Candidates For Norlund-Rice Treatment

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Abstract

The following is a list of Newton series or other suggestive sums or integrals that may be amenable to a Norlund-Rice addle-point treatment.

0.1 Reciprocal Riemann Zeta

The Riemann zeta function has regularly-spaced zeros along the negative real axis. Thus, the reciprocol has poles at the (even) integers, and thus resembles the Norlund-Rice integrand. Viz:

$$\oint_C \frac{ds}{\zeta(s)} \sim \sum \binom{n}{k} \frac{1}{\zeta'(2n)}$$

where the contour encircles n poles. Not clear how to turn the integral into something suitable for a saddle-point method. Of some curisity is the Mobius-inversion/Dirichlet convolution identity

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where $\mu(n)$ is the Mobius function.

0.2 Maslanka/Baez-Duarte

A similar sum appears in [1] as

$$c_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{\zeta(2k+2)}$$

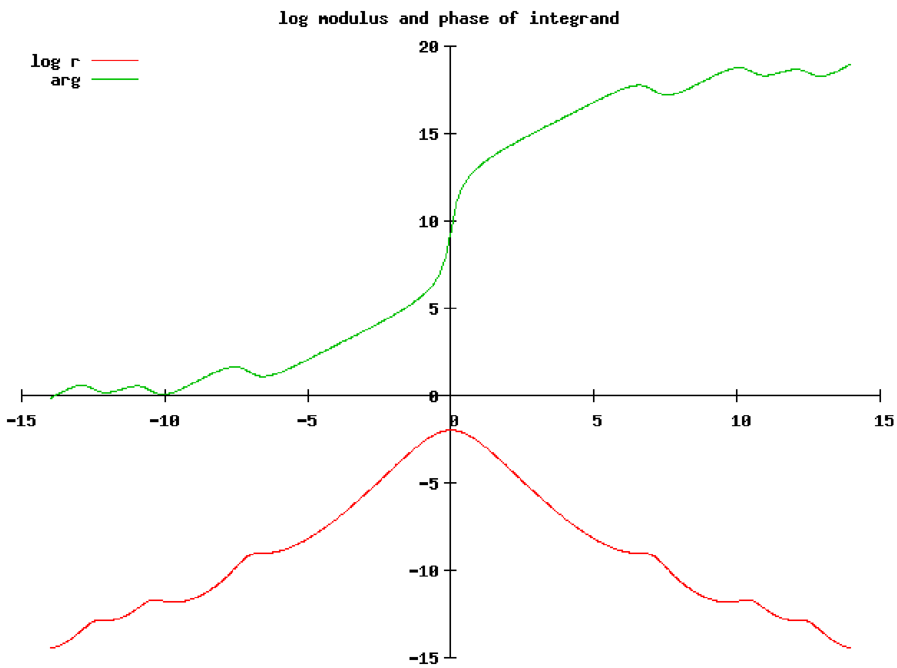
and furthermore, it is claimed that

$$c_n \ll n^{-3/4+\varepsilon} \forall \varepsilon > 0$$

is equivalent to RH. Note the summand is just a ratio of Bernoulli numbers and powers of π. The NR integral is

$$c_n = \frac{(-1)^n}{2\pi i} \int_{-1/4-i\infty}^{-1/4+i\infty} \frac{1}{\zeta(2s+2)} \frac{n!}{s(s-1)\cdots(s-n)} ds + \frac{\delta_{n0}}{2}$$

Figure 1: Log integrand



Graph of log and arg of integrand for $n = 6$. The bumps at 7,11,13 correspond to Riemann zeros at 14, 21, 26. For large n, the real part does not become more parabolic, but retains roughly the same shape. However, for large n, the phase runs more rapidly, pushing apart the saddle points.

The $n = 0$ Cauchy integral has an contribution of $1/2$ coming from the semi-circular contour at infinity on the right, which vanishes for $n \neq 0$. The integrand is suggests a saddle point, bounded by poles at $s = 0, -2$ while getting obviously small for large imaginary s. The problem is the the pole at $s = 0$ has a residue of oppiste sign from that at $s = -2$ and, so, if we are lucky, there is an inflection point between thse two locations, viz. a point where the first derivative vanishes. Its not clear that there's such a place for small n (due to numerical errors in my code). If its there, it seems to be at s=-0.99 for small n and around $s = -0.9$ for larger n,moving slowly to the right. Presumable moving to the right as $O\left(\frac{1}{\log n}\right)$ to some final deestination, if I did my quick sketch right. This doesn't seem terribly tractable.

I'm confused at this point, and am exploring the integral numerically. It appears that I can push the contour at $\sigma = -1/4$ further to the left, which amazingly passes me through the critical strip without changing the value of the integral (I guess that the contribution of all of residues of the poles at the zeros of ζ must add up to zero. I didn't know this, I presume it follows trivially by complex conjugation.).

