

Geometry of Space, Time and Other Things

The Mathematics of Fiber Bundles

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Fiber Bundles

- Central to physics: classical mechanics, electrodynamics, quantum field theory, gravitation, superconductivity.
- It was not always that way!
- Unified (pseudo-)Riemannian Geometry (i.e. Gravitation) with Symplectic Geometry (classical mechanics) with Electrodynamics with Yang-Mills theory with Superconductivity with Fermions (QFT)
- A single, unified framework for (almost) all of the fundamental theories of physics.
- And that is the topic today.

Zen Koans

- There will be a lot of equations today
- More than several semesters worth ...
- Notation is KEY: commonplace, widespread notation
- What do those formulas MEAN? Intuitively ??
- Interpretation of poetry, jokes of Zen koans
- Intuition alone is FAULTY. Formulas are PRECISE!
- Equations are tie-breakers for intuitive ideas
- Creativity and imagination are KEY
- It will be dizzying

Tee-shirt Equations

Before fiber bundles, it was a hot mess:

- Classical mechanics was Hamilton's equations

$$\dot{p} = -\frac{dH}{dq} \quad \dot{q} = \frac{dH}{dp}$$

- Electrodynamics was Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = 0$$

- Gravitation was Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Superconductivity was the Ginzberg–Landau equations

$$\mathcal{L} = \alpha|\phi|^2 + \beta|\phi|^4 + \frac{1}{2m} \left| \left(-i\hbar\vec{\nabla} - 2e\vec{A} \right) \phi \right|^2 + \frac{|\vec{B}|^2}{2}$$

- Standard Model = Yang-Mills + Higgs + Fermions

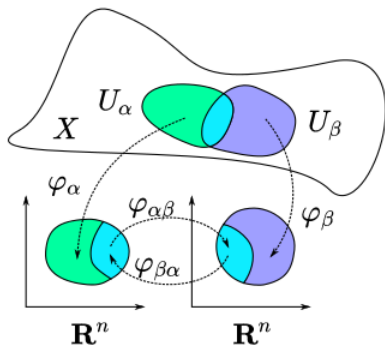
Outline

- Manifold M as gluing of \mathbb{R}^n - coordinate charts
- (Integrable) vector fields as hair/fur that can be combed
- Tangent vector space $T_p M$
- Back to basics: Vector spaces; notation: e_n as basis vector
- A frame field as $e_n(p)$ varying from point to point p .
- Frame fields can twist around, rotate, swirl.
- The rotation matrix A . The connection $A_i = \Gamma_{ij}^k$ aka Christoffel symbol
- Rotations & rotation matrices in 3D
- Curvature as total rotation after walking a loop.
- Parallel transport
- Geodesics

Charts and Manifolds

An atlas is:

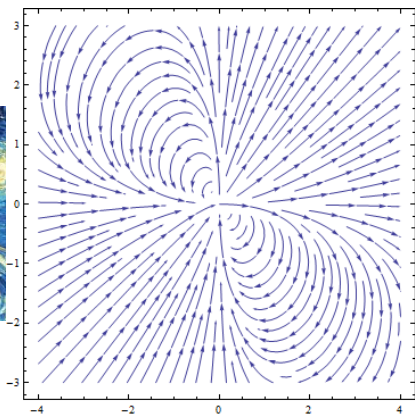
- A collection of regions U_α
- A collection of charts $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$
- A collection of “transition functions” $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$



Vector Fields

A vector field is:

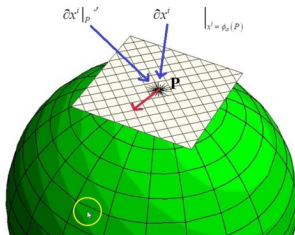
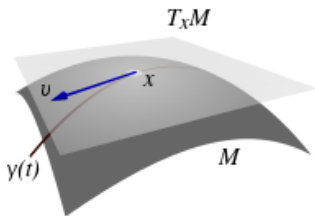
- A collection of vectors \vec{v}_p
- One for each point $p \in U_\alpha$
- Smooth, differentiable, integrable



