8.07 Lecture Slides 3
September 11, 2019

VECTOR CALCULUS
grad, div, curl

DIRAC DELTA FUNCTION

Announcements

Planned Quiz dates:
Monday, October 7, 2019
Wednesday, November 6, 2019

I'm told that there is also an 8.05 quiz on November 6. How many of you are affected?

Review of Lecture 2

The $\vec{\nabla}$ Symbol

The $\vec{\nabla}$ symbol is defined by

$$\vec{\nabla} \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \, .$$

For our purposes, $\vec{\nabla}$ is just a mnemonic — an easy way to remember the formulas for the gradient, curl, divergence, and the Laplacian.

Review of Lecture 2

The Gradient

If φ is a function of position, $\varphi = \varphi(x, y, z)$, then the gradient of φ is defined by

$$\begin{aligned} \vec{\nabla}\varphi &\equiv \left[\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right] \varphi \\ &\equiv \hat{\mathbf{e}}_x \frac{\partial \varphi}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial \varphi}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial \varphi}{\partial z} \, . \end{aligned}$$

Review of Lecture 2

The gradient describes the difference in the function between two infinitesimally separated points, $P \equiv (x,y,z)$ and $Q \equiv (x+dx, y+dy, z+dz)$:

$$\begin{aligned} d\varphi &\equiv \varphi(Q) - \varphi(P) \\ &= \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz \quad \text{(chain rule)} \\ &= \vec{\nabla}\varphi \cdot d\vec{r}\ , \end{aligned}$$

where

$$d\vec{r} \equiv dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z \equiv d\vec{\ell}\ .$$

Review of Lecture 2

"Hillside" Description of the Gradient

For a two dimensional system, $\varphi = \varphi(x,y)$, one can imagine that φ is the height of a hill, above the point (x,y) in the plane at sea level.

Then $\vec{\nabla}\varphi$ points uphill in the steepest direction. $|\vec{\nabla}\varphi|$ is the slope in the steepest direction.

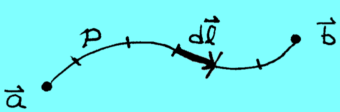
In any dimension, $\vec{\nabla}\varphi$ is orthogonal to surfaces of constant φ. (This follows from $d\varphi = \vec{\nabla}\varphi \cdot d\vec{r}$, which implies that φ does not change if $\vec{\nabla}\varphi$ is orthogonal to $\vec{r}$.

Review of Lecture 2

Fundamental Theorem of the Gradient

Since $d\varphi = \vec{\nabla}\varphi \cdot d\vec{r}$ for infinitesimal separations,

$$\int_{\vec{a},\mathcal{P}}^{\vec{b}} \vec{\nabla}\varphi \cdot d\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})\ .$$



The integral above is called a **line integral** from $\vec{a}$ to $\vec{b}$ along the path $\mathcal{P}$. It is defined by breaking up the path into N segments, each one corresponding to a displacement $d\vec{\ell}$. For each segment one evaluates $\vec{\nabla}\varphi \cdot d\vec{\ell}$, adds this quantity for all N segments, and then takes the limit $N \to \infty$.

The Curl

A vector field $\vec{A}(x,y,z)$ is a vector defined at each point (x,y,z) of space, at least within some region.

Using ∂_i as an abbreviation for $\partial/\partial x_i$, the $\vec{\nabla}$ symbol can be written as $\hat{e}_i\partial_i$, and then the curl of a vector field is defined by

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= (\hat{e}_i\partial_i) \times A_j\hat{e}_j \\ &= \partial_i A_j\, \hat{e}_i \times \hat{e}_j \\ &= \partial_i A_j\, \varepsilon_{ijk}\hat{e}_k \\ &= \varepsilon_{ijk}\partial_j A_k \hat{e}_i \quad \text{[cyclic permutation of } (i,j,k)]\ , \end{aligned}$$

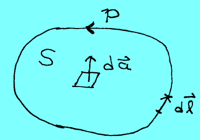
so

$$\left(\vec{\nabla} \times \vec{A}\right)_i = \varepsilon_{ijk}\partial_j A_k\ .$$

Fundamental Theorem of the Curl

Let S be a surface with boundary $\mathcal{P}$. Then the fundamental theorem of the curl, aka Stokes' theorem, says that

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_{\mathcal{P}} \vec{A} \cdot d\vec{\ell} \ .$$

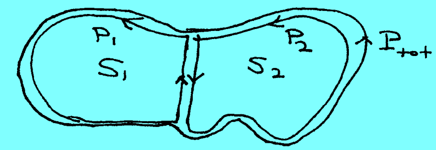


The integral on the left is a **surface integral** over the surface S, defined by breaking up the surface into small surface elements. For each surface element, define an infinitesimal vector $d\vec{a}$ perpendicular to the surface, with length $|d\vec{a}|$ equal to the area of the element. Calculate $(\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$ for each element, add them all, and take the limit as the size of each element approaches zero.

