\text{below}}^\parallel$$

14(c) Linear Dielectrics:

$$\vec{\mathbf{P}} = \epsilon_0 \chi_e \vec{\mathbf{E}}, \quad \chi_e = \text{electric susceptibility}$$

$$\epsilon \equiv \epsilon_0(1 + \chi_e) = \text{permittivity}, \quad \vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e = \text{relative permittivity, or dielectric constant}$$

Clausius-Mossotti equation: $\chi_e = \frac{n\alpha/\epsilon_0}{1 - \frac{n\alpha}{3\epsilon_0}}$, where n = number density of atoms

or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{p} = \alpha\vec{E}$)

Energy: $W = \frac{1}{2} \int \vec{D} \cdot \vec{E} d^3x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla}W$, where W is the potential energy stored in the system. Even if one or more potential differences are held fixed, the force can be found by computing the gradient of W with the total charge on each conductor fixed.

15. Magnetostatics:

15(a) Lorentz Force Law:

$$\vec{F} \equiv \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}), \quad \text{where } \vec{p} = \gamma m_0 \vec{v}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

15(b) Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

where μ_0 = permeability of free space $\simeq 4\pi \times 10^{-7}$ N/A² (to about 8-figure accuracy)

Examples:

Infinitely long straight wire: $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$

Infinitely long tightly wound solenoid: $\vec{B} = \mu_0 n I_0 \hat{z}$, where n = turns per unit length

Loop of current on axis: $\vec{B}(0, 0, z) = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{z}$

16. Table of Legendre Polynomials $P_\ell(x)$:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

17. Table of Spherical Harmonics $Y_{\ell m}(\theta, \phi)$:

$$\begin{aligned}
\ell = 0 \quad & Y_{00} = \frac{1}{\sqrt{4\pi}} \\
\ell = 1 \quad & \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \end{cases} \\
\ell = 2 \quad & \begin{cases} Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} \\ Y_{20} = \frac{1}{2} \sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta - 1) \end{cases} \\
\ell = 3 \quad & \begin{cases} Y_{33} = -\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{3i\phi} \\ Y_{32} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\phi} \\ Y_{31} = -\frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi} \\ Y_{30} = \frac{1}{2} \sqrt{\frac{7}{4\pi}} (5 \cos^3 \theta - 3 \cos \theta) \end{cases} \\
\ell = 4 \quad & \begin{cases} Y_{44} = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \sin^4 \theta e^{4i\phi} \\ Y_{43} = -\frac{3}{2} \sqrt{\frac{35}{2\pi}} \sin^3 \theta \cos \theta e^{3i\phi} \\ Y_{42} = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \sin^2 \theta (7 \cos^2 \theta - 1) e^{2i\phi} \\ Y_{41} = -\frac{3}{2} \sqrt{\frac{5}{2\pi}} \sin \theta (7 \cos^3 \theta - 3 \cos \theta) e^{i\phi} \\ Y_{40} = \frac{3}{\sqrt{4\pi}} (35 \cos^4 \theta - 30 \cos^2 \theta + 3) \end{cases}
\end{aligned}$$

For $m < 0$, use $Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$, which is valid for all m .