MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 23, 2019
Prof. Alan Guth

PROBLEM SET 9

DUE DATE: Friday, December 6, 2019, at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

READING ASSIGNMENT: Griffiths: Chapters 7 (*Electrodynamics*), 8 (*Conservation Laws*), 9 (*Electromagnetic Waves*), and 10 (*Potentials and Fields*). You will be responsible only for the topics discussed in lecture or used on this problem set.

CREDIT: This is a double problem set, incorporating two weeks of work. (It will be the last problem set to be handed in. There will be a set of practice problems for the final exam, which will not be turned in.) This problem set has 215 points of credit, plus the opportunity to earn 90 points of extra credit. Note that only 13 of the 19 problems are required. I have designated Problem 19 as "Read-only," which is the first time I have done this. The solution to this problem is given, so you are expected only to read the problem and solution and to understand them. The final exam will include two problems taken verbatim, or at least almost verbatim, from the problems sets or practice problems; Problem 19 could possibly show up on the final exam.

— PART A —

PROBLEM 1: BAR SLIDING ON TWO RAILS IN A UNIFORM MAGNETIC FIELD *(20 points)*

This problem was carried over from Problem Set 8.

Griffiths Problem 7.7 (p. 310).

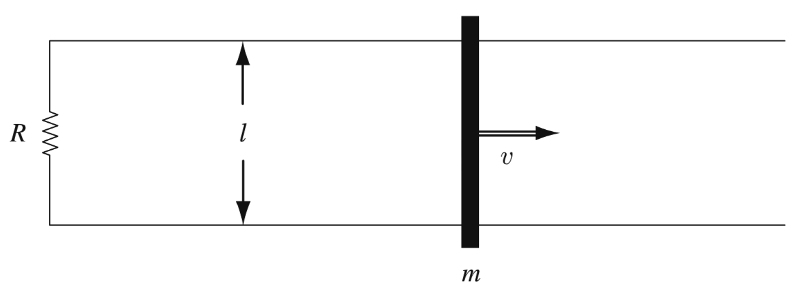

FIGURE 7.17

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance ℓ apart (Fig. 7.17). A resistor R is connected across the rails, and a uniform magnetic field B, pointing into the page, fills the entire region.

(a) [5 pts] If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) [5 pts] What is the magnetic force on the bar? In what direction?

(c) [5 pts] If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) [5 pts] The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor, after an arbitrarily long time, is exactly $\frac{1}{2}mv_0^2$.

PROBLEM 2: DONUT-SHAPED MAGNETS ON A VERTICAL ROD *(15 points extra credit)*

This problem was carried over from Problem Set 8.

Griffiths Problem 6.23 (p. 293). When Griffiths speaks of the ratio of the two heights, he is talking about the ratio x/y, where x and y are defined in the diagram.

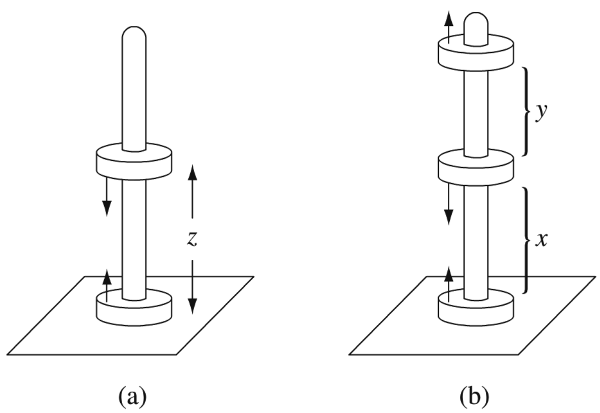

FIGURE 6.31

A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Fig. 6.31). Treat the magnets as dipoles, with mass m_d and dipole moment $\vec{\boldsymbol{m}}$.

(a) [8 pts] If you put two back-to-back magnets on the rod, the upper one will "float"—the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?

(b) [7 pts] If you now add a *third* magnet (parallel to the bottom one), what is the *ratio* of the two heights? (Determine the actual number, to three significant digits.)

[Answer: (a) $[3\mu_0 m^2/2\pi m_d g]^{1/4}$; (b) 0.8501]

PROBLEM 3: PERFECT CONDUCTORS AND SUPERCONDUCTORS *(15 points plus 10 points extra credit)*

Griffiths Problem 7.44 (p. 346).

In a **perfect conductor**, the conductivity is infinite, so $\vec{\boldsymbol{E}} = 0$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) [5 pts] Show that the magnetic field is constant ($\partial\vec{\boldsymbol{B}}/\partial t = 0$), inside a perfect conductor.

(b) [5 pts] Show that the magnetic flux through a perfectly conducting loop is constant.

A **superconductor** is a perfect conductor with the additional property that the (constant) $\vec{\boldsymbol{B}}$ inside is in fact *zero.* (This "flux exclusion" is known as the **Meissner effect**.*)

(c) [5 pts] Show that the current in a superconductor is confined to the surface.

(d) [10 pts extra credit] Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ while cooling it below T_c. Find the induced surface current density $\vec{\boldsymbol{K}}$, as a function of the polar angle θ.

PROBLEM 4: THE EFFECT OF A WEAK MAGNETIC FIELD ON THE RADIUS OF THE ORBIT OF AN ATOMIC ELECTRON *(10 points)*

Griffiths Problem 7.52 (p. 348). You may assume that the magnetic field $d\vec{\boldsymbol{B}}$ is turned on in a circular region, as in Example 7.7 (p. 317) of the textbook, with the nucleus at the center of this region.