Tangent vector spaces

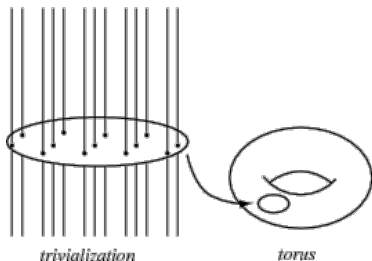
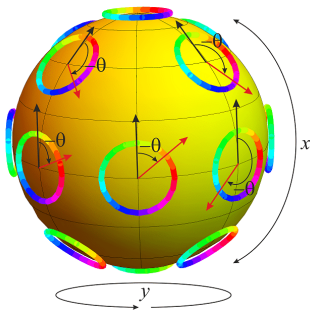
The tangent vector space $T_p M$ is:

- A point $p \in U_\alpha$ (that is, a point in $p \in M$)
- The collection of ALL possible vectors $\vec{v}_p \in T_p M$



Tangent bundles - Fiber bundles

- The tangent bundle TM is the set of all T_pM for all $p \in M$
- The sphere bundle SM is a set of spheres S_pM , one for each $p \in M$
- The circle bundle is a set of circles, one for one for each $p \in M$
- The fiber bundle E is a set of fibers F , one for one for each $p \in M$

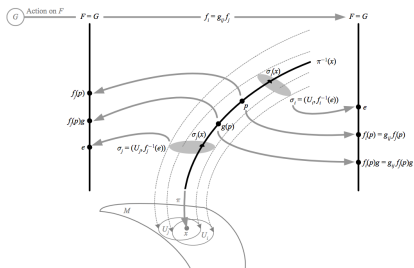


trivialization of a line bundle on a torus

Fiber Bundles

Properties of Fiber bundles

- Locally, they are trivial products $U_\alpha \times F$ of a chart U_α and a fiber F
- Neighboring fibers need to be glued (soldered) together; the connection!
- Works best when fibers have some natural symmetry
- A group G that moves you up and down a fiber F

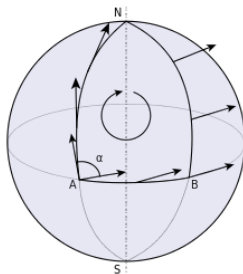
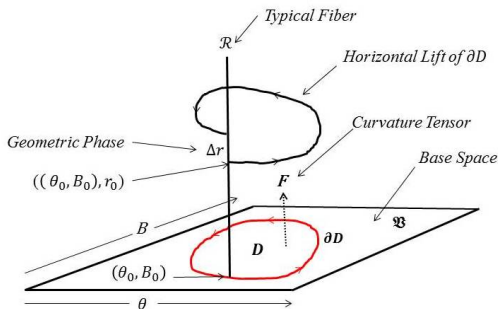


$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \pi \downarrow & \nearrow \text{proj}_1 & \\ U & & \end{array}$$

Horizontal and Vertical Bundles

Gluing together neighboring fibers allows:

- Movement (horizontally) from fiber to fiber
- While carrying a coordinate frame (parallel transport)
- Closed paths in horizontal (base) space typically DON'T close on the bundle!
- That is, curvature!



Fiber bundles in Physics

- Circle bundles – $U(1)$ – Electromagnetism
- Frame bundles – $GL(n, \mathbb{R})$ – General Relativity (Riemannian geometry)
- Lie groups – $SU(3)$ – Quarks & Gluons (strong force)
- Lie groups – $SU(2)$ – Weak force (radioactive decay)
- Tangent bundles – Position and Momenta – Classical Mechanics (Symplectic geometry)
- Spinor bundles – Fermions
- Fisher Information (Kullback-Leibler divergence) – Quantum Mechanics

All fiber bundles have

- Horizontal and Vertical subspaces
- A connection one-form (Christoffel symbols)
- Geodesics (shortest paths)
- Parallel transport (carrying around a coordinate frame)
- Curvature two-form (curvature tensor)
- Torsion and Contorsion tensors

Some have

- A metric
- Ricci and scalar curvature
- Solder form (canonical one-form)

Outline

- Groups
- Actions
- Vectors
- Rotations
- Infinitesimal rotations (generators)
- Derivatives

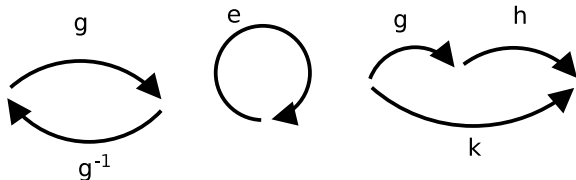
Groups

Examples of Groups:

- Rotation group
- Translation group
- Permutation group

A Group G is a set where:

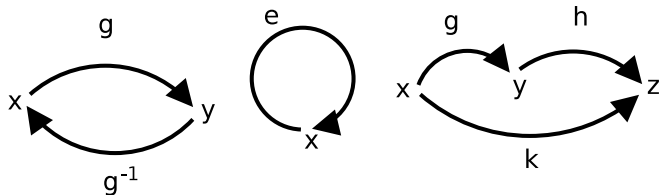
- Inverses: for all $g \in G \exists g^{-1} \in G$ s.t. $gg^{-1} = e$
- Identity element: $e \in G$ s.t. $\forall g \in G e \cdot g = g$
- Closure: For all $g, h \in G \exists k \in G$ s.t. $gh = k$



Group Actions

A group G acting on a set X :

- Notation: $G : X \rightarrow X$ with $g : x \mapsto y$ also written as $g \cdot x = y$ or $x \xrightarrow{g} y$
- Identity: $e \cdot x = x$
- Associative: $(g \cdot (h \cdot x)) = (g \cdot h) \cdot x$
- Invertable: $(g^{-1} \cdot (g \cdot x)) = (g^{-1} \cdot g) \cdot x = e \cdot x = x$
(non-dissipative)



Vectors and Bases

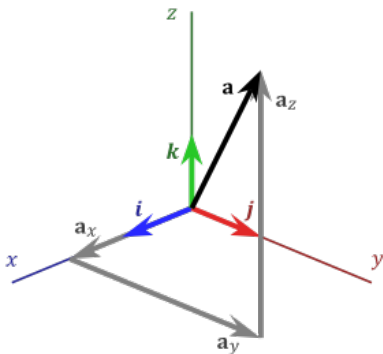
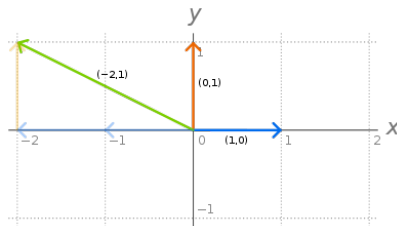
A Vector $\vec{v} \in \mathbb{R}^n$ in n -dimensional space is:

- A collection of n real numbers: $\vec{v} = (v^{(1)}, v^{(2)}, v^{(3)}, \dots, v^{(n)})$

A vector space basis for \mathbb{R}^n is a collection of n vectors

$\{\vec{e}_k : 1 \leq k \leq n\}$:

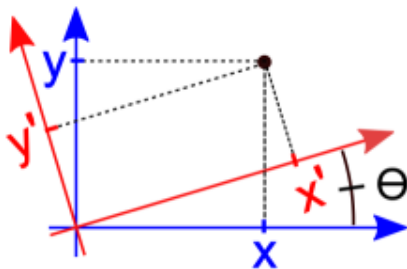
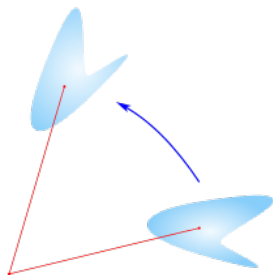
- Where $e_1 = (1, 0, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$ and $e_3 = (0, 0, 1, 0, \dots, 0)$ and ...



Passive and Active Rotations

- A rotation changes the direction of a vector: $\vec{x}' = R\vec{x}$
 - Body coordinates vs. Space coordinates
- A rotation can be represented by a matrix
 - In 2D:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Rotations in N dimensions

A rotation changes the direction of a vector: $\vec{x}' = R\vec{x}$

- In n dimensions :

$$\begin{pmatrix} x^{(1)'} \\ x^{(2)'} \\ \vdots \\ x^{(n)'} \end{pmatrix} = \begin{bmatrix} \cos \theta & 0 & \cdots & 0 & -\sin \theta & \cdots & 0 \\ 0 & 1 & & & 0 & & 0 \\ \vdots & & \ddots & & \vdots & & \\ 0 & & & 1 & 0 & & \vdots \\ \sin \theta & 0 & \cdots & 0 & \cos \theta & & \\ & & & & & 1 & \\ \vdots & & & & & & \ddots & 0 \\ 0 & 0 & & & \cdots & & 0 & 1 \end{bmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(n)} \end{pmatrix}$$

