MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 2, 2019
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PROBLEM SET 3 SOLUTIONS

PROBLEM 1: THE LAPLACIAN AS THE ANTI-LUMPINESS OPERATOR *(15 points)*

(a) [4 pts] Using divergence theorem,

$$\int_{r<R} \mathrm{d}^3x\, \vec{\boldsymbol{\nabla}} \cdot \left(g\vec{\boldsymbol{\nabla}}\varphi\right) = \oint_S \left(g\vec{\boldsymbol{\nabla}}\varphi\right) \cdot \mathrm{d}\vec{\boldsymbol{a}} = 0 \; . \tag{1.1}$$

The surface integral gives zero, because $g(R) = 0$ on the surface.

(b) [4 pts] Using ∂_i to denote $\partial/\partial x_i$, we can write,

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot (g\vec{\boldsymbol{\nabla}}\varphi) &= \partial_i(g\partial_i\varphi) = \partial_i g \partial_i \varphi + g\partial_i\partial_i\varphi \\ &= \vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}}\varphi + g\nabla^2\varphi \; . \end{aligned} \tag{1.2}$$

(c) [7 pts] Combining Eqs. (1.1) and (1.2), we have

$$\int_{r<R} \mathrm{d}^3x\, \vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}}\varphi + \int_{r<R} \mathrm{d}^3x\, g\nabla^2\varphi = 0 \; . \tag{1.3}$$

To evaluate the first integral, use

$$\vec{\boldsymbol{\nabla}} g(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}}\left(\frac{1}{r} - \frac{1}{R}\right) = -\frac{1}{r^2}\hat{\boldsymbol{e}}_{\boldsymbol{r}} \tag{1.4}$$

to express $\vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}}\varphi$ in polar coordinates:

$$\vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}}\varphi = -\frac{1}{r^2}\hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \left(\frac{\partial\varphi}{\partial r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\boldsymbol{e}}_{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right) = -\frac{1}{r^2}\frac{\partial\varphi}{\partial r} \; . \tag{1.5}$$

The first integral in Eq. (1.3) can then be evaluated:

$$\begin{aligned} \int_{r<R} d^3x \vec{\boldsymbol{\nabla}} g \cdot \vec{\boldsymbol{\nabla}}\varphi &= -\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^R r^2\, dr\, \frac{1}{r^2}\frac{\partial\varphi}{\partial r} \\ &= \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \left[\varphi(\vec{\mathbf{0}}) - \varphi(R,\theta,\phi)\right] \\ &= 4\pi\varphi(\vec{\mathbf{0}}) - \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \varphi(R,\theta,\phi) \\ &= 4\pi\left[\varphi(\vec{\mathbf{0}}) - \bar{\varphi}(R)\right] \; . \end{aligned} \tag{1.6}$$

Putting this result into Eq. (1.3),

$$4\pi\left[\varphi(\vec{\mathbf{0}})-\bar{\varphi}(R)\right]+\int_{r<R}\mathrm{d}^3x\left(\frac{1}{r}-\frac{1}{R}\right)\nabla^2\varphi=0$$

$$\Longrightarrow\quad \varphi(\vec{\mathbf{0}})-\bar{\varphi}(R)=-\frac{1}{4\pi}\int_{r<R}d^3x\left(\frac{1}{r}-\frac{1}{R}\right)\nabla^2\varphi. \tag{1.7}$$

PROBLEM 2: SPHERES AND IMAGE CHARGES *(10 points)*

Griffiths 3.9 (p. 129). We use the same notation as in Example 3.2 of Griffiths: a sphere of radius R and a charge q at a distance $a > R$ from the center. The image charge $q' = -Rq/a$ lies on the line connecting the center and the outside charge, a distance b away from the center, where $b = R^2/a$.

(a) [5 pts] The above charges produce zero potential on the surface of the sphere. To produce a constant potential V_0 at the sphere we need an extra charge q'' at the center, of magnitude defined by

$$V_0=\frac{1}{4\pi\epsilon_0}\frac{q''}{a}\qquad\Longrightarrow\qquad q''=(4\pi\epsilon_0)\,aV_0\,.$$

(b) [5 pts] The force on the charge q is equal to the force that would be applied by the two image charges q' and q'', even though these charges do not actually exist. The reason is that the electric field outside the sphere is identical to that which would be produced by the image charges, and the electric field in turn determines the force on q. So the force on q is given by

$$\begin{aligned}\vec{\boldsymbol{F}}_q&=\frac{1}{4\pi\epsilon_0}\left[\frac{qq''}{a^2}+\frac{qq'}{(a-b)^2}\right]\hat{\boldsymbol{r}}\\&=\frac{1}{4\pi\epsilon_0}\left[\frac{q(4\pi\epsilon_0)aV_0}{a^2}-\frac{q^2R}{a\left(a-\frac{R^2}{a}\right)^2}\right]\hat{\boldsymbol{r}}\\&=\left[\frac{qV_0}{a}-\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}\right]\hat{\boldsymbol{r}}\;.\end{aligned}$$

