THE GAUSS-KUZMIN-WIRSING OPERATOR

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ABSTRACT. This paper presents a review of the Gauss-Kuzmin-Wirsing (GKW) operator. The GKW operator is the transfer operator of the Gauss map, and thus has connections to the theory of continued fractions – specifically, it is the shift operator for continued fractions. The operator appears to have a reasonably smooth, well-behaved structure, however, no closed-form analytic solutions are known, and these are not easy to obtain. Eigenvalues and eigenfunctions can be obtained numerically, but little else is known in the mathematical literature. While this paper does attempt to be a review, it is incomplete; it is more of a diary of research results.

Connections to the Minkowski Question Mark Function are probed. In particular, the Question Mark is used to define a transfer operator which is conjugate to the GKW. This conjugate operator is solvable, and can be shown to have fractal eigenfunctions. However, the spectrum of this operator is not at all the same as that of the GKW. This is because the Jacobian of the transformation relating the two is given by $(?'\circ ?^{-1})(x)$, which is well-known as the prototypical "multi-fractal measure". Nonetheless, conjugacy allows the eigenfunctions of the one to be used to construct eigenfunctions of the other; in this sense, a "solution" of the GKW operator is undertaken.

The presentation given here assumes little math background beyond basic linear algebra and analytic function theory. This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.x

1. THE GAUSS-KUZMIN-WIRSING OPERATOR

This text is a diary of ongoing research results. As such, it not always coherent, and is somewhat disorganized. It is only sporadically updated. It starts with a review of basic ideas, and moves on to a series of original results.

The overall layout is as follows:

- Present the Gauss-Kuzmin-Wirsing (GKW) operator, including basic facts, theorems, relationships, numerical studies. Explore an asymptotic expansion for the GKW operator. Introduce the Ruelle-Mayer (transfer) operator.
- Show that the Minkowski Question Mark converts the GKW into a sawtooth.
- Discuss how the Cantor set is a model for the unit interval, and is the appropriate setting for discussing the question mark, fractals, dyadic monoid self similarity, and also for solving the GKW.
- Solve the two sawtooth transfer operators (these are exactly solvable). Provide a discrete spectrum of polynomial solutions. For dyadic sawtooth, provide a complete set of fractal eigenfunctions, possessing a continuous spectrum.
- Show how to get GKW eigenfunctions from the dyadic sawtooth eigenfunctions; present a complete(?) set of fractal eigenfunctions for the GKW operator.
- Show how the continuous solutions arise as the kernel of an operator; discuss differentiability.
- Review the Farey Map

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- Appendixes providing details for various results.

2. The Gauss-Kuzmin-Wirsing Operator

The map that that acts as the shift operator for continued fractions is

$$h(x)=\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor$$

That is, if one writes out the continued-fraction expansion for $x\in[0,1]$:

$$x=\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\cdots}}}\equiv[a_1,a_2,a_3,\cdots] \tag{2.1}$$

then one has that

$$h(x)=[a_2,a_3,\cdots]$$

whence the name "shift operator". This map is often called the Gauss Map. Note that shift operators, as linear operators, are studied as a subtopic of Banach Space theory, and often appear in applied mathematics texts devoted to the engineering topics of control theory, stability theory and filter design[21]; they are studied in Operator Theory as a topic in pure mathematics[22]. However, in these texts, shift operators are typically applied to sequences of functions defined on Hardy spaces, or more generally on Hilbert spaces. It appears that the shift as applied to continued fractions is very nearly unstudied – and no wonder – it appears nearly intractable when approached with standard analytic tools. The usual tools and techniques seem unapplicable; thus, much of this paper is devoted to finding tools and techniques that are relevant.

The Ruelle-Frobenius-Perron or transfer operator associated with the Gauss map is known as the Gauss-Kuzmin-Wirsing (GKW)[17, 32] operator $\mathscr{L}_h$. It is the pushback of h, and as such, is a linear map between spaces of functions on the unit interval (topological vector spaces)[30]. That is, given the vector space of functions from the closed unit interval to the real numbers

$$\mathscr{F}=\{f\,|\,f:[0,1]\to\mathbb{R}\}$$

then $\mathscr{L}_h$ is a linear operator mapping $\mathscr{F}$ to $\mathscr{F}$. Given $f\in\mathscr{F}$, it is represented by

$$\left[\mathscr{L}_h f\right](x)=\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}f\left(\frac{1}{n+x}\right) \tag{2.2}$$

This operator is bounded; its largest eigenvalue is 1.

The GKW operator $\mathscr{L}_h$ is a special case of what is sometimes called the Ruelle-Mayer operator[13]

$$\left[G_s f\right](z)=\sum_{n=1}^{\infty}\frac{1}{(n+z)^s}f\left(\frac{1}{n+z}\right)$$

for general complex s and z. This operator, with a value of $s=4$, occurs in the study of the Gaussian reduction algorithm applied to modular lattices[13].

Neither the GKW nor the Mayer-Ruelle operators have been "solved", in the sense that there is no known closed-form analytic solution expressing its all of its eigenfunctions and eigenvectors. The GKW operator has one classically known eigenvector, $f(x)=1/(1+x)$, which corresponds to the unit eigenvalue; this solution was given by Gauss.

Kuzmin considers iterating this operator, and shows that given any continuous, differentiable function $g(x)$ with bounded derivative on the unit interval, that the iterate converges uniformly to $f(x) = C/(1+x)$. That is, by defining

$$g_{k+1}(x) = \left[\mathscr{L}_h g_k\right](x)$$

as the k'th iterate of $g(x)$, then $g_k(x) \to C/(1+x)$ uniformly, for all bounded, differentiable g. Thus, as a corollary, this eigenvector is unique[15, section 15]. An alternative way of understanding this result is via the Frobenius-Perron theorem, which asserts that the eigenfunction associated with the maximal eigenvalue is unique.

The operator is not normal (*i.e.* $\mathscr{L}_h \mathscr{L}_h^T$ is not equal to $\mathscr{L}_h^T \mathscr{L}_h$); this is typically the case for transfer operators. Thus, the left and right eigenvectors are distinct, although they share common eigenvalues. To use proper matrix algebra language, these should be called "singular values", although we will persist in calling them eigenvalues below; and likewise diagonalization should properly be called "singular value decomposition", and the left and right eigenvectors are properly called the left and right singular vectors. Alternately, if one considers the operator as acting on a Banach space, then right singular vectors form a basis for for the Banach space, and the left vectors are the dual.

When the domain of the Mayer-Ruelle operator is restricted to certain Banach spaces, then the operator is a nuclear operator – that is, it has a discrete spectrum, and its eigenvectors form a basis for Banach space. By considering the operator restricted to a Hardy space, Daudé *etal* show that the spectrum is real when s is real[13].

The first eigenvalue below 1.0 is approximately 0.3036, and is known as the GKW constant [6, 12, 13][xxx need Babenko ref for original discussion]. As is typically the case for transfer operators, when the right eigenvectors are smooth functions, then the left eigenvectors are linear combinations of derivatives of Dirac delta functions, located at 0 and 1. All of this is explored in greater detail, in the rest of this paper.

Aside from the analytic solutions, there is also large class of fractal, discontinuous-everywhere functions associated with eigenvalue 1. The prototypical such solution is the derivative of the Minkowski Question Mark function $?(x)$. That is,

$$\left[\mathscr{L}_h ?'\right](x) = ?'(x)$$

A proper construction for the everywhere-discontinuous function $?'$, and the derivation of the above identity, is given in [30], together with a construction of a class of other similar solutions.

2.1. **Relation to the Riemann Zeta Function.** The Gauss map is connected to the Riemann zeta function by a Mellin Transform:

$$\zeta(s) = \frac{1}{s-1} - s\int_0^1 h(x)x^{s-1}dx$$

The Riemann zeta can be written under a change of variable $y = h(x)$ as

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x\left[\mathscr{L}_h x^{s-1}\right] \tag{2.3}$$

and thus it seems possible that a better understanding of GKW may shed light on the Riemann Hypothesis and/or the Berry conjecture[xxx need ref].

Curiously, the above equation can be understood to be a kind of linear equation specifying the Riemann zeta, because the operator $\mathscr{L}_h$ is a constant, independent of the integration variable, and can be pulled out of the integral. For a suitable basis, the integral can become easy to evaluate, leaving a linear equation, albeit depending on a parameter s in a non-linear way. Sketching this out briefly, suppose one has a basis $\{e_n\}$ for the unit interval, so that one can write $f(x) = \sum_n f_n e_n(x)$. In this basis, the operator $\mathscr{L}_h$ will have matrix elements L_{jk} given by $[\mathscr{L}_h f](x) = \sum_{jk} L_{jk} f_k e_j(x)$. Supposing now that $f(x)$ is just x^{s-1}, and that the order of summation and integration can be exchanged, this leaves only the integral $\int x e_j(x)\, dx$ to evaluate, which can then be replaced in the sum: $\int dx\, x\left[\mathscr{L}_h x^{s-1}\right] = \sum_{jk} L_{jk} f_k \int x e_j(x)\, dx$. This last is then a linear equation, although the f_k depend on s in a non-linear way. The collection $\{f_k\}$ can be thought of as a vector in an infinite-dimensional space; this vector depends on a single parameter s. As the value of s is varied, there are special values of s where its product times the linear operator is equal to $1/(s-1)$: these special values of s are nothing but the zeroes of the Riemann zeta.

The section 2.6 provides explicit expressions for the matrix elements L_{jk} for the polynomial basis $e_n(x) = x^n$. In this case, the integral becomes trivial to evaluate, and one discovers a series expansion for the Riemann zeta in terms of the falling factorial (equivalently, binomial coefficients). This series has many interesting properties, and is explored in greater detail in [26, 11]. The series generalizes; it can be used to formulate a class of criteria on various number-theoretic series (such as those constructed from the Mobius function, or the totient function of the Liouville series) that are equivalent to the Riemann Hypothesis. This class of RH-equivalent hypothesis were already noted by Báez-Duarte[4, 5], Maslanka[18], and Flajolet and Vallée[10], and are explored in detail in [29].

Far more promising, however, are basis functions derived from or structured on the Cantor set. These are explored below, and in other related papers [xxx need ref].

2.2. **Lack of simple solutions.** Aside from the classical solution, $1/(1+x)$, there do not seem to be any "easy" polynomial series solutions to the operator, where a "solution" would be a closed-form specification of the eigenvectors.

The author performed a combinatorial search of simple combinations and summations of rational functions and various classical functions, such as the exponential, the gamma, the digamma, the dilogarithm, Bessel functions and the exponential integral. No eigenvectors were found in this way, although, of course, various close approximations can be so obtained.

The naivest approaches to solving the GKW operator, which is suggested by the eqn 2.2, is blocked by the next two theorems.

Theorem 1. *There is no (non-trivial) polynomial $f(\tau)$ such that the equation*

$$f\left(\frac{1}{\tau+n}\right) = \lambda_n (\tau+n)^2 f(\tau)$$

holds for all integer values of n and arbitrary values of λ_n.

Proof. Assume that there does exist such a solution. Then it could be written as

$$f(\tau) = \sum_{k=0}^{\infty} a_k \tau^k$$

for some unknown values of a_k (which are independent of n). Inserting this into the hypothetical form leads to the equation

$$\lambda_n \sum_{k=0}^{\infty} \tau^k \left[a_{k-2} + 2na_{k-1} + n^2 a_k\right] =$$
$$\sum_{k=0}^{\infty} \tau^k \frac{(-1)^k}{n^k} \sum_{j=0}^{\infty} a_j \binom{j+k-1}{j-1} \frac{1}{n^j}$$

Setting $\tau = 1$ in the above allows it to be re-written as

$$\lambda_n (n-1)^2 \sum_{k=0}^{\infty} (-1)^k a_k = a_0 + \sum_{k=1}^{\infty} n^{-k} \sum_{j=0}^{k} \binom{k}{j} a_{j+1}$$

Since the a_k are independent of n by assumption, one must then have $a_0 = 0$, and for each term in the series on the right hand side, one must have, individually, that

$$0 = \sum_{j=0}^{k} \binom{k}{j} a_{j+1}$$

or $a_k = 0$ for all k. Thus the theorem is proved. □

Were it not for this theorem, a solution would have been provided by looking for quasi-modular-form-like functions f. Another naive avenue is also blocked:

Theorem 2. *There is only one series solution to*

$$f\left(\frac{1}{\tau+n}\right) = \frac{(\tau+n)(\tau+1)}{(\tau+n+1)} \lambda f(\tau)$$

and it is $\lambda = 1$ *and* $f(\tau) = a_0/(1+\tau)$ *for any constant* a_0.

Proof. As in the previous proof, assume a series solution. Substituting this into the above, and performing a straightforward but tedious expansion in powers of τ, n, and then comparing terms, reveals that $\lambda = 1$, and that $a_k = (-1)^k a_0$. □

If the above had allowed solutions for something other than $\lambda = 1$, then one would have also had that $\mathcal{L}_h f = \lambda f$.

2.3. **Assorted Algebraic Identities.** This section lists an assortment of random algebraic results, none particularly deep; some are vaguely suggestive of deeper relations. These are listed here mostly for the sake of completeness. These are the sorts of identities one obtains by means of knuckle-headed persistence in the hope that that maybe one little algebraic twist will yield a closed-form solution. These were obtained by the author long before he knew that $\mathcal{L}_h$ had a name or had been previously studied by others.

First, notice that adjacent terms in the series can be made to cancel by shifting the series by one:

$$[\mathcal{L}_h f](x) - [\mathcal{L}_h f](x+1) = \frac{1}{(1+x)^2} f\left(\frac{1}{1+x}\right)$$

which holds for any function $f(x)$. Thus, if $\rho(x)$ is an eigenvector, so that $\mathcal{L}_h \rho = \lambda \rho$, then it would also solve

$$\frac{1}{(1+x)^2} \rho\left(\frac{1}{1+x}\right) = \lambda \left(\rho(x) - \rho(x+1)\right)$$

This can be solved easily to get the zeroth eigenvector

$$\rho_0(x) = \frac{1}{\ln 2}\,\frac{1}{1+x}$$

which satisfies $[\mathscr{L}_h \rho_0](x) = \rho_0(x)$ and the normalization is given by requiring

$$\int_0^1 \rho_0(x)\,dx = 1$$

There is a reflection identity: $f(x) = 1-(1+x)^{-2}$ satisfies $\mathscr{L}_h f = 1-f$.

