8.07 Lecture Slides 9
October 2, 2019

ELECTRIC POTENTIAL: SEPARATION OF VARIABLES (CARTESIAN AND SPHERICAL)

Announcements

Quiz 1 is this coming Monday, October 7.

- All material referenced in Problem Sets 1, 2, and 3 will be a fair subject for problems on the quiz.
- Calculators will not be needed and will not be allowed.
- I have posted 10 Practice Problems for Quiz 1, with solutions, on the Exams tab of the class website.
- I have also posted a Formula Sheet for Quiz 1 on the Exams tab. It is a very complete formula sheet, intended to be a tool for review. A copy of the formula sheet will be included with the quiz, so of course you need not memorize any of the formulas. But I recommend that you go over the formula sheet carefully. If you understand the meaning of each of the formulas, you are very well-prepared for the quiz.

- One of the problems on the quiz will be taken verbatim (or at least almost verbatim) from either the problem sets (extra credit problems are possible), or from the practice problems.
- Yitian Sun will run a Review Session for the quiz on Sunday (Oct 6) from 3:00 to 5:00 pm, in Room 2-131.
- Recitations this Thursday, with Marin Soljačić, will focus on reviewing for the Quiz.
- Upcoming office hours:

  Today, 5:00 - 6:00 pm: Me, Room 6-322. Please come!

  Thurs, Yitian, 5:30 - 6:30 pm, Room 8-320

  Fri, Yitian, 1:30 - 3:30 pm, Room 8-320

Good luck on the quiz.

Announcements:
Special Lecture by Marin Soljacic

One week from today, on Wednesday, October 9, Marin will give a special lecture on modern developments in electricity and magnetism.

The lecture is to broaden your horizons — the material will not be included in problem sets or quizzes.

The lecture will not be video-recorded, so if you want to see it, you have to come to the lecture.

Separation of Variables in Cartesian Coordinates

How to solve $\nabla^2 V = 0$?

In Cartesian coordinates,

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \ .}$$

Try a solution of the form

$$\boxed{V(x, y, z) = X(x)Y(y)Z(z) \ .}$$

(Even if this is not general enough, sums of solutions of this form might work.)

Laplace's equation implies

$$\boxed{YZ\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+XZ\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}+XY\frac{d^2Z}{\mathrm{d}z^2}=0\ .}$$

Now divide by $V = XYZ$:

$$\boxed{\underbrace{\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}}_{C_1}+\underbrace{\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}}_{C_2}+\underbrace{\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2}}_{C_3}=0\ .}$$

Since the 1st term depends only on x, the 2nd depends only on y, and the 3rd depends only on z, each term must be a constant. So

$$\boxed{C_1+C_2+C_3=0\ .}$$

General Form for $X(x)$

We look first at $X(x)$. The other two functions $Y(y)$ and $Z(z)$ are analogous. We have

$$\boxed{\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = C_1 X \ .}$$

There are three possibilities:

1) If $C_1 > 0$, write $C_1 \equiv \alpha^2$. Then the general solution can be written as $\boxed{X(x) = Ae^{\alpha x} + Be^{-\alpha x}}$, or $\boxed{X(x) = A\sinh(\alpha x) + B\cosh(\alpha x)}$, where A and B are constants.

2) If $C_1 < 0$, write $C_1 = -\alpha^2$. The general solution can then be written as $\boxed{X(x) = Ae^{i\alpha x} + Be^{-i\alpha x}}$, or $\boxed{X(x) = A\sin(\alpha x) + B\cos(\alpha x)}$, where A and B are constants.

3) If $C_1 = 0$, then the general solution can be written as $\boxed{X(x) = A + Bx}$.

Separation of Variables Example: Infinitely Long Rectangular Pipe

This is Griffiths's Example 3.5, pp. 134–136.

