MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 20, 2019
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# PROBLEM SET 4 SOLUTIONS

## PROBLEM 1: TRACELESS SYMMETRIC PART OF $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n$ *(15 points)*

The terms that can appear in the expansion of $\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\,\}_{\rm TS}$ can be classified by the number of Kronecker $\delta$-functions that they contain. With 5 indices, there can be terms with no Kronecker $\delta$-functions, terms with one Kronecker $\delta$-function, and terms with two. Once the number of $\delta$-functions is specified, the symmetry in the indices dictates the form of the terms up to an unknown coefficient for each classification. So we can begin by writing

$$\begin{aligned}\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\,\}_{\rm TS} = \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n &+ c_1(\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{mn} + 9\text{ other permutations})\\ &+ c_2(\hat{\boldsymbol{n}}_i\delta_{jk}\delta_{mn} + 14\text{ other permutations})\ .\end{aligned} \tag{1.1}$$

There are 10 terms with one $\delta$-function, since a term can be identified by specifying the two indicies on the $\delta$-function. The first of these indices is one choice out of 5 possibilities, while the second is a choice out of 4 possibilities. This would give $5\cdot 4 = 20$ possible combinations, but the order of the two indices does not matter, so there are 10 possibilities. They can be listed in full:

$$\begin{aligned}\text{Terms with 1 }\delta\text{-function} = \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{mn} &+ \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\delta_{in} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\delta_{jn} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m\delta_{kn}\\ &+ \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_n\delta_{im} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_n\delta_{jm} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_n\delta_{km}\\ &+ \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\delta_{ik} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\delta_{jk} + \hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\delta_{ij}\ .\end{aligned} \tag{1.2}$$

The terms with two $\delta$-functions can be identified by first specifying the index on $\hat{\boldsymbol{n}}$, which is one choice out of 5 possibilities. For each choice of the index on $\hat{\boldsymbol{n}}$, there are 3 ways of arranging the other indices. For example, suppose that the index on $\hat{\boldsymbol{n}}$ is $i$, so that the remaining indices are $j$, $k$, $m$ and $n$. The $\delta$-functions can then pair $j$ with either $k$, $m$, or $n$, and the term is completely specified once that choice is made. If we wish to write out all 15 terms, they are:

$$\begin{aligned}\text{Terms with 2 }\delta\text{-functions} = \hat{\boldsymbol{n}}_i\delta_{jk}\delta_{mn} &+ \hat{\boldsymbol{n}}_i\delta_{jm}\delta_{kn} + \hat{\boldsymbol{n}}_i\delta_{jn}\delta_{km}\\ &+ \hat{\boldsymbol{n}}_j\delta_{ik}\delta_{mn} + \hat{\boldsymbol{n}}_j\delta_{im}\delta_{kn} + \hat{\boldsymbol{n}}_j\delta_{in}\delta_{km}\\ &+ \hat{\boldsymbol{n}}_k\delta_{ij}\delta_{mn} + \hat{\boldsymbol{n}}_k\delta_{im}\delta_{jn} + \hat{\boldsymbol{n}}_k\delta_{in}\delta_{jm}\\ &+ \hat{\boldsymbol{n}}_m\delta_{ij}\delta_{kn} + \hat{\boldsymbol{n}}_m\delta_{ik}\delta_{jn} + \hat{\boldsymbol{n}}_m\delta_{in}\delta_{jk}\\ &+ \hat{\boldsymbol{n}}_n\delta_{ij}\delta_{km} + \hat{\boldsymbol{n}}_n\delta_{ik}\delta_{jm} + \hat{\boldsymbol{n}}_n\delta_{im}\delta_{jk}\ .\end{aligned} \tag{1.3}$$

Now we contract Eq. (1.1) on two indices. I will choose these indices as $m$ and $n$, so I multiply the left- and right-hand sides by $\delta_{mn}$, insisting that the trace vanish. The first term on the right-hand side gives

$$\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\delta_{mn} = \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k \ , \tag{1.4}$$

since $\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\delta_{mn} = \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}} = 1$. The second term (proportional to $c_1$) is more complicated. For the first term shown in Eq. (1.2), $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{km}\delta_{km} = \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{kk} = 3\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k$. For the 6 terms for which either $m$ or $n$ appears on the $\delta$-function, the result is $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k$ for each case. For the 3 terms for which both $m$ and $n$ appear on the $\hat{\boldsymbol{n}}$'s, the result is $\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij}$. So finally,

$$c_1(\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{mn} + 9 \text{ other permutations})\delta_{mn} = c_1\Big[9\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k + (\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})\Big] \ . \tag{1.5}$$

