MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 19, 2019
Prof. Alan Guth

# PROBLEM SET 5

**DUE DATE:** Friday, October 25, 2019, at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

**READING ASSIGNMENT:** *Lecture Notes 3: Traceless Symmetric Tensor Approach to Legendre Polynomials and Spherical Harmonics.*

**CREDIT:** This problem set has 70 points of credit, plus the option of earning 20 points extra credit.

## — PART A —

### PROBLEM 1: INTEGRATION OF THE LEGENDRE POLYNOMIALS *(15 points)*

We learned in lecture and in Lecture Notes 2 (Eq. (2.50)) that

$$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\rm TS}\ . \tag{1.1}$$

In Lecture Notes 3 (Eq. (3.6)), and also in Problem Set 4, Problem 3, we used Eq. (1.1) to show that the normalized Legendre polynomials can be written

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ . \tag{1.2}$$

Further in Lecture Notes 3 (Eq. (3.31)) we learned how to integrate the product of two functions expressed in terms of symmetric traceless tensors, finding that

$$\begin{aligned}\int d\Omega\,&\left[C^{(\ell)}_{i_1\ldots i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\rm TS}\right]\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}}\{\,\hat{\boldsymbol{n}}_{j_1}\ldots\hat{\boldsymbol{n}}_{j_{\ell'}}\,\}_{\rm TS}\right]\\ &= \frac{4\pi\,2^\ell(\ell!)^2}{(2\ell+1)!}C^{(\ell)}_{i_1\ldots i_\ell}C'^{(\ell')}_{i_1\ldots i_{\ell'}}\,\delta_{\ell,\ell'}\ ,\end{aligned} \tag{1.3}$$

where $d\Omega = \sin\theta\,d\theta\,d\phi$.

(a) [5 pts] Use Eqs. (1.2) and (1.3) to show that

$$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx = \int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = 0 \tag{1.4}$$

whenever $\ell$ and $\ell'$ are not equal.

(b) [10 pts] Use Eqs. (1.1)-(1.3) to show that

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta\, d\theta = \frac{2}{2\ell+1}\, \delta_{\ell,\ell'} \ . \tag{1.5}$$

This result can be proven many different ways, starting from many different definitions of the Legendre polynomials, but here you are asked to use the definition of Eq. (1.2). To get full credit, you must start with this definition.

Hint: note that

$$\{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS} \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS} = \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \ . \tag{1.6}$$

If you can explain why this is true, you can then iteratively apply Eq. (1.1), as you did in Problem 3 of Problem Set 4.

## PROBLEM 2: INTEGRATION OF THE SPHERICAL HARMONICS *(25 points)*

In lecture and in Lecture Notes 3, Section 11, we learned that the spherical harmonics $Y_{\ell m}(\theta,\phi)$, for $m \geq 0$, can be written as

$$Y_{\ell,m}(\theta,\phi) = N(\ell,m)\, \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \tag{2.1}$$

where

$$\hat{\boldsymbol{u}}^+ = \frac{1}{\sqrt{2}} \left( \hat{\boldsymbol{e}}_{\boldsymbol{x}} + i \hat{\boldsymbol{e}}_{\boldsymbol{y}} \right) \tag{2.2}$$

and

$$\hat{\boldsymbol{n}} = \sin\theta \sin\phi\, \hat{\boldsymbol{e}}_{\boldsymbol{x}} + \sin\theta\cos\phi\, \hat{\boldsymbol{e}}_{\boldsymbol{y}} + \cos\theta\, \hat{\boldsymbol{e}}_{\boldsymbol{z}} \ . \tag{2.3}$$

(a) [10 pts] Show, using the definitions above, that

$$\int d\Omega\, Y^*_{\ell',m'}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) = 0 \tag{2.4}$$

unless $\ell' = \ell$ and $m' = m$.

(b) [15 pts] In lecture we showed that

$$\{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{z}}_{i_\ell} = \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)} \{\, \hat{\boldsymbol{u}}^+_{i_1} \ldots \hat{\boldsymbol{u}}^+_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_{\ell-1}} \,\}_{\rm TS} \ . \tag{2.5}$$

Using this fact, show that the desired normalization for $Y_{\ell,m}(\theta,\phi)$,

$$\int d\Omega\, Y^*_{\ell',m'}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) = \delta_{\ell',\ell}\, \delta_{m',m} \ , \tag{2.6}$$

implies that

$$N^2(\ell,m) = \frac{(2\ell+1)[(2\ell)!]^2}{4\pi \cdot 2^{2\ell-m} (\ell!)^2 (\ell+m)!(\ell-m)!} \ . \tag{2.7}$$