(c) [5 pts] For a neutral conducting sphere we need an extra image at the center of the sphere with charge $q'' = -q' = Rq/a$, so the total image charge is zero. The force due to the first image q' is attractive, and the force due to the second image q'' is repulsive, but lower in magnitude since the second image is further away. Thus, the net force felt by the outside charge q is attractive, and given by the vector expression

$$\begin{aligned}\vec{\boldsymbol{F}}_q &= \frac{1}{4\pi\epsilon_0} q q' \left[\frac{1}{(a-b)^2} - \frac{1}{a^2}\right] \hat{\boldsymbol{r}} \\ &= \frac{q^2}{4\pi\epsilon_0}\frac{R}{a}\left[\frac{-1}{\left(a-\frac{R^2}{a}\right)^2} + \frac{1}{a^2}\right]\hat{\boldsymbol{r}} \\ &= -\frac{q^2}{4\pi\epsilon_0}\frac{R^3}{a^3}\,\frac{2a^2-R^2}{(a^2-R^2)^2}\hat{\boldsymbol{r}}\ .\end{aligned}$$

Note that the force is infinite at $a = R$, where the distance to the image charge goes to zero. The force decreases monotonically with a, falling off as a^{-5} for large a.

PROBLEM 3: IMAGE CHARGES WITH A PLANE AND HEMISPHERICAL BULGE *(15 points)*

In this problem, we have a conducting plane with a hemispherical bulge of radius a. We use coordinates for which the conducting plane is the x-y plane, and the hemispherical bulge is centered at the origin. A charge q is placed at $(0, 0, z_0)$; i.e., it is placed on the line going though the center of the bulge perpendicularly to the surface; the distance between the charge and the center of the bulge is z_0, $z_0 > a$. We need to find the set of image charges describing the field outside of the conductor and the force between the charge q and the surface.

We will solve this problem in three steps:

1) put an image charge to cancel the potential of the charge q on the bulge,
2) do the same for the planar part of the conductor,
3) find the remaining image charges.

1. The image charge of the bulge.

Here, we can use the results for an image charge of a complete spherical conductor: the same image charge would cancel the potential of an external charge q on the surface of a hemisphere.

Thus, we place an image charge $q_b = -\frac{a}{z_0} q$ on the line connecting the center of the bulge and the charge (the z axis), at the distance $z_b = \frac{a^2}{z_0}$ away from the bulge's center, above the level of the plane. I.e., the charge q_b is located at $(0, 0, z_b) = (0, 0, a^2/z_0)$.

2. The image charge of the rest of the plane.

The potential of the external charge q on the plane outside of the bulge is cancelled by an image charge $q_p = -q$, positioned on the z axis below the plane, $z_p = -z_0$. I.e., the charge q_p is located at $(0, 0, z_p) = (0, 0, -z_0)$.

3. The balancing image charge.

The charges q_b and q_p do not make a complete set of image charges. Indeed, after we introduced the charge q_p, it created a nonzero potential on the bulge, and similarly the charge q_b distorts the potential of the plane. The complete set of image charges thus requires the additions of one or perhaps several additional charges.

Luckily, for this particular configuration, a single "balancing" charge can fix the potentials on both the bulge and the plane: we have to put a charge $q_{\text{bal}} = \frac{a}{z_0} q$ on the z axis, at $z_{\text{bal}} = \frac{-a^2}{z_0}$ below the plane. I.e., q_{bal} is placed at $(0, 0, z_{\text{bal}}) = (0, 0, -a^2/z_0)$.

Indeed, since $q_{\text{bal}} = -q_b$ and $z_{\text{bal}} = -z_b$, the balancing charge q_{bal} cancels the potential of the charge q_b on the plane:

$$V(\vec{\boldsymbol{r}}) = V_0 + V_b + V_p + V_{\text{bal}} = (V_0 + V_p) + (V_b + V_{\text{bal}}) = 0 + 0 = 0 \ .$$

And, because $q_{\text{bal}} = -\frac{a}{z_p} q_p$ and $z_{\text{bal}} = \frac{-a^2}{z_p}$, it will cancel the potential of the charge q_p on the surface of the bulge:

$$V(\vec{\boldsymbol{r}}) = V_0 + V_b + V_p + V_{\text{bal}} = (V_0 + V_b) + (V_p + V_{\text{bal}}) = 0 + 0 = 0 \ .$$

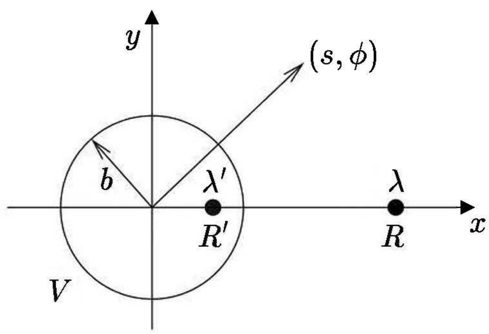
We conclude that the charges q_b, q_p and q_{bal} make a complete set of image charges.