Griffiths' text:

An atomic electron (charge q) circles about the nucleus (charge Q) in an orbit of radius r; the centripetal acceleration is provided, of course, by the Coulomb attraction of opposite charges. Now a small magnetic field $\mathrm{d}\vec{\boldsymbol{B}}$ is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy, dT , imparted by the induced electric field, is just right to sustain circular motion *at the same radius* r. (That's why, in my discussion of diamagnetism, I assumed the radius is fixed. See Sect. 6.1.3 and the references cited there.)

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* The Meissner effect is sometimes referred to as "perfect diamagnetism," in the sense that the field inside is not merely *reduced*, but canceled entirely. However, the surface currents responsible for this are *free*, not bound, so the actual *mechanism* is quite different.

PROBLEM 5: CAPACITANCE AND INDUCTANCE FOR A SIMPLE TRANSMISSION LINE *(20 points)*

Griffiths Problem 7.62 (p. 352).

A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case, it spreads out uniformly over the surface of the ribbon.

(a) [5 pts] Find the capacitance per unit length, $\mathcal{C}$.

(b) [5 pts] Find the inductance per unit length, $\mathcal{L}$.

(c) [5 pts] What is the product $\mathcal{LC}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{LC}}$.]

(d) [5 pts] If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeability μ, what then is the product $\mathcal{LC}$? What is the propagation speed? [*Hint*: see Ex. 4.6; by what factor does L change when an inductor is immersed in linear material of permeability μ?]

PROBLEM 6: ALFVEN'S THEOREM: FROZEN FLUX IN A PERFECTLY CONDUCTING FLUID *(15 points)*

Griffiths Problem 7.63 (p. 352).

Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) [6 pts] Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\vec{\boldsymbol{J}}$ is finite, then

$$\frac{\partial \vec{\boldsymbol{B}}}{\partial t} = \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \ .$$

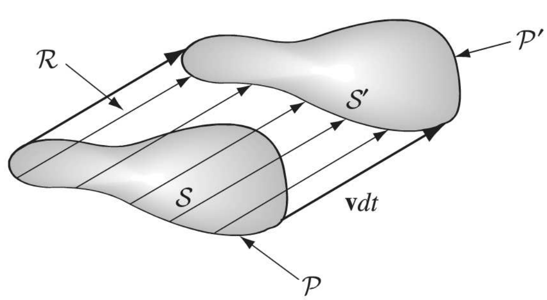


FIGURE 7.58

(b) [9 pts] Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + \mathrm{d}t$ (see Fig. 7.58). The change in flux is

$$\mathrm{d}\Phi = \int_{\mathcal{S}'} \vec{\boldsymbol{B}}(t + \mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{a}} \ .$$

Use $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0$ to show that

$$\int_{\mathcal{S}'} \vec{\boldsymbol{B}}(t + \mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} + \int_{\mathcal{R}} \vec{\boldsymbol{B}}(t + \mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}} = \int_{\mathcal{S}} \vec{\boldsymbol{B}}(t + \mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$\mathrm{d}\Phi = \mathrm{d}t \int_{\mathcal{S}} \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \mathrm{d}\vec{\boldsymbol{a}} - \int_{\mathcal{R}} \vec{\boldsymbol{B}}(t + \mathrm{d}t) \cdot \mathrm{d}\vec{\boldsymbol{a}}$$

(for infinitesimal $\mathrm{d}t$). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$\mathrm{d}t \oint_{\mathcal{P}} (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{v}}) \cdot \mathrm{d}\vec{\boldsymbol{\ell}} \ ,$$

and invoke Stokes' theorem to conclude that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int_{\mathcal{S}} \left(\frac{\partial \vec{\boldsymbol{B}}}{\partial t} - \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \right) \cdot \mathrm{d}\vec{\boldsymbol{a}} \ .$$

Together with the result in (a), this proves the theorem.

PROBLEM 7: A LONG, TIGHTLY WOUND SOLENOID, INDUCTANCE, AND THE ENERGY DENSITY OF A MAGNETIC FIELD *(20 points)*

(a) [5 pts] Begin by considering an infinitely long tightly wound solenoid, in the form of a cylinder with a circular cross section of radius R, centered on the z axis. Let n be the number of turns per length, and assume that a steady current I flows through the wire. By "tightly wound," we mean that the current in the wires can be treated as a uniform surface current in the tangential direction, ignoring the granularity of the wires and ignoring the component of the surface current in the z direction. Show that Maxwell's equations are satisfied by a uniform $\vec{\boldsymbol{B}}$ field inside the solenoid, in the z direction, with $\vec{\boldsymbol{B}} = 0$ outside the solenoid. Find $B \equiv B_z$, assuming that the current circulates counterclockwise as seen from above (i.e., from the positive z direction). [*Hint*: this was Example 5.9 in the textbook, starting on p. 235.]

(b) [5 pts] Now consider a segment of the above solenoid that extends from $z = 0$ to $z = \ell$, where $\ell \gg R$. Assume that end effects can be neglected, so the $\vec{\boldsymbol{B}}$ field is well-approximated by the solution in part (a) for $0 < z < \ell$, with $\vec{\boldsymbol{B}} = 0$ for $z < 0$ or $z > \ell$. Find the inductance L of this device.

(c) [5 pts] Using the fact that the energy stored in an inductor is given by

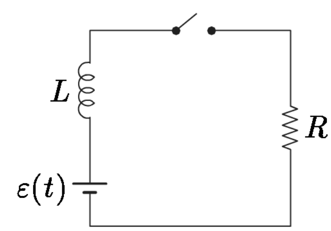
$$W = \frac{1}{2} L I^2 \ ,$$

which is Eq. (7.30) in the textbook, at p. 328, use the result of part (b) to find the energy density inside the solenoid. Expressing the energy density in terms of B, show that you obtain the standard result for the energy density,

$$u = \frac{|\vec{\boldsymbol{B}}|^2}{2\mu_0} \ .$$

(d) [5 pts] Consider the LR circuit shown at the right, with a time-dependent voltage source $\varepsilon(t)$. Suppose that the switch is closed at $t = 0$, and that $\varepsilon(t)$ has the form

$$\varepsilon(t) = \begin{cases} \varepsilon_0 & \text{if } t < 2L/R \\ 0 & \text{if } t > 2L/R \ . \end{cases}$$



Find the current $I(t)$ as a function of time.

