MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 4, 2019
Prof. Alan Guth

PROBLEM SET 1

DUE DATE: Friday, September 13, 2019. Due at 4:45 pm in the 8.07 homework boxes. The homework boxes are at the intersection of buildings 8 and 16, on the third floor of bldg. 8 and the 4th floor of bldg. 16. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B.

READING ASSIGNMENT: Chapter 1 of Griffiths: *Vector Analysis*.

CREDIT: This problem set has 95 points of credit, plus the option of earning 15 points extra credit. (The Course Information description, on the General Info tab of the webpage, describes how extra credit grades will be handled.)

— PART A —

PROBLEM 1: VECTOR IDENTITIES INVOLVING CROSS PRODUCTS *(20 points)*

In manipulating cross products, it is useful to define ϵ_{ijk} (the Levi-Civita antisymmetric symbol) to be:

$$\epsilon_{ijk} = \begin{cases} +1 \text{ if } ijk = (123, 231, 312) \\ -1 \text{ if } ijk = (213, 321, 132) \\ \quad 0 \text{ otherwise} . \end{cases} \tag{1.1}$$

That is, ϵ_{ijk} is nonzero only when all three indices are different; it is then equal to +1 if ijk is a cyclic permutation of 123, and -1 if ijk is an anti-cyclic permutation. Note that ϵ_{ijk} is totally antisymmetric, in the sense that it changes sign if any two indices are interchanged:

$$\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij} \ . \tag{1.2}$$

With this definition, the i^{th} component of the cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as

$$\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk} A_j B_k \ , \tag{1.3}$$

where we have used the summation convention that repeated indices are summed over (that is, $\epsilon_{ijk} A_{jl} B_{km} \equiv \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_{jl} B_{km}$). For the rest of this problem set, we will always assume that this summation convention is implied, unless explicitly stated otherwise.

(a) [4 pts] From the definition in Eq. (1.1), show that

$$\epsilon_{ijk} \epsilon_{inm} = \delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn} \ , \tag{1.4}$$

where of course there is an implied sum over the i index in Eq. (1.4), but the indices j, k, n, and m are free. We are physicists, so we have no formal rules for what constitutes a mathematical proof, but give a solid argument.

(b) [4 pts] Using Eqs. (1.3) and (1.4), show that for any vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B}) \ . \tag{1.5}$$

(c) [4 pts] Using Eqs. (1.3) and (1.4), show that for any vector fields $\vec{A}(\vec{r})$ and $\vec{B}(\vec{r})$,

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \ . \tag{1.6}$$

(d) [4 pts] Using Eqs. (1.3) and (1.4), show that for any vector $\vec{A}$,

$$\vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2}\vec{\nabla} A^2 - \left(\vec{A} \cdot \vec{\nabla}\right)\vec{A} \ . \tag{1.7}$$

Note that, in Cartesian coordinates,

$$\vec{A} \cdot \vec{\nabla} = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \ .$$

(e) [4 pts] Using Eqs. (1.3) and (1.4), show that for any vectors $\vec{A}$ and $\vec{B}$,

$$\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot \vec{\nabla}\right)\vec{A} - \left(\vec{A} \cdot \vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\vec{\nabla} \cdot \vec{A}\right) \ . \tag{1.8}$$

PROBLEM 2: TRIPLE CROSS PRODUCTS *(10 points)*

(a) [5 pts] Griffiths Problem 1.2 (p. 4): Is the cross product associative?

$$(\vec{A} \times \vec{B}) \times C \stackrel{?}{=} \vec{A} \times (\vec{B} \times \vec{C}) \ . \qquad 2.1$$

If so, prove it; if not, provide a counterexample (the simpler the better).

(b) [5 pts] Griffiths Problem 1.6 (p. 8): Prove that

$$[\vec{A} \times (\vec{B} \times \vec{C})] + [\vec{B} \times (\vec{C} \times \vec{A})] + [\vec{C} \times (\vec{A} \times \vec{B})] = 0 \ . \qquad 2.2$$

Under what conditions does $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$?

