8.07 Lecture Slides 14
October 28, 2019

ELECTRIC DIPOLES; ELECTRIC FIELDS IN MATTER

Announcements

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Quiz 2 will be given on Wednesday, November 13, two weeks from this Wednesday. Problem Set 6 is due this Friday, 11/1/19, and Problem Set 7 will be due the next Friday, 11/8/19. The quiz will include material through Problem Set 7.

Detailed Form of the Trace Decomposition Theorem

For any symmetric tensor $S_{i_1 \ldots i_\ell}$, the traceless symmetric part can be written as

$$\boxed{\begin{aligned} \{ S_{i_1 \ldots i_\ell} \}_{\mathrm{TS}} = S_{i_1 \ldots i_\ell} + \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \Big[& a_{1,\ell} \delta_{i_1 i_2} S_{j_1 j_1 i_3 \ldots i_\ell} \\ & + a_{2,\ell} \delta_{i_1 i_2} \delta_{i_3 i_4} S_{j_1 j_1 j_2 j_2 i_5 \ldots i_\ell} + \ldots \Big] \ , \end{aligned}}$$

where

$$\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[T_{i_1 \ldots i_\ell} \right] \equiv \frac{1}{\ell!} \sum_{\substack{\text{all } \ell! \text{ index} \\ \text{orderings of } i_1 \ldots i_\ell}} T_{i_1 \ldots i_\ell} \ ,$$

and

$$\boxed{a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n!(\ell - 2n)!(\ell - n)!(2\ell)!} \ .}$$

In the last lecture we derived $a_{1,\ell}$.

Identity for $\hat{n}_{i_\ell}\{\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\,\}_{\rm TS}$

$$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}=\frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}\ .$$

Integration over Spherical Harmonics

For the conventional Legendre polynomials:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,\mathrm{d}x = \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\,\mathrm{d}\theta = \frac{2}{2\ell+1}\,\delta_{\ell,\ell'} \ .$$

For the conventional spherical harmonics,

$$\int \mathrm{d}\Omega\, Y^*_{\ell',m'}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm} \ .$$

Intermediate step, for traceless symmetric tensor formalism:

$$
\begin{aligned}
I_{i_1 \ldots i_{2\ell}} &\equiv \int \mathrm{d}\Omega \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{2\ell}} \\
&= 4\pi \frac{2^\ell \ell!}{(2\ell+1)!} \sum_{\text{all pairings}} \delta_{i_1,i_2} \, \delta_{i_3,i_4} \ldots \delta_{i_{2\ell-1},i_{2\ell}} \; .
\end{aligned}
$$

Final result:

$$
\begin{aligned}
&\int d\Omega \left[C^{(\ell)}_{i_1 \ldots i_\ell} \{ \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \}_{\mathrm{TS}} \right] \left[C'^{(\ell')}_{j_1 \ldots j_{\ell'}} , \{ \hat{\boldsymbol{n}}_{j_1} \ldots \hat{\boldsymbol{n}}_{j_{\ell'}} \}_{\mathrm{TS}} \right] \\
&\quad = 4\pi \frac{2^\ell \ell!^2}{(2\ell+1)!} C^{(\ell)}_{i_1 \ldots i_\ell} \, C'^{(\ell)}_{i_1 \ldots i_\ell} \quad \text{if } \ell' = \ell .
\end{aligned}
$$

And it equals zero if $\ell' \neq \ell$.

Construction of the Conventional Spherical Harmonics

For azimuthal symmetry (no dependence on ϕ),

$$C^{(\ell)}_{i_1 \ldots i_\ell} \propto \{ \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \}_{\rm TS} \ ,$$

where

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \{ \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \}_{\rm TS} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ .$$

To include ϕ-dependence, use a complete basis of unit vectors:

$$\hat{\boldsymbol{u}}^{(1)} \equiv \hat{\boldsymbol{u}}^{+} \equiv \frac{1}{\sqrt{2}} (\hat{\boldsymbol{x}} + i \hat{\boldsymbol{y}}) \ , \quad \hat{\boldsymbol{u}}^{(2)} \equiv \hat{\boldsymbol{u}}^{-} \equiv \frac{1}{\sqrt{2}} (\hat{\boldsymbol{x}} - i \hat{\boldsymbol{y}}) \ ,$$

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We use complex unit vectors, because the conventional spherical harmonics are complex.