The force on the charge q is:

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{tot}} &= \frac{q}{4\pi\epsilon_0}\left\{\frac{q_p}{|z_0 - z_p|^2} + \frac{q_b}{|z_0 - z_b|^2} + \frac{q_{\text{bal}}}{|z_0 - z_{\text{bal}}|^2}\right\}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{-q}{(2z_0)^2} + \frac{-\frac{a}{z_0}q}{\left(z_0 - \frac{a^2}{z_0}\right)^2} + \frac{\frac{a}{z_0}q}{\left(z_0 + \frac{a^2}{z_0}\right)^2}\right\}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{-\frac{q}{4z_0^2} - a z_0 q\left[\frac{1}{(z_0^2 - a^2)^2} - \frac{1}{(z_0^2 + a^2)^2}\right]\right\}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{-\frac{q}{4z_0^2} - a z_0 q\left[\frac{(z_0^2 + a^2)^2 - (z_0^2 - a^2)^2}{(z_0^2 - a^2)^2(z_0^2 + a^2)^2}\right]\right\}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{-\frac{q}{4z_0^2} - a z_0 q\left[\frac{4z_0^2 a^2}{(z_0^4 - a^4)^2}\right]\right\}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \boxed{-\frac{1}{4\pi\epsilon_0}\left[\frac{q^2}{4z_0^2} + \frac{4q^2 a^3 z_0^3}{(z_0^4 - a^4)^2}\right]\hat{\boldsymbol{e}}_{\boldsymbol{z}} \ .}
\end{aligned}$$

PROBLEM 4: IMAGES FOR A CONDUCTING CYLINDER *(15 points)*

(a) [10 pts] We attempt to solve the problem by introducing an image line of charge with a linear charge density λ' at a distance R' from the z axis along the line connecting the z axis to the line of charge with linear charge density λ. Symmetry requires that the image line of charge must be parallel to the z axis, due to translation invariance in the z direction, and along the line joining the axis with the external line of charge, due to the reflection symmetry $y \leftrightarrow -y$.



The electric field of a straight line charge is simply $\frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$, where r is the distance from the line and $\hat{r}$ is a unit vector pointing outward from the line, as can be found from Gauss's law. Then the integral yields a logarithmic potential which is infinite in magnitude at both $r = 0$ and $r = \infty$. It is therefore necessary to pick a reference point at some arbitrary distance.

We choose an arbitrary reference point for the potential for each of the line sources, taking the potential to be zero at a distance r_0 from the line. We can add the potentials of the two line charges to find an expression for the total potential:

$$\begin{aligned} V(s,\phi) &= \frac{1}{2\pi\epsilon_0}\left(\lambda \ln\frac{r_0}{r} + \lambda' \ln\frac{r_0}{r'}\right) + V_c = \\ &= \frac{1}{4\pi\epsilon_0}\left[\lambda \ln\left(\frac{r_0^2}{s^2 + R^2 - 2sR\cos\phi}\right) + \lambda' \ln\left(\frac{r_0^2}{s^2 + R'^2 - 2sR'\cos\phi}\right)\right] + V_c \ , \end{aligned}$$

where r and r' are the distances from λ and λ' respectively, and V_c is an arbitrary constant, which gives us the most general possible expression for the potential of the two lines of charge. Since we are given that the potential difference between the cylinder and $s = \infty$ is finite, we require $\lim_{s\to\infty} V(s,\phi) - V(b,0) < \infty$. By using

$$\begin{aligned} \ln\left(\frac{s^2 + R^2 - 2sR\cos\phi}{r_0^2}\right) &= 2\ln\frac{s}{r_0} + \ln\left(1 - 2\frac{R}{s}\cos\phi + \frac{R^2}{s^2}\right) \\ &= 2\ln\frac{s}{r_0} - 2\left(\frac{R}{s}\right)\cos\phi + O(R/s)^2 \ , \end{aligned}$$

one can see that for large s,

$$V(s) - V(b,0) \sim -\frac{\lambda + \lambda'}{2\pi\epsilon_0} \ln(s/r_0) + V_c - V(b,0) + O(R/s) \ .$$

