8.07 Lecture Slides 22

December 2, 2019

ELECTRODYNAMICS, CONSERVATION LAWS, and ~~ELECTROMAGNETIC WAVES~~

(Next time)

The Complete Maxwell Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

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- $-\frac{\partial \vec{B}}{\partial t}$ added by Faraday. It restores Galilean relativity: if the magnetic flux through a loop changes, with this addition it does not matter whether $\vec{B}$ changes, or the loop moved.

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- $\mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ added by Maxwell to make RHS divergenceless.

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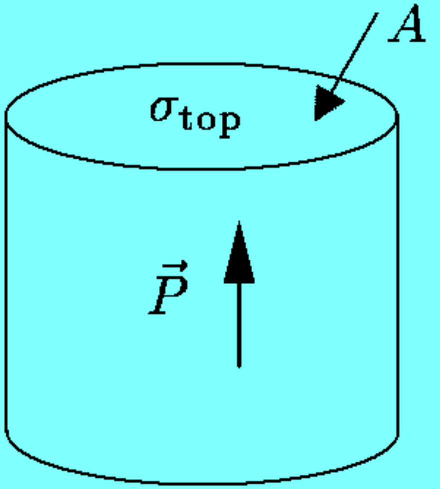
The Complete Maxwell Equations In Matter

There is one more bound charge effect: if $\vec{\boldsymbol{P}}$ changes with time, currents flow.

$$\sigma_{\rm top} = \vec{\boldsymbol{P}} \cdot \hat{\boldsymbol{n}} \ ,$$



where $\hat{\boldsymbol{n}}$ is the upward normal, the direction of $\vec{\boldsymbol{P}}$. So $Q = PA$, where $P = |\vec{\boldsymbol{P}}|$, so

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}t} A \quad \Longrightarrow \quad |\vec{J}| = \frac{\mathrm{d}P}{\mathrm{d}t} \ .$$

So

$$\boxed{\vec{J}_b = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}} + \frac{\partial \vec{\boldsymbol{P}}}{\partial t} \ .}$$

If polarized material moves, there are other terms, but we'll assume that all our matter is static.

Starting with

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

and using

$$\rho = \rho_{\text{free}} - \vec{\nabla} \cdot \vec{P} \, , \quad \vec{J} = \vec{J}_{\text{free}} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \, ,$$

and the definitions

$$\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M} \, , \quad \vec{D} \equiv \epsilon_0 \vec{E} + \vec{P} \, ,$$

one finds

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{D} = \rho_{\text{free}} & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\[2ex] \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}} + \dfrac{\partial \vec{D}}{\partial t} \end{array}}$$

Conductivity: Ohm's Law

$$\boxed{\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \ ,}$$

where $\vec{v}$ is the velocity of the medium, and

$$\boxed{\sigma \equiv \text{conductivity} \ ,}$$

$$\boxed{\rho \equiv \frac{1}{\sigma} = \text{resistivity.}}$$

Units:

$$\left[\vec{J}\right] = \frac{\mathrm{C}}{\mathrm{s} \cdot \mathrm{m}^2} \ , \qquad \left[\vec{E}\right] = \frac{\mathrm{N}}{\mathrm{C}} \qquad \Longrightarrow \qquad [\sigma] = \frac{\mathrm{C}}{\mathrm{s} \cdot \mathrm{m}^2}\frac{\mathrm{C}}{\mathrm{N}} = \frac{\mathrm{C}^2}{\mathrm{J\,m\,s}}$$

$$1 \ \mathrm{Ohm} \ (\Omega) \equiv \frac{1 \ \mathrm{volt}}{\mathrm{ampere}} = \frac{\mathrm{V}}{\mathrm{A}} = \frac{\mathrm{J/C}}{\mathrm{C/s}} = \frac{\mathrm{J\,s}}{\mathrm{C}^2} \qquad \Longrightarrow \qquad [\rho] = \Omega \cdot \mathrm{m} \ .$$

Drude Model of Resistivity

This is a very crude classical model. Assume:

1) Interatomic distance $= \lambda$.
2) Thermal velocity of electrons is given by
$$\frac{1}{2} m_e v_{\rm th}^2 = \frac{3}{2} kT \ ,$$
where $k =$ Boltzmann constant, $T =$ temperature. For $T = 300\,\mathrm{K}$, $v_{\rm th} = 1.17 \times 10^7$ cm/s $= 2.6 \times 10^5$ mile/hour.
3) $\Delta t = \lambda / v_{\rm th} =$ mean time between collisions,
4) $\langle \vec{\boldsymbol{v}} \rangle = \frac{1}{2} \vec{\boldsymbol{a}}\, \Delta t$, where $\vec{\boldsymbol{a}} = q \vec{\boldsymbol{E}} / m_e$.
5) Let f = number of free electrons per atom or molecule.
6) Let n = number density of atoms or molecules. Then

$$\boxed{\vec{\boldsymbol{J}} = fnq \langle \vec{\boldsymbol{v}} \rangle = \left(\frac{n f \lambda q^2}{2 m v_{\rm th}} \right) \vec{\boldsymbol{E}} \quad \Longrightarrow \quad \sigma = \frac{n f \lambda q^2}{2 m v_{\rm th}} \ .}$$

