MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 25, 2019
Prof. Alan Guth

PROBLEM SET 3

Revised September 27, 2019 *

DUE DATE: Wednesday, October 2, 2019. Due at 6:30 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

READING ASSIGNMENT: Griffiths Sections 3.1 (*Potentials: Laplace's Equation*), 3.2 (*The Method of Images*), and 3.3.1 (*Separation of Variables, Cartesian Coordinates).*

CREDIT: This problem set has 130 points of credit.

— PART A —

PROBLEM 1: THE LAPLACIAN AS THE ANTI-LUMPINESS OPERATOR *(15 points)*

In this problem you will prove a relation that was stated in lecture. Let $\varphi(\vec{\boldsymbol{r}})$ be any scalar function of position $\vec{\boldsymbol{r}}$. We are interested in relating the value of φ at an arbitrary point $\vec{\boldsymbol{r}}_0$ to the average value of φ on a sphere that is centered at $\vec{\boldsymbol{r}}_0$. While the point $\vec{\boldsymbol{r}}_0$ is arbitrary, we can simpify our notation by choosing a coordinate system so that $\vec{\boldsymbol{r}}_0$ is the origin $\vec{0}$. Then the relation to be proved can be written

$$\varphi(\vec{\mathbf{0}}) - \bar{\varphi}(R) = -\frac{1}{4\pi}\int_{r<R} \mathrm{d}^3x \left(\frac{1}{r} - \frac{1}{R}\right) \nabla^2\varphi \ , \tag{1.1}$$

where we note that the quantity in parentheses is positive. Here $\bar{\varphi}(R)$ represents the average value of φ on the surface of a sphere of radius R, which can be written explicitly as

$$\bar{\varphi}(R) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \varphi(R,\theta,\phi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{1.2}$$

The integration in Eq. (1.1) is over the volume of a sphere of radius R, centered at the origin. The relation to lumpiness can be seen by thinking of φ as the density of a pudding. The equation implies that if $\nabla^2\varphi = 0$, then the value of φ at the origin is the same as the average value of its surroundings (no lumpiness). But if $\nabla^2\varphi < 0$, then the value of φ at the origin is higher than the average value of its surroundings (i.e., there is a lump).

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* The wording of Problem 6(d) was modified to improve the clarity.

(a) [4 pts] Defining

$$g(r) \equiv \frac{1}{r} - \frac{1}{R} \tag{1.3}$$

for compactness, use the divergence theorem to show that

$$\int_{r<R} \mathrm{d}^3x \, \vec{\boldsymbol{\nabla}} \cdot \left(g \, \vec{\boldsymbol{\nabla}} \varphi\right) = 0 \tag{1.4}$$

for any well-behaved function $\varphi(\vec{\boldsymbol{r}})$.

(b) [4 pts] Use index notation (i.e., $\vec{\boldsymbol{\nabla}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{i}} \partial_i$) to show that for arbitrary scalar functions $g(\vec{\boldsymbol{r}})$ and $\varphi(\vec{\boldsymbol{r}})$

$$\vec{\boldsymbol{\nabla}} \cdot (g \, \vec{\boldsymbol{\nabla}} \varphi) = \vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}} \varphi + g \, \nabla^2 \varphi \ . \tag{1.5}$$

(c) [7 pts] Use the identity of Eq. (1.5) to rewrite the integrand of the integral of Eq. (1.4). We suggest that the integral of $\vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}} \varphi$ be expressed in spherical polar coordinates. Show that the vanishing of the integral in Eq. (1.4), re-expressed in this way, implies Eq. (1.1).

PROBLEM 2: SPHERES AND IMAGE CHARGES *(15 points)*

Griffiths Problem 3.9 (p. 129):

This problem is essentially Griffiths 3.9 (p. 129), which follows example 3.2.

In Example 3.2, we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential.

(a) [5 pts] If the sphere is held at potential $V = V_0$, where should the second image charge q'' be placed, and what value should q'' take?

(b) [5 pts] Find the force between the charge q and the sphere at potential V_0. (This is slightly different than the question Griffiths asks.)

(c) [5 pts] What value should q'' take if the sphere is to be *neutral*? Compute the force in this case.

PROBLEM 3: IMAGE CHARGES FOR A PLANE WITH A HEMISPHERICAL BULGE *(15 points)*

Consider a conducting plane that occupies the x-y plane of a coordinate system, but with the circular disk $x^2 + y^2 < a^2$ removed. The circular disk is replaced by a conducting hemisphere of radius a, described by the equation

$$x^2 + y^2 + z^2 = a^2 \ , \qquad z > 0 \ . \tag{3.1}$$

You may assume that the plane is grounded, meaning that $V = 0$. A charge q is placed on the z-axis at $(0, 0, z_0)$, with $z_0 > a$. Find a suitable set of image charges for this configuration. Show that the charge is attracted toward the plate with a force

$$|\vec{\boldsymbol{F}}| = \frac{1}{4\pi\epsilon_0} \left[\frac{q^2}{4z_0^2} + \frac{4q^2 a^3 z_0^3}{(z_0^4 - a^4)^2} \right] \ . \tag{3.2}$$

PROBLEM 4: IMAGES FOR A CONDUCTING CYLINDER *(15 points)*

This problem is based on Problem 2.11 of Jackson: **Classical Electrodynamics**, *3rd edition.*

A line of charge with linear charge density λ is placed parallel to, and at a distance R away from, the axis of a conducting cylinder of radius b (where $b < R$). Use cylindrical coordinates (s, ϕ, z), as described in Sec. 1.4.2 of Griffiths, choosing the axis of the cylinder to be the axis of the coordinate system. We consider a situation in which the total linear charge density λ' on the cylinder is determined by the criterion that the potential difference between the cylinder and $s = \infty$ is finite.

