The Transfer Operator

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Abstract

This chapter develops some of the tools for studying chaotic maps, specifically the idea of studying the Frobenius-Perron (FP) or Transfer operator of a chaotic map. The presntation here is rather simplified, and assumes little prior acquaintance with concepts in math beyond a typical college engineering math background; nor does it introduce or explain higher math concepts. The presentation is also very incomplete, leaving out a discussion of related important concepts. There may also be some misleading statements in here, that might casuse readers to draw incorrect conclusions; this needs a big re-write.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

1 Transfer Operators

THIS IS A DRAFT WORK IN PROGRESS. The intro hasn't been written yet.

This chapter presents the concept of the transfer operator, or Frobenius-Perron operator, in general terms.

The general layout is:

– present the Frobenius-Perron operator, which is our core tool

To do: describe the following:

- General concept of Hilbert space
- Riesz representation thm
- "Solvability" is a synonym for a triangular matrix. In particular, a triangular matrix corresponds to polynomial eigenvectors; this bridges over to the topic of the representation of analytic functions by means of polynomial basis. Reference Boas and Buck for details. The point is that the space of functions representable by any given "complete" set of polynomials has a structure that is perhaps un-intuitive: questions about convergence, analyticity, etc. cannot be taken for granted. Example: Carlson's theorem for Newton series shows how things can go wrong.
- Change notation used here to be more standard along the lines of polynomial bases of function spaces.

- Discuss Sheffer sequences, as these form a broad class of polynomial bases.
- Discusss resolvent formalism.
- Discuss how polynomials correspond to a discrete spectrum; how interpolated polynmoials correspond to a continouus spectrum. For example, the Bernoulli polynomials are the Hurwitz zeta at the integers.

2 Introduction: The Frobenius-Perron Operator for Iterated Maps

This section provides a basic review of the Frobenius-Perron operator and its use in the description of fractals and chaotic iterated maps. No results are presented here; rather the goal is to provide the notation and general concepts that will be used in later sections. This review assumes no prior encounter with these concepts, and keeps the development simple, avoiding the language of higher mathematics. More sophisticated developments build on concepts such as Borel Sigma Algebras and define the Frobenius-Perron operator on Banach Spaces. In the following, we avoid this sophisticated language in order to keep the presentation accessible. However, we do so at some peril: many of the quantities we'll work with are potentially ill-defined or divergent, and so the validity of some of the transformations and equations in such foggy surrounds can be questionable. A more rigorous treatment with appeals to higher math would help clarify where the rocky shoals are. As a substitute, we try to maintain a physicist's attitude, and keep our heads about us when faced with something dangerous. Be aware that not all extrapolations from the following may be warranted.

The Frobenius-Perron operator of a function, sometimes called the Transfer Operator of that function, provides a tool for studying the dynamics of the iteration of that function. If one only studies how a point value jumps around during iteration, one gets a very good sense of the point dynamics but no sense of how iteration acts on non-point sets. If the iterated function is applied on a continuous, possibly even smooth density, then one wants to know how that smooth density evolves over repeated iteration.

If we consider a smooth density $\rho(x)$ as a set of values on a collection of points, we can take each point and iterate it to find its new location, and then assign the old value to the new location. Of course, after iteration, several points may end up at the same location, at which point we need to add their values together. Lets write the new density as $\rho_1(x)$, with the subscript 1 denoting we've iterated once. We can express this idea of iterating the underlying points, and then assigning their old values to new locations as

$$\rho_1(x) = \int dy\, \delta\left(x - g(y)\right)\, \rho(y) \tag{1}$$

where $g(x)$ is the iterated function. To get $\rho_n(x)$, one simply repeats the procedure n times. In more abstract notation, one writes

$$\left[U_g \rho\right](x) = \rho_1(x) \tag{2}$$

to denote this time evolution. The notation here emphasizes that $U_g : f \mapsto U_g f$ is an operator that maps functions to functions: written formally, we have $U_g : \mathcal{F} \to \mathcal{F}$ where $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ is the set of all functions. In analyzing U_g, we will often be interested in how it acts on the subset of square-integrable functions, or possibly just C^∞ functions or polynomials or the like. Repeated iteration just gives the time-evolution of the the density; that is,

$$U_g^n \rho \equiv \underbrace{U_g \circ U_g \circ \ldots \circ U_g}_{n \text{ times}} \circ \rho = \rho_n \tag{3}$$

where iteration is just ordinary operator multiplication.