Infinitesimal Rotations

- An infinitesimal rotation:

$$\vec{x} + \delta\vec{x} = (I + \delta R)\vec{x} = \vec{x} + \delta R\vec{x} = \vec{x} + \left(\left. \frac{dR}{d\theta} \right|_{\theta=0} \delta\theta \right) \vec{x}$$

- In 2D:

$$\vec{x} = \begin{pmatrix} x' \\ y' \end{pmatrix} = R\vec{x} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- but:

$$\left. \frac{d \cos \theta}{d\theta} \right|_{\theta=0} = 0 \quad \text{and} \quad \left. \frac{d \sin \theta}{d\theta} \right|_{\theta=0} = 1$$

- so

$$\delta\vec{x} = \delta R\vec{x} = \left. \frac{dR}{d\theta} \right|_{\theta=0} \delta\theta \vec{x} = \delta\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} = \delta\theta L\vec{x}$$

- The matrix L is called the “the infinitesimal generator of rotations” AKA “the angular momentum operator”.

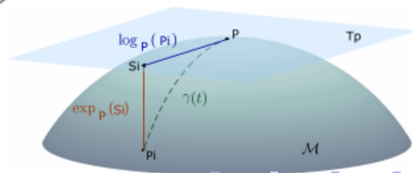
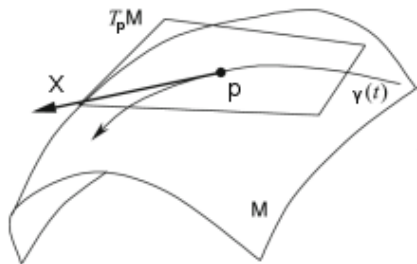
Partial derivatives

Given a curve $\gamma(t)$ in a manifold M , such that the curve is tangent to the vector X at $p \in M$, the Lie derivative of a function f on M is:

$$\mathcal{L}_X f(p) = \left. \frac{f(\gamma(t)) - f(\gamma(0))}{t} \right|_{p=\gamma(0) \text{ and } X=\gamma'(0)}$$

Notation: the vector (field) X is written as

$$X = X^\mu \frac{\partial}{\partial x^\mu} = X^\mu \partial_\mu = X^\mu e_\mu$$



Differential forms

The dual basis: $e^\mu(e_\nu) = e^\mu e_\nu = \delta^\mu_\nu$

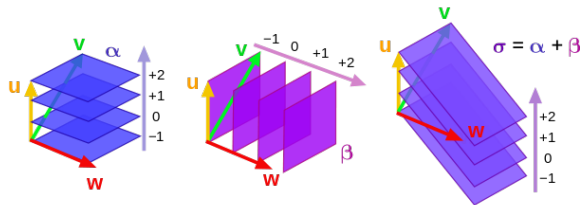
The Kronecker delta: $\delta^\mu_\nu = \begin{cases} 1 & \text{when } \mu = \nu \\ 0 & \text{when } \mu \neq \nu \end{cases}$

Partial derivatives: $\partial_\mu = e_\mu$

Differential forms: $dx^\mu = e^\mu$

They are dual: $dx^\mu(\partial_\nu) = \delta^\mu_\nu$

A function that takes a vector and spits out a number (“counting surfaces”):



$$\langle \alpha, u \rangle = 3$$

$$\langle \alpha, v \rangle = 3$$

$$\langle \alpha, w \rangle = 0$$

$$\langle \beta, u \rangle = 0$$

$$\langle \beta, v \rangle = 0$$

$$\langle \beta, w \rangle = 2.5$$

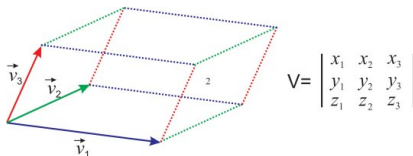
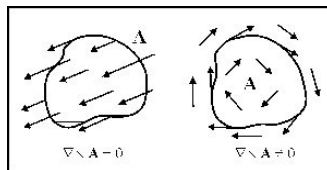
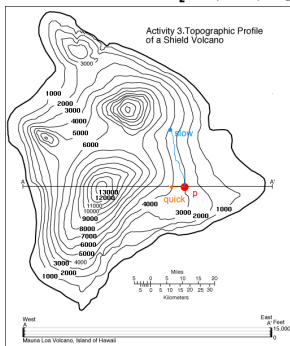
$$\langle \sigma, u \rangle = 3$$