PROBLEM 3: PROPERTIES OF THE ROTATION MATRIX $\boldsymbol{R}$ *(15 points)*

Griffiths Eq. (1.31), p. 11, is

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j \ ,$$

where R_{ij} are components of the rotation matrix that transforms from the unbarred to the barred coordinate system, and A_i are components of some arbitrary vector $\vec{A}$. If we use the convention that repeated indices are summed over, then this can be written as

$$\bar{A}_i = R_{ij} A_j \ . \tag{3.1}$$

(a) [5 pts] Show that the elements (R_{ij}) of the three-dimensional rotation matrix must satisfy the constraint

$$R_{ij} R_{ik} = \delta_{jk} \tag{3.2}$$

in order to preserve the length of $\vec{A}$ for all $\vec{A}$. Matrices satisfying Eq. (3.2) are called *orthogonal*. Here δ_{jk} is the Kronecker delta (δ_{jk} is 1 if $j = k$ and 0 otherwise), and we use the summation convention described above. (Note that Eq. (3.2) can be written simply using matrix notation. If R_{ij} is the (i, j) component of the matrix $\boldsymbol{R}$, then Eq. (3.2) can be written as $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix.)

(b) [5 pts] Using the orthogonality constraint (3.2), show that

$$A_i = R_{ji} \bar{A}_j \ . \tag{3.3}$$

Note that we can now show that $R_{ji} R_{ki} = \delta_{jk}$ using this relation, in a manner similar to the procedure in (a) (you do not have to show this).

(c) [5 pts] Using the chain rule for partial differentiation and the results of (b), show that if f is scalar function of $\vec{r} \equiv (x_1, x_2, x_3)$, then $\vec{\nabla} f(\vec{r})$ transforms as a vector; i.e., show that if

$$\bar{f}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = f(x_1, x_2, x_3) \ , \tag{3.4}$$

where $\bar{x}_i = R_{ij} x_j$, then

$$\frac{\partial \bar{f}}{\partial \bar{x}_i} = R_{ij} \frac{\partial f}{\partial x_j} \ . \tag{3.5}$$

PROBLEM 4: USE OF THE GRADIENT *(10 points)*

(a) Griffiths Problem 1.12 (p. 15): The height of a certain hill (in feet) is given by

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12) \ ,$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(i) [3 pts] Where is the top of the hill located?

(ii) [2 pts] How high is the hill?

(iii) [2 pts] How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

(b) [3 pts] Griffiths Problem 1.13 (p. 15), part (a) only: Let $\vec{\boldsymbol{\mathscr{r}}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

$$\vec{\nabla}(\mathscr{r}^2) = 2\vec{\boldsymbol{\mathscr{r}}} \ .$$

— PART B —

PROBLEM 5: THE DIRAC DELTA FUNCTION AND $\boldsymbol{\nabla^2(1/4\pi r)}$ *(20 points)*

(This problem is similar to Griffiths' Problem 1.64.)

One of the most useful identities in this course is the relation

$$-\nabla^2 \frac{1}{4\pi r} \equiv -\vec{\nabla} \cdot \left[\vec{\nabla} \frac{1}{4\pi r} \right] = \vec{\nabla} \cdot \left[\frac{\hat{r}}{4\pi r^2} \right] = \delta^3(\vec{\boldsymbol{r}}) \equiv \delta(x)\, \delta(y)\, \delta(z) \ . \tag{5.1}$$

It is straightforward to show that (see Griffiths Sec. 1.5.1, p. 45)

$$-\nabla^2 \frac{1}{4\pi r}$$

is zero everywhere except at the origin, where it is ill-defined, at least in the context of ordinary functions. In the context of generalized functions (also called distributions), however, the expression is well-defined and given by Eq. (5.1). While a delta-function is not defined as the limit of ordinary functions, it is sometimes helpful to consider such a limit. One approach is to replace the function $-\frac{1}{4\pi r}$ by the function

$$f_a(r) = -\frac{1}{4\pi} \frac{1}{\sqrt{r^2 + a^2}} \ . \tag{5.2}$$

For a nonzero, $f_a(r)$ is well-behaved everywhere, and

$$\lim_{a\to 0} f_a(r) = -\frac{1}{4\pi r} \quad \text{if} \quad r \neq 0 \ . \tag{5.3}$$

(a) [10 pts] Calculate $g_a(r) = \nabla^2 f_a(r)$ and show that it is also well behaved for all r. Sketch $g_a(r)$ for some value of a as a function of r/a.

