

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II
Prof. Alan Guth

December 8, 2019

FORMULA SHEET FOR FINAL EXAM

Revised December 18, 2019*

Exam Date: December 19, 2019

Table of Contents:

1. Index Notation
2. Vector Calculus
3. Fundamental Theorems of Vector Calculus
4. Vector Identities
5. Spherical Coordinates
6. Cylindrical Coordinates
7. Delta Functions
8. Electrostatics
9. Electrostatic Energy
10. Conductors
11. Separation of Variables for Laplace's Equation in Cartesian Coordinates
12. Separation of Variables for Laplace's Equation in Spherical Coordinates
 - (a). Traceless Symmetric Tensor expansion
 - (i). General solution to Laplace's equation
 - (ii). Azimuthal Symmetry
 - (b) Legendre Polynomial / Spherical Harmonic expansion
 - (i) Azimuthal Symmetry
 - (ii) General solution to Laplace's equation
 - (iii) Orthonormality
 - (iv) Spherical Harmonics in terms of Traceless Symmetric Tensors
 - (v) More Information about Spherical Harmonics
13. Electric Multipole Expansion
 - (a) First several terms
 - (b) Traceless Symmetric Tensor version
 - (c) Griffiths version (azimuthal symmetry only)
 - (d) Spherical Harmonic version
 - (e) Properties of Traceless Symmetric Tensors
 - (i) Trace Decomposition
 - (ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1 \dots i_\ell}$:
 - (iii) Integration
 - (iv) Other identities

* In Sec. 29(e), the coefficient in the equation for $\vec{A}(\vec{r}, t)$ has been corrected. In Sec. 30(b), a minus sign in the equation for \vec{E} has been deleted.

14. Electric Fields in Matter
 - (a) Electric Dipoles
 - (b) Electrically Polarizable Materials
 - (c) Linear Dielectrics
15. Magnetostatics
 - (a) Lorentz Force Law
 - (b) Current Density
 - (c) Biot-Savart Law
 - (d) Vector Potential
 - (e) Ampère's Law
16. Magnetic Multipole Expansion
 - (a) Traceless Symmetric Tensor version
 - (b) Griffiths version (azimuthal symmetry only, current in wires)
17. Magnetic Fields in Matter
 - (a) Magnetic Dipoles
 - (b) Magnetically Polarizable Materials
 - (c) Linear Magnetic Materials
18. Magnetic Monopoles
19. Maxwell's Equations
20. Maxwell's Equations in Matter
21. Boundary Conditions
22. Maxwell's Equations with Magnetic (Monopole) Charge
23. Current, Resistance, and Ohm's Law
24. Inductance
25. Circuit Elements
26. Conservation Laws
27. Wave Equation in 1 Dimension
28. Electromagnetic Waves
 - (a) Linearly Polarized Plane Waves
 - (b) Reflection and Transmission at Normal Incidence
29. Electromagnetic Potentials
 - (a) The fields
 - (b) Gauge transformations
 - (c) Coulomb gauge
 - (d) Lorenz gauge
 - (e) Retarded time solutions (Lorenz gauge)
 - (f) Liénard-Wiechert Potentials (potentials of a point charge)
 - (g) Fields of a point charge
30. Radiation
 - (a) Radiation from an oscillating electric dipole along the z axis
 - (b) Magnetic Dipole Radiation
 - (c) General Electric Dipole Radiation
 - (d) Multipole Expansion for Radiation
 - (e) Radiation from a Point Particle

31. Radiation Reaction
 - (a) Abraham-Lorentz formula
32. Table of Legendre Polynomials $P_\ell(x)$
33. Table of Spherical Harmonics $Y_{\ell m}(\theta, \phi)$

A few items below are marked with asterisks, ***. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

1. Index Notation:

Unit Vectors: $\hat{x} \equiv \hat{i} \equiv \hat{e}_x \equiv \hat{e}_1$, $\hat{y} \equiv \hat{j} \equiv \hat{e}_y \equiv \hat{e}_2$, $\hat{z} \equiv \hat{k} \equiv \hat{e}_z \equiv \hat{e}_3$, $\vec{A} \equiv A_i \hat{e}_i$

General vector: $\vec{r} = x \hat{x} + y \hat{y} + z \hat{z} = x_i \hat{e}_i$

Dot Product, Cross Product, and Determinant:

$$\vec{A} \cdot \vec{B} = A_i B_i, \quad \vec{A} \times \vec{B}_i = \epsilon_{ijk} A_j B_k, \quad \epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \dots i_n} A_{1, i_1} A_{2, i_2} \dots A_{n, i_n} \quad ***$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j, \quad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \quad (R^T T = I)$$

$$\text{Rotation about } z\text{-axis by } \phi: R_z(\phi)_{ij} = \begin{matrix} & \begin{matrix} j=1 & j=2 & j=3 \end{matrix} \\ \begin{matrix} i=1 \\ i=2 \\ i=3 \end{matrix} & \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Rotation about axis \hat{n} by ϕ : ***

$$R(\hat{n}, \phi)_{ij} = \delta_{ij} \cos \phi + \hat{n}_i \hat{n}_j (1 - \cos \phi) - \epsilon_{ijk} \hat{n}_k \sin \phi.$$

2. Vector Calculus:

$$\text{Gradient: } (\vec{\nabla} \varphi)_i = \partial_i \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y} + \frac{\partial \varphi}{\partial z} \hat{z}, \quad \partial_i \equiv \frac{\partial}{\partial x_i}$$

$$\text{Divergence: } \vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{Curl: } (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}$$

$$\text{Laplacian: } \nabla^2 \varphi = \vec{\nabla} \cdot (\vec{\nabla} \varphi) = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

3. Fundamental Theorems of Vector Calculus:

Gradient:
$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} \varphi \cdot d\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence:
$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{A} d^3x = \oint_S \vec{A} \cdot d\vec{a}$$

where S is the boundary of \mathcal{V}

Curl:
$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_P \vec{A} \cdot d\vec{\ell}$$

where P is the boundary of S

4. Vector Identities:

Triple Products:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Product Rules:

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$$

$$\vec{\nabla} \cdot (f\vec{A}) = f\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla}f$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times (f\vec{A}) = f\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla}f$$

