MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 16, 2019

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PROBLEM SET 8

Revised November 16, 2019 *

DUE DATE: Friday, November 22, 2019, at 4:45 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

READING ASSIGNMENT: Griffiths: Sections 5.3 (*The Divergence and Curl of B*), 5.4 (*Magnetic Vector Potential*), and Chapter 6 (*Magnetic Fields in Matter*).

CREDIT: This problem set has 105 points of credit, plus the opportunity to earn 45 points of extra credit.

— PART A —

PROBLEM 1: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

This problem was carried over from Problem Set 7.

Griffiths Problem 5.60 (p. 264). In part (b), you are expected to use Eq. (5.93), from Problem 5.59, without proving it in any way. In part (c), where you are asked to find the approximate vector potential, you are expected to find it in the dipole approximation.

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) [5 pts] What is the magnetic dipole moment of the sphere?

(b) [5 pts] Find the average magnetic field within the sphere (see Prob. 5.59).

(c) [5 pts] Find the approximate vector potential at a point (r, θ) where $r \gg R$.

(d) [5 pts] Find the *exact* potential at a point (r, θ) outside the sphere, and check that it is consistent with (c). [*Hint*: refer to Ex. 5.11.]

(e) [5 pts] Find the magnetic field at a point (r, θ) inside the sphere (Prob. 5.30), and check that it is consistent with (b).

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* Added labels for Part A and Part B.

PROBLEM 2: THE COMPLETE MAGNETIC FIELD OF A MAGNETIC DIPOLE: *(10 points)*

This problem was carried over from Problem Set 7.

In Problem Set 6 you proved the identity

$$\boxed{\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\vec{\boldsymbol{r}}) \ .} \tag{2.1}$$

Using this identity, show that the vector potential for a magnetic dipole,

$$\vec{\boldsymbol{A}}_{\rm dip}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{\vec{\boldsymbol{m}} \times \hat{\boldsymbol{r}}}{r^2} \ , \tag{2.2}$$

can be used to find immediately that the magnetic field of a magnetic dipole is given by

$$\vec{\boldsymbol{B}}_{\rm dip}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \frac{\mu_0}{4\pi} \frac{3(\vec{\boldsymbol{m}} \cdot \hat{\boldsymbol{r}})\, \hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{r^3} + \frac{2\mu_0}{3} \vec{\boldsymbol{m}}\, \delta^3(\vec{\boldsymbol{r}}) \ . \tag{2.3}$$

PROBLEM 3: SQUARE CURRENT LOOP ON AXIS: BIOT-SAVART AND THE MAGNETIC DIPOLE APPROXIMATION *(15 points)*

This problem was carried over from Problem Set 7.

Griffiths Problem 5.36 (p. 255).

Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.

— PART B (To be handed in separately from Part A) —

PROBLEM 4: A BAR MAGNET IN THE SHAPE OF A RIGHT CIRCULAR CYLINDER *(20 points)*

This problem is based on Griffiths Problem 6.9 (p. 277) and Jackson Problem 5.19 (p. 230).

A bar magnet is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0 uniform throughout its volume and parallel to its axis.

(a) [5 pts] Calculate $\vec{H}$ and $\vec{B}$ at all points on the axis of the cylinder, both inside and outside the magnet. Use a coordinate system in which the z axis is the axis of the cylinder, with the cylinder extending from $z = -\frac{1}{2}L$ to $z = \frac{1}{2}L$.

(b) [5 pts] Sketch in a plot the quantities $|\vec{B}|/\mu_0 M_0$ and $|\vec{H}|/M_0$ as a function of z/a, for $L/a = 5$.

(c) [5 pts] Find the bound current.

(d) [5 pts] The $\vec{B}$ field far away from the magnet is approximately that of a magnetic dipole. What is the dipole moment?

PROBLEM 5: CURRENT TRAVELING ON A LONG STRAIGHT WIRE MADE OF A MATERIAL WITH LINEAR MAGNETIZATION *(15 points)*

Griffiths Problem 6.17 (p. 287).

A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the net bound current flowing down the wire?

PROBLEM 6: BAR SLIDING ON TWO RAILS IN A UNIFORM MAGNETIC FIELD *(20 points)*

Griffiths Problem 7.7 (p. 310).

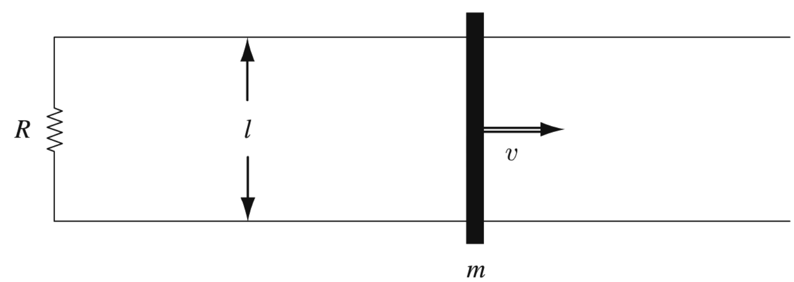

FIGURE 7.17

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance ℓ apart (Fig. 7.17). A resistor R is connected across the rails, and a uniform magnetic field B, pointing into the page, fills the entire region.

(a) [5 pts] If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) [5 pts] What is the magnetic force on the bar? In what direction?

(c) [5 pts] If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) [5 pts] The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor, after an arbitrarily long time, is exactly $\frac{1}{2}mv_0^2$.

