8.07 Lecture Slides 24
December 9, 2019

POTENTIALS, FIELDS,
and
RADIATION

Announcements

Practice Problems for the Final Exam, and a Formula Sheet for the Final Exam, have been posted. The Formula Sheet is very thorough, and is intended as a tool for reviewing the course. If you understand all the formulas, you are in great shape for the Final Exam.

The Final Exam will be given on Thursday, December 19, from 1:30 pm - 4:30 pm, in this room (6-120). The final exam will include material from the entire course, but will emphasize material since the last quiz.

Two of the problems on the Final Exam will be taken verbatim, or at least almost verbatim, from Problem Sets 8 and 9, or the Practice Problems for the Final Exam. Extra credit problems on the homework will be possible choices.

Office hours will continue through Friday of this week, at the usual times and places. They are listed on the Staff tab of the website.

Yitian Sun will hold a review session for the Final Exam next weekend: Saturday, 1:00 pm. The ending time is flexible, but 2 hours is an estimate of how long it might last.

During final exam week, there will be no regular office hours or review sessions. But if you have questions, feel free to email Marin, Yitian, and me, and we will try to arrange a time to meet with you. Or you can just send your questions to us by email, and we will try to either answer you, or set up a meeting if the answer is hard to answer by email. In some cases we may decide it would be helpful to send the answer to the entire class. Please let us know, with your question, if it is okay for us to tell the class who asked the question, or if you would prefer to remain anonymous.

Last Friday I emailed to you a link to a survey about the video capture. Please respond!

You have also received a link to the end-of-term course evaluations, which are open until Monday December 16, at 9 am. Please respond! We very much value your feedback. Remember, it is your feedback that helps to keep the quality of teaching at MIT high.

POTENTIALS AND FIELDS

Maxwell's Equations with Sources:

$$
\begin{aligned}
&\text{(i)} \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho &&\text{(iii)} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ , \\
&\text{(ii)} \quad \vec{\nabla} \cdot \vec{B} = 0 &&\text{(iv)} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \ ,
\end{aligned}
\tag{1}
$$

Question: If we are given the sources $\rho(\vec{r}, t)$ and $\vec{J}(\vec{r}, t)$, can we find $\vec{E}$ and $\vec{B}$? If we accept the proposition that all integrals are in principle doable (at least numerically), then the answer is **YES**.

Electromagnetic Potentials

If $\vec{\boldsymbol{B}}$ depends on time, then $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} \neq \vec{\mathbf{0}}$, so we cannot write $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V$. BUT: we can still write

$$\boxed{\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\ .} \tag{2}$$

Then notice that

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial}{\partial t}(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \quad \Longrightarrow \quad \vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{E}} + \frac{\partial \vec{\boldsymbol{A}}}{\partial t}\right) = 0\ . \tag{3}$$

so we can write

$$\vec{\boldsymbol{E}} + \frac{\partial \vec{\boldsymbol{A}}}{\partial t} = -\vec{\boldsymbol{\nabla}} V \quad \Longrightarrow \quad \boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t}\ .} \tag{4}$$

With

$$\boxed{\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} \ , \quad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t} \ ,} \tag{5}$$

the source-free Maxwell equations (ii) and (iii),

$$\text{(i)} \quad \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{1}{\epsilon_0}\rho \qquad \text{(iii)} \quad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \ ,$$

$$\text{(ii)} \quad \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0 \qquad \text{(iv)} \quad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} + \frac{1}{c^2}\frac{\partial \vec{\boldsymbol{E}}}{\partial t} \ ,$$

are automatically satisfied. We must therefore deal with the two other Maxwell equations, (i) and (iv).

Maxwell's Other Equations:

$$\text{(i)} \quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho \; . \tag{6}$$

$$\text{(iv)} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\Longrightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Longrightarrow \quad \left(\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \tag{7}$$

where we used

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \; . \tag{8}$$

Gauge Transformations

We have already discussed gauge transformations for statics. But it easily generalizes to the full theory of electrodynamics.