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

Second Derivatives:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

5. Spherical Coordinates:

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right)$$

$$\phi = \tan^{-1} (y/x)$$

$$\hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$$

$$\hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$$

$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$$

Point separation: $d\vec{\ell} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$

Volume element: $d^3x \rightarrow r^2 \sin \theta dr d\theta d\phi$

Gradient: $\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \varphi}{\partial \phi} \hat{\phi}$

Divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$

Curl:
$$\begin{aligned} \vec{\nabla} \times \vec{A} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} \\ & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

Laplacian: $\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$

6. Cylindrical Coordinates:

$x = s \cos \phi$ $s = \sqrt{x^2 + y^2}$

$y = s \sin \phi$ $\phi = \tan^{-1}(y/x)$

$z = z$ $z = z$

$\hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y}$ $\hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi}$

$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$ $\hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi}$

$\hat{z} = \hat{z}$ $\hat{z} = \hat{z}$

Point separation: $d\vec{\ell} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$

Volume element: $d^3x \rightarrow s ds d\phi dz$

Gradient: $\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial \varphi}{\partial \phi} \hat{\phi} + \frac{\partial \varphi}{\partial z} \hat{z}$

Divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} (s A_s) + \frac{1}{s} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

Curl:
$$\begin{aligned} \vec{\nabla} \times \vec{A} = & \left[\frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right] \hat{\phi} \\ & + \frac{1}{s} \left[\frac{\partial}{\partial s} (s A_\phi) - \frac{\partial A_s}{\partial \phi} \right] \hat{z} \end{aligned}$$

Laplacian: $\nabla^2 \varphi = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial \varphi}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2}$

7. Delta Functions:

$$\int \varphi(x) \delta(x - x') dx = \varphi(x'), \quad \int \varphi(\vec{r}) \delta^3(\vec{r} - \vec{r}') d^3x = \varphi(\vec{r}')$$

$$\int \varphi(x) \frac{d}{dx} \delta(x - x') dx = - \left. \frac{d\varphi}{dx} \right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \quad \text{where } i \text{ is summed over all points for which } g(x_i) = 0$$

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -\vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = -4\pi \delta^3(\vec{r} - \vec{r}')$$

$$\partial_i \partial_j \left(\frac{1}{r} \right) = -\partial_i \left(\frac{\hat{r}_j}{r^2} \right) = -\partial_i \left(\frac{x_j}{r^3} \right) = \frac{3\hat{r}_i \hat{r}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3} \delta_{ij} \delta^3(\vec{r})$$

8. Electrostatics:

$\vec{F} = q\vec{E}$, where

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{(\vec{r} - \vec{r}') q_i}{|\vec{r} - \vec{r}'|^3} = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d^3x'$$

ϵ_0 = permittivity of free space = 8.854×10^{-12} C²/(N·m²)

$$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \text{ N·m}^2/\text{C}^2$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}') \cdot d\vec{\ell}' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x'$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = 0, \quad \vec{E} = -\vec{\nabla} V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (\text{Poisson's Eq.}), \quad \rho = 0 \implies \nabla^2 V = 0 \quad (\text{Laplace's Eq.})$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

9. Electrostatic Energy:*

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{\substack{ij \\ i \neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int d^3x d^3x' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$W = \frac{1}{2} \int d^3x \rho(\vec{r}) V(\vec{r}) = \frac{1}{2} \epsilon_0 \int |\vec{E}|^2 d^3x$$

* See Sec. 14(c) for energy in the presence of dielectrics.

10. Conductors:

Just outside, $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$

Pressure on surface: $\frac{1}{2} \sigma |\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2} CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j, \quad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j, \quad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a : Image of Q at R is $q = -\frac{a}{R}Q$, $r = \frac{a^2}{R}$

11. Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix} \begin{Bmatrix} \cos \beta y \\ \sin \beta y \end{Bmatrix} \begin{Bmatrix} \cosh \gamma z \\ \sinh \gamma z \end{Bmatrix} \quad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

or, more generally,

$$V = \begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix} \begin{Bmatrix} \cos \beta y \\ \sin \beta y \end{Bmatrix} \begin{Bmatrix} \cos \gamma z \\ \sin \gamma z \end{Bmatrix}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$, each of α , β , and γ can be real or imaginary, with $\sin(i\theta) = i \sinh \theta$ and $\cos(i\theta) = \cosh \theta$. $\begin{Bmatrix} \cos \alpha x \\ \sin \alpha x \end{Bmatrix}$ means any linear combination of $\cos \alpha x$ or $\sin \alpha x$, but usually one or the other suffices.

12. Separation of Variables for Laplace's Equation in Spherical Coordinates:**12(a) Traceless Symmetric Tensor expansion:**

$$\nabla^2 \varphi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \nabla_{\text{ang}}^2 \varphi = 0,$$

where the angular part is given by

$$\nabla_{\text{ang}}^2 \varphi \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$$

$$\nabla_{\text{ang}}^2 C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \hat{n}_{i_2} \dots \hat{n}_{i_\ell} = -\ell(\ell+1) C_{i_1 i_2 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \hat{n}_{i_2} \dots \hat{n}_{i_\ell},$$

where $C_{i_1 i_2 \dots i_\ell}^{(\ell)}$ is a symmetric traceless tensor and

$$\hat{n} = \sin \theta \cos \phi \hat{e}_1 + \sin \theta \sin \phi \hat{e}_2 + \cos \theta \hat{e}_3 \quad (\text{unit vector in } (\theta, \phi) \text{ direction})$$

12(a)(i) General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C_{i_1 i_2 \dots i_\ell}^{(\ell)} r^\ell + \frac{C_{i_1 i_2 \dots i_\ell}'^{(\ell)}}{r^{\ell+1}} \right) \hat{n}_{i_1} \hat{n}_{i_2} \dots \hat{n}_{i_\ell}, \quad \text{where } \vec{r} = r \hat{n}$$