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Serendipity: $\boxed{\hat{\boldsymbol{u}}^{+}_i \hat{\boldsymbol{u}}^{+}_i = 0}$, so strings of $\hat{\boldsymbol{u}}^{+}$'s (or $\hat{\boldsymbol{u}}^{-}$'s) are traceless!

$$\boxed{\hat{\boldsymbol{u}}^{(1)} \equiv \hat{\boldsymbol{u}}^{+} \equiv \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}) \ , \quad \hat{\boldsymbol{u}}^{(2)} \equiv \hat{\boldsymbol{u}}^{-} \equiv \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}) \ , \quad \hat{\boldsymbol{u}}^{(3)} \equiv \hat{\boldsymbol{z}} \ .}$$

$$\boxed{\hat{\boldsymbol{u}}^{+} \cdot \hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}} \sin\theta e^{i\phi} \ ,}$$

from which it follows that

$$\boxed{\{\, \hat{\boldsymbol{u}}^{+}_{i_1} \ldots \hat{\boldsymbol{u}}^{+}_{i_m} \, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} \, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \propto e^{im\phi} \ ,}$$

which is wonderful, because $Y_{\ell m}(\theta, \phi) \propto e^{im\phi}$.

$$\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\propto e^{im\phi}\;,$$

which is wonderful, because $Y_{\ell m}(\theta,\phi)\propto e^{im\phi}$. Define

$$Y_{\ell,m}(\theta,\phi)=C^{(\ell,m)}_{i_1\ldots i_\ell}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\;,$$

where

$$C^{(\ell,m)}_{i_1\ldots i_\ell}=\begin{cases}N(\ell,m)\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\geq 0\;,\\ N(\ell,m)\{\,\hat{\boldsymbol{u}}^-_{i_1}\ldots\hat{\boldsymbol{u}}^-_{i_{|m|}}\,\hat{\boldsymbol{z}}_{i_{|m|+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\leq 0\;,\end{cases}$$

Find $N(\ell,m)$ by normalizing, and checking sign conventions with the standard definition of $Y_{\ell,m}$. For $m\geq 0$:

$$N(\ell,m)=\frac{(-1)^m(2\ell)!}{2^\ell\ell!}\sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}\;.$$

For $m<0$, the factor of $(-1)^m$ is not included. $Y_{\ell,-m}(\theta,\phi)=(-1)^mY^*_{\ell,m}(\theta,\phi)$.

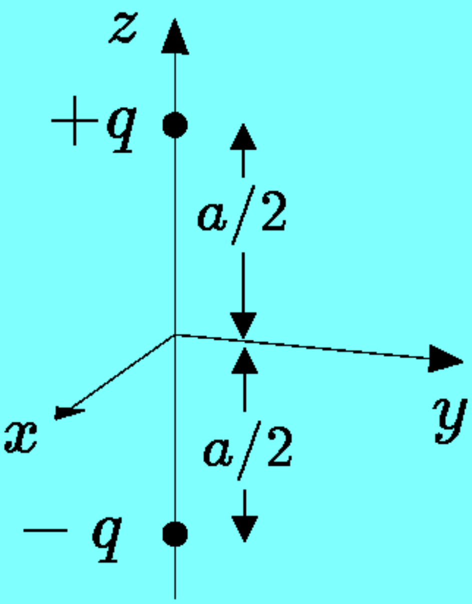
Electric Dipoles

As we have seen, the dipole term of the multipole expansion looks like

$$V_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \ , \quad \text{where} \quad \vec{\boldsymbol{p}} = \int \rho(\vec{\boldsymbol{r}}\,')\vec{\boldsymbol{r}}\,' \, \mathrm{d}^3x' \ .$$

A **physical dipole** is defined to be two charges, $+q$ and $-q$, separated by a distance a. Then, for the orientation in the diagram, $\vec{\boldsymbol{p}} = aq\hat{\boldsymbol{z}}$. A physical dipole is not a pure dipole, because it also has higher (odd ℓ) moments.

