

Weekly Assignment 5

Leong Kai Ler

15334636

March 5, 2019

Question 1

A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you lose \$1.00. Calculate:

Problem Part a:

The expected value of the amount you win.

Solution Part a:

There are two possibilities that satisfy the event, either we draw two red marbles or two blue ones. Let say we want to draw two red marbles, there are $\frac{5}{10}$ chance to get a red marble in first draw and $\frac{4}{9}$ of getting a second one. Same calculations apply for the getting two blue marbles. Hence, probability of getting two marbles of same color is $\frac{5}{10} * \frac{4}{9} * 2 = \frac{4}{9}$.

$$\begin{aligned} E[X] &= P(E) * 1.1 + (1 - P(E)) * (-1.0) \\ &= \frac{4}{9} * 1.1 + \frac{5}{9} * (-1.0) \\ &= -\frac{1}{15} \\ &\approx -0.0667 \end{aligned}$$

Problem Part b:

The variance of the amount you win.

Solution Part b:

$$\begin{aligned} E[X^2] &= \frac{4}{9} * 1.1^2 + \frac{5}{9} * (-1.0)^2 \\ &= \frac{82}{75} \\ Var[X] &= E[X^2] - (E[X])^2 \\ &= \frac{82}{75} - \left(-\frac{1}{15}\right)^2 \\ &= \frac{49}{45} \\ &\approx 1.08889 \end{aligned}$$

Question 2

Suppose you carry out a poll following an election. You do this by selecting n people uniformly at random and asking whether they voted or not, letting $X_i = 1$ if person i voted and $X_i = 0$ otherwise. Suppose the probability that a person voted is 0.6.

Problem Part a:

Calculate $E[X_i]$ and $Var(X_i)$.

Solution Part a:

Problem Part b:

Let $Y = \sum_{i=1}^n X_i$.

What is $E[Y]$? Is it the same as $E[X]$ or different, and why?

Solution Part b:

Problem Part c:

What is $E[\frac{1}{n}Y]$?

Solution Part c:

Problem Part d:

What is the variance of $\frac{1}{n}Y$ (express in terms of $Var(X)$)?

Hints: use linearity of the expectation and the fact that people are sampled independently.

Solution Part d:

Question 3

Suppose that 2 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i^{th} ball selected is white, and let it equal 0 otherwise.

Problem Part a:

Give the joint probability mass function of X_1 and X_2

Solution Part a:

To calculate joint probability mass function of X_1 and X_2 , we need to compute their distributions.

- $$P(X_1 = 0, X_2 = 0) = \frac{8}{13} * \frac{7}{12} = \frac{14}{39}$$
- $$P(X_1 = 1, X_2 = 0) = \frac{5}{13} * \frac{8}{12} = \frac{10}{39}$$
- $$P(X_1 = 0, X_2 = 1) = \frac{8}{13} * \frac{5}{12} = \frac{10}{39}$$
- $$P(X_1 = 1, X_2 = 1) = \frac{5}{13} * \frac{4}{12} = \frac{5}{39}$$

Hence, the tabular form of joint pmf would look as illustrated below:

	$X_1 = 0$	$X_1 = 1$	$P(X_1 = x_1)$
$X_2 = 0$	$\frac{14}{39}$	$\frac{10}{39}$	$\frac{24}{39}$
$X_2 = 1$	$\frac{10}{39}$	$\frac{5}{39}$	$\frac{15}{39}$
$P(X_2 = x_2)$	$\frac{24}{39}$	$\frac{15}{39}$	1

Problem Part b:

Are X_1 and X_2 independent ? (Use the formal definition of independence to determine this)

Solution Part b:

For X_1 and X_2 to be independent, we need to prove:

$$P(X_1 = x_1 \cap X_2 = x_2) = P(X_1 = x_1) * P(X_2 = x_2)$$

for all $x_1 \in R_{X_1}$ and $x_2 \in R_{X_2}$.

Take an example of $P(X_2 = 1)$ and $P(X_2 = 1|X_1 = 1)$:

$$\begin{aligned} P(X_2 = 1) &= \frac{15}{39} \\ P(X_2 = 1|X_1 = 0) &= \frac{P(X_1 = 0|X_2 = 1)}{P(X_1 = 0)} \\ &= \frac{\frac{10}{39}}{\frac{24}{39}} \\ &= \frac{5}{12} \end{aligned}$$

Since $P(X_2 = 1) \neq P(X_2 = 1|X_1 = 0)$, X_1 and X_2 are not independent.

Problem Part c:

Calculate $E[X_2]$

Solution Part c:

$$\begin{aligned} E[X_2] &= \frac{15}{39} * 1 + \frac{24}{39} * 0 \\ &= \frac{5}{13} \\ &\approx 0.3846 \end{aligned}$$

Problem Part d:

Calculate $E[X_2|X_1 = 1]$

Solution Part d:

$$\begin{aligned} E[X_2|X_1 = 1] &= \frac{P(X_2 = x_2)}{P(X_1 = 1)} \\ &= \frac{\frac{5}{39} * 1 + \frac{10}{39} * 0}{\frac{15}{39}} \\ &= \frac{1}{3} \\ &\approx 0.3333 \end{aligned}$$