8.07 Lecture Slides 2
September 9, 2019

REVIEW OF VECTORS, CONTINUED

VECTOR CALCULUS: grad, div, curl

Class Contact Page

The Class Contact Page is up and running, linked to the class website, web.mit.edu/8.07/www.

It is your way to make yourself known to your fellow students, whether you are interested in forming study groups or not.

To sign up, see my email of 9/7/19, subject: *Class Contact and Feedback pages are now up.*

Access requires an MIT certificate, with your name on the list of registered students and listeners. If you have trouble accessing the page, send me an email.

Feedback Page

A feedback page is up and running, linked to the class website.

I invite you to send emails to me with complaints, comments, or suggestions about the class, and I promise that such emails will not affect your grades.

However, you may want your complaints, comments, or suggestions to be anonymous. If so, the Feedback Page is for you!

You can check boxes to send to Marin Soljačić, Yitian Sun, or me, or any combination of us.

To verify that you are a member of the class, first link to the Class Contact Page, which at the bottom shows a Feedback Page Code. On the Feedback page, there is a place to type in this code. This will show us that the sender is a member of the class, without giving any identifying information.

Announcements

Office hours:

Monday	Marin Soljačić	4:00-5:00 pm	Room 6C-419
Tuesday	(No office hours)		
Wednesday	Alan Guth	5:00-6:00 pm	Room 6-322
Thursday	Yitian Sun	5:30-7:30 pm	Room 8-306
Friday	Yitian Sun	2:00-3:00 pm	Room 8-306

Planned Quiz dates:

Monday, October 7, 2019
Wednesday, November 6, 2019

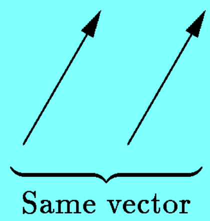
If you have conflicts, let me know by Wednesday.

Definition

DEFINITION: A *vector* is a quantity that has magnitude and direction.

Examples: displacement, velocity, acceleration, force, momentum, electric and magnetic fields.

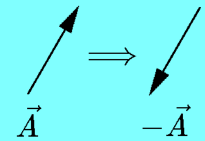
Vectors do not have position:



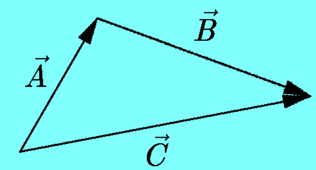
Operations

Magnitude: $\|\vec{\boldsymbol{A}}\| \equiv$ magnitude of $\vec{\boldsymbol{A}}$. (Here $\equiv$ means "is defined to be".) Often A is used to denote $\|\vec{\boldsymbol{A}}\|$.

Negation:



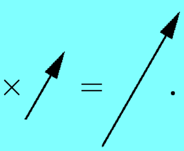
Addition:



$\vec{A} + \vec{B} = \vec{C}$

Subtraction: $\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} \equiv \vec{\boldsymbol{A}} + (-\vec{\boldsymbol{B}})$.

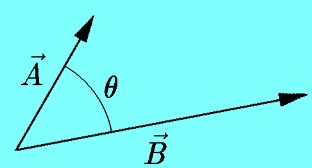
Multiplication by a scalar: $2 \times$



Property— Distributive: $a(\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}) = a\vec{\boldsymbol{A}} + a\vec{\boldsymbol{B}}$.

Dot Product

Dot product of two vectors: $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \equiv |\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}| \cos\theta$, where θ is the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$:



always tail to tail!

Properties:

Rotational invariance. The value of the dot product does not change if both of the vectors are rotated together.

Cummutative: $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}}$.

Distributive: $\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}) = \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}}$.

Scalar multiplication: $(a\vec{\boldsymbol{A}}) \cdot \vec{\boldsymbol{B}} = a(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})$.

Review of Lecture 1

Why Cosθ?

Query: Why $\cos\theta$??? If I defined a Guth-dot product by

$$\vec{A}\cdot\vec{B}\Big|_{\text{Guth}} \equiv |\vec{A}||\vec{B}|\sin\theta \ ,$$

and hired a really good advertising agency, could my product (note the pun!) compete?

Answer: Maybe a really good advertising agency can do anything, but I would have a serious marketing problem. My dot product would not be distributive. In fact, one can show that if $\vec{A}\cdot\vec{B}$ obeys rotational invariance, commutativity, the scalar multiplication law, and the distributive law, then

$$\vec{A}\cdot\vec{B} = \text{const}|\vec{A}||\vec{B}|\cos\theta \ .$$

We'll come back to this later.

