MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 7, 2019
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FORMULA SHEET FOR QUIZ 2

Exam Date: November 13, 2019

Revised November 11, 2019 *

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* The normalization of the electric quadrupole tensor Q_{ij}, in Sec. 13(a), was revised to match the convention of Griffths's 4th edition, which is the normalization used in Problem Set 4, Problem 2. The original version of the Formula Sheet for Quiz 2 used the normalization from Griffiths's 3rd edition, where $Q_{ij}^{\text{3rd Ed.}} = 2Q_{ij}^{\text{4th Ed.}}$. With the new convention, $Q_{ij}^{\text{4th Ed.}} = C_{ij}^{(2)}$, where $C_{i_1 \ldots i_\ell}^{(\ell)}$ is defined in Sec. 13(b).

A few items below are marked with asterisks, $***$. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

1. Index Notation:

Unit Vectors: $\hat{\boldsymbol{x}} \equiv \hat{\boldsymbol{\imath}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{x}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{1}}\ ,\ \hat{\boldsymbol{y}} \equiv \hat{\boldsymbol{\jmath}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{y}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{2}}\ ,\ \hat{\boldsymbol{z}} \equiv \hat{\boldsymbol{k}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{z}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{3}}\ ,\ \vec{\boldsymbol{A}} \equiv A_i \hat{\boldsymbol{e}}_{\boldsymbol{i}}$

General vector: $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} = x_i\,\hat{\boldsymbol{e}}_{\boldsymbol{i}}$

Dot Product, Cross Product, and Determinant:

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_i B_i\ , \qquad \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}_i = \epsilon_{ijk} A_j B_k\ , \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}\ ***$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j\ , \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

$$\text{Rotation about } z\text{-axis by } \phi\text{: } R_z(\phi)_{ij} \;=\; \begin{array}{c c} & \begin{array}{ccc} \scriptstyle j=1 & \scriptstyle j=2 & \scriptstyle j=3 \end{array} \\ \begin{array}{c} \scriptstyle i=1 \\ \scriptstyle i=2 \\ \scriptstyle i=3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{\boldsymbol{n}}$ by ϕ: $***$

$$R(\hat{\boldsymbol{n}}, \phi)_{ij} = \delta_{ij} \cos\phi + \hat{\boldsymbol{n}}_i \hat{\boldsymbol{n}}_j (1 - \cos\phi) - \epsilon_{ijk} \hat{\boldsymbol{n}}_k\ \sin\phi\ .$$

2. Vector Calculus:

Gradient: $$(\vec{\boldsymbol{\nabla}}\varphi)_i = \partial_i \varphi = \frac{\partial \varphi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial \varphi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial \varphi}{\partial z}\hat{\boldsymbol{z}}\ , \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

Divergence: $$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} \equiv \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl: $$(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})_i = \epsilon_{ijk} \partial_j A_k$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Laplacian: $$\nabla^2 \varphi = \vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}}\varphi) = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

3. Fundamental Theorems of Vector Calculus:

Gradient: $$\int_{\vec{\boldsymbol{a}}}^{\vec{\boldsymbol{b}}} \vec{\boldsymbol{\nabla}}\varphi \cdot \mathrm{d}\vec{\boldsymbol{\ell}} = \varphi(\vec{\boldsymbol{b}}) - \varphi(\vec{\boldsymbol{a}})$$

Divergence: $$\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}\,\mathrm{d}^3x = \oint_S \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{a}}$$

where S is the boundary of $\mathcal{V}$

Curl: $$\int_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) \cdot \mathrm{d}\vec{\boldsymbol{a}} = \oint_P \vec{\boldsymbol{A}} \cdot \mathrm{d}\vec{\boldsymbol{\ell}}$$

where P is the boundary of S

4. Vector Identities:

Triple Products:

$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{C}} \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})$$

$$\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}}) - \vec{\boldsymbol{C}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})$$

Product Rules:

$$\vec{\boldsymbol{\nabla}}(fg) = f\vec{\boldsymbol{\nabla}}g + g\vec{\boldsymbol{\nabla}}f$$

$$\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) + \vec{\boldsymbol{B}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) + (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}}$$

$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} + \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}}f$$

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}})$$

$$\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{A}}) = f\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} - \vec{\boldsymbol{A}} \times \vec{\boldsymbol{\nabla}}f$$

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = (\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}})$$

Second Derivatives:

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = 0$$

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}}f) = 0$$

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}}) - \nabla^2\vec{\boldsymbol{A}}$$