To understand U_g, one typically tries to understand its spectrum, that is, its eigenvalues and eigenfunctions. In most cases, one finds that U_g is contractive in that it has one eigenvalue equal to one and all the other eigenvalues are real and smaller than one. However, one must be terribly careful here, as there are land-mines strewn about: the actual spectrum, and the nature of the eigenvalues, depends very much on the function space chosen. Typically, when acting on polynomials, one gets discrete, real eigenvalues for U_g. When acting on square-integrable functions, one seems to usually get a continuous set of complex-valued eigenvalues. This is because one can often find shift-states ψ_n such that $U_g \psi_n = \psi_{n-1}$, in which case one can construct eigenfunctions $\phi(z) = \sum_n z^n \psi_n$ whose complex eigenvalues z form the unit disk. Sometimes, $\phi(z)$ can be meromorphically extended to a larger region, and sometimes it cannot. It is often considered to be a mistake to try to analyze U_g acting on a finite grid of discrete points, such as one might try on a computer: it is all to easy to turn this into an exercise of analyzing the permutation group on a set of k elements, of which any student knows that the eigenvalues are the k'th roots of unity.

Since U_g is a linear operator, it induces a homomorphism in its mapping, and so one should study its kernel $Ker\ U_g = \{f \mid U_g f = 0\}$ to gain insights into its symmetry as well as to express more correctly the quotient space. Insofar as the iterated map might represent a dynamical system, one knows that symmetries lead to conserved currents, via Noether's theorem, and sometimes to topologically-conserved (quantum) numbers, winding numbers or other invariants.

Finally, we note that since U_g looks like a time-evolution operator, we are tempted to write

$$U_g^t = \exp -tH_g \tag{4}$$

for some other operator H_g. Since U is in general not unitary, H is not (anti-)Hermitian. However, for many systems, the eigenvalues of U_g are real and less than or equal to one, and thus, one would expect that H_g would be positive-definite. If H_g is Hermitian, then one is lead to look for an associated Heisenberg Algebra, which would point to a dynamical system that can be understood through the map iteration.

Also, any group of symmetries on U should express themselves as an algebra on H and these might provide an alternate path for exploring and describing the fractal in question.

In practice, when one is given an iterated map $g(x)$, one computes the Frobenius-Perron operator as

$$[U_g \rho](x) = \sum_{x' : x = g(x')} \frac{\rho(x')}{|dg(x')/dx'|} \tag{5}$$

which provides an expression for U_g acting on a general function ρ.

2.1 Polynomial Representation

If one is interested in U acting on polynomial functions, then one immediately writes the Taylor (or Maclaurin) series

$$\rho(x) = \sum_{n=0}^{\infty} \frac{\rho^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n \tag{6}$$

and substitutes this in to get the matrix form of U:

$$[U\rho](x) = \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} x^m \sum_{n=0}^{\infty} U_{mn} a_n \tag{7}$$

Equating each power of x^m we get

$$\frac{1}{m!} \left. \frac{d^m [U\rho](x)}{dx^m} \right|_{x=0} = \sum_{n=0}^{\infty} U_{mn} \frac{1}{n!} \left. \frac{d^n \rho(x)}{dx^n} \right|_{x=0} \tag{8}$$

as the matrix equation for the transformation of polynomials, expressed in classical notation.

There are a variety of different notations that one can use when working with matrix operators, all of which are, at a certain level, completely equivalent. However, certain notations are handier than others depending on what representation one is working with, and what point one is trying to emphasize. Note in particular that the Dirac bra-ket notation is both very useful, and is also sometimes a source for confusion, especially when mixed with other notations. Thus, in the following, we take some pains to clarify this notation, giving a prolonged remedial presentation.