Let $\Lambda(\vec{\boldsymbol{r}}, t)$ be an arbitrary scalar function. Then, if we are given $V(\vec{\boldsymbol{r}}, t)$ and $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t)$, we can define new potentials by a **gauge transformation:**

$$\boxed{\vec{\boldsymbol{A}}' = \vec{\boldsymbol{A}} + \vec{\boldsymbol{\nabla}}\Lambda \ , \quad V' = V - \frac{\partial \Lambda}{\partial t} \ .} \tag{9}$$

Then

$$\vec{\boldsymbol{B}}' = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}' = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}}\Lambda = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{B}} \ . \tag{10}$$

$$\vec{\boldsymbol{E}}' = -\vec{\boldsymbol{\nabla}}V' - \frac{\partial \vec{\boldsymbol{A}}'}{\partial t} = -\vec{\boldsymbol{\nabla}}V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t} + \vec{\boldsymbol{\nabla}}\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\vec{\boldsymbol{\nabla}}\Lambda = \vec{\boldsymbol{E}} \ . \tag{11}$$

Choice of Gauge:

Can use gauge freedom, $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, to make $\vec{\nabla}\cdot\vec{A}$ whatever we want.

Coulomb Gauge: $\boxed{\vec{\nabla}\cdot\vec{A} = 0\ .}$ (12)

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho \quad\Longrightarrow\quad \nabla^2 V = -\frac{1}{\epsilon_0}\rho\ . \tag{13}$$

V is easy to find, but $\vec{A}$ is hard. V responds instantaneously to changes in ρ, but V is not measurable. $\vec{E}$ and $\vec{B}$ receive information only at the speed of light.

Lorenz Gauge: $\boxed{\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}\ .}$ (14)

$$\Longrightarrow \quad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho\ . \tag{15}$$

(Aside: Until recently, this gauge condition was called **Lorentz gauge**, named for Hendrik A. Lorentz, the same Dutch physicist whose name is attached to the Lorentz transformation. Starting with the 4th Edition, Griffiths has adopted the new name **Lorenz gauge**, referring to the earlier use of this gauge by the Danish physicist Ludwig V. Lorenz, which goes back to 1867. Lorenz was also the first person to derive the Clausius-Mossotti equation, in 1869, so this equation is sometimes called the Lorentz-Lorenz equation, since it was rediscovered by Lorentz in 1878.)

Define

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \text{D'Alembertian} \ . \tag{16}$$

Then, in Lorenz gauge,

$$\boxed{\Box^2 V = -\frac{1}{\epsilon_0}\rho \ .} \tag{17}$$

In general, $\vec{\boldsymbol{A}}$ obeys Eq. (7):

$$\left(\nabla^2 \vec{\boldsymbol{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\boldsymbol{A}}}{\partial t^2}\right) - \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \vec{\boldsymbol{J}} \ .$$

In Lorenz gauge,

$$\boxed{\Box^2 \vec{\boldsymbol{A}} = -\mu_0 \vec{\boldsymbol{J}} \ .} \tag{18}$$

Solution to $\Box^2 V = -\dfrac{1}{\epsilon_0}\rho$

Method: Guess a solution and then show that it works.

We know that

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,', t)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ . \tag{19}$$

We try the guess

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \quad \Longrightarrow \quad \boxed{V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ ,} \tag{20}$$

where

$$\boxed{t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} = \text{retarded time.}} \tag{21}$$

Testing the trial solution:

$$\vec{\boldsymbol{r}} = x_i \hat{\boldsymbol{e}}_i \ , \quad \partial_i |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = \frac{x_i - x'_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{\boldsymbol{r}}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{\boldsymbol{r}}\,', t_r)\frac{x_i - x'_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x'_i}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = 4\pi \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \ .$$

Testing the trial solution:

$$\vec{\boldsymbol{r}} = x_i \hat{\boldsymbol{e}}_i \ , \quad \partial_i |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{\boldsymbol{r}}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{\boldsymbol{r}}\,', t_r)\frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = 4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \ .$$

Then

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ .$$

Testing the trial solution:

$$\vec{\boldsymbol{r}} = x_i \hat{\boldsymbol{e}}_i \ , \quad \partial_i |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{\boldsymbol{r}}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{\boldsymbol{r}}\,', t_r)\frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = 4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \ .$$

