MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — December 19, 2019

Prof. Alan Guth

FINAL EXAM

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FORMULA SHEETS ARE BEING HANDED OUT SEPARATELY

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Your Name

1. There are **FIVE** problems — be sure not to miss any! This exam is worth 100 points. The problems are **not** equally weighted.
2. A separate formula sheet will be distributed.
3. The exam is **closed book** and **closed notes**. Cell phones and calculators will not be needed, and are not allowed.
4. Write all of your responses in this booklet. Extra pages are provided at the end of this booklet, but if you need more, please ask.
5. **Please** raise your hand and **ask** if there is anything about a question that you find unclear or confusing. Perhaps you have simply misread something, but it is possible that there is an error we need to correct. In either case, it would be good to know for certain, and we are happy to help.

Maybe useful trig ID's:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

PROBLEM 1: SHORT QUESTIONS *(30 points)*

(a) [6 pts] Which of the following field expressions are physically possible equations for an electric field, when all fields are static, assuming that all constants c_i are nonzero. Answer yes or no for each, and give a brief reason.

(i) $\vec{\boldsymbol{E}} = c_1(y\,\hat{\boldsymbol{z}} + z\,\hat{\boldsymbol{y}}) + c_2\,\hat{\boldsymbol{x}}$.

(ii) $\vec{\boldsymbol{E}} = c_1(y\,\hat{\boldsymbol{z}} - z\,\hat{\boldsymbol{y}}) + c_2\,\hat{\boldsymbol{x}}$.

(iii) $\vec{\boldsymbol{E}} = c_1(2r\cos\theta\,\hat{\boldsymbol{r}} - 3r\sin\theta\,\hat{\boldsymbol{\theta}})$ (spherical coordinates).

(b) [3 pts] If we drop the requirement that all fields are static, which of the above expressions are possible equations for an electric field at some given time t_0?

(c) [4 pts] Which of the following field expressions are physically possible equations for a magnetic field, when all fields are static, assuming that all constants c_i are nonzero. Answer yes or no for each, and give a brief reason.

(i) $\vec{\boldsymbol{B}} = c_1(x\,\hat{\boldsymbol{x}} - z\,\hat{\boldsymbol{z}})$.

(ii) $\vec{\boldsymbol{B}} = c_1(s^2\cos\phi\,\hat{\boldsymbol{s}} - 3sz\cos\phi\,\hat{\boldsymbol{z}} + 2z^2\sin\phi\,\hat{\boldsymbol{\phi}})$ (cylindrical coordinates).

Do these answers change if we drop the requirement that all fields are static? Is so, what do they become?

(d) [5 pts] Evaluate $I \equiv \int_0^1 \mathrm{d}x\, x^2\delta(1-\alpha x^2)$, for $\alpha > 1$. What is the value of I if $\alpha = \frac{1}{2}$?

(e) [6 pts] Suppose that a uniform linear medium, of permittivity ϵ and permeability μ, has a very thin wafer-shaped cavity, as shown at the right.

Wafer

(i) Suppose that inside the material there is a uniform electric field $\vec{\boldsymbol{E}}$, oriented perpendicular to the wafer. What is the value of $\vec{\boldsymbol{E}}_{\text{cav}}$ inside the cavity?

(ii) Supoose that inside the material there is a uniform magnetic field $\vec{\boldsymbol{B}}$, oriented perpendicular to the wafer. What is the value of $\vec{\boldsymbol{B}}_{\text{cav}}$ inside the cavity?

(f) [6 pts] Consider a cube that fills the region $0 \le x \le a$, $0 \le y \le a$, and $0 \le z \le a$. Suppose that the Maxwell stress tensor has the values

$$\begin{aligned} T_{xx} &= T_1 \quad \text{at } x = a, \\ T_{xx} &= T_2 \quad \text{at } x = 0, \\ T_{xy} &= T_{yx} = T_3 \quad \text{at } y = a, \end{aligned}$$

and all other components of T_{ij} vanish, on all of the walls of the box. Calculate the total electromagnetic force on the box.