5. Spherical Coordinates:

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$$

$$z = r\cos\theta \qquad \phi = \tan^{-1}(y/x)$$

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{x}} = \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \qquad \hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

Point separation: $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}$

Volume element: $\mathrm{d}^3x \to r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}}$$
$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

6. Cylindrical Coordinates:

$x = s\cos\phi$	$s = \sqrt{x^2+y^2}$
$y = s\sin\phi$	$\phi = \tan^{-1}(y/x)$
$z = z$	$z = z$
$\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}} = \cos\phi\,\hat{\boldsymbol{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{y}} = \sin\phi\,\hat{\boldsymbol{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$

Point separation: $\mathrm{d}\vec{\boldsymbol{\ell}} = \mathrm{d}s\,\hat{\boldsymbol{s}} + s\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\boldsymbol{z}}$

Volume element: $\mathrm{d}^3x \to s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z$

Gradient: $$\vec{\boldsymbol{\nabla}}\varphi = \frac{\partial\varphi}{\partial s}\,\hat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial\varphi}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial\varphi}{\partial z}\,\hat{\boldsymbol{z}}$$

Divergence: $$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{A}} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_s\right) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Curl: $$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\boldsymbol{\phi}}$$
$$+\ \frac{1}{s}\left[\frac{\partial}{\partial s}\left(sA_\phi\right) - \frac{\partial A_s}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

Laplacian: $$\nabla^2\varphi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\varphi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}$$

7. Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{\boldsymbol{r}})\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x = \varphi(\vec{\boldsymbol{r}}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad \text{where } i \text{ is summed over all points for which } g(x_i)=0$$

$$\nabla^2\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|} = -\vec{\boldsymbol{\nabla}}\cdot\left(\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'|^3}\right) = -4\pi\delta^3(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')$$

$$\partial_i\partial_j\left(\frac{1}{r}\right) = -\partial_i\left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i\left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j-\delta_{ij}}{r^3} - \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{\boldsymbol{r}})$$

8. Electrostatics:

$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}$, where

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')\,q_i}{\left|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'\right|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,')}{\left|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'\right|^3}\,\rho(\vec{\boldsymbol{r}}\,')\,\mathrm{d}^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times10^{-12}\ \mathrm{C^2/(N{\cdot}m^2)}$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^9\ \mathrm{N{\cdot}m^2/C^2}$$

$$V(\vec{\boldsymbol{r}}) = V(\vec{\boldsymbol{r}}_0) - \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}\,')\cdot\mathrm{d}\vec{\boldsymbol{\ell}}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}\,')}{\left|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'\right|}\mathrm{d}^3x'$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = 0\,, \qquad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0\ \ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

9. Electrostatic Energy:*

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{\boldsymbol{r}})\rho(\vec{\boldsymbol{r}}\,')}{\left|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}\,'\right|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{\boldsymbol{r}})V(\vec{\boldsymbol{r}}) = \frac{1}{2}\epsilon_0\int\left|\vec{\boldsymbol{E}}\right|^2\mathrm{d}^3x$$

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* See Sec. 14(c) for energy in the presence of dielectrics.

10. Conductors:

Just outside, $\vec{\boldsymbol{E}} = \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{\boldsymbol{E}}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij}Q_j\ , \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij}V_j\ , \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

11. Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cosh\gamma z \\ \sinh\gamma z \end{Bmatrix} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

or, more generally,

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cos\gamma z \\ \sin\gamma z \end{Bmatrix}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$, each of α, β, and γ can be real or imaginary, with $\sin(i\theta) = i\sinh\theta$ and $\cos(i\theta) = \cosh\theta$. $\begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix}$ means any linear combination of $\cos\alpha x$ or $\sin\alpha x$, but usually one or the other suffices.

12. Separation of Variables for Laplace's Equation in Spherical Coordinates:

12(a) Traceless Symmetric Tensor expansion:

$$\nabla^2\varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\nabla^2_{\text{ang}}\,\varphi = 0\ ,$$

where the angular part is given by

$$\nabla^2_{\text{ang}}\,\varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\nabla^2_{\text{ang}}\, C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell} = -\ell(\ell+1)C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{\boldsymbol{n}}_{i_1}\hat{\boldsymbol{n}}_{i_2}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_{\mathbf{1}} + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_{\mathbf{2}} + \cos\theta\,\hat{\boldsymbol{e}}_{\mathbf{3}} \quad \text{(unit vector in } (\theta,\phi) \text{ direction)}$$