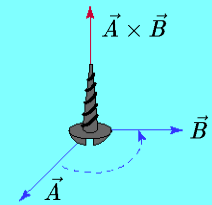
Review of Lecture 1

The Vector Cross-Product $\vec{A}\times\vec{B}$

Cross product of two vectors:

$$\vec{A}\times\vec{B} \equiv |\vec{A}||\vec{B}|\sin\theta\,\hat{n} \ ,$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{A}$ and perpendicular to $\vec{B}$. The choice of the two (opposite) directions that are perpendicular to both $\vec{A}$ and $\vec{B}$ is determined by the right-hand rule:



Right-hand Rule

(Source: Modified from chortle.ccsu.edu/vectorlessons/vch12/rightHandRule.gif.)

Review of Lecture 1

Properties:

Rotational invariance: If both vectors are rotated by the same rotation R, then the result of the cross product is also rotated by R.

Anticommutative: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

Distributive: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$.

Scalar multiplication: $(a\vec{A})\times\vec{B} = a(\vec{A}\times\vec{B})$.

Review of Lecture 1

Why Sinθ?

Query: Why $\sin\theta$???

Answer: Again, the function of θ is required for rotationl invariance and distributivity. If we assume that the cross product is rotationally invariant, and obeys the distributive law and the scalar multiplication law on both the right and the left (i.e., $(\vec{A}+\vec{B})\times\vec{C} = \vec{A}\times\vec{C}+\vec{B}\times\vec{C}$ and $\vec{A}\times(a\vec{B}) = a(\vec{A}\times\vec{B})$ as well as the identities on the previous slide), then one can show that

$$\vec{A}\times\vec{B} = \text{const}|\vec{A}||\vec{B}|\sin\theta\,\hat{n} \ .$$

I'll come back to this.

Component Notation

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \ ,$$

where

$\hat{x}$ is a unit vector in the x direction

$\hat{y}$ is a unit vector in the y direction

$\hat{z}$ is a unit vector in the z direction.

(Various notations are in use. Griffiths uses $\hat{x}$, $\hat{y}$, and $\hat{z}$, while many other books use $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.)

Index Notation: Components by Number

Let $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, and $\hat{e}_3 \equiv \hat{z}$, so

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \ ,$$

can be written

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 \ ,$$

which can be abbreviated as $\vec{A} = \sum_{i=1}^{3} A_i\hat{e}_i$, which can be further abbreviated as $\vec{A} = A_i\hat{e}_i$. This final expression uses the "Einstein summation convention": if an index is repeated within one term, it is implicitly summed from 1 to 3. That is, A_iB_i is summed, but $A_i + B_i$ is not (these are two different terms).

Vector Addition in Components

Vector Addition:

$$\begin{aligned} \vec{C} = \vec{A} + \vec{B} \quad &\Longrightarrow \quad C_i\hat{e}_i = A_i\hat{e}_i + B_i\hat{e}_i \\ &\Longrightarrow \quad C_i\hat{e}_i = (A_i + B_i)\hat{e}_i \\ &\Longrightarrow \quad \boxed{C_i = A_i + B_i \ .} \end{aligned}$$

Vector Dot Product in Components

Vector Dot Product:

$$\begin{aligned} \hat{e}_i \cdot \hat{e}_j &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\ &= \delta_{ij} \ , \end{aligned}$$

where δ_{ij} is called the Kronecker δ-function.

Then

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_i\hat{e}_i \cdot B_j\hat{e}_j \\ &= A_iB_j\hat{e}_i \cdot \hat{e}_j = A_iB_j\delta_{ij} \\ &= \sum_{i=1}^{3} A_i \left(\sum_{j=1}^{3} B_j\delta_{ij} \right) = \sum_{i=1}^{3} A_i\,(B_i) = A_iB_i \ . \end{aligned}$$

Thus,

$$\boxed{\vec{A} \cdot \vec{B} = A_iB_i \equiv A_xB_x + A_yB_y + A_zB_z \ .}$$

Vector Cross Product in Components

Cross products of unit vectors:

$$\boxed{\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\ , \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}\ , \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\ .}$$

"Cyclic Permutations": $x \to y$, $y \to z$, and $z \to x$.

Note that each term in the boxed equation is a cyclic permutation of the previous term. Can replace the boxed equation by

$$\boxed{\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\ , \quad \text{and cyclic permutations.}}$$

In the opposite order, there are minus signs: e.g., $\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\hat{\boldsymbol{z}}$.

The Levi-Civita Symbol

Named for Italian mathematician/physicist Tullio Levi-Civita (1873-1941).

Definition:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3) \text{ or a cyclic permutation} \\ -1 & \text{if } (i,j,k) = (3,2,1) \text{ or a cyclic permutation} \\ 0 & \text{otherwise.} \end{cases}$$

It can be equivalently defined by saying the $\varepsilon_{123} = 1$ and that ε_{ijk} is totally antisymmetric.

