Weekly Assignment 5
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March 5, 2019

Question 1

A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $\$1.10$; if they are different colors, then you lose $\$1.00$. Calculate:

Problem Part a:

The expected value of the amount you win.

Solution Part a:

There are two possibilities that satisfy the event, either we draw two red marbles or two blue ones. Let say we want to draw two red marbles, there are $\frac{5}{10}$ chance to get a red marble in first draw and $\frac{4}{9}$ of getting a second one. Same calculations apply for the getting two blue marbles. Hence, probability of getting two marbles of same color is $\frac{5}{10} * \frac{4}{9} * 2 = \frac{4}{9}$.

$$\begin{aligned}
E[X] &= P(E) * 1.1 + (1 - P(E)) * (-1.0) \\
&= \frac{4}{9} * 1.1 + \frac{5}{9} * (-1.0) \\
&= -\frac{1}{15} \\
&\approx -0.0667
\end{aligned}$$

Problem Part b:

The variance of the amount you win.

Solution Part b:

$$\begin{aligned}
E[X^2] &= \frac{4}{9} * 1.1^2 + \frac{5}{9} * (-1.0)^2 \\
&= \frac{82}{75} \\
Var[X] &= E[X^2] - (E[X])^2 \\
&= \frac{82}{75} - (-\frac{1}{15})^2 \\
&= \frac{49}{45} \\
&\approx 1.08889
\end{aligned}$$

Question 2

Suppose you carry out a poll following an election. You do this by selecting n people uniformly at random and asking whether they voted or not, letting Xi = 1 if person i voted and Xi = 0 otherwise. Suppose the probability that a person voted is 0.6.

Problem Part a:

Calculate $E[X_i]$ and $Var(X_i)$.

Solution Part a:

$$\begin{aligned}
E[X_i] &= 0.6 * 1 + 0.4 * 0 \\
&= 0.6 \\
E[(X_i)^2] &= 0.6 * 1^2 + 0.4 * 0 \\
&= 0.6 \\
Var[X_i] &= E[(X_i)^2] - E[X_i]^2 \\
&= 0.6 - 0.6^2 \\
&= 0.24
\end{aligned}$$

or Using Bernoulli random variable method:

$$\begin{aligned}
E[X_i] &= p \\
&= 0.6 \\
Var[X_i] &= p * (1 - p) \\
&= 0.6 * (1 - 0.6) \\
&= 0.6 * 0.4 \\
&= 0.24
\end{aligned}$$

Problem Part b:

Let $Y = \sum_{i=1}^{n} X_i$.
What is $E[Y]$? Is it the same as $E[X]$ or different, and why ?

Solution Part b:

(Not sure if $E[X]$ is actually a typo for $E[X_i]$, so i will do both versions.)

$E[X_i]$ **version:**

$$
\begin{aligned}
E[Y] &= E[\sum_{i=1}^{n} X_i] \\
&= \sum_{i=1}^{n} E[X_i] \\
&= n * E[X_i] \\
&= n * 0.6
\end{aligned}
$$

Since $E[Y] \neq E[X_i]$, i.e. E[Y] is the sum of $E[X_i]$ by n times, E[Y] is different compared to $E[X_i]$.

$E[X]$ **version:**

By linearity,

$$E[X] = \sum_{i=1}^{n} E[X_i] = E[\sum_{i=1}^{n} X_i] = E[Y]$$

Hence, $E[X]$ is the same as $E[Y]$.

Problem Part c:

What is $E[\frac{1}{n}Y]$?

Solution Part c:

Using linearity of expectation:

$$
\begin{aligned}
E[\frac{1}{n}Y] &= E[\frac{1}{n}\sum_{i=1}^{n} X_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\
&= \frac{1}{n} * n * E[X_i] \\
&= 0.6 \\
&= E[X_i]
\end{aligned}
$$

Hence, $E[\frac{1}{n}Y]$ can be simplified into $E[X_i]$.

(For this section, there seems to be no difference however I treat $E[X]$ like before as $E[\frac{1}{n}Y] = E[\frac{1}{n}X] = \frac{1}{n}E[X]$. Hence, no need for two versions of solutions.)