PROBLEM 8: CALCULATING THE FORCE BETWEEN TWO POINT CHARGES USING THE MAXWELL STRESS TENSOR *(15 points)*

Griffiths Problem 8.4 (p. 366).

(a) [9 pts] Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) [6 pts] Do the same for charges that are opposite in sign.

— PART B (To be handed in separately from Part A) —

PROBLEM 9: MOMENTUM STORAGE IN A PARALLEL-PLATE CAPACITOR IN A UNIFORM MAGNETIC FIELD *(15 points)*

Griffiths Problem 8.6 (p. 369).

A charged parallel-plate capacitor (with uniform electric field $\vec{\boldsymbol{E}} = E\hat{\boldsymbol{z}}$) is placed in a uniform magnetic field $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{x}}$, as shown in Fig. 8.6.

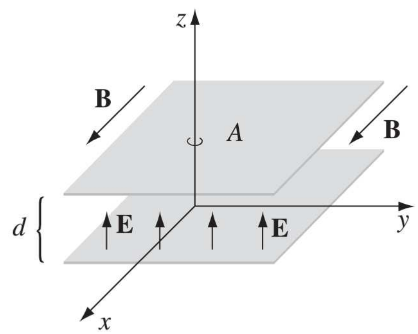


FIGURE 8.6

(a) [6 pts] Find the electromagnetic momentum in the space between the plates.

(b) [9 pts] Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?*

PROBLEM 10: ANGULAR MOMENTUM AND A ROTATING SHELL OF CHARGE *(15 points plus 10 points extra credit)*

A total charge Q is uniformly distributed over the surface of a sphere of radius R. The sphere rotates about the z axis with angular velocity ω.

(a) [5 pts] Write down, using your book, previous notes, or homework, the magnetic field inside and outside the sphere. Write down also the electrostatic field inside and outside the sphere.

(b) [5 pts] Now consider the case where $\dot{\omega} \neq 0$. Calculate the Faraday induced electric field at the surface of the sphere as a function of θ. Calculate also the torque this field produces on the sphere.

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* There is *much* more to be said about this problem, so don't get too excited if your answers to (a) and (b) appear to be consistent. See D. Babson, S. P. Reynolds, R. Bjorkquist and D. J. Griffiths, *Hidden momentum, field momentum, and electromagnetic impulse,* Am. J. Phys. **77**, 826 (2009).

(c) [5 pts] Suppose now the sphere has a mechanical moment of inertia I about the z axis. Show that if an external torque $\vec{\boldsymbol{\tau}}$ is applied, the sphere undergoes an angular acceleration as if it had an additional moment of inertia I_{mag} due to the magnetic contributions. Calculate I_{mag}.

Now assume again that ω is constant:

(d) [5 pts extra credit] Calculate the energy stored in the magnetic field. Show that two-thirds is inside the sphere and one-third is outside the sphere. Verify that the total magnetic energy coincides with $\frac{1}{2} I_{mag}\omega^2$.

(e) [5 pts extra credit] Calculate the angular momentum stored in the fields. Explain why it should point in the z-direction. Verify that the magnitude of the angular momentum coincides with $I_{mag}\omega$.

PROBLEM 11: THE CURRENT DENSITY FOR AN IDEAL MAGNETIC DIPOLE *(20 points extra credit)*

In lecture we found the current density of an ideal magnetic dipole is given by

$$\vec{\boldsymbol{J}}_{\rm dip} = -\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}}\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \ , \tag{11.1}$$

where $\vec{\boldsymbol{m}}$ is the magnetic dipole moment, and $\vec{\boldsymbol{r}}_d$ is the position of the dipole. Recall that the magnetic dipole moment of an arbitrary current density configuration is defined by

$$\vec{\boldsymbol{m}} = \frac{1}{2}\int d^3x\, \vec{\boldsymbol{r}} \times \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}) \ . \tag{11.2}$$

The derivation in lecture used a particular model of the ideal dipole, treating it as a square current loop that is contracted to zero size, as its current is scaled up so that the dipole moment is fixed. Since that derivation is not in the textbook, it is attached as an appendix to this problem set.

The derivation in lecture, however, leaves open the question of whether the answer we obtained might depend on the shape of the current distribution that we started with. In this problem we will show that we can obtain the same answer by starting with any localized divergenceless distribution of current $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$, contracting it as we scale up the current so that the dipole moment remains fixed. (We insist that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{J}} = 0$, since $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{J}} = -\partial\rho/\partial t$, and we are interested in time-independent situations. We insist that it be localized—that it approaches zero at large distances at least fast enough so that the integral in Eq. (11.2) is absolutely convergent—so that the dipole moment is well-defined.)

(a) [5 pts] To describe the contraction and scaling up, we introduce a rescaled current

$$\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda) \equiv \lambda^n \vec{\boldsymbol{J}}(\lambda\vec{\boldsymbol{r}}) \ . \tag{11.3}$$

So, for example, if $\lambda = 3$, the current density is 3 times smaller in linear size, and larger by a factor of 3^n, where n is an integer that you are asked to find. For what value of n will the dipole moment be independent of λ? *Hint:* evaluate the integral of Eq. (11.2) for $\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda)$, changing the variable of integration to

$$\vec{\boldsymbol{u}} \equiv \lambda \vec{\boldsymbol{r}} \ . \tag{11.4}$$

Since the rescaling contracts the current distribution around the origin, this procedure will describe a dipole at the origin.

