A Set of Identities for a Class of Alternating Binomial Sums Arising in Computing Applications

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Abstract

We perform certain alternating binomial summations with parameters that occur in the analysis of algorithms. A combination of integral and special function and special number representations is used. The results are sufficiently general to subsume several previously known cases. Extensions of the method are apparent and are outlined.

Key words and phrases

binomial summation, Stirling numbers, Beta function, polygamma function, generalized harmonic numbers, generating function, Pochhammer symbol, binomial coefficient, Bell polynomial

Introduction

Alternating binomial sums arise frequently in computer science and data processing in the design and analysis of algorithms (e.g., [6, 9, 10, 11, 15]). The asymptotic form of such sums is often of interest in connection with determining the average- or worst-case run time. Because of the sign alternation of the summands, there may be substantial cancellation, masking the dominant behaviour. Recently there has been additional interest in certain alternating binomial sums and their connection with harmonic numbers and representation in terms of the Bell polynomials [8, 14]. In fact, the question of a broader range of validity of such identities has been posed [14]. In this article, we demonstrate a method to reach alternating binomial sum representations with a domain extended to the complex plane.

Our approach is to obtain exact analytic relations. To these, known asymptotic relations may be applied if desired. Our results help to elucidate the connections between certain Bell polynomial representations, generalized harmonic numbers, Stirling numbers, and special values of the polygamma functions, and should be helpful in the analysis of either deterministic or probabilistic algorithms. Various integral representations provide a convenient centerpoint of our development, but this is by no means necessary. There are many complementary approaches, including the use of finite difference operators [14, 4]. Afterall, $\Delta^n f(x) = (E-I)^n f(x) = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} f(x+k)$ where $\Delta f(x) \equiv f(x+1) - f(x)$, $Ef(x) \equiv f(x+1)$, and I is the identity operator. A way to think of the underlying combinatorics is in terms of number partitioning needed in the course of differentiating composite functions [19].

In this paper, we calculate the alternating binomial sums

$$S(x, N, m) \equiv \sum_{k=0}^{N} \binom{N}{k} \frac{(-1)^k}{(x+k)^m}, \tag{1}$$

in multiple fashion for positive integers N and m and complex $x \in C/\{0, -1, -2, \ldots, -N\}$. The special cases $x = \pm K$ for K a positive integer recover results of Kirschenhofer [8] and Larcombe et al. [14]. The special case of $x = 1$ in Eq. (1) has further applications in quantum information science [3] and this case is evident in our scheme. We point out additional special cases for x a rational number.

Our work is more illustrative than exhaustive as there are many possible extensions. Especially when the summand in question contains a function for which Lemma 1 below applies, there will always be an equivalent representation in terms of the Bell polynomials $Y_m(x_1,\ldots,x_m)$. Moreover, as Eq. (10) below demonstrates, any time an expression contains Stirling numbers of the first kind, these may be replaced with Bell polynomials with generalized harmonic number arguments. Of the Bell polynomials, we note that they may be written as a lower triangular determinant save for a superdiagonal of -1's:

$$Y_n(x_1,\ldots,x_n) = \begin{vmatrix} x_1 & -1 & 0 & 0 & \ldots & 0 \\ x_2 & x_1 & -1 & 0 & \ldots & 0 \\ x_3 & 2x_2 & x_1 & -1 & \ldots & 0 \\ x_4 & 3x_3 & 3x_2 & x_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & -1 \\ x_n & \binom{n-1}{1}x_{n-1} & \binom{n-1}{2}x_{n-2} & \binom{n-1}{3}x_{n-3} & \ldots & \binom{n-1}{n-1}x_1 \end{vmatrix}. \tag{2}$$

They satisfy the recursion relation

$$Y_{n+1}(x_1,x_2,\ldots,x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1,x_2,\ldots,x_{n-k})x_{k+1}, \tag{3a}$$

and

$$Y_n(x_1+y_1,\ldots,x_n+y_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(x_1,x_2,\ldots,x_{n-k})Y_k(y_1,y_2,\ldots,y_k), \quad n \geq 0. \tag{3b}$$

Reference [18] contains a background section on the Bell polynomials, wherein Proposition 2 repeats the well known determinant expression (2). Those authors also denote by B_n^- "inverse Bell polynomials" that have usually been called logarithmic polynomials L_n [4] (p. 140). Proposition 1 of Ref. [18] covers the parity of Bell polynomials when the even- or odd-indexed variables are put to zero and may instead by obtained from the determinantal expression (2). For further information on Y_n we refer to standard works [4, 16, 17].