For the 15 terms proportional to $c_2$, 3 of them are proportional to $\delta_{mn}$. The result of multiplying these three terms by $\delta_{mn}$ is then $3(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})$. For 6 of the terms, $m$ appears on one $\delta$-function and $n$ appears on the other. The result for these 6 terms is then $2(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})$. Finally, for 6 of the terms either $m$ or $n$ appears as the index of $\hat{\boldsymbol{n}}$, and the other appears as an index on one of the $\delta$-functions. These 6 terms also give the result $2(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})$, so

$$c_2(\hat{\boldsymbol{n}}_i\delta_{jk}\delta_{mn} + 14 \text{ other permutations})\,\delta_{mn} = 7(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij}) \ . \tag{1.6}$$

Collecting the results of Eqs. (1.4), (1.5), and (1.6), we have

$$\begin{aligned} \{\, \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n \,\}_{\text{TS}}\,\delta_{mn} &= 0 \\ &= \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k + c_1\Big[9\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k + (\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})\Big] \\ &\quad + 7c_2(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij}) \ . \end{aligned} \tag{1.7}$$

To insure that the coefficient of $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k$ equals zero, we find

$$c_1 = -\frac{1}{9} \ . \tag{1.8}$$

To insure that the coefficient of $(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij})$ is zero, we find

$$c_2 = -\frac{1}{7}c_1 = \frac{1}{63} \ . \tag{1.9}$$

So finally,

$$\boxed{\begin{aligned} \{\, \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n \,\}_{\text{TS}} &= \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n - \frac{1}{9}(\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{mn} + 9 \text{ other permutations}) \\ &\quad + \frac{1}{63}(\hat{\boldsymbol{n}}_i\delta_{jk}\delta_{mn} + 14 \text{ other permutations}) \ . \end{aligned}} \tag{1.10}$$

As a check, we can rewrite Eq. (1.10) for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$,

$$\begin{aligned}\{\, \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m \hat{\boldsymbol{z}}_n \,\}_{\rm TS} = \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m \hat{\boldsymbol{z}}_n - \frac{1}{9}(\hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \delta_{mn} + 9 \text{ other permutations}) \\ + \frac{1}{63}(\hat{\boldsymbol{z}}_i \delta_{jk}\delta_{mn} + 14 \text{ other permutations}) \ ,\end{aligned} \tag{1.11}$$

as the coefficients do not depend on what unit vector is used in the construction. Then we can use Eq. (1.51) of Lecture Notes 1 to evaluate

$$\begin{aligned} P_5(\cos\theta) &= \frac{10!}{2^5 \cdot (5!)^2} \{\, \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m \hat{\boldsymbol{z}}_n \,\}_{\rm TS}\, \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \hat{\boldsymbol{n}}_k \hat{\boldsymbol{n}}_m \hat{\boldsymbol{n}}_n \\ &= \frac{63}{8}\left[\cos^5\theta - \frac{10}{9}\cos^3\theta + \frac{15}{63}\cos\theta\right] \\ &= \frac{1}{8}\left[63\cos^5\theta - 70\cos^3\theta + 15\cos\theta\right] \ , \end{aligned} \tag{1.12}$$

where the last line agrees with the standard expression for $P_5(x)$, as can be found, for example, in the Wikipedia. It can also be computed in Mathematica as LegendreP[5,x].

## PROBLEM 2: QUADRUPOLE AND OCTOPOLE TERMS OF THE MULTIPOLE EXPANSION *(20 points)*

(a) [5 pts] The preamble said that this problem can be attacked in any one of three methods, so we give three different answers.