Relation to cross product:

$$\boxed{\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = \varepsilon_{ijk} \hat{\boldsymbol{e}}_k\ .}$$

So

$$\begin{aligned} \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} &= A_i \hat{\boldsymbol{e}}_i \times B_j \hat{\boldsymbol{e}}_j \\ &= A_i B_j (\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j) \\ &= A_i B_j \varepsilon_{ijk} \hat{\boldsymbol{e}}_k\ , \end{aligned}$$

and then

$$(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})_k = \varepsilon_{ijk} A_i B_j\ .$$

Renaming the indices by a cyclic permutation, $k \to i$, $i \to j$, $j \to k$, the equation can be written more neatly as

$$\boxed{(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}})_i = \varepsilon_{ijk} A_j B_k\ .}$$

Levi-Civita Identity

$$\boxed{\varepsilon_{ijk}\varepsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}\ .}$$

You get to prove this on Problem Set 1. It is proven essentially by looking at all possible cases for (j, k, n, m), of which there are $3^4 = 81$ cases. But they can be discussed in groups: it matters whether two indices are equal or unequal, but it doesn't really matter what the particular numbers are.

Uniqueness of the Vector Cross Product

We assume the cross product is rotationally invariant, and obeys the distributive law and the scalar multiplication law on both the right and the left (i.e., $(\vec{A}+\vec{B})\times\vec{C}=\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$ and $\vec{A}\times(a\vec{B})=a(\vec{A}\times\vec{B})$ as well as the identities written earlier.

Consider $\hat{e}_x\times\hat{e}_y$. What can it be? Try

$$\hat{e}_x\times\hat{e}_y=a\hat{e}_x+b\hat{e}_y+c\hat{e}_z\ .$$

Consider rotation by 180° about the z axis, so $\hat{e}'_x=-\hat{e}_x$, $\hat{e}'_y=-\hat{e}_y$, and $\hat{e}'_z=\hat{e}_z$. So in the rotated system,

$$(-\hat{e}'_x)\times(-\hat{e}'_y)=a(-\hat{e}'_x)+b(-\hat{e}'_y)+c\hat{e}'_z \quad\Longrightarrow\quad \hat{e}'_x\times\hat{e}'_y=-a\hat{e}'_x-b\hat{e}'_y+c\hat{e}'_z\ .$$

By rotational invariance, the cross product obeys the same rules in the rotated coordinate system, so $a=-a$ and $b=-b$ imply that $a=b=0$, but c is undetermined.

–20–

The same equation must hold for all cyclic permutations, since they are related by rotations:

$$\hat{e}_x\times\hat{e}_y=c\hat{e}_z\ ,\quad \hat{e}_y\times\hat{e}_z=c\hat{e}_x\ ,\quad \hat{e}_z\times\hat{e}_x=c\hat{e}_y\ .$$

To understand the reverse order, consider a rotation by 90° about the z axis, which gives $\hat{e}'_x=\hat{e}_y$, $\hat{e}'_y=-\hat{e}_x$, and $\hat{e}'_z=\hat{e}_z$. Thus the first of the 3 equations above becomes

$$(-\hat{e}'_y)\times\hat{e}'_x=c\hat{e}'_z,$$

so, remembering that rotational invariance means that the formulas must be the same in the primed and unprimed frames, we have

$$\hat{e}_y\times\hat{e}_x=-c\hat{e}_z\ .$$

Finally, consider what happens when the two unit vectors are the same:

$$\hat{e}_x\times\hat{e}_x=d\hat{e}_x+e\hat{e}_y+f\hat{e}_z\ .$$

A 180° rotation about the x axis implies $e=f=0$. A 180° rotation about the y axis implies that $d=0$. So $\hat{e}_x\times\hat{e}_x=0$, and the same must be true for $\hat{e}_y\times\hat{e}_y$ and $\hat{e}_z\times\hat{e}_z$. Putting together these results, we have shown that

$$\hat{e}_i\times\hat{e}_j=c\varepsilon_{ijk}\hat{e}_k\ ,$$

which is the standard definition multiplied by an arbitrary constant c. By convention $c=1$.

–21–

Rotation of a Vector

Active vs. Passive Rotations:

Active: rotate the object

Passive: rotate the coordinate system

An active rotation by θ about a specified axis has the same effect as a passive rotation by $-\theta$ about the same axis.

Advantage of passive: it is well-defined — just define new coordinates in terms of old.

Downside of passive: I find passive rotations hard to visualize, and it is even harder to visualize successive passive rotations.

Advantage of active: describing successive rotations is easier to visualize, since one has a fixed coordinate system in which to visualize them.

Downside of active: it is not clear how to implement — to actually rotate an object, one must apply forces, which then distort the object in a way that depends on where the forces were applied.

