MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 25, 2019
Prof. Alan Guth

# PROBLEM SET 3

**DUE DATE:** Wednesday, October 2, 2019. Due at 6:30 pm in the 8.07 homework boxes. The problem set has two parts, A and B. Please write your recitation section, R01 (2:00 pm Thurs) or R02 (3:00 pm Thurs) on each part, and turn in Part A to homework box A and Part B to homework box B. Thanks!

**READING ASSIGNMENT:** Griffiths Sections 3.1 (*Potentials: Laplace's Equation*), 3.2 (*The Method of Images*), and 3.3.1 (*Separation of Variables, Cartesian Coordinates*).

**CREDIT:** This problem set has 130 points of credit.

## — PART A —

### PROBLEM 1: THE LAPLACIAN AS THE ANTI-LUMPINESS OPERATOR *(15 points)*

In this problem you will prove a relation that was stated in lecture. Let $\varphi(\vec{\boldsymbol{r}})$ be any scalar function of position $\vec{\boldsymbol{r}}$. We are interested in relating the value of $\varphi$ at an arbitrary point $\vec{\boldsymbol{r}}_0$ to the average value of $\varphi$ on a sphere that is centered at $\vec{\boldsymbol{r}}_0$. While the point $\vec{\boldsymbol{r}}_0$ is arbitrary, we can simpify our notation by choosing a coordinate system so that $\vec{\boldsymbol{r}}_0$ is the origin $\vec{0}$. Then the relation to be proved can be written

$$\varphi(\vec{\mathbf{0}}) - \bar{\varphi}(R) = -\frac{1}{4\pi}\int_{r<R} \mathrm{d}^3x \left(\frac{1}{r} - \frac{1}{R}\right) \nabla^2\varphi \ , \tag{1.1}$$

where we note that the quantity in parentheses is positive. Here $\bar{\varphi}(R)$ represents the average value of $\varphi$ on the surface of a sphere of radius $R$, which can be written explicitly as

$$\bar{\varphi}(R) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \varphi(R,\theta,\phi)\, \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{1.2}$$

The integration in Eq. (1.1) is over the volume of a sphere of radius $R$, centered at the origin. The relation to lumpiness can be seen by thinking of $\varphi$ as the density of a pudding. The equation implies that if $\nabla^2\varphi = 0$, then the value of $\varphi$ at the origin is the same as the average value of its surroundings (no lumpiness). But if $\nabla^2\varphi < 0$, then the value of $\varphi$ at the origin is higher than the average value of its surroundings (i.e., there is a lump).

(a) [4 pts] Defining

$$g(r) \equiv \frac{1}{r} - \frac{1}{R} \tag{1.3}$$

for compactness, use the divergence theorem to show that

$$\int_{r<R} \mathrm{d}^3x\, \vec{\nabla} \cdot \left(g\, \vec{\nabla}\varphi\right) = 0 \tag{1.4}$$

for any well-behaved function $\varphi(\vec{r})$.

(b) [4 pts] Use index notation (i.e., $\vec{\nabla} \equiv \hat{e}_i \partial_i$) to show that for arbitrary scalar functions $g(\vec{r})$ and $\varphi(\vec{r})$

$$\vec{\nabla} \cdot (g\, \vec{\nabla}\varphi) = \vec{\nabla} g \cdot \vec{\nabla}\varphi + g\, \nabla^2\varphi\ . \tag{1.5}$$

(c) [7 pts] Use the identity of Eq. (1.5) to rewrite the integrand of the integral of Eq. (1.4). We suggest that the integral of $\vec{\nabla} g \cdot \vec{\nabla}\varphi$ be expressed in spherical polar coordinates. Show that the vanishing of the integral in Eq. (1.4), re-expressed in this way, implies Eq. (1.1).

## PROBLEM 2: SPHERES AND IMAGE CHARGES *(15 points)*

Griffiths Problem 3.9 (p. 129):

This problem is essentially Griffiths 3.9 (p. 129), which follows example 3.2.

In Example 3.2, we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential.

(a) [5 pts] If the sphere is held at potential $V = V_0$, where should the second image charge $q''$ be placed, and what value should $q''$ take?

(b) [5 pts] Find the force between the charge $q$ and the sphere at potential $V_0$. (This is slightly different than the question Griffiths asks.)

(c) [5 pts] What value should $q''$ take if the sphere is to be *neutral*? Compute the force in this case.

## PROBLEM 3: IMAGE CHARGES FOR A PLANE WITH A HEMISPHERICAL BULGE *(15 points)*

Consider a conducting plane that occupies the $x$-$y$ plane of a coordinate system, but with the circular disk $x^2+y^2 < a^2$ removed. The circular disk is replaced by a conducting hemisphere of radius $a$, described by the equation

$$x^2 + y^2 + z^2 = a^2\ , \qquad z > 0\ . \tag{3.1}$$

You may assume that the plane is grounded, meaning that $V = 0$. A charge $q$ is placed on the $z$-axis at $(0, 0, z_0)$, with $z_0 > a$. Find a suitable set of image charges for this configuration. Show that the charge is attracted toward the plate with a force

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2}{4z_0^2} + \frac{4q^2a^3z_0^3}{(z_0^4 - a^4)^2}\right]\ . \tag{3.2}$$

## PROBLEM 4: IMAGES FOR A CONDUCTING CYLINDER *(15 points)*

*This problem is based on Problem 2.11 of Jackson:* **Classical Electrodynamics***, 3rd edition.*

A line of charge with linear charge density $\lambda$ is placed parallel to, and at a distance $R$ away from, the axis of a conducting cylinder of radius $b$ (where $b < R$). Use cylindrical coordinates $(s, \phi, z)$, as described in Sec. 1.4.2 of Griffiths, choosing the axis of the cylinder to be the axis of the coordinate system. We consider a situation in which the total linear charge density $\lambda'$ on the cylinder is determined by the criterion that the potential difference between the cylinder and $s = \infty$ is finite.

