Evidence in favor of the Baez-Duarte criterion for the Riemann Hypothesis

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Abstract

We give formulae allowing calculation of numerical values of the numbers c_k appearing in the Baez-Duarte criterion for the Riemann Hypothesis for arbitrary large k. We present plots of c_k for $k \in (1, 10^9)$.

1. Introduction.

In 1997 K. Maślanka [1] proposed a new formula for the zeta Riemann function valid on the whole complex plane $\mathbb{C}$ except a point $s = 1$:

$$\zeta(s) = \frac{1}{1-s} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-\frac{s}{2})}{\Gamma(1-\frac{s}{2})} \frac{A_k}{k!} \tag{1}$$

where coefficients A_k are given by

$$A_k = \sum_{j=0}^{k} \binom{k}{j} (2j-1)\zeta(2j+2).$$

This formula was rigorously proved by L. Baez-Duarte in 2003 [2]. In the subsequent preprint [3] the same author proved the new criterion for the Riemann Hypothesis (RH), the journal version of it appeared two years later [4]. Baez-Duarte considered the sequence of numbers c_k defined by:

$$c_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}. \tag{2}$$

He proved that RH is equivalent to the following rate of decreasing to zero of the above sequence:

$$c_k = \mathcal{O}(k^{-\frac{3}{4}+\epsilon}) \qquad \text{for each} \;\; \epsilon > 0. \tag{3}$$

Furthermore, if ϵ can be put zero, i.e. if $c_k = \mathcal{O}(k^{-\frac{3}{4}})$, then the zeros of $\zeta(s)$ are simply. Baez-Duarte also proved in [4] that it is not possible to replace $\frac{3}{4}$ by $\frac{3}{4}+\epsilon$.

There appeared papers [11], [12], [13] where generalizations of the sequence c_k were considered, in particular to the Hurwitz zeta and other functions. In this latter paper also generalizations of the Maślanka representation (1) are introduced and discussed.

Neither in [4] nor in [6] it is explicitly written whether the sequence c_k starts from $k = 0$ or $k = 1$. However in [4] a few formulas contain $k = 0$, i.e. summation starts from c_0. The point is that if we allow $k = 0$, for which $c_0 = 6/\pi^2$, then the inversion formula (see e.g. [7]) is fulfilled:

$$\frac{1}{\zeta(2k+2)} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} c_j. \tag{4}$$

However I do not see application of the above formula, except the possibility of checking some of the statements made in [11]. Furthermore, if the Baez-Duarte sequence c_k starts from $k = 0$ then the following identity holds:

$$\sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = e^x \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\zeta(2k+2)}. \tag{5}$$

It is an application of the general formal identity:

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} a_j \right) \frac{x^k}{k!} = e^x \sum_{k=0}^{\infty} \frac{a_k x^k}{k!}, \tag{6}$$

where a_k should not increase to fast with k to ensure convergence of series. [1] Putting here $a_j = (-1)^j b_j$ gives the usual formula appearing in the finite difference theory (see [15] §1):

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} b_j \right) \frac{x^k}{k!} = e^x \sum_{k=0}^{\infty} \frac{(-1)^k b_k x^k}{k!}. \tag{7}$$

The identity (5) can be used to establish the connection with the Riesz criterion for RH (original paper [9], discussed in [4], [10]). Riesz has considered the function:

$$R(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k!\zeta(2k+2)}.$$

[1] Indeed, collecting on the l.h.s. terms multiplying a_j we get: $a_j \sum_{k=j}^{\infty} \binom{k}{j} \frac{x^k}{k!} = a_j \sum_{k=j}^{\infty} \frac{k!}{j!(j-k)!} \frac{x^k}{k!} = a_j \frac{x^j}{j!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_j \frac{x^j}{j!} e^x$ and summing over j gives r.h.s.

Unconditionally it can be proved that $R(x) = \mathcal{O}(x^{1/2+\epsilon})$, see [8] §14.32. Riesz has proved that the Riemann Hypothesis is equivalent to slower increasing of the function $R(x)$:

$$RH \Leftrightarrow R(x) = \mathcal{O}\left(x^{1/4+\epsilon}\right). \tag{8}$$

But from (5) we get:

$$\sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = \frac{e^x}{x} R(x) \tag{9}$$

thus the generating function for c_k can be expressed by $R(x)$. The original Riesz criterion can now be reformulated as:

$$RH \Leftrightarrow \frac{1}{e^x} \sum_{k=0}^{\infty} \frac{c_k x^{k+1}}{k!} = \mathcal{O}\left(x^{1/4+\epsilon}\right), \tag{10}$$

or as

$$RH \Leftrightarrow \sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = \mathcal{O}\left(x^{-3/4+\epsilon} e^x\right). \tag{11}$$