12(a)(i) General solution to Laplace's equation:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}} \right) \hat{\boldsymbol{n}}_{i_1} \hat{\boldsymbol{n}}_{i_2} \ldots \hat{\boldsymbol{n}}_{i_\ell}\,, \quad \text{where } \vec{\boldsymbol{r}} = r\hat{\boldsymbol{n}}$$

12(a)(ii)Azimuthal Symmetry:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell}$$

where $\{\ldots\}_{\text{TS}}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{\, 1 \,\}_{\text{TS}} = 1$$

$$\{\, \hat{\boldsymbol{z}}_i \,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i$$

$$\{\, \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j - \tfrac{1}{3}\delta_{ij}$$

$$\{\, \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \,\}_{\text{TS}} = \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k - \tfrac{1}{5}\big(\hat{\boldsymbol{z}}_i \delta_{jk} + \hat{\boldsymbol{z}}_j \delta_{ik} + \hat{\boldsymbol{z}}_k \delta_{ij} \big)$$

$$\begin{aligned} \{\, \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m \,\}_{\text{TS}} = {} & \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m - \tfrac{1}{7}\big(\hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_j \delta_{km} + \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_k \delta_{mj} + \hat{\boldsymbol{z}}_i \hat{\boldsymbol{z}}_m \delta_{jk} \\ & + \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_k \delta_{im} + \hat{\boldsymbol{z}}_j \hat{\boldsymbol{z}}_m \delta_{ik} + \hat{\boldsymbol{z}}_k \hat{\boldsymbol{z}}_m \delta_{ij} \big) + \tfrac{1}{35}\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk} \big) \end{aligned}$$

12(b) Legendre Polynomial / Spherical Harmonic expansion:

12(b)(i) Azimuthal Symmetry:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell (\ell!)^2} \{\, \hat{\boldsymbol{z}}_{i_1} \ldots \hat{\boldsymbol{z}}_{i_\ell} \,\}_{\text{TS}}\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell}$$

12(b)(ii) General solution to Laplace's equation:

$$V(\vec{\boldsymbol{r}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m}\, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi)$$

12(b)(iii) Orthonormality: $\displaystyle\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\,\mathrm{d}\theta\, Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm}$

12(b)(iv) Spherical Harmonics in terms of Traceless Symmetric Tensors:

$$Y_{\ell,m}(\theta,\phi) = C^{(\ell,m)}_{i_1\ldots i_\ell}\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\ ,$$

where

$$C^{(\ell,m)}_{i_1\ldots i_\ell} = \begin{cases} N(\ell,m)\{\,\hat{\boldsymbol{u}}^{+}_{i_1}\ldots\hat{\boldsymbol{u}}^{+}_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\geq 0\ , \\ N(\ell,m)\{\,\hat{\boldsymbol{u}}^{-}_{i_1}\ldots\hat{\boldsymbol{u}}^{-}_{i_{|m|}}\,\hat{\boldsymbol{z}}_{i_{|m|+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\rm TS} & \text{for } m\leq 0\ , \end{cases}$$

$$\hat{\boldsymbol{u}}^{+}\equiv\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}}+i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right)\ ,\quad \hat{\boldsymbol{u}}^{-}\equiv\frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}}-i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right)$$

$$N(\ell,m)=\frac{(-1)^m(2\ell)!}{2^\ell\ell!}\sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}\qquad (\text{for } m\geq 0).$$

Connection between m and $-m$: $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$, which holds for all m.

12(b)(v) More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P^m_\ell(\cos\theta)e^{im\phi}$$

where $P^m_\ell(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P^m_\ell(x)=\frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

13. Electric Multipole Expansion:

13(a) First several terms:

$$V(\vec{\boldsymbol{r}})=\frac{1}{4\pi\epsilon_0}\Big[\,\frac{Q}{r}+\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}{r^2}+\frac{\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j}{r^3}\,Q_{ij}+\cdots\Big]\ \text{, where}$$

$$Q=\int \mathrm{d}^3x\,\rho(\vec{\boldsymbol{r}})\,,\ p_i=\int d^3x\,\rho(\vec{\boldsymbol{r}})\,x_i\,,\ Q_{ij}=\frac{1}{2}\int d^3x\,\rho(\vec{\boldsymbol{r}})(3x_ix_j-\delta_{ij}|\vec{\boldsymbol{r}}|^2)$$

$$\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}})=-\frac{1}{4\pi\epsilon_0}\vec{\boldsymbol{\nabla}}\left(\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}{r^2}\right)=\frac{1}{4\pi\epsilon_0}\frac{3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\vec{\boldsymbol{p}}}{r^3}-\frac{1}{3\epsilon_0}p_i\delta^3(\vec{\boldsymbol{r}})$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}})=0\ ,\qquad \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}}_{\rm dip}(\vec{\boldsymbol{r}})=\frac{1}{\epsilon_0}\rho_{\rm dip}(\vec{\boldsymbol{r}})=-\frac{1}{\epsilon_0}\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}\delta^3(\vec{\boldsymbol{r}})$$

