MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 7, 2019

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QUIZ 1

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______________________________ Your Name

2:00 pm OR 3:00 pm — Recitation

1. There are **FOUR** problems — be sure not to miss any! This exam is worth 100 points. The problems are **not** equally weighted.
2. A separate formula sheet will be distributed.
3. The exam is **closed book** and **closed notes**. Cell phones and calculators will not be needed, and are not allowed.
4. Write all of your responses in this booklet. Extra pages are provided in this booklet, but if you need more, please ask.
5. **Please** raise your hand and **ask** if there is anything about a question that you find unclear or confusing. Perhaps you have simply misread something, but it is possible that there is an error we need to correct. In either case, it would be good to know for certain.

PROBLEM 1: SOME SHORT EXERCISES *(25 points)*

(a) [5 pts] The formula sheet includes the identity

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) - \nabla^2 \vec{\boldsymbol{A}} \ .$$

Using index notation, prove this identity. You may use any formula from the formula sheet *other than* the formula that you are proving.

(b) [5 pts] Evaluate $\displaystyle\int_{-1}^{5} 2x^3\, \delta(x-2)\, dx$.

(c) [5 pts] Evaluate $\displaystyle\int_{0}^{\infty} x^3\, \delta(x^3-8)\, \mathrm{d}x$.

(d) [5 pts] Evaluate $\displaystyle\int_{-\infty}^{\infty} \cos x \frac{d}{\mathrm{d}x} \delta(x-a)\, dx$, where a is a real constant.

(e) [5 pts] Suppose that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ is a vector field, described in spherical coordinates as

$$\vec{\boldsymbol{A}}(r,\theta,\phi) = f(\theta,\phi)\,\hat{\boldsymbol{r}} \ ,$$

where $f(\theta,\phi)$ is an arbitrary smooth function. (By “smooth”, we mean that it can be differentiated as many times as might be needed.) Calculate $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$.

PROBLEM 2: A SPHERICAL CONDUCTOR AND A CONDUCTING PLANE *(30 points)*

This problem was Problem 6 of Problem Set 3.

Consider a solid spherical conductor of radius R, with center on the positive z-axis at $z = z_0$, with $z_0 > R$. Suppose that the x-y plane is conducting, and is held at potential $V = 0$, while the sphere is held at potential V_0.

To first approximation, we can think of the field as that of a point charge q_0 at the center of the sphere, with q_0 related to V_0 by

$$V_0 = \frac{q_0}{4\pi\epsilon_0 R} \ . \tag{2.1}$$

The field due to this charge gives a potential V_0 on the surface of the sphere, as desired. But now the potential on the x-y plane is not zero.

(a) [5 pts] The potential on the x-y plane can be restored to zero by placing an image charge below the x-y plane (i.e., at negative z). What charge q' should this image have, and where should it be placed?

(b) [10 pts] The potential on the surface of the spherical conductor is now no longer constant, but it can be made constant by adding another image charge q''. The potential on the x-y plane can be restored to zero by adding another image charge q''', and the potential on the sphere can be restored to a constant by adding yet another image charge q''''. The series will continue forever, but it does converge fairly quickly. Calculate the positions and charges of the image charges q'', q''', and q''''.

(c) [5 pts] After all the image charges are added through q'''', what is the potential V of the spherical conductor?

(d) [5 pts] What is the total potential energy of the physical system, i.e., the sphere at potential V_0 and the conducting plane? Express your answer as the first three terms of an infinite series. *(Hint: the series will be related to the image charges, and the first three terms will correspond to the image charges through q''''.)*

(e) [5 pts] Would the fields outside the conductors be different if the solid spherical conductor were replaced by a spherical conducting shell, with the same outer radius?

PROBLEM 3: ELECTRIC FIELDS IN A SPHERICAL GEOMETRY *(30 points)*

Consider a spherically symmetric distribution of charge, which we describe in spherical coordinates. It includes a shell of charge at radius R, centered at the origin, with an unknown surface charge density σ. There is also an unknown charge density $\rho(r)$, defined for all $r \neq R$. We are told that the electric field is given by

$$\vec{\boldsymbol{E}}(r,\theta,\phi) = \begin{cases} A r e^{-r^2/b^2}\,\hat{\boldsymbol{r}} & \text{if } r > R \\ 0 & \text{if } r \leq R \end{cases} \,, \tag{3.1}$$

where A is a positive constant.

(a) [5 pts] What is the surface charge density σ on the shell?

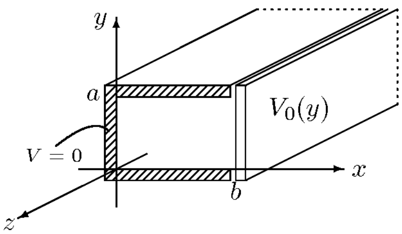
(b) [10 pts] Find the charge density $\rho(r)$, for all values of r other than $r = R$.

(c) [10 pts] Find the potential $V(r)$ for all values of r, defining the potential at spatial infinity to be zero. Is it well-defined at $r = R$?

(d) [5 pts] What is the total charge of this system, integrated over all space?

PROBLEM 4: AN INFINITE RECTANGULAR PIPE *(15 points)*

A rectangular pipe, running parallel to the z-axis (from $-\infty$ to ∞), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$. The potential inside the pipe does not depend on z, so it can be written as $V(x, y)$.



We seek a solution of the form

$$V(x,y) = \sum_{\alpha} X_\alpha(x) Y_\alpha(y) \ , \tag{4.1}$$

where α represents some unspecified type of label.

(a) [5 pts] Given the boundary conditions on $Y_\alpha(y)$, write the set of allowed functions $Y_\alpha(y)$.

(b) [10 pts] Given the boundary condition on $X_\alpha(x)$ at $x = 0$, write out an explicit version of Eq. (4.1). In the explicit version, the sum over α should be an explicit sum over some specified set of values, and the functions $X_\alpha(x)$ and $Y_\alpha(y)$ should be explicit functions. The expression might also contain a constant C_α for each value α.

You are not asked to find the values of the constants C_α.

Problem	**Maximum**	**Score**	**Initials**
1	25		
2	30		
3	30		
4	15		
TOTAL	100		