There is a hint of a relationship between period-doubling and the GKW in the identity

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{1}{2^n}\left[\frac{2}{x+n} - \frac{1}{x+n+1}\right]$$

Acting on the monomial, one gets

$$\left[\mathscr{L}_h x^k\right](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^{k+2}} = \frac{(-1)^{k+2}}{(k+1)!}\,\psi^{(k+1)}(1+x)$$

where $\psi^{(k)}(x)$ is the k'th derivative of the Gamma function. The true difficulty of finding the solution to GKW becomes clear when the search leads one to start discovering complicated identities, such as

$$\sum_{m=1}^{\infty} \frac{1}{m^2}\psi^{(1)}\left(1+\frac{1}{m}+x\right) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\,\psi^{(1)}\left(\frac{1}{n+x}+1\right)$$

or to finding curiosities such as $f(x) = (1+ax)^2$ gives $\mathscr{L}_h f = \psi^{(1)}(1+x+a)$.

For $f(x) = (1+nx)^{-2} - 1$ one gets $\mathscr{L}_h f = -\sum_{k=1}^{n}(x+k)^{-2}$

Acting on a general power, the map gives the Hurwitz Zeta:

$$\left[\mathscr{L}_h x^s\right](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^{s+2}} = \zeta_H(s+2, x+1)$$

This allows eqn 2.3 to be written as

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x\zeta_H(s+1, x+1)$$

2.4. **Non-differentiable Identities.** The above identities involve smooth, differentiable functions. In addition to these, there are a number of relations involving the Minkowski Question Mark function[20]. This function is fractal, continuous everywhere, and differentiable only on the rationals, where it's derivative is zero. There are many ways to define the Question Mark function $?(x)$; perhaps one of the easiest is as follows. Given the continued fraction expansion $x = [a_1, a_2, a_3, \cdots]$ as defined in eqn 2.1, one has

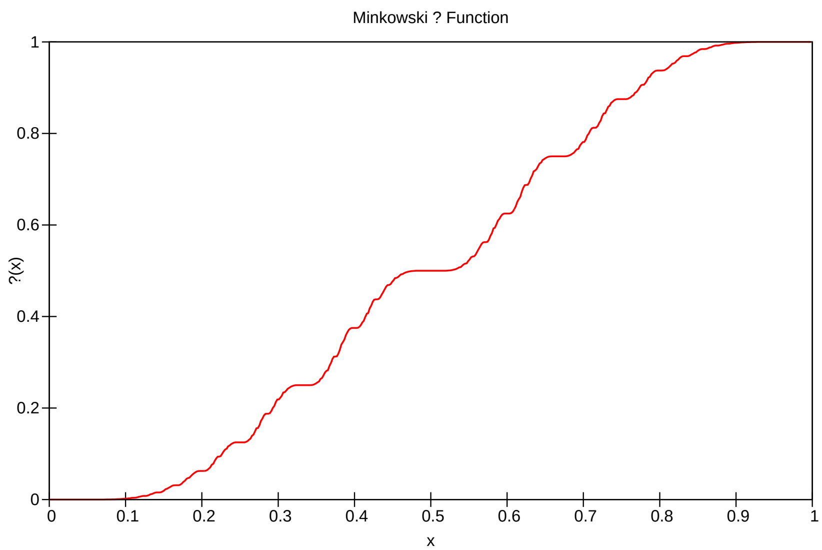
$$?(x) = 2\sum_{k=1}^{\infty} (-1)^{k+1}\; 2^{-(a_1+a_2+\ldots+a_k)} \tag{2.4}$$

where the sum terminates after a finite number of terms when x is rational. A graph of the function is shown in figure 2.4.

This function has many interesting properties and symmetries[30, 28, 25]. The self-symmetry of the curve is generated by two relations, a halving:

$$?\left(\frac{x}{1+x}\right) = \frac{?(x)}{2}$$

FIGURE 2.1. The Minkowski Question Mark Function



and a left-right reflection: $?(1-x) = 1-?(x)$. The full set of symmetries, generating the dyadic monoid, can be exhibited by defining

$$g_C(x) = \frac{x}{1+x} \quad \text{and} \quad g_D(x) = \frac{x}{2} \tag{2.5}$$

and noting that the symmetry relation above may be written as $? \circ g_C = g_D \circ ?$. Likewise, letting $r(x) = 1-x$ one has $? \circ r = r \circ ?$. These two functions can be taken as the generators of a monoid, whose general element is written as $g^m \circ r \circ g^n \circ r \circ g^p \circ \cdots$. This monoid forms an interesting subset of the modular group $SL(2,\mathbb{Z})$. Elements of the monoid are in 1-1 correspondence with the Cantor set; alternately, they are in 1-1 correspondence with the binary tree.

Given the above definitions, one may now derive various identities. Thus, one has

$$\mathcal{L}_h[(1+?(x))/(1+x)^2] = 1-?(x)$$

$$\mathcal{L}_h[?(x)x^{-2}] = 2-?(x)$$

$$\mathcal{L}_h\left[?(x)\left(\frac{1}{(1+x)^2}-2\right)\right] = \frac{?(x)-2}{(1+x)^2}$$

One may continue in this vein indefinitely, but this exercise does not seem to lead to any sort of worthwhile recurrence relations.

The generating function for the moments of the Minkowski Question Mark[2, 1, 3] participates in a curious identity. This generating function obeys the relation

$$\frac{1}{z^2}G\left(\frac{1}{z}\right)+\frac{1}{(z+1)^2}G\left(\frac{1}{z+1}\right)=\frac{1}{z(z+1)}$$

which holds for complex-valued z (such as, for example, $z\mapsto z+n$), and also

$$\frac{1}{z}+\frac{1}{z^2}G\left(\frac{1}{z}\right)=G(z)-2G(z+1)$$

From this, one has the curious shift-over-by-one relationship

$$\left[\mathscr{L}_hG\right](z)=G(1+z)+\left[\mathscr{L}_hK\right](z)$$

where we've defined $K(z)=G(1+z)$.

One can define an object that behaves like the derivative of the Question Mark function; properly speaking, it is a singular measure on the unit interval. For all practical purposes, it can be called the derivative; and it does have one surprising property: it is an eigenvector of the GKW, corresponding to eigenvalue 1. That is,

$$\mathscr{L}_h?'=?'$$

A proof of this relation, including all the required machinery to define and demonstrate this result, is given in [30]. In simple terms, it follows from the self-symmetry relation on the Question Mark:

$$\left(?\circ g_C^{n-1}\circ r\circ g_C\right)(x)=?\left(\frac{1}{n+x}\right)=\frac{1}{2^{n-1}}-\frac{?(x)}{2^n}=\left(g_D^{n-1}\circ r\circ g_D\circ ?\right)(x)$$

which induces a relation on the measure:

$$?'\left(\frac{1}{n+x}\right)=\frac{(x+n)^2}{2^n}?'(x)$$

This last is easily inserted into the definition 2.2 to obtain the desired result. A graph of the measure is shown in figure 4.2.

2.5. **Miscellaneous Series.** Algebraic manipulations of the above algebraic relations leads one to consider various conditionally convergent series, such as

$$\sum_{k=0}^{\infty}(-1)^k\binom{k+m+1}{m}\zeta(k+m+2)=1$$

which holds for any integer m. Additional series, similar to the form given above, are possible, and are reported in [23]. Thus, for example, one has:

$$\sum_{k=0}^{\infty}(-1)^k\binom{k+m+1}{m+1}\zeta(k+m+2)=\zeta(m+2)-1$$

More generally, for integer $n>0$, one has

$$\sum_{k=0}^{\infty}(-1)^k\binom{k+m+1}{m+1}\zeta(k+m+2-n)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\zeta(m+2-j)$$

There are also divergent series which can be evaluated by regulating them, and then taking the limit. Examples include

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-1)^k e^{-tk}=\frac{1}{2}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-1)^k(k+2)e^{-tk}=\frac{3}{4}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-1)^k(k+2)(k+3)e^{-tk}=\frac{7}{4}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-1)^k(k+2)(k+3)(k+4)e^{-tk}=\frac{45}{8}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-1)^k(k+2)(k+3)(k+4)(k+5)e^{-tk}=\frac{93}{4}$$

These are readily obtained[8] by considering the binomial generating function. That is, define

$$\begin{aligned}A_m(x) &= \sum_{k=0}^{\infty}\frac{\Gamma(k+m+2)}{\Gamma(k+2)}(-x)^k\\ &= -\frac{\Gamma(m+1)}{x}\sum_{k=1}^{\infty}\binom{k}{m}(-x)^k\\ &= \frac{\Gamma(m+1)}{x}\left(1-\frac{1}{(1+x)^{m+1}}\right)\end{aligned}$$

and so the above sums are given by

$$A_m\equiv\lim_{x\to 1}A_m(x)=\Gamma(m+1)\left(\frac{2^{m+1}-1}{2^{m+1}}\right)$$

Similarly, let

$$S_m(x)=\sum_{k=0}^{\infty}(-)^k\frac{(k+m+1)!}{(k+1)!}\left[\zeta(k+m+2)-1\right]x^k$$

and

$$S_m\equiv\lim_{x\to 1}S_m(x)$$

then one finds that $S_0=1/2$, $S_1=1/4$, $S_2=1/4$, $S_3=3/8$ and $S_4=3/4$; the general expression is given by $S_m=\Gamma(m+1)/2^{m+1}$. This result can be obtained by simply by examining the zeroth eigenvector of the GKW operator, as given below, in equations 2.6,2.7. Alternately, the general expression for $S_m(x)$ can be obtained by working with the expression for $\mathscr{L}_h x^k$ (XXX do this and provide expression). The above sums appear when considering

$$\psi(1+z)=\frac{-1}{1+z}+1-\gamma+\sum_{m=0}^{\infty}(-)^m\left[\zeta(m+2)-1\right]z^{m+1}$$

and then writing

$$z^{m+1}=(z+1-1)^{m+1}=\sum_{k=0}^{m}(-)^{m-k}\binom{m}{k}(z+1)^k$$

2.6. **Polynomial Representation.** One can attempt to solve GKW by working in the polynomial representation. One possible choice is to make one's Taylor expansion about $x=0$, but this turns out to be a very poor choice. Writing $\mathscr{L}_h f = g$ and substituting a Taylor's expansion for f and g, so that

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}f^{(k)}(0)$$

and likewise for $g(x)$, one gets

$$\frac{g^{(m)}(0)}{m!}=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(-)^m\,\frac{(k+m+1)!}{m!\,(k+1)!}\,\zeta(k+m+2)$$

or, adopting the bra-ket notation introduced earlier[24], we have

$$\langle m|\mathscr{L}_h|k\rangle=(-)^m\left(\begin{array}{c}k+m+1\\ m\end{array}\right)\zeta(k+m+2)$$

where we've replaced the factorials by the binomial coefficient that they form. Unfortunately, this is clearly a very poorly conditioned matrix. One can make some progress, if one wishes, by applying a regulator and using Levin-type sequence acceleration techniques. One can thus find a number of curious identities, some of which we've listed previously. However, the difficulty of working with divergent sums seems to outweigh any advantages given by the relatively simple form of the matrix elements. Thus, one is lead to consider the matrix elements for a polynomial expansion about $x=1$. These are far more complex, but give a very well-conditioned matrix. These are:

$$G_{mn}=\sum_{k=0}^{n}(-)^k\left(\begin{array}{c}n\\ k\end{array}\right)\left(\begin{array}{c}k+m+1\\ m\end{array}\right)[\zeta(k+m+2)-1] \tag{2.6}$$

satisfying

$$(-)^m\frac{g^{(m)}(1)}{m!}=\sum_{n=0}^{\infty}G_{mn}(-)^n\frac{f^{(n)}(1)}{n!}$$

Other authors have chosen to expand about $x=1/2$ [6] but it would appear that expansion about $x=1$ leads to the simplest tractable expansion. The general case, of a Taylor's expansion about some fixed point, is explored in appendix A.

The eigenvector equation to be solved is then

$$\sum_{n=0}^{\infty}G_{mn}v_n=\lambda v_m$$

where $v=\{v_n\}$ is an eigenvector with components v_n. This beast has a discrete spectrum (XXX need reference for proof), and so we may label the eigenvalues and eigenvectors with a label k, so that the k'th eigen equation is

$$\sum_{n=0}^{\infty}G_{mn}v_{nk}=\lambda_k v_{mk} \tag{2.7}$$

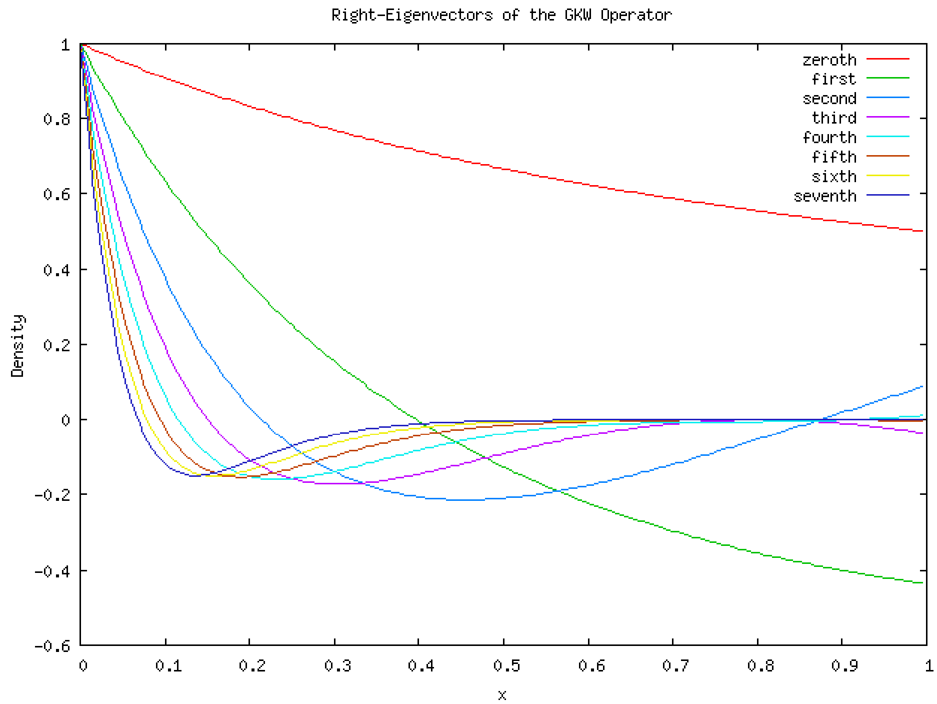
The k'th polynomial eigenfunction of the GKW operator is then given by

$$\rho_k(x)=\sum_{n=0}^{\infty}v_{nk}(1-x)^k$$

The zeroth eigenfunction was given by Gauss as

$$\rho_0(x)=\frac{1}{1+x}$$

FIGURE 2.2. GKW Right Eigenfunctions



This graph shows the right eigenfunctions of the GKW operator. These were computed numerically, by truncating the GKW operator to a submatrix of 55 by 55 elements and then solving for the eigenvectors of the matrix. The elements of the eigenvectors appear to be well-behaved, being oscillatory for small values, and then converging to zero rapidly.

