8.07 Lecture Slides 12
October 21, 2019

ELECTRIC POTENTIAL: LEGENDRE POLYNOMIALS AND SPHERICAL HARMONICS VIA TRACELESS SYMMETRIC TENSORS

Expansion of $F(\hat{n})$

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3 \ ,$$

so $F(\hat{\boldsymbol{n}})$ can also be written $F(\theta, \phi)$.

Can expand any square-integrable $F(\hat{\boldsymbol{n}})$ in a power series:

$$F(\hat{\boldsymbol{n}}) = C^{(0)} + C^{(1)}_i \hat{\boldsymbol{n}}_i + C^{(2)}_{ij} \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j + \ldots + C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} + \ldots \ ,$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates), and each $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is **symmetric and traceless.**

$$\nabla^2_{\mathrm{ang}} F_\ell(\hat{\boldsymbol{n}}) = -\ell(\ell+1) F_\ell(\hat{\boldsymbol{n}}) \ ,$$

where

$$F_\ell(\hat{\boldsymbol{n}}) = C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ .$$

General Solution to Laplace's Equation in Spherical Coordinates

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \;,}$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{n}$.

Trace Decomposition Theorem

Any symmetric matrix $S_{i_1 \ldots i_\ell}$ can be uniquely written in the form

$$\boxed{S_{i_1 \ldots i_\ell} = S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} + \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] \ ,}$$

where $S^{(\mathrm{TS})}_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1 \ldots i_{\ell-2}}$ is a symmetric tensor, and

$$\boxed{\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} [xxx]}$$

means to symmetrize the expression xxx in the indices $i_1 \ldots i_\ell$.

$$\boxed{\{\, S_{i_1 \ldots i_\ell} \,\}_{\mathrm{TS}} \equiv S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} = S_{i_1 \ldots i_\ell} - \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] .}$$

Example: Azimuthal Symmetry

Azimuthal symmetry: symmetry under rotation about z axis. Construct traceless symmetric tensors from $\hat{\boldsymbol{z}}$:

$$\{\,1\,\}_{\rm TS} = 1\ ,$$

$$\{\,\hat{\boldsymbol{z}}_i\,\}_{\rm TS} = \hat{\boldsymbol{z}}_i\ ,$$

$$\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\,\}_{\rm TS} = \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j - \tfrac{1}{3}\delta_{ij}\ ,$$

$$\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\,\}_{\rm TS} = \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k - \tfrac{1}{5}\big(\hat{\boldsymbol{z}}_i\delta_{jk} + \hat{\boldsymbol{z}}_j\delta_{ik} + \hat{\boldsymbol{z}}_k\delta_{ij}\big)\ ,$$

$$\begin{aligned}\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m\,\}_{\rm TS} = \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m - \frac{1}{7}\big(&\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\delta_{km} + \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_k\delta_{mj} + \hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_m\delta_{jk} + \hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\delta_{im}\\ &+ \hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_m\delta_{ik} + \hat{\boldsymbol{z}}_k\hat{\boldsymbol{z}}_m\delta_{ij}\big) + \frac{1}{35}\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big)\ .\end{aligned}$$

Construction of $F(\theta)$

Any square-integrable function $F(\theta)$ can be expanded

$$
\begin{aligned}
F(\theta) = c_0 + c_1\{\,\hat{\boldsymbol{z}}_i\,\}_{\rm TS}\hat{\boldsymbol{n}}_i + c_2\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j + \ldots \\
+ c_\ell\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell} + \ldots\ ,
\end{aligned}
$$

where the c_ℓ's are constants.

Since $\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}\cdot(\sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3) = \cos\theta$,

$$
\begin{aligned}
\{\,1\,\}_{\rm TS} &= 1 \\
\{\,\hat{\boldsymbol{z}}_i\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_i &= \cos\theta \\
\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j &= \cos^2\theta - \frac{1}{3} \\
\{\,\hat{\boldsymbol{z}}_i\hat{\boldsymbol{z}}_j\hat{\boldsymbol{z}}_k\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k &= \cos^3\theta - \frac{3}{5}\cos\theta
\end{aligned}
$$

Connection to Legendre Polynomials

Up to normalization, this is the standard expansion in Legendre polynomials.