Baez-Duarte has proved in [4] that unconditionally $c_k = \mathcal{O}(k^{-\frac{1}{2}})$ and there is a counterpart of it for $R(x)$ mentioned above thus one can suspect that there is a kind of duality:

$$c_k = \mathcal{O}\left(k^{-\alpha+\epsilon}\right) \Leftrightarrow R(x) = \mathcal{O}\left(x^{1-\alpha+\epsilon}\right). \tag{12}$$

As a further corroboration of this duality we give two examples: $c_k = 1 = k^0$ i.e. $\alpha = 0$, $(\epsilon = 0)$ and less trivial $c_k = \frac{1}{k}$ i.e. $\alpha = 1$ $(\epsilon = 0)$. In the first case we get for l.h.s of (9) simply e^x thus on the r.h.s. of this equality $R(x) = x$ and (12) is fulfilled[2]. In the second case for $\alpha = 1$ we get:

$$\sum_{k=1}^{\infty} \frac{x^k}{k k!} = -C - \log(x) + \mathrm{Ei}(x),$$

where the sum begins with $k = 1$ now, $C = 0.5772156649\ldots$ is the Euler constant and $\mathrm{Ei}(x)$ is the exponential integral:

$$\mathrm{Ei}(x) \equiv \int_{-\infty}^{x} \frac{e^t}{t} dt.$$

Because the identity holds

$$\mathrm{Ei}(\log(x)) = \mathrm{Li}(x) \equiv \int_0^x \frac{dt}{\log(t)},$$

[2] The function $R(x)$ has nothing to do with Riesz function now — it is merely a matter of notation: the series on the l.h.s. of (9) is written as a combination $e^x R(x)/x$, i.e. the dependence e^x/x is factored out.

where $\mathrm{Li}(x)$ is the logarithmic integral and it has the well known asymptotic expansion:

$$\mathrm{Li}(x) = x\left(\frac{1!}{\log(x)} + \frac{2!}{\log^2(x)} + \frac{3!}{\log^3(x)} + \dots\right)$$

we get for large x:

$$\sum_{k=1}^{\infty} \frac{x^k}{kk!} \sim \frac{e^x}{x}, \qquad x \to \infty$$

thus now $R(x) = 1$ and (12) is again fulfilled.

Is the established behavior $R(x) = \mathcal{O}(x^{1/2+\epsilon})$ sufficient to deduce from the above (9) the Baez-Duarte criterion (3) is under study now.

2. Computer experiments

The criterion (3) seemed to be very well suited for the computer verification. At the end of [3] Baez-Duarte wrote a sentence "A test for the first c_k up to $k =$ 1000 shows a very pleasant smooth curve". However for larger values of k the true behavior of the sequence turned out to be more complicated: instead of monotonic tending to zero there appeared oscillations and c_k changed the sign at first for[3] $k =$ 19320: $c_{19319} = -1.7870567 \times 10^{-13}$ while $c_{19320} = 9.170232808 \ldots \times 10^{-12}$. The next sign change is: $c_{22526} = 2.2292905301 \ldots \times 10^{-13}$ but $c_{22527} = -6.5057526 \times 10^{-12}$.

Table I

n	$\sum_{j=0}^{n} (-1)^j \binom{1200}{j} \frac{1}{\zeta(2j+2)}$
1000	$8.6575528427959311728 \times 10^{1492}$
2000	$1.0610772171540382076 \times 10^{2346}$
3000	$2.6820721693716011525 \times 10^{2928}$
4000	$8.4511383022435967124 \times 10^{3314}$
5000	$1.8751018390471552047 \times 10^{3537}$
6000	$8.3417729099514988532 \times 10^{3609}$
7000	$1.3393584564622537177 \times 10^{3537}$
8000	$4.2255691511217983562 \times 10^{3314}$
9000	$8.9402405645720038417 \times 10^{2927}$
10000	$2.1221544343080764152 \times 10^{2345}$
11000	$7.8705025843599374298 \times 10^{1491}$
12000	$-1.6973092190852083930 \times 10^{-7}$

To my knowledge the first plot of c_k for k up to 95000 appeared in the book [5] published in Polish. The same plot was reproduced in [4]. Data used to make this plot consisted of c_k calculated every 500-th k — it is very time consuming to get c_k

[3] in fact if the sequence c_k starts from $k = 0$ the first sign change occurs for $c_0 = 6/\pi^2 > 0$ and $c_1 = 6/\pi^2 - 90/\pi^4 = (6\pi^2 - 90)/\pi^4 \approx (-30/\pi^4) < 0$ (more precisely $c_1 = -0.3160113011$...)

directly from (2). Indeed, for large j the values of $\zeta(2j+2)$ very quickly become practically equal to 1, thus the summation of alternating series gives wrong result when not performed with sufficient number of digits accuracy. For example, the Table I below presents values of the partial sums for c_{12000} recorded every thousand summands (the calculation was performed with precision of 9000 digits).