The integral on the right is a line integral, where the circle on the integral sign means that the path is closed.

The choice of which normal to use for $d\vec{a}$ is by convention controlled by right-hand-rule: rotate your right hand with knuckles moving along the circulation of $\mathcal{P}$; the direction of your thumb is the direction of $d\vec{a}$.

Additivity of the Fundamental Theorem:



If two surfaces are joined, then the surface integral of any vector field is additive: the surface integral for $S_1 \cup S_2$ is the sum of the surface areas.

The line integral of any vector field around the combined surface is also additive: if the two line integrals are added, the integrals along the interface cancel, so the sum is the integral around the outer surface.

Consequence: if the fundamental theorem of the curl holds for S_1 and S_2, then it holds for $S_1 \cup S_2$.

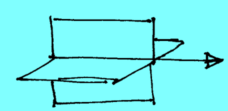
Sketch of Proof of Fundamental Theorem:

1) Show that it holds for infinitesimal loops. Describe the vector field by its first order Taylor expansion.

2) Use the additivity properties to show that the theorem then holds for finite surfaces.

"Paddlewheel" Description of the Curl

To have an intuition for the curl, visualize the vector field as the flow of a liquid. Imagine a paddlewheel in the liquid:



Circular patterns of flow will cause the paddlewheel to rotate. At any location, the curl points in the direction that maximizes the angular velocity of the paddlewheel. The magnitude of the curl is proportional to the angular velocity, ω:

$$\omega = \frac{1}{2} |\vec{\nabla} \times \vec{A}| \ .$$

The Divergence

The divergence of a vector field $\vec{A}(x, y, z)$ is defined by

$$\vec{\nabla} \cdot \vec{A} = (\hat{e}_i \partial_i) \cdot A_j \hat{e}_j = \delta_{ij} \partial_i A_j = \partial_i A_i \ .$$

Fundamental Theorem of the Divergence

Let V be a volume, and let S be its surface. Then the fundamental theorem of the divergence, aka Gauss' theorem, Green's theorem, or the divergence theorem, says that

$$\int_V (\vec{\nabla} \cdot \vec{A})\, d^3x = \oint_S \vec{A} \cdot d\vec{a} \ .$$

The integral on the left is a **volume integral**, defined by breaking up the volume into small volume elements. For each volume element one calculates $(\vec{\nabla} \cdot \vec{A})$ times the volume of the element; then one adds this quantity for all volume elements, and takes the limit as the size of each volume element approaches zero.

The integral on the right is a surface integral. The circle on the integral sign indicates that the surface is closed. The boundary of a volume is always closed.

Sketch of proof of the divergence theorem:

1) Show that the theorem holds for infinitesimal volumes, using a first order Taylor expansion to describe the vector field.
2) Verify that both sides of the divergence theorem equation are additive, so if it holds for infinitesimal volumes, it holds for finite volumes.

"Fluid Flow" Description of the Divergence

Visualize the vector field $\vec{A}$ as the velocity field of an incompressible fluid. Then

$$\int_S \vec{A} \cdot d\vec{a} = \text{flow rate through } S \ .$$

By "flow rate", I mean the volume of fluid that flows through S per unit time.

Furthermore,

$$\vec{\nabla} \cdot \vec{A} = \text{rate of fluid creation} \ .$$

By "rate of fluid creation," I mean the volume of fluid created per unit volume per unit time.

The diverence theorem becomes the statement that the total rate of fluid creation in any volume is equal to the flow rate of the liquid through the boundary.

The Laplacian

The Laplacian of a function of position $\varphi(x,y,z)$ is defined by

$$\begin{aligned}\nabla^2\varphi &\equiv \vec{\nabla}\cdot(\vec{\nabla}\varphi)\\ &= \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\ .\end{aligned}$$

The Laplacian can be thought of as the "anti-lumpiness" operator.

That is, if we imaging that $\varphi(x,y,z)$ describes the density of pudding, a "lump" is a place where $\varphi(\vec{r})$ is larger than the average value of φ in the region. If $\nabla^2\varphi<0$, we have a lump.

A "Lumpiness" Theorem for the Laplacian

Let φ be a function of position, $\varphi(x,y,z)$. Consider any sphere, and choose the origin to be the center of the sphere. Let R be the radius of the sphere, and let $\bar{\varphi}(R)$ be the average value of φ on the surface of the sphere. Then, you will prove on Problem Set 1, that

$$\varphi(\vec{0})=\bar{\varphi}(R)-\frac{1}{4\pi}\int_{\text{sphere}} d^3x\,\nabla^2\varphi\left[\frac{1}{r}-\frac{1}{R}\right]\ .$$

The quantity in square brackets is positive. So, if $\nabla^2\varphi<0$, then $\varphi(\vec{0})>\bar{\varphi}(R)$, and we have a lump at the origin.

