8.07 Lecture Slides 23
December 4, 2019

MAGNETIC FORCES DO NO WORK (!?), and ELECTROMAGNETIC WAVES

Magnetic Forces Do No Work (!?)

Since $\vec{F}_{\text{magnetic}} = q\vec{v} \times \vec{B}$, $\vec{F}_{\text{magnetic}}$ is always perpendicular to the velocity, and so it cannot do work. By the work-energy theorem, the work done is equal to the change in kinetic energy, so magnetic forces cannot impart kinetic energy to the body they act on.

Griffiths is one of the very rare authors who is brave enough to address this question: "What about that magnetic crane lifting the carcass of a junked car? *Somebody* is doing work on the car, and if it's not the magnetic field, who *is* it?"

Case 1: A Magnet Picking Up a Rotating, Insulating Ring of Line Charge

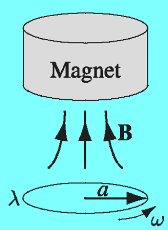

FIGURE 8.8

From Griffiths, *Introduction to Electrodynamics*, 4th Edition

Case 2: Two Wire Loops, With Batteries

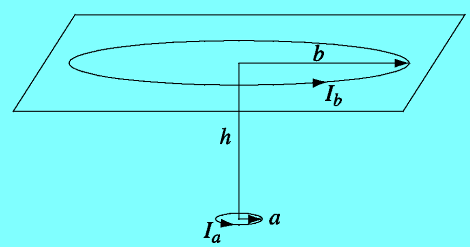

FIGURE 8.10

From Griffiths, *Introduction to Electrodynamics*, 4th Edition

Case 3: Two Boxes Held Together Magnetically Pulled Upwards on a Rope

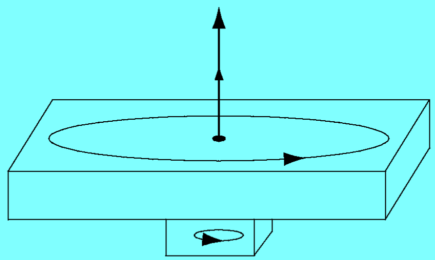

FIGURE 8.11

From Griffiths, *Introduction to Electrodynamics*, 4th Edition

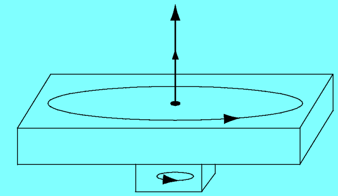

FIGURE 8.11

Griffiths: "Who did the work to lift the car? The person pulling up on the rope, obviously. The role of the magnetic field was merely to transmit this energy to the car, via the vertical component of the magnetic force. But the magnetic field itself (as always) did no work."

What? The magnetic field CANNOT transmit the energy to the car! The transmission of energy to the car is work, which is what magnetic fields can't do!

Right answer: When the boxes move, the magnetic field changes, inducing an electric field. The electric field fully accounts for the energy transferred to the lower box.

Case 4 (Not in Griffths): So How Does a Magnet Cause a Paperclip to Jump Off a Table?

The quantum theory of an electron in an electromagnetic field makes it clear that a static magnetic field cannot change the energy

$$E_{\text{conserved}} = \frac{1}{2} m_e v^2 - \vec{\mu}_e \cdot \vec{B} \ ,$$

where $\vec{\mu}_e$ is the magnetic dipole vector of the electron.

[In quantum notation, $\frac{1}{2} m_e v^2$ would be written as $(\vec{p}_{\text{canonical}} - q\vec{A})^2/(2m_e)$.]

This slide was added after lecture, but the content was discussed on the blackboard.

$$E_{\text{conserved}} = \frac{1}{2} m_e v^2 - \vec{\mu}_e \cdot \vec{B} \ .$$

Does $-\vec{\mu}_e \cdot \vec{B}$ have any connection to kinetic energy?

Answer: Yes, in a way.

If we try to classically model an electron as a spinning ball of charge, spinning about the z-axis, in the limit as its radius $R \to 0$, then the behavior of the ball is qualitatively the same as the rotating ring of Case 1. The ball moves toward to the magnet, but the rotation slows down as the ball moves, so its kinetic energy does not change. The slowing of the rotation causes the angular momentum and the magnetic dipole moment of the ball to decrease in magnitude. This is different from the electron, for which the angular momentum and the magnetic dipole moment is fixed.