So not only is the saddle not a saddle but a slide with maybe a flat spot in it, but there's no (simple) expansion at the flat spot. Not sure where to take this next.

0.3 Hasse-Knoppe

Helmut Hasse and Conrad Knoppe (1930) give a series for the Riemann zeta is convergent everywhere on the complex s-plane, (except at $s = 1$):

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s}$$

It would be curious to explore the associated integral. I'm particularly intrigued by the power-of-2 sum.

0.4 Hasse for Hurwitz

Hasse also gave a similar, globally convergent, expansion for the Hurwitz zeta:

$$\zeta(s,q) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (q+k)^{1-s}$$

Same question as above.

0.5 Dirichlet Beta

The Dirichlet beta is given by

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

and has a functional equation

$$\beta(s) = \left(\frac{\pi}{2}\right)^{s-1} \Gamma(1-s) \cos\frac{\pi s}{2}\, \beta(1-s)$$

This is just the L-function for the second character modulo 4, so we already have this sum, and can read it right off.

0.6 L-function of principal character modulo 2

The L-function of the principal character modulo 2 is given by

$$L(\chi_1, s) = \left(1 - 2^{-s}\right) \zeta(s)$$

and we already ahve expansions for this. The Newton series for this appears in [5] as equation 4.11 and also in [3] equation 16 as the sum

$$S_1(n) = \sum_{k=2}^{n} (-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k)$$

Coffey states the theorem that

$$S_1(n) \geq \frac{n}{2}\ln n + (\gamma - 1)\frac{n}{2} + \frac{1}{2}$$

We can read off the full result instantly from the result on the L-functions.

0.7 Another Coffey sum

Coffey [3] shows interest in another sum:

$$S_3(n) = \sum_{k=2}^{n} (-1)^k \binom{n}{k} 2^k \zeta(k)$$

The reson for the interest in this sum is unclear.

0.8 A Li Criterion-related sum

Bombbieri [2], Lagarias [5] and Coffey [3] provides a sequence $\eta_k$which seems to be of the form $\exp -k$ and thus suggests that the saddle-point techniques should be applicable. These appear in Lagarias equation 4.13 and in Coffey equation 10 as

$$\lambda_n = -\sum_{k=1}^{n} \binom{n}{k} \eta_{k-1} + S_1(n) + 1 - \frac{n}{2}(\gamma + \ln\pi + 2\ln 2)$$

where $S_1(n)$is given above, and λ_n are the Li coefficients

$$\lambda_n = \frac{1}{(n-1)!} \left. \frac{d^n}{ds^n} s^{n-1} \ln\xi(s) \right|_{s=1}$$

and

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Part of what is curious is that the η_k appear in an expression similar to the what is seen for the Stieltjes constants, but involve the von Mangoldt function.

0.9 Prodinger, Knuth

Prodinger considers a curious sum, and provides an answer; Knuth [4] had previously provided a related sum. Doesn't seem to be much to do here, as the leading terms are already given, and,m from the comp-sci point of view, these are enough. The motivation for providing the exponentially small terms is uncertain. The Prodinger sum is [6]:

$$S = \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k - 1} \simeq -\log_2 n + \frac{1}{2} + \delta_2(\log_2 n)$$

where B_k are the Bernoulli numbers. The Norlund-Rice integral is

$$S = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{(-1)^n n!}{((s-1)(s-2)\cdots(s-n))} \cdot \frac{\zeta(1-s)}{2^s-1}\, ds$$

The poles due to $2^z - 1$ lead to the curious term in the asymptotic expansion:

$$\delta_2(x) = \frac{1}{\log 2}\sum_{k\neq 0} \zeta\left(1-\chi_k\right)\Gamma\left(1-\chi_k\right) e^{2\pi i k x}$$

where

$$\chi_k = \frac{2\pi k i}{\log 2}$$

The Knuth sums are similar, but different.

0.10 Lagarias

Lagarias [5] has a a sum over the Hurwitz zeta, equation 5.5:

$$T(n,z) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{2^k} \zeta(k, z+1)$$

We've already computed this sum, so should be able to read off an answer directly, and improve significantly on Lagarias results.

References

[1] Luis Baez-Duarte. A new necessary and sufficient condition for the riemann hypothesis. *arXiv*, math.NT/0307215, 2003.

[2] E. Bombieri and J. C. Lagarias. Complements to li's criterion for the riemann hypothesis. *J. Number Theory*, 77:274–287, 1999.

[3] Mark W. Coffey. Toward verification of the riemann hypothesis: Application of the li criterion. *Mathematical Physics, Analysis and Geometry*, 00:00, 2005. in print.

[4] D. E. Knuth. *The Art of Computer Science*, volume 3. Addison-Wesley, Reading MA, 1973.

[5] Jeffrey C. Lagarias. Li coefficients for automorphic l-functions. *arXiv*, math.NT/0404394, 2005.

[6] Helmut Prodinger. How to select a loser. *Discrete Mathematics*, 120(1-3):149–159, September 1993.