Numerically, for copper

$$n = \frac{\text{density}}{\text{atomic weight}}(\text{Avogadro's number})$$

$$= \frac{(8.96\ \text{g}/\text{cm}^3)}{(63.5\ \text{g}/\text{mole})}\left(6.022 \times 10^{23}\,\frac{\text{atoms}}{\text{mole}}\right)$$

$$= 8.50 \times 10^{22} \,/\, \text{cm}^3\ .$$

$$\lambda = \frac{1}{n^{1/3}} = 2.27 \times 10^{-8}\ \text{cm}.$$

Drude model gives

$$\rho_{\text{Drude}} = 4.3 \times 10^{-7}\ \Omega\text{-m}\ ,$$

while in reality

$$\rho_{\text{Copper}} = 1.68 \times 10^{-8}\ \Omega\text{-m}.$$

So copper conducts about 25 times better than the Drude model predicts.

Typical Values of Resistivity

Griffiths, Table 7.1, resistivities in Ω-m:

Conductors:

Silver: 1.59×10^{-8}

Iron: 9.61×10^{-8}

Graphite: 1.6×10^{-5}

Semiconductors:

Sea water: 0.2

Germanium: 0.46

Silicon: 2500

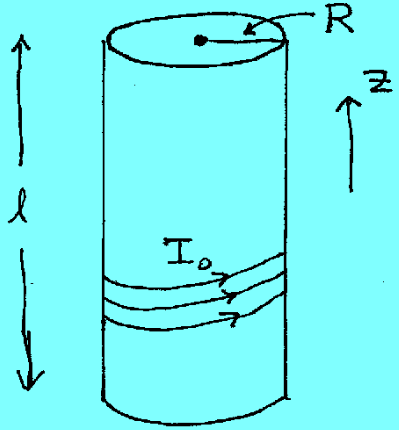
Insulators:

Water (pure): 8.3×10^{3}

Glass: $10^{9} - 10^{14}$

Teflon: $10^{22} - 10^{24}$

Faraday's Law and Inductors



Consider a long thin circular solenoid, of n turns per unit length, length ℓ, radius R, with $\ell \gg R$.

Inside $\vec{\boldsymbol{B}} \approx B_0 \hat{\boldsymbol{z}}$, outside $\vec{\boldsymbol{B}} \approx 0$, since the return flux is spread over an area that grows with ℓ.

Ampere's law, $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 E_{\text{enc}} \quad \Longrightarrow \quad \vec{\boldsymbol{B}} = n\mu_0 I_0 \hat{\boldsymbol{z}}$.

Magnetic flux $\Phi_B = \pi R^2 (n\mu_0 I_0)$ per turn, so

$$\Phi_B = \pi R^2 n^2 \mu_0 I_0 \ell = \mathcal{V} n^2 \mu_0 I_0 \ ,$$

where $\mathcal{V}$ = volume $= \pi R^2 \ell$.

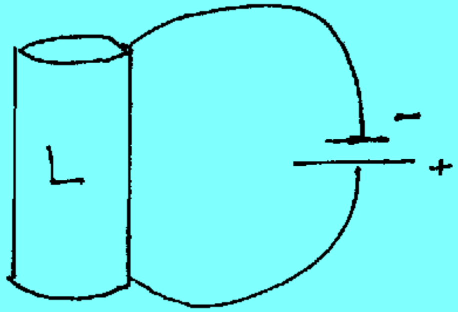
Inductance L is defined so that $\Phi_B = I_0 L$, and then

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt} = -L\frac{dI_0}{dt}}$$

For this system, $\boxed{L = \mu_0 n^2 \mathcal{V}}$

Inductors in Circuits

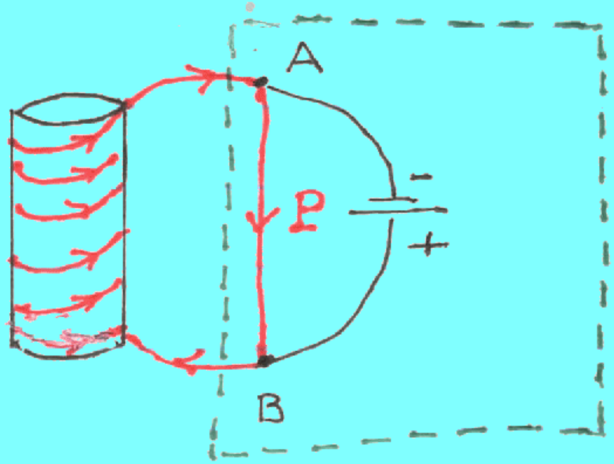
Suppose an inducting solenoid is connected to a battery:



Problem: Since $\vec{B}$ is changing, $\vec{\nabla} \times \vec{E} \neq 0$, so $V(\vec{r})$ is not well-defined.

