MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II
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PROBLEM SET 2 SOLUTIONS

PROBLEM 1: CAPACITANCE OF A CYLINDRICAL CAPACITOR *(15 points)*

(a) [3 pts] Using Gauss's law, find an expression for the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ at points $a < r < b$. Neglect end effects due to the finite length of the capacitor.

Following Griffiths' conventions, the radial coordinate in cylindrical coordinates is s, and a unit vector pointing radially outward from the z axis is $\hat{\boldsymbol{s}}$. So we imagine a Gaussian cylinder of radius s and length ℓ, concentric with the capacitor, between the outer shell and the inner cylinder. The charge enclosed is $Q(s) = \lambda\ell$. By symmetry, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ must have the form

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = E(s)\hat{\boldsymbol{s}} ,$$

and then Gauss' law implies that

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{a}} = 2\pi s\ell E(s) = \frac{Q(s)}{\epsilon_0} = \frac{\lambda\ell}{\epsilon_0} \quad \Longrightarrow \quad E(s) = \frac{\lambda}{2\pi\epsilon_0 s}$$

$$\Longrightarrow \quad \vec{\boldsymbol{E}} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\boldsymbol{s}} \qquad \text{for} \qquad a < s < b .$$

(b) [3 pts] Using your expression for $\vec{\boldsymbol{E}}$ from part (a), find the potential difference ΔV between the outer shell and the inner cylinder.

$$\begin{aligned}\Delta V = V(a) - V(b) &= -\int_b^a \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{s}}\, ds = -\int_b^a \frac{\lambda}{2\pi\epsilon_0 s}\, ds \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a}{b}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) .\end{aligned}$$

(c) [3 pts] Derive an expression for the capacitance of this capacitor in terms of the quantities given. What is the capacitance per unit length?

Consider a length ℓ of the capacitor. Then

$$C = \frac{Q}{\Delta V} = \frac{\lambda\ell}{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon_0\ell}{\ln\left(\frac{b}{a}\right)} \quad \Longrightarrow \quad \frac{C}{\ell} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} .$$

(d) [3 pts] Compute the energy stored in the capacitor by integrating the energy density over the volume where $\vec{\boldsymbol{E}}$ is nonzero, and compare with the result you get using $W = \frac{1}{2}CV^2$. Do they agree? (Hint: they better agree!)

The energy density is given by

$$u = \frac{1}{2}\epsilon_0|\vec{\boldsymbol{E}}|^2 ,$$

so the energy in a length ℓ of the capacitor is given by

$$W = \frac{1}{2}\epsilon_0\ell\int_a^b \left[\frac{\lambda}{2\pi\epsilon_0 s}\right]^2 (2\pi s)\, ds = \frac{\ell\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{b}{a}\right) .$$

To compute the comparison,

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\left[\frac{2\pi\epsilon_0\ell}{\ln\left(\frac{b}{a}\right)}\right]\left[\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)\right]^2 = \frac{\ell\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{b}{a}\right) .$$

They had better agree, and they do!

(e) [3 pts] Let the gap $d = b - a$ between the cylinders be small compared to the radii, a and b. Show that in this case that your answer for part (c) reduces to that for a parallel plate capacitor (see Griffiths Eq. (2.54) on p. 106).

We found that

$$C = \frac{2\pi\epsilon_0\ell}{\ln\frac{a+d}{a}} = \frac{2\pi\epsilon_0\ell}{\ln\left(1+\frac{d}{a}\right)} .$$

When the gap is small compared to the radii,

$$\ln\left(1+\frac{d}{a}\right) \approx \frac{d}{a} , \qquad \text{so} \qquad C = \frac{2\pi\epsilon_0\ell}{\ln\left(1+\frac{d}{a}\right)} \approx \frac{2\pi a\epsilon_0\ell}{d} = \frac{\epsilon_0 A}{d} ,$$

where $A = 2\pi a\ell$ is the area of the equivalent parallel plate capacitor. This agrees with Griffiths' Eq. (2.54).

PROBLEM 2: THE ELECTRIC FIELD, POTENTIAL, AND ENERGY OF A UNIFORM SPHERE OF CHARGE *(16 points)*

(a) [4 pts] A uniformly charged sphere of charge has radius R and total charge Q. Using Gauss's law, calculate the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ everywhere.

