NOTE ON A ZETA OPERATOR EQUATION

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Here is something that perhaps pleases the senses.
From the series formula:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}(s)_k \zeta(s+k) = -1$$

We convert the series to a form involving ρ operators, from difference equation theory, where E is the shift operator.

$$\rho f(s) = sEf(s) = sf(s+1)$$

We now write the above series in operator form and have:

$$\zeta(s) = 1 + e^{-\rho}\zeta(s)$$

Thus $\zeta(s)$ satisfies an infinite differential equation, from the relation $E = e^D$, where D is the differential operator.

By elementary manipulations, this operator equation reveals many basic characteristics of ζ.

First, observe the following:

$$\rho(1) = s$$

and

$$\rho^{-1}(1) = \frac{1}{s-1} \quad ,$$

also

$$\rho^k \frac{1}{\Gamma(s)} = \frac{1}{\Gamma(s)} E^k \quad ,$$

and also

$$\rho^k \Gamma(1-s) = (-1)^k \Gamma(1-s) E^k \quad .$$

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Thus we verify the above operator equation for $\zeta(s)$, by:

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{s-1}dt \\
&= 1+e^{-\rho}\frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-t}}{1-e^{-t}}t^{s-1}dt \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \{e^{-t}+\frac{e^{-2t}}{1-e^{-t}}\}t^{s-1}dt \\
&= \zeta(s)
\end{aligned}$$

We also verify the operator equation for $\zeta(s)$ in terms of it's *Hankel* loop integral representation:

$$\begin{aligned}
\zeta(s) &= \frac{\Gamma(1-s)}{2\pi i}\int_\gamma \frac{e^{t}}{1-e^{t}}t^{s-1}dt \\
&= 1+e^{-\rho}\frac{\Gamma(1-s)}{2\pi i}\int_\gamma \frac{e^{t}}{1-e^{t}}t^{s-1}dt \\
&= \frac{\Gamma(1-s)}{2\pi i}\int_\gamma \{e^{t}+\frac{e^{2t}}{1-e^{t}}\}t^{s-1}dt \\
&= \zeta(s)
\end{aligned}$$

Note, also that:

$$e^{-k\rho}(1) = (k+1)^{-s}$$

Thus by successive iterations of the operator equation we get:

$$\zeta(s) = \frac{1}{1-e^{-\rho}}(1) = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\dots$$

and also

$$\zeta(s) = \frac{1}{s-1}+\frac{1}{2}+\{\frac{1}{1-e^{-\rho}}-\frac{1}{\rho}-\frac{1}{2}\}(1)$$

Immediately one sees a connection to the *Bernoulli Numbers*.
One wonders if there is a contour intergral representation similar to the Norlund formula.

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