MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II September 30, 2019
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FORMULA SHEET FOR QUIZ 1

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Index Notation:

Unit Vectors: $\hat{\boldsymbol{x}} \equiv \hat{\boldsymbol{\imath}} \equiv \hat{\boldsymbol{e}}_{\mathbf{1}}\ ,\quad \hat{\boldsymbol{y}} \equiv \hat{\boldsymbol{\jmath}} \equiv \hat{\boldsymbol{e}}_{\mathbf{2}}\ ,\quad \hat{\boldsymbol{z}} \equiv \hat{\boldsymbol{k}} \equiv \hat{\boldsymbol{e}}_{\mathbf{3}}\ ,\quad \vec{\boldsymbol{A}} \equiv A_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_i B_i\ , \qquad \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}_i = \epsilon_{ijk} A_j B_k\ , \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n} \ ***$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j\ , \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ: $R_z(\phi)_{ij} \ = \ \begin{array}{c c} & \begin{array}{ccc} j{=}1 & j{=}2 & j{=}3 \end{array} \\ \begin{array}{c} i{=}1 \\ i{=}2 \\ i{=}3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$

Rotation about axis $\hat{\boldsymbol{n}}$ by ϕ: ***

$$R(\hat{\boldsymbol{n}}, \phi)_{ij} = \delta_{ij} \cos\phi + \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j (1 - \cos\phi) - \epsilon_{ijk} \hat{\boldsymbol{n}}_k \sin\phi\ .$$

Vector Calculus:

Gradient: $(\vec{\boldsymbol{\nabla}}\varphi)_i = \partial_i \varphi = \dfrac{\partial\varphi}{\partial x}\hat{\boldsymbol{x}} + \dfrac{\partial\varphi}{\partial y}\hat{\boldsymbol{y}} + \dfrac{\partial\varphi}{\partial z}\hat{\boldsymbol{z}}\ , \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} \equiv \partial_i A_i = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

Curl: $(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})_i = \epsilon_{ijk} \partial_j A_k$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Laplacian: $\nabla^2 \varphi = \vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}}\varphi) = \dfrac{\partial^2 \varphi}{\partial x_i \partial x_i} = \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial z^2}$

Fundamental Theorems of Vector Calculus:

Gradient: $\displaystyle\int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \vec{\boldsymbol{\nabla}}\varphi \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \varphi(\vec{\boldsymbol{b}}) - \varphi(\vec{\boldsymbol{a}})$

Divergence: $\displaystyle\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}\, \mathrm{d}^3x = \oint_S \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{a}}$

where S is the boundary of $\mathcal{V}$

Curl: $\displaystyle\int_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot \mathrm{d}\vec{\boldsymbol{a}} = \oint_P \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}$

where P is the boundary of S

Vector Identities:

Triple Products:

$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{C}} \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$
$$\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}}) - \vec{\boldsymbol{C}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})$$

Product Rules:

$$\vec{\boldsymbol{\nabla}}(fg) = f\vec{\boldsymbol{\nabla}}g + g\vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) + \vec{\boldsymbol{B}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) + (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}$$
$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} + \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}})$$
$$\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} - \vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}}f$$
$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}})$$

Second Derivatives:

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = 0$$
$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} f) = 0$$
$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) - \nabla^2\vec{\boldsymbol{A}}$$

Spherical Coordinates:

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$$
$$z = r\cos\theta \qquad \phi = \tan^{-1}(y/x)$$
$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{x}} = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \qquad \hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

Point separation: $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}$

Volume element: $\mathrm{d}^3x \to r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}}$$
$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

Cylindrical Coordinates:

$x = s\cos\phi$	$s = \sqrt{x^2+y^2}$
$y = s\sin\phi$	$\phi = \tan^{-1}(y/x)$
$z = z$	$z = z$
$\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}} = \cos\phi\,\hat{\boldsymbol{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{y}} = \sin\phi\,\hat{\boldsymbol{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$

Point separation: $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}s\,\hat{\boldsymbol{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}}$

Volume element: $\mathrm{d}^3x \to s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial s}\,\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial\varphi}{\partial z}\,\hat{\boldsymbol{z}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_s\right) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\boldsymbol{\phi}}$$
$$+\frac{1}{s}\left[\frac{\partial}{\partial s}\left(sA_\phi\right) - \frac{\partial A_s}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r}-\vec{r}\,')\,\mathrm{d}^3x = \varphi(\vec{r}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad \text{where } i \text{ is summed over all points for which } g(x_i)=0$$

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = -4\pi\delta^3(\vec{r}-\vec{r}\,')$$

Electrostatics:

$\vec{F} = q\vec{E}$, where

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r}-\vec{r}\,')\,q_i}{\left|\vec{r}-\vec{r}\,'\right|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3}\,\rho(\vec{r}\,')\,\mathrm{d}^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times 10^{-12}\ \mathrm{C}^2/(\mathrm{N}\cdot\mathrm{m}^2)$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times 10^9\ \mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0\ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2 \mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j \,, \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j \,, \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\frac{a}{R}Q$, $r = \frac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

Seek solution to $\nabla^2 V = 0$ of the form

$$V(x, y, z) = X(x)Y(y)Z(z) \ ,$$

where

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = C_1 \ , \quad \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = C_2 \ , \quad \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = C_3 \ ,$$

and $C_1 + C_2 + C_3 = 0$.

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = C\,X \quad \Longrightarrow \quad \begin{cases} X \propto e^{\pm\sqrt{C}\,x} \text{ or } X \propto \left\{ \begin{matrix} \sinh(\sqrt{C}\,x) \\ \cosh(\sqrt{C}\,x) \end{matrix} \right\} & \text{if } C > 0 \\ X \propto e^{\pm i\sqrt{-C}\,x} \text{ or } X \propto \left\{ \begin{matrix} \sin(\sqrt{-C}\,x) \\ \cos(\sqrt{-C}\,x) \end{matrix} \right\} & \text{if } C < 0 \\ X \propto A + Bx & \text{if } C = 0 \end{cases}$$

$$\int_0^a \sin\frac{n'\pi x}{a} \sin\frac{n\pi x}{a}\,\mathrm{d}x = \frac{1}{2} a \delta_{n'\,n}$$