An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

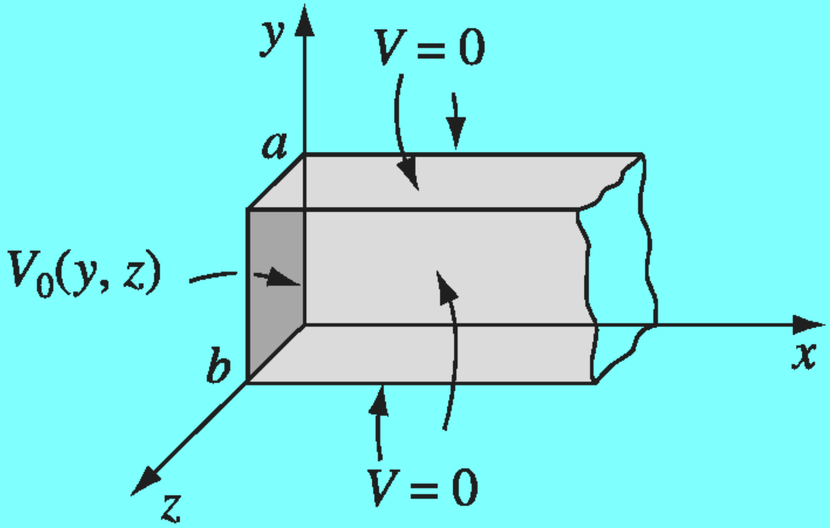

FIGURE 3.22

Choosing for each variable the only solutions to the differential equation with the right boundary conditions, the full solution becomes

$$\boxed{V(x,y,z) = \sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}e^{-\gamma x}\ .}$$

Try a sum of solutions of this form, with as yet unspecified coefficients:

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,x\right]\ .}$$

Summary: Blackboard Discussion Continuing the Calculation

For any choice of the V_{nm}'s, the solution

1) satisfies $\nabla^2 V = 0$;

2) satisfies boundary conditions on the sides of the pipe, at $y = 0, a$ and $z = 0, b$;

3) satisfies the boundary condition $V \to 0$ as $x \to \infty$.

We must still arrange to satisfy the boundary condition at $x = 0$,

$$\boxed{V(0, y, z) = V_0(y, z)\ .}$$

So we need to choose the V_{nm}'s so that

$$\boxed{V(0, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm} \sin \frac{n\pi y}{a} \sin \frac{m\pi z}{b} = V_0(y, z)\ .}$$

So we need to choose the V_{nm}'s so that

$$\boxed{V(0,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm} \sin\frac{n\pi y}{a} \sin\frac{m\pi z}{b} = V_0(y,z) \ .}$$

Can we do this? **YES!**

This is a Fourier series. Fourier's theorem (often called Dirichlet's theorem in its most rigorous formulation) says that this series is complete. Any piecewise continuous bounded function, satisfying the boundary conditions that it vanishes when $y = 0$ or $y = a$ or $z = 0$ or $z = b$, can be expanded this way.

We will not prove completeness, which is not an easy thing to prove — so we will leave it to math classes.

But once one knows that $V_0(y,z)$ can be written as such a series, it is easy to find the V_{nm}'s that do it. We use the fact that

$$\boxed{\int_0^a \sin\frac{n'\pi y}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}y = \frac{1}{2}a\,\delta_{n'n}\ .}$$

For $n' = n$, the integral is evaluated by recognizing that the average value of $\sin^2 w$, when averaged over any number of half periods of $\sin w$, is equal to $1/2$. For $n' \neq n$, trig identities can be used to convert the integral into integrations of $\cos w$ over integral numbers of periods, which vanish.

So we integrate both sides,

$$\boxed{\begin{aligned}&\int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\,\mathrm{d}z\,\mathrm{d}y\\ &\qquad = \int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\,V_0(y,z)\,\mathrm{d}z\,\mathrm{d}y\ .\end{aligned}}$$

Using

$$\int_0^a \sin\frac{n'\pi y}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}y = \frac{1}{2}a\,\delta_{n'n}$$

and the analogous formula for z, we find

$$\frac{1}{4}\,ab\,V_{n'm'} = \int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\,V_0(y,z)\,\mathrm{d}z\,\mathrm{d}y\ .$$

$$\frac{1}{4}\,ab\,V_{n'm'} = \int_0^a \sin\frac{n'\pi y}{a}\int_0^b \sin\frac{m'\pi z}{b}\,V_0(y,z)\,\mathrm{d}z\,\mathrm{d}y\ .$$

So we can write the final solution as

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}\sin\frac{n\pi y}{a}\sin\frac{m\pi z}{b}\exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,x\right]\ ,}$$

where

$$\boxed{V_{nm} = \frac{4}{ab}\int_0^a \sin\frac{n\pi y}{a}\int_0^b \sin\frac{m\pi z}{b}\,V_0(y,z)\,\mathrm{d}z\,\mathrm{d}y\ .}$$

We have dropped the primes in the formula above, which we can do because the equation no longer contains any instances of the original indices m and n.

