The Minkowski Question Mark and the Modular Group $SL(2,\mathbb{Z})$

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Abstract

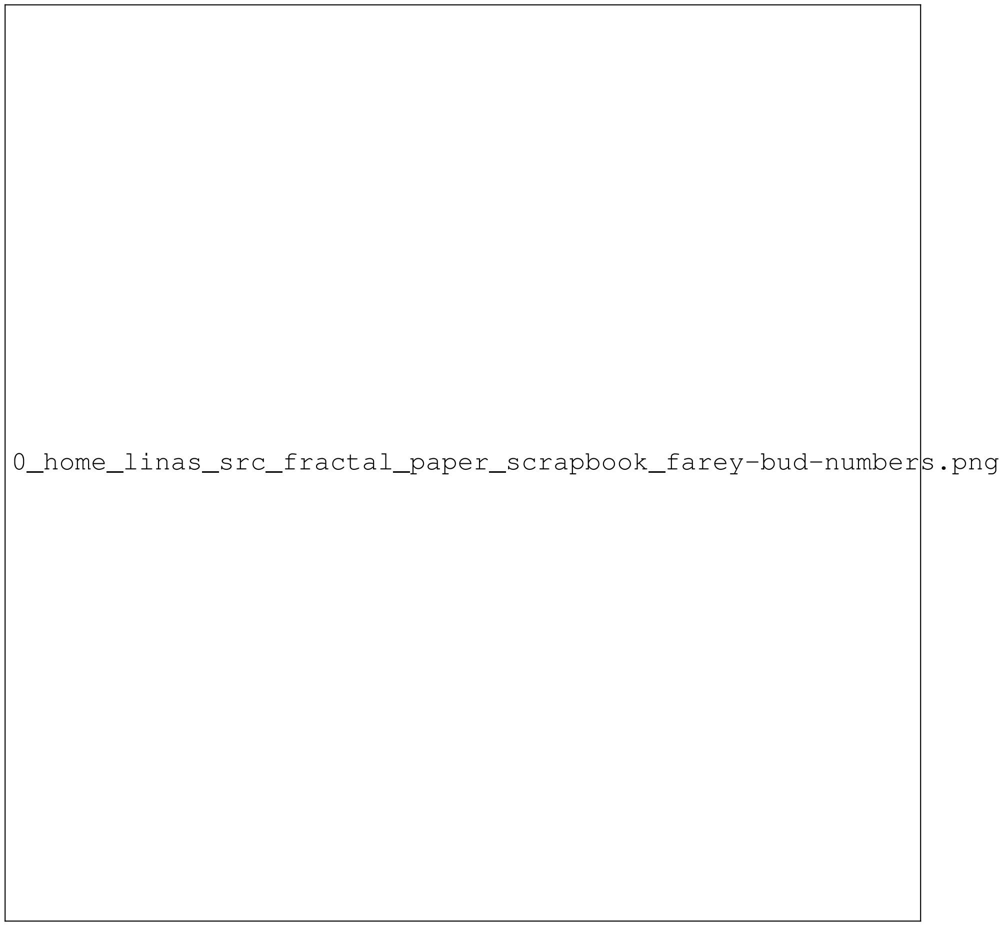
Fractals unfinished scrapbook.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

XXXX This paper is under construction; it is a rough, unfinished draft. Some of the more speculative remarks need to be, ahem, edited. I definitely dislike some of the things I say below. There are probably various mundane errors as well. Caution!! XXX

0.1 Diatribe

It has been widely noticed that Farey Numbers appear naturally in the Mandelbrot Set. For example, here is a picture stolen from Robert L. Devaney's website The Fractal Geometry of the Mandelbrot Set `http://math.bu.edu/DYSYS/FRACGEOM2/node5.html`:



This figure shows how to count the buds on the Mandelbrot set by assigning Farey fractions to them. It is also widely known that Farey numbers are deeply related to continued fractions: for example, Alexander Bogomolny's Stern Brocot Trees `http://www.cut-the-knot.org/blue/Stern.shtml` web page at Cut-the-Knot details this relationship. It is also well known that the natural way to compute with continued fractions are to use a dyadic representation, using the matrices

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{1}$$

to compute the partial convergents of the continued fraction. It also well-known that these matrices generate the Modular Group $SL(2,\mathbb{Z})$. Yet I am personally not aware of any studies that make this complete chain of connections, and seriously explore fractal symmetries as an expression of the Modular Group. It is a deep mystery to me why books on modular forms never mention fractals, and books on fractals never mention modular forms. It just takes one whiff of the j-function, which is normally discussed only in books on modular forms, and constructed out of Jacobi theta functions, to see jaw-dropping similarities between it and the Mandelbrot set, complete with prototype

Pickover stalks. The latest book by Mandelbrot [Man04], published in 2004, comes perilously close, by touching the Minkowski function, but still failing to mention the Modular Group. The goal here is to take some baby steps to remedy this situation.

