MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 2, 2019
Prof. Alan Guth

PROBLEM SET 7

DUE DATE: Friday, November 8, 2019. at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

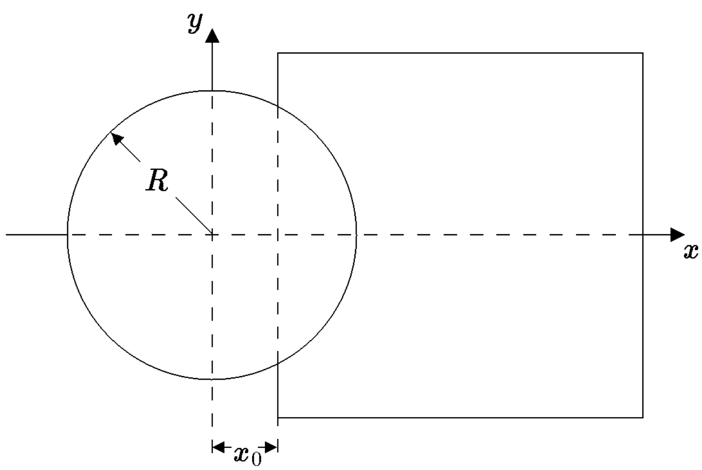
READING ASSIGNMENT: Griffiths: Chapter 5 (*Magnetostatics*).

CREDIT: This problem set has 125 points of credit, plus the opportunity to earn 10 point of extra credit.

— PART A —

PROBLEM 1: FORCE ON A DIELECTRIC SLAB PART WAY INSIDE A CIRCULAR CAPACITOR *(15 points)*

This problem was carried over from Problem Set 6.



Consider a circular capacitor, consisting of two thin circular disks, of radius R, parallel to the x-y plane. Both are centered on the z-axis, one at $z = +\frac{1}{2}d$ and the other at $z = -\frac{1}{2}d$. The two disks are held at a potential difference V_0. A square sheet of dielectric, with thickness just a shade smaller than d and with side larger than $2R$, is placed part way between the two disks, as shown in the diagram. The left-hand edge of the square is at $x = x_0$. Calculate the force on the slab as a function of x_0.

PROBLEM 2: J.J. THOMPSON AND THE CHARGE TO MASS RATIO OF THE ELECTRON *(10 points)*

Griffiths Problem 5.3 (p. 216).

In 1897, J. J. Thomson "discovered" the electron by measuring the charge-to-mass ratio of "cathode rays" (actually, streams of electrons, with charge q and mass m) as follows:

(a) [5 pts] First he passed the beam through uniform crossed electric and magnetic fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?

(b) [5 pts] Then he turned off the electric field, and measured the radius of curvature, R, of the beam, as deflected by the magnetic field alone. In terms of E, B, and R, what is the charge-to-mass ratio (q/m) of the particles?

PROBLEM 3: EXAMPLES OF THE USE OF THE BIOT-SAVART LAW *(15 points)*

Griffiths, Problem 5.8 (p. 228).

(a) [5 pts] Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side (Fig. 5.22).

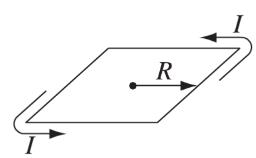


FIGURE 5.22

(b) [5 pts] Find the field at the center of a regular n-sided polygon, carrying a steady current I . Again, let R be the distance from the center to any side.

(c) [5 pts] Check that your formula reduces to the field at the center of a circular loop, in the limit $n \to \infty$.

PROBLEM 4: MAGNETIC FIELD ON THE AXIS OF A TIGHTLY WOUND SOLENOID *(15 points)*

Consider a cylinder of radius of radius R centered on the z-axis, extending from $z = z_1$ to $z = z_2 > z_1$. The cylinder is evenly wrapped with a wire which carries a current I, counterclockwise as seen from the positive z direction. The wire forms a helix, but we can approximate each turn as a circle in the x-y plane, since the number n of turns per unit length is large compared to $1/a$.

(a) [10 pts] Calculate the magnetic field at the origin.