PROBLEM 2: A MAGNETIC MONOPOLE AND A CURRENT LOOP *(10 points)*

(a) [4 pts] A magnetic monopole of magnetic charge q_m, at rest at the origin of coordinates, creates a magnetic field

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \frac{q_m}{r^2}\,\hat{\boldsymbol{r}} \ . \tag{2.1}$$

Suppose that the magnetic monopole is surrounded by a wire loop of radius R in the x-y plane, centered on the origin, and carrying a current I, counterclockwise as viewed from the positive z direction. Calculate the force $\vec{\boldsymbol{F}}$ that the monopole exerts on the wire loop. (Show your calculation.)

(b) [4 pts] Using the Biot-Savart law (see Sec. 15(c) of the formula sheet), calculate the magnetic field of the wire loop at the location of the magnetic monopole. You must show the calculation.

(c) [2 pts] Assuming that a magnetic field exerts a force $\vec{\boldsymbol{F}} = \lambda q_m \vec{\boldsymbol{B}}$ on a magnetic monopole, where λ is a dimensionless constant, give an argument which implies that $\lambda = 1$.

PROBLEM 3: CURRENT TRAVELING ON A LONG STRAIGHT WIRE MADE OF A MATERIAL WITH LINEAR MAGNETIZATION *(15 points)*

Griffiths Problem 6.17 (p. 287). This problem was Problem 5 on Problem Set 8, except that here it is divided into parts.

A current I flows down a long straight wire of radius a. The wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly.

(a) [5 pts] What is the magnetic field a distance s from the axis?

(b) [10 pts] Find all the bound currents. What is the net bound current flowing down the wire?

PROBLEM 4: A ROTATING ELECTRIC DIPOLE *(15 points)*

This problem is adapted from Problem 6 on the Practice Problems for Final Exam, Part A, which was Griffiths Problem 11.4 (p. 473).

A *rotating* electric dipole can be thought of as the superposition of two *oscillating* dipoles, one along the x axis, and the other along the y axis (Fig. 11.7), with the latter out of phase by 90°:

$$\vec{\boldsymbol{p}} = p_0[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}] \ . \tag{4.1}$$

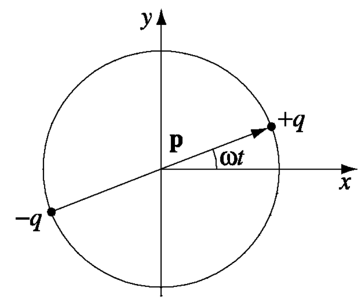

FIGURE 11.7

(a) [8 pts] Find the radiation zone electric field of a rotating dipole. You will probably want to use the principle of superposition, starting with the equations

$$\begin{aligned} \vec{\boldsymbol{E}}_{\text{dip}}(\vec{\boldsymbol{r}}, t) &= -\frac{\mu_0 \bar{p}_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}} \ , \\ \vec{\boldsymbol{B}}_{\text{dip}}(\vec{\boldsymbol{r}}, t) &= \frac{1}{c}\,\hat{\boldsymbol{r}} \times \vec{\boldsymbol{E}}_{\text{dip}}(\vec{\boldsymbol{r}}, t) \ , \end{aligned} \tag{4.2}$$

which describe a simple dipole,

$$\vec{\boldsymbol{p}}_{\text{dip}}(t) = \bar{p}_0 \cos(\omega t)\hat{\boldsymbol{z}} \ . \tag{4.3}$$

In using these formulas, you may want to keep in mind that they can be expressed in a "coordinate-free" form by writing $\bar{p}_0 \cos\theta = \vec{\bar{\boldsymbol{p}}}_{\mathbf{0}} \cdot \hat{\boldsymbol{r}}$, and $\bar{p}_0 \sin\theta\,\hat{\boldsymbol{\theta}} = (\vec{\bar{\boldsymbol{p}}}_{\mathbf{0}} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \vec{\bar{\boldsymbol{p}}}_{\mathbf{0}}$.

(b) [7 pts] In the homework version of this problem, you were asked to evaluate the Poynting vector, and its time average. That calculation involved more algebra than seems appropriate for this exam. Instead, here you are asked only to show that the time-averaged Poynting vector for the rotating electric dipole of Eq. (4.1) is equal to the sum of the time-averaged Poynting vectors that would be obtained separately for each of the two dipoles described by the two terms in Eq. (4.1). It is possible to do this without fully evaluating any of these quantities.