## — PART B (To be handed in separately from Part A) —

### PROBLEM 3: EXPLICIT CALCULATION OF SPHERICAL HARMONICS

*(15 points)*

The standard spherical harmonics, for all values of $m$, can be described in the traceless symmetric tensor formalism by introducing three basis vectors,

$$\begin{aligned} \hat{\boldsymbol{u}}^{(1)} &\equiv \hat{\boldsymbol{u}}^{+} \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} + i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right) \ , \\ \hat{\boldsymbol{u}}^{(2)} &\equiv \hat{\boldsymbol{u}}^{-} \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} - i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right) \ , \\ \hat{\boldsymbol{u}}^{(3)} &\equiv \hat{\boldsymbol{z}} \ . \end{aligned} \tag{3.1}$$

This is an orthonormal basis in the sense that

$$\hat{\boldsymbol{u}}^{(i)*} \cdot \hat{\boldsymbol{u}}^{(j)} = \delta_{ij} \ . \tag{3.2}$$

Then

$$Y_{\ell,m}(\theta,\phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \tag{3.3}$$

where

$$C^{(\ell,m)}_{i_1 \ldots i_\ell} = \begin{cases} N(\ell,m)\{\, \hat{\boldsymbol{u}}^{+}_{i_1} \ldots \hat{\boldsymbol{u}}^{+}_{i_m}\, \hat{\boldsymbol{z}}_{i_{m+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} & \text{for } m \geq 0 \ , \\ N(\ell,m)\{\, \hat{\boldsymbol{u}}^{-}_{i_1} \ldots \hat{\boldsymbol{u}}^{-}_{i_{|m|}}\, \hat{\boldsymbol{z}}_{i_{|m|+1}} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\mathrm{TS}} & \text{for } m \leq 0 \ , \end{cases} \tag{3.4}$$

where

$$N(\ell,m) = \frac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}} \ . \tag{3.5}$$

Note that Eq. (3.5) is consistent with Eq. (2.7), but generalizes it to allow for negative values of $m$, and also includes a sign factor $(-1)^m$ which is part of the standard conventions.

(a) [10 pts] Use this formalism to show that

$$Y_{31}(\theta,\phi) = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\,(5\cos^2\theta - 1)\sin\theta\, e^{i\phi} \ . \tag{3.6}$$

*Hint:* You might benefit from that fact that Eq. (3.3) can alternatively be written as

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}\, \{\, \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\mathrm{TS}} \ , \tag{3.7}$$

since $\{\, \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\mathrm{TS}}$ differs from $\hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}$ only by terms proportional to kronecker $\delta$-functions, and such terms will give no contribution because $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}$ is traceless. Once the expansion of $\{\, \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\mathrm{TS}}$ is written out explicitly, there is no need to take the traceless symmetric part in the evaluation of $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell}$ using Eq. (3.4). It is a little easier to write the traceless symmetric part of $\hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}$ than it is to find the traceless symmetric part of the right-hand side of Eq. (3.4), since $\hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}$ is already symmetric.

(b) [5 pts] Use this formalism to derive a general expression for $Y_{\ell\ell}$ that is valid for all $\ell$.

## PROBLEM 4: A SPHERE WITH OPPOSITELY CHARGED HEMISPHERES *(15 points)*

Griffiths Problem 3.23 (p. 150).

A spherical shell of radius $R$ carries a uniform surface charge $\sigma_0$ on the "northern" hemisphere and a uniform surface charge $-\sigma_0$ on the "southern" hemisphere. Find the potential inside and outside the sphere, calculating the coefficients explicitly up to $A_6$ and $B_6$.

[Note from Alan Guth: $A_6$ and $B_6$ refer to Eq. (3.65) in Griffiths' text,

$$V(r,\theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \ .$$

You can work the problem either using this expansion in Legendre polynomials, or by using traceless symmetric tensors.]

## PROBLEM 5: A CIRCULAR DISK AT A FIXED POTENTIAL *(20 points extra credit)*

*This problem is based on Jackson, Problem 3.3 (challenging!).*

A thin, flat, conducting circular disk of radius $R$ is located in the $x$-$y$ plane with its center at the origin, and is mantained at a fixed potential $V_0$.

(a) [12 pts extra] With the information that the surface charge density on a disk at fixed potential is proportional to $(R^2 - s^2)^{-1/2}$, where $s$ is the distance out from the center of the disk, show that for $r > R$,

$$V(r,\theta) = \frac{2V_0}{\pi} \frac{R}{r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \left( \frac{R}{r} \right)^{2l} P_{2\ell}(\cos\theta) \, .$$

*Hint:* First find the field along the $z$-axis, and then use your knowledge about the general solution to Laplace's equation to infer the angular dependence.

(b) [8 pts extra] Calculate the capacitance of the disk, defined by $Q = CV$, where $Q$ is the total charge on the disk and $V$ is the potential of the disk relative to $|\vec{r}| \to \infty$.