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LECTURE NOTES 4

TRACELESS SYMMETRIC TENSOR APPROACH TO LEGENDRE POLYNOMIALS AND SPHERICAL HARMONICS, PART IV

In this last set of notes on the use of traceless symmetric tensors, I will give a proof of what, in Lecture Notes 1, I called the Trace Decomposition Theorem:

Trace Decomposition Theorem: Any symmetric tensor $S_{i_1 \ldots i_\ell}$ can be written uniquely as the sum of a traceless symmetric tensor and a tensor which can be written in terms of a Kronecker δ-function, as

$$S_{i_1 \ldots i_\ell} = S^{(\mathrm{TS})}_{i_1 \ldots i_\ell} + \underset{i_1 \ldots i_\ell}{\mathrm{Sym}} \left[M_{i_1 \ldots i_{\ell-2}} \delta_{i_{\ell-1}, i_\ell} \right] \ , \tag{4.1}$$

where $S^{(\mathrm{TS})}_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor, $M_{i_1 \ldots i_{\ell-2}}$ is a symmetric tensor, and

$$\underset{i_1 \ldots i_\ell}{\mathrm{Sym}} [xxx] \tag{4.2}$$

means to symmetrize the expression xxx in the indices $i_1 \ldots i_\ell$.

This theorem was crucial to the general definition of the traceless symmetric part of a tensor: we defined the traceless symmetric part of $S_{i_1 \ldots i_\ell}$ to be the tensor $S^{(\mathrm{TS})}_{i_1 \ldots i_\ell}$ appearing on the right-hand side of Eq. (4.1).

In these notes I will state and prove a more detailed form of the trace decompostion theorem, of which the statement above will be a corollary. I will also derive Eqs. (2.35)–(2.37), which show explicitly how to write the traceless symmetric part of a symmetric tensor in terms of the tensor and its multiple traces. In Lecture Notes 2 we derived the first nontrivial term of this series, but here I will derive the general term.

I am including this material for completeness, but for purposes of the course you will not be required to use it in any way. There will be no assigned problems that depend on this material, and it will not appear on Quiz 2 or the final exam. It is pure culture, no requirements. Read it if you are interested. Maybe save it in your achives in case you want to refer to it 10 years from now.

12. PROOF OF THE TRACE DECOMPOSITION THEOREM:

I don't know a really simple proof of the trace decomposition theorem, even though it seems like a very simple statement. It is possible that I am missing something. If any of you invent a simpler proof, please let me know.

I will start with a more detailed version:

Detailed Trace Decomposition Theorem: Any symmetric tensor $S_{i_1 \ldots i_\ell}$ can be uniquely written as

$$\boxed{S_{i_1 \ldots i_\ell} = S^{(0)}_{i_1 \ldots i_\ell} + \operatorname*{Sym}_{i_1 \ldots i_\ell} \left[S^{(1)}_{i_1 \ldots i_{\ell-2}} \, \delta_{i_{\ell-1}, i_\ell} + S^{(2)}_{i_1 \ldots i_{\ell-4}} \, \delta_{i_{\ell-3}, i_{\ell-2}} \, \delta_{i_{\ell-1}, i_\ell} + \ldots \right] \ ,} \tag{4.3}$$

where all the tensors $S^{(k)}$ are traceless, and each term in the sum includes one more Kronecker δ-function than the previous term. The sum continues until no more Kronecker δ-functions are possible, when the number of indices on the accompanying $S^{(k)}$ factor is zero or one.

