MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 7, 2019
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PRACTICE PROBLEMS FOR QUIZ 2

PROBLEM 1: MULTIPOLE MOMENTS OF A THIN INSULATING ROD

Griffiths, Problem 3.46 (p. 163): A thin insulating rod, running from $z = -a$ to $z = +a$, carries the indicated line charges. In each case, find the leading [nonzero] term in the multipole expansion of the potential:

(a) $\lambda = k\cos(\pi z/2a)$,

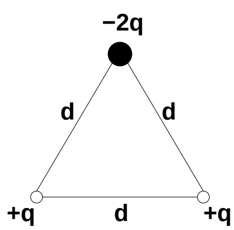
(b) $\lambda = k\sin(\pi z/a)$,

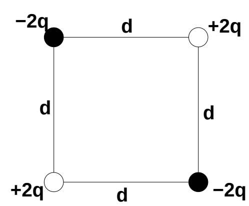
(c) $\lambda = k\cos(\pi z/a)$,

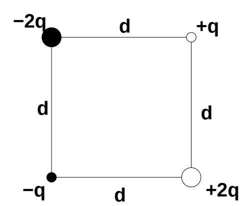
where k is a constant.

PROBLEM 2: DIPOLE MOMENTS OF SIMPLE CONFIGURATIONS

Compute the electric dipole moments of the following three charge configurations:







PROBLEM 3: THICK SPHERICAL SHELL WITH NONUNIFORM POLARIZATION

Griffiths Problem 4.15 (p. 183): A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\vec{\boldsymbol{P}}(\vec{\boldsymbol{r}}) = \frac{k}{r}\hat{\boldsymbol{r}} \ ,$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no free charge in the problem.) Find the electric field in all three regions by two different methods:

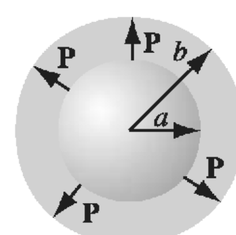


FIGURE 4.18

(a) Locate all the bound charge, and use Gauss's law,

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{a}} = \frac{1}{\epsilon_0} Q_{\text{enc}} \ ,$$

to calculate the field it produces.

(b) Use

$$\oint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{a}} = Q_{f,\mathrm{enc}}$$

to find $\vec{\boldsymbol{D}}$, and then get $\vec{\boldsymbol{E}}$ from

$$\vec{\boldsymbol{D}} \equiv \epsilon_0 \vec{\boldsymbol{E}} + \vec{\boldsymbol{P}} \ .$$

[Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]

PROBLEM 4: CAPACITANCE OF A COAXIAL CABLE

Griffiths Problem 4.21 (p. 192): A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable. [Note added by Alan Guth: I found the diagram confusing. The outer concentric copper tube lies outside the radius c, and is not shown.]

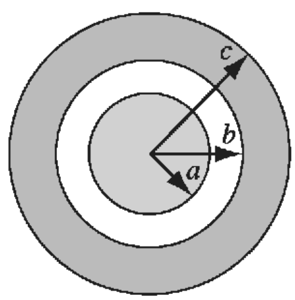


FIGURE 4.26

PROBLEM 5: SPHERE WITH VARIABLE DIELECTRIC CONSTANT

This problem appeared on Quiz 2 in 2012, where it was worth 35 points.

A dielectric sphere of radius R has variable permittivity, so the permittivity throughout space is described by

$$\epsilon(r) = \begin{cases} \epsilon_0 (R/r)^2 & \text{if } r < R \\ \epsilon_0 \ , & \text{if } r > R \ . \end{cases} \tag{5.1}$$

There are no free charges anywhere in this problem. The sphere is embedded in a constant external electric field $\vec{E} = E_0 \hat{\boldsymbol{z}}$, which means that $V(\vec{\boldsymbol{r}}) \equiv -E_0 r \cos\theta$ for $r \gg R$.

(a) *(9 points)* Show that $V(\vec{r})$ obeys the differential equation

$$\nabla^2 V + \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}r}\frac{\partial V}{\partial r} = 0\ . \tag{5.2}$$

(b) *(4 points)* Explain why the solution can be written as

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r)\{\,\hat{\boldsymbol{z}}_{i_1}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\mathrm{TS}}\,\hat{\boldsymbol{r}}_{i_1}\ldots\hat{\boldsymbol{r}}_{i_\ell}\ , \tag{5.3a}$$

or equivalently (your choice)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r) P_\ell(\cos\theta)\ , \tag{5.3b}$$

where $\{\ldots\}_{\mathrm{TS}}$ denotes the traceless symmetric part of $\ldots$, and $P_\ell(\cos\theta)$ is the Legendre polynomial. (Your answer here should depend only on general mathematical principles, and should not rely on the explicit solution that you will find in parts (c) and (d).)

(c) *(9 points)* Derive the ordinary differential equation obeyed by $V_\ell(r)$ (separately for $r < R$ and $r > R$) and give its two independent solutions in each region. *Hint:* they are powers of r. You may want to know that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_\ell(\cos\theta)}{\mathrm{d}\theta}\right) = -\ell(\ell+1)\sin\theta P_\ell(\cos\theta)\ . \tag{5.4}$$

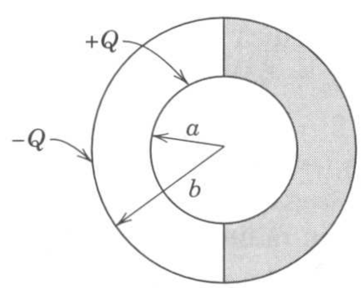
The relevant formulas for the traceless symmetric tensor formalism are in the formula sheets.