**Method (i)** (Using Griffiths' Eq. (3.95)): The starting equation is

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ , \tag{Griffiths 3.95}$$

where I am using $\mathrm{d}^3x'$ where Griffiths uses $\mathrm{d}\tau'$. Using the expression for $P_2(x)$, the quadrupole term becomes

$$\begin{aligned} V_{\rm quad}(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2 P_2(\cos\theta')\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \frac{1}{2}\big(3r'^2\cos^2\theta' - r'^2\big)\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ . \end{aligned} \tag{2.1}$$

We are asked to show that we can rewrite the quadrupole term in terms of the quadrupole moment:

$$V_{\rm quad}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i,j=1}^{3}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j Q_{ij} \ , \tag{2.2}$$

where

$$Q_{ij} = \frac{1}{2}\int [3r'_i r'_j - r'^2\delta_{ij}]\rho(\vec{\boldsymbol{r}}')\mathrm{d}^3x' \ . \tag{2.3}$$

If we write $\cos\theta' = \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}' = \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}'_i$, and then $\cos^2\theta' = \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}'_i\hat{\boldsymbol{r}}'_j$, we see that

$$\left(3r'^2\cos^2\theta' - r'^2\right) = \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\left(3r'_i r'_j - r'^2\delta_{ij}\right) \ , \tag{2.4}$$

and then Eq. (2.1) becomes

$$\boxed{V_{\text{quad}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\left[\frac{1}{2}\int\left(3r'_i r'_j - r'^2\delta_{ij}\right)\rho(\vec{\boldsymbol{r}}')\,\mathrm{d}^3x'\right] \ ,} \tag{2.5}$$

which is the desired result.

**Method (ii)** (Using Griffiths' Eq. (2.29)): The starting equation in this case is

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\,\mathrm{d}^3x \ , \tag{Griffiths 2.29}$$

where I am using $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ where Griffiths uses $\boldsymbol{\mathscr{r}}$. We now want to expand $f(\vec{\boldsymbol{r}}') \equiv 1/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|$ in a power series in $r'_i$, the components of $\vec{\boldsymbol{r}}'$:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\int\rho(\vec{\boldsymbol{r}}')\left[f(\vec{\boldsymbol{0}}) + \left.\frac{\partial f}{\partial r'_i}\right|_{\vec{\boldsymbol{r}}'=\vec{\boldsymbol{0}}} r'_i + \frac{1}{2}\left.\frac{\partial^2 f}{\partial r'_i\partial r'_j}\right|_{\vec{\boldsymbol{r}}'=\vec{\boldsymbol{0}}} r'_i r'_j + \ldots \ .\right]\mathrm{d}^3x' \tag{2.6}$$

The quadrupole term is the term quadratic in $r'_i$:

$$V_{\text{quad}}(\vec{\boldsymbol{r}}) = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int\rho(\vec{\boldsymbol{r}}')\left.\frac{\partial^2 f}{\partial r'_i\partial r'_j}\right|_{\vec{\boldsymbol{r}}'=\vec{\boldsymbol{0}}} r'_i r'_j\,\mathrm{d}^3x' \ . \tag{2.7}$$

To evaluate the partial derivatives of $f$, define $w \equiv |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2 = (r'_k - r_k)(r'_k - r_k)$, so $f(\vec{\boldsymbol{r}}') = 1/w^{1/2}$, and

$$\frac{\partial w}{\partial r'_i} = 2\delta_{ik}(r'_k - r_k) = 2(r'_i - r_i) \ . \tag{2.8}$$

Then

$$\frac{\partial f}{\partial r'_i} = -\frac{1}{2}\frac{1}{w^{3/2}}\frac{\partial w}{\partial r'_i} = -\frac{1}{w^{3/2}}(r'_i - r_i) \ , \tag{2.9}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial r'_j\partial r'_i} &= -\frac{\frac{\partial}{\partial r'_j}(r'_i - r_i)}{w^{3/2}} + \frac{3}{2}\frac{1}{w^{5/2}}(r'_i - r_i)\frac{\partial w}{\partial r'_j} \\ &= \frac{1}{w^{5/2}}\left[3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}\right] \ .\end{aligned} \tag{2.10}$$