(b) [5 pts] Eq. (11.1) is an equality of distributions, which means that it is equivalent to the statement that for any well behaved vector test function $\vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})$,

$$\begin{aligned} \int d^3x\, \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{J}}_{\rm dip}(\vec{\boldsymbol{r}}) &= -\int d^3x\, \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}}) \cdot \Big(\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}}\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \Big) \\ &= -\int d^3x\, \epsilon_{ijk}\, \varphi_i(\vec{\boldsymbol{r}})\, m_j \frac{\partial}{\partial x_k}\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \ . \end{aligned} \tag{11.5}$$

When things get complicated, it is often helpful to express relations involving delta functions in terms of test functions. (To review some of the basics of δ-functions, see Problem 8 of Problem Set 2.) The derivative of a distribution is defined by integrating by parts, with no surface term, so Eq. (11.5) can be rewritten as

$$\begin{aligned} \int d^3x\, \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{J}}_{\rm dip}(\vec{\boldsymbol{r}}) &= \int d^3x\, \epsilon_{ijk} m_j \frac{\partial \varphi_i}{\partial x_k}\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \\ &= \int d^3x\, \vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \\ &= \vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\big|_{\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_d} \ . \end{aligned} \tag{11.6}$$

Here we are defining $\vec{\boldsymbol{J}}_{\rm dip}(\vec{\boldsymbol{r}})$ for a dipole at $\vec{\boldsymbol{r}}_d = \vec{\boldsymbol{0}}$ as the limit of

$$\vec{\boldsymbol{J}}_{\rm dip}(\vec{\boldsymbol{r}}) = \lim_{\lambda \to \infty} \vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda) \ , \tag{11.7}$$

where $\vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda)$ is interpreted as a distribution. (The limit would not exist in the space of ordinary functions, since the limiting function would vanish at all $\vec{\boldsymbol{r}} \neq \vec{\boldsymbol{0}}$, and any ordinary function with this property would have vanishing integrals as well.) Thus, we wish to show that

$$\lim_{\lambda \to \infty} \int d^3x\, \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}}) \cdot \vec{\boldsymbol{J}}_r(\vec{\boldsymbol{r}}, \lambda) = \vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\varphi}}(\vec{\boldsymbol{r}})\big|_{\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}} \ . \tag{11.8}$$

As a first step, show that the left-hand side of Eq. (11.8) can be rewritten by using Eqs. (11.3) and (11.4) and a Taylor expansion of $\varphi(\vec{r})$ to give

$$\lim_{\lambda\to\infty} \int d^3x\, \vec{\varphi}(\vec{r}) \cdot \vec{J}_r(\vec{r},\lambda) = \lambda\, \vec{\varphi}(\vec{0}) \cdot \int d^3u\, \vec{J}(\vec{u}) + \left.\frac{\partial \varphi_i(\vec{r})}{\partial x_j}\right|_{\vec{r}=\vec{0}} \int d^3u\, u_j\, J_i(\vec{u}) \ . \tag{11.9}$$

(c) [5 pts] Use the fact that $\vec{\nabla} \cdot \vec{J} = 0$ to show that

$$\int d^3u\, \vec{J}(\vec{u}) = 0 \tag{11.10}$$

and that

$$\int d^3u\, u_j\, J_i(\vec{u}) = -\int d^3u\, u_i\, J_j(\vec{u}) \ . \tag{11.11}$$

Hint: if $\partial_i J_i = 0$, then $\int d^3u\, u_{i_1} \ldots u_{i_\ell} \partial_i J_i(\vec{u}) = 0$ for every integer ℓ. Try integrating these relations by parts.

(d) [5 pts] Eq. (11.10) allows us to drop the first term on the right-hand side of Eq. (11.9). Use the properties of the Levi-Civita symbol ϵ_{ijk} to show that

$$m_i \epsilon_{i\ell m} = \int d^3x\, x_\ell J_m \ , \tag{11.12}$$

and then use this result to complete the derivation of Eq. (11.8).

PROBLEM 12: POWER TRANSMISSION IN A COAXIAL CABLE *(20 points)*

This problem is based on Griffiths' Example 7.13 (pp. 330–331) and Problem 8.1 (p. 360). Some of the answers are in the textbook, but you are expected to give complete answers that indicate how the results are justified.

A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, of radius a, and back along the outer cylinder, of radius b. The potential of the inner cylinder is higher than that of the outer cylinder by V_0. We consider a length ℓ of this cable. You may write your answers in terms of cylindrical unit vectors $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{s}}$, and $\hat{\boldsymbol{\phi}}$, which point along the axis to the right, radially outward from the axis, and in the direction $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}$, respectively.

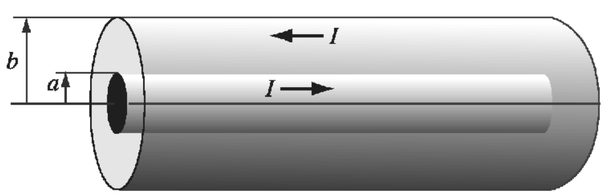


FIGURE 7.40

(a) [4 pts] What is the value of the magnetic field $\vec{B}$ as a function of s, the distance from the axis?

(b) [3 pts] What is the value of the electic field $\vec{\boldsymbol{E}}$ as a function of s, the distance from the axis?

(c) [4 pts] What is the total electromagnetic energy in the length ℓ of the cable?

(d) [4 pts] What is the capacitance and the self-inductance of the length ℓ of the cable?

(d) [5 pts] What is the Poynting vector in this configuration, and how much power does it indicate is being transported? Does the power agree with $P = IV_0$?