(b) [5 pts] What is $\vec{\boldsymbol{B}}$ at the origin in the limit of $z_1 \to -\infty$ and $z_2 \to \infty$, corresponding to the infinite solenoid?

— PART B (To be handed in separately from Part A) —

PROBLEM 5: A CHARGED PARTICLE IN THE FIELD OF A MAGNETIC MONOPOLE *(20 points)*

Griffiths Problem 5.45 (p. 258), parts (a)–(c), with a different continuation.

Consider the motion of a particle with mass m and electric charge q_e in the field of a (hypothetical) stationary magnetic *monopole* q_m at the origin:

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\boldsymbol{r}} \ .$$

(a) [5 pts] Find the acceleration of q_e, expressing your answer in terms of q_e, q_m, m, $\vec{\boldsymbol{r}}$ (the position of the particle), and $\vec{\boldsymbol{v}}$ (its velocity).

(b) [5 pts] Show that the speed $v = |\vec{\boldsymbol{v}}|$ is a constant of the motion.

(c) [5 pts] Show that the vector quantity

$$\vec{\boldsymbol{Q}} \equiv m(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) - \frac{\mu_0 q_e q_m}{4\pi}\hat{\boldsymbol{r}}$$

is a constant of the motion. [Hint: differentiate it with respect to time, and prove—using the equation of motion from (a)—that the derivative is zero.]

(d) [5 pts] Adopt a coordinate system so that the z-axis points in the direction of $\vec{\boldsymbol{Q}}$, with the monopole at the origin. Then $\vec{\boldsymbol{Q}} = Q\,\hat{\boldsymbol{z}}$. Calculate $\vec{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}$, and use the result to show that the motion is confined to a fixed value of the polar angle θ, and hence to a cone. Express the value of θ in terms of q_e, q_m, Q, and μ_0.

PROBLEM 6: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

Griffiths Problem 5.60 (p. 264). In part (b), you are expected to use Eq. (5.93), from Problem 5.59, without proving it in any way. In part (c), where you are asked to find the approximate vector potential, you are expected to find it in the dipole approximation.

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) [5 pts] What is the magnetic dipole moment of the sphere?

(b) [5 pts] Find the average magnetic field within the sphere (see Prob. 5.59).

(c) [5 pts] Find the approximate vector potential at a point (r, θ) where $r \gg R$.

(d) [5 pts] Find the *exact* potential at a point (r, θ) outside the sphere, and check that it is consistent with (c). [*Hint*: refer to Ex. 5.11.]

(e) [5 pts] Find the magnetic field at a point (r, θ) inside the sphere (Prob. 5.30), and check that it is consistent with (b).

PROBLEM 7: THE COMPLETE MAGNETIC FIELD OF A MAGNETIC DIPOLE: *(10 points)*

In Problem Set 6 you proved the identity

$$\boxed{\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{\boldsymbol{r}})\ .} \tag{7.1}$$

Using this identity, show that the vector potential for a magnetic dipole,

$$\vec{\boldsymbol{A}}_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{m}} \times \hat{\boldsymbol{r}}}{r^2}\ , \tag{7.2}$$

can be used to find immediately that the magnetic field of a magnetic dipole is given by

$$\vec{\boldsymbol{B}}_{\rm dip}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}} \cdot \hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{r^3} + \frac{2\mu_0}{3}\vec{\boldsymbol{m}}\,\delta^3(\vec{\boldsymbol{r}})\ . \tag{7.3}$$

PROBLEM 8: SQUARE CURRENT LOOP ON AXIS: BIOT-SAVART AND THE MAGNETIC DIPOLE APPROXIMATION *(15 points)*

Griffiths Problem 5.36 (p. 255).

Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.

PROBLEM 9: VECTOR POTENTIAL FOR A UNIFORM $\vec{B}$ FIELD *(10 points extra credit)*

Griffiths Problem 5.25 (p. 248).

If $\vec{\boldsymbol{B}}$ is *uniform*, show that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = -\frac{1}{2}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}})$ works. That is, check that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$ and $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$. Is this result unique, or are there other functions with the same divergence and curl?