Then

$$\left.\frac{\partial^2 f}{\partial r'_j \partial r'_i}\right|_{\vec{r}\,'=0} r'_i r'_j = \frac{1}{r^5}\left[3r_i r_j - r^2\delta_{ij}\right] r'_i r'_j$$
$$= \frac{1}{r^3}\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j \left[3r'_i r'_j - r'^2\delta_{ij}\right] \ , \tag{2.11}$$

and finally, using Eq. (2.7), we find

$$\boxed{V_{\text{quad}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j \left[\frac{1}{2}\int \left[3r'_i r'_j - r'^2\delta_{ij}\right]\rho(\vec{r}\,')\,\mathrm{d}^3x'\right] \ .} \tag{2.12}$$

**Method (iii)** (Using traceless symmetric tensors): The starting point in this case is Eqs. (2.26) and (2.27) of Lecture Notes 2:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}C^{(\ell)}_{i_1\ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell} \ , \tag{Lecture Notes 2.26}$$

where

$$C^{(\ell)}_{i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!}\int \rho(\vec{r}\,')\,\{\,\vec{r}\,'_{i_1}\ldots\vec{r}\,'_{i_\ell}\,\}_{\text{TS}}\,\mathrm{d}^3x' \ . \tag{Lecture Notes 2.27}$$

Identifying $Q_{ij}$ with $C^{(2)}_{ij}$, we see that $V(\vec{r})$ already has the desired form, so all we have to do is write out the quadrupole term, $\ell = 2$, explicitly:

$$Q_{ij} = \frac{3}{2}\int \rho(\vec{r}\,')\{\,r'_i r'_j\,\}_{\text{TS}}\,\mathrm{d}^3x'$$
$$= \boxed{\frac{1}{2}\int \rho(\vec{r}\,')\left[3r'_i r'_j - r'^2\delta_{ij}\right]\,\mathrm{d}^3x' \ .} \tag{2.13}$$

(b) [5 pts] The square in Griffiths Fig. 3.30 has side $a$ and lies in the $xy$ plane, centered at the origin. From the definition of $Q_{ij}$ given in the problem, or equivalently in

Eq. (2.3),

$$Q_{xx} = Q_{yy} = \frac{1}{2}\left[3(a/2)^2 - (\sqrt{2}a/2)^2\right](q - q + q - q) = \boxed{\ 0\ }$$

$$Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = \boxed{\ 0\ } \quad \text{since } z = 0$$

$$Q_{zz} = -\frac{1}{2}(\sqrt{2}a/2)^2(q - q + q - q) = \boxed{\ 0\ } \tag{2.14}$$

$$\begin{aligned} Q_{xy} = Q_{yx} = \frac{3}{2}\bigg[&\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)q + \left(\frac{a}{2}\right)\left(\frac{-a}{2}\right)(-q) + \left(\frac{-a}{2}\right)\left(\frac{-a}{2}\right)q \\ &+ \left(\frac{-a}{2}\right)\left(\frac{a}{2}\right)(-q)\bigg] = \boxed{\ \frac{3}{2}a^2 q\ .\ } \end{aligned}$$

(c) [5 pts] For a given charge distribution the quadrupole moment is defined as

$$Q_{ij} = \frac{1}{2}\int [3r_i r_j - r^2\delta_{ij}]\rho(\vec{\boldsymbol{r}})\,\mathrm{d}^3x\ . \tag{2.15}$$

When we move the reference frame by $\vec{\boldsymbol{d}}$, the quadrupole moment in the new reference frame becomes:

$$\begin{aligned} Q'_{ij} &= \frac{1}{2}\int \left[3(r_i - d_i)(r_j - d_j) - |\vec{\boldsymbol{r}} - \vec{\boldsymbol{d}}|^2\delta_{ij}\right]\rho(\vec{\boldsymbol{r}})\,\mathrm{d}^3x \\ &= \frac{1}{2}\int [3r_i r_j - r^2\delta_{ij}]\rho\,\mathrm{d}^3x - \frac{3}{2}d_i\int r_j\rho\,\mathrm{d}^3x - \frac{3}{2}d_j\int r_i\rho\,\mathrm{d}^3x + \frac{3}{2}d_i d_j\int \rho\,\mathrm{d}^3x \\ &\qquad + \vec{\boldsymbol{d}}\cdot\int \vec{\boldsymbol{r}}\rho\delta_{ij}\,\mathrm{d}^3x - \frac{1}{2}d^2\delta_{ij}\int \rho\,\mathrm{d}^3x \\ &= Q_{ij} - \frac{3}{2}(d_i p_j + d_j p_i) + \frac{3}{2}d_i d_j Q + \delta_{ij}\vec{\boldsymbol{d}}\cdot\vec{\boldsymbol{p}} - \frac{1}{2}d^2\delta_{ij}Q\ , \end{aligned} \tag{2.16}$$

