MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 11, 2019
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PROBLEM SET 4

DUE DATE: Friday, October 18, 2019, at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

READING ASSIGNMENT: Griffiths Section 3.3.2 (*Separation of Variables, Spherical Coordinates*) and Section 3.4 (*Multipole Expansion*). Also *Lecture Notes 1 and 2: Traceless Symmetric Tensor Approach to Legendre Polynomials and Spherical Harmonics.*

CREDIT: This problem set has 75 points of credit.

— PART A —

PROBLEM 1: TRACELESS SYMMETRIC PART OF $\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n$ *(15 points)*

In lecture and in Lecture Notes 1, we have learned how to extract the traceless symmetric part of tensors constructed from a unit vector $\hat{\boldsymbol{n}}$:

$$\begin{aligned}
\{\,1\,\}_{\rm TS} &= 1 \\
\{\,\hat{\boldsymbol{n}}_i\,\}_{\rm TS} &= \hat{\boldsymbol{n}}_i \\
\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\,\}_{\rm TS} &= \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j - \tfrac{1}{3}\delta_{ij} \\
\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\,\}_{\rm TS} &= \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k - \tfrac{1}{5}\left(\hat{\boldsymbol{n}}_i\delta_{jk} + \hat{\boldsymbol{n}}_j\delta_{ik} + \hat{\boldsymbol{n}}_k\delta_{ij}\right) \\
\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\,\}_{\rm TS} &= \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m - \tfrac{1}{7}\left(\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\delta_{km} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_k\delta_{mj} + \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_m\delta_{jk} + \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\delta_{im}\right. \\
&\qquad \left. + \hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m\delta_{ik} + \hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\delta_{ij}\right) + \tfrac{1}{35}\left(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\right)\ ,
\end{aligned} \tag{1.1}$$

Here $\{\,xxx\,\}_{\rm TS}$ denotes the traceless symmetric part of xxx. Extend this table to include the next entry, $\{\,\hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k\hat{\boldsymbol{n}}_m\hat{\boldsymbol{n}}_n\,\}_{\rm TS}$.

PROBLEM 2: QUADRUPOLE AND OCTOPOLE TERMS OF THE MULTIPOLE EXPANSION *(20 points)*

Griffiths Problem 3.52 (p. 165). For parts (a) and (d), you can find the answers either by (i) starting with Griffiths' Eq. (3.95) (p. 153) for the multipole expansion, (ii) by starting with the integral expression (Griffiths' Eq. (2.29), p. 85) for the potential and then expanding in powers of $1/r$, or (iii) by using the results of Lecture Notes 2, Section 7. Note that the quadrupole moment Q_{ij} is exactly the traceless symmetric tensor that is called $C^{(2)}_{ij}$ in Lecture Notes 2, and that the octopole is generally defined to be $C^{(3)}_{ijk}$. It is common — almost universal — to describe the low ℓ multipoles as traceless symmetric tensors, but most textbooks use traceless symmetric tensors for only the low ℓ multipoles.

The text of the problem is as follows:

(a) [5 pts] Show that the quadrupole term in the multipole expansion can be written

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i,j=1}^{3} \hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j Q_{ij} \ ,$$

(in the notation of Eq. 1.31 [*sic*: should be Eq. 1.21, which defines $\hat{\boldsymbol{r}}$)], where

$$Q_{ij} \equiv \frac{1}{2}\int [3r'_i r'_j - (r')^2\delta_{ij}]\rho(\vec{\boldsymbol{r}}\,')\mathrm{d}\tau' \ .$$

Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the **Kronecker delta**, and Q_{ij} is the **quadrupole moment** of the charge distribution. Notice the hierarchy:

$$V_{\text{mon}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}; \qquad V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{\boldsymbol{r}}_i p_i}{r^2}; \qquad V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j Q_{ij}}{r^3}; \quad \ldots$$

The monopole moment (Q) is a scalar, the dipole moment (p) is a vector, the quadrupole moment (Q_{ij}) is a second-rank tensor, and so on.

(b) [5 pts] Find all nine components of Q_{ij} for the configuration in Fig. 3.30 (assume the square has side a and lies in the xy plane, centered at the origin).

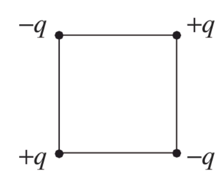

FIGURE 3.30

(c) [5 pts] Show that the quadrupole moment is independent of origin if the monopole and dipole moments both vanish. (This works all the way up the hierarchy—the lowest nonzero multipole moment is always independent of origin.)