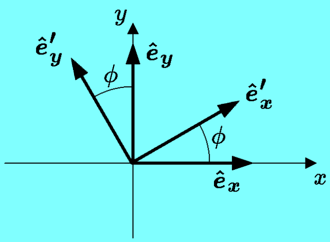
My compromise: describe rotations as active, but keep in mind that we really mean a rotation of the coordinate system in the opposite direction.

Rotations in Coordinates

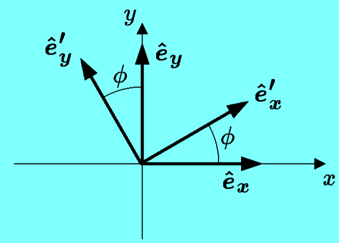
Example: let $R_z(\phi)$ denote a rotation about the z axis by an angle ϕ, counterclockwise as viewed looking "down" the z axis (i.e., looking from positive z towards the origin).

Let $\vec{A}$ be an arbitrary vector, and $\vec{A}'$ be the same vector after rotation by $R_z(\phi)$.

First write equations for rotated basis vectors:



$$\hat{e}'_x = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi$$
$$\hat{e}'_y = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi$$



$$\hat{e}'_x = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi$$
$$\hat{e}'_y = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi$$

$$\boxed{\hat{e}'_j = \hat{e}_i R_{ij}}$$

where

$$R_{z,ij}(\phi) = \begin{array}{c c} & \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\boxed{\hat{e}'_j = \hat{e}_i R_{ij}}$$

But the arbitrary vector $\vec{A}$ can be expanded as

$$\vec{A} = A_i \hat{e}_i \ .$$

If each of the unit vectors is rotated by $R_z(\phi)$, then $\vec{A}$ will be rotated by $R_z(\phi)$. So

$$\begin{aligned} \vec{A}' &= A_j \hat{e}'_j = A_j \hat{e}_i R_{z,ij}(\phi) \\ &= [R_{z,ij}(\phi) A_j]\hat{e}_i \ . \end{aligned}$$

So finally,

$$\boxed{A'_i = R_{z,ij}(\phi) A_j \ .}$$

General Rotations

The rotation matrix $R_{ij}(\hat{n}, \phi)$, which describes an active rotation about the $\hat{n}$ axis by an angle ϕ , counterclockwise as viewed looking into the vector $\hat{n}$, is given by

$$\boxed{R_{ij}(\hat{n}, \phi) = \delta_{ij}\cos\phi - \varepsilon_{ijk}\hat{n}_k \sin\phi + (1-\cos\phi)\hat{n}_i\hat{n}_j \ .}$$

We will not derive this formula, but it is true and can be useful.

Any rotation matrix R_{ij} is **orthogonal**, which means that

$$\boxed{R^{-1} = R^T \ ,}$$

where R^T is the transpose of R, so $R^T_{ij} \equiv R_{ji}$. This property guarantees that the length of a vector is not changed by rotating it.

Vectors and Tensors

For a triplet of numbers N_i to be called a vector, it must transform under rotations as

$$\boxed{N_i' = R_{ij}(\hat{\boldsymbol{n}}, \phi) N_j \ ,}$$

with the same matrix $R_{ij}(\hat{\boldsymbol{n}}, \phi)$ as on the previous slide.

As Griffiths points out, if a barrel of fruit contains N_x pears, N_y apples, and N_z bananas, the quantity $\vec{\boldsymbol{N}} = N_x\hat{\boldsymbol{x}} + N_y\hat{\boldsymbol{y}} + N_z\hat{\boldsymbol{z}}$ is NOT a vector!

A **tensor** is a generalization of a vector, with n indices, where n is called the rank of the tensor. Each index transforms in the same way as a vector index does. For example, a rank 2 tensor transforms as

$$T_{ij}' = R_{ik} R_{j\ell} T_{k\ell} \ .$$

A vector is a rank 1 tensor, and a scalar is a rank 0 tensor.

Examples of tensors:

If v_i is a vector, such as the velocity of a particle, then $M_{ij} \equiv v_i v_j$ is a tensor. Don't ask what it means, but it is a tensor.

We will see later that electromagnetic fields create a stress tensor, T_{ij}, which describes the flow of i-momentum in the j-direction.

The rotation matrix R_{ij} is itself a rank 2 tensor. What does it mean to rotate a rotation matrix? For example, the orientation of one rigid body might be related to the orientation of a second rigid body by a rotation R_{ij}. If both bodies are rotated by another rotation $\bar{R}_{ij}$, then the rotation R_{ij} that relates them will transform as a tensor:

$$R_{ij}' = \bar{R}_{ik} \bar{R}_{j\ell} R_{k\ell}.$$

As I mentioned earlier, we will learn how to describe the spherical harmonics in terms of tensors.