Consider a Gaussian sphere of radius $r < R$. The charge enclosed $Q(r) = \frac{4}{3}\pi r^3\rho$. By Gauss' law,

$$4\pi r^2 E(r) = \frac{Q(r)}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3\rho \quad \Longrightarrow \quad E(r) = \frac{\rho\, r}{3\epsilon_0} , \quad \text{for } r < R .$$

For a Gaussian sphere of radius $r > R$, the charge enclosed is Q, so

$$4\pi r^2 E(r) = \frac{Q}{\epsilon_0} \quad \Longrightarrow \quad E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \,.$$

Thus,

$$\vec{\boldsymbol{E}} = \begin{cases} \dfrac{Qr}{4\pi\epsilon_0 R^3}\hat{\boldsymbol{r}} & \text{for } r < R \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} & \text{for } r > R \,. \end{cases}$$

(b) [4 pts] Using the electric field you calculated in part (a), find the electric potential $V(\vec{\boldsymbol{r}})$ everywhere.

Outside of the sphere, $r > R$,

$$V(\infty) - V(r) = -\int_r^\infty E(r)\, dr = -\int_r^\infty \frac{Q}{4\pi\epsilon_0 r^2}\, dr = -\frac{Q}{4\pi\epsilon_0 r} \,.$$

Inside the sphere, $r < R$,

$$\Delta V = V(R) - V(r) = -\int_r^R E(r)\, dr = -\int_r^R \frac{Qr}{4\pi\epsilon_0 R^3} dr = -\frac{Q(R^2 - r^2)}{8\pi\epsilon_0 R^3} \,.$$

Putting these results together,

$$V(r) = \begin{cases} \dfrac{Q\left(3R^2 - r^2\right)}{8\pi\epsilon_0 R^3} & \text{if } r < R \\ \dfrac{Q}{4\pi\epsilon_0 r} & \text{if } r > R. \end{cases}$$

(c) [4 pts] Using the expression

$$W = \frac{1}{2}\epsilon_0 \int_{\text{all space}} |\vec{\boldsymbol{E}}|^2\, \mathrm{d}^3x \,,$$

for the total work needed to assemble the charge configuration, calculate W using the expressions above.

$$W = \frac{1}{2}\epsilon_0 \int_{\text{all space}} |\vec{\boldsymbol{E}}|^2\, \mathrm{d}^3x = 2\pi\,\epsilon_0 \left[\int_0^R r^2\, dr \left[\frac{Qr}{4\pi\epsilon_0 R^3}\right]^2 + \int_R^\infty r^2\, dr \left[\frac{Q}{4\pi\epsilon_0 r^2}\right]^2\right]$$
$$= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{5R} + \frac{1}{R}\right] = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R} \,.$$

(d) [4 pts] Using the expression

$$W = \frac{1}{2}\int_{\text{all space}} \rho V\, \mathrm{d}^3x \,,$$

calculate W again using your expressions above.

$$W = \frac{1}{2}\int_{\text{all space}} \rho V\, \mathrm{d}^3x = \frac{4\pi}{2}\int_0^R r^2\, dr \left(\frac{3Q}{4\pi R^3}\right)\frac{Q\left(3R^2 - r^2\right)}{8\pi\epsilon_0 R^3}$$
$$= \frac{3Q^2}{16\pi\epsilon_0 R}\left[1 - \frac{1}{5}\right] = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R} \,.$$

PROBLEM 3: POTENTIAL OF A HEMISPHERICAL BOWL *(10 points)*

Griffiths Problem 2.48 (p. 108):

We start by a reminder that the potential due to a point charge at a distance r away is given by:

$$V = \frac{q}{4\pi\epsilon_0 r} \,, \tag{3.1}$$

whereas for a charge distribution we have

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d^3x \,, \tag{3.2}$$

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, da' \,, \tag{3.3}$$

or

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\, d\ell' \,, \tag{3.4}$$

depending on the type of charge distribution, where $\rho(\vec{\boldsymbol{r}}\,')$ is the charge density (charge per volume), $\sigma(\vec{\boldsymbol{r}}\,')$ is the surface charge density (charge per area), and $\lambda(\vec{\boldsymbol{r}}\,')$ is the linear charge density (charge per length). This problem concerns a surface charge density, so it is Eq. (3.3) that is relevant.