The operator, written in the polynomial representation, in space coordinates, is:

$$\begin{aligned} \delta(x - g(y)) = U_g(x,y) &= \langle x | U_g | y \rangle \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle x|m\rangle \langle m | U_g | n \rangle \langle n|y\rangle \end{aligned} \tag{9}$$

where $U_{mn} = \langle m|U|n\rangle$ and $\langle x|m\rangle = x^m$ and $\langle n|y\rangle = (-)^n \delta^{(n)}(y)/n!$, the latter being the n'th derivative of the Dirac delta function. In this basis, U is not diagonal, and the kets $|n\rangle$ are not eigenvectors, and the vector element $\langle x|m\rangle$ is neither the complex conjugate nor the transpose of $\langle n|y\rangle$. These are rather monomials and their inverses, and obey traditional orthogonality and completeness relationships. The inner products demonstrate orthogonality:

$$\begin{aligned} \langle n|m\rangle &= \int dx\, \langle n|x\rangle\, \langle x|m\rangle \\ &= \int dx\, (-)^n \frac{\delta^{(n)}(x)}{n!} x^m \\ &= \delta_{nm} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \langle x|y\rangle &= \sum_{n=0}^{\infty} \langle x|n\rangle\, \langle n|y\rangle \\ &= \sum_{n=0}^{\infty} (-x)^n \frac{\delta^{(n)}(y)}{n!} \\ &= \delta(y-x) \end{aligned} \tag{11}$$

are the orthogonality relationships in polynomial space and coordinate space, respectively. The completeness relationships define the identity operator

$$\mathbb{I} = \sum_{n=0}^{\infty} |n\rangle\, \langle n| = \int dx\, |x\rangle\, \langle x| \tag{12}$$

whose matrix elements in coordinate space are $\langle x|\,\mathbb{I}\,|y\rangle = \delta(y-x)$ and, in polynomial space, $\langle m|\,\mathbb{I}\,|n\rangle = \delta_{mn}$. In this notation, a function is represented by it's Taylor series:

$$\begin{aligned} f(x) &= \langle x|f\rangle \\ &= \sum_{n=0}^{\infty} \langle x|n\rangle\, \langle n|f\rangle \\ &= \sum_{n=0}^{\infty} x^n\, \langle n|f\rangle \\ &= \sum_{n=0}^{\infty} x^n \int dy\, \langle n|y\rangle\, \langle y|f\rangle \\ &= \sum_{n=0}^{\infty} x^n \int dy (-)^n \frac{\delta^{(n)}(y)}{n!} f(y) \\ &= \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} \end{aligned} \tag{13}$$

Lets complete the review by taking the coordinate-space representation of the Frobenius-Perron operator back to its matrix representation. Integrating the coordinate-space operator representation over y, we regain the previous expressions for the operator in

Hilbert space:

$$\begin{aligned}
\left[U_g\rho\right](x) &= \int dy\, U_g(x,y)\rho(y) \\
&= \int dy\, \delta\left(g(x)-y\right)\rho(y) \\
&= \sum_{m,n=0}^{\infty} x^m\, U_{mn} \int dy\, (-)^n \frac{\delta^{(n)}(y)}{n!}\rho(y) \\
&= \sum_{m,n=0}^{\infty} x^m\, U_{mn}\, \frac{1}{n!}\, \left.\frac{d^n\rho(y)}{dy^n}\right|_{y=0} \\
&= \sum_{m,n=0}^{\infty} x^m\, U_{mn}\, \frac{\rho^{(n)}(0)}{n!} \qquad (14)
\end{aligned}$$

Note that when one goes to diagonalize the operator, one will find "right eigenvectors" that will consist solely of a linear combination of $\langle x|m\rangle = x^m$, that is, will be polynomials. The "left eigenstates" will, by definition, be a linear combination solely of $\langle n|y\rangle = (-)^n\delta^{(n)}(y)/n!$ since, in a polynomial Hilbert space, these are the basis functions that are dual to polynomials.

It is critical to understand that the above notation and conventions are applicable only to the polynomial representation, and by construction, yields discrete spectra and polynomial (analytic, C^∞) eigenfunctions. This representation is more-or-less incapable of doing otherwise. The above expressions, although constructed using an equals sign, in fact do a great deal of violence and are in a certain way violently incorrect, because they hide or incorrectly equate the function spaces on which the operator U_g acts. That is, whenever $\rho(x)$ is not differentiable or is otherwise singular, the expansion in derivatives is not justified. As we will see shortly, when considered as acting in the space of square-integrable functions, U_g can and will have fractal eigenfunctions, which will typically be non-differentiable and even possibly continuous-nowhere, and thus not representable by polynomials. This is, of course, the whole point of this exercise!