PROBLEM 7: VECTOR POTENTIAL FOR A UNIFORM $\vec{B}$ FIELD *(10 points extra credit)*

This problem was carried over from Problem Set 7.

Griffiths Problem 5.25 (p. 248).

If $\vec{\boldsymbol{B}}$ is *uniform*, show that $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = -\frac{1}{2}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{B}})$ works. That is, check that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$ and $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$. Is this result unique, or are there other functions with the same divergence and curl?

PROBLEM 8: DONUT-SHAPED MAGNETS ON A VERTICAL ROD *(15 points extra credit)*

Griffiths Problem 6.23 (p. 293). When Griffiths speaks of the ratio of the two heights, he is talking about the ratio x/y, where x and y are defined in the diagram.

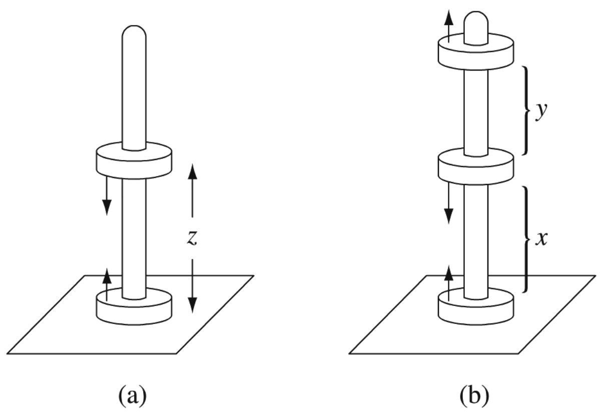

FIGURE 6.31

A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Fig. 6.31). Treat the magnets as dipoles, with mass m_d and dipole moment $\vec{\boldsymbol{m}}$.

(a) [8 pts] If you put two back-to-back magnets on the rod, the upper one will "float"—the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?

(b) [7 pts] If you now add a *third* magnet (parallel to the bottom one), what is the *ratio* of the two heights? (Determine the actual number, to three significant digits.)

[Answer: (a) $[3\mu_0 m^2/2\pi m_d g]^{1/4}$; (b) 0.8501]

PROBLEM 9: THE MAGNETIC DIPOLE MOMENT OF A CURRENT LOOP *(10 points extra credit)*

When I discussed the magnetic dipole moment in lecture, I showed that the dipole moment of a current loop P can be written as

$$\vec{\boldsymbol{m}} = \frac{1}{2} I \int_P \vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{\ell}} \ , \tag{9.1}$$

where I is the current through the loop. I also asserted, without taking the time to prove it, that $\vec{\boldsymbol{m}}$ can be rewritten as

$$\vec{\boldsymbol{m}} = I \int_S \mathrm{d}\vec{\boldsymbol{a}} \ , \tag{9.2}$$

where S is any surface that spans the loop P. Your job in this problem is to justify Eq. (9.2). (*Hint:* The goal is to use Stokes' theorem to convert the line integral into a surface integral. One method is to start by writing the i-component of $\vec{\boldsymbol{m}}$ as

$$m_i = \frac{1}{2} I \int_P \hat{\boldsymbol{e}}_{\boldsymbol{i}} \ \cdot (\vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{\ell}}) \ , \tag{9.3}$$

which can then be rewritten as

$$m_i = \frac{1}{2} I \int_P (\hat{\boldsymbol{e}}_{\boldsymbol{i}} \times \vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{\ell}} \tag{9.4}$$

by using the cyclic property of the triple product, i.e., that

$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}}) \ . \tag{9.5}$$

Then apply Stokes theorem and simplify.)

PROBLEM 10: CURRENT LOOPS AND NEWTON'S THIRD LAW *(10 points extra credit)*

Griffiths Problem 5.50 (p. 259). There is nothing wrong with the hint that Griffiths gives, but I would give a slightly different hint: recall that

$$\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3} = -\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}} \frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ . \tag{10.1}$$

Griffiths' text:

Magnetostatics treats the "source current" (the one that sets up the field) and the "recipient current" (the one that experiences the force) so asymmetrically that it is by no means obvious that the magnetic force between two current loops is consistent with Newton's third law. Show, starting with the Biot-Savart law (Eq. 5.34) and the Lorentz force law (Eq. 5.16), that the force on loop 2 due to loop 1 (Fig. 5.61) can be written as

$$\vec{\boldsymbol{F}}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \mathrm{d}\vec{\boldsymbol{\ell}}_1 \cdot \mathrm{d}\vec{\boldsymbol{\ell}}_2 \ . \tag{10.2}$$

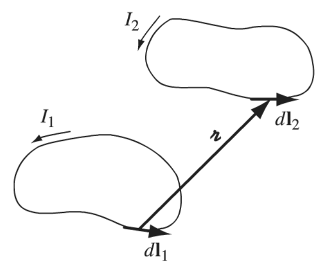

FIGURE 5.61

In this form, it is clear that $\vec{\boldsymbol{F}}_2 = -\vec{\boldsymbol{F}}_1$, since $\vec{\boldsymbol{\mathscr{r}}}$ changes direction when the roles of 1 and 2 are interchanged. (If you seem to be getting an "extra" term, it will help to note that $\mathrm{d}\vec{\boldsymbol{\ell}}_2 \cdot \vec{\boldsymbol{\mathscr{r}}} = \mathrm{d}\mathscr{r}$.)