where $Q = \int \rho\,\mathrm{d}^3x$ and $p_i = \int r_i\rho\,\mathrm{d}^3x$. So if $\vec{\boldsymbol{p}} = 0$ and $Q = 0$ then $Q'_{ij} = Q_{ij}$. Alternatively, using the notation of traceless symmetric tensors, we find

$$\begin{aligned} Q'_{ij} &= \frac{3}{2}\int \rho(\vec{\boldsymbol{r}})\{\,(r_i - d_i)(r_j - d_j)\,\}_{\mathrm{TS}}\,\mathrm{d}^3x \\ &= \frac{3}{2}\int \rho(\vec{\boldsymbol{r}})\{\,r_i r_j - (r_i d_j + r_j d_i) + d_i d_j\,\}_{\mathrm{TS}}\,\mathrm{d}^3x' \\ &= \frac{3}{2}\left[Q_{ij} - \{\,p_i d_j + p_j d_i\,\}_{\mathrm{TS}} + \{\,d_i d_j\,\}_{\mathrm{TS}}\,Q\right]\ , \end{aligned} \tag{2.17}$$

so again we see that $Q'_{ij} = Q_{ij}$ if $p_i = Q = 0$.

(d) [5 pts] As in part (a), the preamble said that this problem can be attacked in any one of three methods:

**Method (i)** (Using Griffiths' Eq. (3.95)): Writing the $n = 3$ term of Griffiths' Eq. (3.95),

$$\begin{aligned} V_{\rm oct}(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int (r')^n P_3(\cos\alpha)\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int (r')^3\left[\frac{1}{2}\left(5\cos^3\alpha - 3\cos\alpha\right)\right]\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ . \end{aligned} \tag{2.18}$$

As in part (a), if we use $\cos\alpha = \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}'_i$, the above equation can be rewritten as

$$\begin{aligned} V_{\rm oct}(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_k\int\left\{\frac{1}{2}\left[5r'_ir'_jr'_k - r'^2(\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k)\right]\right\}\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_kQ_{ijk} \ , \end{aligned} \tag{2.19}$$

where

$$\boxed{Q_{ijk} = \frac{1}{2}\int\left[5r'_ir'_jr'_k - r'^2(\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k)\right]\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x' \ .} \tag{2.20}$$

**Method (ii)** (Using Griffiths' Eq. (2.29)): Continuing from part (a(ii)), the octopole term is the 3rd order term in the power expansion:

$$V_{\rm oct}(\vec{\boldsymbol{r}}) = \frac{1}{3!}\frac{1}{4\pi\epsilon_0}\int \rho(\vec{\boldsymbol{r}}\,')\left.\frac{\partial^3 f}{\partial r'_i\partial r'_j\partial r'_k}\right|_{\vec{\boldsymbol{r}}\,'=\vec{\mathbf{0}}} r'_ir'_jr'_k\,\mathrm{d}^3x' \ . \tag{2.21}$$

Starting from Eq. (2.10),

$$\begin{aligned} \frac{\partial^3 f}{\partial r'_i\partial r'_j\partial r'_k} &= \frac{\partial}{\partial r'_k}\left\{\frac{1}{w^{5/2}}\left[3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}\right]\right\} \\ &= -\frac{5}{2}\frac{1}{w^{7/2}}\left[3(r'_i - r_i)(r'_j - r_j) - w\delta_{ij}\right]2(r'_k - r_k) \\ &\quad + \frac{1}{w^{5/3}}\left[3\delta_{ik}(r'_j - r_j) + 3\delta_{jk}(r'_i - r_i) - 2\delta_{ij}(r'_k - r_k)\right] \\ &= -\frac{15}{w^{7/2}}(r'_i - r_i)(r'_j - r_j)(r'_k - r_k) \\ &\quad + \frac{3}{w^{5/3}}\left[\delta_{ik}(r'_j - r_j) + \delta_{jk}(r'_i - r_i) + \delta_{ij}(r'_k - r_k)\right] \ . \end{aligned} \tag{2.22}$$