Analytic number theory is an additional field where significant alternating sums occur and these applications should not be overlooked. In fact, we take up an important such instance elsewhere.

Summary of results and preparation

Let $N > 0$ and $m > 0$ be integers, $x \in C/\{0, -1, -2, \ldots, -N\}$, Γ the Gamma function, $(a)_n = \Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol, B the Beta function, and ${}_pF_q$ the generalized hypergeometric function [2, 13]. Let $s(n,k)$ and $S(n,k)$ be Stirling numbers of the first and second kind, respectively [1, 4, 7, 16, 17]. We have

Proposition 1

$$S(x,N,m) = \frac{1}{x^m}\ {}_{m+1}F_m(x,\ldots,x,-N;x+1,\ldots,x+1;1) \tag{4}$$

$$= \frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial}{\partial x}\right)^{m-1}\frac{N!\Gamma(x)}{\Gamma(N+x+1)} \tag{5}$$

$$= \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-xt}(1-e^{-t})^N dt \tag{6}$$

$$= \frac{2^{N+m}}{(m-1)!}\int_0^\infty w^{m-1}e^{-(2x+N)w}\sinh^N w\ dw \tag{7}$$

$$= \frac{N!}{(m-1)!}\sum_{n=N}^\infty \frac{S(n,N)}{n!}\frac{(n+m-1)!}{(x+N)^{n+m}} \tag{8}$$

$$= \sum_{n=m-1}^\infty \frac{(-1)^n}{n!}s(n,m-1)B(N+n+1,x) \tag{9}$$

$$= \frac{(-1)^{m-1}}{(m-2)!}\sum_{n=m-1}^\infty \frac{B(n+N+1,x)}{n}Y_{m-2}[H_{n-1}, -H_{n-1}^{(2)}, 2!H_{n-1}^{(3)}, \ldots, (-1)^{m-1}(m-3)!H_{n-1}^{(m-2)}], \tag{10}$$

$$= \frac{(-1)^{m-1}}{(m-1)!}\frac{N!}{(x)_{N+1}}Y_{m-1}\left[g(x), g'(x), \ldots, g^{(m-1)}(x)\right]. \tag{11}$$

In Eq. (11),

$$g(x) = \psi(x) - \psi(x+N+1) = -\sum_{k=0}^N \frac{1}{x+k}, \tag{12a}$$

$$g^{(\ell)}(x) = \psi^{(\ell)}(x) - \psi^{(\ell)}(x+N+1) = -(-1)^\ell \ell! \sum_{k=0}^{N} \frac{1}{(x+k)^{\ell+1}}, \qquad (12b)$$

where $\psi = \Gamma'/\Gamma$ is the digamma function and $\psi^{(j)}$ the polygamma function [1]. The summation expressions in Eq. (12) follow by the use of the functional equations of these functions. In Eq. (10), $H_n^{(\ell)}$ are generalized harmonic numbers,

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} = \frac{(-1)^{r-1}}{(r-1)!} \left[\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)\right]. \qquad (13)$$

The proof of the equivalence given in Eq. (11) makes use of the following.
Lemma 1. For differentiable functions f and g such that $f'(x) = f(x)g(x)$, assuming all higher order derivatives exist, we have

$$\left(\frac{d}{dx}\right)^j f(x) = f(x) Y_j \left[g(x), g'(x), \ldots, g^{(j-1)}(x)\right]. \qquad (14)$$

Proof of Lemma 1. Under the premise, $g(x) = (d/dx)\ln f(x)$ whenever $f \neq 0$ and $(d/dx)^j f = f \exp(-\ln f)(d/dx)^j \exp(\ln f)$. The conclusion then follows as a special case of the Faà di Bruno formula for the derivative of a composite function. The result extends to $x \in C$ when a branch cut for the logarithm is taken from the origin to the point at infinity.