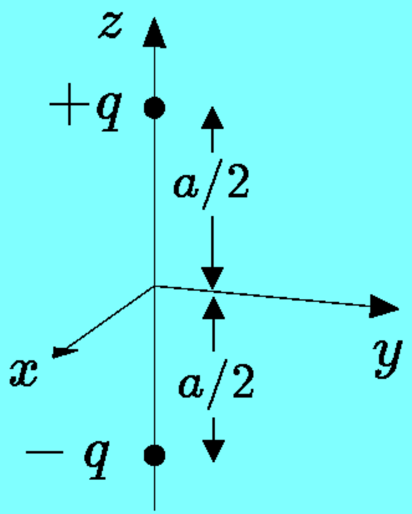
An **ideal dipole** is the limit of a physical dipole as $a \to 0$, $q \to \infty$, with $\vec{\boldsymbol{p}}$ fixed. An ideal dipole is a pure dipole, with no moments other than the dipole moment.



Charge Density of an Ideal Dipole

$$\rho(\vec{r}) = q\delta(x)\delta(y)\delta\left(z - \frac{a}{2}\right) - q\delta(x)\delta(y)\delta\left(z + \frac{a}{2}\right)$$

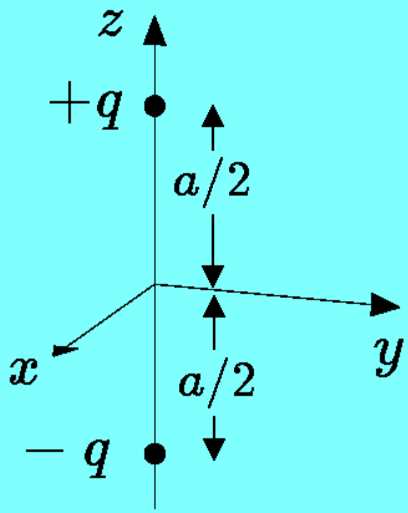
$$= -qa\delta(x)\delta(y)\left[\frac{\delta\left(z + \frac{a}{2}\right) - \delta\left(z - \frac{a}{2}\right)}{a}\right] .$$



In the limit $a \to 0$ with $p = qa$ fixed,

$$\rho(\vec{r}) = -p\delta(x)\delta(y)\frac{\mathrm{d}}{\mathrm{d}z}\delta(z) .$$

For the configuration shown,



$$\rho(\vec{\boldsymbol{r}}) = -p\delta(x)\delta(y)\frac{\mathrm{d}}{\mathrm{d}z}\delta(z) \ .$$

For an arbitrary orientation,

$$\boxed{\rho_{\rm dip}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{r}}\,\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0) \ ,}$$

where $\vec{\boldsymbol{r}}_0$ is the position of the dipole.

This formula does not appear in Griffiths, but we are braver in our use of δ-functions.

Derivation of V from ρ

For a dipole $\vec{\boldsymbol{p}}$ at location $\vec{\boldsymbol{r}}_0$,

$$\begin{aligned}
V(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x' \\
&= -\frac{1}{4\pi\epsilon_0}\int \frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}_{\vec{r}\,'}\delta(\vec{\boldsymbol{r}}\,'-\vec{\boldsymbol{r}}_0)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\,\mathrm{d}^3x' \\
&= \frac{1}{4\pi\epsilon_0}\int \delta^3(\vec{\boldsymbol{r}}\,'-\vec{\boldsymbol{r}}_0)\left[\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}_{\vec{r}\,'}\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\right]\,\mathrm{d}^3x' \\
&= -\frac{1}{4\pi\epsilon_0}\int \delta^3(\vec{\boldsymbol{r}}\,'-\vec{\boldsymbol{r}}_0)\left[\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}_{\vec{r}}\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}\right]\,\mathrm{d}^3x' \\
&= \frac{1}{4\pi\epsilon_0}\int \delta^3(\vec{\boldsymbol{r}}\,'-\vec{\boldsymbol{r}}_0)\left[\vec{\boldsymbol{p}}\cdot\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\right]\,\mathrm{d}^3x' \\
&= \boxed{\frac{1}{4\pi\epsilon_0}\frac{\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_0)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\;.}
\end{aligned}$$