Then

$$\left.\frac{\partial^3 f}{\partial r'_i \partial r'_j \partial r'_k}\right|_{\vec{\boldsymbol{r}}\,'=\vec{\boldsymbol{0}}} = \frac{3}{r^4}\left[5\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_k - (\delta_{jk}\hat{\boldsymbol{r}}_i + \delta_{ik}\hat{\boldsymbol{r}}_j + \delta_{ij}\hat{\boldsymbol{r}}_k)\right] , \tag{2.23}$$

and from Eq. (2.21) we have

$$V_{\text{oct}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_k Q_{ijk} \ , \tag{2.24}$$

where

$$\boxed{Q_{ijk} = \frac{1}{2}\int \rho(\vec{\boldsymbol{r}}\,')\left[5r'_i r'_j r'_k - r'^2(\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k)\right] \mathrm{d}^3x' \ .} \tag{2.25}$$

**Method (iii)** (Using traceless symmetric tensors): Identifying $Q_{ijk}$ with $C^{(3)}_{ijk}$, we see that $V(\vec{\boldsymbol{r}})$ already has the desired form, so all we have to do is write out the octopole term, $\ell = 3$, of Lecture Notes 2 Eq. (2.27), quoted above in the solution to part (a).

$$\begin{aligned} Q_{ijk} &= \frac{5}{2}\int \rho(\vec{\boldsymbol{r}}\,')\{\, r'_i r'_j r'_k \,\}_{\text{TS}}\, \mathrm{d}^3x' \\ &= \boxed{\frac{1}{2}\int \rho(\vec{\boldsymbol{r}}\,')\left[5r'_i r'_j r'_k - r'^2(\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k)\right] \mathrm{d}^3x' \ .} \end{aligned} \tag{2.26}$$

The expansion of $\{\, r'_i r'_j r'_k \,\}_{\text{TS}}$ in the above equation can be found directly from the definition, which says that it is equal to $r'_i r'_j r'_k$, combined with terms proportional to at least one Kronecker delta function to make it traceless. It is the same calculation that was done in Eq. (1.44) of Lecture Notes 1.

## PROBLEM 3: NORMALIZING THE TRACELESS SYMMETRIC TENSOR REPRESENTATION OF THE LEGENDRE POLYNOMIALS *(15 points)*

From Eqs. (3.2) and (3.5) of the problem set, we have

$$N(\ell) = \frac{1}{\{\, \hat{\boldsymbol{z}}_{i_1} \dots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}}\hat{\boldsymbol{z}}_{i_1} \dots \hat{\boldsymbol{z}}_{i_\ell}} \ ,$$

so our goal is to evaluate

$$\{\, \hat{\boldsymbol{z}}_{i_1} \dots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}}\hat{\boldsymbol{z}}_{i_1} \dots \hat{\boldsymbol{z}}_{i_\ell} \ .$$

But Eq. (3.6) of the problem set tells us that

$$\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_\ell} = \frac{\ell}{2\ell-1}\,\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\,\}_{\rm TS}\ ,$$

where I have replaced $\hat{\boldsymbol{n}}$ by $\hat{\boldsymbol{z}}$, since the equation was derived for any unit vector $\hat{\boldsymbol{n}}$. Now we can proceed by iterating this relation:

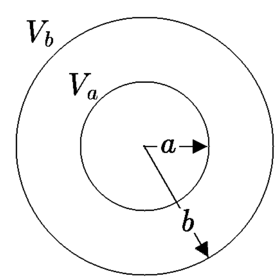
$$\begin{aligned}\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell} &= \frac{\ell}{2\ell-1}\,\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\,\}_{\rm TS}\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{\ell-1}\\ &= \left(\frac{\ell}{2\ell-1}\right)\left(\frac{\ell-1}{2\ell-3}\right)\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_{\ell-2}}\,\}_{\rm TS}\\ &= \left(\frac{\ell}{2\ell-1}\right)\left(\frac{\ell-1}{2\ell-3}\right)\cdots\left(\frac{1}{1}\right)\{\,1\,\}_{\rm TS}\\ &= \frac{\ell!}{(2\ell-1)!!}\ .\end{aligned}$$