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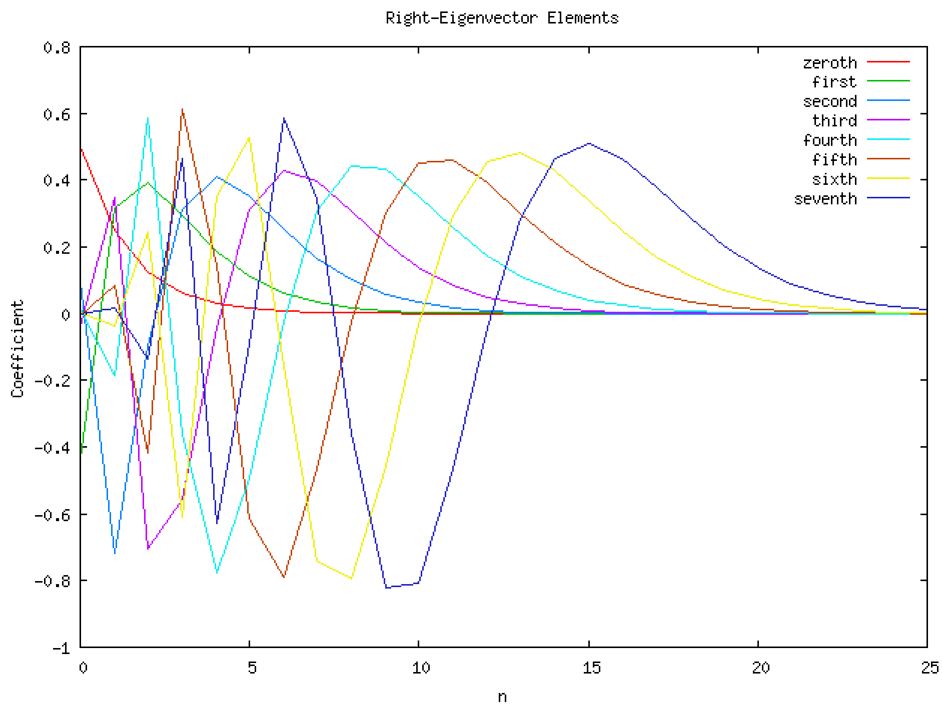
and corresponds to the eigenvector $v_{n0} = 2^{-n}$. Again, note the curious appearance of powers of two. The zeroth eigenvalue is 1, the first eigenvalue is known as the GKW constant, and is about 0.3036. Additional eigenvalues are given in table 1. Graphs of the first few eigenvectors are shown in figure 2.2.

Based on numerical explorations (reported below), the series appears to be easily convergent even for $x = 0$. In particular, it appears that $\lim_{n\to\infty} v_{nk} = 0$, and that furthermore, that $\lim_{n\to\infty} v_{nk}/v_{n+1,k} = 2$. One must not conclude that this implies that $v_{nk}\ \mathcal{O}(2^{-n})$, as this sort of asymptotic behaviour conflicts with the zeroth eigenfunction. Thus, one might guess at $v_{nk}\ \mathcal{O}(n^s 2^{-n})$ but numeric data suggests that $s = 0$. Numeric data also excludes the form $v_{nk}\ \mathcal{O}(2^{-n}\log n)$ although it is possible that the softer $v_{nk}\ \mathcal{O}(2^{-n}\log\log n)$ might hold.

The coefficients are oscillatory, with k half-oscillations in the k'th eigenvector. That is, for $k = 0$, all of the v_{n0} can be taken to be of the same sign. For $k = 1$, the v_{n1} change sign, once, between $n = 0$ and $n = 1$. For $k = 2$, the coefficients change sign twice, and so on. This is shown in the graph 2.3.

FIGURE 2.3. Right Eigenvector Coefficients



This figure shows a graph of the coefficients of the first eight right-eigenvectors of the GKW operator. The red line corresponds to the zeroth eigenvector with components $v_{n0} = 2^{-(n+1)}$. All of these were obtained numerically. The normalization used here is to require that $\sum_{n=0}^{\infty} v_{nk} = 1$.

The left eigenvectors are given by

$$\sum_{m=0}^{\infty} w_{km} G_{mn} = \lambda_k w_{kn}$$

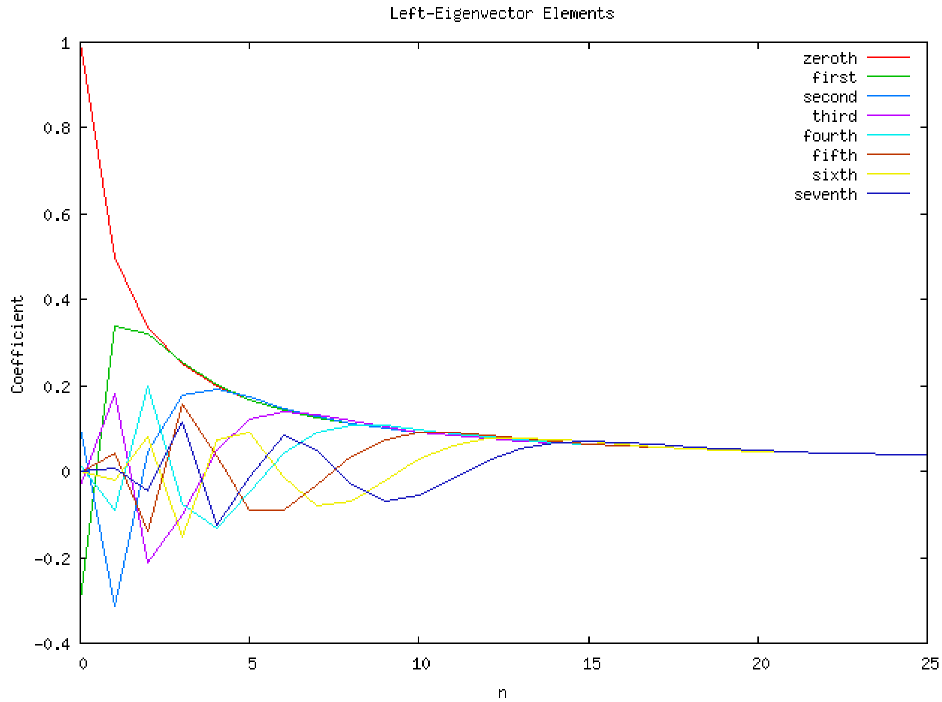
and correspond to left eigenfunctions

$$\ell_k(x) = \sum_{n=0}^{\infty} w_{kn} (-1)^n \delta^{(n)}(1-x)$$

where $\delta^{(n)}(x)$ is the n'th derivative of the Dirac delta function. The zeroth left eigenvector is given by $w_{0n} = 1/(n+1)$. Very curiously, this is the harmonic series. Thus, true to form, it appears that yet again, we are in the presence of another manifestation of the duality between the dyadic rationals and rationals, the duality between the Stern-Brocot tree and the dyadic tree, the duality captured in the Minkowski Question Mark function.

In analogy to the right eigenvectors, the series again appears to be not only convergent in that $\lim_{n\to\infty} w_{kn} = 0$, but also that a strict ratio is maintained in the limit: $\lim_{n\to\infty}(n+2)\, w_{kn}/(n+1)\, w_{k,n+1} = 1$, with strict equality holding for all n, and not just in the limit, when $k = 0$. A similar oscillatory behaviour is seen as well, as shown in figure 2.4.

FIGURE 2.4. Left Eigenvector Coefficients



This figure shows a graph of the coefficients of the first eight left-eigenvectors of the GKW operator. The red line corresponds to the zeroth eigenvector with components $w_{n0} = 1/(n+1)$. All of these were obtained numerically. The normalization used here is to require that w_{nk} $1/(n+1)$ for large values of n.

2.7. **Identity.** We have the identity

$$\sum_{n=0}^{\infty} G_{mn} p^{-n} = p\left[\zeta(m+1) - 1 - \zeta_H\left(m+1, 2+\frac{1}{p-1}\right)\right]$$

In particular, for $p = 2$, the right hand side equals p^{-m}; this corresponds to the known eigenvector. Note that for any value of p, the leading term on the right is $2^{-(m+1)}$. A simple way to arrive at this is to note that

$$\sum_{n=0}^{\infty} p^{-n}(1-x)^n = \frac{p}{p-1+x}$$

and then evaluate this expression under the action of the GKW operator.

2.8. **The Kernel.** The spectrum of the polynomial eigenfunctions of the GKW operator is discrete, and does not include zero. However, if one considers a larger set of functions, say, the set of square-integrable functions, then the spectrum becomes continuous, and includes zero.

The kernel of the GKW operator is defined as the set of functions f for which $\mathscr{L}_h f = 0$. The kernel of the GKW is readily demonstrated: let

$$k_n(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{n+2} \\ \frac{1}{x^2} & \text{for } \frac{1}{n+2} \le x < \frac{1}{n+1} \\ \frac{-1}{x^2} & \text{for } \frac{1}{n+1} \le x < \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le x \le 1 \end{cases}$$

Then clearly, $[\mathscr{L}_h k_n](x) = 0$ for all integer $n \ge 1$. The kernel is in fact much larger. Consider the set of functions

$$c_{n,l}(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{n+2} \\ \frac{\cos((2l+1)\pi/x)}{x^2} & \text{for } \frac{1}{n+2} \le x < \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le x \le 1 \end{cases}$$

Then

$$\left[\mathscr{L}_h c_{n,l}\right](x) = \cos\left((2l+1)\pi(x+n)\right) + \cos\left((2l+1)\pi(x+n+1)\right) = 0$$

and so $c_{n,l}$ belongs to the kernel for all integer $l \ge 1$, as does $s_{n,l}$ when defined analogously for sine instead of cosine.

2.9. **Numeric Attacks.** One can mount numeric attacks on the GKW operator. The matrix elements in 2.6 are easily computed numerically, and the eigenvectors and eigenvalues are easily obtained by applying standard matrix diagonalization software. For example, using the LAPACK DGEEV eigenvalue-finding routine, and working with the upper-left 55×55 entry block of the GKW operator, one obtains the eigenvalues shown in table 1 These compare well to previously published values (xxx need reference for previous numerics - specifically babenko)[12, 6, 13]. The ratio of successive eigenvalues λ_n/λ_{n+1} tends to the square of golden mean $(3+\sqrt{5})/2 = 2.61083398875\cdots$, as given by [12].

Eigenvectors can be guessed at in various ways. One can find, for instance, that the first right-hand-side eigenvector ρ_1 is approximated by

$$\rho_1(x) \approx \frac{-3}{4} + \frac{7}{4}\frac{1}{(1+x)^{5/2}}$$

with the approximation accurate to about one or two percent over the domain $x \in [0,1]$. This is the eigenvector associated with the eigenvalue $\lambda_1 \approx 0.303663$. Numerics suggest that all of the eigenvectors have a pole at $x = -1$. Whether they might have poles at other negative values is unclear; however, the idea that the eigenvectors might be linear combinations of the Hurwitz zeta function suggests itself. Thus, for example, a slightly better approximation is given by:

$$\rho_1(x) \approx 3.078\left[\zeta_H(2,1+x) - \frac{1.32}{(1+x)^{3/4}}\right]$$

where ζ_H is the Hurwitz zeta function:

$$\zeta_H(s,q) = \sum_{n=0}^{\infty}(n+q)^{-s}$$

Graphs of the first seven right-hand-side eigenvectors are shown in figure 2.2. The general oscillatory nature of the eigenfunctions is echoed in the numeric values of the eigenvector coefficients themselves, shown in figures 2.3 and 2.4.

TABLE 1. GKW Eigenvalues

N	Eigenvalue λ_n	Ratio λ_n/λ_{n+1}
0	1	-3.29312425436788
1	-0.303663002898733	-3.0100062440358
2	0.100884509293104	-2.842124671335
3	-0.03549615904	-2.763682528286
4	0.01284379036244	-2.72242392332
5	-0.00471777751158	-2.69791537939
6	0.0017486751243	-2.6819312645
7	-0.0006520208580	-2.67077775
8	0.00024413145	-2.662606
9	-9.16889×10^{-5}	-2.65651
10	3.45147×10^{-5}	-2.6539
11	-1.3005×10^{-5}	
12	4.860×10^{-6}	
13	-1.7×10^{-6}	

This table lists the first dozen eigenvalues of the polynomial representation of the GKW operator. The numbers are certain to about about the last figure or two quoted. They were obtained by numerically inverting the 55 by 55 entry upper-right sub-matrix of the GKW operator using ordinary double-precision floats.