The Legendre polynomials are normalized so that $P_\ell(\cos\theta = 1) = 1$, and you showed on Problem Set 4 that

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ .$$

The Multipole Expansion

The most general solution to Laplace's equation, in spherical coordinates, can be written as

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{r} = r\hat{n}$. For localized charge distributions, $C^{(\ell)}_{i_1 i_2 \ldots i_\ell} = 0$. So

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C'^{(\ell)}_{i_1 \ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ .}$$

Since each term falls off faster, the first few terms are usually sufficient at large r.

How to Find Multipole Moments Method 1 (as in Griffiths)

Combine

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, \mathrm{d}^3x' \ ,}$$

with

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta')$$

to obtain

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{\boldsymbol{r}}\,') P_\ell(\cos\theta')\, \mathrm{d}^3x' \ .}$$

Drawback of Method 1

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int {r'}^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta')\, \mathrm{d}^3x' \ .$$

Since θ' is the angle between $\vec{r}$ and $\vec{r}\,'$, to use this equation directly one has to redo the integral for every direction of $\vec{r}$.

How to Find Multipole Moments
Method 2: with Traceless Symmetric Tensors

Expand

$$f(\vec{\boldsymbol r}\,') \equiv \frac{1}{|\vec{\boldsymbol r} - \vec{\boldsymbol r}\,'|} \ .$$

in a power series in the components of $\vec{\boldsymbol r}\,' = x_i' \hat{\boldsymbol e}_i = r' \hat{\boldsymbol n}_i'$:

$$\begin{aligned} f(\vec{\boldsymbol r}\,') &= f(\vec{0}) + \left.\frac{\partial f}{\partial x_i'}\right|_{\vec{\boldsymbol r}\,'=\vec{0}} x_i' + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x_i' \partial x_j'}\right|_{\vec{\boldsymbol r}\,'=\vec{0}} x_i' x_j' + \dots \ , \\ &= f(\vec{0}) + r' \left.\frac{\partial f}{\partial x_i'}\right|_{\vec{\boldsymbol r}\,'=\vec{0}} \hat{\boldsymbol n}_i' + \frac{r'^2}{2!} \left.\frac{\partial^2 f}{\partial x_i' \partial x_j'}\right|_{\vec{\boldsymbol r}\,'=\vec{0}} \hat{\boldsymbol n}_i' \hat{\boldsymbol n}_j' + \dots \ . \end{aligned}$$

Since f is a function of $\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'$, $\dfrac{\partial f}{\partial x'_i} = -\dfrac{\partial f}{\partial x_i}$, and

$$\left.\frac{\partial^\ell f}{\partial x'_{i_1} \ldots \partial x'_{i_\ell}}\right|_{\vec{\boldsymbol{r}}\,'=\vec{0}} = (-1)^\ell \frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} \ .$$

This quantity is traceless, since

$$\left(\frac{\partial^\ell}{\partial x_j \partial x_j \partial x_3 \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|}\right) = \frac{\partial^\ell}{\partial x_3 \ldots \partial x_{i_\ell}} \nabla^2 \frac{1}{|\vec{\boldsymbol{r}}|} = 0 \ ,$$

because $\nabla^2(1/|\vec{\boldsymbol{r}}|) = 0$ except at $\vec{\boldsymbol{r}} = 0$.

We found that

$$\frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\vec{\boldsymbol{r}}|} = \frac{(-1)^\ell (2\ell-1)!!}{r^{\ell+1}} \{ \hat{\boldsymbol{n}}_{i_i} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \ ,$$

where

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 \ , \ \text{with} \ \ (-1)!! \equiv 1 \ .$$

$$= \frac{(2\ell)!}{2^\ell \ell!}$$

The power series then becomes

$$\boxed{\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\text{TS}} \{ \hat{\boldsymbol{n}}'_{i_1} \ldots \hat{\boldsymbol{n}}'_{i_\ell} \}_{\text{TS}} \ , \ (\text{for } r' < r)}$$

where either one (but not both) of the TS's can be dropped, since the difference is proportional to a Kronecker delta function, which leads to taking a trace of the other TS expression, which vanishes.