Then

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{-\frac{1}{c}\,\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2}(x_i - x_i') - \frac{\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}(x_i - x_i')\right] \ .$$

Testing the trial solution:

$$\vec{\boldsymbol{r}} = x_i \hat{\boldsymbol{e}}_i \ , \quad \partial_i |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{\boldsymbol{r}}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{\boldsymbol{r}}\,', t_r)\frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = 4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \ .$$

Then

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{-\frac{1}{c}\,\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2}(x_i - x_i') - \frac{\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}(x_i - x_i')\right] \ .$$

Testing the trial solution:

$$\vec{\boldsymbol{r}} = x_i \hat{\boldsymbol{e}}_i \ , \quad \partial_i |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ ,$$

$$\partial_i \rho(\vec{\boldsymbol{r}}\,', t_r) = -\frac{1}{c}\dot{\rho}(\vec{\boldsymbol{r}}\,', t_r)\frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \quad \text{where } \dot{\rho} \equiv \frac{\partial \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r} \ , \tag{22}$$

$$\partial_i \frac{x_i - x_i'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = 4\pi\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') \ .$$

Then

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ .$$

$$\partial_i V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{-\frac{1}{c}\,\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2}(x_i - x_i') - \frac{\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}(x_i - x_i')\right] \ .$$

$$\partial_i^2 V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[-4\pi\rho\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') + \frac{\frac{1}{c}\,\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2} + \frac{\frac{1}{c^2}\,\ddot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\right.$$

$$\left. + \frac{\frac{2}{c}\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2} - \frac{\frac{3}{c}\,\dot{\rho}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2}\right]$$

$$\partial_i^2 V = \frac{1}{4\pi\epsilon_0}\int d^3x' \Bigg[-4\pi\rho\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,') + \frac{\frac{1}{c}\dot{\rho}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^2} + \frac{\frac{1}{c^2}\ddot{\rho}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}$$

$$+ \frac{\frac{2}{c}\dot{\rho}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^2} - \frac{\frac{3}{c}\dot{\rho}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^2}\Bigg]$$

$$= -\frac{\rho(\vec{\boldsymbol{r}},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int d^3x' \frac{\ddot{\rho}(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} \qquad \left(t_r = t - \frac{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{\boldsymbol{r}},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int d^3x' \frac{\frac{\partial^2\rho(\vec{\boldsymbol{r}}\,',t_r)}{\partial t_r^2}}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} \qquad \left(t_r = t - \frac{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{\boldsymbol{r}},t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}$$

$$= -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

$$\partial_i^2 V = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[-4\pi\rho\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,') + \frac{\frac{1}{c}\,\dot\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2} + \frac{\frac{1}{c^2}\,\ddot\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}\right.$$

$$\left. + \frac{\frac{2}{c}\dot\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2} - \frac{\frac{3}{c}\,\dot\rho}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^2}\right]$$

$$= -\frac{\rho(\vec{\boldsymbol{r}}, t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int \mathrm{d}^3x' \frac{\ddot\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \qquad \left(t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{\boldsymbol{r}}, t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\int \mathrm{d}^3x' \frac{\frac{\partial^2 \rho(\vec{\boldsymbol{r}}\,', t_r)}{\partial t_r^2}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \qquad \left(t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c}\right)$$

$$= -\frac{\rho(\vec{\boldsymbol{r}}, t)}{\epsilon_0} + \frac{1}{4\pi\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\int \mathrm{d}^3x' \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}$$

$$= -\frac{1}{\epsilon_0}\rho + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

YES!