For the case of the hemisphere, finding the potential at the center is straightforward, as all surface elements are at the same distance R away from the center point, so the integral simplifies:

$$V_{\text{center}} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{\boldsymbol{r}}\,')}{|\vec{\boldsymbol{r}}\,' - \vec{\boldsymbol{r}}|} da' = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}\int da' = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}(2\pi R^2) = \frac{\sigma R}{2\epsilon_0} \,. \tag{3.3}$$

Finding the potential at the center is slightly more complicated. We set up the coordinates as shown in Fig. 1. Based on this definition of the coordinates, we have $da = 2\pi R^2 \sin\theta d\theta$ and $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2 = |R\hat{\boldsymbol{z}} - R\hat{\boldsymbol{r}}|^2 = R^2(|\hat{\boldsymbol{z}}|^2 + |\hat{\boldsymbol{r}}|^2 - 2\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}}) = 2R^2(1 - \cos\theta)$. Thus the integral for the potential becomes:

$$\begin{aligned} V_{\text{pole}} &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} da' = \frac{1}{4\pi\epsilon_0} \frac{\sigma(2\pi R^2)}{R\sqrt{2}} \int_0^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{1-\cos\theta}} \\ &= \frac{\sigma R}{2\sqrt{2}\epsilon_0} (2\sqrt{1-\cos}) \Big|_0^{\pi/2} = \frac{\sigma R}{\sqrt{2}\epsilon_0}(1-0) = \frac{\sigma R}{\sqrt{2}\epsilon_0} . \end{aligned} \tag{3.4}$$

Thus we obtain:

$$V_{\text{pole}} - V_{\text{center}} = \frac{\sigma R}{2\epsilon_0}(\sqrt{2} - 1) . \tag{3.5}$$

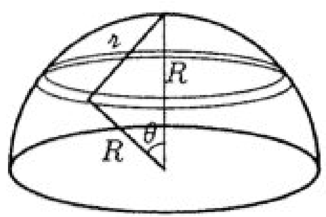


Figure 1: setup of coordinates for finding potential at pole.

PROBLEM 4: CALCULATING FORCES USING VIRTUAL WORK *(10 points)*

In this problem we use the method of virtual work in order to calculate the force F of attraction between the plates of a capacitor. Specifically, we increase the separation between the plates by a small displacement δd, measure the change in the energy of the system ΔE, and then use the law of conservation of energy to calculate the amount of work $W = F\delta d$ needed to move the plates against the force of attraction between the plates.

In this solution, we will use the expressions for the capacitance of a parallel plane capacitor C,

$$C = \frac{\epsilon_0 A}{d} , \tag{4.1}$$

and the energy of the electric field of the capacitor:

$$E = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} . \tag{4.2}$$

(a) [5 pts] **The charges on the plates are fixed.** In this case, the capacitor is isolated from the rest of the world, and the change in the gap d between the two plates will change the capacitance C, but not the charge Q. The work that we do must equal the change in the energy of the capacitor:

$$\begin{aligned} W = F\delta d = \Delta E &= \frac{1}{2}\frac{Q^2}{C_{after}} - \frac{1}{2}\frac{Q^2}{C_{before}} = \frac{Q^2}{2\epsilon_0 A}(d + \delta d - d) = \\ &= \frac{Q^2}{2\epsilon_0 A}\delta d . \end{aligned} \tag{4.3}$$

Thus, the force F between the plates of the capacitor is:

$$F = \frac{1}{2}\frac{Q^2}{\epsilon_0 A} . \tag{4.4}$$

The force F is positive, which means that we apply the force in the same direction as the displacement δd and, therefore, the electrostatic force on the plates is attractive: each plate experiences an electrostatic force F directed toward the other plate.

(b) [5 pts] **The voltage between the plates is fixed.** In order to keep the voltage fixed, we must attach the capacitor to a battery. When we shift one of the plates and change the capacitance C, the battery will move some charge ΔQ between the two plates, so that $V = Q/C$ remains fixed:

$$\Delta Q = Q_{\text{after}} - Q_{\text{before}} = V\Delta C . \tag{4.5}$$

This amount of charge ΔQ is negative (assuming that Q is positive), because the capacitance decreases when we move the plates apart. By moving the charge between the plates, the battery will perform some work too, equal to $V\Delta Q$, which we have to take into account. The total energy input to the capacitor is then

$$\Delta E = F\delta d + V\Delta Q . \tag{4.6}$$

So

$$\begin{aligned} F\delta d = \Delta E - V\Delta Q &= \frac{1}{2}V^2\Delta C - V^2\Delta C = -\frac{1}{2}V^2\Delta C \\ &= -\frac{1}{2}V^2\left(\frac{\epsilon_0 A}{d + \delta d} - \frac{\epsilon_0 A}{d}\right) \approx \frac{1}{2}V^2\frac{\epsilon_0 A}{d}\frac{\delta d}{d} = \\ &= \frac{V^2C^2}{2\epsilon_0 A}\delta d = \frac{Q^2}{2\epsilon_0 A}\delta d . \end{aligned} \tag{4.7}$$

(If you have trouble following the calculation in the middle line of this equation, see the footnote* on the next page.) Thus we have

$$F = \frac{Q^2}{2\epsilon_0 A} , \tag{4.8}$$

in agreement with part (a).