PROBLEM 5: A TIME-VARYING SHEET OF CURRENT *(30 points)*

Suppose that a thin, electrically neutral conducting sheet fills the x-y plane, and carries time-dependent but uniform surface current density $\vec{\boldsymbol{K}} = K(t)\hat{\boldsymbol{y}}$. Here $K(t)$ is some arbitrary function of time that vanishes for times earlier than some $t_1 < 0$.

(a) [8 pts] By translation invariance the vector potential $\vec{\boldsymbol{A}}$ at a point (x, y, z) can depend only on z. Use the Lorenz gauge vector potential formula,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}\,', t_r)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|} \ , \tag{5.1}$$

where the retarded time t_r is given by

$$t_r = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|}{c} \ , \tag{5.2}$$

to write an integral expression for the vector potential $\vec{\boldsymbol{A}}(z, t)$ at a point $P = (0, 0, z)$, with $z > 0$.

(b) [5 pts] Write the integral in part (a) so that the variable of integration is the radial distance $\rho \equiv \sqrt{x^2 + y^2}$ from the origin in the x-y plane. Then show that by a change of integration variable, the vector potential can be written as

$$\vec{\boldsymbol{A}}(z, t) = \frac{1}{2}\mu_0 c \hat{\boldsymbol{y}} \int_0^\infty K\left(t - \frac{z}{c} - u\right) \mathrm{d}u \ . \tag{5.3}$$

(c) [4 pts] Using Eq. (5.3) (whether or not you were able to derive it), show that the magnetic field is given by

$$\vec{\boldsymbol{B}}(z, t) = \frac{1}{2}\mu_0 K\left(t - \frac{z}{c}\right) \hat{\boldsymbol{x}} \qquad \text{(for } z > 0\text{)}. \tag{5.4}$$

Hint: Note that the derivative of K with respect to z can be related to the derivative of K with respect to u.

(d) [8 pts] As an alternative method of solving this problem, we can use the planar symmetry to suggest a trial solution of the form

$$\begin{aligned} \vec{\boldsymbol{E}}(z, t) &= E_y(z, t)\hat{\boldsymbol{y}} \ , \\ \vec{\boldsymbol{B}}(z, t) &= B_x(z, t)\hat{\boldsymbol{x}} \ . \end{aligned} \tag{5.5}$$

For $z > 0$, find the differential equations that $E_y(z, t)$ and $B_x(z, t)$ must obey for Maxwell's equations to be satisfied. Show that these equations imply that $B_x(z, t)$ obeys the one-dimensional wave equation,

$$\frac{\partial^2 B_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 B_x}{\partial t^2} = 0 \ . \tag{5.6}$$

Show that this equation is satisfied by $\vec{\boldsymbol{B}}(z, t)$, as given in Eq. (5.4).

(e) [5 pts] The surface current can be written as

$$\vec{\boldsymbol{J}}(z,t) = K(t)\delta(z)\hat{\boldsymbol{y}}. \tag{5.7}$$

Show that if

$$\vec{\boldsymbol{B}}(z,t) = \frac{1}{2}\mu_0 K\left(t - \frac{z}{c}\right)\begin{cases} \hat{\boldsymbol{x}} & \text{if } z > 0 \ , \\ -\hat{\boldsymbol{x}} & \text{if } z < 0 \ , \end{cases} \tag{5.8}$$

then the Maxwell equation

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} + \frac{1}{c^2}\frac{\partial \vec{\boldsymbol{E}}}{\partial t} \tag{5.9}$$

is satisfied at $z = 0$. You can either use the differential form of the equation, Eq. (5.9), or you might prefer to use the integral form. In either case, you should remember that in the immediate vicinity of $z = 0$, only the surface current density and the discontinuity in $\vec{\boldsymbol{B}}$ are relevant.

Problem	**Maximum**	**Score**	**Initials**
1	30		
2	10		
3	15		
4	15		
5	30		
TOTAL	100		