If one is very lucky, one finds that U_{mn} is upper-triangular, in which case it can be solved immediately for its eigenfunctions, and its eigenvalues already lie on the diagonal. We will find that we get lucky in this way for the Bernoulli operator, and for the "singular sawtooth" operator, but not for the Gauss-Kuzmin-Wirsing operator. Of course, it is known that a complete solution of the GKW should lead directly to a proof of the Riemann Hypothesis, so getting lucky would be truly lucky indeed. XXXX edit the above sentences.

2.2 Fourier Representation

We repeat the above analysis using standard Fourier Series techniques. Although such an analysis may be considered to be old and shop-worn, it is critical to note that in this context, the Fourier representation is not only inequivalent to the polynomial representation, but that attempting to establish an equivalence leads to divergences reminiscent of those seen in more complicated Hilbert spaces, such as those encountered in Quantum Field Theory and elsewhere. In less flowery terms, we provide a simple example where undergraduate “textbook math” leads one to form incorrect conclusions about Hilbert Spaces and the behavior of operators in them. What look like simple statements about orthogonality and completeness of a set of basis functions can lead to serious trouble when analyzing even simple operators, as we shall show. The goal here is to get this “dirty laundry” out in the open, as it affects the development of later sections.

Lets quickly review the standard textbook treatment of a Fourier Series. In traditional notation, for some (periodic) function $f(x)$ one writes the Fourier Series as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \cos 2\pi nx \, + b_n \sin 2\pi nx \tag{15}$$

where the conjugates of f are given by

$$a_n = \int_0^1 f(x)\, \cos(2\pi nx)\, dx \tag{16}$$

and

$$b_n = \int_0^1 f(x)\, \sin(2\pi nx)\, dx \tag{17}$$

Moving over to bra-ket notation, we can define the Fourier-space basis vectors $|em\rangle$ in terms of their components in coordinate space. These components are $\langle x|em\rangle = \exp(i2\pi mx)$. The conjugate vectors $\langle en|$ have an equally simple representation: $\langle en|x\rangle = \exp(-i2\pi nx)$. One has the usual sense of orthogonality over coordinate space in that

$$\langle em|en\rangle = \int_0^1 dx\, \langle em|x\rangle \langle x|en\rangle = \int_0^1 dx\, \exp(2\pi i(n-m)x) = \delta_{nm} \tag{18}$$

and the traditional presentation of the Fourier Series is a statement of completeness over coordinate space, in that for an arbitrary square-integrable coordinate-space function $f(x) = \langle x|f\rangle$ one has

$$\begin{aligned} f(x) = \langle x|f\rangle &= \sum_{n=-\infty}^{\infty} \langle x|en\rangle \langle en|f\rangle \\ &= \sum_{n=-\infty}^{\infty} \exp(i2\pi nx) \int_0^1 dy\, \langle en|y\rangle \langle y|f\rangle \\ &= \sum_{n=-\infty}^{\infty} \exp(i2\pi nx) \int_0^1 dy\, \exp(-i2\pi ny)\, f(y) \\ &= \int_0^1 dy\, \delta(x-y)\, f(y) \end{aligned} \tag{19}$$

Thus, one is accustomed to the notion of having an identity operator of the form $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$ because it has the matrix elements that one expects in both the Fourier space and in coordinate space: that is, $\langle em|\, 1_F\, |en\rangle = \delta_{nm}$ and $\langle x|\, 1_F\, |y\rangle = \delta(x-y)$.

Thus, in light of this perfectly ordinary standard textbook behavior, the following shall be surprising. The matrix elements of this operator, expressed in the polynomial basis, are not only non-trivial, but are divergent. That is, one can be lulled into believing that $\langle m| 1_F |n\rangle = \delta_{nm}$ for the polynomial basis, and indeed, by performing the operations in a certain order, one can certainly show this. However, reversing the order of operations shows that what might seem like simple operations can in fact be quite treacherous.