Let us remark that the partial sums for n and $12000-n$ are of the same order. The binomial coefficients become very large numbers in the middle and to get accurate value of c_k one needs a lot of digits accuracy during the calculation. Maślanka has used `Mathematica` to perform these calculation. Over a year ago I started to calculate c_k using the free package PARI/GP [14] developed especially for number theoretical purposes and which allows practically arbitrary accuracy arithmetics both fixed-point as well as floating-point. I started to calculate consecutive c_k for each k with the help of the following script in Pari:

```
\p 3500    /*     precision set to 3500 digits    */
allocatemem(250000000)
range=10000     /*     the largest subscript in c_k         */
denomin=vector(range);
for (n=1, range, denomin[n]=zeta(2*n));
default(format, "e22.20")
{
for  (k=1, range, c=sum(j=0,k,((-1)^j)*binomial(k,j)/denomin[j+1]);
write("c_k.dat",k," ", c))
}
```

Table II

k	c_k
198000	$-8.18094200179687479 12\times10^{-9}$
198500	$-8.11303972500073791 08\times10^{-9}$
199000	$-8.04311631205752968 23\times10^{-9}$
199500	$3.41225839122053536 16\times10^{49}$
200000	$-1.92766083815985236 88\times10^{200}$

The problem I have encountered during these calculations was that it seems to be not possible to change accuracy of calculation during running the script (the command `\p 3500` above). Thus I had to change the precision by hand. It turned out that when the precision was to small produced values of c_k were obviously wrong, something like ten to the very large power. The rule learned from these examples for precision set to make calculations confident was that the number of digits should be at least enough to distinguish between 1 and $1+\frac{1}{2^k}$ in the zeta appearing in

(2), i.e. the precision set to calculate c_k should be at least $\texttt{\p} = k * \log_{10}(2)$. Table II presents the real example I have met during calculations: when the precision was set to 60000 digits the values of c_k for k between 198000 and 200000 were $(198000 \times \log_{10}(2) = 59603.93914, \quad 200000 \times \log_{10}(2) = 60205.99913)$:

Maślanka kindly send me values of c_k from his calculations up to $k = 95000$ with k jumping in intervals of 500, i.e. $k = 500l$. Last autumn I have started to continue this efforts on the cluster of 8 processors Xeon 2.8 GHz, with 4 GB RAM per node of two processors[4], with the aim to reach $k = 200000$ also every 500-th value of k using PARI/GP computer algebra system [14]. During last five months of computations between 4 and 6 processors I have used to calculate c_k in different intervals of k. When these calculations were running I have learned of the paper [4] where explicit formulae for c_k in terms of zeros of $\zeta(s)$ were given. Quite recently there appeared the paper [6] where the prescription to obtain c_k very quickly were also given. In view of these developments there is no need to continue very time consuming calculations based on the formula (2). The only benefit of these calculation was the possibility to compare c_k obtained by means of formulae presented in [4] and [6] against those c_k obtained from the generic formula (2). It should be stressed that calculations based on (2) does not assume the validity of Riemann Hypothesis in contrast to formulae presented by Maślanka or below. Using these formulae c_k can be calculated very quickly for practically arbitrary k — it is very time consuming to calculate c_k without assuming RH.

3. Explicit formulae.

The formulae presented in [4] and in [6] expressing c_k directly in terms of the zeros of $\zeta(s)$ are essentially the same, they differ in the manner they were derived. Maślanka has used the binomial transforms discussed in [15] while Baez-Duarte is developing the whole machinery by himself. The formulae of these two authors can be written as a sum of two parts: quickly decreasing with k trend $\bar{c}_k$ and oscillations $\tilde{c}_k$:

$$c_k = \bar{c}_k + \tilde{c}_k$$

where:

$$\bar{c}_k = -\frac{1}{(2\pi)^2} \sum_{2}^{\infty} \frac{B(k+1,m)}{\Gamma(2m-1)} \frac{(-1)^m (2\pi)^{2m}}{\zeta(2m-1)} \tag{13}$$

and oscillating part:

$$\tilde{c}_k = \sum_{\rho} \frac{\Gamma(k+1)\Gamma(\frac{1+\rho}{2})}{\Gamma(k+1+\frac{1+\rho}{2})} \frac{1}{\zeta'(1-\rho)} = \sum_{\rho} B\left(k+1, \frac{1+\rho}{2}\right) \frac{1}{\zeta'(1-\rho)} \tag{14}$$

[4] because I have used 32-bits version of PARI/GP I was able to use 2^{31} bytes =2 GB of RAM per process

where the sum is over all (i.e. on the positive as well as negative imaginary axis) nontrivial zeros of the $\zeta(s)$, i.e. $\zeta(\rho) = 0$ and $\Im\rho \neq 0$ and

$$B(w, z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)}$$

is the Beta function. In fact Baez-Duarte is skipping the trend remarking only that it is of the order $o(1/k)$ (Remark 1.6 in [4]). Theoretically the formula for $\tilde{c}_k$ is valid in the limit of large k, but surprisingly the numbers produced from the above formulae (13) and (14) are practically the same as obtained from the generic formula (2) for all k, e.g. already for $k = 2$ we get $c_2 = -2.5699711 \times 10^{-1}$ from (2), while (13) and (14) give $c_2 = -2.56969863 \times 10^{-1}$ and accuracy increases with k. It suggests that the integrals J_k appearing in [4] in the proof of the Theorem 1.5 are decreasing to zero rather fast with k.