Derivation of $\vec{E}$ from V

Very Useful Identity:

$$\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\vec{\boldsymbol{r}}) \ .$$

On Problem Set 6, you will prove this identity, and show that it leads immediately to

$$\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} V_{\rm dip} = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}})\, \hat{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{r^3} - \frac{1}{3\epsilon_0} \vec{\boldsymbol{p}}\, \delta^3(\vec{\boldsymbol{r}}) \ .$$

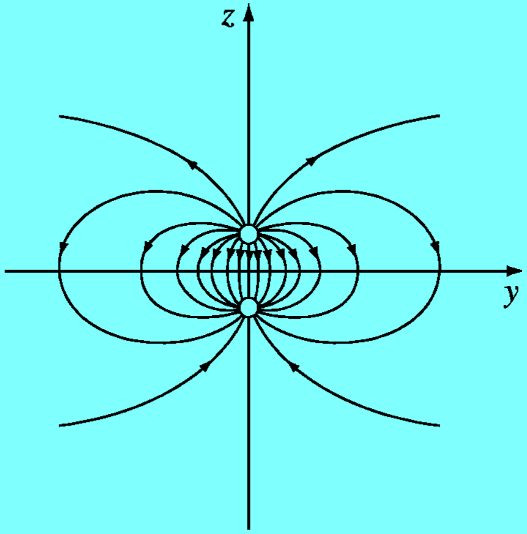
Why Is There a Delta Function in $\vec{E}$?

$$\vec{E}_{\rm dip}(\vec{r}\,) = -\vec{\nabla} V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{r})\,\hat{r}-\vec{p}}{r^3} - \frac{1}{3\epsilon_0}\vec{p}\,\delta^3(\vec{r}\,)\ .$$

The delta function describes the contribution to $\int d^3x\, \vec{E}(\vec{r}\,)$ of the strong $\vec{E}$ field at the center of the dipole, as shown in the diagram.

Even as $a \to 0$ with $\vec{p}$ fixed, the contribution to $\int d^3x\, \vec{E}(\vec{r}\,)$ does not approach zero.

The δ-function has real physical consequences (the Clausius-Mossotti equation, Griffiths Problem 4.41; see also Griffiths Problem 3.48), which we will discuss later.



Field of a “physical” dipole

FIGURE 3.37

Torque on a Dipole

A physical dipole consists of a charge $+q$ at $\vec{\boldsymbol{r}}_+$, and a charge $-q$ at $\vec{\boldsymbol{r}}_-$. Then

$$\vec{\boldsymbol{p}} = q(\vec{\boldsymbol{r}}_+ - \vec{\boldsymbol{r}}_-) \ .$$

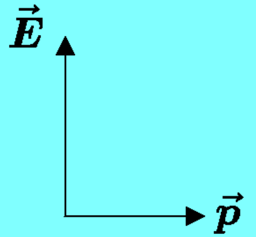
Remember torque? $\vec{\boldsymbol{\tau}} = \mathrm{d}\vec{\boldsymbol{L}}/\mathrm{d}t$, where $\vec{\boldsymbol{L}}$ = angular momentum. Since we are interested in the limit of $|\vec{\boldsymbol{r}}_+ - \vec{\boldsymbol{r}}_-| \to 0$, we take $\vec{\boldsymbol{E}}$ as uniform over the dipole.

$$\vec{\boldsymbol{\tau}} = \sum_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{r}}_+ \times (q\vec{\boldsymbol{E}}) + \vec{\boldsymbol{r}}_- \times (-q\vec{\boldsymbol{E}}) = q(\vec{\boldsymbol{r}}_+ - \vec{\boldsymbol{r}}_-) \times \vec{\boldsymbol{E}} \ ,$$

so

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}} \ .}$$

Example: torque $\vec{\boldsymbol{\tau}}$ is out of the screen. The torque always tends to make $\vec{\boldsymbol{p}}$ align with $\vec{\boldsymbol{E}}$.