So

$$\boxed{N(\ell) = \frac{(2\ell-1)!!}{\ell!} = \frac{(2\ell)!}{2^\ell(\ell!)^2}\ .}$$

**PROBLEM 4: CONCENTRIC SPHERICAL SHELLS OF CHARGE** *(25 points)*

(a) [5 pts] The situation is spherically symmetric, which means that only the $\ell = 0$ solution of the multipole expansion is allowed. Thus, $V(r)$ in any region has at most two terms, a constant and a term proportional to $1/r$. For the potential to vanish at infinity, the potential for $r > b$ can have only a $1/r$ term. Matching the boundary condition at $r = b$ then gives




$$\boxed{V(r) = \frac{bV_b}{r} \qquad (\text{for } r \ge b)\ .} \tag{4.1}$$

Similarly, the potential for $r \le a$ can only have the constant term, since the term proportional to $1/r$ would diverge. The constant must then match the potential at $r = a$, so

$$\boxed{V(r) = V_a \qquad (\text{for } r \le a)\ .} \tag{4.2}$$

For $a \le r \le b$, both terms are allowed, but the potential must match $V_a$ on the inside and $V_b$ on the outside. Writing

$$V(r) = A + \frac{B}{r}$$

for this region, the two boundary conditions give

$$A + \frac{B}{a} = V_a \ , \qquad A + \frac{B}{b} = V_b \ .$$

The problem is then solved by solving these two equations for $A$ and $B$. For example, we can subtract the second equation from the first to find

$$B\left(\frac{1}{a} - \frac{1}{b}\right) = V_a - V_b \ ,$$

which implies that

$$B = \frac{ab}{b-a}(V_a - V_b) \ .$$

Then from the first equation,

$$A = V_a - \frac{B}{a} = V_a - \frac{b}{b-a}(V_a - V_b) = \frac{bV_b - aV_a}{b-a} \ .$$

Finally, then,

$$\boxed{V(r) = \frac{1}{b-a}\left[bV_b - aV_a + \frac{ab}{r}(V_a - V_b)\right] \qquad (\text{for } a \le r \le b) \ .} \tag{4.3}$$

(b) [10 pts] In terms of the charge $Q_a$ on the inner shell, the potential in the region $a \le r \le b$ can be written as

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_a}{r} + \text{const} \ ,$$

which can be compared with Eq. (4.3) found in part (a) to see that

$$\boxed{Q_a = 4\pi\epsilon_0\left(\frac{ab}{b-a}\right)(V_a - V_b) \ .} \tag{4.4}$$

Similarly, the potential outside of both shells can be expressed in terms of the total charge on both shells as

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_a + Q_b}{r} \ ,$$

where in this case we do not need to include a constant term, since we know that we have chosen $V = 0$ at $r = \infty$. Comparing with Eq. (4.1), we see that

$$Q_a + Q_b = 4\pi\epsilon_0 b V_b \ .$$

Using Eq. (4.4), we find

$$\begin{aligned} Q_b &= 4\pi\epsilon_0 \left[ bV_b - \left(\frac{ab}{b-a}\right)(V_a - V_b)\right] \\ &= 4\pi\epsilon_0 \left[\frac{b(b-a)V_b - ab(V_a - V_b)}{b-a}\right] \\ &= \boxed{4\pi\epsilon_0 \left(\frac{b}{b-a}\right)(bV_b - aV_a) \ .} \end{aligned} \tag{4.5}$$

The total electrostatic energy $W$ is then given by

$$\boxed{W = \frac{1}{2}(Q_a V_a + Q_b V_b) \ .} \tag{4.6}$$

Substituting $Q_a$ and $Q_b$ from Eqs. (4.4) and (4.5), one finds

$$\boxed{W = \frac{2\pi\epsilon_0 b}{b-a}\left[aV_a^2 - bV_b^2 - 2aV_aV_b\right]} \tag{4.7a}$$

or, equivalently,

$$\boxed{W = \frac{2\pi\epsilon_0 b}{b-a}\left[a(V_a - V_b)^2 + (b-a)V_b^2\right] \ .} \tag{4.7b}$$