12(a)(ii) Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{ \hat{z}_{i_1} \dots \hat{z}_{i_\ell} \}_{\text{TS}} \hat{n}_{i_1} \dots \hat{n}_{i_\ell}$$

where $\{ \dots \}_{\text{TS}}$ denotes the traceless symmetric part of \dots

Special cases:

$$\{ 1 \}_{\text{TS}} = 1$$

$$\{ \hat{z}_i \}_{\text{TS}} = \hat{z}_i$$

$$\{ \hat{z}_i \hat{z}_j \}_{\text{TS}} = \hat{z}_i \hat{z}_j - \frac{1}{3} \delta_{ij}$$

$$\{ \hat{z}_i \hat{z}_j \hat{z}_k \}_{\text{TS}} = \hat{z}_i \hat{z}_j \hat{z}_k - \frac{1}{5} (\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij})$$

$$\begin{aligned} \{ \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \}_{\text{TS}} = & \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \frac{1}{7} (\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} \\ & + \hat{z}_j \hat{z}_k \delta_{im} + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij}) + \frac{1}{35} (\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) \end{aligned}$$

12(b) Legendre Polynomial / Spherical Harmonic expansion:

12(b)(i) Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

$$P_\ell(\cos \theta) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \{ \hat{z}_{i_1} \dots \hat{z}_{i_\ell} \}_{\text{TS}} \hat{n}_{i_1} \dots \hat{n}_{i_\ell}$$

12(b)(ii) General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

12(b)(iii) Orthonormality: $\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$

12(b)(iv) Spherical Harmonics in terms of Traceless Symmetric Tensors:

$$Y_{\ell,m}(\theta, \phi) = C_{i_1 \dots i_\ell}^{(\ell,m)} \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell} ,$$

where

$$C_{i_1 \dots i_\ell}^{(\ell,m)} = \begin{cases} N(\ell, m) \{ \hat{\mathbf{u}}_{i_1}^+ \dots \hat{\mathbf{u}}_{i_m}^+ \hat{\mathbf{z}}_{i_{m+1}} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} & \text{for } m \geq 0 , \\ N(\ell, m) \{ \hat{\mathbf{u}}_{i_1}^- \dots \hat{\mathbf{u}}_{i_{|m|}}^- \hat{\mathbf{z}}_{i_{|m|+1}} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} & \text{for } m \leq 0 , \end{cases}$$

$$\hat{\mathbf{u}}^+ \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y) , \quad \hat{\mathbf{u}}^- \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y)$$

$$N(\ell, m) = \frac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\frac{2^{|m|} (2\ell+1)}{4\pi (\ell+m)! (\ell-m)!}} \quad (\text{for } m \geq 0).$$

Connection between m and $-m$: $Y_{\ell,-m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$, which holds for all m .

12(b)(v) More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$$

where $P_\ell^m(\cos \theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

13. Electric Multipole Expansion:

13(a) First several terms:

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}_i \hat{\mathbf{r}}_j}{r^3} Q_{ij} + \dots \right] , \text{ where}$$

$$Q = \int d^3x \rho(\vec{\mathbf{r}}) , \quad p_i = \int d^3x \rho(\vec{\mathbf{r}}) x_i , \quad Q_{ij} = \frac{1}{2} \int d^3x \rho(\vec{\mathbf{r}}) (3x_i x_j - \delta_{ij} |\vec{\mathbf{r}}|^2)$$

$$\vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \vec{\mathbf{p}}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{\mathbf{r}})$$

$$\vec{\nabla} \times \vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = 0 , \quad \vec{\nabla} \cdot \vec{\mathbf{E}}_{\text{dip}}(\vec{\mathbf{r}}) = \frac{1}{\epsilon_0} \rho_{\text{dip}}(\vec{\mathbf{r}}) = -\frac{1}{\epsilon_0} \vec{\mathbf{p}} \cdot \vec{\nabla} \delta^3(\vec{\mathbf{r}})$$

13(b) Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C_{i_1 \dots i_\ell}^{(\ell)} \hat{n}_{i_1} \dots \hat{n}_{i_\ell} ,$$

where

$$C_{i_1 \dots i_\ell}^{(\ell)} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \{x_{i_1} \dots x_{i_\ell}\}_{\text{TS}} d^3x \quad (\vec{r} \equiv r\hat{n} \equiv x_i \hat{e}_i)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{ \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \}_{\text{TS}} \hat{n}'_{i_1} \dots \hat{n}'_{i_\ell} , \quad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5) \dots 1 = \frac{(2\ell)!}{2^\ell \ell!} , \quad \text{with } (-1)!! \equiv 1 .$$

Reminder: $\{\dots\}_{\text{TS}}$ denotes the traceless symmetric part of \dots .

13(c) Griffiths version (azimuthal symmetry only):

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^\ell \rho(\vec{r}') P_\ell(\cos \theta') d^3x$$

where θ' = angle between \vec{r} and \vec{r}' .

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} P_\ell(\cos \theta') , \quad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell (x^2 - 1)^\ell , \quad (\text{Rodrigues' formula})$$

$$P_\ell(1) = 1 \quad P_\ell(-x) = (-1)^\ell P_\ell(x) \quad \int_{-1}^1 dx P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

13(d) Spherical Harmonic version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

$$\text{where } q_{\ell m} = \int Y_{\ell m}^* r'^\ell \rho(\vec{r}') d^3x'$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^\ell}{r^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) , \quad \text{for } r' < r$$

13(e) Properties of Traceless Symmetric Tensors:

13(e)(i) Trace Decomposition: Any symmetric tensor $S_{i_1 \dots i_\ell}$ can be written uniquely as

$$S_{i_1 \dots i_\ell} = S_{i_1 \dots i_\ell}^{(\text{TS})} + \text{Sym}_{i_1 \dots i_\ell} [M_{i_1 \dots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell}] ,$$

where $S_{i_1 \dots i_\ell}^{(\text{TS})}$ is a traceless symmetric tensor, $M_{i_1 \dots i_{\ell-2}}$ is a symmetric tensor, and $\text{Sym}_{i_1 \dots i_\ell} [xxx]$ symmetrizes xxx in the indices $i_1 \dots i_\ell$. $S_{i_1 \dots i_\ell}^{(\text{TS})}$ is called the *traceless symmetric part* of $S_{i_1 \dots i_\ell}$.