Force on a Dipole

A nonzero force requires a non-uniform $\vec{\boldsymbol{E}}$.

$$\vec{\boldsymbol{F}}_{\rm tot} = \vec{\boldsymbol{F}}_{+} + \vec{\boldsymbol{F}}_{-} = q\left[\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}_{+}) - \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}_{-})\right] \equiv q\Delta\vec{\boldsymbol{E}} \ .$$

Taylor expand one component at a time:

$$\Delta E_x = E_x(\vec{\boldsymbol{r}}_{+}) - E_x(\vec{\boldsymbol{r}}_{-}) = \vec{\boldsymbol{\nabla}} E_x \cdot (\vec{\boldsymbol{r}}_{+} - \vec{\boldsymbol{r}}_{-}) \ ,$$

so

$$\vec{\boldsymbol{F}}_{{\rm tot},x} = \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}} E_x \ .$$

As a vector equation,

$$\boxed{\vec{\boldsymbol{F}}_{\rm tot} = (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{E}} \ .}$$

Electric Fields in Matter

Microscopically, matter is made out of protons, neutrons, and electrons.

Protons and neutrons are made out of quarks. Quarks are permanently confined inside of protons, neutrons, mesons, and other strongly interacting particles. They cannot exist as free particles.

Quarks and electrons might be truly fundamental, but they might be made of strings or something else.

If we treat protons, neutrons, and electrons as point particles, then

$$\boxed{\rho_{\text{micro}}(\vec{\boldsymbol{r}}) = \sum_i q_i \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \ ,}$$

where q_i = charge of i'th particle, and $\vec{\boldsymbol{r}}_i$ = position of i'th particle.

Microscopic and Macroscopic Fields

In matter,

$$\rho_{\text{micro}}(\vec{\boldsymbol{r}}) = \sum_i q_i \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i) \ ,$$

Define a macroscopic field by

$$\begin{aligned} \rho_{\text{macro}}(\vec{\boldsymbol{r}}) &= \text{average of } \rho_{\text{micro}} \text{ in small sphere centered at } \vec{\boldsymbol{r}}. \\ &= \langle \rho_{\text{micro}}(\vec{\boldsymbol{r}}) \rangle \ . \end{aligned}$$

Choose radius of sphere to be

(a) large compared to the size of atoms.

(b) small compared to macroscopic dimensions (i.e., the sizes of physical objects).

We will treat $\rho_{\text{macro}}(\vec{\boldsymbol{r}})$ as a smooth function.

$$\rho_{\rm macro}(\vec{\boldsymbol r}) = \text{average of } \rho_{\rm micro} \text{ in small sphere centered at } \vec{\boldsymbol r}.$$
$$= \langle \rho_{\rm micro}(\vec{\boldsymbol r}) \rangle \ .$$

Similarly, in matter we distinguish between $\vec{\boldsymbol E}_{\rm micro}(\vec{\boldsymbol r})$ and $\vec{\boldsymbol E}_{\rm macro}(\vec{\boldsymbol r})$.

Convention: in matter, with no subscript, $\rho(\vec{\boldsymbol r})$ and $\vec{\boldsymbol E}(\vec{\boldsymbol r})$ refer to the macro quantities.

Bound Charges

Matter can become "polarized," meaning that it acquires a nonzero density of dipoles.

$$\boxed{\vec{P}(\vec{\boldsymbol{r}}) = \text{dipole moment per unit volume.}}$$

$\vec{P}(\vec{\boldsymbol{r}})$ is just a particular way of describing a distribution of charge. In principle, one can equivalently use $\rho(\vec{\boldsymbol{r}})$

Given $\vec{P}(\vec{\boldsymbol{r}})$, what is $\rho(\vec{\boldsymbol{r}})$?

Answer:

$$\boxed{\rho_b(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} \cdot \vec{P}(\vec{\boldsymbol{r}}) \ ,}$$

and on the surface of a polarized material,

$$\boxed{\sigma_b = \vec{P} \cdot \hat{\boldsymbol{n}} \ ,}$$

where $\hat{\boldsymbol{n}}$ is the outward unit normal.

Derivation of Bound Charge Density

I will show three ways to derive this important relation, on the blackboard.

Note added after class: we only discussed the first of these three methods, the method of using the potential. This method followed closely the derivation in Griffiths, Sec. 4.2.1, pp. 173-174.