Separation of Variables for Spherical Coordinates: Traceless Symmetric Tensor Approach

Laplace's Equation in Spherical Coordinates:

$$\nabla^2 \varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\nabla^2_{\rm ang}\,\varphi = 0\ ,$$

where

$$\nabla^2_{\rm ang}\,\varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\ .$$

Radial part can be written 2 alternative ways:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) \ .$$

Seek solution of form:

$$\boxed{\varphi(r,\theta,\phi) = R(r)F(\theta,\phi) \ .}$$

Write Laplace's equation as

$$0 = \frac{r^2}{RF}\nabla^2\varphi = \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{F}\nabla^2_{\text{ang}}F \ .$$

Since 1st term depends only on r, and 2nd only on θ and ϕ, each must be a constant:

$$\boxed{\begin{aligned} \frac{1}{F}\nabla^2_{\text{ang}}F &= C_{\text{ang}} \ , \\ \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) &= -C_{\text{ang}} \ . \end{aligned}}$$

Expansion of $F(\theta, \phi)$

We wish to find the most general solution to

$$\boxed{\nabla^2_{\rm ang} F = C_{\rm ang} F \ .}$$

Vocabulary: F is an *eigenfunction* of the operator $\nabla^2_{\rm ang}$, with *eigenvalue* $C_{\rm ang}$.

Instead of θ and ϕ, we will indicate directions by the unit vector $\hat{\boldsymbol{n}}$:

$$\boxed{\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3 \ ,}$$

where $\hat{\boldsymbol{e}}_i$ is the unit vector in the i-direction (where $i = 1, 2$, or 3).

Write $F(\theta, \phi)$ as a power series in the components of $\hat{\boldsymbol{n}}$. (This is equivalent to the standard spherical harmonic expansion, known to converge for square-integrable piece-wise continuous functions.)

$$\boxed{F(\hat{\boldsymbol{n}}) = C^{(0)} + C^{(1)}_i \hat{\boldsymbol{n}}_i + C^{(2)}_{ij} \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j + \ldots + C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} + \ldots \;,}$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates).

The general term $C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}$ is indicated by the label ℓ. It has ℓ indices i_1, i_2, $\ldots$, i_ℓ, which are each summed from 1 to 3.

Vocabulary: The $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are called *tensors*, and the number of indices is called the rank of the tensor. $C^{(0)}$, $C^{(1)}_i$, and $C^{(2)}_{ij}$ are special cases of tensors — they can also be called a scalar, a vector, and a matrix.

Restrictions on the $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$

Without loss of generality, we can insist:

1) $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are symmetric under any reordering of the indices:

$$\boxed{C^{(\ell)}_{i_1 i_2 \ldots i_\ell} = C^{(\ell)}_{j_1 j_2 \ldots j_\ell} \ ,}$$

where $\{j_1, j_2, \ldots, j_\ell\}$ is any permutation of $\{i_1, i_2, \ldots, i_\ell\}$.

2) The $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are traceless: if any two indices are set equal to each other and summed, the result is equal to zero. Since they are symmetric, it does not matter which indices are summed.

$$\boxed{C^{(\ell)}_{i_1 i_2 \ldots i_{\ell-2} j j} = 0 \ .}$$

Evaluation of $\nabla^2 F_\ell(\hat{n})$

Define

$$F_\ell(\hat{\boldsymbol{n}}) = C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ .$$

That is, $F_\ell(\hat{\boldsymbol{n}})$ is the ℓ'th term of the series.