13(e)(ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1 \dots i_\ell}$:

$$\begin{aligned} \{S_{i_1 \dots i_\ell}\}_{\text{TS}} = & S_{i_1 \dots i_\ell} + \text{Sym}_{i_1 \dots i_\ell} \left[a_{1,\ell} \delta_{i_1 i_2} \delta^{j_1 j_2} S_{j_1 j_2 i_3 \dots i_\ell} \right. \\ & \left. + a_{2,\ell} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta^{j_1 j_2} \delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5 \dots i_\ell} + \dots \right] \end{aligned}$$

where

$$a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n! (\ell - 2n)! (\ell - n)! (2\ell)!}$$

The series terminates when the number of i indices on S is zero or one.

13(e)(iii) Integration:

$$\int d\Omega \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_{2\ell}} = 4\pi \frac{2^\ell \ell!}{(2\ell + 1)!} \sum_{\text{all pairings}} \delta_{i_1, i_2} \delta_{i_3, i_4} \dots \delta_{i_{2\ell-1}, i_{2\ell}}$$

$$\text{where } \int d\Omega \equiv \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi$$

The integral vanishes if the number of $\hat{\mathbf{n}}$ factors is odd.

$$\begin{aligned} \int d\Omega \left[C_{i_1 \dots i_\ell}^{(\ell)} \{ \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell} \}_{\text{TS}} \right] \left[C_{j_1 \dots j_{\ell'}}^{(\ell')} \{ \hat{\mathbf{n}}_{j_1} \dots \hat{\mathbf{n}}_{j_{\ell'}} \}_{\text{TS}} \right] \\ = \begin{cases} 4\pi \frac{2^\ell \ell!^2}{(2\ell + 1)!} C_{i_1 \dots i_\ell}^{(\ell)} C_{i_1 \dots i_\ell}^{(\ell)} & \text{if } \ell' = \ell \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

13(e)(iv) Other identities:

$$\hat{\mathbf{n}}_{i_\ell} \{ \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_\ell} \}_{\text{TS}} = \frac{\ell}{2\ell - 1} \{ \hat{\mathbf{n}}_{i_1} \dots \hat{\mathbf{n}}_{i_{\ell-1}} \}_{\text{TS}} \quad (\hat{\mathbf{n}} = \text{any unit vector})$$

$$\hat{\mathbf{z}}_{i_\ell} \{ \hat{\mathbf{u}}_{i_1}^+ \dots \hat{\mathbf{u}}_{i_m}^+ \hat{\mathbf{z}}_{i_{m+1}} \dots \hat{\mathbf{z}}_{i_\ell} \}_{\text{TS}} = \frac{(\ell + m)(\ell - m)}{\ell(2\ell - 1)} \{ \hat{\mathbf{u}}_{i_1}^+ \dots \hat{\mathbf{u}}_{i_m}^+ \hat{\mathbf{z}}_{i_{m+1}} \dots \hat{\mathbf{z}}_{i_{\ell-1}} \}_{\text{TS}}$$

$$\text{where } \hat{\mathbf{u}}^+ \equiv \frac{1}{\sqrt{2}} (\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y)$$

*** For any symmetric traceless tensor $S_{i_1 \dots i_\ell}$,

$$\delta_{i_{\ell+2n-1}, i_{\ell+2n}} \left\{ \text{Sym}_{i_1 \dots i_{\ell+2n}} \left[S_{i_1 \dots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \dots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}}_{n \text{ Kronecker } \delta\text{-functions}} \right] \right\}$$

$$= F(n, \ell) \text{Sym}_{i_1 \dots i_{\ell+2n-2}} \left[S_{i_1 \dots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \dots \delta_{i_{\ell+2n-3}, i_{\ell+2n-2}}}_{n-1 \text{ Kronecker } \delta\text{-functions}} \right]$$

where

$$F(n, \ell) = \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)}$$

14. Electric Fields in Matter:

14(a) Electric Dipoles:

$$\vec{p} = \int d^3x \rho(\vec{r}) \vec{r}$$

$$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla}_{\vec{r}} \delta^3(\vec{r} - \vec{r}_d), \text{ where } \vec{r}_d = \text{position of dipole}$$

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E}) \quad (\text{force on a dipole})$$

$$\vec{\tau} = \vec{p} \times \vec{E} \quad (\text{torque on a dipole})$$

$$U = -\vec{p} \cdot \vec{E} \quad (\text{potential energy})$$

14(b) Electrically Polarizable Materials:

$$\vec{P}(\vec{r}) = \text{polarization} = \text{electric dipole moment per unit volume}$$

$$\rho_{\text{bound}} = -\nabla \cdot \vec{P}, \quad \sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$$

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}, \quad \vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}, \quad \vec{\nabla} \times \vec{E} = 0$$

Boundary conditions:

$$E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0} \quad D_{\text{above}}^\perp - D_{\text{below}}^\perp = \sigma_{\text{free}}$$

$$\vec{E}_{\text{above}}^\parallel - \vec{E}_{\text{below}}^\parallel = 0 \quad \vec{D}_{\text{above}}^\parallel - \vec{D}_{\text{below}}^\parallel = \vec{P}_{\text{above}}^\parallel - \vec{P}_{\text{below}}^\parallel$$

14(c) Linear Dielectrics:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}, \quad \chi_e = \text{electric susceptibility}$$

$$\epsilon \equiv \epsilon_0(1 + \chi_e) = \text{permittivity}, \quad \vec{D} = \epsilon \vec{E}$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e = \text{relative permittivity, or dielectric constant}$$

Clausius-Mossotti equation: $\chi_e = \frac{n\alpha/\epsilon_0}{1 - \frac{n\alpha}{3\epsilon_0}}$, where n = number density of atoms

or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{p} = \alpha\vec{E}$)

Energy: $W = \frac{1}{2} \int \vec{D} \cdot \vec{E} d^3x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla}W$, where W is the potential energy stored in the system. Even if one or more potential differences are held fixed, the force can be found by computing the gradient of W with the total charge on each conductor fixed.