(d) *(9 points)* Using appropriate boundary conditions on $V(r,\theta)$ at $r = 0$, $r = R$, and $r \to \infty$, determine $V(r,\theta)$ for $r < R$ and $r > R$.

(e) *(4 points)* What is the net dipole moment of the polarized sphere?

PROBLEM 6: CONCENTRIC SPHERICAL CAPACITOR, HALF-FILLED WITH DIELECTRIC *(15 points)*

Problem 4.10 of J.D. Jackson, **Classical Electrodynamics**, *3rd Edition (John Wiley & Sons, 1999).*



Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\epsilon_r = \epsilon/\epsilon_0$), as shown in the figure.

(a) [5 pts] Find the electric field everywhere between the spheres.

(b) [5 pts] Calculate the surface charge distribution on the inner sphere. By "on the inner sphere," we mean to include charges on the inner surface of the dielectric and on the surface of the inner conducting sphere.

(c) [5 pts] Calculate the polarization charge density [i.e., the bound charge density] induced on the surface of the dielectric at $r = a$.

PROBLEM 7: THE ELECTRIC FIELD OF AN IDEAL QUADRUPOLE

This problem was a short-answer queston on the Final Exam of 2012, where it was worth 7 points.

The potential energy function for an ideal (static) electric quadrupole is given by

$$V(\vec{\boldsymbol{r}}) = \frac{1}{8\pi\epsilon_0} Q_{ij} \frac{\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} \ , \tag{7.1}$$

where Q_{ij} is the quadrupole moment tensor, which is traceless and symmetric. Use Poisson's equation to find the charge density $\rho(\vec{\boldsymbol{r}})$ for an ideal quadrupole. Your answer should be expressed in terms of Q_{ij}, and some kind of derivative of a delta function. (*Hint:* Start by using the identity

$$-\partial_i \partial_j \left(\frac{1}{r}\right) = \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} + \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\vec{r}) \tag{7.2}$$

from the formula sheet to express $\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j / r^3$ in terms of other quantities.)

PROBLEM 8: MANIPULATING TRACELESS SYMMETRIC TENSORS

Suppose that $S_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor, and A_{jk} is a symmetric tensor. Show that $A_{i_{\ell+1}, i_{\ell+2}} \underset{i_1 \ldots i_{\ell+2}}{\mathrm{Sym}} \left[S_{i_1 \ldots i_\ell} \delta_{i_{\ell+1}, i_{\ell+2}} \right]$ can be written as

$$\begin{aligned} A_{i_{\ell+1}, i_{\ell+2}} \underset{i_1 \ldots i_{\ell+2}}{\mathrm{Sym}} \left[S_{i_1 \ldots i_\ell} \delta_{i_{\ell+1}, i_{\ell+2}} \right] &= C(\ell)\, \mathrm{Tr}(A) S_{i_1 \ldots i_\ell} + D(\ell) \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[S_{i_1 \ldots i_{\ell-1} j} A_{j, i_\ell} \right] \\ &\quad + E(\ell) \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[S_{i_1 \ldots i_{\ell-2} jk} A_{jk} \delta_{i_{\ell-1}, i_\ell} \right] \ , \end{aligned}$$

where $C(\ell)$, $D(\ell)$, and $E(\ell)$ are functions that you are asked to evaluate. Notation: here

$$\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} [xxx]$$

means to symmetrize xxx in the indices $i_1 \ldots i_\ell$, repeated indices are summed from 1 to 3, $\mathrm{Tr}(A) \equiv A_{ii}$, and δ_{ij} is the Kronecker δ-function. Hint: write

$$\begin{aligned} & A_{i_{\ell+1}, i_{\ell+2}} \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[S_{i_1 \ldots i_\ell} \delta_{i_{\ell+1}, i_{\ell+2}} \right] \\ & \quad = \frac{1}{(\ell+2)!} A_{i_{\ell+1}, i_{\ell+2}} \sum_{\substack{\text{all } (\ell+2)! \text{ index} \\ \text{orderings of } i_1 \ldots i_{\ell+2}}} \left[S_{i_1 \ldots i_\ell} \delta_{i_{\ell+1}, i_{\ell+2}} \right] \ , \end{aligned}$$

and then consider the various ways that the indices $i_{\ell+1}$ and $i_{\ell+2}$ can be positioned in the different index orderings (permutations) of $\left[S_{i_1 \ldots i_\ell} \delta_{i_{\ell+1}, i_{\ell+2}} \right]$, as was done several times in lecture and in Lecture Notes 2, 3, and 4.

PROBLEM 9: PRACTICE WITH THE LAW OF BIOT AND SAVART

Griffiths, Problem 5.9 (p. 228): Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23.

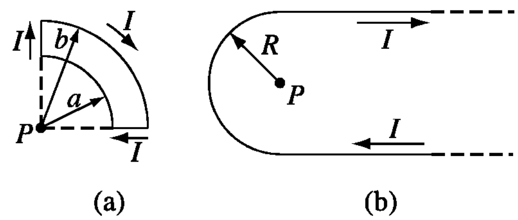


FIGURE 5.23

PROBLEM 10: MORE PRACTICE WITH THE LAW OF BIOT AND SAVART

Griffiths, Problem 5.10 (p. 228):

(a) Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).

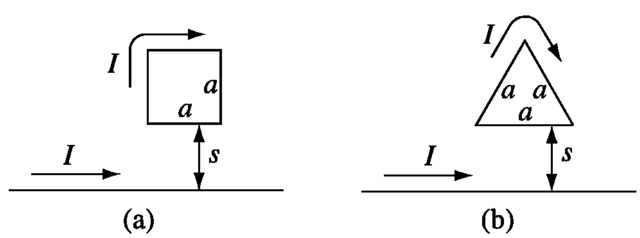


FIGURE 5.24