(a) [10 pts] Find the magnitude and position of the image charge(s).

(b) [5 pts] Find the potential V_0 of the cylinder in terms of R, b, and λ, where the potential is defined to be zero at $s = \infty$.

— PART B (To be handed in separately from Part A) —

PROBLEM 5: CAPACITANCE OF A SINGLE CONDUCTOR *(20 points)*

(a) [4 pts] Consider a single conductor, and define its capacitance by $Q = CV$, where Q is the charge on the conductor, and V is the potential of the conductor defined so that $V = 0$ at infinity. Show that C can be expressed as

$$C = \frac{\epsilon_0}{V_0^2} \int_{\mathcal{V}} |\vec{\nabla} V|^2 \, \mathrm{d}^3x \ , \tag{5.1}$$

where $\mathcal{V}$ is the space outside the conductor, and $V(\vec{r})$ is the solution for the potential when the conductor is held at $V = V_0$.

(b) [4 pts] Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi(\vec{r})] = \frac{\epsilon_0}{V_0^2} \int_{\mathcal{V}} |\vec{\nabla} \Psi|^2 \mathrm{d}^3x \ , \tag{5.2}$$

where $\Psi(\vec{r})$ is any trial function satisfying the boundary condition $\Psi = V_0$ at the conductor, and $\Psi = 0$ at infinity. (Note that Ψ is *not* required to satisfy Laplace's equation, or any other equation.)

(c) [7 pts] Prove that the capacitance C' of a conductor with surface S' is smaller than the capacitance C of a conductor whose surface S encloses S'.

(d) [5 pts] Use part (c) to find upper and lower limits for the capacitance of a conducting cube of side a. Write your answer in the form: $\alpha(4\pi\epsilon_0 a) < C_{\text{cube}} < \beta(4\pi\epsilon_0 a)$ and find the constants α and β. A numerical calculation* gives $C \simeq 0.661(4\pi\epsilon_0 a)$. Compare this answer with your limits.

PROBLEM 6: A SPHERICAL CONDUCTOR AND A CONDUCTING PLANE *(30 points)*

Consider a solid spherical conductor of radius R, with center on the positive z-axis at $z = z_0$, with $z_0 > R$. Suppose that the x-y plane is conducting, and is held at potential $V = 0$, while the sphere is held at potential V_0.

To first approximation, we can think of the field as that of a point charge q_0 at the center of the sphere, with q_0 related to V_0 by

$$V_0 = \frac{q_0}{4\pi\epsilon_0 R} \ . \tag{6.1}$$

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* C.-O. Hwang and M. Mascagni, *Journal of Applied Physics* **95**, 3798 (2004).

The field due to this charge gives a potential V_0 on the surface of the sphere, as desired. But now the potential on the x-y plane is not zero.

(a) [5 pts] The potential on the x-y plane can be restored to zero by placing an image charge below the x-y plane (i.e., at negative z). What charge q' should this image have, and where should it be placed?

(b) [10 pts] The potential on the surface of the spherical conductor is now no longer constant, but it can be made constant by adding another image charge q''. The potential on the x-y plane can be restored to zero by adding another image charge q''', and the potential on the sphere can be restored to a constant by adding yet another image charge q''''. The series will continue forever, but it does converge fairly quickly. Calculate the positions and charges of the image charges q'', q''', and q''''.

(c) [5 pts] After all the image charges are added through q'''', what is the potential V of the spherical conductor?

(d) [5 pts] What is the total potential energy of the physical system, i.e., the sphere at potential V_0 and the conducting plane? Express your answer as the first three terms of an infinite series. *(Hint: the series will be related to the image charges, and the first three terms will correspond to the image charges through q''''.)*

(e) [5 pts] Would the fields outside the conductors be different if the solid spherical conductor were replaced by a spherical conducting shell, with the same outer radius?

PROBLEM 7: LAPLACE'S EQUATION IN A BOX *(20 points)*

Griffiths Problem 3.16 (p. 141).

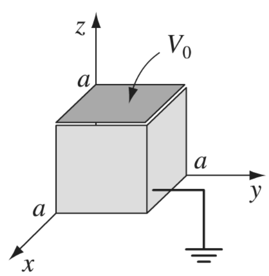

FIGURE 3.23

A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center ($a/2$, $a/2$, $a/2$) be? Check numerically that your formula is consistent with this value.]