First let us consider trend. It can be calculated directly from (13):

$$\bar{c}_k = -\frac{1}{(2\pi)^2}\sum_{m=2}^{\infty}\frac{1}{(k+1)(k+2)\dots(k+m)m(m+1)\dots(2m-2)}\frac{(-1)^m(2\pi)^{2m}}{\zeta(2m-1)} \tag{15}$$

Table III

k	$\bar{c}_k$ from eq.(15)	$\bar{c}_k$ from eq.(16)	$\bar{c}_k$ from eq.(17)
1	$-2.60052406393 \times 10^{-1}$	$-4.0752814729 \times 10^{3}$	$-1.6421193331 \times 10^{1}$
10	$-6.9069591105 \times 10^{-2}$	$-2.0455052855 \times 10^{1}$	$-1.6421193331 \times 10^{-1}$
10^2	$-1.4804264464 \times 10^{-3}$	$-5.9943727867 \times 10^{-3}$	$-1.6421193331 \times 10^{-3}$
10^3	$-1.6248041420 \times 10^{-5}$	$-1.6824923096 \times 10^{-5}$	$-1.6421193331 \times 10^{-5}$
10^4	$-1.6403755367 \times 10^{-7}$	$-1.6418022398 \times 10^{-7}$	$-1.6421193331 \times 10^{-7}$
10^5	$-1.6419448299 \times 10^{-9}$	$-1.6420436474 \times 10^{-9}$	$-1.6421193331 \times 10^{-9}$
10^6	$-1.6421018816 \times 10^{-11}$	$-1.6421113244 \times 10^{-11}$	$-1.6421193331 \times 10^{-11}$
10^7	$-1.6421175880 \times 10^{-13}$	$-1.6421185279 \times 10^{-13}$	$-1.6421193331 \times 10^{-13}$
10^8	$-1.6421191586 \times 10^{-15}$	$-1.6421192526 \times 10^{-15}$	$-1.6421193331 \times 10^{-15}$
10^9	$-1.6421193157 \times 10^{-17}$	$-1.6421193251 \times 10^{-17}$	$-1.6421193331 \times 10^{-17}$

Using this formula I was able to produce every 500-th value of $\bar{c}_k$ for $k = 500, 1000, \dots 10^9$ performing calculations in Pari with 100 digits accuracy in about 4 hours. For large k I have used following asymptotic expansion of (15):

$$\bar{c}_k = -\frac{1}{(k+1)(k+2)}\left(\frac{(2\pi)^2}{2\zeta(3)} - \frac{(2\pi)^4}{12(k+3)\zeta(5)} + \frac{(2\pi)^6}{120(k+3)(k+4)\zeta(7)}\right). \tag{16}$$

It can be further simplified to:

$$\bar{c}_k = -\frac{1}{k^2}\frac{(2\pi)^2}{2\zeta(3)} \tag{17}$$

The comparison of these formulae is given in Table III.

Now we consider the oscillating part $\tilde{c}_k$. Since PARI/GP does not have built in $B(x,y)$ function, I had to use $\Gamma(z)$ functions instead. Because of the fast growth of the $\Gamma(x)$ function even in PARI/GP it was not possible to pursue with formula (14) for large k. Namely it crashes for $k = 356000$ because of overflow. But there is a following asymptotic formula (see e.g. [16], §1.8.7):

$$\frac{\Gamma(x)}{\Gamma(x+a)} \sim x^{-a}, \qquad x \to \infty$$

thus we have

$$B(a,x) \sim x^{-a}\Gamma(a) \qquad \text{for } \ x \ \text{ large.} \tag{18}$$

Using it for large k and assuming the Riemann Hypothesis: $\rho_l = \frac{1}{2} + i\gamma_l$, $\bar{\rho}_l = \frac{1}{2} - i\gamma_l (= 1 - \rho_l)$ after collecting together in pairs conjugate zeros we get:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}} \sum_{l=1}^{\infty} \alpha_l \cos\left(\frac{1}{2}\gamma_l \log(k+1)\right) - \beta_l \sin\left(\frac{1}{2}\gamma_l \log(k+1)\right), \tag{19}$$

where I have denoted:

$$\alpha_l = \Re\left(\frac{\Gamma(\frac{1+\rho_l}{2})}{\zeta'(\bar{\rho}_l)}\right), \tag{20}$$

$$\beta_l = \Im\left(\frac{\Gamma(\frac{1+\rho_l}{2})}{\zeta'(\bar{\rho}_l)}\right). \tag{21}$$