Problem Part d:

What is the variance of $\frac{1}{n}Y$ (express in terms of $Var(X)$)?
Hints: use linearity of the expectation and the fact that people are sampled independently.

Solution Part d:

$E[X_1]$ version:

As Y is a binomial variable:

$$\begin{aligned} Var[Y] &= n * Var[X_i] \\ &= n * 0.24 \\ Var[\frac{1}{n}Y] &= n * Var[\frac{1}{n} * X_i] \\ &= \frac{1}{n}^2 * n * Var[X_i] \\ &= \frac{Var[X_i]}{n} \\ &= \frac{0.24}{n} \end{aligned}$$

$E[X]$ version:

if $E[X] = E[Y]$, then, by definition, $Var[X] = Var[Y]$, hence:

$$Var[\frac{1}{n}Y] = Var[\frac{1}{n}X] = (\frac{1}{n})^2 * Var[X]$$

Question 3

Suppose that 2 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i^{th} ball selected is white, and let it equal 0 otherwise.

Problem Part a:

Give the joint probability mass function of X_1 and X_2

Solution Part a:

To calculate joint probability mass function of X_1 and X_2, we need to compute their distributions.

- $$\begin{aligned} P(X_1 = 0, X_2 = 0) =& \frac{8}{13} * \frac{7}{12} \\ =& \frac{14}{39} \end{aligned}$$

- $$\begin{aligned} P(X_1 = 1, X_2 = 0) =& \frac{5}{13} * \frac{8}{12} \\ =& \frac{10}{39} \end{aligned}$$

- $$\begin{aligned} P(X_1 = 0, X_2 = 1) =& \frac{8}{13} * \frac{5}{12} \\ =& \frac{10}{39} \end{aligned}$$

- $$\begin{aligned} P(X_1 = 1, X_2 = 1) =& \frac{5}{13} * \frac{4}{12} \\ =& \frac{5}{39} \end{aligned}$$

Hence, the tabular form of joint pmf would look as illustrated below:

	$X_1 = 0$	$X_1 = 1$	$P(X_1 = x_1)$
$X_2 = 0$	$\frac{14}{39}$	$\frac{10}{39}$	$\frac{24}{39}$
$X_2 = 1$	$\frac{10}{39}$	$\frac{5}{39}$	$\frac{15}{39}$
$P(X_2 = x_2)$	$\frac{24}{39}$	$\frac{15}{39}$	1

Problem Part b:

Are X_1 and X_2 independent ? (Use the formal definition of independence to determine this)

Solution Part b:

For X_1 and X_2 to be independent, we need to prove:

$$P(X_1 = x_1 \cap X_2 = x_2) = P(X_1 = x_1) * P(X_2 = x_2)$$

for all $x_1 \in R_{X_1}$ and $x_2 \in R_{X_2}$.

Take a counter example of $P(X_2 = 1 \cap X_1 = 1)$:

$$\begin{aligned}
P(X_2 = 1 \cap X_1 = 1) &= \frac{5}{39} \\
P(X_1 = 1) &= \frac{15}{39} \\
P(X_2 = 1) &= \frac{15}{39} \\
P(X_2 = 1) * P(X_1 = 1) &= \frac{15}{39} * \frac{15}{39} \\
&= \frac{25}{169}
\end{aligned}$$

Since $P(X_2 = 1 \cap X_1 = 1) \neq P(X_2 = 1) * P(X_1 = 1)$, X_1 and X_2 are not independent.

Problem Part c:

Calculate $E[X_2]$

Solution Part c:

$$\begin{aligned}
E[X_2] &= P(X_2 = x_2) \\
&= \frac{15}{39} * 1 + \frac{24}{39} * 0 \\
&= \frac{5}{13} \\
&\approx 0.3846
\end{aligned}$$

Problem Part d:

Calculate $E[X_2|X1 = 1]$

Solution Part d:

$$\begin{aligned}
E[X_2|X1 = 1] &= \frac{P(X_2 = x_2)}{P(X_1 = 1)} \\
&= \frac{\frac{5}{39} * 1 + \frac{10}{39} * 0}{\frac{15}{39}} \\
&= \frac{1}{3} \\
&\approx 0.3333
\end{aligned}$$