There is a simple reason why the two answers agree. Even though the fixed-voltage calculation involves introducing a new element, the battery, we know that the battery will not affect the force on the capacitor plates. Once the capacitor is charged to the same voltage as the battery, we could imagine disconnecting and reconnecting the wires between the battery and the capacitor. No charge will flow, since the the capacitor is at the same voltage as the battery. But if no charges move when the wires are disconnected and reconnected, there can be no change in the force, since the force is ultimately determined by the Coulomb forces between the charges.

PROBLEM 5: MUTUAL CAPACITANCE *(15 points)*

(a) [5 pts] To add charge Q_i to conductor i, we need to gradually change the charge q_i on conductor i from 0 to Q_i. At any point during the change, the potential of conductor i will be

$$V_i = \sum_{k=1}^{n} P_{ik} q_k = P_{ii} q_i \ , \tag{5.1}$$

where the sum collapses because only $q_i \neq 0$. (Note that we are not using a summation convention here — if there is no $\sum$ sign, there is no sum.) The work needed to move each charge element $\mathrm{d}q_i$ from infinity ($V = 0$) to the conductor is then

$$\mathrm{d}W = V_i \, \mathrm{d}q_i = P_{ii} \, q_i \, \mathrm{d}q_i \ , \tag{5.2}$$

so the total work is

$$W = \int_0^{Q_i} P_{ii} \, q_i \, \mathrm{d}q_i = \frac{1}{2} P_{ii} Q_i^2 \ . \tag{5.3}$$

Now we hold fixed the charge on conductor i, and increase the charge q_j on conductor j from 0 to Q_j. The potential of conductor j is

$$V_j = \sum_{k=1}^{n} P_{jk} q_k = P_{ji} Q_i + P_{jj} q_j \ . \tag{5.4}$$

* The Taylor series for $1/(1+x)$ is given by

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \ .$$

Therefore

$$\frac{1}{d + \delta d} = \frac{1}{d\left(1 + \frac{\delta d}{d}\right)} \approx \frac{1}{d}\left(1 - \frac{\delta d}{d}\right) \ .$$

The work to move a charge element $\mathrm{d}q_j$ from infinity to conductor j is then

$$\mathrm{d}W = V_j \, \mathrm{d}q_j = \left[P_{ji} Q_i + P_{jj} q_j\right] \mathrm{d}q_j \ , \tag{5.5}$$

so the total work for this operation is

$$W = \int_0^{Q_j} \left[P_{ji} Q_i + P_{jj} q_j\right] \mathrm{d}q_j = P_{ji} Q_i Q_j + \frac{1}{2} P_{jj} Q_j^2 \ . \tag{5.6}$$

The total work done to charge the two conductors is then

$$W_{\mathrm{tot}} = \frac{1}{2} P_{ii} Q_i^2 + P_{ji} Q_i Q_j + \frac{1}{2} P_{jj} Q_j^2 \ . \tag{5.7}$$

If we had charged the capacitors in the opposite order, we would have obtained a similar expression, except that we would have P_{ij} appearing instead of P_{ji}. But these two sequences both started with the charges at infinity, and ended with Q_i on conductor i and Q_j on conductor j. Since electrostatics is known to be conservative, the work done must be the same in both cases, and hence $P_{ij} = P_{ji}$.

Finally, the matrix C_{ij} is the matrix inverse of P_{ij}. It can be easily proven that if a symmetric matrix is invertible then its inverse is also symmetric. One way to see this is to look at the explicit expression for the matrix inverse given by Cramer's rule.

(b) [5 pts] The matrix equations are

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \ . \tag{5.8}$$

Inverting we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{\det \boldsymbol{C}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \ . \tag{5.9}$$

If we set $Q_1 = Q$ and $Q_2 = -Q$ then, the capacitance C is defined from

$$Q = C(V_1 - V_2) \ . \tag{5.10}$$

Putting these values of Q_1 and Q_2 into Eq. (5.9), we get

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{\det \boldsymbol{C}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} Q \\ -Q \end{pmatrix} , \tag{5.11}$$

and therefore

$$V_1 - V_2 = \frac{1}{\det \boldsymbol{C}} Q (C_{22} + C_{12} + C_{21} + C_{11}) \ . \tag{5.12}$$