(c) [10 pts] The expression for $V(r{=}b, \theta, \phi)$ can be rewritten as

$$V(r{=}b, \theta, \phi) = V_0 \hat{\boldsymbol{n}}_x \hat{\boldsymbol{n}}_y \ ,$$

or $V_0 \hat{\boldsymbol{r}}_x \hat{\boldsymbol{r}}_y$, depending on notation. This can be rewritten as

$$V(r{=}b, \theta, \phi) = C_{ij} \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \ ,$$

where

$$C_{ij} = \frac{1}{2} V_0 (\hat{\boldsymbol{x}}_i \hat{\boldsymbol{y}}_j + \hat{\boldsymbol{y}}_i \hat{\boldsymbol{x}}_j) \ .$$

One sees that $\delta_{ij} C_{ij} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = 0$, so $C_{ij}$ is traceless and symmetric. Thus, the angular dependence is described entirely by the $\ell = 2$ term of the general expansion (on the formula sheets),

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \left( C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{\boldsymbol{r}}_{i_1} \hat{\boldsymbol{r}}_{i_2} \ldots \hat{\boldsymbol{r}}_{i_\ell} \ , \quad \text{where } \vec{\boldsymbol{r}} = r \hat{\boldsymbol{r}} \ .$$

For those who prefer standard spherical harmonics, one can see from the tables in the formula sheet that

$$V(r{=}b, \theta, \phi) = \frac{1}{2} V_0 \left( 4 \sqrt{\frac{2\pi}{15}} \right) \frac{Y_{2,2} - Y_{2,-2}}{2i} \ ,$$

so again it is pure $\ell = 2$. Thus, the allowed radial dependence, for solutions to Laplace's equation, is either $r^2$ or $1/r^3$.

For $r \leq a$, we see that the boundary condition of $V = 0$ at $r = a$ is trivially matched by

$$\boxed{V(r) = 0 \qquad (\text{for } r \leq a) \ .} \tag{4.8}$$

For $r \geq b$, a term proportional to $r^2$ would not be allowed, since it blows up at infinity, so $V$ must be proportional to $1/r^3$. It is then seen that the boundary condition at $r = b$ is easily matched by writing

$$\boxed{V(r) = \left( \frac{b}{r} \right)^3 V_0 \sin^2\theta \sin\phi \cos\phi \qquad (\text{for } r \geq b) \ .} \tag{4.9}$$

Finally, the region $a \leq r \leq b$ is the most complicated, since it can include both terms. We write

$$V(r) = V_0 \sin^2\theta \sin\phi \cos\phi \left[ A r^2 + \frac{B}{r^3} \right] \ , \tag{4.10}$$

where the boundary conditions at $r = a$ and $r = b$ require that

$$Aa^2 + \frac{B}{a^3} = 0 \ , \qquad Ab^2 + \frac{B}{b^3} = 1 \ . \tag{4.11}$$

Multipying the first equation above by $b^2$, and the second equation above by $a^2$, and then subtracting the second equation from the first, one finds

$$B\left(\frac{b^2}{a^3} - \frac{a^2}{b^3}\right) = -a^2 \ ,$$

from which one finds

$$B = -\frac{a^5 b^3}{b^5 - a^5} \ .$$

Then, from the first of Eqs. (4.11),

$$A = -\frac{B}{a^5} = \frac{b^3}{b^5 - a^5} \ .$$

Substituting these values into Eq. (4.10), one finds

$$\boxed{V(r) = \frac{V_0 \sin^2\theta \sin\phi\cos\phi}{b^5 - a^5}\left[b^3 r^2 - \frac{a^5 b^3}{r^3}\right] \ ,} \tag{4.12}$$

which could equivalently be written

$$\boxed{V(r) = V_0 \sin^2\theta \sin\phi\cos\phi\left(\frac{b^3}{b^5 - a^5}\right)\left(r^2 - \frac{a^5}{r^3}\right) \ .} \tag{4.13}$$