The only way to prevent the potential V from growing to infinity as $s \to \infty$ is to choose $\lambda' = -\lambda$. Using this, we can rewrite the expression for the potential as:

$$V(s,\phi) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{s^2 + R'^2 - 2sR'\cos\phi}{s^2 + R^2 - 2sR\cos\phi}\right) + V_c \ .$$

To determine the location R', we impose the constant voltage boundary condition $V(b,\phi) = V_0$. This implies

$$\frac{b^2 + R'^2 - 2bR'\cos\phi}{b^2 + R^2 - 2bR\cos\phi} = c \ ,$$

where c is an unknown constant. Rearranging the terms, we get

$$(1-c)b^2 + R'^2 - cR^2 = 2b(R' - cR)\cos\phi \ .$$

Since we need this equation to hold for any angle ϕ along the cylinder, we see that both sides have to independently vanish. The RHS of the equation gives $R' = cR$. Putting $c = R'/R$ into the equation we get a quadratic equation for R' with two solutions:

$$R'_1 = \frac{b^2}{R} \quad , \quad R'_2 = R \ .$$

Both solutions give V = constant on the surface of the cylinder, which is the requirement that we used in deriving the quadratic equation. The second solution ($R' = R$), however, puts the second line charge on top of the first, canceling the electric field entirely! This is not the image charge that we seek, which must always lie outside the region in which we are trying to determine the fields. Therefore only the $R'_1 = b^2/R$ solution is relevant to our problem.

This problem could alternatively be solved using the electric fields from the two line charges. Since the cylinder is a conductor, the electric field just outside its surface must be normal to the surface. Equating the tangential electric field $\vec{\boldsymbol{E}}_\parallel$ to zero yields the same results.

(b) [5 pts] The condition $\lim\limits_{s\to\infty} V(s,\phi) = 0$ allows us to find V_c:

$$V(s \to \infty) \sim V_c + O(R/s) \ ,$$

which gives us $V_c = 0$. The potential V_0 is then given as:

$$V(b,\phi) = V_0 = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{b^2 + R'^2 - 2bR'\cos\phi}{b^2 + R^2 - 2bR\cos\phi}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln c = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{R}.$$

PROBLEM 5: CAPACITANCE OF A SINGLE CONDUCTOR *(20 points)*

(a) [4 pts] The potential energy W of the capacitor is $W = \frac{1}{2}QV_0 = \frac{1}{2}CV_0^2$. This energy is electrostatic energy given by the expression: $W = \frac{\epsilon_0}{2}\int_{\mathcal{V}} |\vec{\boldsymbol{E}}|^2 d^3x = \frac{\epsilon_0}{2}\int_{\mathcal{V}} |\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}})|^2 d^3x$, where $V(\vec{\boldsymbol{r}})$ is the potential on the volume $\mathcal{V}$ outside the conductor. Comparing the two expressions gives:

$$C = \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} |\vec{\boldsymbol{\nabla}}V|^2 d^3x\,. \tag{5.1}$$

(b) [4 pts] Write the trial function Ψ as

$$\Psi = V + \delta V\ . \tag{5.2}$$

Here Ψ satisfies the boundary conditions, thus $\delta V = 0$ on the boundary S of the conductor, and also at infinity.

$$\begin{aligned} C[\Psi] &= \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} |\vec{\boldsymbol{\nabla}}\Psi|^2 d^3x = \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} \left(|\vec{\boldsymbol{\nabla}}V|^2 + 2\vec{\boldsymbol{\nabla}}V\cdot\vec{\boldsymbol{\nabla}}\delta V + |\vec{\boldsymbol{\nabla}}\delta V|^2\right) d^3x \\ &= C + \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} \vec{\boldsymbol{\nabla}}\delta V|^2 d^3x + \underbrace{\frac{2\epsilon_0}{V_0^2}\int_{\mathcal{V}}\left[\vec{\boldsymbol{\nabla}}\cdot(\delta V\,\vec{\boldsymbol{\nabla}}V) - \delta V\,\nabla^2 V\right]\mathrm{d}^3x}_{I_1}\ . \end{aligned} \tag{5.3}$$

To deal with the last integral above, denoted by I_1, note that $V(\vec{\boldsymbol{r}})$ is the potential outside the conductor when the conductor is held at potential V_0, so $\nabla^2 V = 0$. The remaining integral can be turned into a surface integral by use of the divergence theorem:

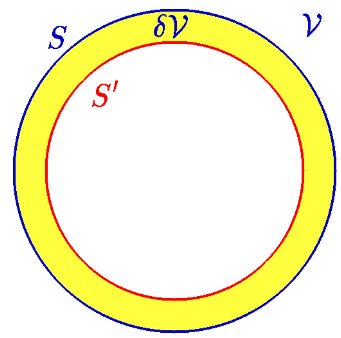
$$I_1 = \frac{2\epsilon_0}{V_0^2}\int_S \delta V\,\vec{\boldsymbol{\nabla}}V\cdot\mathrm{d}\vec{\boldsymbol{a}} + \frac{2\epsilon_0}{V_0^2}\int_\infty \delta V\,\vec{\boldsymbol{\nabla}}V\cdot\mathrm{d}\vec{\boldsymbol{a}}\ , \tag{5.4}$$

where the second term is an integral over a large spherical surface at infinity. The first term clearly vanishes, since $\delta V = 0$ on the boundary of the conductor S. $\delta V = 0$ at infinity also, but we need to be more careful, since the integration is over an infinity area. If we let R denote the radius of the sphere, which will be taken to infinity, then the area grows as R^2. But $\vec{\boldsymbol{\nabla}}V$ is the negative of the electric field, so its magnitude falls off as $1/R^2$, canceling the infinity of the area. In fact we know that $\int_\infty \vec{\boldsymbol{\nabla}}V\cdot\mathrm{d}\vec{\boldsymbol{a}} = -Q/\epsilon_0$, where Q is the charge on the conductor. Since the infinity of the surface area is canceled by the falling off of $\vec{\boldsymbol{\nabla}}V$, the integral will vanish if $\delta V \to 0$ as $R \to \infty$, no matter how slowly. Thus $I_1 = 0$, and Eq. (5.4) becomes

$$C[\Psi] = C + \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} |\vec{\boldsymbol{\nabla}}\delta V|^2 d^3x \geq C\ . \tag{5.5}$$

The integrand above is nonnegative, so $C[\Psi]$ can equal C only if $\vec{\nabla}\delta V = 0$ everywhere. Since $\delta V = 0$ on S, $C[\Psi]$ is in fact strictly bigger than C whenever δV is nonzero anywhere in $\mathcal{V}$. In other words, $C[\Psi] > C$ whenever $\Psi(\vec{x})$ is not equal to the true potential function.

(c) [7 pts] Let $\mathcal{V}'$ be the volume outside of the surface S' of the smaller conductor, which we imagine fits entirely inside the larger conductor whose boundary is S. Assume that we hold the smaller conductor at potential V_0. We will use $\delta\mathcal{V}$ to denote the volume enclosed between S' and S, so $\delta\mathcal{V} = \mathcal{V}' - \mathcal{V}$, as shown in the diagram at the right. To find an upper bound for C', let's use the trial function:



$$\Psi(\vec{x}) = \begin{cases} V(\vec{x}) & \text{for } \vec{x} \in \mathcal{V} \\ V_0 & \text{for } \vec{x} \in \delta\mathcal{V}\ . \end{cases} \tag{5.6}$$

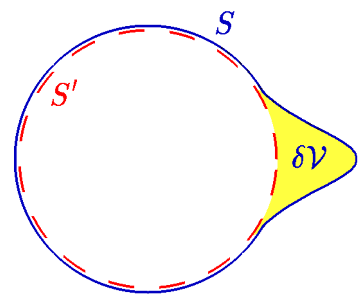
Note that for the region outside the surface S of the larger conductor, this trial function is equal to the true potential for the larger conductor, when the conductor is held at potential V_0. In the region between S' and S, it has the constant value V_0. Thus the trial function $\Psi(\vec{x})$ satisfies the correct boundary condition ($\Psi = V_0$) at S', and it is continuous at S and throughout the region $\mathcal{V}'$ outside of the smaller conductor. Using this trial function we get

$$C' \le C'[\Psi] \equiv \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}'} |\vec{\nabla}\Psi|^2 \mathrm{d}^3x = \frac{\epsilon_0}{V_0^2}\int_{\delta\mathcal{V}} |\vec{\nabla}\Psi|^2 \mathrm{d}^3x + \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} |\vec{\nabla}\Psi|^2 \mathrm{d}^3x\ . \tag{5.7}$$

Since $\vec{\nabla}\Psi = 0$ in $\delta\mathcal{V}$ we get

$$C' \le \frac{\epsilon_0}{V_0^2}\int_{\mathcal{V}} |\vec{\nabla}V|^2 \mathrm{d}^3x = C\ . \tag{5.8}$$

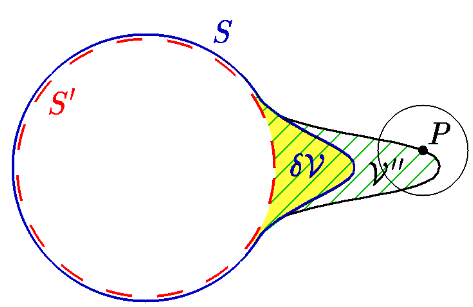
As discussed at the end of the answer to (b), C' will be strictly less than C unless the trial function $\Psi(\vec{x})$ is exactly equal to the true potential function $V'(\vec{x})$ for the smaller conductor. While it seems obvious that $\Psi(\vec{x})$ is not the true potential, we will nonetheless give a proof. We suppose that S' is completely in the interior of S, as shown in the diagram above, so that S' is completely surrounded by the region we called $\delta\mathcal{V}$. In this region $\vec{\nabla}\Psi = 0$, so if Ψ were the potential we would have $\vec{E}(\vec{x}) = 0$ over the surface S', and then by Gauss's law the total charge on the (smaller) conductor would be zero. Since $Q = CV_0 \neq 0$, $\Psi(\vec{x})$ cannot be the true potential $V'(\vec{x})$, and therefore the capacitance C' must be strictly less than C.