Fro Eq. (5.10) we know that $C = Q/(V_1 - V_2)$, so

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{22} + C_{12} + C_{21} + C_{11}} . \tag{5.13}$$

The above equation is a perfectly acceptable answer, but we could use $C_{21} = C_{12}$ to rewrite it as

$$C = \frac{C_{11}C_{22} - C_{12}^2}{C_{22} + 2C_{12} + C_{11}} . \tag{5.14}$$

(c) [5 pts] We must calculate the charges on the shells when the potentials have arbitrary values V_1 and V_2 for the inner and outer conductor respective. We first solve the inverse problem, finding the potentials V_1 and V_2 in terms of the charges Q_1 and Q_2. Using Gauss's law for spherical Gaussian surfaces concentric with the spherical conducting shells, we find that the electric field is given by

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \begin{cases} \vec{\boldsymbol{0}} & \text{for } r < a \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q_1}{r^2}\,\hat{\boldsymbol{e}}_{\boldsymbol{r}} & \text{for } a < r < b \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q_1+Q_2}{r^2}\,\hat{\boldsymbol{e}}_{\boldsymbol{r}} & \text{for } r > b. \end{cases} \tag{5.15}$$

Defining the potential to be 0 at infinity,

$$V_2 = -\int_\infty^b \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \int_b^\infty \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \frac{Q_1+Q_2}{4\pi\epsilon_0}\int_b^\infty \frac{\mathrm{d}r}{r^2} = \frac{Q_1+Q_2}{4\pi\epsilon_0 b} , \tag{5.16}$$

which is the same answer (and same calculation) that one would have for a point charge $Q_1 + Q_2$ at the origin. To find V_1, we can use a point on the outer shell as a reference point:

$$\begin{aligned} V_1 = V_2 - \int_b^a \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{\ell}} &= \frac{Q_1+Q_2}{4\pi\epsilon_0 b} + \int_a^b \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{\ell}} \\ &= \frac{Q_1+Q_2}{4\pi\epsilon_0 b} + \frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{a} + \frac{Q_2}{b}\right) . \end{aligned} \tag{5.17}$$

We now must solve these equations for Q_1 and Q_2. We can eliminate Q_2 by subtracting Eq. (5.16) from Eq. (5.17), finding

$$V_1 - V_2 = \frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q_1}{4\pi\epsilon_0}\frac{b-a}{ab} . \tag{5.18}$$

Inverting, this gives

$$Q_1 = \tilde{C}(V_1 - V_2) , \tag{5.19}$$

where for convenience we have defined

$$\tilde{C} \equiv 4\pi\epsilon_0 \frac{ab}{b-a} . \tag{5.20}$$

Having found Q_1, we can find Q_2 from Eq. (5.17):

$$\begin{aligned} Q_2 = b\left(4\pi\epsilon_0 V_1 - \frac{Q_1}{a}\right) &= 4\pi\epsilon_0 b\left[V_1 - \frac{b}{b-a}(V_1 - V_2)\right] \\ &= 4\pi\epsilon_0 b\left[\frac{bV_2 - aV_1}{b-a}\right] = -\tilde{C}V_1 + \frac{b}{a}\tilde{C}V_2 . \end{aligned} \tag{5.21}$$

The C_{ij} are defined so that

$$Q_i = \sum_{j=1}^n C_{ij}V_j , \tag{5.22}$$

so comparing with Eqs. (5.19) and (5.21) we see that

$$\boldsymbol{C} = \begin{pmatrix} \tilde{C} & -\tilde{C} \\ -\tilde{C} & \frac{b}{a}\tilde{C} \end{pmatrix} . \tag{5.23}$$

Substituting these matrix elements into Eq. (5.13), we find

$$C = \frac{\frac{b}{a}\tilde{C}^2 - \tilde{C}^2}{\frac{b}{a}\tilde{C} - \tilde{C}} = \tilde{C} . \tag{5.24}$$

This agrees with Griffiths' example 2.11 (p. 106).

PROBLEM 6: A CAVITY IN A CONDUCTING SPHERE *(8 points)*

(a) [2 pts] Since region I is filled with a conductor, it must be the case that $\vec{\boldsymbol{E}} = 0$.

(b) [2 pts] We know from Gauss's law that the total electric flux through any surface enclosing the entire sphere will be q/ϵ_0. Furthermore, we know that the electric field in region II should be spherically symmetric — since there is no $\vec{\boldsymbol{E}}$ in region I, the irregularity of the cavity and the off-center position of the charge q have no way of influencing the distribution of charge on the surface of the sphere, or the electric field outside the sphere. Thus the electric field in region II will be the same as that of a point charge q at the origin:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{q}{4\pi\epsilon_0}\,\hat{\boldsymbol{r}} ,$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing radially outward from the origin.