$$\langle \sigma, v \rangle = 3$$

$$\langle \sigma, w \rangle = 2.5$$

Examples of differential forms

- The 1-form: df is like the gradient $\vec{\nabla} f$
- The 2-form: $dx \wedge dy$ is like the curl: $\vec{\nabla} \times \vec{v}$
- The 3-form $dx \wedge dy \wedge dz$ is like the volume determinant
- $\det I = \det [e_1, e_2, e_3]$

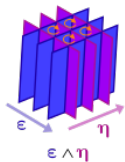


Wedge products



$$\epsilon = \epsilon_\mu dx^\mu$$

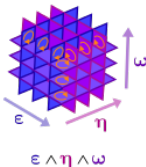
$$\eta = \eta_\mu dx^\mu$$



$$\epsilon \wedge \eta = \epsilon_\mu \eta_\nu dx^\mu \wedge dx^\nu$$

$$\text{Antisymmetric: } dx \wedge dy = -dy \wedge dx$$

$$\text{Linear: } adx + bdx = (a + b)dx$$



$$\text{Tensorial: } T_{\mu\nu\cdots\rho} dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\rho$$

Covariant derivative

Joins neighboring fibers: $D = d + A$

- Alternate notation: $D^\mu = dx^\mu + A^\mu$ when moving in direction μ
- A is an infinitesimal rotation matrix: $A^\mu = [A^\mu]_{ij} = \Gamma^\mu_{ij}$
- Connection=Christoffel symbols
- Fiber coordinates: index i, j act on the fiber
- Base space coordinates: μ is a direction in the base space.

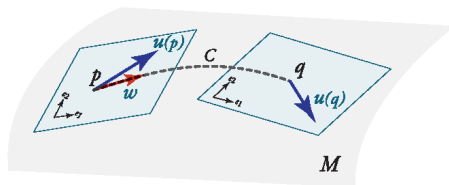
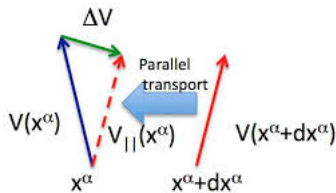


Fig. 1. Covariant derivative. On a curved manifold, a connection indicates



Moving (alternately) in two directions:

- Notation: Field strength 2-form: $F = D \wedge D = dA + A \wedge A$

- Notation: Curvature tensor:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X, Y]$$

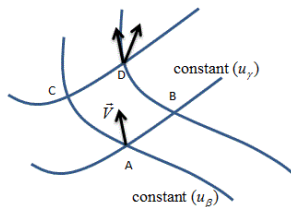
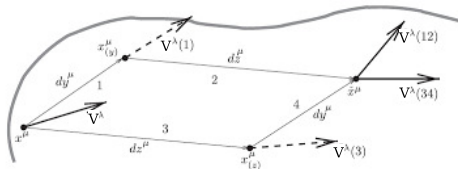
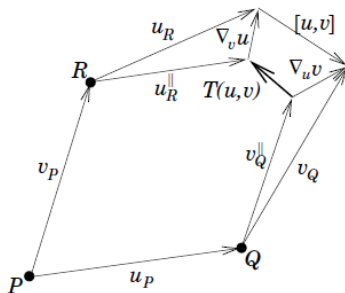
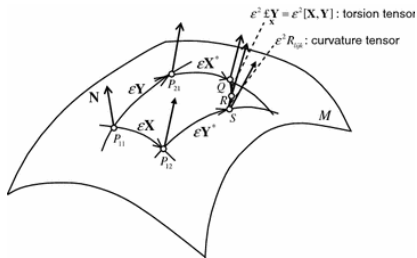


Fig 1.2



Moving (alternately) in two directions:

- Notation: Torsion form: $\Theta = D\theta = d\theta + A \wedge \theta$
- ... where θ is the solder form: $\theta = \sum_i p_i dq_i$
- Notation: Torsion tensor: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
- There is one unique torsionless connection: the Levi-Civita connection



Electromagnetism

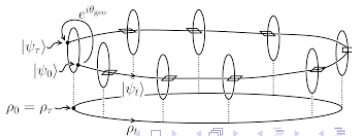
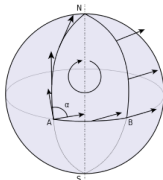
E&M is (just) a circle bundle!