The analogous equation for the standard spherical harmonics is

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) \;, \qquad (\text{for } r' < r).$$

Inserting the above boxed equation into

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \, \mathrm{d}^3x' \;,$$

we find the final result

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \;,}$$

where the multipole moments $C^{(\ell)}_{i_1 \ldots i_\ell}$ are given by

$$\boxed{C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{\boldsymbol{r}}\,') \left\{ \vec{\boldsymbol{r}}\,'_{i_1} \ldots \vec{\boldsymbol{r}}\,'_{i_\ell} \right\}_{\mathrm{TS}} \mathrm{d}^3x' \;.}$$

The analogous formulas for the standard spherical harmonic treatment are

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \ ,$$

where the multipole moments $q_{\ell m}$ are given by

$$q_{\ell m} = \int \rho(\vec{\boldsymbol{r}}\,') \, r'^{\ell} \, Y^*_{\ell m}(\theta',\phi') \, \mathrm{d}^3 x' \ .$$

Once we calculate the $C^{(\ell)}_{i_1\ldots i_\ell}$ or the $q_{\ell m}$ for a given charge distribution $\rho(\vec{\boldsymbol{r}}\,')$, from

$$C^{(\ell)}_{i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{\boldsymbol{r}}\,') \left\{ \vec{\boldsymbol{r}}\,'_{i_1} \ldots \vec{\boldsymbol{r}}\,'_{i_\ell} \right\}_{\rm TS} \mathrm{d}^3x' \ ,$$

$$q_{\ell m} = \int \rho(\vec{\boldsymbol{r}}\,')\, r'^{\ell}\, Y^*_{\ell m}(\theta',\phi')\, \mathrm{d}^3x' \ ,$$

then we can calculate $V(\vec{\boldsymbol{r}})$ for any $\vec{\boldsymbol{r}}$:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1\ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ ,$$

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \ ,$$

avoiding the drawback of "Method 1".

Detailed Form of the Trace Decomposition Theorem

For an symmetric tensor $S_{i_1 \ldots i_\ell}$, the traceless symmetric part can be written as

$$\{\, S_{i_1 \ldots i_\ell} \,\}_{\rm TS} = S_{i_1 \ldots i_\ell} + \mathop{\rm Sym}_{i_1 \ldots i_\ell} \Big[a_{1,\ell}\, \delta_{i_1 i_2} \delta^{j_1 j_2} S_{j_1 j_2 i_3 \ldots i_\ell} + a_{2,\ell}\, \delta_{i_1 i_2} \delta_{i_3 i_4} \delta^{j_1 j_2} \delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5 \ldots i_\ell} + \ldots \Big] \;,$$

where

$$\mathop{\rm Sym}_{i_1 \ldots i_\ell} \big[T_{i_1 \ldots i_\ell} \big] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} T_{i_1 \ldots i_\ell} \;,$$

and

$$a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n! (\ell - 2n)! (\ell - n)! (2\ell)!} \;.$$

Today we will derive $a_{1,\ell}$.

$$\{\, S_{i_1 \dots i_\ell} \,\}_{\mathrm{TS}} = S_{i_1 \dots i_\ell} + \mathop{\mathrm{Sym}}_{i_1 \dots i_\ell} \Big[a_{1,\ell} \delta_{i_1 i_2} \delta^{j_1 j_2} S_{j_1 j_2 i_3 \dots i_\ell} + a_{2,\ell} \delta_{i_1 i_2} \delta_{i_3 i_4} \delta^{j_1 j_2} \delta^{j_3 j_4} S_{j_1 j_2 j_3 j_4 i_5 \dots i_\ell} + \dots \Big] \; ,$$

The coefficients $a_{n,\ell}$ are determined by the requirement that the trace of $\{\, S_{i_1 \dots i_\ell} \,\}_{\mathrm{TS}}$ equals zero. We can trace on any two indicies, since they are symmetric. Choose i_1 and i_2. The trace of the RHS, through the $a_{1,\ell}$ term, is

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3 \dots i_\ell} + a_{1,\ell} \delta_{i_1 i_2} \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \dots i_\ell}} \Big[\delta_{i_1 i_2} S_{jji_3 \dots i_\ell} \Big] \; .$$

Evaluation of this trace is tricky, since the right-hand piece is summed over all $\ell!$ orderings of the indices. So the indices are not fixed at the locations shown. For example, for $\ell = 3$,

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3} + a_{1,3} \delta_{i_1 i_2} \frac{1}{6} \Big[\delta_{i_1 i_2} S_{jji_3} + \delta_{i_1 i_3} S_{jji_2} + \delta_{i_2 i_3} S_{jji_1} + \delta_{i_2 i_1} S_{jji_3} + \delta_{i_3 i_1} S_{jji_2} + \delta_{i_3 i_2} S_{jji_1} \Big] \; .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

To carry out the sum, we have to consider 4 possible cases of where the indices i_1 and i_2 can be.