13(b) Traceless Symmetric Tensor version:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}}\, C^{(\ell)}_{i_1\ldots i_\ell} \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \ ,$$

where

$$C^{(\ell)}_{i_1\ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{\boldsymbol{r}}) \,\{\, x_{i_1} \ldots x_{i_\ell} \,\}_{\rm TS}\, \mathrm{d}^3x \qquad (\vec{\boldsymbol{r}} \equiv r\hat{\boldsymbol{n}} \equiv x_i \hat{\boldsymbol{e}}_{\boldsymbol{i}})$$

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \,\{\, \hat{\boldsymbol{n}}_{i_1} \ldots \hat{\boldsymbol{n}}_{i_\ell} \,\}_{\rm TS}\, \hat{\boldsymbol{n}}'_{i_1} \ldots \hat{\boldsymbol{n}}'_{i_\ell} \ , \qquad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 = \frac{(2\ell)!}{2^\ell \ell!} \ , \ \text{with} \ \ (-1)!! \equiv 1 \ .$$

Reminder: $\{\ldots\}_{\rm TS}$ denotes the traceless symmetric part of $\ldots$.

13(c) Griffiths version (azimuthal symmetry only):

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^\ell \rho(\vec{\boldsymbol{r}}\,') P_\ell(\cos\theta')\, \mathrm{d}^3x$$

where θ' = angle between $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}\,'$.

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}}\, P_\ell(\cos\theta') \, , \qquad \frac{1}{\sqrt{1-2\lambda x+\lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell \, , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x\, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell'\ell}$$

13(d) Spherical Harmonic version:

$$V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}}\, Y_{\ell m}(\theta,\phi)$$

where $q_{\ell m} = \int Y^*_{\ell m} r'^\ell \rho(\vec{\boldsymbol{r}}\,')\, \mathrm{d}^3x'$

$$\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \ , \qquad \text{for } r' < r$$

13(e) Properties of Traceless Symmetric Tensors:

13(e)(i) Trace Decomposition: Any symmetric tensor $S_{i_1\ldots i_\ell}$ can be written uniquely as

$$S_{i_1\ldots i_\ell} = S^{(\mathrm{TS})}_{i_1\ldots i_\ell} + \operatorname*{Sym}_{i_1\ldots i_\ell}\left[M_{i_1\ldots i_{\ell-2}}\delta_{i_{\ell-1},i_\ell}\right] \ ,$$

where $S^{(\mathrm{TS})}_{i_1\ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1\ldots i_{\ell-2}}$ is a symmetric tensor tensor, and $\operatorname*{Sym}_{i_1\ldots i_\ell}[\, xxx\,]$ symmetrizes xxx in the indices $i_1\ldots i_\ell$. $S^{(\mathrm{TS})}_{i_1\ldots i_\ell}$ is called the *traceless symmetric part* of $S_{i_1\ldots i_\ell}$.

13(e)(ii) Extraction of the traceless part of an arbitrary symmetric tensor $S_{i_1\ldots i_\ell}$:

$$\begin{aligned}\{\, S_{i_1\ldots i_\ell}\,\}_{\mathrm{TS}} = S_{i_1\ldots i_\ell} + \operatorname*{Sym}_{i_1\ldots i_\ell}\Big[& a_{1,\ell}\,\delta_{i_1 i_2}\,\delta^{j_1 j_2}\, S_{j_1 j_2 i_3\ldots i_\ell} \\ &+ a_{2,\ell}\,\delta_{i_1 i_2}\,\delta_{i_3 i_4}\,\delta^{j_1 j_2}\,\delta^{j_3 j_4}\, S_{j_1 j_2 j_3 j_4 i_5\ldots i_\ell} + \ldots\Big]\end{aligned}$$

where

$$a_{n,\ell} = (-1)^n \frac{\ell!^2(2\ell-2n)!}{n!(\ell-2n)!(\ell-n)!(2\ell)!}$$

The series terminates when the number of i indices on S is zero or one.