PROBLEM 13: MANIPULATING AMPLITUDES AND PHASES *(10 points)*

This is based on Griffiths' Example 9.1 and Problem 9.3 (pp. 387-388), but it is reworded slightly to make the problem self-contained.

Suppose you want to combine two sinusoidal waves:

$$f_3 = f_1 + f_2 = \Re(\tilde{f}_1) + \Re(\tilde{f}_2) = \Re(\tilde{f}_1 + \tilde{f}_2) = \Re(\tilde{f}_3) \ ,$$

with $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$. You simply add the corresponding *complex* wave functions, and then take the real part. In particular, if they have the same frequency and wave number,

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)} \ ,$$

where

$$\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2 \ , \text{ or } A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2} \ . \tag{13.1}$$

In other words, you just add the (complex) amplitudes. The combined wave still has the same frequency and wavelength,

$$f_3(z,t) = A_3 \cos(kz - \omega t + \delta_3) \ .$$

Determine A_3 and δ_3 in terms of A_1, A_2, δ_1 and δ_2. Note: Griffiths points out that if you try doing this *without* using the complex notation, you will find yourself looking up trig identities and slogging through nasty algebra.

PROBLEM 14: THE MAXWELL STRESS TENSOR FOR A MONOCHROMATIC LINEARLY POLARIZED PLANE WAVE *(15 points)*

Griffiths Problem 9.13 (p. 400). Do all your calculations with time-averaged quantities. Griffiths reminds the reader that $-\overleftrightarrow{\boldsymbol{T}}$ represents the momentum flux density. In index notation, this means that $-T_{ij}$ represents the flux of the i'th component of momentum in the j-direction.

Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction:

$$\vec{\boldsymbol{E}}(z,t) = E_0 \cos(kz - \omega t + \delta)\, \hat{\boldsymbol{x}} \ , \qquad \vec{\boldsymbol{B}}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\, \hat{\boldsymbol{y}} \ .$$

Does your answer make sense? (Remember that $-\overleftrightarrow{\boldsymbol{T}}$ represents the momentum flux density.) How is the momentum flux density related to the energy density, in this case?

PROBLEM 15: REFLECTION AND TRANSMISSION OF A PLANE WAVE AT NORMAL INCIDENCE *(15 points)*

Griffiths Problem 9.14 (p. 405).

Calculate the *exact* reflection and transmission coefficients [for the situation described in Sec. 9.3.2, p. 403], *without* assuming $\mu_1 = \mu_2 = \mu_0$. Confirm that $R + T = 1$.

PROBLEM 16: TRANSMISSION OF LIGHT THROUGH THREE MEDIA *(25 points)*

Griffiths Problem 9.36 (p. 433).

Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right] .$$

[*Hint*: To the *left*, there is an incident wave and a reflected wave; to the *right*, there is a transmitted wave; inside the slab there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.]

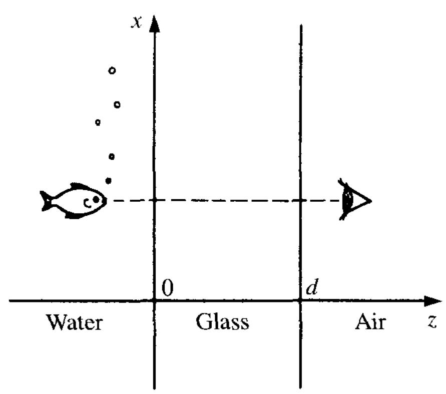


Figure 9.27

PROBLEM 17: ENERGY CONSERVATION IN THE PRESENCE OF MAGNETIC MONOPOLES *(15 points extra credit)*

In Eq. (7.44) (p. 339), Griffiths gives the form of Maxwell's equations as extended to include the possibility of magnetic monopoles:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\mu_0\vec{J}_m - \frac{\partial\vec{B}}{\partial t} \ ,$$
$$\vec{\nabla}\cdot\vec{B} = \mu_0\rho_m \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_e + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \ .$$

In Eq. (7.69) (p. 354), he gives the Lorentz force law for monopoles:

$$\vec{F} = q_m\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right) \ .$$

Note that the factor of $1/c^2$ in the second term (or something with the same units) is needed for the term to have the correct dimensionality.

Generalize Poynting's theorem to this case. You will find that there is no need to modify the formulas for the Poynting vector or for the energy density of an electromagnetic field. You should be able to show that with the extra terms in Maxwell's equations, and with the inclusion of a term for the work done on magnetic monopoles, energy is still conserved. Would energy conservation still work if the force on a magnetic monopole were given by

$$\vec{F} = q_m\left(\vec{B} + \frac{1}{c^2}\vec{v}\times\vec{E}\right) \qquad \text{(wrong!) ?}$$

PROBLEM 18: MORE FUN WITH δ-FUNCTIONS *(20 points extra credit)*

In this problem you will carry out a variety of mathematical exercises intended to illustrate some of the do's and don'ts about δ-functions. I recommend that you consult the "Delta Functions" section of the Quiz 2 Formula Sheet to remind yourself of the key definitions.