Remarks and an example. (i) The (exponential) complete Bell polynomials Y_n may be obtained as a sum over the (exponential) partial Bell polynomials $B_{n,k}$: $Y_n(x_1, \ldots, x_n) = \sum_{k=1}^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$, $Y_0 = 1$, $Y_1(x_1) = x_1$. (ii) The condition of the Lemma occurs often in practice and a nice example is given by the Gamma function with $\Gamma' = \Gamma\psi$ ([4], p. 175). Then $\Gamma^{(n)}(1) = Y_n(-\gamma, x_2, x_3, \ldots, x_n) = \int_0^\infty e^{-x} \ln^n x \, dx$, where $\gamma = -\psi(1)$ and $x_j = (-1)^j (j-1)! \zeta(j)$ with $\zeta(s)$ the Riemann zeta function.

(iii) The equality of Eqs. (3) and (4) is obvious. The original binomial series form (1) may be returned from Eq. (4) by the following two steps. First, depending upon whether N is even or odd we substitute into Eq. (4) either the expansion ([5], p. 25)

$$\sinh^{2n} x = \frac{(-1)^n}{2^{2n}} \left[\sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n}\right], \qquad (15a)$$

or

$$\sinh^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} 2 \binom{2n-1}{k} \sinh(2n-2k-1)x. \quad (15b)$$

We then apply tabulated integrals ([5], p. 360) to find Eq. (1).

(iv) The upper limit on the summation in Eq. (1) could just as well be put to ∞ due to the property $\binom{n}{k} = 0$ for $k > n$.

(v) We are using notation for the Stirling numbers as followed by Comtet [4] and Riordan [16] and the reader should be aware of other conventions. Indeed the notation for these numbers has never been standardized [1]. (vi) The Stirling numbers of the second kind are of rank one while of the first kind are of rank two. I.e., the latter numbers require a double summation in order to be expressed in terms of elementary factors [4]. (vii) We do not require them here, but mention that asymptotic forms of the Stirling numbers and functions are known.

Proof of Proposition 1

We now proceed systematically through the list of equivalences given in Proposition 1. In writing Eq. (1) in the form (4) we use the power series form of ${}_{m+1}F_m$ [2] and apply $(x)_k/(x+1)_k = x/(x+k)$. The terminating hypergeometric series in Eq. (4) is $m+N$-balanced since the sums of numerator and denominator parameters differ by this positive integer.

We next recognize that

$$\sum_{k=0}^{N} \binom{N}{k} \frac{(-1)^k}{x+k} = \frac{N!\Gamma(x)}{\Gamma(x+N+1)} = \frac{N!}{(x)_{N+1}} = \frac{N!}{x(x+1)_N} = B(x, N+1), \quad (16)$$

that is equivalent to a partial fractional decomposition. Then

$$\left(\frac{\partial}{\partial x}\right)^{m-1} \frac{N!\Gamma(x)}{\Gamma(N+x+1)} = (-1)^{m-1}(m-1)!S(x,N,m)$$

$$= \left(\frac{\partial}{\partial x}\right)^{m-1} \sum_{k=0}^{N} \binom{N}{k} (-1)^k \int_0^\infty e^{-(x+k)t}\, dt$$

$$= (-1)^{m-1} \sum_{k=0}^{N} \binom{N}{k} (-1)^k \int_0^\infty t^{m-1} e^{-(x+k)t}\, dt$$

$$= (-1)^{m-1} \int_0^\infty t^{m-1} e^{-xt} (1-e^{-t})^N dt. \tag{17}$$

In the above the interchange of differentiation and integration is justified by the absolute convergence of the integral. We have shown the equality of Eq. (1) and (4)-(6). Equation (7) follows from (6) by using the definition of the hyperbolic sine function in terms of exponentials.

To obtain the form of Eq. (8) we write Eq. (6) as

$$S(x,N,m) = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-(x+N)t} (e^t - 1)^N dt, \tag{18}$$

and apply a generating function for the Stirling numbers of the second kind [1]:

$$(e^x - 1)^m = m! \sum_{n=m}^\infty S(n,m) \frac{x^n}{n!}. \tag{19}$$

In obtaining Eq. (9) we first make the change of variable $v(t) = 1 - e^{-t}$ in Eq. (6), giving

$$S(x,N,m) = \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 v^N (1-v)^{x-1} \ln^{m-1}(1-v)\ dv. \tag{20}$$

We then use a generating function for Stirling numbers of the first kind,

$$\ln^m(1+x) = m! \sum_{n=m}^\infty s(n,m) \frac{x^n}{n!}, \tag{21}$$

and carry out the integration with the Beta function.