Case I: i_1 and i_2 can appear on the Kronecker δ-function in the square brackets. There are two possible orderings of i_1 and i_2. There are $(\ell-2)!$ orderings of the other indices, $i_3, \ldots, i_\ell$. The value includes a factor of $\delta_{i_1 i_2}\delta_{i_1 i_2} = 3$. Summary:

Case I: $\quad$ Multiplicity $= 2(\ell-2)!$, $\quad$ Value $= 3S_{jji_3\ldots i_\ell}\dfrac{a_{1,\ell}}{\ell!}$.

$$\mathrm{Tr}\left(\mathrm{RHS}_1\right) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

Case II: i_1 can appear on the Kronecker δ-function in the square brackets, while i_2 appears as one of the indices of S. There are two possible positions for i_1, and $\ell - 2$ positions for i_2. Again, there are $(\ell-2)!$ orderings for the other indices. The value can be seen by looking at a sample term:

$$a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\left[\delta_{i_1 i_3} S_{jji_2 i_4\ldots i_\ell}\right] = \frac{a_{1,\ell}}{\ell!} S_{jji_3\ldots i_\ell} .$$

Summarizing,

$$\boxed{\text{Case II:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)! \; , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!} \; .}$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

Case III: i_2 can appear on the Kronecker δ-function in the square brackets, while i_1 appears as one of the indices of S. This case is equivalent to Case II:

$$\boxed{\text{Case III:} \qquad \text{Multiplicity} = 2(\ell-2)(\ell-2)!\ , \qquad \text{Value} = S_{jji_3\ldots i_\ell}\frac{a_{1,\ell}}{\ell!}\ .}$$

$$\mathrm{Tr}\left(\mathrm{RHS}_1\right) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\,\delta_{i_1 i_2}\,\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

Case IV: Both i_1 and i_2 can appear on S. i_1 has $\ell-2$ possible places, and i_2 has $\ell-3$ places. The other indices again have $(\ell-2)!$ orderings. The value can again be seen by looking at a representative term:

$$a_{1,\ell}\,\delta_{i_1 i_2}\,\frac{1}{\ell!}\left[\delta_{i_3 i_4} S_{jji_1 i_2 i_5\ldots i_\ell}\right] = \frac{a_{1,\ell}}{\ell!}\,\mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] .$$

Summary:

$$\boxed{\begin{array}{ll} \text{Case IV:} & \text{Multiplicity} = (\ell-2)(\ell-3)(\ell-2)! \\ & \text{Value} = \mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right]\dfrac{a_{1,\ell}}{\ell!} \ . \end{array}}$$

$$\mathrm{Tr}\left(\mathrm{RHS}_1\right) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] \ .$$

Check on multiplicities. Remember, multiplicity = number of terms in above sum for each case, so they should sum to $\ell!$. Check:

$$(\ell-2)!\left[2 + 2(\ell-2) + 2(\ell-2) + (\ell-2)(\ell-3)\right] = \ell! \ .$$

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1\ldots i_\ell}}\left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

Check on multiplicities. Remember, multiplicity = number of terms in above sum for each case, so they should sum to $\ell!$. Check:

$$(\ell-2)!\,[2+2(\ell-2)+2(\ell-2)+(\ell-2)(\ell-3)] = \ell!\ .$$

TERRIFIC!

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell} + a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] .$$

Check on multiplicities. Remember, multiplicity = number of terms in above sum for each case, so they should sum to $\ell!$. Check:

$$(\ell-2)!\,[2 + 2(\ell-2) + 2(\ell-2) + (\ell-2)(\ell-3)] = \ell!\ .$$

TERRIFIC!