(a) [3 pts] The Heaviside step function is defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \ , \end{cases} \tag{18.1}$$

where $\theta(0)$ is often defined to be $\frac{1}{2}$. If we think of $\theta(x)$ as a distribution, an object that is defined only by the result of multiplying by a test function $\varphi(x)$ and then integrating, the value of $\theta(0)$ is irrelevant. Treating $\theta(x)$ as a distribution, prove the identity

$$\boxed{\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = \delta(x) \ .} \tag{18.2}$$

In more detail, you are to assume a test function $\varphi(x)$ that is arbitrarily smooth, and falls off rapidly as $|x| \to \infty$. (The mathematicians often restrict $\varphi(x)$ to belong to the space of Schwartz functions, which are infinitely differentiable, and which have the property that the function and all its derivatives fall off faster than any power at large $|x|$. Here we do not expect any rigorous mathematical analysis, but we will just assume that $\varphi(x)$ is smooth enough, and falls off fast enough at infinity, to justify any argument we need.) Furthermore, recall that if $\Psi(x)$ is a distribution defined in terms of the integral

$$\int_{-\infty}^{\infty} \varphi(x)\Psi(x)\,\mathrm{d}x \ ,$$

then its derivative is defined by

$$\int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}\Psi(x)}{\mathrm{d}x}\,\mathrm{d}x \equiv -\int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\Psi(x)\,\mathrm{d}x \ . \tag{18.3}$$

With these definitions, show that

$$\int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\delta(x)\,\mathrm{d}x \ . \tag{18.4}$$

(*Hint:* Once you figure out what you are really being asked to prove, it takes only one or two lines.)

(b) [3 pts] Show that

$$\boxed{\theta^n(x) = \theta(x) \ ,} \tag{18.5}$$

in the sense of distributions, for any positive integer n. (That is, show that if each side of the above equation is multiplied by an arbitrary smooth test function $\varphi(x)$, and then integrated from $-\infty$ to ∞, the two results are always equal.)

(c) [4 pts] The derivative of a distribution is *always* well-defined, by Eq. (18.3). Thus we might expect that we could differentiate Eq. (18.5), obtaining

$$n\theta^{n-1}(x)\delta(x) \stackrel{?}{=} \delta(x) \ . \tag{18.6}$$

Show that when combined with Eq. (18.5), Eq. (18.6) can lead to a contradiction. The bottom line is that while the derivative of a distribution is always well-defined, the product of two distributions, such $\theta(x)\delta(x)$ or $\delta^2(x)$, is generally not well-defined. ($\delta^3(\vec{\boldsymbol{r}}) \equiv \delta(x)\delta(y)\delta(z)$ is however okay, because the arguments of the delta functions are independent coordinates — $\delta^3(\vec{\boldsymbol{r}})$ can be thought of as a single δ-function defined as a distribution for functions on three-dimensional space.) What is

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta^n(x)\,?$$

(d) [3 pts] Show that for any smooth function $f(x)$,

$$\boxed{f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)\ ,} \tag{18.7}$$

in the sense of distributions.

(e) [4 pts] If we differentiate the above identity with respect to x, we find

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}\delta(x-x_0) + f(x)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x} = f(x_0)\frac{\mathrm{d}\delta(x-x_0)}{\mathrm{d}x}\ . \tag{18.8}$$

The two sides do not look like they are equal, but show that they are.

(f) [3 pts] Evaluate

$$G[\varphi(x)] \equiv \int_{-\infty}^{\infty} \varphi(x)\delta(x^2-a^2)\,\mathrm{d}x\ , \tag{18.9}$$

where a is a positive constant.

PROBLEM 19: SPHERICAL WAVES AND THE POYNTING VECTOR

This problem is READ-ONLY. The solution follows, and you are only expected to read and understand the problem and its solution. The final exam will include two problems that are at least almost verbatim from the problem sets and practice problems, and this problem is eligible to be one of them.

Griffiths Problem 9.35 (p. 432). In part (a), show that $\vec{\boldsymbol{E}}$ and the associated $\vec{\boldsymbol{B}}$ (which you must find) obey all four Maxwell equations.

Suppose

$$\vec{\boldsymbol{E}}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}[\cos(kr-\omega t] - (1/kr)\sin(kr-\omega t)]\hat{\boldsymbol{\phi}}\ \text{, with } \frac{\omega}{k} = c\ .$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr-\omega t) \equiv u$ in your calculations.)

(a) [16 pts] Show that $\vec{\boldsymbol{E}}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) [8 pts] Calculate the Poynting vector. Average $\vec{\boldsymbol{S}}$ over a full cycle to get the intensity vector $\vec{\boldsymbol{I}}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) [6 pts] Integrate $\vec{\boldsymbol{I}}\cdot\mathrm{d}\vec{\boldsymbol{a}}$ over a spherical surface to determine the total power radiated. [*Answer:* $4\pi A^2/3\mu_0 c$]

SOLUTION:

For this problem it will be useful to keep in mind the formulas for the divergence and curl in spherical coordinates. For an arbitrary vector field $\vec{\boldsymbol{A}}(r,\theta,\phi)$,

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{19.1a}$$

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\ .\end{aligned} \tag{191.b}$$

(a) [16 pts] First consider the equation $\nabla\cdot E = 0$. In spherical coordinates, this equation becomes (since $E_r, E_\theta = 0$):

$$\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0\ . \tag{19.2}$$

We have

$$E_\phi = \frac{A\sin\theta}{r}\left(\cos\left(kr-\omega t\right) - \frac{1}{kr}\sin\left(kr-\omega t\right)\right)\ . \tag{19.3}$$

Since E_ϕ is independent of ϕ, we have

$$\boxed{\nabla\cdot E = 0\ .} \tag{19.4}$$

Next consider $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}$, which in spherical coordinates becomes:

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right)\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\boldsymbol{\theta}} \\ &= \frac{2}{r}\cot\theta E_\phi\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\boldsymbol{\theta}} \\ &= \frac{2A\cos\theta}{r^2}\left(\cos\left(kr-\omega t\right) - \frac{1}{kr}\sin\left(kr-\omega t\right)\right)\hat{\boldsymbol{r}} \\ &\quad + \frac{Ak\sin\theta}{r}\left(\left(1-\frac{1}{k^2r^2}\right)\sin\left(kr-\omega t\right) + \frac{1}{kr}\cos\left(kr-\omega t\right)\right)\hat{\boldsymbol{\theta}}\ .\end{aligned} \tag{19.5}$$