The theorem can be proven by using recursion and some brute force, but it helps to introduce some simplifying notation. First, we recognize that a symmetric tensor of rank ℓ has ℓ indices, which we can generically take as $i_1 \ldots i_\ell$. As long as the indices are taken as $i_1 \ldots i_\ell$, we can suppress them without loss of information, referring to the tensor $S_{(\ell), i_1 \ldots i_\ell}$ simply as $S_{(\ell)}$. To simplify the notation of the right-hand side of Eq. (4.3), we can define a shorthand notation for the operation of appending a Kronecker δ-function to a tensor and then symmetrizing:

$$\left[S_{(\ell)} \circ \delta \right]_{i_1 \ldots i_{\ell+2}} \equiv \operatorname*{Sym}_{i_1 \ldots i_{\ell+2}} \left[S_{(\ell), i_1 \ldots i_\ell} \, \delta_{i_{\ell+1}, i_{\ell+2}} \right] \ . \tag{4.4}$$

The operation can be applied multiple times, so

$$\Big[S_{(\ell)} \circ \underbrace{\delta \circ \ldots \circ \delta}_{n \text{ times}} \Big]_{i_1 \ldots i_{\ell+2n}} = \operatorname*{Sym}_{i_1 \ldots i_{\ell+2n}} \Big[S_{(\ell), i_1 \ldots i_\ell} \, \underbrace{\delta_{i_{\ell+1}, i_{\ell+2}} \ldots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}}_{n \text{ times}} \Big] \ . \tag{4.5}$$

Note that in this compact notation, square brackets [ ] implicitly symmetrize over all the indices of the object inside them. Then I can further abbreviate the notation by defining

$$\left[S_{(\ell)} \circ \delta^n \right] \equiv \Big[S_{(\ell)} \circ \underbrace{\delta \circ \ldots \circ \delta}_{n \text{ times}} \Big] \ . \tag{4.6}$$

With this notation, the equation of the detailed trace theorem, Eq. (4.3), can be written compactly as

$$\boxed{S_{(\ell)} = S^{(0)}_{(\ell)} + \left[S^{(1)}_{(\ell-2)} \circ \delta\right] + \left[S^{(2)}_{(\ell-4)} \circ \delta^2\right] + \ldots\ .} \tag{4.7}$$

The rank of a tensor can be increased by appending a Kronecker δ-function as shown above, but it can also be reduced by the operation of tracing on two of the indices. Since we are dealing exclusively with symmetric tensors, it does not matter which two indices are traced, so we can denote the operation simply by Tr:

$$\left[\operatorname{Tr} S_{(\ell)}\right]_{i_1 \ldots i_{\ell-2}} \equiv S_{(\ell), i_1 \ldots i_\ell}\, \delta_{i_{\ell-1}, i_\ell}\ . \tag{4.8}$$

It will be useful to know the result of taking the trace of $\left[S_{(\ell)} \circ \delta^n\right]$, for the special case where $S_{(\ell)}$ is traceless. That is, we want to evaluate

$$T_{(\ell+2n-2)} \equiv \operatorname{Tr}\left[S_{(\ell)} \circ \delta^n\right]\ . \tag{4.9}$$

Writing it with explicit indices,

$$\begin{aligned}
\left[T_{(\ell+2n-2)}\right]_{i_1 \ldots i_{\ell+2n-2}} &\equiv \delta_{i_{\ell+2n-1}, i_{\ell+2n}} \operatorname*{Sym}_{i_1 \ldots i_{\ell+2n}} \left[S_{(\ell), i_1 \ldots i_\ell}\, \delta_{i_{\ell+1}, i_{\ell+2}} \ldots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}\right] \\
&= \delta_{i_{\ell+2n-1}, i_{\ell+2n}} \frac{1}{(\ell+2n)!} \\
&\qquad \times \sum_{\substack{\text{all } (\ell+2n)! \text{ index} \\ \text{orderings of } i_1 \ldots i_{\ell+2n}}} \left(S_{(\ell), i_1 \ldots i_\ell}\, \delta_{i_{\ell+1}, i_{\ell+2}} \ldots \delta_{i_{\ell+2n-1}, i_{\ell+2n}}\right)\ .
\end{aligned} \tag{4.10}$$

The calculation is similar to two calculations that we have done before: the evaluations of Eq. (2.38) and (3.48). In both cases, we wrote the symmetrization as a sum over all orderings, as we did above, and then we divided the orderings into convenient sets of cases.