But the limit $R \to 0$ is tricky. The energy (and hence the mass) approaches ∞ as $R \to 0$, so we have to imagine adding a negative "bare mass," of infinite magnitude, to keep the total energy equal to $m_e c^2$.

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For a given change in rotational kinetic energy E_{rot}, we can calculate the change in the z-components of the angular momentum and the magnetic dipole moment, L_z and μ_z. Let $\Delta\omega_z$ be the change in angular velocity. Since $E_{\text{rot}} \propto \omega_z^2$, while L_z and μ_z are proportional to ω_z, it follows that $\Delta E_{\text{rot}} \propto \omega_z \, \Delta\omega_z$, while ΔL_z and $\Delta\mu_z$ are proportional to $\Delta\omega_z$. So ΔL_z and $\Delta\mu_z$ are proportional to $\Delta E_{\text{rot}}/\omega_z$. But $\omega_z \to \infty$ as $R \to 0$ with L_z and μ_z fixed, so

$$\boxed{\frac{\Delta L_z}{\Delta E_{\text{rot}}} \to 0 \, , \qquad \frac{\Delta \mu_z}{\Delta E_{\text{rot}}} \to 0}$$

as $R \to 0$.

So, as the classical ($R \to 0$) spinning ball moves through a static magnetic field, $|\vec{L}|$ and $|\vec{\mu}|$ remain fixed, and its total kinetic energy,

$$\boxed{E_k = \frac{1}{2}mv^2 + E_{\text{rot}}}$$

also remains fixed.

This slide was added after lecture, but the content was discussed on the blackboard.

Furthermore, we can identify changes in E_{rot} with changes in $-\vec{\mu}_e \cdot \vec{B}$. To see this, let's consider a planar ring of charge described by a vector $\vec{a}$, where $|\vec{a}|$ is the area, and $\vec{a}$ is directed normal to the plane of the ring. (The spinning ball can be built from many such rings.) The dipole moment of the ring is then $\vec{\mu} = I\vec{a}$, where I is the current flowing counterclockwise when looking into the vector $\vec{a}$. Since we are interested in arbitrarily small rings, we can take $\vec{B}_{\text{ring}}$ at the loop as constant in space. If $\vec{B}_{\text{ring}}$ is changing with time, whether the change is due to the motion of the ring or a change in $\vec{B}(\vec{r}, t)$, the kinetic energy of the ring will change at a rate

$$\boxed{\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t} = \text{Power} = \varepsilon I = -\frac{\mathrm{d}}{\mathrm{d}t}\left[I\vec{a}\cdot\vec{B}_{\text{ring}}\right] = -\frac{d}{\mathrm{d}t}\left[\vec{\mu}\cdot\vec{B}\right] ,}$$

This slide was added after lecture, but the content was discussed on the blackboard.

$$\boxed{\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t} = \text{Power} = \mathcal{E} I = -\frac{\mathrm{d}}{\mathrm{d}t}\left[I\vec{a}\cdot\vec{B}_{\text{ring}}\right] = -\frac{d}{\mathrm{d}t}\left[\vec{\mu}\cdot\vec{B}\right] ,}$$

where we used Faraday's law for the emf, $\mathcal{E} = -\mathrm{d}\Phi_B/\mathrm{d}t$. Thus, as the classical spinning ball moves through a static magnetic field, the conservation of its kinetic energy is equivalent to the conservation of

$$\boxed{\frac{1}{2}mv^2 - \vec{\mu}_e \cdot \vec{B} \, ,}$$

exactly like the quantum-mechanical electron.

What is a Wave?

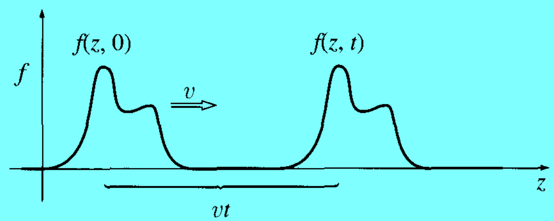
This is "intrinsically somewhat vague" — Griffiths.

A wave is a disturbance of a continuous medium that propagates with a fixed shape [or almost] at constant velocity [or almost].

Examples: waves on a stretched string, sound waves, water waves (especially low amplitude, gentle waves), and electromagnetic waves.