Retarded Time Solutions

$$\begin{aligned} V(\vec{\boldsymbol{r}},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,',t_r)}{\boldsymbol{\mathscr{r}}} \\ \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) &= \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{\mathscr{r}} \quad , \end{aligned} \tag{23}$$

where

$$\vec{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,' \; , \quad \mathscr{r} = |\vec{\boldsymbol{\mathscr{r}}}| \; , \quad t_r = t - \frac{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|}{c} = t - \frac{\mathscr{r}}{c} \; . \tag{24}$$

The Fields of a Point Charge

From the retarded time solution (Eq. (23)),

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(\vec{\boldsymbol{r}}\,', t_r)}{\mathscr{r}} \; .$$

For a point charge q moving on a trajectory $\vec{\boldsymbol{r}}_p(t)$,

$$\rho(\vec{\boldsymbol{r}}, t) = q\delta^3\Big(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p(t)\Big) \; , \tag{25}$$

so

$$\begin{aligned} V(\vec{\boldsymbol{r}}, t) &= \frac{q}{4\pi\epsilon_0} \int d^3x' \, \frac{\delta^3\Big(\vec{\boldsymbol{r}}\,' - \vec{\boldsymbol{r}}_p(t_r)\Big)}{\mathscr{r}} \\ &= \frac{q}{4\pi\epsilon_0 |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p(t_r)|} \int d^3x' \, \delta^3\Big(\vec{\boldsymbol{r}}\,' - \vec{\boldsymbol{r}}_p(t_r)\Big) \; . \end{aligned} \tag{26}$$

But, perhaps surprisingly,

$$Z \equiv \int \mathrm{d}^3x' \, \delta^3\Big(\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}_p(t_r)\Big) \neq 1 \ , \tag{27}$$

where I am calling the integral Z for future reference. Remember,

$$\delta\Big(g(x)\Big) = \sum_i \frac{\delta(x - x_i)}{|\mathrm{d}g(x)/\mathrm{d}x|_{x=x_i}} \ , \quad \text{where} \quad g(x_i) = 0 \ , \tag{28}$$

and

$$t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}{c} \ .$$

To make things simple, suppose that the particle velocity at t_r points in the x-direction. Then

$$\begin{aligned} Z &= \int \mathrm{d}^3x' \delta\Big(x' - x_p(t_r)\Big) \delta\Big(y' - y_p(t_r)\Big) \delta\Big(z' - z_p(t_r)\Big) \\ &= \int \mathrm{d}x' \, \delta\Big(x' - x_p(t_r)\Big) \ , \end{aligned} \tag{29}$$

where the integrals over y' and z' were simple, since $\mathrm{d}y_p(t_r)/\mathrm{d}t_r = \mathrm{d}z_p(t_r)/\mathrm{d}t_r = 0$.

So we need to evaluate

$$Z = \int \mathrm{d}x'\, \delta\Big(x' - x_p(t_r)\Big) \ , \quad \text{where } t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ . \tag{30}$$

So, to use our formula,

$$g(x') = x' - x_p\left(t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c}\right) \ , \tag{31}$$

and then

$$\frac{\mathrm{d}g(x')}{\mathrm{d}x'} = 1 - \frac{\mathrm{d}x_p}{\mathrm{d}t_r}\frac{\mathrm{d}t_r}{\mathrm{d}x'} = 1 + \frac{1}{c}\frac{\mathrm{d}x_p}{\mathrm{d}t_r}\frac{\mathrm{d}}{\mathrm{d}x'}|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'| = 1 - \frac{1}{c}\frac{\mathrm{d}x_p}{\mathrm{d}t_r}\frac{x - x'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \tag{32}$$

and $Z = 1/(\mathrm{d}g(x')/\mathrm{d}x')$. Generalizing,

$$Z = \left(1 - \frac{\vec{\boldsymbol{v}}}{c} \cdot \hat{\boldsymbol{\imath}}\right)^{-1} \ . \tag{33}$$

The Liénard-Wiechert Potentials

Finally,

$$\boxed{V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}\left(1 - \frac{\vec{\boldsymbol{v}}_p}{c} \cdot \hat{\boldsymbol{\mathscr{r}}}\right)}\ ,} \tag{34}$$

where $\vec{\boldsymbol{\mathscr{r}}} \equiv \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p$, and $\vec{\boldsymbol{r}}_p$ and $\vec{\boldsymbol{v}}_p$ are the position and velocity of the particle at t_r. Similarly, starting with