In (19) the decreasing of c_k like $k^{-\frac{3}{4}}$ is obtained as an overall amplitude of the "waves" composed of the cosines and sines with the "frequencies" proportional to imaginary parts of the nontrivial zeros of $\zeta(s)$. The coefficients α_l and β_l decrease to zero very fast with l. Namely using the Hadamard product for $\zeta(s)$:

$$\zeta(s) = \frac{(2\pi)^s e^{-(1+C/2)s}}{2(s-1)\Gamma(s/2+1)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where $C = 0.57721566490153286\ldots$ is the Euler constant, the derivative of $\zeta(s)$ at zeros can be computed. Taking into account miraculous simplifications, $\rho\rho_l + \bar{\rho}\rho_l = \rho_l$ and the identity

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

I have obtained that:

$$|\alpha_l| \propto e^{-\pi\gamma_l/4}, \qquad |\beta_l| \propto e^{-\pi\gamma_l/4}. \tag{22}$$

Because imaginary parts of zeros take large values it suffices to sum in (19) over a few first zeros. I have used 10 zeros and the table below gives coefficients α_l and β_l.

Table IV

l	α_l	β_l	$e^{-\pi\gamma_l/4}$
1	2.029173866 $\times 10^{-5}$	$-3.315924256 \times 10^{-5}$	1.50914×10^{-5}
2	-3.333265938 $\times 10^{-8}$	$-1.298336420 \times 10^{-7}$	6.75315×10^{-8}
3	2.886139424 $\times 10^{-9}$	$-4.153918097 \times 10^{-9}$	2.94404×10^{-9}
4	4.813880001 $\times 10^{-11}$	$-6.332017430 \times 10^{-11}$	4.19039×10^{-11}
5	7.546769513 $\times 10^{-12}$	$7.526891498 \times 10^{-12}$	5.83506×10^{-12}
6	6.162524600 $\times 10^{-14}$	$1.942118979 \times 10^{-13}$	1.51209×10^{-13}
7	-1.578482027 $\times 10^{-14}$	$1.184829593 \times 10^{-14}$	1.10374×10^{-14}
8	-1.07138189 2$\times 10^{-15}$	$-2.209146437 \times 10^{-15}$	1.66491×10^{-15}
9	9.328038737 $\times 10^{-19}$	$-7.472197226 \times 10^{-17}$	4.22403×10^{-17}
10	1.747829093 $\times 10^{-17}$	$1.122667624 \times 10^{-17}$	1.05303×10^{-17}

However already calculations with the first zero γ_1 give numbers which differ much less than 1% (see $l = 1$ and $l = 2$ in the above table) from those calculated with larger number of terms in (19) as well as with c_k for $k < 200000$ obtained directly from (2) without assuming RH. The plots of c_k for k up to 10^9 obtained from these formulae are given in the Fig.1 and Fig.2. In Fig.2 there is logarithmic k-axis and thus the plot has a constant "wavelength", not depending on k like on the Fig.1. The envelope is given by:

$$y = \pm\frac{A}{k^{3/4}}\ , \qquad A = 0.777506276445256 \times 10^{-5} \tag{23}$$

and was obtained in the following way: First I have maintained in (19) only the first zero $\rho_1 = \frac{1}{2} + \gamma_1$:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}}(\alpha_1 \cos\left(\frac{1}{2}\gamma_1 \log(k+1)\right) - \beta_1 \sin\left(\frac{1}{2}\gamma_1 \log(k+1)\right). \tag{24}$$

Next I made use of the identity:

$$a\cos(\theta) - b\sin(\theta) = B\sin(\phi - \theta) \quad \text{where} \quad B = \sqrt{a^2+b^2}, \quad \phi = \arctan\left(\frac{a}{b}\right) \tag{25}$$

to obtain:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}}\sqrt{\alpha_1^2+\beta_1^2}\sin\left(\phi - \frac{1}{2}\gamma_1 \log(k+1)\right). \tag{26}$$

from which (23) follows and numerical value of $A = 2\sqrt{\alpha_1^2+\beta_1^2}$ is obtained from α_1 and β_1 in Table IV. Let us remark that this value of A agrees very well with amplitude reported by Beltraminelli and Merlini [12].