Waves in One Dimension

For some waves, the wave form propagates with a fixed shape at constant velocity:



From Griffiths, *Introduction to Electrodynamics*, 4th Edition

$$f(z,t) = f(z - vt, 0) \equiv g(z - vt) \; . \tag{1}$$

Here $f(z,t)$ could (accurately, but not exactly) represent the transverse displacement of a stretched string running along the z-axis, or the pressure of the air as a sound wave propagates along a tube.

$$f(z,t) = f(z - vt, 0) \equiv g(z - vt) \; . \tag{1}$$

But waves can move in both directions:

$$f(z,t) = g_1(z - vt) + g_2(z + vt) \; . \tag{2}$$

Perhaps surprisingly, we can find a differential equation that any function of this form will satisfy. Let's calculate some derivatives:

$$\begin{aligned} \frac{\partial f}{\partial z} &= g_1'(z - vt) + g_2'(z + vt) \\ \frac{\partial^2 f}{\partial z^2} &= g_1''(z - vt) + g_2''(z + vt) \\ \frac{\partial f}{\partial t} &= -v g_1'(z - vt) + v g_2'(z + vt) \\ \frac{\partial^2 f}{\partial t^2} &= v^2 g_1''(z - vt) + v^2 g_2''(z + vt) \; . \end{aligned} \tag{3}$$

We see that $f(z,t)$ necessarily satisfies the Wave Equation:

$$\boxed{\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \; .} \tag{4}$$

The reverse is true, too:

Any function $f(z,t)$ that satisfies the wave equation,

$$\boxed{\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \; ,} \tag{4}$$

can always be written as

$$\boxed{f(z,t) = g_1(z - vt) + g_2(z + vt) \; .} \tag{2}$$

Sinusoidal Waves:

$$\begin{aligned} f(z,t) &= A\cos\left[k(z - vt) + \delta\right] \\ &= A\cos\left[kz - \omega t + \delta\right] \; , \end{aligned} \tag{5}$$

where

$$\begin{aligned} v &= \frac{\omega}{k} = \text{phase velocity} \\ \omega &= \text{angular frequency} = 2\pi\nu \\ \nu &= \text{frequency} \\ \delta &= \text{phase (or phase constant)} \\ k &= \text{wave number} \\ \lambda &= 2\pi/k = \text{wavelength} \\ T &= 2\pi/\omega = \text{period} \\ A &= \text{amplitude.} \end{aligned} \tag{6}$$

Any wave can be constructed by superimposing sinusoidal waves (Fourier's Theorem, aka Dirichlet's Theorem).

Complex Notation

Let $\tilde{A} = Ae^{i\delta}$. Then

$$f(z,t) = A\cos[kz - \omega t + \delta] = \mathrm{Re}\left[\tilde{A}e^{i(kz-\omega t)}\right] , \qquad (7)$$

where we used

$$e^{i\theta} = \cos\theta + i\sin\theta \ . \qquad (8)$$

Conventions: drop "Re", and drop $\tilde{}$ on $\tilde{A}$.

$$f(z,t) = Ae^{i(kz-\omega t)} \ . \qquad (9)$$

General solution to wave equation:

$$f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kz-\omega t)}\, \mathrm{d}k \ , \qquad (10)$$

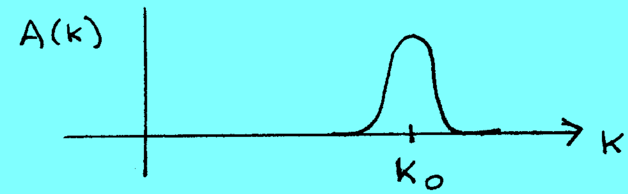
where $\omega/k = v$, v = wave speed = phase velocity.

Group Velocity and Phase Velocity:

In Jackson, pp. 324–325. Griffiths describes group velocity on pp. 417–419, but does not derive Eq. (13) below.

$v = \omega/k$ can sometimes depend on k: dispersion.