(c) [2 pts] Yes, there will have to be a layer of charge on the surface of the cavity, since the electric field $\vec{\boldsymbol{E}}$ in region I must be zero. By Gauss's law, the total charge inside any surface that encloses the cavity must be zero, and so there must be a charge $-q$ distributed on the surface of the cavity, to cancel the charge q inside the cavity. Since the surface is irregular, the surface charge density will presumably also be irregular, arranging itself in just the right configuration so that the total electric field, from the point charge and the surface charge distribution, cancels out for all points in region I.

(d) *(2 points)* Since the conductor was (presumably) uncharged before the region was cut out and the charge was introduced, it must still be uncharged. Since we argued above that there must be a charge $-q$ on the surface of the cavity, it follows that there must be a charge $+q$ on the outer surface. It will be uniformly distributed for the reason discussed in part (b).

PROBLEM 7: SPACE CHARGE, VACUUM DIODES, AND THE CHILD-LANGMUIR LAW *(20 points)*

(a) [3 pts] In general Poisson's equation is $\nabla^2 V = -\rho/\epsilon_0$. In this case we know that V and ρ depend only on x, so we can write

$$\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = -\frac{\rho(x)}{\epsilon_0}\,. \tag{7.1}$$

(b) [3 pts] By energy conservation we know that the change of electrostatic energy $\Delta W_{\rm elec}$ of the electron, as it travels from the cathode to a point x, goes into a change of kinetic energy ΔK. Letting $v(x)$ denote the velocity of the electron at x, we have

$$\Delta W_{\rm elec} + \Delta K = 0 \quad\Longrightarrow\quad (-e)[V(x) - V(0)] + \frac{1}{2}m[v^2(x) - v^2(0)] = 0\,.$$

But the potential at the cathode is zero, $V(0) = 0$, and the velocity of the electrons at the cathode is also zero, $v(0) = 0$. Therefore

$$\frac{1}{2}mv^2(x) = eV(x) \quad\Longrightarrow\quad v(x) = \sqrt{\frac{2eV(x)}{m}}\,. \tag{7.2}$$

(c) [4 pts] If a fluid has velocity v and charge density ρ, then the current I moving across an area A normal to the velocity is $I = \rho v A$ (note that since ρ is negative the current I is also negative). Thus,

$$\frac{I}{A} = \rho v \quad\Longrightarrow\quad \rho(x)\,v(x) = \frac{I}{A}\,. \tag{7.3}$$

The above equation, with I independent of x, is the requested relation between $\rho(x)$ and $v(x)$. They are both x-dependent, but their product is not!

(d) [3 pts] From Eqs. (7.1) and (7.3) we have

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -\frac{\rho}{\epsilon_0} = -\frac{I}{\epsilon_0 A\, v(x)}\,,$$

and using Eq. (7.2) we find

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2e}}\,[V(x)]^{-1/2}\,.$$

This is the desired differential equation. Calling

$$\alpha \equiv -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2e}}\,, \tag{7.4}$$

the equation reads

$$\frac{d^2 V}{dx^2} = \alpha\, V^{-1/2}\,. \tag{7.5}$$

(e) [6 pts] One method is to write Eq. (7.5) as $V'' = \alpha V^{-1/2}$ and multiply by V' to find

$$V'V'' = \alpha V^{-1/2} V' \quad\Longrightarrow\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}V'^2 = 2\alpha\,\frac{\mathrm{d}}{\mathrm{d}x}V^{1/2} \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(V'^2 - 4\alpha V^{1/2}\right) = 0\,.$$

This shows that the expression in parentheses is a constant:

$$V'^2 - 4\alpha V^{1/2} = E\,.$$

(Note that this method is also used in Newtonian mechanics to show that $m\ddot{x} = -V'(x)$ implies that $m\dot{x}\ddot{x} = -\dot{x}V'(x)$, which implies that $\frac{\mathrm{d}}{\mathrm{d}t}[\frac{1}{2}m\dot{x}^2 + V(x)] = 0$.) The constant E can be evaluated using the boundary conditions at $x = 0$. Since $V(0) = 0$ and $V'(0) = 0$ we have $E = 0$. Thus the differential equation becomes quite simple:

$$V'^2 = 4\alpha V^{1/2} \quad\Longrightarrow\quad \frac{\mathrm{d}V}{\mathrm{d}x} = 2\alpha^{1/2} V^{1/4}\,. \tag{7.6}$$