It is useful to introduce a new variable r, and define

$$\vec{\boldsymbol{r}} \equiv r\hat{\boldsymbol{n}} \equiv x_i \hat{\boldsymbol{e}}_i = x_1 \hat{\boldsymbol{e}}_1 + x_2 \hat{\boldsymbol{e}}_2 + x_3 \hat{\boldsymbol{e}}_3 \ .$$

We further define

$$F_\ell(\vec{\boldsymbol{r}}) \equiv C^{(\ell)}_{i_1 i_2 \ldots i_\ell} x_{i_1} x_{i_2} \ldots x_{i_\ell} = r^\ell F_\ell(\hat{\boldsymbol{n}}) \ .$$

Why do we do all this? Because

$$\nabla^2_{\text{ang}} F_\ell(\vec{\boldsymbol{r}}) = 0 \ .$$

[$\nabla^2_{\text{ang}} F_\ell(\vec{\boldsymbol{r}}) = 0$ was derived on the blackboard.]

So what does this tell us about $\nabla^2_{\text{ang}} F_\ell(\hat{\boldsymbol{n}})$?

$$
\begin{aligned}
0 = \nabla^2 F_\ell(\vec{\boldsymbol{r}}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F_\ell(\vec{\boldsymbol{r}})}{\partial r}\right) + \frac{1}{r^2}\nabla^2_{\text{ang}} F_\ell(\vec{\boldsymbol{r}}) \\
&= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}r^\ell}{\mathrm{d}r}\right) F_\ell(\hat{\boldsymbol{n}}) + \frac{1}{r^2} r^\ell \nabla^2_{\text{ang}} F_\ell(\hat{\boldsymbol{n}}) \\
&= r^{\ell-2}\Big[\ell(\ell+1) F_\ell(\hat{\boldsymbol{n}}) + \nabla^2_{\text{ang}} F_\ell(\hat{\boldsymbol{n}})\Big] \ ,
\end{aligned}
$$

and therefore

$$
\boxed{\nabla^2_{\text{ang}} F_\ell(\hat{\boldsymbol{n}}) = -\ell(\ell+1) F_\ell(\hat{\boldsymbol{n}}) \ .}
$$

General Solution to Laplace's Equation in Spherical Coordinates

Recall

$$\frac{1}{F}\nabla^2_{\rm ang} F = C_{\rm ang} \ ,$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -C_{\rm ang} \ .$$

We now know that $C_{\rm ang}$ must have the form $-\ell(\ell+1)$, where ℓ is an integer.

Look at radial equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)R \ .$$

Look at radial equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)R\ .$$

Try a solution $R(r) = r^p$. We find consistency provided that $p(p+1) = \ell(\ell+1)$. This quadratic equation has two roots, $p = \ell$ and $p = -(\ell+1)$. Since we found two solutions to a second order linear differential equation, we know that any solution can be written as a linear sum of these two. Thus we can write

$$\boxed{R_\ell(r) = r^\ell \quad \text{or} \quad R_\ell(r) = \frac{1}{r^{\ell+1}}\ .}$$

The most general solution to Laplace's equation, in spherical coordinates, can then be written as

$$\boxed{\Phi(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\, r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ ,}$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ and $C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are arbitrary traceless symmetric tensors, and $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{n}}$.

Connection to Standard $Y_{\ell m}(\theta,\phi)$'s

Later we will be precise, but for now we state that the ℓ'th term, $F_\ell(\hat{\boldsymbol{n}}) \equiv C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}$ is equivalent to the sum over all $Y_{\ell m}$'s for a given ℓ.

Since m runs from $-\ell$ to ℓ, there are $2\ell+1$ possible values, so we expect that it must take $2\ell+1$ independent parameters to specify the general traceless symmetric tensor $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$. Let's check that.

[The derivation was given on the blackboard.]

Trace Decomposition Theorem

Any symmetric matrix $S_{i_1 \ldots i_\ell}$ can be uniquely written in the form

$$\boxed{S_{i_1 \ldots i_\ell} = S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} + \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] \ ,}$$

where $S^{(\mathrm{TS})}_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1 \ldots i_{\ell-2}}$ is a symmetric tensor, and

$$\boxed{\operatorname*{Sym}_{i_1 \ldots i_\ell} [xxx]}$$

means to symmetrize the expression xxx in the indices $i_1 \ldots i_\ell$.