2.10. **Gross structure.** The GKW operator has a fairly simple coarse-grained structure, which is explored in this section. The nature of this structure is best illustrated graphically. The figure 2.5 shows the entries of the matrix G_{mn} in a color-coded fashion, using black to code negative values, and varying colors to code positive values.

The hyperbola-like structure visible in 2.5 suggests that G_{mn} is approximately constant when the product of the indexes mn is held constant, and oscillatory as a function of the product mn. A closer numerical look shows that the curves are not quite hyperbolas, but are roughly approximated by

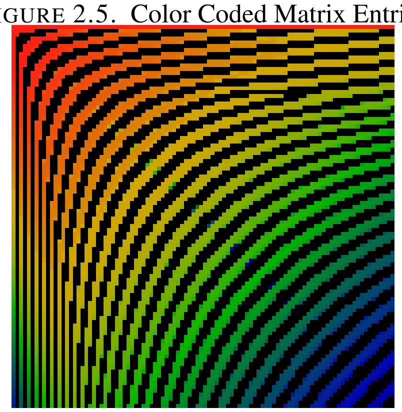
$$(m+n)^{1.7} - |m-n|^{1.7} \approx const$$

Furthermore, the limiting behaviour for $m \ll n$ and $n \ll m$ doesn't seem to be hyperbolic either, as can seen in a pixel-aliasing/Moire type effect visible in the graph, at about $m \approx 5n$ and $n \approx 4m$. In particular, some of the contours appear to merge along the line $n \approx 4m$. As will be shown in a later section, where the asymptotic expansion of the matrix elements is considered, this corresponds to the non-trivial zeros of the Riemann zeta function in the critical strip.

Note that there is a very superficial resemblance of the figure to the so-called "Hadamard matrix" (sometimes also called the Walsh matrix), in that the Hadamard matrix also has a prominent hyperbola-like structure in the sign changes of its matrix elements[31]. It differs from the above, in that, for the GKW, the matrix elements alternate periodically, whereas in the Hadamard matrix, the frequency along the diagonal increases exponentially. None-the-less this begs the question of whether there exists some permutation-like reordering of the GKW that might untangle the marix in some way.

The behavior of the matrix elements along the diagonal is shown in figure 2.6.

FIGURE 2.5. Color Coded Matrix Entries



This figure shows the 100 by 100 submatrix of the entries of G_{mn}, with row,column (0,0) in the upper left. Each square pixel represents one entry in the matrix; this this picture is 100 by 100 pixels in size. Black areas represent negative values of G_{mn} while colored areas are positive. The coloration is such that red corresponds to large values, moving through the rainbow to smaller values. The color scale is logarithmic, so red-orange represents values of $G_{mn} \sim 10^{-3}$, yellow-green to $G_{mn} \sim 10^{-20}$ and blue to $G_{mn} \sim 10^{-40}$. Visually, the hyperbolic curves suggest that G_{mn} is approximately constant when the product of the indexes mn is held constant. However, the curves are not true hyperbolas; a closer numerical examination shows that a better fit is given by
$(m+n)^{1.7} - |m-n|^{1.7} \approx const.$

2.11. **Asymptotic Expansion.** The asymptotic behaviour of the matrix elements for large m,n can be obtained by converting the Newton series 2.6 to Nörlund–Rice integral, and then using saddle-point methods to obtain the large m,n behaviour. A detailed exposition of this procedure is given in [11]; what follows is an abbreviated application of those techniques.

There are several key steps to this process. The first is the observation that a Newton series can be re-written as a Nörlund–Rice integral[9]:

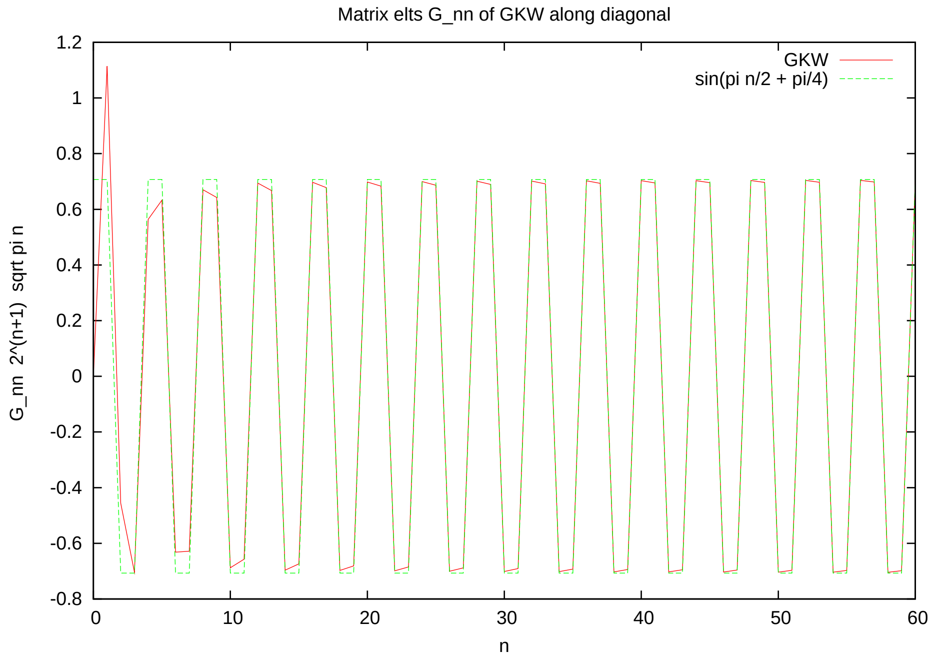
$$\sum_{k=n_0}^{n} (-1)^k \binom{n}{k} \phi(k) = \frac{(-1)^n}{2\pi i} \oint_c \phi(s) \frac{n!}{s(s-1)\cdots(s-n)} ds$$

where the contour c surrounds the poles at $\{n_0, \cdots, n\}$ and $\phi(s)$ is holomorphic in the region $\Re s > n_0 - \frac{1}{2}$. Substitution of 2.6 into the above yields

$$G_{mn} = \frac{(-1)^n}{2\pi i} \frac{n!}{m!} \oint_C \frac{(s+m+1)(s+m)\cdots(s+2)}{s(s-1)\cdots(s-n)} \left[\zeta(s+m+2) - 1\right] ds \tag{2.8}$$

Written in this form, the zeros and poles of the integrand are clearly evident. It also becomes clear that the right-hand closure of the contour integral contributes nothing, since, for $\Re s \to \infty$, one has that the integrand decays exponentially, *viz.* $\zeta(s+m+2) - 1 \to 2^{-s} + \mathscr{O}(3^{-s})$ which over-powers the polynomial numerator and drives the integrand to zero. Thus, the closed contour can be replaced by a line integral running from $c - i\infty$ to $c + i\infty$ with $\Re c$ to the left of the poles:

FIGURE 2.6. GKW along the diagonal



Shown are the matrix elements G_{nn} of the GKW operator along the diagonal. As these are exponentially vanishing, the figure rescales these: thus, the red line shows the values of $G_{nn}2^{-n-1}\sqrt{\pi n}$. The matrix elements are also oscillatory: the overall structure is easily accounted for, as shown by the green line, which graphs $\sin(\pi n/2+\pi/4)$. The period, phase and amplitude of the oscillations appears to be exact: a numerical exploration out to $n=1000$ indicates that this oscillation holds very precisely.

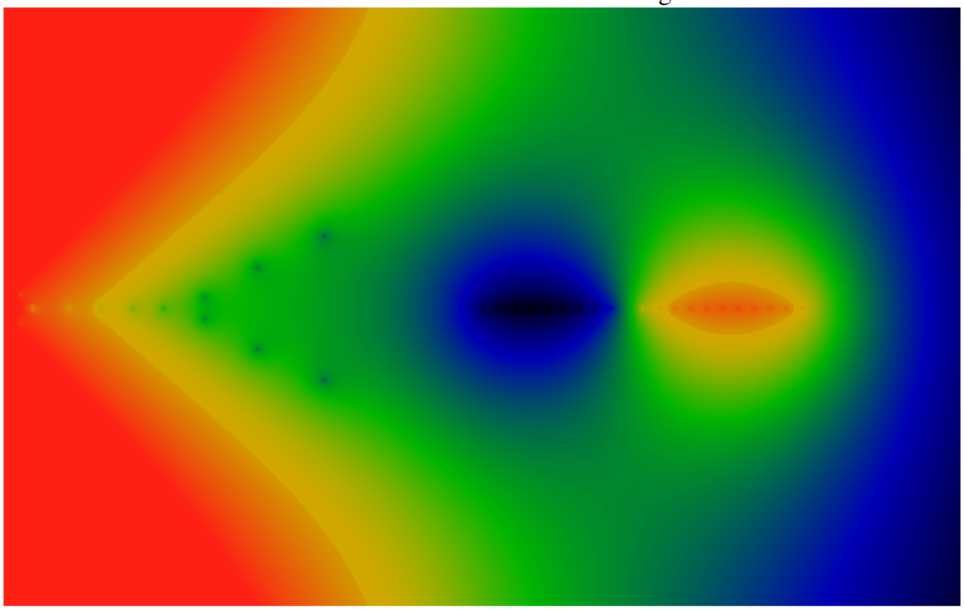
$$G_{mn}=\frac{(-1)^{n+1}}{2\pi i}\frac{\Gamma(n+1)}{\Gamma(m+1)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s+m+2)\Gamma(s-n)}{\Gamma(s+2)\Gamma(s+1)}\left[\zeta(s+m+2)-1\right]ds$$

The integrand becomes exponentially large for $\Re s\to-\infty$ and this leads to a pair of saddle points above and below the real axis: each saddle is situated between the exponential rise to the left and the poles, and straddles a hump between the zeros and the exponentially decay to the right. For $m=n$, it will be shown that the saddle points occur at $s=\pm in+\mathscr{O}(1)$. The integrand and its saddle points is illustrated in the figure 2.7.

The integral may be estimated for $m,n\to\infty$ by applying the method of steepest descent. That is, one deforms the integration contour so that it passes through the saddle-point, following the steepest path through it. There, on applies Laplaces method to approximate the integral as

$$\int e^{Nf(x)}dx=e^{Nf(x_0)}\sqrt{\frac{2\pi}{-Nf''(x_0)}}\left(1+\mathscr{O}\left(\frac{1}{N}\right)\right) \tag{2.9}$$

FIGURE 2.7. Saddle Points of the Integrand



This figure illustrates the integrand of the integral 2.8 on the complex-s plane, for $m=n=10$. The real axis runs along the middle of the figure. Blue and black areas represent small or zero values of the modulus, while yellow and red represent large values. Thus, the exponential decay to the right is shown in blue-black, while the exponential rise to the left is red. Arrayed along the real axis are eleven poles, at $s=0,\cdots,10$ and ten zeros, at $s=-2,\cdots,-11$. Some additional zeros, forming an arrowhead, can be seen the the left. The saddle-points are located at $s=\pm10i$. The illustration runs over the interval $\Re s\in[-40,+20]$. The integration contour can be taken to run in the green region, to the left of the poles, and through the saddle points.

with the integration contour running through the saddle point $f'(x_0)=0$. To be able to apply this, the logarithm of the integrand, expanded in powers of N, is required. As mentioned, the saddle points will be discovered at $s\approx\pm in$, thus suggesting the change of variable $s=nx$ and an expansion in powers of $N=n$, while holding x fixed. To avoid a concurrent expansion in powers of m, write $m=Kn$ for some fixed constant K. One may then proceed with the expansion. The zeta function terms expands as

$$\log\left[\zeta\left(s+m+2\right)-1\right]=-\left[n\left(x+K\right)+2\right]\log2+\mathscr{O}\left(\left(\frac{2}{3}\right)^{n(x+K)}\right)$$

The gamma functions are expanded by applying Stirling's formula, in leading powers of n,

$$\log\Gamma\left(ny\right)=ny\log\left(ny\right)-ny-\frac{1}{2}\log\left(ny\right)+\frac{1}{2}\log\left(2\pi\right)+\frac{1}{12ny}+\mathscr{O}\left(\frac{1}{n^3}\right)$$

which is applicable for $\Re y>0$. This can be applied in a straightforward way, although to do so, one must first write

$$\Gamma\left(s-n\right)=\frac{\pi\left(-1\right)^n}{\sin\left(\pi s\right)\Gamma\left(n-s+1\right)}$$

which follows from the reflection formula $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$. For the saddle point in the upper half-plane, that is, for $\Im x>0$, one expands

$$\begin{aligned}\log\sin\pi xn =& \log\left(\frac{-e^{-i\pi nx}}{2i}+\mathscr{O}\left(e^{-\pi n\Im x}\right)\right)\\ =& -i\pi nx+\frac{i\pi}{2}-\log 2+\mathscr{O}\left(e^{-2\pi n\Im x}\right)\end{aligned}$$

Combining all of these elements, one then obtains the following asymptotic expansion in n:

$$\begin{aligned}Nf(x) =& n(K-1)\log n\\ &+n\left[i\pi x-(x+K)\log 2-2x\log x-(1-x)\log(1-x)+(K+x)\log(K+x)+1-K\right]\\ &-\log n\\ &-2\log x-\frac{1}{2}\log(1-x)+\frac{3}{2}\log(K+x)-\log 8\pi\\ &+\mathscr{O}\left(\frac{1}{n}\right)\end{aligned}$$

The first derivative is

$$\begin{aligned}Nf'(x) =& n\left[i\pi-\log 2+\log\frac{(1-x)(K+x)}{x^2}\right]\\ &+\frac{1}{2(1-x)}+\frac{3}{2(K+x)}-\frac{2}{x}+\mathscr{O}\left(\frac{1}{n}\right)\end{aligned}$$

while the second derivative is

$$\begin{aligned}Nf''(x) =& n\left[\frac{1}{K+x}-\frac{1}{1-x}-\frac{2}{x}\right]\\ &+\frac{2}{x^2}+\frac{1}{2(1-x)^2}-\frac{3}{2(K+x)^2}+\mathscr{O}\left(\frac{1}{n}\right)\end{aligned}$$