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}, t) = q\vec{\boldsymbol{v}}\delta^3\Big(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p(t)\Big) \tag{35}$$

for a point particle, we find

$$\boxed{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \frac{q\vec{\boldsymbol{v}}_p}{\mathscr{r}\left(1 - \frac{\vec{\boldsymbol{v}}_p}{c} \cdot \hat{\boldsymbol{\mathscr{r}}}\right)} = \frac{\vec{\boldsymbol{v}}_p}{c^2} V(\vec{\boldsymbol{r}}, t)\ .} \tag{36}$$

The Fields of a Point Charge

Differentiating the Liénard-Wiechert potentials, after several pages, one finds

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{(\vec{u}\cdot\vec{\boldsymbol{\mathscr{r}}})^3}\left[(c^2 - v_p^2)\vec{u} + \vec{\boldsymbol{\mathscr{r}}}\times(\vec{u}\times\vec{a}_p)\right] \ ,} \tag{37}$$

where

$$\boxed{\vec{u} = c\hat{\boldsymbol{\mathscr{r}}} - \vec{v}_p \ .} \tag{38}$$

And

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\vec{E}(\vec{r},t) \ .} \tag{39}$$

Here $\vec{r}_p$, $\vec{v}_p$, and $\vec{a}_p$ are the position, velocity, and acceleration, respectively, of the particle at the retarded time.

This slide was added after lecture.

What about $\vec{u} \cdot \vec{\mathscr{r}}$ in the denominator?

Can it vanish, leading to an infinite $\vec{E}$? Answer, no:

$$\begin{aligned}\vec{\mathscr{r}} \cdot \vec{u} &= \vec{\mathscr{r}} \cdot (c\hat{\mathscr{r}} - \vec{v}_p) \\ &= c|\vec{\mathscr{r}}| - \vec{v}_p \cdot \vec{\mathscr{r}} \\ &= c\mathscr{r} - v_p \mathscr{r} \cos\Theta \\ &= c\mathscr{r}\left(1 - \frac{v_p}{c}\cos\Theta\right) > 0 \ ,\end{aligned}$$

where Θ is the angle between $\vec{v}_p$ and $\vec{\mathscr{r}}$. But one should not try to infer the angular dependence from this equation, since $\mathscr{r}$ also depends on angle.

If the particle is moving at constant velocity, then the acceleration term in Eq. (37) is absent, and the electric field points along $\vec{\boldsymbol{u}}$. Note that $\vec{\boldsymbol{u}}$ can also be written as

$$\begin{aligned} \vec{\boldsymbol{u}} &= c\hat{\boldsymbol{r}} - \vec{\boldsymbol{v}}_p \\ &= \frac{c}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_p|}\left[\vec{\boldsymbol{r}} - \Big(\vec{\boldsymbol{r}}_p + \vec{\boldsymbol{v}}_p(t - t_r)\Big)\right] \; . \end{aligned} \tag{40}$$

In this form one can see that, for the case of constant velocity, $\vec{\boldsymbol{u}}$ points outward from the current position of the particle, which is $\vec{\boldsymbol{r}}_p + \vec{\boldsymbol{v}}_p(t - t_r)$.

To summarize, for a particle moving at constant velocity, we have

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\mathscr{r}}})^3} \left[(c^2 - v_p^2) \vec{\boldsymbol{u}} \right] ,} \tag{41}$$

where

$$\boxed{\vec{\boldsymbol{u}} = \frac{c}{\mathscr{r}} \left[\vec{\boldsymbol{r}} - \Big(\vec{\boldsymbol{r}}_p + \vec{\boldsymbol{v}}_p (t - t_r) \Big) \right] .} \tag{42}$$

Thus, the electric field of a particle moving at constant velocity looks like:

(Diagram taken from D.W. Griffiths, **Introduction to Electrodynamics,** *4th edition.)*

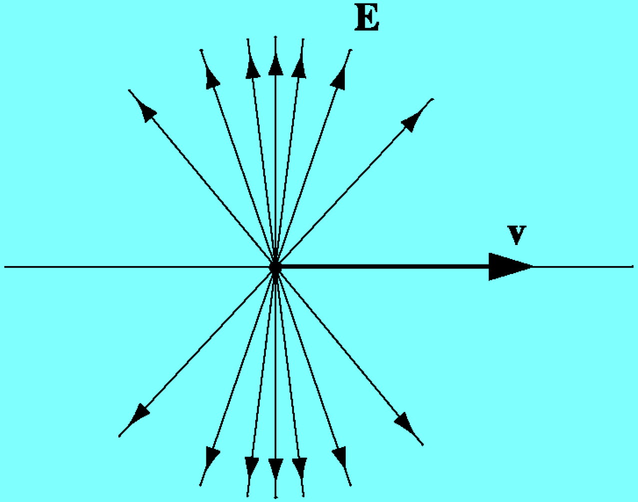


FIGURE 10.10

This slide was added after lecture.