Case I: Both $i_{\ell+2n-1}$ and $i_{\ell+2n}$ appear on the Kronecker δ-functions inside the sum, and they appear on the same δ-function. In that case the two δ-functions contract to give a factor of 3, and the remaining factors give, after symmetrization, $\left[S_{(\ell)} \circ \delta^{n-1}\right]$. The number of orderings that fall into this case is given by a factor of n, since $i_{\ell+2n-1}$ and $i_{\ell+2n}$ can appear on any of n δ-functions, times a factor of 2, because $i_{\ell+2n-1}$ and $i_{\ell+2n}$ can match the indices on the δ function in either order, and finally times a factor of $(\ell + 2n - 2)!$, since the remaining $\ell + 2n - 2$ indices can occur in any order. Summarizing,

$$\text{Case I:} \qquad \text{Multiplicity} = 2n(\ell + 2n - 2)! \qquad \text{Value} = \frac{3}{(\ell + 2n)!}\left[S_{(\ell)} \circ \delta^{n-1}\right] . \tag{4.11}$$

Case II: Both $i_{\ell+2n-1}$ and $i_{\ell+2n}$ appear on Kronecker δ-functions inside the sum, but they occur on different Kronecker δ-functions. In this case we do not find the factor of 3 from δ_{ii}, so the value is simply $\left[S_{(\ell)} \circ \delta^{n-1}\right]$. The multiplicity includes a factor of $n(n-1)$, since $i_{\ell+2n-1}$ can appear on any of n Kronecker δ-functions, and $i_{\ell+2n}$ can then appear on any of $n-1$. There is then a factor of 4, since each of $i_{\ell+2n-1}$ and $i_{\ell+2n}$ can appear as either the first or second subscript of the δ-function. And finally there is a factor of $(\ell + 2n - 2)!$, to account for all perturbations of the remaining indices. Summarizing,

$$\begin{gathered} \text{Case II:} \qquad \text{Multiplicity} = 4n(n-1)(\ell + 2n - 2)! \\ \text{Value} = \frac{1}{(\ell + 2n)!}\left[S_{(\ell)} \circ \delta^{n-1}\right] . \end{gathered} \tag{4.12}$$

Case III: One of $i_{\ell+2n-1}$ and $i_{\ell+2n}$ appears on a Kronecker δ-function, and the other appears on $S_{(\ell)}$. After summing over the repeated indices and symmetrizing over the remaining indices, the value is $\left[S_{(\ell)} \circ \delta^{n-1}\right]$, the same as the previous case. The multiplicity contains a factor of 2 from the choice of which of the two indices $i_{\ell+2n-1}$ and $i_{\ell+2n}$ are on the Kronecker δ-function, and which is on $S_{(\ell)}$. The index on the Kronecker δ-functions can be any of $2n$ choices, and the index on $S_{(\ell)}$ can be any of ℓ choices. And, as in all cases, there are $(\ell+2n-2)!$ possible orderings of the remaining indices. So the multiplicity is $4n\ell(\ell + 2n - 2)!$. So finally,

$$\begin{gathered} \text{Case III:} \qquad \text{Multiplicity} = 4n\ell(\ell + 2n - 2)! \\ \text{Value} = \frac{1}{(\ell + 2n)!}\left[S_{(\ell)} \circ \delta^{n-1}\right] . \end{gathered} \tag{4.13}$$

Case IV: Both $i_{\ell+2n-1}$ and $i_{\ell+2n}$ appear on $S_{(\ell)}$. Since $S_{(\ell)}$ is traceless, the value is clearly zero. Nonetheless I will calculate the multiplicity, so that I can check that the sum of my computed multiplicities adds up to $(\ell + 2n)!$. The multiplicity includes a factor of $\ell(\ell - 1)$, because $i_{\ell+2n-1}$ can be any of ℓ choices, and then $i_{\ell+2n}$ can be any of $(\ell - 1)$ choices. Then there is, as in the other cases, a factor of $(\ell + 2n - 2)!$, since the remaining $\ell + 2n - 2$ indices can occur in any order. Thus,

$$\text{Case IV:} \qquad \text{Multiplicity} = \ell(\ell - 1)(\ell + 2n - 2)! \qquad \text{Value} = 0 . \tag{4.14}$$