For Eq. (10) we apply Theorem B of Ch. V of Ref. [4] (p. 217) for the unsigned Stirling number of the first kind. We write this result in the form

$$|s(n+1,k+1)| = (-1)^{n+k} s(n+1,k+1) = \frac{n!}{k!} Y_k[H_n, -H_n^{(2)}, 2! H_n^{(3)}, \ldots, (-1)^{k-1}(k-1)! H_n^{(k)}]. \tag{22}$$

We then substitute for $s(n,m-1)$ in Eq. (9).

In order to obtain the form (11) we use Eq. (5) and apply Lemma 1 with the function $f(x) = N!/(x)_{N+1}$, such that

$$\frac{d}{dx} f(x) = f(x)[\psi(x) - \psi(x+N+1)], \tag{23}$$

providing the function $g(x)$ presented in Eq. (12a). Hence with this particular $g(x)$ we have

$$\left(\frac{d}{dx}\right)^j f(x) = f(x) Y_j \left[g(x), g'(x), \ldots, g^{(j-1)}(x)\right], \tag{24}$$

and the rest of Proposition 1 follows.

Remarks. (i) There are many other variations on the possible generating functions that may be introduced into the integrand of Eq. (6) to produce equivalent forms with the Stirling numbers. Ref. [3] provides examples of these alternatives. (ii) The lower limit of summation in Eqs. (8) and (9) could just as well be put to 0 due to the property $s(n,k) = S(n,k) = 0$ for $n < k$. (iii) Equations (9) and (20) make it strikingly apparent how the case of $x = 1$ is special, when the Beta function is no longer required. (iv) Single or repeated integration by parts in Eq. (6) permits the derivation of other forms of $S(x,N,m)$ and of recursion relations for these sums. For instance, from $e^{-xt} = -(1/x)(d/dt)e^{-xt}$ and $[e^t/(N+1)](d/dt)(1-e^{-t})^{N+1}$ we immediately have

$$S(x,N,m) = \frac{1}{x}[S(x,N,m-1) + NS(x-1,N-1,m)], \tag{25a}$$

and

$$S(x,N,m) = \frac{1}{N+1}[(x-1)S(x-1,N+1,m) - S(x-1,N+1,m-1)], \tag{25b}$$

respectively. These relations have been obtained subject to Re $x > 0$ for Eq. (25a) and Re $x > 1$ for Eq. (25b).

Special cases

We very briefly mention cases where x is an integer or a rational number in Eqs. (1) and (11). When $x = 1$ we have for Eq. (12)

$$g(1) = \psi(1) - \psi(N+2) = -\sum_{k=1}^{N+1} \frac{1}{k} \equiv -H_{N+1}, \tag{26a}$$

and

$$g^{(\ell)}(1) = -(-1)^\ell \ell! \sum_{k=1}^{N+1} \frac{1}{k^{\ell+1}} \equiv -(-1)^\ell \ell! H_{N+1}^{(\ell+1)}, \tag{26b}$$

where $H_p^{(r)}$ are generalized harmonic numbers. These polygamma values are well known to relate to differences of the Riemann zeta function at integer argument, as $\psi^{(\ell)}(1) = -(-1)^\ell \ell! \zeta(\ell+1)$. Here $f(1) = N!/(N+1)! = 1/(N+1)$ and then

$$\left(\frac{d}{dx}\right)^j f(x)\Bigg|_{x=1} = \frac{1}{N+1} Y_j[-H_{N+1}, H_{N+1}^{(2)}, -2!H_{N+1}^{(3)}, \ldots, (-1)^j (j-1)! H_{N+1}^{(j)}]. \tag{27}$$

When x is a rational number it is possible to re-express the necessary derivatives $g^{(\ell)}(x)$. This can be done either in terms of the polygamma function or in terms of the Hurwitz and Riemann zeta functions (e.g., [12]). For instance we have $\psi^{(\ell)}(1/2) = (-1)^{\ell+1}\ell!(2^{\ell+1}-1)\zeta(\ell+1)$.