(a) [10 pts] Find the magnitude and position of the image charge(s).

(b) [5 pts] Find the potential $V_0$ of the cylinder in terms of $R, b$, and $\lambda$, where the potential is defined to be zero at $s = \infty$.

## — PART B (To be handed in separately from Part A) —

### PROBLEM 5: CAPACITANCE OF A SINGLE CONDUCTOR *(20 points)*

(a) [4 pts] Consider a single conductor, and define its capacitance by $Q = CV$, where $Q$ is the charge on the conductor, and $V$ is the potential of the conductor defined so that $V = 0$ at infinity. Show that $C$ can be expressed as

$$C = \frac{\epsilon_0}{V_0^2} \int_{\mathcal{V}} |\vec{\nabla} V|^2 \, \mathrm{d}^3 x \ , \tag{5.1}$$

where $\mathcal{V}$ is the space outside the conductor, and $V(\vec{r})$ is the solution for the potential when the conductor is held at $V = V_0$.

(b) [4 pts] Show that the true capacitance $C$ is always less than or equal to the quantity

$$C[\Psi(\vec{r})] = \frac{\epsilon_0}{V_0^2} \int_{\mathcal{V}} |\vec{\nabla} \Psi|^2 \mathrm{d}^3 x \ , \tag{5.2}$$

where $\Psi(\vec{r})$ is any trial function satisfying the boundary condition $\Psi = V_0$ at the conductor, and $\Psi = 0$ at infinity. (Note that $\Psi$ is *not* required to satisfy Laplace's equation, or any other equation.)

(c) [7 pts] Prove that the capacitance $C'$ of a conductor with surface $S'$ is smaller than the capacitance $C$ of a conductor whose surface $S$ encloses $S'$.

(d) [5 pts] Use part (c) to find upper and lower limits for the capacitance of a conducting cube of side $a$. Write your answer in the form: $\alpha(4\pi\epsilon_0 a) < C_{\text{cube}} < \beta(4\pi\epsilon_0 a)$ and find the constants $\alpha$ and $\beta$. A numerical calculation* gives $C \simeq 0.661(4\pi\epsilon_0 a)$. Compare this answer with your limits.

### PROBLEM 6: A SPHERICAL CONDUCTOR AND A CONDUCTING PLANE *(30 points)*

Consider a solid spherical conductor of radius $R$, with center on the positive $z$-axis at $z = z_0$, with $z_0 > R$. Suppose that the $x$-$y$ plane is conducting, and is held at potential $V = 0$, while the sphere is held at potential $V_0$.

To first approximation, we can think of the field as that of a point charge $q_0$ at the center of the sphere, with $q_0$ related to $V_0$ by

$$V_0 = \frac{q_0}{4\pi\epsilon_0 R} \ . \tag{6.1}$$

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* C.-O. Hwang and M. Mascagni, *Journal of Applied Physics* **95**, 3798 (2004).

The field due to this charge gives a potential $V_0$ on the surface of the sphere, as desired. But now the potential on the $x$-$y$ plane is not zero.

(a) [5 pts] The potential on the $x$-$y$ plane can be restored to zero by placing an image charge below the $x$-$y$ plane (i.e., at negative $z$). What charge $q'$ should this image have, and where should it be placed?

(b) [10 pts] The potential on the surface of the spherical conductor is now no longer constant, but it can be made constant by adding another image charge $q''$. The potential on the $x$-$y$ plane can be restored to zero by adding another image charge $q'''$, and the potential on the sphere can be restored to a constant by adding yet another image charge $q''''$. The series will continue forever, but it does converge fairly quickly. Calculate the positions and charges of the image charges $q''$, $q'''$, and $q''''$.

(c) [5 pts] After all the image charges are added through $q''''$, what is the potential $V$ of the spherical conductor?

(d) [5 pts] What is the total potential energy of this configuration? Express your answer as the first terms of an infinite series, showing those terms corresponding to the image charges through $q''''$.

(e) [5 pts] Would the fields outside the conductors be different if the solid spherical conductor were replaced by a spherical conducting shell, with the same outer radius?

## PROBLEM 7: LAPLACE'S EQUATION IN A BOX *(20 points)*

Griffiths Problem 3.16 (p. 141).

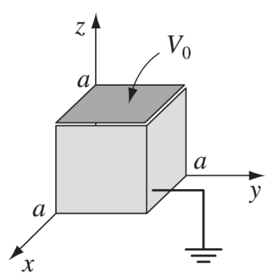


**FIGURE 3.23**

A cubical box (sides of length $a$) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential $V_0$. Find the potential inside the box. [What should the potential at the center ($a/2$, $a/2$, $a/2$) be? Check numerically that your formula is consistent with this value.]