Solving for the saddle points $f'(x_0)=0$ gives

$$x_0=\frac{K-1}{2}\pm\frac{1}{2}\sqrt{1-6K+K^2}+\frac{1}{n}\left(\frac{7Kx_0-3x_0-6K}{2\left(K-x_0^2\right)}\right)+\mathscr{O}\left(\frac{1}{n^2}\right) \tag{2.10}$$

For the special case of $m=n$, that is $K=1$, one gets the simple form

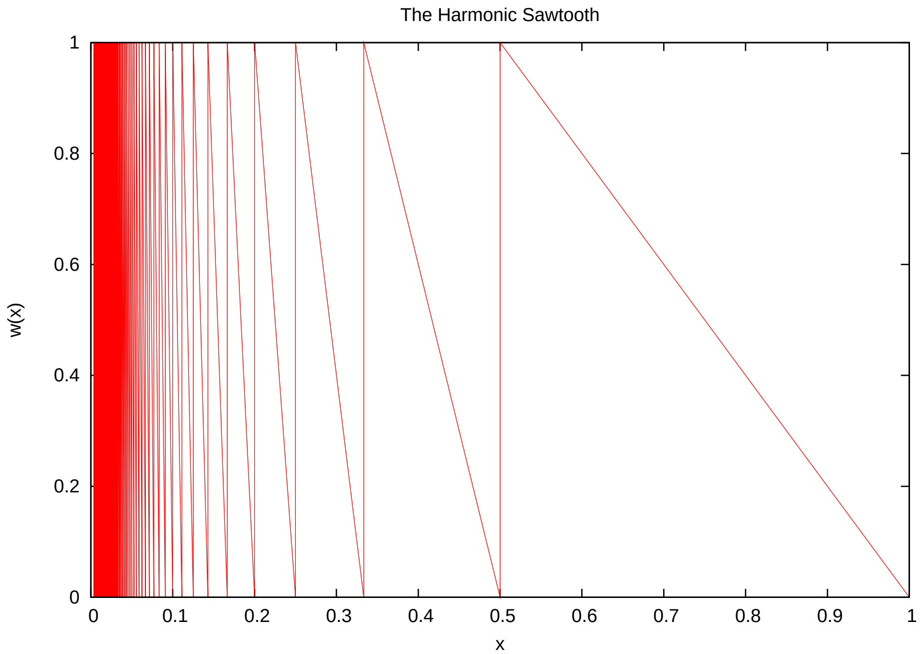
$$x_{0,m=n}=\pm i+\frac{1}{n}\left(\frac{-3}{2}\pm i\right)+\mathscr{O}\left(\frac{1}{n^2}\right)$$

To obtain asymptotic expansion for G_{mn}, it is now enough to insert the saddle point 2.10 into 2.9. The algebra is considerably simplified by setting $K=1$. The final result is precisely that already discovered numerically:

$$G_{nn}=\frac{1}{2^{n+1}\sqrt{\pi n}}\sin\frac{\pi}{2}\left(n+\frac{1}{2}\right)\left[1+\mathscr{O}\left(\frac{1}{n}\right)\right]$$

This is an exact, analytic result for the asymptotic expansion of the GKW matrix elements as $n\to\infty$, obtained by using the method of steepest descent applied to the Nörlund–Rice integral of the Newton series of the GKW operator matrix elements 2.6.

FIGURE 3.1. The Harmonic Sawtooth



The harmonic sawtooth function joins values of $1/n$ with straight lines.

3. The Harmonic Sawtooth

The Gauss map $h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ has a distinctive sawtooth shape; the transfer operator of the Gauss map is the GKW operator. This section presents the harmonic sawtooth, which resembles the Gauss map, but with straight-line edges. The goal here is to develop a model for the GKW, in order to better understand it. In this case. the transfer operator is considerably simpler than GKW; it is solvable. The spectrum is similar, in that it is a countable, discrete set of values, but is otherwise in-equivalent.

The harmonic sawtooth uses straight lines arranged between values of $1/n$ for integer n:

$$w(x) = \begin{cases} 2-2x & \text{for} & \frac{1}{2} < x \le 1 \\ 3-6x & \text{for} & \frac{1}{3} < x \le \frac{1}{2} \\ 4-12x & \text{for} & \frac{1}{4} < x \le \frac{1}{3} \\ n+1-n(n+1)x & \text{for} & \frac{1}{n+1} < x \le \frac{1}{n} \end{cases}$$

The sawtooth is singular at $x = 0$. This is pictured in figure 3.1.

The Frobenius-Perron operator for this sawtooth, acting on a general function $f(x)$, is given by

$$\left[\mathscr{L}_w f\right](x) = \sum_{y:w(y)=x} \frac{f(y)}{|dw(y)/dy|} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} f\left(\frac{n+1-x}{n(n+1)}\right)$$

The spectrum for the polynomial basis Banach space is given below.

3.1. **Relation to the Riemann Zeta function.** The harmonic spacing of the sawtooth edges implies that the harmonic sawtooth will be related to the Riemann zeta in much the same way as the Gauss map is. Specifically, the Mellin transform gives:

$$\zeta(s) = \frac{s+1}{s-1}\left[1 - s\int_0^1 w(x)\, x^{s-1} dx\right]$$

The above can be obtained in a very straightforward manner by direct substitution.

3.2. **The Polynomial Eigenfunctions.** The polynomial eigenfunctions can be obtained in a straight-forward manner, by means of Taylor's expansion. Expanding $f(x)$ as a Taylor's expansion about $x = 0$ gives

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

and likewise

$$[\mathscr{L}_w f](x) = \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} x^m = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \left(\frac{n+1-x}{n(n+1)}\right)^k$$

Rearranging the sums, and equating terms with the same power of x, on obtains matrix elements W_{mk} so that

$$\frac{g^{(m)}(0)}{m!} = \sum_{k=0}^{\infty} W_{mk} \frac{f^{(k)}(0)}{k!}$$

with

$$W_{mk} = \begin{cases} (-1)^m \begin{pmatrix} k \\ m \end{pmatrix} \sum_{n=1}^{\infty} n^{-k-1}(n+1)^{-m-1} & \text{for} \quad k \geq m \\ 0 & \text{for} \quad k < m \end{cases}$$

where $\begin{pmatrix} k \\ m \end{pmatrix}$ denotes the binomial coefficient. This matrix is upper-triangular, and thus has its eigenvalues along the diagonal. These are

$$\lambda_k = (-1)^k \sum_{n=1}^{\infty} \frac{1}{n^{k+1}(n+1)^{k+1}}$$

so that $\lambda_0 = 1$ and $\lambda_1 = 3 - 2\zeta(2)$ where $\zeta(x)$ is the Riemann zeta. Numerically, one finds that the first few eigenvalues are $\lambda_1 = -0.289868...$ and $\lambda_2 = 0.130396...$ and $\lambda_3 = -0.0633278...$ and $\lambda_4 = 0.031383...$ and $\lambda_5 = -0.0156468...$ In the limit of large k, the first term in the summation will dominate, and so $\lambda_k \to (-1)^k / 2^{k+1}$; the ratio of eigenvalues settles down to $\lambda_k/\lambda_{k+1} \to -2$ in the limit. The alternating sign of the eigenvalues, as well as the ratio of successive eigenvalues, is quite unlike the GKW.

The function $w(x)$ is singular at $x = 0$, and this might suggest that a polynomial expansion around a different point might be warranted, as it was for the GKW operator. However, this is not needed: since the resulting matrix is solvable, a transformation to a different point does not offer much, if anything. That is, performing the Taylor's expansion about a point other than $x = 0$ just amounts to multiplying W on the left and right by a binomial transform; this will not make W somehow "more triangular" or "more solvable". It might be possible to improve numerical stability in this way, but there do not seem to be any other gains.

The matrix elements of W_{mk} are easily computed by means of recurrence relations on the indexes m,k. This is done by defining Z_{mk} and observing that

$$Z_{mk} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{m+1} n^{k+1}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^m n^k} \left[\frac{1}{n} - \frac{1}{n+1} \right] = Z_{m-1,k} - Z_{m,k-1}$$

These recursion relations are bounded on the edges by $Z_{00} = 1$, $Z_{10} = 2 - \zeta(2)$ and thus

$$Z_{m0} = Z_{m-1,0} - (\zeta(m+1) - 1) = 1 - \sum_{j=1}^{m} [\zeta(j+1) - 1]$$

and $Z_{01} = \zeta(2) - 1$ so that

$$Z_{0k} = \zeta(k+1) - Z_{0,k-1} = (-)^k \left[1 + \sum_{j=1}^{k} (-)^j \zeta(j+1) \right]$$

From these, one then has $W_{mk} = (-1)^m \binom{k}{m} Z_{mk}$ for $m \le k$.

The first few eigenfunctions are

$$e_0(x) = 1$$

$$e_1(x) = 2x - 1$$

XXX double check e_2 below, I think its wrong

$$\begin{aligned} e_2(y) = & \frac{15 - 13\zeta(2) - 9\zeta(3) + 2\zeta(2)[\zeta(2) + 3\zeta(3)]}{3(13\zeta(2) - 8\zeta(3))(3 - 2\zeta(2))} + \\ & + \frac{6\zeta(2) + 2\zeta(3) - 12}{13 - 8\zeta(2)} y + y^2 \end{aligned}$$

We see that although the eigenfunctions are polynomials and are exactly solvable, there is no particularly simple way of writing down the closed-form solution.

XXX To Do: Double-check e_2 Provide the closed-form finite-sum matrix elements. Provide graphs of the first dozen polynomials. Discuss the similarity transform that takes $w(x)$ to $h(x)$ and discuss why this fails to preserve the eigenvalues. What are the shift-states of this operator? What are the continuous-eigenvalue (square-integrable) eigenfunctions? Graph these eigenfunctions, see what kind of fractals they look like.

4. The Dyadic Sawtooth

The dyadic sawtooth is given by the dyadic-space conjugate of the continued-fraction shift function $h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$, that is,

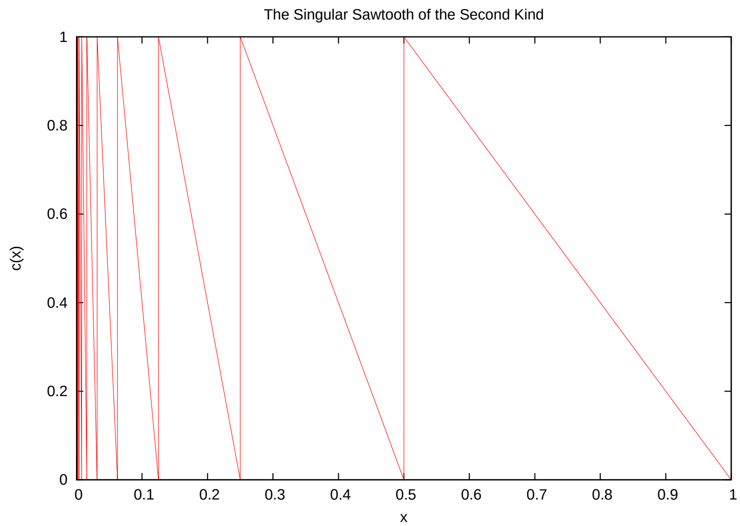
$$c(x) = ?\left(\frac{1}{?^{-1}(x)} - \left\lfloor \frac{1}{?^{-1}(x)} \right\rfloor \right) = (? \circ h \circ ?^{-1})(x)$$

where $?(x)$ is the Minkowski Question Mark, presented in earlier chapters. This map consists of straight-line segments between values of $1/2^k$, as pictured in figure 4.1, and can be written as

$$c(x) = 2 - 2^n x \ \text{ for } \ \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$$

Just as the Gauss Map is able to lop off the leading term of the continued fraction expansion for x, so this map is able to lop off all of the leading zeros of the binary expansion for x. The downward slope of the sawtooth just reflects the binary expansion, exchanging 1's for 0's, so that the next iteration can chop of the next contiguous chunk of identical digits.

FIGURE 4.1. Dyadic Sawtooth



This figure illustrates the second kind of sawtooth, given by equation **??**. It consists entirely of straight-line segments between reciprocal powers of two.

Thus, the orbits of points under this map are completely isomorphic to the orbits of points under the Gauss Map. This is indeed the very idea of a "conjugate map".

The transfer operator of this function provides a second model of the Gauss-Kuzmin-Wirsing operator. It can be solved exactly; unfortunately, it is not trivially conjugate to the GKW operator, as one might naively hope. Normally, when there exists a *smooth* function ϕ such that $\alpha = \phi^{-1} \circ \beta \circ \phi$, then there is a similarity transform that connects the transfer operator for α with that for β, as developed in the appendix B. However, the Question Mark function is not smooth. With some effort, one can define it's derivative in a rigorous way[30], but one finds that the derivative is continuous no-where – vanishing on the rationals, infinite on "most" reals. The similarity transform is given by the Jacobian $\left|\left(?' \circ ?^{-1}\right)(x)\right|$ of the Question Mark, pictured in figure 4.2 – which can be seen to be terribly singular. Put another way, although the point dynamics of this sawtooth map are completely isomorphic to the point dynamics of the Gauss Map, the distribution of these orbits, with respect to the natural measure on the unit interval, is *not* isomorphic by means of any differentiable function. The spectra of the associated transfer operators are *not* identical.