From $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t}$ and Eq. (19.5), we obtain:

$$\begin{aligned}\vec{\boldsymbol{B}}(r,\theta,\phi,t) &= \vec{\boldsymbol{B}}_0(r,\theta,\phi) + \frac{2A\cos\theta}{ckr^2}\left(\frac{1}{kr}\cos\left(kr-\omega t\right) + \sin\left(kr-\omega t\right)\right)\hat{\boldsymbol{r}} \\ &\quad + \frac{A\sin\theta}{cr}\left[\left(\frac{1}{k^2r^2}-1\right)\cos\left(kr-\omega t\right) + \frac{1}{kr}\sin\left(kr-\omega t\right)\right]\hat{\boldsymbol{\theta}}\ .\end{aligned} \tag{19.6}$$

Here $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ is an integration "constant" which needs to be determined later. (It is a constant in the sense that it is independent of t, but it can vary with position.) Students who ignored $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ should get full credit, because in the end we will ignore it, but we will nonetheless discuss it for the sake of thoroughness.

Next we consider $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$. We could directly calculate $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$, but we don't need to. From $\frac{\partial}{\partial t}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}\right) = -\vec{\boldsymbol{\nabla}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}\right) = 0$, we see that $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}$ is time independent. If we imagine taking the divergence of both sides of Eq. (19.6), it is clear that it will have the form

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0(r,\theta,\phi) + f_1(r,\theta)\cos(kr-\omega t) + f_2(r,\theta)\sin(kr-\omega t)\ , \tag{19.7}$$

where $f_1(r,\theta)$ and $f_2(r,\theta)$ must vanish by the previous argument, so we don't need to calculate them. Thus we have $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0$, and therefore

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0(r,\theta,\phi) = 0\ . \tag{19.8}$$

Now, let us consider $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}$:

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} &= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\phi}\right)\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial B_r}{\partial\theta}\hat{\boldsymbol{\phi}} - \frac{1}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi}\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_\theta)\right)\hat{\boldsymbol{\phi}} \\
&= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{Ak\sin\theta}{cr}\left(\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right)\hat{\boldsymbol{\phi}} \\
&= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{c^2}\frac{\partial E_\phi}{\partial t}\hat{\boldsymbol{\phi}} \\
&= \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0 + \frac{1}{c^2}\frac{\partial\vec{\boldsymbol{E}}}{\partial t}
\end{aligned} \tag{19.9}$$

Therefore we must have

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}_0(r,\theta,\phi) = 0\ . \tag{19.10}$$

Eqs. (19.8) and (19.10) imply that $\vec{\boldsymbol{B}}_0(r,\theta,\phi)$ must be a solution to the source-free equations of magnetostatics, which have an identical form to the source-free equations of electrostatics. Eq. (5.9) guarantees that we can write

$$\vec{\boldsymbol{B}}_0(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}V_B(\vec{\boldsymbol{r}})\ , \tag{19.11}$$

where V_B is a magnetic scalar potential which is applicable only when $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = 0$. Since $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}}_0 = 0$, it follows that $\nabla^2 V_B = 0$. There is actually a large class of solutions to these equations, corresponding to the multipole expansion with positive powers of r:

$$V_B(\vec{\boldsymbol{r}}) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} C^{(\ell)}_{i_1\ldots i_\ell} x_{i_1} \ldots x_{i_\ell} \\ \text{or} \\ \displaystyle\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} r^\ell Y_{\ell,m}(\theta,\phi)\ , \end{cases} \tag{19.12}$$

where the $C^{(\ell)}_{i_1\ldots i_\ell}$ are arbitrary traceless symmetric tensors, and the $a_{\ell,m}$ are arbitrary coefficients. Either of the two expressions on the right-hand side describe a complete set of solutions.

Maxwell's equations allow the existence of a static $\vec{\boldsymbol{B}}_0(\vec{\boldsymbol{r}})$ in this situation, but we are not interested in it. We can eliminate this possibility by imposing the (realistic) boundary condition that $|\vec{\boldsymbol{B}}| \to 0$ at infinity, which implies that $\vec{\boldsymbol{B}}_0 = 0$. (To see that this is the only solution in this case, remember that $\nabla^2 V_B = 0$ implies that V_B cannot have any local minima or maxima. So if it is required to approach a constant at infinity, it must be a constant everywhere.) For the rest of the problem, we will take $\vec{\boldsymbol{B}}_0 = 0$.

Therefore $\vec{\boldsymbol{B}}$ is given by:

$$\boxed{\begin{aligned} \vec{\boldsymbol{B}}(r,\theta,\phi,t) = {}& \frac{2A\cos\theta}{ckr^2}\left(\frac{1}{kr}\cos(kr-\omega t) + \sin(kr-\omega t)\right)\hat{\boldsymbol{r}} \\ &+ \frac{A\sin\theta}{cr}\left[\left(\frac{1}{k^2r^2}-1\right)\cos(kr-\omega t) + \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\boldsymbol{\theta}}\ . \end{aligned}} \tag{19.13}$$

(b) [8 pts]

$$\boxed{\begin{aligned} \vec{\boldsymbol{S}} &= \frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} \\ &= \frac{A^2\sin\theta}{\mu_0 ckr^2}\Bigg\{\sin\theta\left[k\cos^2(kr-\omega t) + \left(\frac{1}{2k^2r^3}-\frac{1}{r}\right)\sin\big(2(kr-\omega t)\big)\right. \\ &\qquad\qquad \left. - \frac{1}{kr^2}\cos\big(2(kr-\omega t)\big)\right]\hat{\boldsymbol{r}} \\ &\qquad + \frac{2\cos\theta}{r}\left[\frac{1}{2}\left(1-\frac{1}{k^2r^2}\right)\sin\big(2(kr-\omega t)\big) + \frac{1}{kr}\cos\big(2(kr-\omega t)\big)\right]\hat{\boldsymbol{\theta}}\Bigg\} \end{aligned}} \tag{19.14}$$