You were not asked to go beyond the answer given above, but the theorem also applies to cases in which the smaller conductor never extends outside the larger, but for which there are places where the two surfaces coincide. The smaller conductor might, for example, be a sphere, and the larger conductor can be the same sphere, but with a bump sticking out, as shown in the diagram at the right. We assume of course that the two surfaces are not identical, so the region $\delta\mathcal{V}$ is nonempty. For this situation it is still possible to use the trial function given by Eq. (5.6), and the argument leading to Eq. (5.8) remains valid. So we again have shown that

$$C' \leq C \ . \tag{5.9}$$

For this case, however, it is harder to prove that C' is strictly less than C. As in the previous case, however, we can infer that $C' < C$ if we can prove that $\Psi(\vec{x})$ cannot be equal to the true potential $V'(\vec{x})$. This fact can be proven by using the Laplacian mean value theorem: the statement that for any solution to $\nabla^2 V' = 0$, the value of V' at any point is equal to the average value of V' on any sphere centered on that point. From this theorem, we know that the maximum of $V'(\vec{x})$ (taking $V_0 > 0$) must be on the conductor surface S', the boundary of the region where $\nabla^2 V' = 0$, because the theorem prevents the function from having a maximum or minimum in the interior. Thus $V'(\vec{x}) \leq V_0$ throughout $\mathcal{V}'$. Now we can give a proof by contradiction that $V'(\vec{x})$ cannot be equal to $\Psi(\vec{x})$. If we suppose that they are equal, then we know that $V'(\vec{x}) = V_0$ in the region $\delta\mathcal{V}$. Let $\mathcal{V}''$ be the region for which $V'(\vec{x}) = V_0$, which includes $\delta\mathcal{V}$ but which could conceivably be larger. In the diagram at the right, $\mathcal{V}''$ is shown as the hatched region. Now consider a point P on the outer boundary of $\mathcal{V}''$, and consider a sphere that is centered at P, but small enough to lie outside S'. $V'(P) = V_0$ by construction, but the sphere around P contains points inside $\mathcal{V}''$ with $\Psi = V_0$, but also points outside $\mathcal{V}''$ for which $V' < V_0$. Thus the average value of V' on the sphere is less than V_0, so V' could not satisfy $\nabla^2 V' = 0$. Since the true potential satisfies $\nabla^2 V' = 0$, we have the desired contradiction.



(d) [5 pts] A cube of side a can be contained in a sphere of radius $R_1 = \frac{\sqrt{3}}{2}a$ and it can contain a sphere of radius $R_2 = \frac{1}{2}a$. Moreover, the capacitance of a conducting sphere is given by $C = 4\pi\epsilon_0 R$ (this is a simple exercise). Thus

$$4\pi\epsilon_0 \left(a \, \frac{\sqrt{3}}{2} \right) > C_{\rm cube} > 4\pi\epsilon_0 \left(a \, \frac{1}{2} \right)$$

$$0.866\,(4\pi\epsilon_0 a) > C_{\rm cube} > 0.5\,(4\pi\epsilon_0 a)$$

The average of the two bounds is $0.683(4\pi\epsilon_0 a)$. This is about 3.3% off the precise numerical estimate for the capacitance $0.661(4\pi\epsilon_0 a)$.

PROBLEM 6: A SPHERICAL CONDUCTOR AND A CONDUCTING PLANE *(30 points)*

(a) [5 pts] The potential on the x-y plane can be restored to zero by placing an image charge below the x-y plane. The image charge $q' = -q_0$ must be placed at $z = -z_0$:

$$\boxed{q' = -q_0, \qquad \text{at} \quad z' = -z_0 \ .} \tag{6.1}$$

(b) [10 pts] The potential on the surface of the spherical conductor is now no longer constant, but it can be made constant by adding another image charge q''. The distance from q' to the center of conducting sphere is $2z_0$. Recalling the image charge problem of a point charge and conducting sphere, we cancel the potential due to q' on the surface of the sphere by an image charge

$$\boxed{q'' = -q' \frac{R}{2z_0} = q_0 \frac{R}{2z_0}, \qquad \text{at} \quad z'' = z_0 - \frac{R^2}{2z_0} = z_0 \left(1 - \frac{R^2}{2z_0^2} \right) \ .} \tag{6.2}$$