Integrating we find

$$\frac{\mathrm{d}V}{V^{1/4}} = 2\alpha^{1/2} x \quad\Longrightarrow\quad \frac{4}{3}V^{3/4} = 2\alpha^{1/2} x + F\,,$$

where F is another constant of integration. Again, $V(0) = 0$ implies that the constant of integration is zero, so

$$V^{3/4} = \frac{3}{2}\alpha^{1/2} x \quad \Longrightarrow \quad V(x) = \left(\frac{9\alpha}{4}\right)^{2/3} x^{4/3} . \tag{7.7}$$

As an alternative method, we can try to invent a trial solution. If it satisfies the differential equation (7.5) and boundary conditions $V(0) = V'(0) = 0$, then we would have the desired solution. Since the right-hand side of Eq. (7.5) is a power of V, it is natural to try a power-law solution of the form

$$V = Cx^{\beta} , \tag{7.8}$$

where C and $\beta > 0$ are constants. Inserting this trial solution into Eq. (7.5), we find

$$C\beta(\beta-1)x^{\beta-2} = \alpha C^{-1/2} x^{-\beta/2} .$$

Equating the powers of x we find $\beta = 4/3$ and then

$$C\,\frac{4}{9} = \alpha C^{-1/2} \quad \Longrightarrow \quad C = \left(\frac{9\alpha}{4}\right)^{2/3} .$$

We therefore find

$$V(x) = \left(\frac{9\alpha}{4}\right)^{2/3} x^{4/3} , \tag{7.9}$$

in agreement with Eq. (7.7). Since $\beta > 1$ this solution satisfies not only $V(0) = 0$ but also the condition that the field at the cathode vanishes, $-V'(0) \equiv -\frac{dV}{dx}\big|_{x=0} = 0$, so this is the solution we wanted.

The problem tells us that $V(d) = V_0$, so using the fact that $V(x) \propto x^{4/3}$ we can write

$$\boxed{V(x) = V_0 \left(\frac{x}{d}\right)^{4/3} .} \tag{7.10}$$

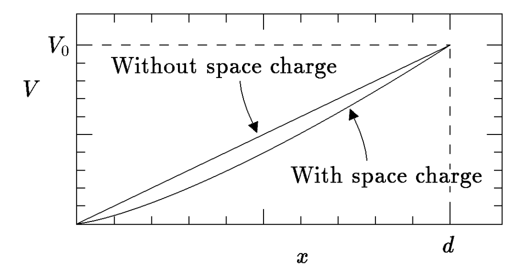
This result is consistent with Eqs. (7.7) or (7.9), since α depends on the unknown I and is therefore not yet determined. For this part, however, it suffices to use Eq. (7.10). $\rho(x)$ can be found directly from Poisson's equation in the form (7.1), with Eq. (7.10):

$$\rho(x) = -\epsilon_0 \frac{d^2}{dx^2}\left[V_0 \left(\frac{x}{d}\right)^{4/3}\right] = \boxed{-\frac{4\epsilon_0 V_0}{9(d^2\,x)^{2/3}} .} \tag{7.12}$$

And $v(x)$ can be found directly from Eqs. (7.2) and (7.10):

$$v(x) = \sqrt{\frac{2eV(x)}{m}} = \boxed{\sqrt{\frac{2eV_0}{m}}\left(\frac{x}{d}\right)^{2/3} .} \tag{7.13}$$

The potential $V(x)$ looks like in this figure. The straight line represents the potential without space charge.



(The above graph was drawn accurately, but for purposes of grading we will accept any graph with the same qualitative features.)

(f) [3 pts] Comparing either Eqs. (7.7) or (7.9) with Eq. (7.10), we determine the value of α:

$$\left(\frac{9\alpha}{4}\right)^{2/3} = V_0 \quad \Longrightarrow \quad \alpha = \frac{4V_0^{3/2}}{9d^2} .$$

Then using Eq. (7.4), we can determine the value of I:

$$\boxed{I = -\frac{4}{9}\frac{\epsilon_0 A V_0^{3/2}}{d^2}\sqrt{\frac{2e}{m}} .} \tag{7.14}$$

Thus we can write $I = KV_0^{3/2}$, with

$$\boxed{K = -\frac{4}{9}\frac{\epsilon_0 A V_0^{3/2}}{d^2}\sqrt{\frac{2e}{m}} .} \tag{7.15}$$