15. Magnetostatics:

15(a) Lorentz Force Law:

$$\vec{F} \equiv \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}), \quad \text{where } \vec{p} = \gamma m_0 \vec{v}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F} = \int I d\vec{\ell} \times \vec{B} = \int \vec{J} \times \vec{B} d^3x$$

15(b) Current Density:

$$\text{Current through a surface } S: I_S = \int_S \vec{J} \cdot d\vec{a}$$

$$\text{Charge conservation: } \frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

$$\text{Moving density of charge: } \vec{J} = \rho \vec{v}$$

15(c) Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

where μ_0 = permeability of free space $\simeq 4\pi \times 10^{-7}$ N/A² (to about 8-figure accuracy)

Examples:

$$\text{Infinitely long straight wire: } \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

$$\text{Infinitely long tightly wound solenoid: } \vec{B} = \mu_0 n I_0 \hat{z}, \text{ where } n = \text{turns per unit length}$$

$$\text{Loop of current on axis: } \vec{B}(0, 0, z) = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{z}$$

15(d) Vector Potential:

$$\vec{A}(\vec{r})_{\text{coul}} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x' , \quad \vec{B} = \vec{\nabla} \times \vec{A} , \quad \vec{\nabla} \cdot \vec{A}_{\text{coul}} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \text{ (Subject to modification if magnetic monopoles are discovered)}$$

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

15(e) Ampère's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} , \text{ or equivalently } \int_P \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$$

16. Magnetic Multipole Expansion:

16(a) Traceless Symmetric Tensor version:

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}_{j;i_1 i_2 \dots i_\ell}^{(\ell)} \frac{\{\hat{r}_{i_1} \dots \hat{r}_{i_\ell}\}_{\text{TS}}}{r^{\ell+1}}$$

$$\text{where } \mathcal{M}_{j;i_1 i_2 \dots i_\ell}^{(\ell)} = \frac{(2\ell-1)!!}{\ell!} \int d^3x J_j(\vec{r}) \{x_{i_1} \dots x_{i_\ell}\}_{\text{TS}}$$

$$\text{Current conservation restriction: } \int_{i_1 \dots i_\ell} d^3x \text{Sym}(x_{i_1} \dots x_{i_{\ell-1}} J_{i_\ell}) = 0$$

where $\text{Sym}_{i_1 \dots i_\ell}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1 \dots i_\ell$

Special cases:

$$\ell = 1: \int d^3x J_i = 0$$

$$\ell = 2: \int d^3x (J_i x_j + J_j x_i) = 0$$

$$\text{Leading term (dipole): } \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} ,$$

where

$$m_i = -\frac{1}{2} \epsilon_{ijk} \mathcal{M}_{j;k}^{(1)}$$

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times d\vec{\ell} = \frac{1}{2} \int d^3x \vec{r} \times \vec{J} = I \vec{a} ,$$

$$\text{where } \vec{a} = \int_S d\vec{a} \text{ for any surface } S \text{ spanning } P$$

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3} \vec{m} \delta^3(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}_{\text{dip}}(\vec{r}) = 0, \quad \vec{\nabla} \times \vec{B}_{\text{dip}}(\vec{r}) = \mu_0 \vec{J}_{\text{dip}}(\vec{r}) = -\mu_0 \vec{m} \times \vec{\nabla} \delta^3(\vec{r})$$

16(b) Griffiths version (azimuthal symmetry only, current in wires):

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \oint (r')^{\ell} P_{\ell}(\cos \theta') d\vec{\ell}', \quad \text{where } \theta' = \text{angle between } \vec{r} \text{ and } \vec{r}'.$$

17. Magnetic Fields in Matter:

17(a) Magnetic Dipoles:

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times d\vec{\ell} = \frac{1}{2} \int d^3x \vec{r} \times \vec{J} = I \vec{a}, \quad \text{where } \vec{a} = \int_S d\vec{a} \text{ for any } S \text{ spanning } P$$

$$\vec{J}_{\text{dip}}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}} \delta^3(\vec{r} - \vec{r}_d), \quad \text{where } \vec{r}_d = \text{position of dipole}$$

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) \quad (\text{force on a dipole})$$

$$\vec{\tau} = \vec{m} \times \vec{B} \quad (\text{torque on a dipole})$$

$$U = -\vec{m} \cdot \vec{B} \quad (\text{potential energy})$$

17(b) Magnetically Polarizable Materials:

$\vec{M}(\vec{r})$ = magnetization = magnetic dipole moment per unit volume

$$\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}, \quad \vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$$

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}, \quad \vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

Boundary conditions:

$$B_{\text{above}}^{\perp} - B_{\text{below}}^{\perp} = 0 \quad H_{\text{above}}^{\perp} - H_{\text{below}}^{\perp} = -(M_{\text{above}}^{\perp} - M_{\text{below}}^{\perp})$$

$$\vec{B}_{\text{above}}^{\parallel} - \vec{B}_{\text{below}}^{\parallel} = \mu_0(\vec{K} \times \hat{n}) \quad \vec{H}_{\text{above}}^{\parallel} - \vec{H}_{\text{below}}^{\parallel} = \vec{K}_{\text{free}} \times \hat{n}$$

17(c) Linear Magnetic Materials:

$$\vec{M} = \chi_m \vec{H}, \quad \chi_m = \text{magnetic susceptibility}$$

$$\mu = \mu_0(1 + \chi_m) = \text{permeability}, \quad \vec{B} = \mu \vec{H}$$

18. Magnetic Monopoles:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{r}; \quad \text{Force on a static monopole: } \vec{F} = q_m \vec{B}$$

Angular momentum of monopole/charge system: $\vec{L} = \frac{\mu_0 q_e q_m}{4\pi} \hat{r}$, where \hat{r} points from q_e to q_m

$$\text{Dirac quantization condition: } \frac{\mu_0 q_e q_m}{4\pi} = \frac{1}{2} \hbar \times \text{integer}$$

19. Maxwell's Equations:

$$\begin{aligned}
\text{(i)} \quad \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho & \text{(iii)} \quad \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\text{(ii)} \quad \vec{\nabla} \cdot \vec{B} &= 0 & \text{(iv)} \quad \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

$$\text{where } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\text{Lorentz force law: } \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\text{Charge conservation: } \frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

20. Maxwell's Equations in Matter:

Polarization \vec{P} and magnetization \vec{M} :

$$\rho_b = -\vec{\nabla} \cdot \vec{P}, \quad \vec{J}_P = \frac{\partial \vec{P}}{\partial t}, \quad \vec{J}_b = \vec{\nabla} \times \vec{M}, \quad \rho = \rho_f + \rho_b, \quad \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P$$