Curvilinear Coordinates

Here I will ask you to read the textbook, pp. 38–45. We want to be familiar with spherical polar coordinates and cylindrical coordinates,

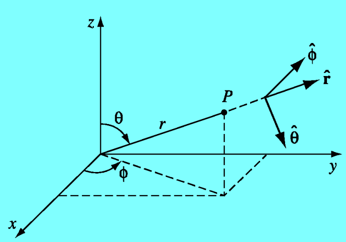


FIGURE 1.36

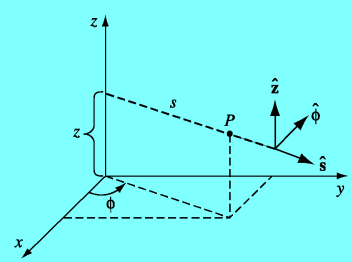


FIGURE 1.42

(Figures from Griffiths, *Electrodnamics*.)

The goal is to understand the **volume element** and the **gradient** in these coordinates.

You should be able to **use**, but need not be able to **derive**, the fomulas for the **divergence**, **curl**, and **Laplacian** in these coordinates.

The Dirac Delta Function

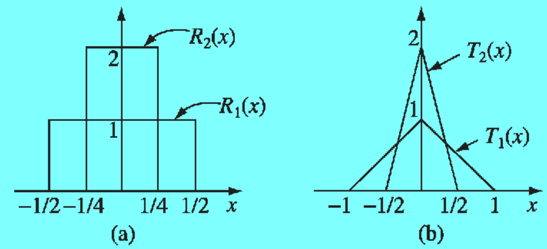
Quote from Griffiths' Textbook:

"Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$; in the mathematical literature it is known as a **generalized function**, or **distribution**."

But what is a distribution?

Griffiths Continues . . .

"It is, if you like, the *limit of a sequence* of functions, such as rectangles $R_n(x)$, of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$."



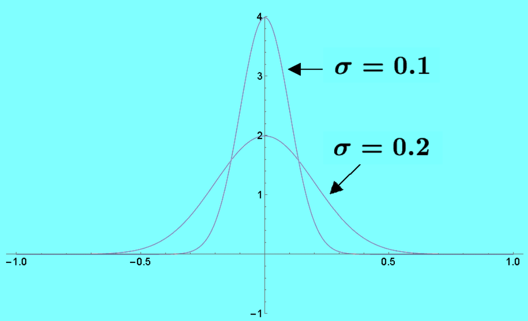
But what do you mean, "if you like"???

Actually, the δ-function **cannot** be defined as the limit of a sequence of functions.

Other Books Use the Limit of Gaussians

For any value of σ, the Gaussian integrates to one:

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} \quad \Longrightarrow \quad \int_{-\infty}^{\infty} g_\sigma(x)\, dx = 1 \ .$$



Can We Define $\delta(x) \equiv \lim_{\sigma\to 0} g_\sigma(x)$?

Unfortunately, we **can't!**

Why not?

Because even though

$$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} g_\sigma(x)\, dx = 1 \ ,$$

it nonetheless turns out that

$$\int_{-\infty}^{\infty} \left[\lim_{\sigma\to 0} g_\sigma(x) \right] dx = 0 \ .$$

It is crucial that we integrate first, and then take the limit!

False Argument for Correct Result

Griffiths goes on to write:

> "If $f(x)$ is some 'ordinary' function (that is, not another delta function—in fact, just to be on the safe side, let's say that $f(x)$ is continuous), then the product $f(x)\delta(x)$ is zero everywhere except at $x=0$. It follows that
>
> $$f(x)\delta(x) = f(0)\delta(x)\ . \tag{1.88}$$
>
> (This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway [anywhere?] except at $x=0$, we may as well replace $f(x)$ by the value it assumes at the origin.) In particular
>
> $$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0)\ . \tag{1.89}$$
>
> Under an integral, then, the delta function 'picks out' the value of $f(x)$ at $x=0$."

Discussion of Griffiths' Statement

Griffiths' conclusion, that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0) \tag{1.89}$$

is correct, but it does not follow from the fact that $f(x)\delta(x)=0$ except at $x=0$. We will see shortly that $f(x)\delta'(x)$, where $\delta'(x)$ is the derivative of the delta function, is also zero except at $x=0$, but

$$\int_{-\infty}^{\infty} f(x)\delta'(x)\,dx = -f'(0)\int_{-\infty}^{\infty}\delta(x)\,dx = -f'(0)\ .$$

Griffiths Comes Very Close to the Target

"Although δ itself is not a legitimate function, integrals over δ are perfectly acceptable. In fact, it's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx\ ,$$

for all ("ordinary") functions $f(x)$."

Issue: how can a mathematical definition depend on someone's "intentions"???

Added after the lecture

References

M. J. Lighthill: *An Introduction to Fourier Analysis and Generalised Functions* (Cambridge University Press, 1958).

A. H. Zemanian: *Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications* (McGraw-Hill Book Company, New York, 1965, and Dover Publications, New York, 1987).