13(e)(iii) Integration:

$$\int d\Omega\, \hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{2\ell}} = 4\pi\frac{2^\ell \ell!}{(2\ell+1)!}\sum_{\text{all pairings}}\delta_{i_1,i_2}\,\delta_{i_3,i_4}\ldots\delta_{i_{2\ell-1},i_{2\ell}}$$

$$\text{where} \int d\Omega \equiv \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

The integral vanishes if the number of $\hat{\boldsymbol{n}}$ factors is odd.

$$\begin{aligned}&\int d\Omega\,\left[C^{(\ell)}_{i_1\ldots i_\ell}\,\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}}\right]\,\left[C'^{(\ell')}_{j_1\ldots j_{\ell'}},\{\,\hat{\boldsymbol{n}}_{j_1}\ldots\hat{\boldsymbol{n}}_{j_{\ell'}}\,\}_{\mathrm{TS}}\right] \\ &\quad = \begin{cases} 4\pi\dfrac{2^\ell \ell!^2}{(2\ell+1)!}\, C^{(\ell)}_{i_1\ldots i_\ell}\, C'^{(\ell)}_{i_1\ldots i_\ell} & \text{if } \ell'=\ell \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

13(e)(iv) Other identities:

$$\hat{\boldsymbol{n}}_{i_\ell}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{\ell}{2\ell-1}\{\,\hat{\boldsymbol{n}}_{i_1}\ldots\hat{\boldsymbol{n}}_{i_{\ell-1}}\,\}_{\mathrm{TS}} \qquad (\hat{\boldsymbol{n}} = \text{any unit vector})$$

$$\hat{\boldsymbol{z}}_{i_\ell}\,\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_\ell}\,\}_{\mathrm{TS}} = \frac{(\ell+m)(\ell-m)}{\ell(2\ell-1)}\{\,\hat{\boldsymbol{u}}^+_{i_1}\ldots\hat{\boldsymbol{u}}^+_{i_m}\,\hat{\boldsymbol{z}}_{i_{m+1}}\ldots\hat{\boldsymbol{z}}_{i_{\ell-1}}\,\}_{\mathrm{TS}}$$

where $\hat{\boldsymbol{u}}^{+} \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_{\boldsymbol{x}} + i\hat{\boldsymbol{e}}_{\boldsymbol{y}}\right)$

$***$ For any symmetric traceless tensor $S_{i_1 \ldots i_\ell}$,

$$\delta_{i_{\ell+2n-1}, i_{\ell+2n}} \left\{ \underset{i_1 \ldots i_{\ell+2n}}{\mathrm{Sym}} \left[S_{i_1 \ldots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \ldots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}}_{n \text{ times}} \right] \right\}$$

$$= F(n, \ell) \underset{i_1 \ldots i_{\ell+2n-2}}{\mathrm{Sym}} \left[S_{i_1 \ldots i_\ell} \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \ldots \delta_{i_{\ell+2n-3}, i_{\ell+2n-2}}}_{n-1 \text{ times}} \right]$$

where

$$F(n, \ell) = \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)}$$

14. Electric Fields in Matter:

14(a) Electric Dipoles:

$$\vec{\boldsymbol{p}} = \int d^3x \, \rho(\vec{\boldsymbol{r}}) \, \vec{\boldsymbol{r}}$$

$\rho_{\text{dip}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{r}}} \, \delta^3(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_d)$, where $\vec{\boldsymbol{r}}_d$ = position of dipole

$\vec{\boldsymbol{F}} = (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}})$ (force on a dipole)

$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}$ (torque on a dipole)

$U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}$ (potential energy)

14(b) Electrically Polarizable Materials:

$\vec{\boldsymbol{P}}(\vec{\boldsymbol{r}})$ = polarization = electric dipole moment per unit volume

$\rho_{\text{bound}} = -\nabla \cdot \vec{\boldsymbol{P}}$, $\sigma_{\text{bound}} = \vec{\boldsymbol{P}} \cdot \hat{\boldsymbol{n}}$

$\vec{\boldsymbol{D}} \equiv \epsilon_0 \vec{\boldsymbol{E}} + \vec{\boldsymbol{P}}$, $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{D}} = \rho_{\text{free}}$, $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0$

Boundary conditions:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}$$

$$\vec{\boldsymbol{E}}^{\parallel}_{\text{above}} - \vec{\boldsymbol{E}}^{\parallel}_{\text{below}} = 0 \qquad \vec{\boldsymbol{D}}^{\parallel}_{\text{above}} - \vec{\boldsymbol{D}}^{\parallel}_{\text{below}} = \vec{\boldsymbol{P}}^{\parallel}_{\text{above}} - \vec{\boldsymbol{P}}^{\parallel}_{\text{below}}$$