When x is an integer we are able to express the sums of Eq. (12) in terms of generalized harmonic numbers. For instance we have for K a positive integer

$$g^{(\ell)}(-K) = (-1)^{\ell+1}\ell! \sum_{\substack{k=0 \\ k\neq K}}^{N} \frac{1}{(k-K)^{\ell+1}} = (-1)^{\ell+1}\ell![H_{N-K}^{(\ell+1)} + (-1)^\ell H_K^{(\ell+1)}], \tag{28a}$$

and

$$g^{(\ell)}(K) = (-1)^{\ell+1}\ell![H_{N+K}^{(\ell+1)} - H_{K-1}^{(\ell+1)}]. \tag{28b}$$

Extensions

The approach of this article may be extended to a great many other integrals. We outline some of this using the Beta function as the base. However, one could just as well apply the techniques to the confluent hypergeometric function ${}_1F_1$, the Gauss hypergeometric function ${}_2F_1$, and then to ${}_pF_q$ more generally.

We consider

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du = 2\int_0^{\pi/2} \sin^{2x-1}\phi \cos^{2y-1}\phi \, d\phi = B(y,x),$$
$$\min[\text{Re } x, \text{Re } y] > 0, \tag{29}$$

so that

$$S(x,y,m,n) \equiv \int_0^1 u^{x-1}(1-u)^{y-1} \ln^{m-1} u \ln^{n-1}(1-u) \; du$$

$$= \left(\frac{\partial}{\partial x}\right)^{m-1} \left(\frac{\partial}{\partial y}\right)^{n-1} B(x,y). \tag{30}$$

Just as generalized binomial expansion gives

$$B(x,y) = \sum_{j=0}^{\infty} (-1)^j \binom{y-1}{j} \frac{1}{x+j}, \tag{31}$$

we have

$$S(x,y,m,n) = (-1)^{m-1}(m-1)! \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial y}\right)^{n-1} (1-y)_j \frac{1}{(x+j)^m}. \tag{32}$$

In obtaining this equation we wrote $\binom{y-1}{j} = (-1)^j (1-y)_j/j!$. Of course the sum terminates when $y-1$ is a positive integer and we again exclude nonpositive integer values for x. The Pochhammer polynomial occurring in Eq. (32) has derivative

$$\frac{d}{dy}(1-y)_j = (1-y)_j[\psi(1-y) - \psi(j+1-y)]. \tag{33}$$

Therefore, for instance, Lemma 1 may be used in finding all higher order derivatives.

We may next change variable in Eq. (30) to obtain

$$S(x,y,m,n) = (-1)^{n-1} \int_0^{\infty} v^{n-1} \ln^{m-1}(1-e^{-v})(1-e^{-v})^{x-1} e^{-yv} dv. \tag{34}$$

Otherwise, we may first put $z = 1/u$ in Eq. (29) and $v = \ln z$, giving

$$B(x,y) = \int_0^{\infty} e^{-(x+y-1)v}(e^v - 1)^{y-1} dv. \tag{35}$$

Then we have

$$\begin{aligned} S(x,y,m,n) = \int_0^{\infty} e^{-(x+y-1)v}(e^v - 1)^{y-1} \big[&(-1)^{m-1} v^{m-1} \ln^{n-1}(e^v - 1) \\ &+ (-1)^{m+n} v^{m+n-2}\big]\, dv. \end{aligned} \tag{36}$$

Equations (34) and (36) are forms suitable for re-expression as a summation with Stirling number coefficients.

By the same token, one may repeatedly integrate the Beta function, thereby obtaining binomial summation expressions for integrals of the form

$$I(x,y,p,m) = \int_0^1 \frac{(1-u^p)^x u^{y-1}}{\ln^m u} du, \tag{37}$$

for Re $x > 0$, Re $y > 0$, and m an integer. In this way, we obtain extensions of tabulated integrals such as given in Sections 4.267 and 4.268 of Ref. [5].

Final remarks. We could also develop series representations with Stirling number coefficients using divided difference formulas. For example, we have [1]

$$\left(\frac{d}{dx}\right)^m f(x) = m! \sum_{n=m}^{\infty} \frac{s(n,m)}{n!} \Delta^n f(x), \qquad (38a)$$

and

$$\Delta^m f(x) = m! \sum_{n=m}^{\infty} \frac{S(n,m)}{n!} f^{(n)}(x), \qquad (38b)$$

these formulas also exhibiting the inverse relations possible with the Stirling numbers. Especially for the Gamma and hence the Beta function, the finite differences and derivatives are relatively easily determined, due to their respective functional equations.

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