For the case of constant velocity, the most informative way to express the electric field is to write it in terms of the velocity $\vec{v}$ and $\vec{R} \equiv \vec{r} - \left(\vec{r}_p + \vec{v}_p(t - t_r)\right)$, which is the vector from the *current* position of the particle to the point of observation $\vec{r}$.

Since $t_r = t - |\vec{r} - \vec{r}_p|/c$, the time difference $\Delta t \equiv t - t_r$ obeys the equation $\Delta t = |\vec{R} + \vec{v}\Delta t|/c$, which leads to the quadratic equation

$$c^2\,\Delta t^2 - (R^2 + 2\vec{v}\cdot\vec{R}\,\Delta t + v^2\,\Delta t^2) = 0\ ,$$

which can be solved for Δt.

With some algebra one finds that

$$\vec{u}\cdot\vec{\mathscr{r}} = Rc\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}\ ,$$

where θ is the angle beteen $\vec{R}$ and $\vec{v}$. Finally,

$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}\ .}$$

RADIATION!

Radiation: Electromagnetic fields that carry energy off to infinity.

At large distances, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fall off only as $1/r$, so the Poynting vector falls off as $1/r^2$. If the Poynting vector is then integrated over a large sphere, of area $4\pi r^2$, the contribution approaches a constant as $r \to \infty$.

Recall the Liénard-Wiechert potentials:

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}\left(1 - \frac{\vec{v}_p}{c} \cdot \hat{\mathscr{r}}\right)}} \ , \tag{43}$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}_p}{\mathscr{r}\left(1 - \frac{\vec{v}_p}{c} \cdot \hat{\mathscr{r}}\right)} = \frac{\vec{v}_p}{c^2} V(\vec{r},t)} \ , \tag{44}$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at t_r,

$$t_r = t - \frac{\mathscr{r}}{c} \ , \tag{45}$$

and

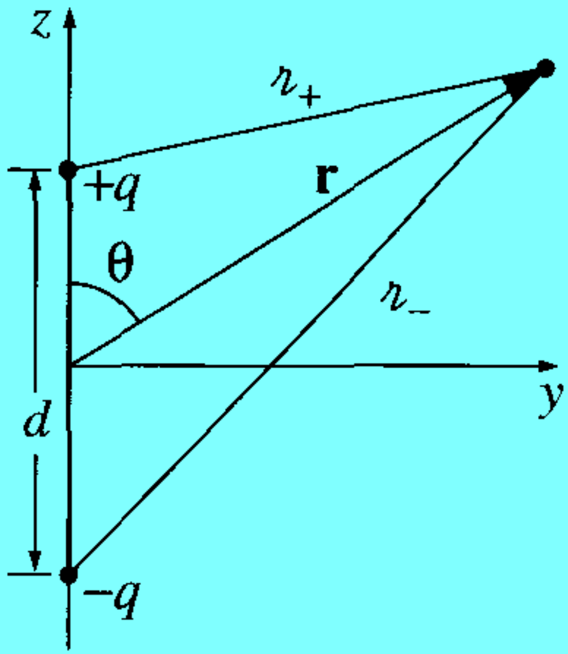
$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{r}_p}{|\vec{r} - \vec{r}_p|} \ , \qquad \mathscr{r} = |\vec{r} - \vec{r}_p| \ . \tag{46}$$

Electric Dipole Radiation

Simplest dipole: two tiny metal spheres separated by a distance d along the z-axis, connected by a wire, with charges

$$q(t) = q_0 \cos(\omega t) \tag{47}$$

on the top sphere, and $q(t) = -q_0 \cos(\omega t)$ on the bottom sphere.