0.2 compact operator

The bernoulii operator is bounded (largest eignevalue is 1) is it a compact operator? The spectrum is that of a compact operator, so it probably is ... The bernoulli operator is a Hilbert-Schmidt operator (sum of its eigenvalues is finite).

0.3 Unfinished – eigenvalues of laplacian on fun domain

what are the eigenvalues of the laplacian on the fundamental domain? Unifinishced – spectrum looks continuous, boundary condition requires solution of complicated linear eqn. What about eigenvalues on general rieman surface?

0.4 bakers map on elliptic curve

what is the bakers map on the torus like? what are its eigenvalues?

0.5 Duoady hubbard rays for julia sets

what is the general theory and mapp of duady hubbard rays for julia sets? In particular, any set gened by binary process can be labelled with real numbers. Fractal dusts in particular could be interesting; the dust seems to resemble aalytic maps; can a conformal equivalence be shown to analytic syrfaces (rieman surfaces with infinite number of cusps?) In particular, for fractal dusts, landing s function of angle is very possible.

0.6 Step functions

take the integral along a circular contour centered on the origin, of the j-invariant. As radius increases, it will encompass more and more poles, leading to a step function. What is that step function? Does it have number-theoretic properties (e.g. will its difference form be a multiplicative function?)

0.7 Blah Blah Blah

Note that $x_+(w)$ is the concatenation operator for continued fractions. If $x = [a_1, a_2, ..., a_N]$ and $w = [b_1, b_2, ...]$ then $x_+(w) = [a_1, a_2, ..., a_N, b_1, b_2, ...]$. Thus, x_+ can be used to construct code-words. Iterating $x_+(w)$ obviously does not yield any periodic orbits, and instead converges to a quadratic irrational that is the solution to $y = x_+(y)$. From the theory of Pellian equations, if a_N is even, then $y = -a_N/2 + \sqrt{D}$ for some integer D where $a_N^2/4 + a_N + 1 > D > a_N^2/4$.

Pre-periodic stuff .. Misiurewitz points.

ToDo

0.8 Enumerating the rest of the modular group

So far, we've only enumerated some, but not all of the modular group. The inverse of our generator g is $g^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. For the group element $\gamma = g^{a_1} r g^{a_2} r ... r g^{a_N}$ we can quickly see that the inverse is $\gamma^{-1} = g^{-a_N} r g^{-a_{N-1}} r ... r g^{-a_1}$. Its easy to see that these define the inverse mappings of smaller intervals onto the unit interval. Note that one needs to be careful when working with $\gamma^{-1}(x)$ because this has poles that are just outside of the interval of the mapping. That is, the domain of the isomorphism is defined only on the interval of the mapping, and not the whole unit interval (its the range, not the domain, that is the whole unit interval). These inverse mappings can also be enumerated by $\mathbb{Q} \otimes \mathbb{N} \otimes C_2$, of course.

All of the remaining group elements that we haven't yet discussed are those with a mixture of positive and negative values for the a_k. These group elements can be obtained by alternately concatenating interval-shrinking and interval-expanding group elements. One has to be considerably more careful in keeping track of the domains and ranges of each mapping. We claim (without proof at this time) that these types of mappings do not introduce any new types of intervals, and so enumeration originally provided, *viz.* $(\mathbb{Q} \otimes \mathbb{N} \otimes C_2) \oplus (\mathbb{Q} \otimes \mathbb{N} \otimes C_2)$ is essentially unchanged.

ToDo: give for example $(rg^{-2}rg^2)(x) = x + 2$ and discuss what this means for the question mark, and how we should treat this (to modulo or not to modulo?)

Note also that if we limit ourselves to alternately shrinking and growing interval maps that are strictly contained one inside the other, then we find a proper subgroup of the modular group, whose group elements are now enumerated by pairs of intervals or .xxx. be careful here . Again, the domain and range of these sub-symmetries are now limited.

Thus I think we can now conclude that γ is an isomorphism of ?, mapping the unit interval into a subinterval, completing the proof. xxxx finish writing these conclusions.