- Each group element: $g = e^{i\theta}$

- Vector potential: $A_\mu = g^{-1} \partial_\mu g = (\vec{A}, \phi)$

- Curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$

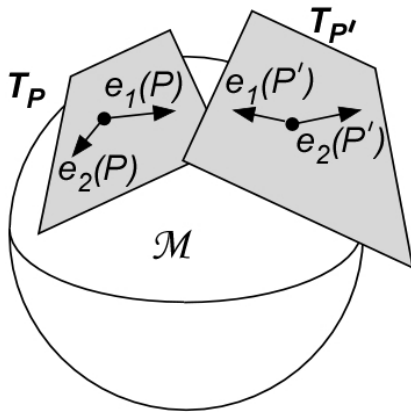
- ...or $\vec{E} = \vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$
- Choice of gauge == choice of coordinates on the circle!
- Geodesics go “splat” on an electric charge!
- Holonomy is the Bohm-Aharonov effect!



Frame fields (Vierbeins)

A frame field is:

- A collection of basis vectors $\{\vec{e}_k(p)\}$
- One for each point $p \in U_\alpha$



Metric Tensor

The metric is (just) a product of vierbeins (frames)

$$g_{\mu\nu} = e_{\mu} \cdot e_{\nu} = e_{\mu}^a \eta_{ab} e_{\nu}^b$$

$$\text{where } \eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Gives the length of a vector:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v^a v^b \eta_{ab}} = \sqrt{(v^{(x)})^2 + (v^{(y)})^2 + (v^{(z)})^2 - (v^{(t)})^2}$$

General Relativity

General Relativity is (just) a frame bundle!

- Each group element: $g \in SO(3, 1)$
- Connection: $A_\mu = g^{-1} \partial_\mu g = \Gamma_\mu^{\rho\sigma}$
- Curvature: $F = dA + A \wedge A$
- ... that is, curvature $R_{\mu\nu}^{\rho\sigma} = \partial_\mu A_\nu^{\rho\sigma} - \partial_\nu A_\mu^{\rho\sigma} + \frac{1}{2} [A_\mu, A_\nu]^{\rho\sigma}$
- Choice of gauge == choice of coordinate frame!
- Geodesics go “splat” on a black hole singularity!

Unification of Physics

- Fiber bundles unify all of the fundamental physics theories
- So what is there left to unify?
- Well, why/how $U(1) \times SU(2) \times SU(3) \times SO(3,1)$?
- Kaluza-Klein theory (the 5-sphere)
- Affine Lie groups (string theory)
- Supersymmetry (fermions)

Geometry with Formulas

The hardest part with formulas is (1) there are so many (2) there are many different ways of writing down the **same** equations, using wildly different notation.

- Introduce Lie derivative $L_X f$
- Introduce covariant derivative $D = d + A$ – Rosetta stone of different notations
- Geodesics as solutions of Hamilton's equations i.e. as linear, first-order diffeq NOT second order!

$$\dot{p} = -\frac{dH}{dq} \quad \dot{q} = \frac{dH}{dp}$$

where H =squared-length-of-curve

- exp as the map that moves along geodesics
- Geodesic completeness

Metric Differential Geometry

- Introduce metric as inner product of frame fields
$$g_{\mu\nu} = e_\mu \cdot e_\nu = e_\mu^a e_\nu^b \eta_{ab}$$
- Metric was NOT needed to define curvature, geodesics, parallel transport
- (metric is almost kind-of useless except that its a standard touch-stone for GR)
- Provide (repeat) Einstein eqns.
- Replace frame field by generic fiber bundle
- e.g. U(1) for electromagnetism, SU(n) for yang-mills
- Maxwell's equations are nothing more than Hamilton's eqns on U(1) + Bianchi identities

$$F = dA \quad d * F = 0$$

- Yang-Mills/Einstein

$$F = dA + A \wedge A \quad D * F = 0$$

is the same as

- Maxwell's eqn's have singularities called "electric charges" and geodesics go "splat" on an electric charge
- Swarzschild BH's are just like electric charges: geodesics go splat when they get there.