4. Final remarks

In Fig.3 the plot of $k^{\frac{3}{4}}c_k$ is presented. The Baez-Duarte criterion requires this "wave" to be contained in the strip of parallel lines for all k. The violation of the RH would manifest as an increase of the amplitude of the combination $k^{\frac{3}{4}}c_k$ for large k. This point is elaborated in more detail by Maślanka in [6]. Here I will make some further comments on this issue. First it should be remarked that the r.h.s. of (19) consists of products of three terms: the first depending only on k (the overall factor $k^{\frac{3}{4}}$), the second depending only on imaginary parts of nontrivial zeros of ζ (the coefficients α_l and β_l) and third ingredient depending both on k and l (the trigonometric functions). Assume there are some zeros of ζ off critical line. We can split the sum over zeros ρ in (14) in two parts: one over zeros on critical line and second over zeros off critical line. This second sum should violate the overall term $k^{-3/4}$ present in the first sum. Let $\gamma_i^{(o)}$ denote the imaginary parts of the zeros lying off critical line ("o" stands for "off"). It is not clear whether asymptotic similar to (22) will be valid for zeros off critical line, but it seems to be reasonable to assume that it should not differ significantly from (22). Then the contribution to c_k of such zeros off critical line should contain a factor of the order $e^{-\gamma_l^{(o)}}$. Because value of the imaginary part $\gamma_l^{(o)}$ of the hypothetical zero off critical line should be extremely large, perhaps even as large as 10^{100} can be expected, the combined contribution to c_k coming from the second sum seems to be extremely small, thus to see violation of the Baez-Duarte criterion the values of k should be larger than famous Skewes number and look something like $10^{10^{\cdot^{\cdot^{\cdot}}}}$. Such a big index k should cause the first term in (19) to overcame the smallness of the second term depending only on $\gamma_l^{(o)}$.

The plot in Fig.3 is a perfect sine of one wavelength thus it gives visual justification of the above statement that $\tilde{c}$ is determined in fact by the first zero γ_1. The same phenomenon was mentioned by Maślanka in [6].

The formula (19) together with a few coefficients α_l and β_l taken from Table IV allows to compute values of c_k for arbitrary large k. Other criteria for RH, like the value of the de Bruijn-Newman constant [17], are vulnerable to the Lehmer pairs of zeros of $\zeta(s)$. It is hard to see the reason for violation of the inequality $|c_k| <$ const $k^{-\frac{3}{4}}$ due to the Lehmer phenomenon. I have checked, that at the first Lehmer pair $\rho_{6709} = 0.5 + 7005.06286617i$ and $\rho_{6710} = 0.5 + 7005.1005646i$ the derivative has value $\zeta'(0.5 + 7005.06286617i) = 3.2229849698 + 0.74179951875i$ and similar value for second zero, thus there is no chance to get values of c_k violating (3) in this way. It seems to be an open problem how to connect the value of the largest k for which $|c_k| < \text{const}k^{-\frac{3}{4}}$ to the number of zeros lying on the critical line. Let us mention that for the Li's criterion [18] which states that if some numbers $\lambda_n > 0$ then RH is true it is known that if the first n Li's constants λ_n are positive then every zero ρ of $\zeta(s)$ with $|\Im\rho| < \sqrt{n}$ lies on the critical line $\Re\rho = \frac{1}{2}$ [19].

After a few months of computer experiments with c_k I believe Baez-Duarte sequence is one the most important and mysterious sequences in the whole mathematics.