It is then just a matter of algebra to check that

$$\sum_{C=I,II,III,IV} \text{Multiplicity}(C) = (\ell + 2n)! \ , \tag{4.15}$$

as it should be. We then find that

$$\begin{aligned} \operatorname{Tr}\left[S_{(\ell)} \circ \delta^n\right] &= \sum_{C=I,II,III,IV} \text{Multiplicity}(C) \cdot \text{Value}(C) \\ &= \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)} \left[S_{(\ell)} \circ \delta^{n-1}\right] \ , \end{aligned} \tag{4.16}$$

which holds for any traceless symmetric tensor $S_{(\ell)}$.

With Eq. (4.16) in hand, we can now prove the detailed trace theorem by using recursion and an explicit construction.

First, I rewrite Eq. (4.16) in a way that will make it easier to use. For any traceless symmtric tensor $S_{(\ell)}$,

$$\boxed{\operatorname{Tr}\left[S_{(\ell)} \circ \delta^n\right] = F(n, \ell)\left[S_{(\ell)} \circ \delta^{n-1}\right] \ ,} \tag{4.17}$$

where

$$\boxed{F(n, \ell) \equiv \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)} \ .} \tag{4.18}$$

As far as the proof of the theorem is concerned, all that will matter is that Eq. (4.17) holds for some $F(n, \ell)$, and that $F(n, \ell) \neq 0$ for all $\ell \geq 0$ and $n > 0$.

To prove the detailed trace decomposition theorem, we will first prove the existence of an expansion of the form of Eq. (4.7), and we will later prove that it is unique. The existence part of the theorem will be proven by recursion. We begin by noting that it is trivially true for $\ell = 0$ and $\ell = 1$, because $S_{(0)}$ and $S_{(1)}$ are by definition traceless. Thus the existence of the expansion is demonstrated by taking $S^{(0)}_{(\ell)} = S_{(\ell)}$, and all other $S^{(k)}_{(\ell)} = 0$.

To prove the existence of the expansion (4.7) for some value of ℓ, we assume that it has already been shown for all smaller values of ℓ. The assumption guarantees that $\operatorname{Tr} S_{(\ell)}$ can be expanded in this way, so we can write

$$\operatorname{Tr} S_{(\ell)} = T^{(0)}_{(\ell-2)} + \left[T^{(1)}_{(\ell-4)} \circ \delta\right] + \left[T^{(2)}_{(\ell-6)} \circ \delta^2\right] + \ldots \ , \tag{4.19}$$

where each $T^{(k)}_{(m)}$ is traceless. We can now construct the expansion for $S_{(\ell)}$ by writing

$$S_{(\ell)} = S^{(0)}_{(\ell)} + \frac{\left[T^{(0)}_{(\ell-2)} \circ \delta\right]}{F(1,\ell-2)} + \frac{\left[T^{(1)}_{(\ell-4)} \circ \delta^2\right]}{F(2,\ell-4)} + \frac{\left[T^{(2)}_{(\ell-6)} \circ \delta^3\right]}{F(3,\ell-6)} + \dots \ , \tag{4.20}$$

where $S^{(0)}_{(\ell)}$ is defined to make the above equality hold. It remains to be shown that $S^{(0)}_{(\ell)}$, defined in this way, is traceless. To show this, take the trace of both sides of the above equation, and use Eq. (4.17):

$$\operatorname{Tr} S_{(\ell)} = \operatorname{Tr} S^{(0)}_{(\ell)} + T^{(0)}_{(\ell-2)} + \left[T^{(1)}_{(\ell-4)} \circ \delta\right] + \left[T^{(2)}_{(\ell-6)} \circ \delta^2\right] + \dots \ . \tag{4.21}$$

By comparing Eqs. (4.19) and (4.21), one can see that $\operatorname{Tr} S^{(0)}_{(\ell)} = 0$. This completes the proof of the existence of an expansion in the form of Eq. (4.7).