We begin by writing the components of the vector $|em\rangle$ in the polynomial-space representation:

$$\begin{aligned}
\langle n|em\rangle &= \int_0^1 dx \, \langle n|x\rangle \, \langle x|em\rangle \\
&= \int_0^1 dx \, \frac{(-)^n}{n!} \delta^{(n)}(x) \, e^{i2\pi mx} \\
&= \int_0^1 dx \, \frac{\delta(x)}{n!} \, \frac{d^n}{dx^n} \, e^{i2\pi mx} \\
&= \frac{(i2\pi m)^n}{n!}
\end{aligned} \tag{20}$$

Essentially, this is nothing more than a plain-old Taylor's Series expansion of the exponential function. The conjugate vectors have a slightly trickier form. They are the Fourier components of monomials. For $m \neq 0$

$$\begin{aligned}
\langle em|n\rangle &= \int_0^1 dy \, \langle em|y\rangle \, \langle y|n\rangle \\
&= \int_0^1 \exp(-2\pi imy) \, y^n \, dy \\
&= \frac{-1}{2\pi im} + \frac{n}{2\pi im} \int_0^1 \exp(-2\pi imy) \, y^{n-1} \, dy \\
&= -\frac{1}{2\pi im} \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} \left(\frac{1}{2\pi im} \right)^k
\end{aligned} \tag{21}$$

and, for $m = 0$, $\langle e0|n\rangle = 1/(n+1)$. Let us now try to explicitly evaluate the matrix elements of the Fourier identity operator in the polynomial representation. That is, we attempt to write the matrix elements of $1_F = \sum_{m=-\infty}^{\infty} |em\rangle \langle em|$

$$\begin{aligned}
\langle p| 1_F |n\rangle &= \sum_{m=-\infty}^{\infty} \langle p|em\rangle \, \langle em|n\rangle \\
&= \sum_{m=-\infty}^{\infty} \left[\delta_{p0} + (1-\delta_{p0}) \frac{(2\pi im)^p}{p!} \right] \left[\frac{\delta_{m0}}{n+1} - \frac{(1-\delta_{m0})}{2\pi im} \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} \left(\frac{1}{2\pi im} \right)^k \right]
\end{aligned} \tag{22}$$

We need only to look at the relatively simple matrix element $n = 1$, $p \neq 0$ to see the misery of this expression:

$$\langle p \neq 0| 1_F |n=1\rangle = \frac{(2\pi i)^p}{p!} \sum_{m=1}^{\infty} \frac{m^p}{2\pi im} \tag{23}$$

One can try to rescue the situation by making the Ansatz that the summation should have been replaced by $\zeta(1-p)$ which is regular, but already this is dangerous. What is perhaps the more surprising is that one might have expected this kind of trouble from the polynomial completeness relationship $\mathbb{I}_A = \sum_{n=0}^{\infty} |n\rangle \langle n|$ because it ranges only over analytic functions: its essentially a statement of the idea that analytic functions are expressible through a series expansion in a variable. Functions that are not infinitely differentiable more-or-less lie in the kernel of $\mathbb{I}_A$. However, we'd expect 1_F to be more faithful, as it would seem to venture over square-integrable functions. Thus, such a simple failing is surprising.

The goal here is to simply present a signpost warning, as we make heavy use of these techniques in the sections that follow, where we work with functions that are differentiable-nowhere or worse.

2.3 The Koopman Operator

The Koopman operator is in a certain sense conjugate to the Frobenius-Perron operator, and defines how observables evolve. Given a density $\rho(x)$ we say that the observation of a function $f(x)$ by ρ is

$$\langle f \rangle_\rho = \int_0^1 f(x)\rho(x)\,dx \tag{24}$$

The term "observable" comes from usage in Quantum Mechanics, where $f(x)$ is associated with the eigenvalues of an operator. We do not need to appeal to these operator equations for the following development. The Koopman operator K gives the change in f when U acts on ρ, thus:

$$K_g : \langle f \rangle_\rho \to \langle K_g f \rangle_\rho = \int_0^1 [K_g f](x)\rho(x)\,dx = \int_0^1 f(x)[U_g\rho](x)\,dx \tag{25}$$

In Dirac bra-ket notation, we have

$$\begin{aligned} \int_0^1 f(x)[U_g\rho](x)\,dx &= \int_0^1 \langle x|U_g|\rho\rangle \langle x|f\rangle\, dx \\ &= \int_0^1 dx \int_0^1 dy\, \langle x|U_g|y\rangle \langle y|\rho\rangle \langle x|f\rangle\, dx \end{aligned} \tag{26}$$

and so we have

$$[K_g f]\,(y) = \int_0^1 \langle x|U_g|y\rangle \langle x|f\rangle\, dx = \int_0^1 U_g(x,y) f(x)\,dx = \int_0^1 \delta\,(x - g(y))\, f(x)\,dx \tag{27}$$

This gives the action of the Koopman operator in a coordinate-space representation. As is the recurring theme, different representations can lead to different results. In the coordinate-space representation, the Koopman operator appears to be the transpose of the Frobenius-Perron operator, in that $K(x,y) = U(y,x)$. However, in a general representation, whether the Koopman operator is the transpose or the complex conjugate or something else needs to be determined on a case-by-case basis, with an appeal to the particular operator $g(x)$ and the representations on which it works.