0.9 The Topology of the Modular Group

The relationship between continued fractions and elements of the modular group hint at an interesting topology of the modular group. A subset of the modular group can be mapped to the rationals, and thus any metric on the rationals, whether the normal one or a p-adic one, can be 'lifted' back up to the modular group. We then ask if this metric can be extended onto the whole group. This mapping also means that we can draw a continuous curve in the modular group: that a subset of the modular group is diffeomorphic to the rationals. Thus, we are lead to ask, where else can we draw continuous curves in the modular group? What is the connectedness of those curves? Is the manifold simply connected, or in fact punctured an infinite number of times? But we also saw that the more natural mapping is between the contractive group elements and intervals, so that the set of contractive group elements is essentially an open covering of the unit interval. Thus, we have a number of confusing, inter-related ideas, and ask how they can be sorted out. The goal is to define and explore the topology of the modular group, by extending what we've learned for the contractive elements onto the whole group.

Sooo ....

Step 1 extend the metric.

0.10 Sierpinski Gasket

Free group of Z_2 and Z_3

More generally ... Binomial coefficients are on a tree, and are thus mappable to real number line. ditto for stirling numbers. But binomial coeffs are also have sierpinski-gasket divisibility features. Divisibility is about Ramanjuan sums. So establish the relationships there.

0.11 Misc junk

In the above sections, we have found the first steps of a set-theoretic representation of the modular group. If we consider the function $\tilde{\gamma}(x) = \gamma(1/x)$ then we see that it is the desired group homomorphism: for $r = [a_1, a_2, ..., a_N] \in \mathbb{Q}$ and $s = [b_1, b_2, ..., b_M] \in \mathbb{Q}$ we have $(\tilde{\gamma}_r \circ \tilde{\gamma}_s)(x) = \tilde{\gamma}_r(\tilde{\gamma}_s(x)) = [a_1, a_2, ..., a_N, b_1, b_2, .., b_M, 1/x]$. And so we have demonstrated a mapping $\rho : SL(2,\mathbb{Z}) \to \mathscr{F}_{\mathbb{R}}$ that maps $\gamma \mapsto \tilde{\gamma}(x) = [a_1, a_2, ..., a_N, x]$. However, we are far from done: this map is not defined for all of the group elements, but only for the subset of elements that are contractive: that is for those elements that have all of the $a_k > 0$.

this property on some but not all of the group elements of SL(2,Z). (We've only done it for the contracting group elements, So we really need to extend this, possibly through topological arguments about the coverings, where we cover each occurrence of a negative a_k in the continued fraction with an open subset, and do the standard topological arguments about open coverings. We really need to get the full topological structure working here.

Note also that in the above, the set X is the rationals $\mathbb{Q}$and so the symmetric group S_X is really kind-of like the power-set 2^X which is of course kind-of like the reals $\mathbb{R}$ and so this is gobbledy-gook about extending the rationals to the reals. (Never mind the p-adic closures). So we are forced to contemplate a seemingly rather deep relationship between the real numbers and the Modular Group, and are prompted to assert that the real numbers (and not just the rationals) have a marvelous fractal symmetry. Furthermore, the natural conclusion of this argument is that the fractals are really the expression of what would otherwise be this deeper underlying but hidden symmetry.

Put in a plug for that educator who is promoting the idea that Farey arithmetic is the "fifth operation", after addition, subtraction, etc. and should be taught in grade-school curricula as such. Just as the reals are closed under addition and multiplication, they are also closed under this weird, freaky modular group aka Farey number operation.

1 Symmetries Induced on $\mathbb{Q}$ and $\mathbb{R}$ (and $\mathbb{Q}_p$?)

Note that the correspondences presented here induce fractal isomorphisms of $\mathbb{Q}$ onto itself. This was already hinted at in the previous section, where we were trying to just plain-old count the rationals, and not getting what every middle and high-school teacher teaches students to expect. This symmetry would also imply a symmetry on the closure in reals $\mathbb{R}$. Exactly what it implies for the non-Archimedean closures $\mathbb{Q}_p$ is not yet clear to me. This symmetry is never made use of in classical analysis.

The field structure of the rationals in turn induces a field structure on a certain subgroup of the modular group, specifically, the interval-mapping subgroup. Albeit this would be a highly fractal structure ... I have no clue of what this field structure implies for modular forms or goodies like the Weierstrass elliptic integrals, etc. This would be a great TBD with a whole-lotta partyin goin on.

2 Conclusions

To be written.

References

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