To prove uniqueness, we imagine that some tensor $S_{(\ell)}$ has two such expansions, one written exactly as Eq. (4.7), and another written as

$$S_{(\ell)} = R^{(0)}_{(\ell)} + \left[R^{(1)}_{(\ell-2)} \circ \delta\right] + \left[R^{(2)}_{(\ell-4)} \circ \delta^2\right] + \dots \ . \tag{4.22}$$

Subtracting the two equations, we would have

$$0 = \Delta^{(0)}_{(\ell)} + \left[\Delta^{(1)}_{(\ell-2)} \circ \delta\right] + \left[\Delta^{(2)}_{(\ell-4)} \circ \delta^2\right] + \dots \ , \tag{4.23}$$

where

$$\Delta^{(k)}_{(m)} \equiv S^{(k)}_{(m)} - R^{(k)}_{(m)} \ . \tag{4.24}$$

If we could prove that each $\Delta^{(k)}_{(m)}$ must vanish, we would prove that the expansion is unique. We know that each $\Delta^{(k)}_{(m)}$ is traceless and symmetric, because each $S^{(k)}_{(m)}$ and each $R^{(k)}_{(m)}$ is traceless and symmetric. To show that all the Δ's vanish, look at Eq. (4.23), and let n be the number of δ's in the last term. If ℓ is even, then $n = \ell/2$, and if ℓ is odd, $n = (\ell-1)/2$. In either case, we can take n successive traces of the entire equation. According to Eq. (4.17), each trace removes one δ, and multiplies by a nonzero factor, as long at the argument of the trace includes at least one δ. When there are no δ's in the expression (the case $n = 0$ in Eq. (4.17)), taking the trace causes the expression to vanish. So, when we take n successive traces of the entire equation, all but the last term will vanish, and we will be left with an equation of the form

$$0 = \text{nonzero factor} \times \Delta^{(n)}_{(\ell-2n)} \ , \tag{4.25}$$

which of course implies that $\Delta^{(n)}_{(\ell-2n)} = 0$.

Knowing that the last term vanishes, we can now go back to the original Eq. (4.23), and take $n-1$ successive traces of the entire equation. By the same argument as before, we will obtain an equation of the form

$$0 = \text{nonzero factor} \times \Delta^{(n-1)}_{(\ell-2n+2)} \ , \tag{4.26}$$

which of course implies that $\Delta^{(n-1)}_{(\ell-2n+2)} = 0$. Continuing, the same procedure shows that every Δ vanishes, as we hoped to show. This completes the proof that the expansion of Eq. (4.7) is unique, and therefore completes the proof of the Detailed Trace Decomposition Theorem.

We can now see that the original Trace Decomposition Theorem follows as an immediate corollary. In our compact notation, the expansion of Trace Decomposition Theorem, Eq. (4.1), can be written as

$$S_{(\ell)} = S^{(\mathrm{TS})}_{(\ell)} + \left[M_{(\ell-2)} \circ \delta\right] \ . \tag{4.27}$$

To show the existence of such an expansion, we can start with the expansion of Eq. (4.7) and take

$$\begin{aligned} S^{(\mathrm{TS})}_{(\ell)} &= S^{(0)}_{(\ell)} \ , \\ M_{(\ell-2)} &= S^{(1)}_{(\ell-2)} + \left[S^{(2)}_{(\ell-4)} \circ \delta\right] + \ldots \ . \end{aligned} \tag{4.28}$$

It is similarly straightforward to show the uniqueness of the expansion. If there existed two such expansions of the same tensor $S_{(\ell)}$, we could subtract the two expansion equations and obtain an equation of the form

$$0 = \Delta^{(\mathrm{TS})}_{(\ell)} + \left[\Delta_{(\ell-2)} \circ \delta\right] \ , \tag{4.29}$$

where $\Delta^{(\mathrm{TS})}_{(\ell)}$ is traceless and symmetric, and $\Delta_{(\ell-2)}$ is symmetric. If we can show that both Δ's in this equation must vanish, then we have shown that the expansion is unique. To show this, expand $\Delta_{(\ell-2)}$ according to the detailed trace decomposition (Eq. (4.7)), and then the entire right-hand side of Eq. (4.29) would have the form of a detailed trace decomposition expansion. Since such expansions have been proven to be unique, it follows that every term must vanish, therefore both Δ's in Eq. (4.29) must vanish. This completes the demonstration that the original Trace Decomposition Theorem is a corollary of the Detailed Trace Decomposition Theorem.

