MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 13, 2019
Prof. Alan Guth

QUIZ 2

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__________________________________ 2:00 pm OR 3:00 pm
Your Name — Recitation

1. There are **FOUR** problems — be sure not to miss any! This exam is worth 100 points. The problems are **not** equally weighted.
2. A separate formula sheet will be distributed.
3. The exam is **closed book** and **closed notes**. Cell phones and calculators will not be needed, and are not allowed.
4. Write all of your responses in this booklet. Extra pages are provided at the end of this booklet, but if you need more, please ask.
5. **Please** raise your hand and **ask** if there is anything about a question that you find unclear or confusing. Perhaps you have simply misread something, but it is possible that there is an error we need to correct. In either case, it would be good to know for certain.

PROBLEM 1: SHORT ANSWER PROBLEMS *(30 points)*

(a) [10 pts] Consider a physical dipole, with a charge q at $(0,0,h)$ and a charge $-q$ at $(0,0,-h)$. As we mentioned in class, a physical dipole has nonzero values for other moments besides the dipole. Calculate the dipole, quadrupole, and octopole moments of this configuration, defined by

$$\begin{aligned} p_i &\equiv \int \mathrm{d}^3x \rho(\vec{\boldsymbol{r}}) x_i \ , \\ Q_{ij} &\equiv \frac{1}{2} \int \mathrm{d}^3x \rho(\vec{\boldsymbol{r}}) \left[3x_i x_j - \delta_{ij} \left|\vec{\boldsymbol{r}}\right|^2 \right] \ , \\ Q_{ijk} &\equiv \frac{1}{2} \int \mathrm{d}^3x \rho(\vec{\boldsymbol{r}}) \left[5x_i x_j x_k - (x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}) \left|\vec{\boldsymbol{r}}\right|^2 \right] \ , \end{aligned} \tag{1.1}$$

(Note that p_i, Q_{ij}, and Q_{ijk} are equal to $C_i^{(1)}$, $C_{ij}^{(2)}$, and $C_{ijk}^{(3)}$, respectively, as defined in section 13(b) of the 11/11/19 revised formula sheets.)

(b) [10 pts] Let $\vec{\boldsymbol{a}} = a_i \hat{\boldsymbol{e}}_i$, $\vec{\boldsymbol{b}} = b_i \hat{\boldsymbol{e}}_i$, and $\vec{\boldsymbol{c}} = c_i \hat{\boldsymbol{e}}_i$ be three arbitrary nonzero vectors, let α and β be arbitrary nonzero real constants, and let $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{n}}$. For each of the following functions $V(\vec{\boldsymbol{r}})$, state whether it is **always, never,** or **conditionally** a possible expression for the electric potential in empty space. If it is conditionally such a potential, state the conditions on the relevant parameters $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{b}}$, $\vec{\boldsymbol{c}}$, α and/or β for which the expression would be a possible expression for the electric potential in empty space. (2 points each, no explanation needed, and no partial credit, except if the answer is conditional.)

(i) $V(\vec{\boldsymbol{r}}) = \sin\alpha x \, \sin\alpha y \, \cos 2\alpha z$.

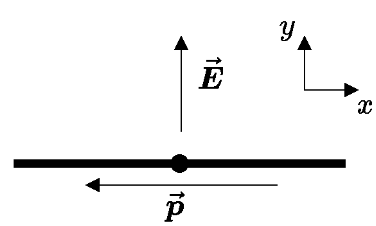
(ii) $V(\vec{\boldsymbol{r}}) = (a_i + 2\beta\hat{\boldsymbol{z}}_i)\,\hat{\boldsymbol{n}}_i/r^2$.

(iii) $V(\vec{\boldsymbol{r}}) = \left[a_i b_j - \frac{1}{3}(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}})\delta_{ij} \right] \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j / r^3$.

(iv) $V(\vec{\boldsymbol{r}}) = [a_i b_j]\, \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j / r^3$.

(v) $V(\vec{\boldsymbol{r}}) = [a_i b_j c_k]\, \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j \hat{\boldsymbol{n}}_k / r^4$.

(c) [10 pts] A rod is pivoted frictionlessly about its center, so that it can turn freely but cannot translate. It has an electric dipole moment of magnitude p, along the rod, and moment of inertia I for rotations about the pivot. The rod is oriented with the dipole in the $-\hat{\boldsymbol{x}}$ direction, initially at rest, in the presence of a uniform electric field $\vec{\boldsymbol{E}} = E_0\hat{\boldsymbol{y}}$. When the rod is released, what is the maximum angular speed $\omega = |\vec{\boldsymbol{\omega}}|$ that the rod experiences. (We have not talked about radiation, but in this problem you should neglect any energy that might be lost to radiation.)



PROBLEM 2: CHARGE DENSITY AND FORCE FOR AN ELECTRIC QUADRUPOLE *(20 points)*

Part (a) of this problem is a version of Practice Problem 7, but part (b) is new.

As you can see from the (11/11/19 revised) formula sheet, the electric potential of an ideal electric quadrupole at the origin is given by

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j}{r^3} Q_{ij} \ ,$$

where Q_{ij} is a traceless symmetric tensor. (Please note that the definition of Q_{ij} used here is half as large as the definition used in Practice Problem 7, so do not expect your answer to exactly match that of Practice Problem 7.)