Solution: If $\vec{B} \approx 0$ outside the inductor, there is a simple solution. ($\vec{B}$ outside is typically small because the return flux spreads out, especially if the length is much larger than the radius. Also, a large number of turns leads to signficant inductance, even if the field inside is not very strong.)

If $\vec{\boldsymbol{B}}$ can be ignored outside the inductor, consider the path shown in red as P, and the region surrounded by the dashed line:



Inside the dashed line, $V(\vec{\boldsymbol{r}})$ can be defined, since $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} \approx 0$, and the region is simply connected (all loops are contractible).

The loop P runs along the center of the wire, except for segment between A and B. If the wire is a good conductor, $\vec{\boldsymbol{E}}$ will vanish inside the wire, so

$$\varepsilon = \oint_P \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = \int_A^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = \frac{\partial \Phi_B}{\partial t} = L\frac{dI}{dt} \ .$$

So

$$\boxed{V(A) - V(B) = -L\frac{dI}{dt} \ .}$$

Boundary Conditions from Maxwell Equations

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}} \quad \Longrightarrow \quad D_1^{\perp} - D_2^{\perp} = \sigma_{\text{free}} \ ,$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \Longrightarrow \quad E_1^{\perp} - E_2^{\perp} = \frac{\sigma}{\epsilon_0} \ ,$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Longrightarrow \quad B_1^{\perp} - B_2^{\perp} = 0 \ ,$$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M} \quad \Longrightarrow \quad H_1^{\perp} - H_2^{\perp} = M_2^{\perp} - M_1^{\perp} \ .$$

where 1 = outside material, 2 = inside material.

$$\vec{\nabla}\cdot\vec{D}=\rho_{\rm free} \quad\Longrightarrow\quad D_1^\perp - D_2^\perp = \sigma_{\rm free}\ ,$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \quad\Longrightarrow\quad E_1^\perp - E_2^\perp = \frac{\sigma}{\epsilon_0}\ ,$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad\Longrightarrow\quad B_1^\perp - B_2^\perp = 0\ ,$$

$$\vec{H}\equiv\frac{1}{\mu_0}\vec{B}-\vec{M} \quad\Longrightarrow\quad H_1^\perp - H_2^\perp = M_2^\perp - M_1^\perp\ .$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\neq\infty \quad\Longrightarrow\quad \vec{E}_1^{\parallel}-\vec{E}_2^{\parallel}=0\ ,$$

$$\vec{D}=\epsilon_0\vec{E}+\vec{P} \quad\Longrightarrow\quad \vec{D}_1^{\parallel}-\vec{D}_2^{\parallel}=\vec{P}_1^{\parallel}-\vec{P}_2^{\parallel}\ ,$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_{\rm free}+\frac{\partial\vec{D}}{\partial t} \quad\Longrightarrow\quad \vec{H}_1^{\parallel}-\vec{H}_2^{\parallel}=-\hat{n}\times\vec{K}_{\rm free}\ ,$$

$$\vec{\nabla}\times\vec{B}-\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad\Longrightarrow\quad \vec{B}_1^{\parallel}-\vec{B}_2^{\parallel}=-\mu_0\hat{n}\times\vec{K}\ ,$$

where $\hat{n}$ = unit outward vector, $\vec{K}$ = surface current density.

Conservation of Energy: Poynting's Theorem

Conjecture: the electromagnetic energy stored in a volume $\mathcal{V}$ is given by

$$\boxed{U_{\mathrm{EM},\mathcal{V}} = \frac{1}{2}\int_{\mathcal{V}}\left[\epsilon_0|\vec{\boldsymbol{E}}|^2 + \frac{1}{\mu_0}|\vec{\boldsymbol{B}}|^2\right]\mathrm{d}^3x,}$$

which is what we found by looking at the energy stored in capacitors and in solenoids. If so, then

$$\frac{\mathrm{d}U_{\mathrm{EM}}}{\mathrm{d}t} = -P_{\mathrm{particles}} - P_{\mathrm{flux}}\ ,$$

where $P_{\mathrm{particles}}$ is the power transferred to charged particles, and P_{flux} is the power transmitted electromagnetically through the boundary S of $\mathcal{V}$. We assume that no particles cross the boundary.