14(c) Linear Dielectrics:

$\vec{\boldsymbol{P}} = \epsilon_0 \chi_e \vec{\boldsymbol{E}}$, χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1 + \chi_e)$ = permittivity, $\vec{\boldsymbol{D}} = \epsilon \vec{\boldsymbol{E}}$

$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{n\alpha/\epsilon_0}{1 - \frac{n\alpha}{3\epsilon_0}}$, where n = number density of atoms or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{\boldsymbol{p}} = \alpha\vec{\boldsymbol{E}}$)

Energy: $W = \dfrac{1}{2}\displaystyle\int \vec{\boldsymbol{D}} \cdot \vec{\boldsymbol{E}}\, \mathrm{d}^3x$ (linear materials only)

Force on a dielectric: $\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}}W$, where W is the potential energy stored in the system. Even if one or more potential differences are held fixed, the force can be found by computing the gradient of W with the total charge on each conductor fixed.

15. Magnetostatics:

15(a) Lorentz Force Law:

$$\vec{\boldsymbol{F}} \equiv \frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \ , \qquad \text{where } \vec{\boldsymbol{p}} = \gamma m_0 \vec{\boldsymbol{v}} \ , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

15(b) Biot-Savart Law:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\boldsymbol{\ell}}' \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3}$$

where μ_0 = permeability of free space $\simeq 4\pi \times 10^{-7}$ N/A^2 (to about 8-figure accuracy)

Examples:

Infinitely long straight wire: $\vec{\boldsymbol{B}} = \dfrac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}$

Infintely long tightly wound solenoid: $\vec{\boldsymbol{B}} = \mu_0 n I_0\, \hat{\boldsymbol{z}}$, where n = turns per unit length

Loop of current on axis: $\vec{\boldsymbol{B}}(0,0,z) = \dfrac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}}\,\hat{\boldsymbol{z}}$

16. Table of Legendre Polynomials $P_\ell(x)$:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \dfrac{1}{2}(3x^2 - 1)$

$P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$

$P_4(x) = \dfrac{1}{8}(35x^4 - 30x^2 + 3)$

17. Table of Spherical Harmonics $\boldsymbol{Y_{\ell m}(\theta,\phi)}$:

$\ell = 0$ $\quad Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$$\ell = 1 \qquad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\, \sin\theta e^{i\phi} \\[2ex] Y_{10} = \sqrt{\dfrac{3}{4\pi}}\, \cos\theta \end{cases}$$

$$\ell = 2 \qquad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\, \sin^2\theta e^{2i\phi} \\[2ex] Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\, \sin\theta\cos\theta e^{i\phi} \\[2ex] Y_{20} = \dfrac{1}{2}\sqrt{\dfrac{5}{4\pi}}\,(3\cos^2\theta - 1) \end{cases}$$

$$\ell = 3 \qquad \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}}\, \sin^3\theta e^{3i\phi} \\[2ex] Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}}\, \sin^2\theta\cos\theta e^{2i\phi} \\[2ex] Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}}\, \sin\theta(5\cos^2\theta - 1)e^{i\phi} \\[2ex] Y_{30} = \dfrac{1}{2}\sqrt{\dfrac{7}{4\pi}}\,(5\cos^3\theta - 3\cos\theta) \end{cases}$$

$$\ell = 4 \qquad \begin{cases} Y_{44} = \dfrac{3}{16}\sqrt{\dfrac{35}{2\pi}}\, \sin^4\theta e^{4i\phi} \\[2ex] Y_{43} = -\dfrac{3}{2}\sqrt{\dfrac{35}{2\pi}}\, \sin^3\theta\cos\theta e^{3i\phi} \\[2ex] Y_{42} = \dfrac{3}{8}\sqrt{\dfrac{5}{2\pi}}\sin^2\theta(7\cos^2\theta - 1)e^{2i\phi} \\[2ex] Y_{41} = -\dfrac{3}{2}\sqrt{\dfrac{5}{2\pi}}\, \sin\theta(7\cos^3\theta - 3\cos\theta)e^{i\phi} \\[2ex] Y_{40} = \dfrac{3}{\sqrt{4\pi}}(35\cos^4\theta - 30\cos^2\theta + 3) \end{cases}$$

For $m < 0$, use $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$, which is valid for all m.