Then

$$a_{1,\ell}\delta_{i_1 i_2}\frac{1}{\ell!}\sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} \left[\delta_{i_1 i_2} S_{jji_3\ldots i_\ell}\right] =$$

$$\sum_{\text{Case}} \mathrm{Multiplicity(Case)} \times \mathrm{Value(Case)}$$

Finally,

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\left(2\cdot 3 + 4(\ell-2)\right)\right]$$
$$+\,\mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] a_{1,\ell}\,\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)}\ .$$

Recall,

$$\boxed{\begin{aligned}\{\, S_{i_1\ldots i_\ell}\,\}_{\mathrm{TS}} = S_{i_1\ldots i_\ell} + \mathop{\mathrm{Sym}}_{i_1\ldots i_\ell}\Big[& a_{1,\ell}\,\delta_{i_1 i_2}\,\delta^{j_1 j_2}\, S_{j_1 j_2 i_3\ldots i_\ell} \\ &+ a_{2,\ell}\,\delta_{i_1 i_2}\,\delta_{i_3 i_4}\,\delta^{j_1 j_2}\,\delta^{j_3 j_4}\, S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big]\ .\end{aligned}}$$

We must require that the trace of the full RHS vanish. However, note that all terms not yet included have 2 or more Kronecker delta functions, so when traced they will have 1 or more. So, they cannot contribute to the terms with no Kronecker delta functions. So the term in square brackets in the top equation must vanish.

$$\mathrm{Tr}\,(\mathrm{RHS}_1) = S_{jji_3\ldots i_\ell}\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3 + 4(\ell-2))\right]$$

$$+\,\mathrm{Sym}\left[\delta_{i_3 i_4} S_{jjkki_5\ldots i_\ell}\right] a_{1,\ell}\,\frac{(\ell-2)(\ell-3)}{\ell(\ell-1)}\ .$$

$$\left[1 + \frac{a_{1,\ell}}{\ell(\ell-1)}\,(2\cdot 3 + 4(\ell-2))\right] = 0 \qquad \Longrightarrow \qquad \boxed{a_{1,\ell} = -\frac{\ell(\ell-1)}{2(2\ell-1)}\ .}$$

Application to $\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\rm TS}$

$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}$ is traceless and symmetric and constructed from $\hat{\boldsymbol{n}}$, so it must be proportional to $\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}$. Then

$$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}=\hat{\boldsymbol{n}}_{i_\ell}\left\{\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}-\frac{\ell(\ell-1)}{2(2\ell-1)}\mathop{\rm Sym}_{i_1\ldots i_\ell}\left[\delta_{i_1i_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right]+\ldots\right\}$$

Clearly

$$\hat{\boldsymbol{n}}_{i_\ell}\left(\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right)=\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\ .$$

Also

$$\hat{\boldsymbol{n}}_{i_\ell}\mathop{\rm Sym}_{i_1\ldots i_\ell}\left[\delta_{i_1i_2}\hat{\boldsymbol{n}}_{i_3}\ldots\hat{\boldsymbol{n}}_{i_\ell}\right]=\frac{2}{\ell}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\ .$$

Then

$$\begin{aligned}\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}&=\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\left[1-\frac{\ell(\ell-1)}{2(2\ell-1)}\frac{2}{\ell}\right]+\ldots\\&=\frac{\ell}{2\ell-1}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}+\ldots\ .\end{aligned}$$

$$\hat{\boldsymbol{n}}_{i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_\ell} \}_{\rm TS} = \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_{\ell-1}} \left[1 - \frac{\ell(\ell-1)}{2(2\ell-1)} \frac{2}{\ell} \right] + \dots$$

$$= \frac{\ell}{2\ell-1} \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_{\ell-1}} + \dots \ .$$

The omitted terms cannot contribute to the term with no Kronecker delta functions, so

$$\boxed{\hat{\boldsymbol{n}}_{i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_\ell} \}_{\rm TS} = \frac{\ell}{2\ell-1} \{ \hat{\boldsymbol{n}}_{i_1} \dots \hat{\boldsymbol{n}}_{i_{\ell-1}} \}_{\rm TS} \ .}$$

Integration over Legendre Polynomials

Result:

$$\int d\Omega\, \left[C^{(\ell)}_{i_1\ldots i_\ell}\,\{\,\hat{\boldsymbol{n}}_{i_1}\,\ldots\,\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\right]\,\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\,\hat{\boldsymbol{n}}_{j_1}\,\ldots\,\hat{\boldsymbol{n}}_{j_{\ell'}}\,\}_{\rm TS}\right]$$

$$= 4\pi\frac{2^\ell \ell!^2}{(2\ell+1)!}\,C^{(\ell)}_{i_1\ldots i_\ell}\,C'^{(\ell)}_{i_1\ldots i_\ell} \quad \text{if } \ell' = \ell.$$

And it equals zero if $\ell' \neq \ell$.

Note added after class: we started this derivation on the blackboard, and will finish it on Wednesday.