Here we have used the trigonometric identities $\sin(2x) = 2\sin x \cos x$ and $\cos(2x) = \cos^2 x - \sin^2 x$. The average of $\sin(2x)$ and $\cos(2x)$ is 0 over a period, while the averages of $\sin^2(x)$ and $\cos^2(x)$ are $\frac{1}{2}$. Therefore, for intensity we obtain:

$$\boxed{\vec{\boldsymbol{I}} = \frac{A^2 \sin^2\theta}{2\mu_0 c r^2}\hat{\boldsymbol{r}}\ .} \tag{19.15}$$

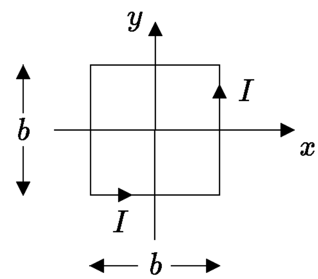
From Eq. (19.15) we see that intensity vector points our $\hat{\boldsymbol{r}}$, as expected. It also falls of as $\frac{1}{r^2}$ as it should.

(c) [6 pts]

$$\begin{aligned}
P &= \int \vec{\boldsymbol{I}} \cdot d\vec{\boldsymbol{a}} \\
&= \int\limits_{\theta=0}^{\pi} \int\limits_{\phi=0}^{2\pi} \sin\theta d\theta d\phi \, I(\theta, r) r^2 \\
&= \frac{A^2\pi}{\mu_0 c} \int_{\theta=0}^{\pi} \sin^3\theta d\theta \\
&= -\frac{A^2\pi}{\mu_0 c} \int_{\theta=0}^{\pi} \sin^2\theta d\left(\cos\theta\right) \\
&= -\frac{A^2\pi}{\mu_0 c} \int_{\theta=0}^{\pi} \left(1 - \cos^2\theta\right) d\left(\cos\theta\right) \qquad (\text{let } u = \cos\theta) \\
&= \frac{A^2\pi}{\mu_0 c} \int_{u=0}^{1} (1 - u^2) du \\
&= \boxed{\frac{4}{3}\frac{A^2\pi}{\mu_0 c}.}
\end{aligned} \tag{19.16}$$

— PEDAGOGICAL APPENDIX —

The Current Density of an Ideal Magnetic Dipole



In lecture we considered a square loop of current, in the x-y plane and centered at the origin, with side b and current I, directed counterclockwise as seen when looking down the z-axis. The magnetic dipole moment is given by

$$\vec{\boldsymbol{m}} = I\vec{\boldsymbol{a}} = Ib^2\,\hat{\boldsymbol{z}}.$$

We then considered the limit of an ideal dipole, taking $b \to 0$ while I is increased so that $|\vec{\boldsymbol{m}}| = Ib^2$ is held fixed. Our goal was to write an expression for the current density of this ideal dipole.

Here I will review this calculation, since it does not appear in the textbook. We start by considering a finite value of b, where we can write $J_x(\vec{\boldsymbol{r}})$ as follows:

$$J_x(\vec{\boldsymbol{r}}) = \begin{cases} I\,\delta(z)\left[\delta\left(y+\frac{b}{2}\right) - \delta\left(y-\frac{b}{2}\right)\right] & \text{if } |x| < \frac{b}{2} \\ 0 & \text{otherwise .}\end{cases}$$

If we set $I = m/b^2$, and then take the limit $b \to 0$, we find

$$J_x(\vec{\boldsymbol{r}}) \to \begin{cases} \dfrac{m}{b}\delta(z)\,\dfrac{d}{dy}\delta(y) & \text{if } |x| < \frac{b}{2} \\ 0 & \text{otherwise ,}\end{cases}$$

which can also be written as

$$J_x(\vec{\boldsymbol{r}}) \to m\delta(z)\frac{d\delta(y)}{dy}\,g(x)\ ,$$

where

$$g(x) = \begin{cases} \frac{1}{b} & \text{if } |x| < \frac{b}{2} \\ 0 & \text{otherwise .}\end{cases}$$

It can be seen that in the limit $b \to 0$, $g(x)$ vanishes for any $x \neq 0$, and that its integral over x is 1. These are the properties that define a Dirac delta function, so

$$\lim_{b\to 0} g(x) = \delta(x)\ ,$$

and

$$\begin{aligned} J_x(\vec{\boldsymbol{r}}) \to m\delta(x)\delta(z)\frac{d\delta(y)}{dy} &= m\frac{\partial}{\partial y}[\delta(x)\delta(y)\delta(z)] \\ &= m\frac{\partial}{\partial y}\delta^3(\vec{\boldsymbol{r}})\ . \end{aligned}$$

Similarly,

$$J_y(\vec{\boldsymbol{r}}) \to -m \frac{\partial}{\partial x} \delta^3(\vec{\boldsymbol{r}}) \ .$$

Putting these together into a vector equation, we can write the current density of an ideal dipole (i.e., the limit $b \to 0$) as

$$\vec{\boldsymbol{J}}_{\rm dip} = -\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}} \delta^3(\vec{\boldsymbol{r}}) \ .$$

The dipole described above is located at the origin, but the formula can easily be generalized for the current density of an ideal dipole at an arbitrary location $\vec{\boldsymbol{r}}_d$:

$$\boxed{\vec{\boldsymbol{J}}_{\rm dip} = -\vec{\boldsymbol{m}} \times \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}} \, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d) \ .}$$