(a) [10 pts] By calculating $\nabla^2 V$, find the charge density $\rho(\vec{\boldsymbol{r}})$ for an ideal electric quadrupole. Your answer should have the form

$$\rho(\vec{\boldsymbol{r}}) = A\, Q_{ij}\, \partial_i \partial_j \delta^3(\vec{\boldsymbol{r}}) \ , \tag{2.1}$$

where

$$\partial_i \equiv \frac{\partial}{\partial x_i}$$

and A is a constant that you are expected to find. [*Hint:* Remember the identity

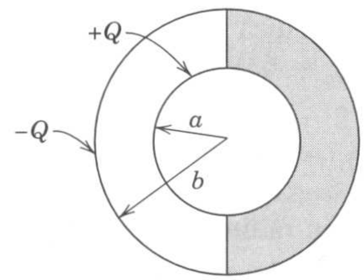
$$\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3} \delta_{ij}\, \delta^3(\vec{r}) \ .$$

Using this identity, try to express $V(\vec{\boldsymbol{r}})$ in terms of $\partial_i \partial_j (1/r)$. In this form it should be relatively easy to apply ∇^2.]

(b) [10 pts] Using Eq. (2.1), whether or not you have found the value of A, find F_k, the k component of the force on the quadrupole, in the presence of a (non-uniform) electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$.

PROBLEM 3: CONCENTRIC SPHERICAL CAPACITOR, HALF-FILLED WITH DIELECTRIC *(25 points)*

Problem 4.10 of J.D. Jackson, **Classical Electrodynamics***, 3rd Edition (John Wiley & Sons, 1999). It was also Practice Problem for Quiz 2, Problem 6. The wording here has been modified slightly in an attempt to improve the clarity.**



Two concentric conducting spherical surfaces of radii a and b, with $a < b$, carry charges $\pm Q$. The empty space between the spherical surfaces is half-filled by a hemispherical shell of dielectric (of dielectric constant $\epsilon_r = \epsilon/\epsilon_0$), as shown in the figure.

(a) [10 pts] Find the electric field everywhere between the spherical surfaces.

(b) [5 pts] Calculate the surface charge distribution on the inner spherical surface, at $r = a$. That is, for the left half where there is no dielectric, you should calculate the surface charge density on the inner conducting spherical surface. For the right half, you should calculate the sum of the surface charge densities on the inner surface of the dielectric and on the inner conducting spherical surface.

(c) [10 pts] Calculate the bound surface charge density induced on the surface of the dielectric at $r = a$.

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* I reworded the problem to conform to Griffiths's wonderful description of how physicists talk about spheres, with which I strongly concur: "In proper mathematical jargon, 'sphere' denotes the surface, and 'ball' the volume it encloses. But physicists are (as usual) sloppy about this sort of thing, and I use the word 'sphere' for both the surface and the volume. Where the meaning is not clear from the context, I will write 'spherical surface' or 'spherical volume.' The language police tell me that the former is redundant and the latter an oxymoron, but a poll of my physics colleagues reveals that this is (for us) the standard usage."

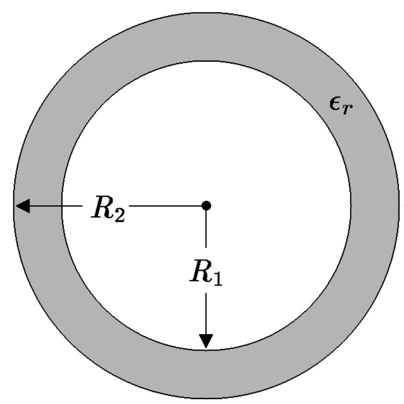
PROBLEM 4: ELECTRIC POTENTIAL DETERMINED BY A BOUNDARY CONDITION ON A SPHERE *(25 points)*

(a) [5 pts] Consider a sphere of radius R_1, located at the origin of a spherical coordinate system, as described in Sec. 5 of the formula sheet. The potential on the surface of the sphere is fixed to be

$$V_a(R_1, \theta, \phi) = V_0 \left(7\cos^2\theta - 1\right) \sin^2\theta \, \cos 2\phi \ , \tag{4.1}$$

where V_0 is a constant. The region outside the sphere is empty. Find the potential $V(r, \theta, \phi)$ everywhere outside the sphere. (You may find it helpful to know that $\cos 2\phi = \cos^2\phi - \sin^2\phi$.)

(b) [10 pts] Now suppose that the same sphere, held at the same potential, is surrounded by a region of dielectric with dielectric constant ϵ_r, extending out to a radius R_2, beyond which there is only vacuum. Again we wish to find the potential everywhere outside the sphere of radius R_1, but to avoid some algebra you are not asked to completely determine the answer. Instead, for each of the two regions $R_1 < r < R_2$ and $R_2 < r < \infty$, write expressions for $V(r, \theta, \phi)$ which include some number of undetermined constants. Then write enough equations so that all of the constants could be determined. You need not solve these equations.



(c) [10 pts] Now consider a single sphere of radius R_1, with no dielectric, as in part (a). This time, however, the sphere is held at potential

$$V_c(R_1, \theta, \phi) = V_0 \left(3\cos^2\theta - 1\right) \sin\theta \, \sin\phi \ , \tag{4.2}$$

where again V_0 is a constant, and again the region outside is empty. For this case, find the potential $V(r, \theta, \phi)$ everywhere outside the sphere.

Problem	**Maximum**	**Score**	**Initials**
1	30		
2	20		
3	25		
4	25		
TOTAL	100		