13. EXPLICIT FORMULA FOR THE TRACELESS PART OF A SYMMETRIC TENSOR:

In Eqs. (2.35)–(2.37) of Lecture Notes 2, I stated a general formula for extracting the traceless part of an arbitrary symmetric tensor $S_{i_1 \ldots i_\ell}$. In the compact notation,

$$\boxed{\begin{aligned} \{\, S_{(\ell)} \,\}_{\mathrm{TS}} &= S_{(\ell)} + a_{1,\ell}\Big[\big[\mathrm{Tr}\, S_{(\ell)}\big] \circ \delta\Big] + a_{2,\ell}\Big[\big[\mathrm{Tr}^2 S_{(\ell)}\big] \circ \delta^2\Big] + \ldots \\ &= \sum_{n=0}^{n_{\max}} a_{n,\ell}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big] \circ \delta^n\Big] \ , \end{aligned}} \tag{4.30}$$

where

$$n_{\max} = \begin{cases} \frac{1}{2}\ell & \text{if } \ell \text{ is even} \\ \frac{1}{2}(\ell - 1) & \text{if } \ell \text{ is odd,} \end{cases} \tag{4.31}$$

Tr^n means to take n successive traces, and

$$\boxed{a_{n,\ell} = (-1)^n \frac{\ell!^2 (2\ell - 2n)!}{n!(\ell - 2n)!(\ell - n)!(2\ell)!} \ .} \tag{4.32}$$

To derive this equation, it will be useful to first generalize Eqs. (4.17) and (4.18) to allow $S_{(\ell)}$ to be an arbitrary symmetric tensor, not necessarily a traceless one. Looking over the derivation in Eqs. (4.10)–(4.14), one sees that the only change occurs in the value for Case IV:

$$\text{Case IV:} \qquad \text{Multiplicity} = \ell(\ell - 1)(\ell + 2n - 2)! \qquad \text{Value} = \frac{1}{(\ell + 2n)!}\Big[\big[\mathrm{Tr}\, S_{(\ell)}\big] \circ \delta^n\Big] \ . \tag{4.33}$$

The identity then becomes, for any symmetric tensor $S_{(\ell)}$,

$$\boxed{\mathrm{Tr}\big[S_{(\ell)} \circ \delta^n\big] = F(n,\ell)\Big[S_{(\ell)} \circ \delta^{n-1}\Big] + G(n,\ell)\Big[\big[\mathrm{Tr}\, S_{(\ell)}\big] \circ \delta^n\Big] \ ,} \tag{4.34}$$

where

$$\boxed{\begin{aligned} F(n,\ell) &\equiv \frac{2n(2\ell + 2n + 1)}{(\ell + 2n)(\ell + 2n - 1)} \\ G(n,\ell) &\equiv \frac{\ell(\ell - 1)}{(\ell + 2n)(\ell + 2n - 1)} \ . \end{aligned}} \tag{4.35}$$

To justify Eqs. (4.30) and (4.32), we need to show that the expression on the right-hand side of Eq. (4.30) is traceless. To calculate this trace, we begin by noting that Eq. (4.34) is valid for any symmetric tensor, so in particular it is valid when $S_{(\ell)}$ is replaced by $\mathrm{Tr}^n S_{(\ell)}$, giving

$$\mathrm{Tr}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^n\Big] = F(n,\ell-2n)\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^{n-1}\Big] + G(n,\ell-2n)\Big[\big[\mathrm{Tr}^{n+1} S_{(\ell)}\big]\circ\delta^{n}\Big]\ . \tag{4.36}$$