Consider a wave packet $f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i\left(kz-\omega(k)t\right)}\, \mathrm{d}k$ centered on k_0:



$$\omega(k) = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)(k-k_0) + \ldots = \omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k} + k\frac{\mathrm{d}\omega}{\mathrm{d}k} + \ldots \ . \qquad (11)$$

$$f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i\left(kz-\omega(k)t\right)}\, \mathrm{d}k$$

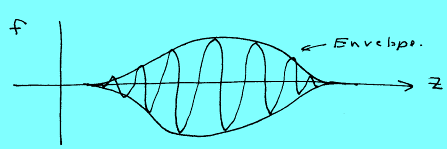
$$\omega(k) = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)(k-k_0) + \ldots \qquad (11)$$

$$= \omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k} + k\frac{\mathrm{d}\omega}{\mathrm{d}k} + \ldots \ .$$

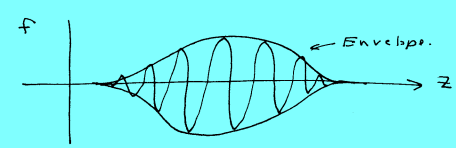
$$f(z,t) \approx e^{i\left[\omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k}\right]t} \int_{-\infty}^{\infty} \mathrm{d}k\, A(k) e^{ik\left(z - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right)} \ . \qquad (12)$$

The integral describes a wave which moves with

$$\boxed{v_{\mathrm{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0) \ .} \qquad (13)$$



$$f(z,t) = e^{i\left[\omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k}\right]t} \int_{-\infty}^{\infty} \mathrm{d}k\, A(k) e^{ik\left(z - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right)} \ . \qquad (12)$$



Envelope moves with $v = v_{\mathrm{group}} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$.

Waves inside envelope move with $v_{\mathrm{phase}} = v = \omega(k)/k$.

If $v_{\mathrm{phase}} > v_{\mathrm{group}}$, then waves appear at the left of the envelope and move forward through the envelope, disappearing at the right.

Electromagnetic Plane Waves

Maxwell Equations in Empty Space:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\,, \qquad\qquad (14)$$
$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\,,$$

where we wrote $\mu_0\epsilon_0$ as $1/c^2$. Initially we can take this as the definition of c, be we will see immediately that it really is the wave velocity. Manipulating,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0}-\nabla^2\vec{E}$$
$$=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)=-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\,, \qquad (15)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{=0}-\nabla^2\vec{E}$$
$$=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)=-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\,, \qquad (15)$$

so

$$\boxed{\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0\,.} \qquad (16)$$

This is the wave equation in 3 dimensions. An identical equation holds for $\vec{B}$:

$$\boxed{\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0\,.} \qquad (17)$$

Each component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation. This implies that waves travel at speed c!

Maxwell discovered the agreement of the electromagnetic wave speed and the speed of light, and commented (as quoted in Griffiths):

> "We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena."

From Griffiths, *Introduction to Electrodynamics*, 4th Edition

The Electromagnetic Spectrum

Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x-rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range

Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

TABLE 9.1

But the wave equation is not all: $\vec{E}$ and $\vec{B}$ are still related by Maxwell's equations, which contain more information than the wave equation.

Try to construct a plane wave solution of the form

$$\boxed{\vec{E}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n}\ ,} \tag{18}$$

where $\tilde{E}_0$ is a complex amplitude, $\hat{n}$ is a unit vector, and $\omega/|\vec{k}| = v_{\text{phase}} = c$. Then

$$\vec{\nabla}\cdot\vec{E} = i\hat{n}\cdot\vec{k}E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ , \tag{19}$$

so we require

$$\boxed{\hat{n}\cdot\vec{k} = 0 \qquad \text{(transverse wave)}\ .} \tag{20}$$

The magnetic field satisfies

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = -i\vec{k}\times\vec{E} = -i\vec{k}\times\hat{n}\,\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ . \tag{21}$$

Integrating,

$$\vec{B} = \frac{\vec{k}}{\omega}\times\hat{n}\,\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ , \tag{22}$$

so, remembering that $|\vec{k}| = \omega c$,

$$\boxed{\vec{B} = \frac{1}{c}\,\hat{k}\times\vec{E}\ .} \tag{23}$$

From Griffiths, *Introduction to Electrodynamics*, 4th Edition

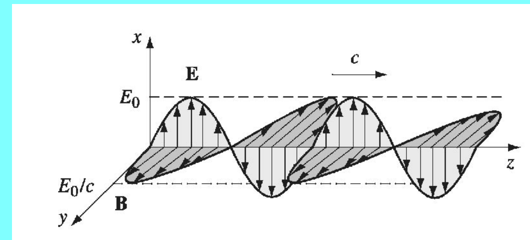


FIGURE 9.10

Energy and Momentum of EM Waves

Energy density:

$$u = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right]\ . \tag{24}$$

The $\vec{E}$ and $\vec{B}$ contributions are equal.