$$
\begin{aligned}
P_{\text{particles}} &= \sum_n \vec{\boldsymbol{F}}_n \cdot \vec{\boldsymbol{v}}_n \\
&= \sum_n q_n(\vec{\boldsymbol{E}}_n + \vec{\boldsymbol{v}}_n \times \vec{\boldsymbol{B}}_n) \cdot \vec{\boldsymbol{v}}_n \\
&= \sum_n q_n \vec{\boldsymbol{E}}_n \cdot \vec{\boldsymbol{v}}_n \ .
\end{aligned}
$$

For continuous matter, $q_n \to \rho\, \mathrm{d}^3x$, so

$$
\boxed{P_{\text{particles}} = \int_{\mathcal{V}} \rho \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{E}}\, \mathrm{d}^3x = \int_{\mathcal{V}} \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{E}}\ , \mathrm{d}^3x\ .}
$$

If the original conjecture is true,

$$
\begin{aligned}
P_{\text{flux}} &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\left[\epsilon_0|\vec{\boldsymbol{E}}|^2+\frac{1}{\mu_0}|\vec{\boldsymbol{B}}|^2\right]\,\mathrm{d}^3x-\int_{\mathcal{V}}\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{E}}\,\mathrm{d}^3x\\
&= -\int_{\mathcal{V}}\mathrm{d}^3x\,\left[\epsilon_0\vec{\boldsymbol{E}}\cdot\frac{\partial\vec{\boldsymbol{E}}}{\partial t}+\frac{1}{\mu_0}\vec{\boldsymbol{B}}\cdot\frac{\partial\vec{\boldsymbol{B}}}{\partial t}+\frac{1}{\mu_0}\vec{\boldsymbol{E}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}}-\mu_0\epsilon_0\frac{\partial\vec{\boldsymbol{E}}}{\partial t}\right)\right]\\
&= -\int_{\mathcal{V}}\mathrm{d}^3x\left[-\frac{1}{\mu_0}\vec{\boldsymbol{B}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}})+\frac{1}{\mu_0}\vec{\boldsymbol{E}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}})\right]\\
&= \frac{1}{\mu_0}\int_{\mathcal{V}}\mathrm{d}^3x\,\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\ .
\end{aligned}
$$

$$P_{\rm flux} = \frac{1}{\mu_0}\int_{\mathcal{V}} \mathrm{d}^3x\,\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}) \ ,$$

so,

$$\boxed{P_{\rm flux} = \oint_S \vec{\boldsymbol{S}}\cdot \mathrm{d}\vec{\boldsymbol{a}} \ ,}$$

where

$$\boxed{\vec{\boldsymbol{S}} = \frac{1}{\mu_0}\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} \ ,}$$

where $\vec{\boldsymbol{S}}$ is called the Poynting vector. $\vec{\boldsymbol{S}}$ describes the flow of energy in space, with units of joules per meter^2 per second.

Forms of energy conservation:

$$\boxed{\frac{\mathrm{d}U_{\mathrm{EM},\mathcal{V}}}{\mathrm{d}t} = -\int_{\mathcal{V}} \vec{\boldsymbol{J}} \cdot \vec{E}\,\mathrm{d}^3x - \oint_S \vec{\boldsymbol{S}} \cdot \mathrm{d}\vec{\boldsymbol{a}}\ ,}$$

or in its differential form,

$$\boxed{\frac{\partial u_{\mathrm{EM}}}{\partial t} = -\vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{E}} - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{S}}\ .}$$

The work-energy theorem implies that

$$\int_{\mathcal{V}} \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{E}}\, d^3x = \frac{\mathrm{d}U_{\mathrm{mech}}}{\mathrm{d}t}\ ,$$

where U_{mech} is the mechanical energy of the particles, meaning the total kinetic energy plus any non-electromagnetic potential energy (e.g., gravitational energy, spring energy, etc.). Thus we can write

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{\mathrm{EM},\mathcal{V}} + U_{\mathrm{mech},\mathcal{V}}\right] = -\oint_S \vec{\boldsymbol{S}} \cdot \mathrm{d}\vec{\boldsymbol{a}}\ .}$$

From

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{\mathrm{EM},\mathcal{V}} + U_{\mathrm{mech},\mathcal{V}}\right] = -\oint_S \vec{\boldsymbol{S}} \cdot \mathrm{d}\vec{\boldsymbol{a}}\ ,}$$

we can write the differential form

$$\boxed{\frac{\partial u}{\partial t} = -\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{S}}\ ,}$$

where

$$u = u_{\mathrm{EM}} + u_{\mathrm{mech}}\ ,$$

where u_{mech} is the density of mechanical energy.

Power Transmission

On the blackboard

Conservation of Momentum

On the blackboard