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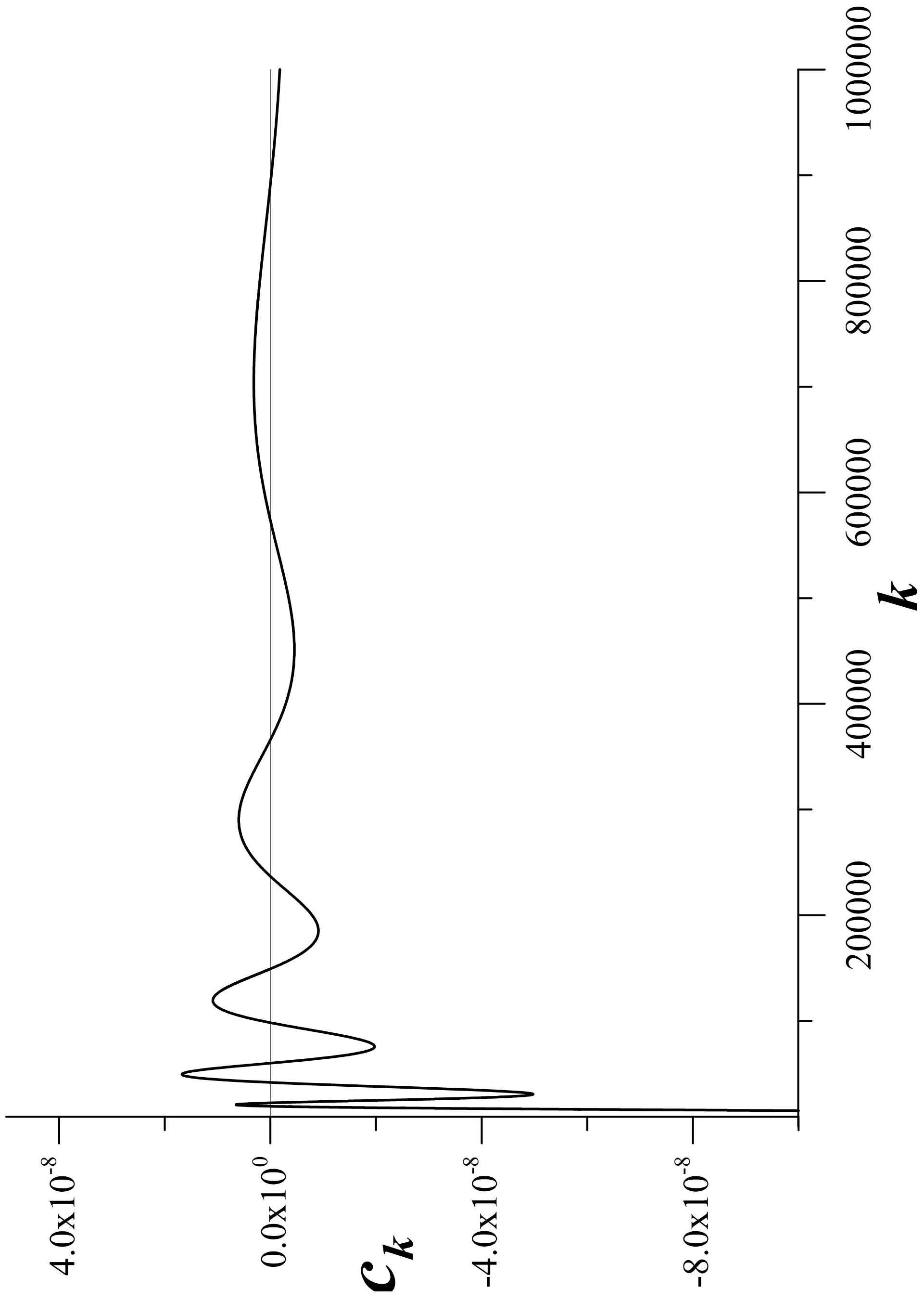

Fig. 1: The plot of c_k for $k \in (1, 10^6)$.