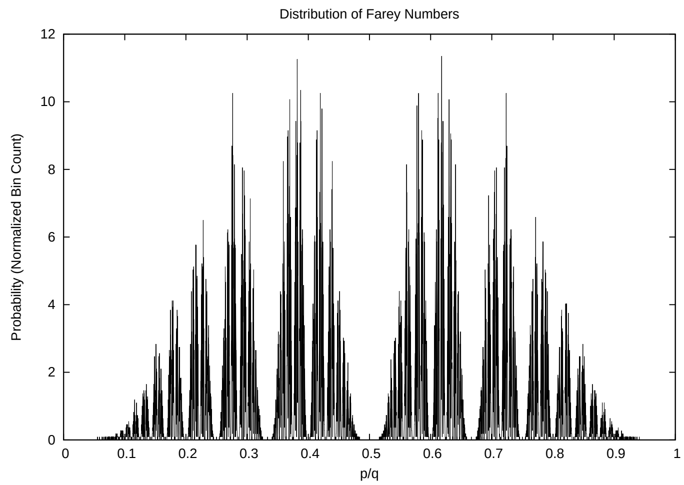
However, with suitable tools, it is possible to construct functions on the Cantor set for which one can obtain similarity transforms which render these operators conjugate. Insofar as the Cantor set is "very nearly" the unit interval, these functions pass over to functions on the unit interval, and thus allow solutions to the GKW operator to be obtained. The development of these tools and their application take up most of what follows.

The transfer operator for the dyadic sawtooth can be easily seen to be

$$[\mathscr{L}_c f](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\frac{2-x}{2^n}\right) \tag{4.1}$$

The following sections develop this operator in different function spaces.

FIGURE 4.2. The Derivative of the Question Mark



This figure shows the derivative of the Minkowski Question Mark; or, more precisely, the weight distribution of a measure on the real-number line, whose integral is well-defined and is exactly equal to $?(x)$.

4.1. **The Polynomial Basis Eigenfunctions.** The polynomial eigenfunctions of $\mathscr{L}_c$ may be found in the same way as before. Write the Taylor's expansion as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

and likewise for $h = \mathscr{L}_c f$. Substituting and matching monomial terms gives

$$\frac{h^{(m)}(0)}{m!} = \sum_{k=0}^{\infty} C_{mk} \frac{f^{(k)}(0)}{k!}$$

where the matrix elements are given by

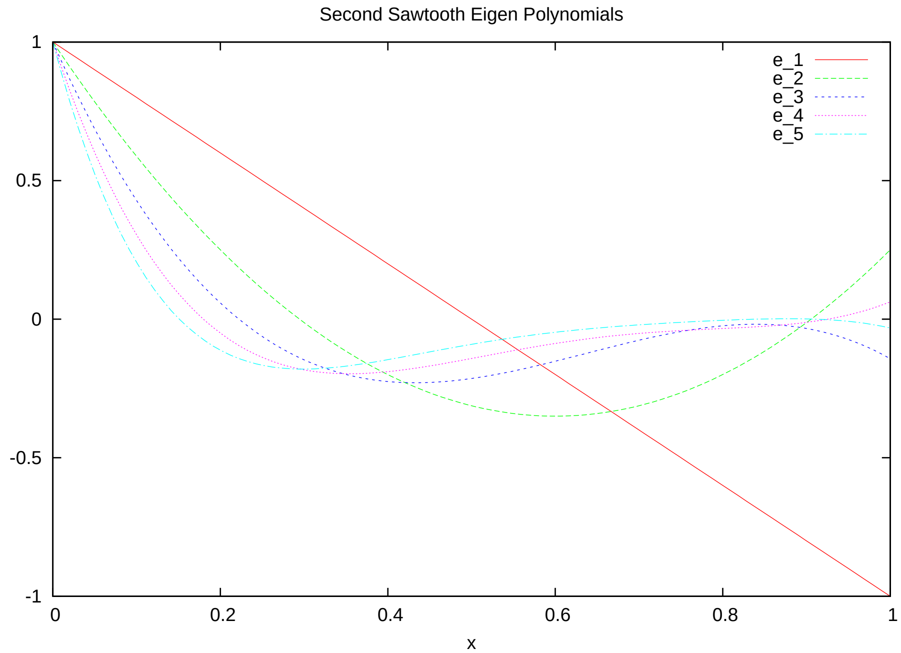
$$C_{mk} = (-1)^m \binom{k}{m} \frac{2^{k-m}}{(2^{k+1}-1)} \tag{4.2}$$

for $k \geq m$ and zero otherwise. This matrix is upper-triangular and thus solvable. Because it is solvable, there is no advantage gained by performing the polynomial expansion at points other than $x = 0$; the matrix cannot become "more solvable". A quick exploration at $x = 1$ does not suggest that the matrix becomes "more diagonal" (*i.e.* more heavily weighted near the diagonal).

As is the case for upper-triangular matrices, the eigenvalues lie along the diagonal. The first few are $\lambda_0 = 1$, $\lambda_1 = -1/3$, $\lambda_2 = 1/7$, *etc.* with the ratio of successive eigenvalues tending to -2. The first few eigenvectors are

$$e_0 = 1$$

FIGURE 4.3. Dyadic Sawtooth Polynomials



This figure shows the eigenvectors of the dyadic sawtooth, as given by equation 4.3. The normalization is such that $e_n(x=0)=1$.

$$e_1 = 2x-1$$

$$e_2 = 4-18x+15x^2$$

$$e_3 = -7+48x-84x^2+44x^3$$

$$e_4 = 16-\frac{5400}{37}x+\frac{14280}{37}x^2-\frac{15300}{37}x^3+\frac{5865}{37}x^4 \tag{4.3}$$

which solve the eigenvector equation $\mathscr{L}_c e_n = \lambda_n e_n$. These are illustrated in figure 4.3.

Because the matrix is upper-triangular, the eigenvectors can be solved for directly, simply by making the ansatz

$$e_n(x) = \sum_{k=0}^{n} e_n^{(k)} \frac{x^k}{k!}$$

substituting directly into eqn 4.2, and solving. The result is a recursion relation

$$\frac{e_n^{(k)}}{k!} = \frac{(-1)^n}{2^k} \frac{\left(2^{n+1}-1\right)\left(2^{k+1}-1\right)}{\left((-1)^k\left(2^{k+1}-1\right)-(-1)^n\left(2^{n+1}-1\right)\right)} \sum_{p=k+1}^{n} \binom{p}{k} \frac{2^p}{2^{p+1}-1} \frac{e_n^{(p)}}{p!}$$

4.2. **The Failure of the Similarity Transform for the Polynomial Basis.** Under normal circumstances, whenever one has a pair of maps $\alpha(x)$ and $\beta(x)$ that are conjugate to each other through a smooth, invertible function $\phi(x)$ such that $\beta(x) = (\phi \circ \alpha \circ \phi^{-1})(x)$, then there exists a similarity transform S_ϕ such that the Frobenius-Perron operators are also conjugate; that is, $\mathscr{L}_\beta = S_\phi \mathscr{L}_\alpha S_\phi^{-1}$ with $S_\phi^{-1} = S_{\phi^{-1}}$. The transform S_ϕ is given by $S_\phi f = \left(f \circ \phi^{-1}\right) / \left|\phi' \circ \phi^{-1}\right|$, where the prime denotes differentiation: $\phi'(x) = d\phi(x)/dx$. A detailed derivation of this is given in appendix B. Since the continued-fraction shift function is conjugate to the sawtooth, one might hope that GKW would be conjugate to $\mathscr{L}_c$, that is, $\mathscr{L}_c = S_? \mathscr{L}_h S_?^{-1}$. Unfortunately, the Minkowski Question Mark is highly singular and is not traditionally differentiable, and so we cannot build such a similarity transform using the polynomial function basis. Another way to deduce this is to note that the similarity transform S_ϕ, working as an ordinary, bounded operator, normally preserves the eigenvalues of a nuclear operator; that is, the eigenvalues of $\mathscr{L}_\alpha$ equal those of $\mathscr{L}_\beta$. In the current case, we see trouble in that the eigenvalues of $\mathscr{L}_c$ are not those of GKW. They are not even 'close', in that for large k, the ratio of the eigenvalues λ_k/λ_{k+1} tends to -2, whereas, for the GKW, the ratio is 2.61803... (the square of golden mean, see [12]).

Although the manipulations required to construct eigenfunctions by means of similarity transforms break down when one insists on working in the polynomial basis, this is not the case if one considers a larger basis of functions. By properly defining the derivative, one can validly obtain solutions to the GKW eigen-equation from solutions to the dyadic sawtooth. This is developed in the next section.

4.3. **Constructing GKW Eigenfunctions.** The manipulations required to define the conjugacy operator that makes GKW and the dyadic sawtooth conjugate require manipulation of the derivative of the Minkowski question mark function. This derivative is not well-defined if one sticks to the natural topology on the real number line, but it can be given a precise expression by working on the Cantor set. Insofar as the Cantor set is in many ways nearly equivalent to the real number line, and that there are many operations that can be "safely" carried over from one to the other, one can thus have a workable set of tools for operating on an object that, for all "practical purposes" can be identified with the derivative of the question mark.

The primary theorem and mechanics establishing this is given in [30], and briefly recapitulated here. The Cantor set can be written as the infinite product $\{0,1\}^{\mathbb{N}}$ of the set $\{0,1\}$ of two elements. As such, points in this product space are naturally identified with real numbers expanded in base-2 or binary – a real number may be written as an infinity long string of zeros and ones. So:

$$x = \sum_{n=1}^{\infty} \frac{b_k}{2^k} \tag{4.4}$$

where $b_k \in \{0,1\}$ is the k'th binary digit in the expansion of the real number x. The natural topology for a Cartesian product space is the so-called "product topology"; it's open sets are called "cylinder sets". For the Cantor set, a basis for the topology is just given by the standard Rademacher functions[7] of Banach Space theory. That is, let $\sigma \in \{0,1\}^{\mathbb{N}}$be an infinite string of binary digits; let b_k, or, alternately, σ_k be the k'th digit in this string. Consider the set of all possible strings, but with one digit held constant. There are two such sets,

$$C_{k,0} = \{\sigma | \sigma_k = 0\} \quad \text{and} \quad C_{k,1} = \{\sigma | \sigma_k = 1\}$$

These two sets are open in the product topology, and are complementary: $C_{k,0} \cap C_{k,1} = \varnothing$ and $C_{k,0} \cup C_{k,1} = \{0,1\}^{\mathbb{N}}$. The collection of these sets form a basis for the product topology; the topology itself is the set of all finite intersections and arbitrary unions of these sets. It is not hard to see that the indicator functions for these sets are (up to a constant) the Rademacher functions. Thus, for example:

$$\begin{aligned} \mathrm{id}_{C_{k,0}}(x) &= \begin{cases} 0 & \text{if } b_k = 1 \\ 1 & \text{if } b_k = 0 \end{cases} \\ &= \frac{1 + \operatorname{sgn}\left(\sin\left(2^k \pi x\right)\right)}{2} \end{aligned}$$

is just a square wave.

The open sets of the product topology on the Cantor set form a sigma algebra on the real-number line; they are Borel sets, and are suitable for constructing measures on the real-number line. Let $\mathscr{A}$ denote this sigma algebra, and consider a set function $\mu : \mathscr{A} \to [0,\infty]$. When this function μ obeys certain properties, such as sigma-additivity, it becomes a measure. The above collection of definitions and maps allow the construction of a measure $?'$ on the unit interval, that, for "all practical purposes", behaves as the derivative of the Minkowski question mark. This measure is constructed by considering the one-to-one correspondence between the dyadic binary tree and the tree of Farey fractions. These trees are identical in structure, and differ only in the labels assigned to their nodes: in the one case, dyadic fractions, and in the other, rationals. As they differ only by labels, then the extension theorem for measures applies. The function that maps the labels from the one tree to the other is just the Minkowski question mark; resulting measure is the definition of $?'$. It is illustrated in figure 4.2.

That this measure deserves to be identified with, or thought of as the "derivative" of the question mark becomes apparent when one switches over to classical notation: one has that

$$\int_a^b ?'(x)\, dx = ?(b) - ?(a)$$

The above skims over a fair amount of machinery from measure theory; the upshot, however, is to defend the use of the simpler, classical notation. Thus, in what follows, we will write $?'(x)$ as if it were well-defined for a real number x. It is not. However, by always reverting to the language of measure theory, and always working with the cylinder sets formed on the Cantor set, and then considering a filter of cylinder sets converging to the point x given by equation 4.4, then one always has an object $?'(x)$ that behaves more or less as if it were a classical function, and can be more or less safely manipulated as one.

For what follows, the most important property of this measure is that it too has a set of self-similarity properties, induced by the action of the dyadic monoid on the question mark. Of the various self-similarities, the most important one is

$$?'\left(\frac{1}{n+x}\right) = \frac{(x+n)^2}{2^n} ?'(x)$$

which follows from

$$?\left(\frac{1}{n+x}\right) = \frac{2 - ?(x)}{2^n}$$

This may be directly employed with the GKW operator to great effect. Suppose that f is an eigenfunction of the transfer operator $\mathscr{L}_c$ of the dyadic sawtooth with eigenvalue λ; that is, $\mathscr{L}_c f = \lambda f$. Then $q = ?' \cdot f \circ ?$ is an eigenfunction of the GKW operator $\mathscr{L}_h$, so that $\mathscr{L}_h q = \lambda q$. The functions f and q that can be manipulated in this way are in general functions on the Cantor set, and not, strictly speaking, functions on the unit interval, except insofar as equation 4.4 allows the Cantor set and the unit interval to be confused with one-another. That is, rather than imagining that $f : [0,1] \to \mathbb{R}$, one should instead keep in mind $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ or even possibly that f is a set function on the product topology (equivalently, a set function on a sigma algebra), as needed. Considered as functions on the unit interval, these will typically be non-differentiable on the rationals or the dyadic fractions; or they may even be discontinuous at these points. Although one is tempted to say them, it is important to avoid phrases like "differentiable nowhere" or "discontinuous everywhere", as these are misleading: the discontinuities happen only on the rationals, and the rationals are simply not that big a set. Many examples of such eigenfunctions are developed in the sections that follow.