Then the potential on the x-y plane should be restored to zero by adding another image charge q''',

$$\boxed{q''' = -q'' = -q_0 \frac{R}{2z_0}, \qquad \text{at} \quad z''' = -z'' = -z_0 \left(1 - \frac{R^2}{2z_0^2} \right) \ .} \tag{6.3}$$

Again the potential on the sphere can be restored to a constant by adding yet another image charge q''''. The distance of q''' to the center of the conducting sphere is $|z'''| + z_0 = 2z_0 \left(1 - \frac{R^2}{4z_0^2}\right)$, so

$$\boxed{q'''' = -q''' \frac{R}{2z_0 \left(1 - \frac{R^2}{4z_0^2}\right)} = q_0 \frac{R^2}{4z_0^2 \left(1 - \frac{R^2}{4z_0^2}\right)}, \ \text{at} \ z'''' = z_0 - \frac{R^2}{2z_0 \left(1 - \frac{R^2}{4z_0^2}\right)} \ .} \tag{6.4}$$

(c) [5 pts] The potential of the conducting sphere is the superposition of the potentials of all the image charges. But the image charges have been arranged to give cancelations, so on the surface of the sphere $V_{q'} + V_{q''} = 0$ and $V_{q'''} + V_{q''''} = 0$. Thus we find that, on the surface of the sphere,

$$\boxed{V_{\text{sphere}} = V_{q_0} = V_0 \ .} \tag{6.5}$$

(d) [5 pts] The total energy can be found by using the general formula

$$W = \frac{1}{2} \int \rho V \, \mathrm{d}^3 x \ . \tag{6.6}$$

Before we begin thinking about image charges, we can apply this formula to the stated problem. Only real charges are to be included in Eq. (6.6), so for this problem we need to consider the charge on the surface of the sphere, Q_{sphere}, and the charge on the surface of the conductor. But the charge on the conductor is at $V = 0$, so Eq. (6.6) reduces to

$$W = \frac{1}{2} Q_{\text{sphere}} V_0 \ . \tag{6.7}$$

The charge on the surface of the sphere is equal to the sum of the charges of the image charges inside the sphere, so

$$Q_{\text{sphere}} = q_0 + q'' + q'''' + q'''''' + \dots \ , \tag{6.8}$$

and so

$$\boxed{\begin{aligned} W &= \frac{1}{2} V_0 (q_0 + q'' + q'''' + \dots) \\ &= \frac{q_0^2}{8\pi\epsilon_0 R} \left[1 + \frac{R}{2z_0} + \frac{R^2}{4z_0^2 - R^2} + \dots \right] \ . \end{aligned}} \tag{6.9}$$

It is tempting to try to find W by treating the image charges as if they were real point charges, and then calculating half the potential energy of the point charge configuration. We found that this works for the case of a point charge and a conducting plane, and for the case of a point charge and a grounded sphere. But it is not a general result, and it is not valid in this case. (It is valid for the case of a conducting plane, with any charge distribution, and it is also valid for any problem in which there is one real point charge and one image point charge.) By contrast, Eq. (6.6) is a universally valid relation for electrostatics.

The force on the conducting sphere can be calculated by the method of virtual work, which relates it to the derivative of W with respect to z_0. But there is a paradox that needs to be resolved: Eq. (6.9) implies that $\mathrm{d}W/\mathrm{d}z_0 < 0$, which suggests that the force is upward. But the induced charge on the plate has the opposite charge as the sphere, so the force must really be downward. Can you figure out what is missing in this discussion? (Hint: see Problem 4 from Problem Set 2.)

(e) [5 pts] Having a conducting shell instead of a sphere doesn't change the boundary conditions: $V = V_0$ on the surface of the sphere and $V = 0$ on the conducting plane. Since there are no charges in the region bounded by the conducting plane and the sphere, the first uniqueness theorem in Griffiths' treatment assures that the potential in the two cases is the same.

PROBLEM 7: LAPLACE'S EQUATION IN A BOX *(20 points)*

Griffiths Problem 3.16 (p. 141).