PROBLEM 8: $\nabla^2(1/r)$ IN THE LANGUAGE OF DISTRIBUTIONS *(20 points)*

(a) [10 pts] We want to evaluate $F[\varphi(\vec{r})]$, which is defined by

$$F[\varphi(\vec{r})] \equiv \int \nabla^2\varphi(\vec{r}) \left(\frac{1}{r}\right) \mathrm{d}^3x \ . \tag{8.1}$$

As pointed out in the statement of the problem, we can begin by integrating by parts, which takes us back to Eq. (8.10b) of the problem statement,

$$F[\varphi(\vec{r})] = -\int \partial_i\varphi(\vec{r})\,\partial_i \left(\frac{1}{r}\right) \mathrm{d}^3x \ . \tag{8.2}$$

But

$$\partial_i \left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r}_i \ , \tag{8.3}$$

so

$$F[\varphi(\vec{r})] = +\int \frac{1}{r^2}\hat{r}\cdot\vec{\nabla}\varphi(\vec{r})\,\mathrm{d}^3x \ . \tag{8.4}$$

If we write this in polar coordinates, the radial component of the gradient is just $\partial/\partial r$, and the volume element is $d^3x = r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, so

$$F[\varphi(\vec{r})] = \int_0^\infty \mathrm{d}r \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \frac{\partial\varphi(r,\theta,\phi)}{\partial r} \ . \tag{8.5}$$

The integral over r is an integral of a derivative, so if we integrate first over r we find

$$F[\varphi(\vec{r})] = \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\,[\varphi(\infty,\theta,\phi) - \varphi(0,\theta,\phi)] \ . \tag{8.6}$$

But the test function $\varphi(\vec{r})$ is required to approach zero at $r=\infty$, and continuity requires that $\varphi(0,\theta,\phi)$ is independent of θ and ϕ. The integral over angles then gives a factor of 4π, and we have

$$F[\varphi(\vec{r})] = -4\pi\varphi(\vec{0}) \ , \tag{8.7}$$

which is exactly what we were asked to prove.

(b) [10 pts] To evaluate $\nabla^2 \ln r$ as a distribution in two dimensions, we introduce a test function $\varphi(x,y)$, and evaluate

$$F[\varphi(x,y)] = \int \varphi(x,y)\nabla^2(\ln r)\,d^2x \ , \tag{8.8}$$

where $r \equiv \sqrt{x^2+y^2}$ and $\nabla^2(\ln r)$ is interpreted as a Lapalacian of the distribution $\ln r$. The Laplacian (or any derivative of a distribution) is defined by formally integrating by parts, omitting any boundary terms, so

$$F[\varphi(x,y)] = \int \nabla^2\varphi\,\ln r\,\mathrm{d}^2x \ , \tag{8.9}$$

which is a well-defined ordinary integral. Starting from Eq. (8.9) we can integrate by parts, finding

$$F[\varphi(x,y)] = -\int \partial_i\varphi\,\partial_i \ln r\,\mathrm{d}^2x = -\int \partial_i\varphi \left(\frac{1}{r}\hat{r}_i\right) \mathrm{d}^2x \ . \tag{8.10}$$

Using polar coordinates r and ϕ, we can write $\hat{r}_i\partial_i\varphi = \partial\varphi/\partial r$ and $\mathrm{d}^2x = r\,\mathrm{d}r\,\mathrm{d}\phi$, so

$$\begin{aligned} F[\varphi(x,y)] &= -\int_0^{2\pi} \mathrm{d}\phi \int_0^\infty r\,\mathrm{d}r\,\frac{1}{r}\frac{\partial\varphi}{\partial r} \\ &= -\int_0^{2\pi} \mathrm{d}\phi\,\left[\varphi(r{=}\infty,\phi) - \varphi(r{=}0,\phi)\right] = 2\pi\varphi(\vec{0}) \ , \end{aligned} \tag{8.11}$$

where as in part (a) we used the fact that φ is required to approach zero as $r\to\infty$, and that it cannot depend on angle at $r=0$. Thus we have

$$\int \varphi(\vec{r})\nabla^2(\ln r)\,d^2x = 2\pi\varphi(\vec{0}) \ , \tag{8.12}$$

and by definition

$$\int \varphi(\vec{r})\,\delta^2(\vec{r})\,\mathrm{d}^2x \equiv \varphi(\vec{0}) \ . \tag{8.13}$$

Since distributions are defined solely in terms of the generalized integral that they produce (i.e., by how they map test functions $\varphi(\vec{r})$ to the number given by these integrals), we see that

$$\nabla^2 \ln r = 2\pi\delta^2(\vec{r}) \ , \tag{8.14}$$

in the sense that the two sides of this equation represent the same distribution.