By using either a sigma algebra or a topology as the base space for a function space, the notion of the transfer operator changes as well. That is, when one focuses on point dynamics, then one might define a function space as $\mathscr{F} = \{f \mid f : [0,1] \to \mathbb{R}\}$, that is, as the set of all functions on the unit interval. Lets call this space $\mathscr{F}_p$ with the subscript p emphasizing that the base space is a point-set. The "ordinary" transfer operator is then a map between such spaces $\mathscr{F}_p$, as previously defined. However, the measure-theoretic approach instead requires that the function space be defined as $\mathscr{F}_\sigma = \{f \mid f : \mathscr{A} \to \mathbb{R}\}$ where $\mathscr{A}$ is a sigma algebra. Here, the subscript σ is a reminder that the base space for this $\mathscr{F}$ is a sigma algebra. Note that, in a certain sense, the elements of $\mathscr{F}_p$ are "derivatives" of corresponding elements of $\mathscr{F}_\sigma$, and that the space $\mathscr{F}_\sigma$ is much "larger" than $\mathscr{F}_p$. In particular, the elements of $\mathscr{F}_\sigma$ can be thought of as integrals, over some set $U \in \mathscr{A}$, of the elements of $\mathscr{F}_p$. The elements of $\mathscr{F}_\sigma$ are sometimes called "generalized functions", in that the Dirac delta function finds a comfortable home there: $\delta(x)$ is simply the indicator function for all sets U that contain the point x. To relate these two spaces, one then typically uses the language of topological filters to establish a many-to-one mapping of elements of $\mathscr{F}_\sigma$ to $\mathscr{F}_p$. One curious thing happens to the transfer operator when working with $\mathscr{F}_\sigma$ instead of $\mathscr{F}_p$: the transfer operator is now nothing more than a pushforward. This interpretation gives a nice theoretical grounding; pushforwards are fairly well understood category-theoretic objects. This preceding paragraph is again a whirlwind review of some concepts from topology and measure theory; more concrete details can be found in textbooks such as [16], while theorems that recast the transfer operator ass a pushforward are given in [30, 24].

Keeping in mind that the proper setting for both the transfer operator and for the Minkowski measure $?'$ is on sigma algebras, we none-the-less revert to classical point-set notation to supply a proof that a similarity transform connects solutions of the dyadic sawtooth and the GKW operator:

$$\begin{aligned}
\left[\mathscr{L}_h ?' \cdot f\circ ?\right](x) &= \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}\, ?'\left(\frac{1}{x+n}\right)(f\circ ?)\left(\frac{1}{x+n}\right) \\
&= ?'(x) \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\frac{2-?(x)}{2^n}\right) \\
&= ?'(x)\cdot \left[\mathscr{L}_c f\right](?(x)) \\
&= \lambda\, ?'(x)\, f(?(x)) \\
&= \lambda \left(?' f\circ ?\right)(x)
\end{aligned}$$

The above point-set notation "works", in that, at each step, there is a corresponding measure-theoretic expression for which the equality holds. Notationally, this classical point-set notation is simply easier to read for a wider audience, without loosing the overall fidelity of the argument. Given an eigenfunction f of $\mathscr{L}_c$, the practical way to work with this will be to work with the quantity

$$\int_a^b \left(?' f \circ ?\right)(x)\,dx = \int_{?(a)}^{?(b)} f(y)\,dy$$

The quantity on the left may be understood to denote an element of $\mathscr{F}_\sigma$ valued on the set $U = [a,b] \in \mathscr{A}$, while that on the right can be taken as an ordinary, college-calculus integral, for which the usual manipulations are allowed. Both sides of the above equality are understood to give an eigenfunction of the GKW operator. In general, this eigenfunction will, in practice, be found to be discontinuous on the rationals. However, suppose instead one now had not just a single eigenfunction, but in fact a complete set of them, spanning a suitable function space. One may then ask if there are linear combinations of these that are continuous, or even differentiable, with regards to the natural topology on the unit interval. This last question then spells out the program for the subsequent sections.

A practical method for picking out the continuous eigenfunctions out of the haystack of all possible eigenfunctions is to look for the bounded eigenfunctions. Specifically, one would look for those f which satisfy

$$\lim_{b-a\to 0} \frac{1}{b-a} \int_{?(a)}^{?(b)} f(y)\,dy < \infty$$

This is harder than it seems, because, as a approaches b, then $?(a)$ approaches $?(b)$ exponentially fast when a and b straddle a rational number, while 'staying apart' exponentially fast when a and b straddle a quadratic irrational. Thus, f has to be very "badly behaved" in order for $?' f \circ ?$ to be well-behaved.

4.4. **Axiom of Choice.** A few words are in order about the axiom of choice. First, recall its definition. Given an indexed family of sets $\{E_\alpha\}_{\alpha\in A}$, with each of the sets E_α non-empty, the axiom states that the Cartesian product $\prod_{\alpha\in A} E_\alpha$ is also non-empty, or more specifically, that one can choose an element that belongs to the product. If we take each E_α to be the set $\{0,1\}$, then the Cartesian product can be understood to be a real number; selecting a specific real number requires the exercise of the Axiom of Choice. Similarly, if we take each $E_\alpha$to be $\mathbb{N}$, then the Cartesian product can be understood as a continued fraction – again, a specific point on the real number line. By contrast, employing the cylinder set topology can avoid an appeal to the Axiom of Choice: to define a cylinder set, one need only pick a finite number of explicit elements (which can always be done); the rest of the Cartesian product is left indefinite, and does not require an infinite number of choices.

4.5. **The Kernel of the Dyadic Sawtooth.** The kernel of the dyadic sawtooth is defined as the set of functions k such that $\mathscr{L}_c k = 0$. It is clear that none of the polynomial eigenfunctions belong to the kernel; the polynomial spectrum is discrete, and zero is not a part of the spectrum. However, if one considers a larger set of functions, say the square-integrable functions, then the kernel is readily demonstrated. Let

$$k_n(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2^{n+1}} \\ -1 & \text{for } \frac{1}{2^{n+1}} \le x < \frac{1}{2^n} \\ 0 & \text{for } \frac{1}{2^n} \le x \le 1 \end{cases}$$

then clearly one has that $\mathscr{L}_c k_n = 0$ for all $n \geq 0$. Clearly, each k_n is linearly independent of the others. Note that the k_n shift under a doubling of the argument: $k_n(2x) = k_{n+1}(x)$, or alternately, $k_n \circ g_D = k_{n-1}$ where g_D is as defined in 2.5.

4.6. **Fractal Eigenfunctions of the Dyadic Sawtooth.** The Takagi curve can be used to build an alternate set of eigenfunctions for the dyadic sawtooth, possessing continuous-spectrum eigenvalues. These eigenfunctions are not differentiable, and thus cannot be obtained through polynomials, and thus are not visible when working with the operator in a polynomial-basis Hilbert Space. They can be used to build an alternate function space, in which the dyadic sawtooth remains exactly solvable.

The Takagi curve has a set of self-similarities generated by the dyadic monoid[27]; this monoid was already introduced above, in the paragraph surrounding eqn 2.5. One of the elements from the monoid is

$$\left[g_D^{k-1} r_D g_D\right](x) = \frac{1}{2^{k-1}} - \frac{x}{2^k} \tag{4.5}$$

and it appears in the definition 4.1 of the transfer operator:

$$\left[\mathscr{L}_c f\right](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\left[g_D^{n-1} r_D g_D\right](x)\right)$$

From this, one may surmise that functions $f(x)$ that are self-similar under the dyadic monoid might be used to construct solutions to the operator $\mathscr{L}_C$. In this case, the candidates are of course the family of Takagi curves.

The Takagi curve, shown in figure 4.4, may be defined as

$$t_w(x) = \sum_{k=0}^{\infty} w^k \tau\left(2^k x - \left\lfloor 2^k x \right\rfloor\right)$$

where $\tau(x)$ is the triangle wave:

$$\tau(x) = \begin{cases} 2x & \text{when} \quad 0 \leq x \leq 1/2 \\ 2(1-x) & \text{when} \quad 1/2 \leq x \leq 1 \end{cases}$$

This curve is self-similar to itself under the dyadic monoid generated by the two elements g and r. The specific self-similarity is given by a 3-dimensional matrix representation of this monoid. Writing t_w as a part of a 3-vector:

$$\left(\begin{array}{c} 1 \\ x \\ t_w(x) \end{array}\right)$$

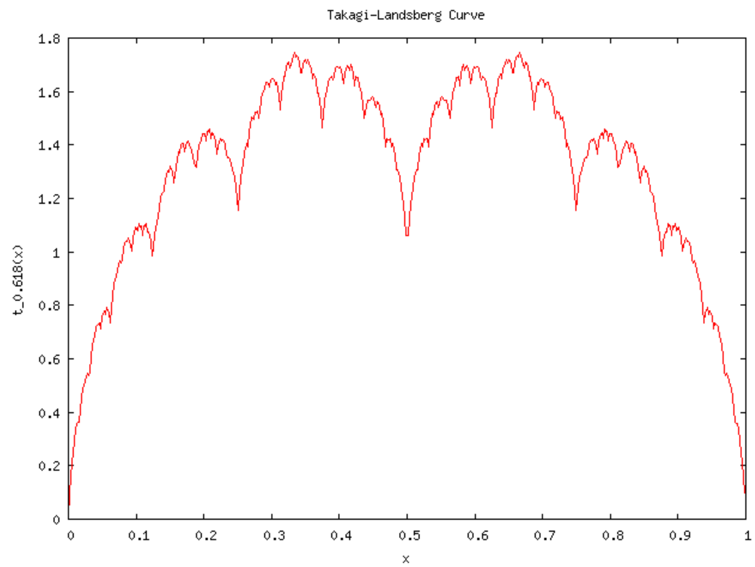
one finds that the monoid acts on the vector with the matrix transformations

$$g_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & w \end{array}\right) \quad \text{and} \quad r_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right) \tag{4.6}$$

The self-similarity may be expressed as a monoid action isomorphism: $t_w \circ g_D = g_3 t_w$ and likewise $t_w \circ r_D = r_3 t_w$. To obtain the behaviour of t_w when inserted into eqn 4.1, one needs the action of the element $g^{k-1} r g$ so as to obtain the monoid action isomorphism $t_w g_D^{k-1} r_D g_D = g_3^{k-1} r_3 g_3 t_w$. This can be assembled in pieces. First, note that

$$g_3^n = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2^{-n} & 0 \\ 0 & q_n(w) & w^n \end{array}\right)$$

FIGURE 4.4. Takagi Curve



where $q_n(w)$ is the polynomial

$$q_n(w) = \frac{1}{2^{n-1}} \sum_{k-0}^{n-1} (2w)^k = \frac{1}{2^{n-1}} \left(\frac{1-(2w)^n}{1-2w} \right)$$

Multiplying by r_3 and $g_3$and applying to the vector, one obtains

$$t_w\left(\frac{1}{2^{k-1}} - \frac{x}{2^k}\right) = q_{k-1}(w) + x\left(w^{k-1} - q_{k-1}(w)/2\right) + w^k t_w(x)$$

Inserting the above back into the definition 4.1 for the sawtooth operator, and performing the sum, one obtains

$$\left[\mathscr{L}_c t_w\right](x) = \frac{4}{3(2-w)} + \frac{x}{3(2-w)} + \frac{w t_w(x)}{2-w}$$

From this, we can immediately read off the eigenvalue as $w/(2-w)$. To get the eigenfunction, we need to complete the diagonalization by using $\left[\mathscr{L}_C 1\right](x) = 1$ and $\left[\mathscr{L}_C x\right](x) = (2-x)/3$ to get the eigenfunction

$$E_2(x) = \frac{2-w}{2(w+1)(w-1)} + \frac{x}{2(w+1)} + t_w(x)$$

The above is not the only fractal solution that transforms under the three-dimensional representation of the dyadic monoid. A complete set of linearly independent solutions spanning the space are constructed from

$$t_{w,l}(x) = t_w((2l+1)x)$$

Theorem: above provide a complete set spanning the space. Proof: XXX details to be done. The proof is special case/variant from theory of shift operators: take the kernel, apply the shift ad-infinitum, the union of the resulting spaces is the full space (this holds true in general for shifts). More narrowly: Sketch of proof is that these can be re-expressed as unique linear combinations of cosine waves; the specific linear combinations being given in [24]. The transform is invertible: $\cos 2\pi nx$ can be expressed as a unique linear combination of $t_{w,l}$. By contrast, certain linear combinations of $\sin 2\pi nx$ are used to construct the kernel of $\mathscr{L}_c$. Since sine and cosine span the space L_2 of square-integrable functions on the unit interval, the above do likewise. An alternate way of reaching the same conclusion is to consider the Haar basis functions for Banach spaces on the unit interval, which are known to form a complete set. One then notes that the triangle wave is just an integral of the Haar basis functions. QED.

It should be clear, from the above presentation, that results are possible for other Takagi curves, constructed from piece-wise polynomials, are possible. In general, a Takagi curve constructed from a polynomial of order n transforms under an $n+1$ dimensional representation of the dyadic monoid[27]. The next section develops some the general framework for these solutions.

4.7. **Practical Tools.** Practical computations with the complete set of fractal eigenfunctions require additional tools. These are developed here. If $f(x)$ is a function possessing dyadic self-similarity, then given γ_D as some product of g_D and r_D, and f self-similar under the action of g_f and r_f, then one has the commuting diagram $\gamma_f f = f \circ \gamma_D$. Computations with $\mathscr{L}_c$ require the evaluation of expressions of the form $f((2l+1)(2-x)2^{-n})$. If the argument is re-expressed in terms of the transforms g_D and r_D, analogously to eqn 4.5, then one may use the the commutation relation $\gamma_f f = f \circ \gamma_D$ to easily compute eigenvectors of $\mathscr{L}_c$. This motivates the development of the machinery below.