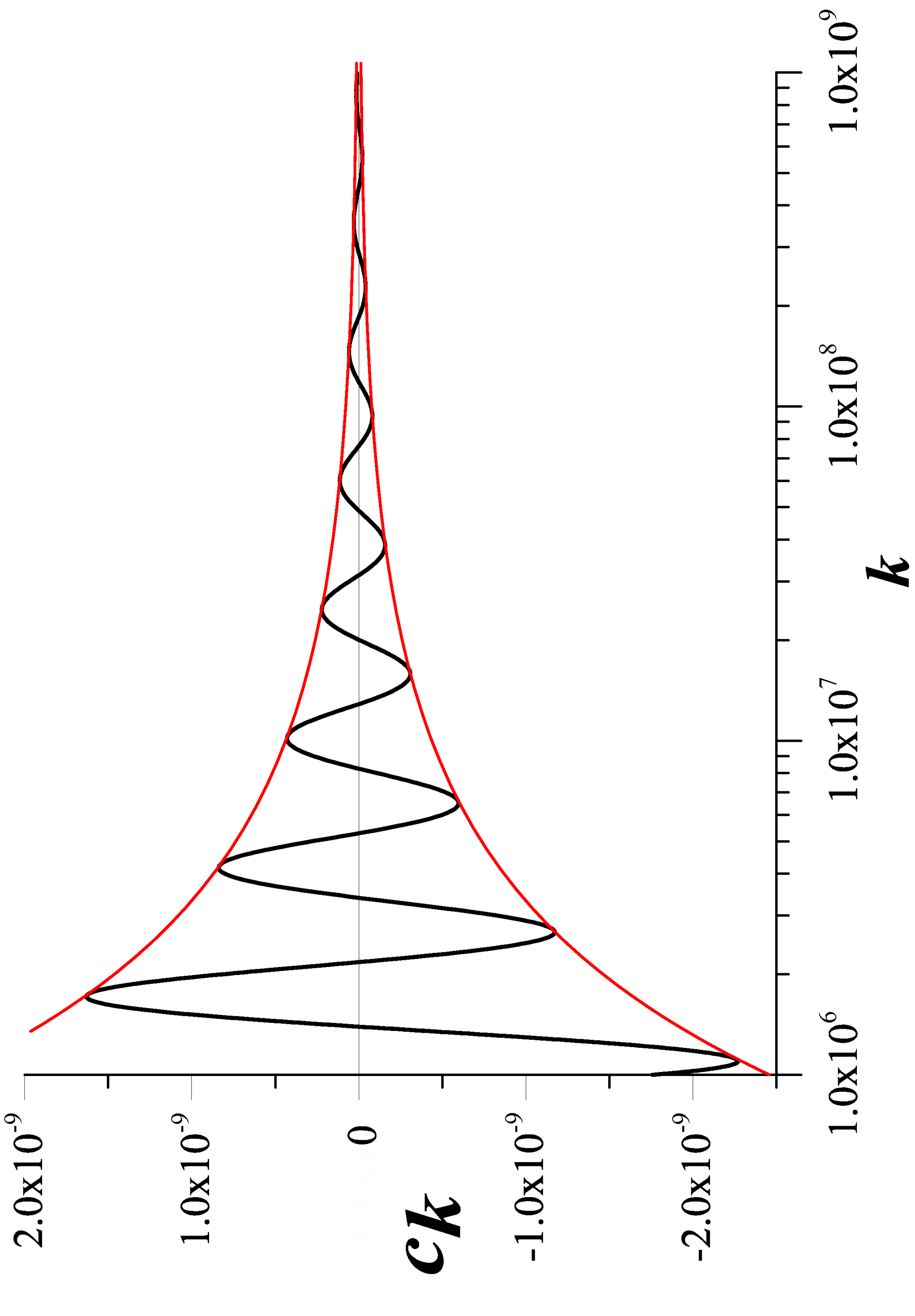

Fig. 2: The plot of c_k for $k \in (10^6, 10^9)$.

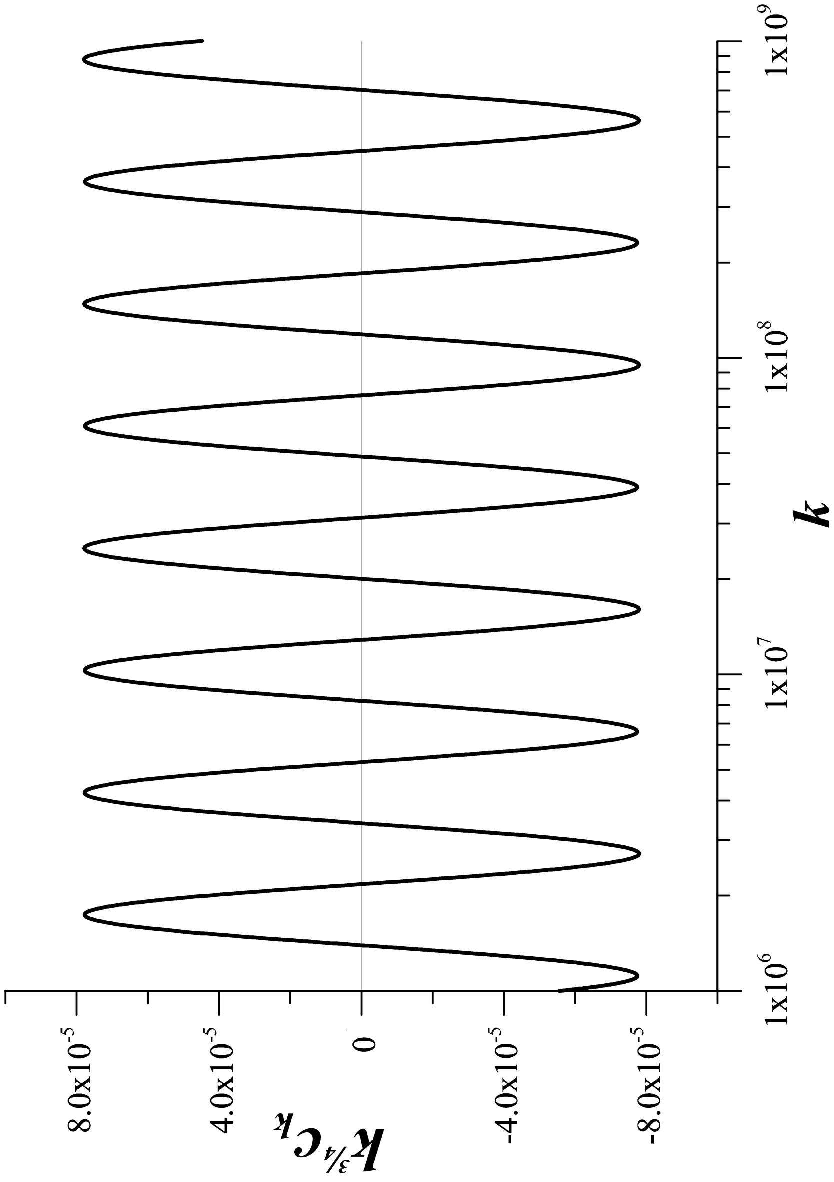

Fig. 3: The plot of $k^{3/4}c_k$ for $k \in (10^6, 10^9)$. The amplitude is $A = 0.777506276445256 \times 10^{-5}$