The second argument of F and G is $\ell-2n$, because $\mathrm{Tr}^n S_{(\ell)}$ is a rank $\ell-2n$ tensor. Thus,

$$\begin{aligned}\mathrm{Tr}\left\{\sum_{n=0}^{n_{\max}} a_{n,\ell}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^n\Big]\right\} &= \sum_{n=0}^{n_{\max}} a_{n,\ell}F(n,\ell-2n)\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^{n-1}\Big]\\ &\quad+\sum_{n=0}^{n_{\max}} a_{n,\ell}G(n,\ell-2n)\Big[\big[\mathrm{Tr}^{n+1} S_{(\ell)}\big]\circ\delta^{n}\Big]\ .\end{aligned} \tag{4.37}$$

For the first sum on the right-hand side, note that there is no contribution for $n=0$, since $F(0,\ell)=0$. For the second sum on the right-hand side, there is no contribution for $n=n_{\max}$, since $\mathrm{Tr}^{n_{\max}+1}S_{(\ell)}=0$ (i.e., the value of $n_{\max}$ was defined to be the maximum number of traces that could be taken before there are no longer any pairs of indices to contract). Thus we can rewrite the equation as

$$\begin{aligned}\mathrm{Tr}\left\{\sum_{n=0}^{n_{\max}} a_{n,\ell}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^n\Big]\right\} &= \sum_{n=1}^{n_{\max}} a_{n,\ell}F(n,\ell-2n)\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^{n-1}\Big]\\ &\quad+\sum_{n=0}^{n_{\max}-1} a_{n,\ell}G(n,\ell-2n)\Big[\big[\mathrm{Tr}^{n+1} S_{(\ell)}\big]\circ\delta^{n}\Big]\ .\end{aligned} \tag{4.38}$$

To be able to combine the two sums, we rewrite the second sum in terms of a new variable, $n'\equiv n+1$, so

$$\begin{aligned}\mathrm{Tr}\left\{\sum_{n=0}^{n_{\max}} a_{n,\ell}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^n\Big]\right\} &= \sum_{n=1}^{n_{\max}} a_{n,\ell}F(n,\ell-2n)\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^{n-1}\Big]\\ &\quad+\sum_{n'=1}^{n_{\max}} a_{n'-1,\ell}G(n'-1,\ell-2n'+2)\Big[\big[\mathrm{Tr}^{n'} S_{(\ell)}\big]\circ\delta^{n'-1}\Big]\ .\end{aligned} \tag{4.39}$$

We can now drop the prime in the second sum, since n' is a "dummy" summation variable for which the name is irrelevant, and then the two sums can be combined:

$$\begin{aligned}&\mathrm{Tr}\left\{\sum_{n=0}^{n_{\max}} a_{n,\ell}\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^n\Big]\right\}\\ &\quad=\sum_{n=1}^{n_{\max}}\big[a_{n,\ell}F(n,\ell-2n)+a_{n-1,\ell}G(n-1,\ell-2n+2)\big]\Big[\big[\mathrm{Tr}^n S_{(\ell)}\big]\circ\delta^{n-1}\Big]\ .\end{aligned} \tag{4.40}$$

Thus, the trace of this expression will vanish provided that $a_{n,\ell}$ obeys the recursion relation

$$\begin{aligned} a_{n,\ell} &= -\frac{G(n-1,\ell-2n+2)}{F(n,\ell-2n)}\, a_{n-1,\ell} \\ &= -\frac{(\ell-2n+1)(\ell-2n+2)}{2n(2\ell-2n+1)}\, a_{n-1,\ell}\ , \end{aligned} \tag{4.41}$$

with the initial condition $a_{0,\ell} = 1$, since the first term on the right-hand side of Eq. (4.30) has coefficient 1. It is then just a matter of algebra to show that $a_{n,\ell}$, as defined by Eq. (4.32), satisfies the recursion relation with the right initial condition.