Polarization current if the matter is in motion, with velocity $\vec{v}(\vec{r}, t)$: ***

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{P})$$

$$\text{Auxiliary Fields: } \vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}, \quad \vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$$

Maxwell's Equations:

$$\begin{aligned}
\text{(i)} \quad \vec{\nabla} \cdot \vec{D} &= \rho_f & \text{(iii)} \quad \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\text{(ii)} \quad \vec{\nabla} \cdot \vec{B} &= 0 & \text{(iv)} \quad \vec{\nabla} \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t}
\end{aligned}$$

For linear media:

$$\vec{D} = \epsilon \vec{E}, \quad \vec{H} = \frac{1}{\mu} \vec{B}$$

where ϵ = dielectric constant, μ = relative permeability

$$\vec{J}_d \equiv \frac{\partial \vec{D}}{\partial t} = \text{displacement current}$$

21. Boundary Conditions:

$$\begin{aligned}
D_1^\perp - D_2^\perp &= \sigma_f & \vec{E}_1^\parallel - \vec{E}_2^\parallel &= 0 \\
E_1^\perp - E_2^\perp &= \frac{1}{\epsilon_0} \sigma & \vec{D}_1^\parallel - \vec{D}_2^\parallel &= \vec{P}_1^\parallel - \vec{P}_2^\parallel
\end{aligned}$$

$$\begin{aligned}
B_1^\perp - B_2^\perp &= 0 & \vec{H}_1^\parallel - \vec{H}_2^\parallel &= -\hat{n} \times \vec{K}_f \\
H_1^\perp - H_2^\perp &= M_2^\perp - M_1^\perp & \vec{B}_1^\parallel - \vec{B}_2^\parallel &= -\mu_0 \hat{n} \times \vec{K}
\end{aligned}$$

22. Maxwell's Equations with Magnetic (Monopole) Charge:

$$\begin{aligned}
 \text{(i)} \quad \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho_e & \text{(iii)} \quad \vec{\nabla} \times \vec{E} &= -\mu_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t} , \\
 \text{(ii)} \quad \vec{\nabla} \cdot \vec{B} &= \mu_0 \rho_m & \text{(iv)} \quad \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}_e + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}
 \end{aligned}$$

Magnetic Lorentz force law: $\vec{F} = q_m \left(\vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E} \right)$

23. Current, Resistance, and Ohm's Law:

$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$, where σ = conductivity. $\rho = 1/\sigma$ = resistivity

Resistors: $V = IR$, $P = IV = I^2 R = V^2/R$

Resistance in a wire: $R = \frac{\ell}{A} \rho$, where ℓ = length, A = cross-sectional area, and ρ = resistivity

Charging an RC circuit: $I = \frac{V_0}{R} e^{-t/RC}$, $Q = CV_0 [1 - e^{-t/RC}]$

Emf (Electromotive force): $\mathcal{E} \equiv \oint (\vec{f}_s + \vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell}$. Here \vec{f}_s is the force per unit charge of any external power source, such as a battery. \vec{v} can be either the velocity of the wire, or the velocity of the charge carriers (since the velocity of the charge carriers relative to the wire points along the wire, and gives no contribution to $(\vec{v} \times \vec{B}) \cdot d\vec{\ell}$).

24. Inductance:

Universal flux rule: Whenever the flux through a loop changes, whether due to a changing \vec{B} or motion of the loop, $\mathcal{E} = -\frac{d\Phi_B}{dt}$, where Φ_B is the magnetic flux through the loop. Here \mathcal{E} is the emf induced by the magnetic field, and does not include possible contributions from sources \vec{f}_s .

Mutual inductance: $\Phi_2 = M_{21} I_1$, M_{21} = mutual inductance


(Franz) Neumann's formula: $M_{21} = M_{12} = \frac{\mu_0}{4\pi} \oint_{P_1} \oint_{P_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$


Self inductance: $\Phi = LI$, $\mathcal{E} = -L \frac{dI}{dt}$; L = inductance

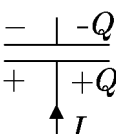
Self inductance of a solenoid: $L = n^2 \mu_0 \mathcal{V}$, where n = number of turns per length, \mathcal{V} = volume

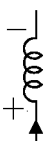
Rising current in an RL circuit: $I = \frac{V_0}{R} [1 - e^{-\frac{R}{L} t}]$

25. Circuit Elements:

Voltage source: V_0  $V_+ - V_- \equiv V_0$

Resistor: R  $V_+ - V_- \equiv V_R = IR$

Capacitor: C  $V_+ - V_- \equiv V_C$, $Q = CV_C$, $\frac{dQ}{dt} = I$.

Inductor: L  $V_+ - V_- \equiv V_L = L \frac{dI}{dt}$

26. Conservation Laws:

Energy density: $u_{\text{EM}} = \frac{1}{2} \left[\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right]$

Poynting vector (flow of energy): $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

Conservation of energy:

Integral form: $\frac{d}{dt} [U_{\text{EM}} + U_{\text{mech}}] = - \int \vec{S} \cdot d\vec{a}$

Differential form: $\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S}$, where $u = u_{\text{EM}} + u_{\text{mech}}$

Momentum density: $\vec{\wp}_{\text{EM}} = \frac{1}{c^2} \vec{S}$; $\frac{1}{c^2} S_i$ is the density of momentum in the i 'th direction

Maxwell stress tensor: $T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} |\vec{E}|^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} |\vec{B}|^2 \right)$

where $-T_{ij} = -T_{ji}$ = flow in j 'th direction of momentum in the i 'th direction

Conservation of momentum:

Integral form: $\frac{d}{dt} \left(P_{\text{mech},i} + \frac{1}{c^2} \int_{\mathcal{V}} S_i d^3x \right) = \oint_S T_{ij} da_j$, for a volume \mathcal{V} bounded by a surface S

Differential form: $\frac{\partial}{\partial t} (\wp_{\text{mech},i} + \wp_{\text{EM},i}) = \partial_j T_{ji}$

Angular momentum:

Angular momentum density (about the origin):

$$\vec{\ell}_{\text{EM}} = \vec{r} \times \vec{\wp}_{\text{EM}} = \epsilon_0 [\vec{r} \times (\vec{E} \times \vec{B})]$$

27. Wave Equation in 1 Dimension:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0, \text{ where } v \text{ is the wave velocity}$$

Sinusoidal waves:

$$f(z, t) = A \cos[k(z - vt) + \delta] = A \cos[kz - \omega t + \delta]$$

where

$$\begin{aligned} \omega &= \text{angular frequency} = 2\pi\nu & \nu &= \text{frequency} \\ v &= \frac{\omega}{k} = \text{phase velocity} & \delta &= \text{phase (or phase constant)} \\ k &= \text{wave number} & \lambda &= 2\pi/k = \text{wavelength} \\ T &= 2\pi/\omega = \text{period} & A &= \text{amplitude} \end{aligned}$$

$$\text{Euler identity: } e^{i\theta} = \cos \theta + i \sin \theta$$

Complex notation: $f(z, t) = \Re[\tilde{A}e^{i(kz - \omega t)}]$, where $\tilde{A} = Ae^{i\delta}$; “ \Re ” is usually dropped.

Wave velocities: $v = \frac{\omega}{k}$ = phase velocity; $v_{\text{group}} = \frac{d\omega}{dk}$ = group velocity

28. Electromagnetic Waves:

$$\text{Wave Equations: } \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \quad \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

28(a) Linearly Polarized Plane Waves:

$$\vec{E}(\vec{r}, t) = \tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \hat{n}, \quad \text{where } \tilde{E}_0 \text{ is a complex amplitude, } \hat{n} \text{ is a unit vector, and } \omega/|\vec{k}| = v_{\text{phase}} = c.$$

$$\hat{n} \cdot \vec{k} = 0 \quad (\text{transverse wave})$$

$$\vec{B} = \frac{1}{c} \hat{k} \times \vec{E}$$

Energy and Momentum:

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz - \omega t + \delta)}_{\text{averages to } 1/2}, \quad (\vec{k} = k \hat{z})$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = uc \hat{z}, \quad I \text{ (intensity)} = \langle |\vec{S}| \rangle = \frac{1}{2} \epsilon_0 E_0^2$$

$$\vec{\wp}_{\text{EM}} = \frac{1}{c^2} \vec{S} = \frac{u}{c} \hat{z}$$

Electromagnetic Waves in Matter:

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{index of refraction}$$

$$v = \text{phase velocity} = \frac{c}{n}$$

$$u = \frac{1}{2} \left[\epsilon |\vec{E}|^2 + \frac{1}{\mu} |\vec{B}|^2 \right]$$

$$\vec{B} = \frac{n}{c} \hat{\mathbf{k}} \times \vec{E}$$

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B} = \frac{uc}{n} \hat{\mathbf{z}}$$

28(b) Reflection and Transmission at Normal Incidence:

Boundary conditions:

$$\begin{aligned} \epsilon_1 E_1^\perp &= \epsilon_2 E_2^\perp & \vec{E}_1^\parallel &= \vec{E}_2^\parallel, \\ B_1^\perp &= B_2^\perp & \frac{1}{\mu_1} \vec{B}_1^\parallel &= \frac{1}{\mu_2} \vec{B}_2^\parallel. \end{aligned}$$

Incident wave ($z < 0$):

$$\begin{aligned} \vec{E}_I(z, t) &= \tilde{E}_{0,I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}} \\ \vec{B}_I(z, t) &= \frac{1}{v_1} \tilde{E}_{0,I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}. \end{aligned}$$

Transmitted wave ($z > 0$):

$$\begin{aligned} \vec{E}_T(z, t) &= \tilde{E}_{0,T} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}} \\ \vec{B}_T(z, t) &= \frac{1}{v_2} \tilde{E}_{0,T} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}. \end{aligned}$$

Reflected wave ($z < 0$):

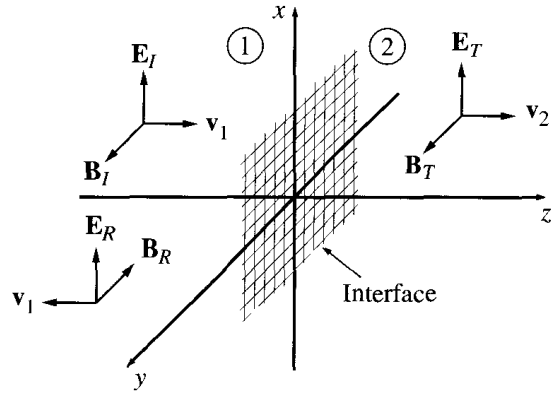
$$\begin{aligned} \vec{E}_R(z, t) &= \tilde{E}_{0,R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}} \\ \vec{B}_R(z, t) &= -\frac{1}{v_1} \tilde{E}_{0,R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}. \end{aligned}$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1}, \quad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2}$$

Applying boundary conditions and solving, approximating $\mu_1 = \mu_2 = \mu_0$,

$$\tilde{E}_{0,R} = \frac{n_1 - n_2}{n_1 + n_2} \tilde{E}_{0,I} \quad E_{0,T} = \left(\frac{2n_1}{n_1 + n_2} \right) \tilde{E}_{0,I}$$



29. Electromagnetic Potentials:

29(a) The fields: $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$

29(b) Gauge transformations: $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, $V' = V - \frac{\partial \Lambda}{\partial t}$

29(c) Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0 \implies \nabla^2 V = -\frac{1}{\epsilon_0} \rho$ (but \vec{A} is complicated)

29(d) Lorenz gauge: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \implies$

$$\square^2 V = -\frac{1}{\epsilon_0} \rho , \quad \square^2 \vec{A} = -\mu_0 \vec{J} , \quad \text{where } \square^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

$$\square^2 \equiv \text{D'Alembertian}$$

29(e) Retarded time solutions (Lorenz gauge):

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} , \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|}$$

where

$$t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

29(f) Liénard-Wiechert Potentials (potentials of a point charge):