2.4 Topologically Conjugate Maps

Conjugation of the function that generates the map will provide, in general, another map that behaves exactly the same as the first, as long as the conjugating function is a 1-1 and onto diffeomorphism. That is, if ϕ is invertible, so that

$$\gamma = \phi \circ g \circ \phi^{-1} \tag{28}$$

then γ will iterate the same way that g does: $\gamma^n = \phi \circ g^n \circ \phi^{-1}$. The orbit of any point x under the map g is completely isomorphic to the orbit of a point $y = \phi(x)$ under the map γ. Because the (chaotic) point dynamics of these two maps are isomorphic, we expect just about any related construction and analysis to show evidence of this isomorphism.

In particular, we expect that the Koopman and Frobenius-Perron operators for γ are conjugate to those for g:

$$U_\gamma = U_\phi^{-1} U_g U_\phi \tag{29}$$

XXX ToDo derive the above. Show that eigenvalues are preserved. The most trivial way to see that the eigenvalues are unchanged is through the formal definition of the characteristic polynomial for this operator, which is

$$p_U(\lambda) = \det\left[U_g - \lambda \mathbb{I}\right] \tag{30}$$

Just as in the finite-dimensional case, a similarity transform commutes inside the determinant, leaving the characteristic polynomial unchanged. XXX ToDo a more correct, non-formal proof that the eigenvalues are preserved.

Note that in the construction of this proof, we invoke the Jacobian $\left|d\phi(y)/dy\right|_{y=\phi^{-1}(x)}$ and thus, in order to preserve the polynomial-rep eigenvalues, the conjugating function must be a diffeomorphism; a homeomorphism does not suffice. We will show an example below of a conjugating function that is highly singular, and thus the Jacobian does not exist (in the ordinary sense). When the conjugating function is sufficiently singular, then U_ϕ cannot be coherently defined. As a result, one can have conjugate maps with completely isomorphic point dynamics, but the eigenvalue spectra associated with these maps will *not* be identical.

2.5 The Topological Zeta

Another interesting quantity is the topological zeta function associated with the transfer operator. It is formally defined by

$$\zeta_{U_g}(t) = \frac{1}{\det\left[\mathbb{I} - tU_g\right]} \tag{31}$$

and embeds number-theoretic information about the map. Using standard formal manipulations on operators, one can re-write the above as the operator equation

$$\zeta_{U_g}(t) = \exp \sum_{k=1}^{\infty} \frac{t^k}{k} \operatorname{Tr} U_g^k \tag{32}$$

Of associated interest is the Maclaurin Series

$$t\frac{d}{dt}\log\zeta_{U_g}(t)=\sum_{k=1}^{\infty}n_k t^k \tag{33}$$

where we can read off $n_k = \mathrm{Tr}U_g^k$. From graph theory and the theory of dynamical systems, it is known that the n_k correspond to the number of periodic orbits of length k. In the context of dynamical systems, this zeta is often referred to as the Artin-Mazur Zeta function. In the context of graph theory, it is referred to as the Ihara Zeta. Both are connected to the Selberg Zeta.

The standard definition of the Ihara Zeta applies only to the adjacency matrix of finite-sized graphs. Adjacency matrices only have (non-negative) integer entries as matrix elements. Thus, we ask: given an appropriate basis, can an inifinite-dimensional transfer operator be written so as to have integer entries as matrix elements?

The standard definition of the Artin-Mazur Zeta function requires that the number of fixed points (periodic orbits) be a finite number. For the operators that we are studying, there will in general be (countably) infinite number of periodic orbits. Yet the zeta will still be well defined, although the coefficients of the Maclaurin expansion will not be integers. Can these be reinterpreted as a density or measure?

3 Conclusions

Apologies for the format of this paper. Its utterly incomplete, Sorry.

References

[asdf] Here is a very similarly titled paper with a very different subject matter: Continued Fractions and Chaos `http://www.cecm.sfu.ca/organics/papers/corless/confrac/html/confrac.html` by Robert M. Corless

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