(d) [5 pts] How would you define the **octopole moment**? Express the octopole term in the multipole expansion in terms of the octopole moment.

— PART B (To be handed in separately from Part A) —

PROBLEM 3: NORMALIZING THE TRACELESS SYMMETRIC TENSOR REPRESENTATION OF THE LEGENDRE POLYNOMIALS *(15 points)*

As we saw in lecture and in the lecture notes, in the case of azimuthal symmetry, the general solution to Laplace's equation can be expanded as

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)} r^{\ell} + \frac{C'^{(\ell)}}{r^{\ell+1}} \right) \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ , \tag{3.1}$$

where the coefficients $C^{(\ell)}$ and $C'^{(\ell)}$ are arbitrary, and $\{\ xxx\ \}_{\rm TS}$ denotes the traceless symmetric part of xxx. The functions

$$F_\ell(\hat{\boldsymbol{n}}) \equiv \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \tag{3.2}$$

are proportional to the standard Legendre polynomial functions $P_\ell(\cos\theta)$. That is,

$$P_\ell(\cos\theta) = N(\ell) F_\ell(\hat{\boldsymbol{n}}) \ , \tag{3.3}$$

where θ is the polar angle of $\hat{\boldsymbol{n}}$, i.e., the angle between $\hat{\boldsymbol{n}}$ and the z axis. The normalization of the Legendre polynomials is fixed by the convention that

$$P_\ell(\cos\theta{=}1) = 1 \ . \tag{3.4}$$

Note that $\cos\theta = 1$ implies that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$, so the normalization condition implies that

$$N(\ell) = \frac{1}{F_\ell(\hat{\boldsymbol{z}})} \ . \tag{3.5}$$

In lecture and in Lecture Notes 2, Section 8, we will show that for any unit vector $\hat{\boldsymbol{n}}$,

$$\{\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{n}}_{i_\ell} = \frac{\ell}{2\ell - 1} \{\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_{\ell-1}} \,\}_{\rm TS} \ . \tag{3.6}$$

Use this relation to show that

$$N(\ell) = \frac{2\ell - 1)!!}{\ell!} \ . \tag{3.7}$$

The identity

$$(2\ell - 1)!! = \frac{(2\ell)!}{2^\ell \ell!} \tag{3.8}$$

is sometimes used to reexpress the normalization as

$$N(\ell) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \ . \tag{3.9}$$

PROBLEM 4: CONCENTRIC SPHERICAL SHELLS OF CHARGE *(25 points)*

Two concentric spherical shells, of radii a and b, with $a < b$, are centered at the origin of our coordinate system. The space is otherwise empty. The potential on the inner shell (at $r = a$) is V_a, while the potential on the outer shell is V_b. We define the potential at infinity to be zero.

(a) [5 pts] Find the potential for all values of the radial coordinate r.

(b) [10 pts] Find the total charge on each of the two shells, and find the total electrostatic energy of the system.

(c) [10 pts] Now suppose that the potential on the outer shell is held at

$$V(r = b, \theta, \phi) = V_0 \sin^2\theta \sin\phi \cos\phi \ ,$$

and the potential on the inner shell is held at zero. For this case, find the potential everywhere. Note that we are using the usual spherical polar coordinates, related to the Cartesian coordinates by

$$\begin{aligned} x &= r \sin\theta \cos\phi \ , \\ y &= r \sin\theta \sin\phi \ , \\ z &= r \cos\theta \ . \end{aligned}$$

(*Hint:* this problem can be solved using traceless symmetric tensors, or if you prefer you can use standard spherical harmonics. A table of the low-ℓ Legendre polynomials and spherical harmonics is included at the end of the problem set. You may also wish to know that $2 \sin\phi \cos\phi = \sin 2\phi$.)

Table of Legendre Polynomials $P_\ell(x)$:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Table of Spherical Harmonics $Y_{\ell m}(\theta, \phi)$:

$$\ell = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\ell = 1 \qquad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\,\sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}}\,\cos\theta \end{cases}$$

$$\ell = 2 \qquad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\,\sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\,\sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \dfrac{1}{2}\sqrt{\dfrac{5}{4\pi}}\,(3\cos^2\theta - 1) \end{cases}$$

$$\ell = 3 \qquad \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}}\,\sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}}\,\sin^2\theta\cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}}\,\sin\theta(5\cos^2\theta - 1)e^{i\phi} \\ Y_{30} = \dfrac{1}{2}\sqrt{\dfrac{7}{4\pi}}\,(5\cos^3\theta - 3\cos\theta) \end{cases}$$

For $m < 0$, $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$