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz-\omega t+\delta)}_{\text{averages to }1/2}\ , \qquad (\vec{k} = k\,\hat{z})\ . \tag{25}$$

Energy flux:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = uc\,\hat{z}\ . \tag{26}$$

Momentum density:

$$\vec{\mathcal{P}}_{\text{EM}} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\,\hat{z}\ . \tag{27}$$

Intensity:

$$I = \left\langle|\vec{S}|\right\rangle = \frac{1}{2}c\epsilon_0 E_0^2\ . \tag{28}$$

Electromagnetic Waves in Matter

For linear, homogeneous materials, Maxwell's equations are unchanged except for the replacement $\mu_0\epsilon_0 \to \mu\epsilon$. Define

$$\boxed{n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{index of refraction.}} \tag{29}$$

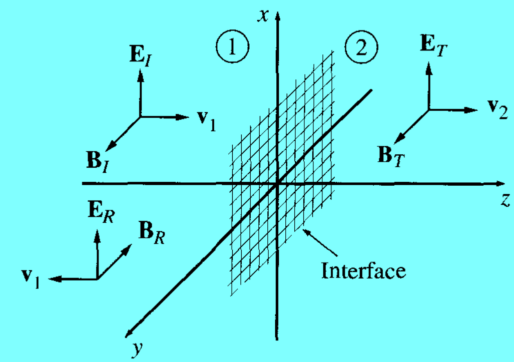
Then

$$\boxed{v = \text{phase velocity} = \frac{c}{n}\ .} \tag{30}$$

When expressed in terms of $\vec{E}$ and $\vec{B}$, everything carries over, with these substitutions:

$$u = \frac{1}{2}\left[\epsilon|\vec{E}|^2 + \frac{1}{\mu}|\vec{B}|^2\right]$$
$$\vec{B} = \frac{n}{c}\,\hat{k}\times\vec{E} \qquad (31)$$
$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{uc}{n}\,\hat{z}\ .$$

Boundary Conditions, Transmission and Reflection



From Griffiths, *Introduction to Electrodynamics*, 4th Edition

Boundary Conditions

At the interface shown on the previous slide, there are no free charges or currents, so:

$$\vec{\nabla}\cdot\vec{D} = 0 \implies D_1^\perp = D_2^\perp \implies \boxed{\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp}$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\approx 0 \implies \boxed{\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\ ,}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \implies \boxed{B_1^\perp = B_2^\perp\ ,} \qquad (32)$$
$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}\approx 0 \implies \boxed{\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}\ .}$$

Incident wave ($z<0$):

$$\vec{E}_I(z,t) = \tilde{E}_{0,I}\,e^{i(k_1 z-\omega t)}\,\hat{x}$$
$$\vec{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}\,e^{i(k_1 z-\omega t)}\,\hat{y}\ . \qquad (33)$$

Transmitted wave ($z>0$):

$$\vec{E}_T(z,t) = \tilde{E}_{0,T}\,e^{i(k_2 z-\omega t)}\,\hat{x}$$
$$\vec{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0,T}\,e^{i(k_2 z-\omega t)}\,\hat{y}\ . \qquad (34)$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1}\ , \qquad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2}\ . \qquad (35)$$

Reflected wave ($z < 0$):

$$\vec{\boldsymbol{E}}_R(z,t) = \tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{\boldsymbol{x}}$$
$$\vec{\boldsymbol{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{\boldsymbol{y}}\ . \tag{36}$$

Boundary conditions:

$$\vec{\boldsymbol{E}}_1^{\|} = \vec{\boldsymbol{E}}_2^{\|} \quad \Longrightarrow \quad \tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}\ , \tag{37}$$

$$\frac{1}{\mu_1}\vec{\boldsymbol{B}}_1^{\|} = \frac{1}{\mu_2}\vec{\boldsymbol{B}}_2^{\|} \quad \Longrightarrow \quad \frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0,I} - \frac{1}{v_1}\tilde{E}_{0,R}\right) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0,T}\ . \tag{38}$$

Two equations in two unknowns: $\tilde{E}_{0,R}$ and $\tilde{E}_{0,T}$.

Solution:

$$\tilde{E}_{0,R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| \tilde{E}_{0,I} \qquad E_{0,T} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{0,I}\ . \tag{39}$$