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathcal{R} \left(1 - \frac{1}{c} \vec{v}_p \cdot \hat{\mathcal{R}} \right)}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q \vec{v}_p}{\mathcal{R} \left(1 - \frac{1}{c} \vec{v}_p \cdot \hat{\mathcal{R}} \right)} = \frac{\vec{v}_p}{c^2} V(\vec{r}, t)$$

where \vec{r}_p and \vec{v}_p are the position and velocity of the particle at the retarded time t_r , and

$$\vec{\mathcal{R}} = \vec{r} - \vec{r}_p , \quad \mathcal{R} = |\vec{r} - \vec{r}_p| , \quad \hat{\mathcal{R}} = \frac{\vec{r} - \vec{r}_p}{|\vec{r} - \vec{r}_p|}$$

29(g) Fields of a point charge (from the Liénard-Wiechert potentials):

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathcal{R}}{(\vec{u} \cdot \vec{\mathcal{R}})^3} \left[(c^2 - v_p^2) \vec{u} + \vec{\mathcal{R}} \times (\vec{u} \times \vec{a}_p) \right]$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{\mathcal{R}} \times \vec{E}(\vec{r}, t)$$

$$\text{where } \vec{u} = c \hat{\mathcal{R}} - \vec{v}_p$$

30. Radiation:

30(a) Radiation from an oscillating electric dipole along the z axis:

$$p(t) = p_0 \cos(\omega t) , \quad p_0 = q_0 d$$

Approximations: $d \ll \lambda \ll r$,

$$V(r, \theta, t) = -\frac{p_0 \omega}{4\pi \epsilon_0 c} \left(\frac{\cos \theta}{r} \right) \sin[\omega(t - r/c)]$$

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t - r/c)] \hat{z}$$

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\theta} , \quad \vec{B}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}(\vec{r}, t)$$

$$\text{Poynting vector: } \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{\mu_0}{c} \left\{ \frac{p_0 \omega^2}{4\pi} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \right\}^2 \hat{r}$$

$$\text{Intensity: } \vec{I} = \langle \vec{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2 \theta}{r^2} \hat{r} , \quad \text{using } \langle \cos^2 \rangle = \frac{1}{2}$$

$$\text{Total power: } \langle P \rangle = \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

30(b) Magnetic Dipole Radiation:

Dipole moment: $\vec{m}(t) = m_0 \cos(\omega t) \hat{z}$, at the origin

$$\vec{E} = \frac{\mu_0 m_0 \omega^2}{4\pi c} \left(\frac{\sin \theta}{r} \right) \cos[\omega(t - r/c)] \hat{\phi} , \quad \vec{B}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}(\vec{r}, t)$$

Compared to the electric dipole radiation, $p_0 \rightarrow \frac{m_0}{c}$, $-\hat{\theta} \rightarrow \hat{\phi}$

30(c) General Electric Dipole Radiation:

$$\vec{E}(\vec{r}, t) = \frac{\mu_0}{4\pi r} [(\hat{r} \cdot \ddot{\vec{p}}) \hat{r} - \ddot{\vec{p}}] , \quad \vec{B}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}(\vec{r}, t) = -\frac{\mu_0}{4\pi r c} [\hat{r} \times \ddot{\vec{p}}]$$

30(d) Multipole Expansion for Radiation:

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, ... radiation, and also magnetic dipole, quadrupole, ... radiation.

30(e) Radiation from a Point Particle:

When the particle is at rest at the retarded time,

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi \epsilon_0 c^2 |\vec{r} - \vec{r}'|} [\hat{n} \times (\hat{n} \times \vec{a}_p)]$$

$$\text{Poynting vector: } \vec{S}_{\text{rad}} = \frac{1}{\mu_0 c} |\vec{E}_{\text{rad}}|^2 \hat{n} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left(\frac{\sin^2 \theta}{r^2} \right) \hat{n}$$

where θ is the angle between \vec{a}_p and \hat{n} .

$$\text{Total power (Larmor formula): } P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

(valid for $\vec{v}_p = 0$ or $|\vec{v}_p| \ll c$)

Liénard's Generalization if $\vec{v}_p \neq 0$: ***

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| \frac{\vec{v} \times \vec{a}}{c} \right|^2 \right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c} \frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau}}_{\text{For relativists only}}$$

31. Radiation Reaction:***

31(a) Abraham-Lorentz formula:

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}$$

The Abraham-Lorentz formula is guaranteed to give the correct average energy loss for periodic or nearly periodic motion, but one would like a formula that works under general circumstances. The Abraham-Lorentz formula leads to runaway solutions which are clearly unphysical. The problem of radiation reaction for point particles in classical electrodynamics apparently remains unsolved.

32. Table of Legendre Polynomials $P_\ell(x)$:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

33. Table of Spherical Harmonics $Y_{\ell m}(\theta, \phi)$:

$$\begin{aligned}
\ell = 0 \quad & Y_{00} = \frac{1}{\sqrt{4\pi}} \\
\ell = 1 \quad & \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \end{cases} \\
\ell = 2 \quad & \begin{cases} Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} \\ Y_{20} = \frac{1}{2} \sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta - 1) \end{cases} \\
\ell = 3 \quad & \begin{cases} Y_{33} = -\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{3i\phi} \\ Y_{32} = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\phi} \\ Y_{31} = -\frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi} \\ Y_{30} = \frac{1}{2} \sqrt{\frac{7}{4\pi}} (5 \cos^3 \theta - 3 \cos \theta) \end{cases} \\
\ell = 4 \quad & \begin{cases} Y_{44} = \frac{3}{16} \sqrt{\frac{35}{2\pi}} \sin^4 \theta e^{4i\phi} \\ Y_{43} = -\frac{3}{2} \sqrt{\frac{35}{2\pi}} \sin^3 \theta \cos \theta e^{3i\phi} \\ Y_{42} = \frac{3}{8} \sqrt{\frac{5}{2\pi}} \sin^2 \theta (7 \cos^2 \theta - 1) e^{2i\phi} \\ Y_{41} = -\frac{3}{2} \sqrt{\frac{5}{2\pi}} \sin \theta (7 \cos^3 \theta - 3 \cos \theta) e^{i\phi} \\ Y_{40} = \frac{3}{\sqrt{4\pi}} (35 \cos^4 \theta - 30 \cos^2 \theta + 3) \end{cases}
\end{aligned}$$

For $m < 0$, use $Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi)$, which is valid for all m .