We use separation of variables in Cartesian coordinates: $V(x,y,z) = X(x)Y(y)Z(z)$, where Laplace's equation becomes

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = C_1\ , \quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = C_2\ , \quad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = C_3\ , \quad \text{with } C_1 + C_2 + C_3 = 0. \tag{7.1}$$

Since the potential is required to vanish for $x = 0$ and $x = a$, we use the functions

$$\sin\left(\frac{m\pi x}{a}\right)$$

to describe $X(x)$, with an analogous choice for $Y(y)$. This implies that C_1 and C_2 are negative, so $C_3 \equiv \gamma^2$ must be positive. Thus $Z(z)$ must be expressed as a linear combination of $e^{\gamma z}$ and $e^{-\gamma z}$, or as a linear combination of $\sinh\gamma z$ and $\cosh\gamma z$. The boundary condition that $V = 0$ at $z = 0$ is most easily enforced by the latter choice, in which case the boundary condition simply implies that the coefficient of the cosh term must be zero. Thus we can choose basis functions

$$\Phi_{mn}(x,y,z) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\frac{\sinh(\gamma_{mn}z)}{\sinh(\gamma_{mn}a)}, \quad \gamma_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\ . \tag{7.2}$$

Note that we divided by a constant in the factor relevant to the z-direction, in order to have a simple behavior at $z = a$.

The desired potential $V(x,y,z)$ is a superposition of the above ones, with coefficients V_{mn} to be determined:

$$V(x,y,z) = \sum_{m,n\geq 1} V_{mn}\Phi_{mn}(x,y,z)\ . \tag{7.3}$$

The boundary conditions on all the faces except for the one at $z = a$ have already been satisfied. The boundary condition at $z = a$ is satisfied if

$$V_0 = \sum_{m,n\geq 1} V_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) , \tag{7.4}$$

where V_0 is a the constant specified in the problem. The V_{mn} are readily calculated by using the orthogonality relation

$$\int_0^a \mathrm{d}x \, \sin\left(\frac{m'\pi x}{a}\right)\left(\frac{m\pi x}{a}\right) = \frac{a}{2}\delta_{m'm} \, . \tag{7.5}$$

We multiply both sides of Eq. (7.4) by $\sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{a}\right)$ and integrate over x and y:

$$\begin{aligned} &\int_0^a \mathrm{d}x \, \sin\left(\frac{m'\pi x}{a}\right) \int_0^a \mathrm{d}y \, \sin\left(\frac{n'\pi y}{a}\right) V_0 \\ &\quad = \sum_{m,n\geq 1} V_{mn} \int_0^a \mathrm{d}x \, \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \int_0^a \mathrm{d}y \, \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \\ &\quad = \frac{a^2}{4} \sum_{m,n\geq 1} V_{mn}\delta_{m'm}\delta_{n'n} = \frac{a^2}{4} V_{mn} \, . \end{aligned} \tag{7.6}$$

Thus

$$V_{mn} = \frac{4V_0}{a^2} \int_0^a \mathrm{d}x \, \sin\left(\frac{m\pi x}{a}\right) \int_0^a \mathrm{d}y \, \sin\left(\frac{n\pi y}{a}\right) = \begin{cases} \dfrac{16V_0}{\pi^2}\dfrac{1}{mn} & \text{for } m,\, n \text{ odd} \\ 0 & \text{otherwise} \, . \end{cases} \tag{7.7}$$

Therefore, the full potential is given by Eq. (7.3), with V_{nm} given above:

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{\text{odd } m,n} \frac{1}{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \frac{\sinh(\gamma_{mn} z)}{\sinh(\gamma_{mn} a)} , \tag{7.8}$$

where γ_{mn} is given in Eq. (7.2).

Finally, we can find the potential in the center of the box $V(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$ using the superposition principle and the symmetry of the system. The superposition principle — which here is simply the fact that the Laplacian is a linear operator — implies that the sum of any two solutions to Laplace's equation is also a solution. Using the cubical symmetry of this problem, for each face of the cube we can easily construct a solution for which the chosen face would be at potential V_0, and the other 5 faces would be at $V = 0$. These solutions would simply be rotations of the solution we found. Again by symmetry, each of these 6 solutions would have the same value of V at the center. By superposition, the sum of these 6 solutions must also be a solution. But the sum of all 6 would have $V = V_0$ on each of the 6 walls, and would therefore be the trivial solution $V = V_0$ everywhere inside the cube. If the sum of the solutions gives $V = V_0$ at the center, and each contributes equally, then each solution must have $V = \frac{1}{6}V_0$ at the center. So

$$V\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right) = \frac{V_0}{6} \ . \tag{7.9}$$

To check this numerically, we can evaluate $6V(a/2, a/2, a/2)/V_0$ from the first terms of Eq. (7.8), to see if we come close to finding 1. We can build a small table, summing from $(m, n) = (1, 1)$ to $(m_{\rm max}, n_{\rm max})$:

$m_{\rm max}$	$n_{\rm max}$	$\frac{6}{V_0} V\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$
1	1	1.04264
1	3	1.02007
3	1	1.02007
3	3	0.998873
5	5	1.00004
7	7	0.999999
9	9	1.000000

So, the numerical evaluation is very much consistent with the answer of $V_0/6$.