Then

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 \cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0 \cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-} \right\} . \tag{48}$$

Approximation 1: $d \ll r$.

$$\imath_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$$

$$\Longrightarrow \quad \begin{cases} \frac{1}{\imath_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \\ \cos[\omega(t - \imath_\pm/c)] \simeq \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right] \end{cases} . \tag{49}$$

$d \ll r$ is ALWAYS valid for radiation, which is defined in the $r \to \infty$ limit.

Approximation 2: $d \ll \frac{c}{\omega}$.

Since $\lambda = 2\pi c/\omega$, this is equivalent to $d \ll \lambda$. This is the IDEAL DIPOLE APPROXIMATION. This is really the first term in a power expansion in d/λ, the multipole expansion for radiation, but we will go no further than the dipole. Implies

$$\cos[\omega(t - \imath_\pm/c)] \simeq \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta\sin[\omega(t - r/c)] \, . \tag{50}$$

Then, defining $p_0 = q_0\, d$,

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\left\{\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right\}. \tag{51}$$

Approximation 3: $r \gg \lambda$.

The region $r \gg \lambda$ is called the *radiation zone.* This approximation is ALWAYS valid for discussing radiation. Implies that the first term in curly brackets can be dropped:

$$\boxed{V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]\ .} \tag{52}$$

Summary of approximations: $d \ll \lambda \ll r$.

Need also $\vec{\boldsymbol{A}}$, which is due to the current in the wire. Recall

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,',t_r)}{\scriptr} \ . \tag{53}$$

Here

$$\mathrm{d}^3x' \, \vec{\boldsymbol{J}} = I\mathrm{d}\vec{\boldsymbol{\ell}} = \frac{\mathrm{d}q}{\mathrm{d}t}\,\mathrm{d}z\,\hat{\boldsymbol{z}} \ , \tag{54}$$

so

$$\begin{aligned}\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}},t) &= \frac{\mu_0}{4\pi}\hat{\boldsymbol{z}}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin[\omega(t-\scriptr/c)]}{\scriptr}\,\mathrm{d}z \\ &= \boxed{-\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{z}} \ .}\end{aligned} \tag{55}$$

Differentiating, and keeping terms that fall off as $1/r$, while dropping terms that fall off as $1/r^2$,

$$\boxed{\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\boldsymbol{A}}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)]\, \hat{\boldsymbol{\theta}}\ .} \tag{56}$$

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c} \hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t)\ .} \tag{57}$$

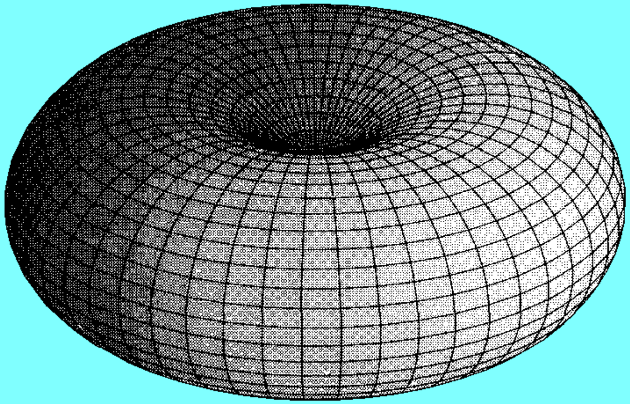
Poynting Vector:

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0}(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\boldsymbol{r}} \ . \tag{58}$$

Intensity:

Average the Poynting vector over a complete cycle: $\langle \cos^2 \rangle = 1/2$.

$$\left\langle \vec{\boldsymbol{S}} \right\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2}\, \hat{\boldsymbol{r}} \ . \tag{59}$$



All diagrams are from D.J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition

Total Power:

Integrate over a sphere at large r.

$$\langle P \rangle = \int \left\langle \vec{\boldsymbol{S}} \right\rangle \cdot \mathrm{d}\vec{\boldsymbol{a}} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \int \frac{\sin^2\theta}{r^2} \, r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= \boxed{\frac{\mu_0 p_0^2 \omega^4}{12\pi c}\ .} \tag{60}$$