(b) [7 pts] Show that

$$\int_{\textit{all space}} g_a(r)\ d^3x = 1 \tag{5.4}$$

for any value of a, from which it follows that

$$\lim_{a\to 0} \int_{\textit{all space}} g_a(r)\ d^3x = 1 \ . \tag{5.5}$$

(c) [3 pts] Show that

$$\lim_{a\to 0} g_a(r) = 0 \quad \text{if} \quad r \neq 0 \ . \tag{5.6}$$

Thus in the limit that a goes to zero, our well-behaved function $g_a(r)$ exhibits the properties we expect of a three-dimensional delta function. Note, however, that the ordering of the limit and the integration in Eq. (5.5) is crucial. In the other order, it will be shown in lecture that

$$\int_{\textit{all space}} \lim_{a\to 0} g_a(r)\ d^3x = 0 \ , \tag{5.7}$$

which is why one cannot define the delta function by

$$\delta^3(\vec{\boldsymbol{r}}) \equiv \lim_{a\to 0} g_a(r) \qquad \text{(WRONG!)}. \tag{5.8}$$

A delta function can be understood as the limit as $a \to 0$ of $g_a(r)$, but only if one remembers that one has to integrate *before* taking the limit.

PROBLEM 6: EXERCISES WITH δ-FUNCTIONS *(10 points)*

(a) [2 pts] Evaluate $\int_0^5 \cos x\, \delta(x-\pi)\, dx$.

(b) [2 pts] Evaluate $\int_0^2 (x^3 + 3x + 1)\, \delta(1-x)\, dx$.

(c) [2 pts] Evaluate $\int_0^1 9x^2\, \delta(3x+1)\, dx$.

(d) [4 pts] In Example 1.15 (p. 48), Griffiths shows that $\delta(kx) = \delta(x)/|k|$. Generalize this to show that

$$\delta[f(x)] = \frac{1}{\left| \left. \frac{df}{dx} \right|_{x_0} \right|} \, \delta(x - x_0) \ , \tag{6.1}$$

where we assume that $f(x)$ has only one zero in the range of integration, at x_0, and that it is a simple zero (i.e., df/dx does not vanish at $x = x_0$). Hint: consider the integral

$$\int \phi(x) \, \delta[f(x)] \, dx \ , \tag{6.2}$$

where $\phi(x)$ is an arbitrary smooth test function, and assume that the range of integration includes $x = x_0$. Evaluate this integral by using the change of variable $y \equiv f(x)$.

PROBLEM 7: CHARGE DISTRIBUTIONS AND δ-FUNCTIONS *(10 points)*

(a) [2 pts] A charge Q is spread uniformly over a spherical shell of radius R. Express the volume charge density $\rho(\vec{\boldsymbol{r}})$ using a delta function in spherical coordinates.

(b) [2 pts] A charge λ per unit length is distributed uniformly over a cylindrical surface of radius b. Give the volume charge density $\rho(\vec{\boldsymbol{r}})$ using a delta function in cylindrical coordinates.

(c) [2 pts] In cartesian coordinates, we can write $\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') = \delta(x - x')\delta(y - y')\delta(z - z')$. How would one express $\delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')$ in cylindrical coordinates (s, ϕ, z).

(d) [4 pts] What is $\nabla^2 \ln r$ in two dimensions? (Here r is the radial coordinate, $r = \sqrt{x^2 + y^2}$.)

PROBLEM 8: COROLLARIES OF THE FUNDAMENTAL INTEGRAL THEOREMS *(15 points extra credit)*

This problem is closely related to Problem 1.61, p. 56 of Griffiths. You will find useful hints there — but try without hints first!!. Show that:

(a) [4 pts] $\int_V \vec{\nabla}\psi \, d^3x = \int_S \psi \, d\vec{a}$, where S is the surface bounding the volume V. Show that as a consequence of this, $\int_S d\vec{a} = 0$ for a closed surface S.

(b) [4 pts] $\int_V \vec{\nabla} \times \vec{A} \, d^3x = -\int_S \vec{A} \times d\vec{a}$, where S is the surface bounding the volume V.

(c) [4 pts] $\int_S \vec{\nabla}\psi \times d\vec{a} = -\oint_\Gamma \psi \, d\vec{l}$, where Γ is the boundary of the surface S.

(d) [3 pts] For a closed surface S, one has $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = 0$.