To perform this re-write, it is sufficient to re-express $(M-y)2^{-n}$ in terms of g_D and r_D. We begin by noting that $(M-y)2^{-n} = (M-1+r_D y)2^{-n}$ since $r_D y = 1-y$. Next, one needs the binary expansion for $M-1$; so write

$$M-1 = \sum_{k=0}^{\infty} \beta_k 2^k \tag{4.7}$$

with $\beta_k \in \{0,1\}$ the k'th bit in the base-2 representation. Of course, all $\beta_k = 0$ for $k > \log_2(M-1)$. The desired expression is

$$\frac{M-y}{2^n} = B_n B_{n-1} \cdots B_1 B_0 r_D y$$

where each B_k is given by

$$B_k = \begin{cases} L & \text{for } \beta_k = 0 \\ R & \text{for } \beta_k = 1 \end{cases}$$

and L and R are left and right sub-trees of the dyadic tree, respectively:

$$L(x) = g_D(x) = \frac{x}{2} \quad \text{and} \quad R(x) = (r_D g_D r_D)(x) = \frac{1+x}{2}$$

Functions written next to one-another are understood to mean function composition: $pqrx = (p \circ q \circ r)(x) = p(q(r(x)))$. For a fixed value of $M-1$, the transfer operator for the dyadic sawtooth may then be written as

$$\mathscr{L}_c f = \frac{1}{2} f B_0 r + \frac{1}{4} f B_1 B_0 r + \frac{1}{8} f B_2 B_1 B_0 r + \cdots$$

If the function f is self-similar under the dyadic monoid, *i.e.* if it commutes with L, R then the above provides a convenient, relatively simple way of explicitly computing the action of $\mathscr{L}_c$. *i.e.* one has

$$\mathscr{L}_c f = \left[\frac{1}{2}B_0 r + \frac{1}{4}B_1 B_0 r + \frac{1}{8}B_2 B_1 B_0 r + \cdots\right] f$$

which is straight-forward to evaluate when the B_k are square matrices, such as, for example, those of eqn 4.6.

Several examples can help ground and clarify this. Consider, for example $(6-y)/8$. The binary expansion for $6-1=5$ is 101 and so one has

$$\frac{6-y}{8} = RLRry = rgr\, g\, rgr\, ry = (rg)^3 y$$

The bits to the left are all zero, and so one has

$$\frac{6-y}{2^{n+3}} = L^n RLRry = g^n (rg)^3 y$$

For readability, the subscript D was dropped above two equations.

Consider then the action of $\mathscr{L}_c$ on $f_l(x) = f((2l+1)x)$ for $l=1$ and $0 \le x \le 1/3$. One then has

$$\begin{aligned}
[\mathscr{L}_c f_1](x) =& \frac{1}{2} f\left(\frac{6-3x}{2}\right) + \frac{1}{4} f\left(\frac{6-3x}{4}\right) + \frac{1}{8} f\left(\frac{6-3x}{8}\right) + \cdots \\
=& \frac{1}{2} f\, Rr\, 3x + \frac{1}{4} f\, LRr\, 3x + \frac{1}{8} f\, RLRr\, 3x + \cdots + \frac{1}{2^n} f\, L^{n-3} RLRr\, 3x + \cdots
\end{aligned}$$

while, for $l=1$ and $\frac{1}{3} \le x \le \frac{2}{3}$, one uses RLL:

$$\begin{aligned}
[\mathscr{L}_c f_1](x) =& \frac{1}{2} f\left(\frac{5-3\left(x-\frac{1}{3}\right)}{2}\right) + \frac{1}{4} f\left(\frac{5-3\left(x-\frac{1}{3}\right)}{4}\right) + \frac{1}{8} f\left(\frac{5-3\left(x-\frac{1}{3}\right)}{8}\right) + \cdots \\
=& \frac{1}{2} f\, Lr\, 3\left(x-\frac{1}{3}\right) + \frac{1}{4} f\, LLr\, 3\left(x-\frac{1}{3}\right) + \frac{1}{8} f\, RLLr\, 3\left(x-\frac{1}{3}\right) + \cdots \\
& \cdots + \frac{1}{2^n} f\, L^{n-3} RLLr\, 3\left(x-\frac{1}{3}\right) + \cdots
\end{aligned}$$

So, for example, if $f = t_{w,l}$ the Takagi function defined in the previous section, one then has, for $k \ge 3$, $l=1$ and $0 \le x < 1/3$, that

$$t_{w,1}\left(\frac{2-x}{2^k}\right) = t_w\left(3\left(\frac{2-x}{2^k}\right)\right) = g^{k-3}(rg)^3 t_w(3x) = g^{k-3}(rg)^3 t_{w,1}(x)$$

This method is applied in earnest in the next section.

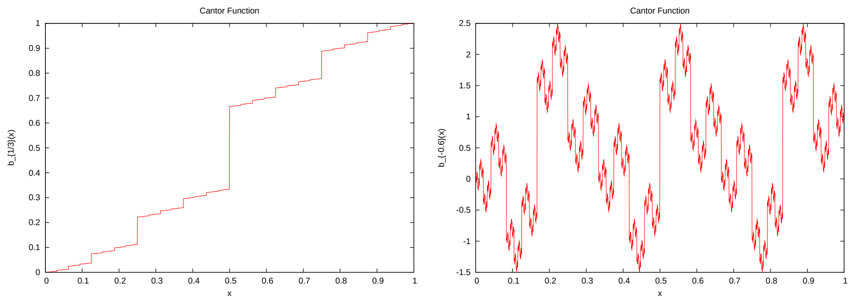
4.8. **The Two-Dimensional Representation.** The Cantor function transforms under a two-dimensional representation of the dyadic monoid. It is constructed from the step function

$$b(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le x < 1 \end{cases}$$

and a weight w (equal to 1/3 for the canonical Cantor function):

$$b_w(x) = (1-w)\sum_{n=0}^{\infty} w^n b\left(2^n x - \lfloor 2^n x \rfloor\right)$$

FIGURE 4.5. Cantor Function



Two examples of the Cantor function $b_w(x)$. The figure on the right shows the "canonical" cantor function obtained by removing middle-thirds, and so shows $b_{1/3}(x)$. The figure on the left shows $b_{-0.6}(3x)$.

This function is shown in figure 4.5. It is self-similar under the action of the dyadic monoid: the generators are $b_w(x/2) = wb_w(x)$ and $b_w(1-x) = 1-b_w(x)$. As a matrix representation, one has

$$1 \to e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$t_w \to e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that

$$g_b = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}$$

$$r_b = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

are the generators of the dyadic monoid in this representation. Applying the previous developments, one then has that

$$[\mathscr{L}_c b_w](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} b_w\left(\frac{2-x}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_b^{n-1} r_b g_b b_w(x)$$

Since

$$g_b^{n-1} r_b g_b = \begin{bmatrix} 1 & 0 \\ w^{n-1} & -w^n \end{bmatrix}$$

one promptly obtains that

$$[\mathscr{L}_c b_w](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(w^{n-1} - w^n b_w(x)\right) = \frac{1-wb_w(x)}{2-w}$$

From this, one can then promptly obtain an eigenfunction $e_w = b_w - 1/2$ which satisfies

$$\mathscr{L}_c e_w = \frac{-w}{2-w} e_w$$

Additional eigenfunctions transforming according to the same representation are given by $b_{w,l}(x) = b_w((2l+1)x)$. Applying the techniques above, one finds that

$$\mathscr{L}_c b_{w,l} = \frac{1}{2-w} c_{w,l} - \frac{w}{2-w} b_{w,l}$$

where $c_{w,l}$ is piecewise-constant depending on w and l . Eigenfunctions are then given by

$$e_{w,l} = \frac{-c_{w,l}}{2} + b_{w,l}$$

which satisfy

$$\mathscr{L}_c e_{w,l} = \frac{-w}{2-w} e_{w,l}$$

We've already seen that $c_{w,0} = 1$. For $l = 1$, one has

$$c_{w,1}(x) = \begin{cases} \frac{5-w}{4} & \text{for } 0 \le x < \frac{1}{3} \\ \frac{1+3w}{4} & \text{for } \frac{1}{3} \le x < \frac{2}{3} \\ \frac{3-w}{2} & \text{for } \frac{2}{3} \le x < 1 \end{cases}$$

For $l = 2$ one has

$$c_{w,2}(x) = \begin{cases} \frac{9-w}{8} & \text{for } 0 \le x < \frac{1}{5} \\ \frac{1+7w}{8} & \text{for } \frac{1}{5} \le x < \frac{2}{5} \\ \frac{7-3w}{4} & \text{for } \frac{2}{5} \le x < \frac{3}{5} \\ \frac{3+w}{4} & \text{for } \frac{3}{5} \le x < \frac{4}{5} \\ \frac{5-w}{4} & \text{for } \frac{4}{5} \le x < 1 \end{cases}$$

The general expression is given by the piece-wise flat function

$$c_{w,l}(x) = w + (1-w) \sum_{k=0}^{\infty} \frac{\beta_k}{2^k} \qquad \text{for } \frac{4l+1-(M-1)}{2l+1} \le x < \frac{4l+2-(M-2)}{2l+1}$$

where $M-1$ and the binary bits β_k are related as given in eqn 4.7. The above formula may be derived by applying the techniques from section 4.7, commuting so as to use the 2×2 matrices $L_b = g_b$ and $R_b = r_b g_b r_b$, and expanding in powers of w, and tracking each power separately. Note that the sum has the effect of "reversing" the order of the bits in the binary expansion for $M-1$. Note that $\int_0^1 c_{w,l}(x)\,dx = 2l+1$. In computing the above, it is useful to have a certain polynomial: let γ be a product of L and R, so that

$$\gamma = L^{m_0} R^{m_1} L^{m_2} \cdots L^{m_{N-1}} R^{m_N}$$

Then one has that

$$\gamma_b(w) = \begin{bmatrix} 1 & 0 \\ q_\gamma(w) & w^M \end{bmatrix}$$

where $M = m_0 + m_1 + \cdots + m_N$ and $q_\gamma$is the polynomial

$$q_\gamma(w) = w^{m_0} - w^{m_0+m_1} + w^{m_0+m_1+m_2} - \cdots + (-1)^N w^M$$

Since the m_k are counting the number of repeated digits in a binary expansion, we once again see the insidious presence of integer series (and thus the connection to continued fractions, these being given by integer series).

4.9. **The three dimensional representation.** As noted above, the Takagi curve transforms under a 3D representation of the dyadic monoid. A few more factoids about this are below. Under the action of $\mathscr{L}_c$, the Takagi curves transform as

$$\left[\mathscr{L}_c t_{w,l}\right](x) = \frac{1}{2-w}\left[\alpha_l(w) + x\beta_l(w) + w t_{w,l}(x)\right]$$

where $\alpha_l(w)$ and $\beta_l(w)$ are polynomials in w. The $l=0$ case was already given above; for $l=1$ and $0 \le x < 1/3$, one has

$$\alpha_1(w) = \frac{3}{4} \text{ and } \beta_1(w) = \frac{19}{24}$$

which is obtained using the methods above. The corresponding eigenfunctions are given by

$$E_{w,l}(x) = \frac{-1}{1-w}\left[\frac{\alpha_l(w)}{2} + \frac{1}{3}\right] + x\frac{3}{2}\frac{\beta_l(w)}{1+w} + t_{w,l}(x)$$

so that

$$\left[\mathscr{L}_c E_{w,l}\right](x) = \frac{w}{2-w} E_{w,l}(x)$$

Since the eigenvalue is $w/(2-w)$ as before, the general solution may be written as a linear combination of these solutions.

Previously, we saw that the Takagi Curves served as basis vectors for a space of degenerate eigenfunctions of the Bernoulli Map, associated with arbitrary eigenvalue. We saw that this space could also be spanned by the Hurwitz Zeta, through a change of basis. Thus, we expect that we can extend these results to this map as well. That is, there is a linear combination of the $E_{w,l}$ that is differentiable in x for each w.

4.10. **Walsh functions.** The Walsh functions provide an orthonormal basis for the Banach space $L_2[0,1]$ of square-integrable functions on the unit interval[7]. These can be used to construct eigen-solutions to $\mathscr{L}_c$. This section defines the Walsh functions, and constructs a set of self-similar functions from them, transforming under a finite-dimensional representation of the dyadic monoid. These are in turn used to construct the eigen-solutions.

The Walsh functions are built from the Rademacher functions. Define the step function

$$s_1(x) = \begin{cases} +1 & \text{for } 0 \le x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

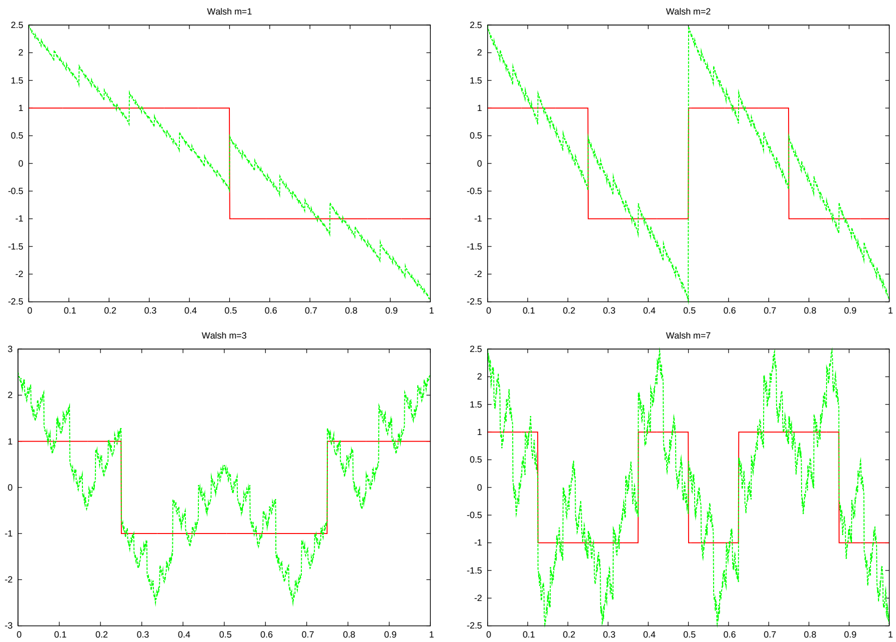
and extend it to the whole real-number line by making it be periodic (*i.e.* so that its a square wave). The Rademacher functions $r_n(x)$ are defined as $r_n(x) = s_1(2^n x)$. Given a finite sequence of positive integers $n_1 < n_2 < n_3 < \cdots < n_k$, the corresponding Walsh function is defined as the product of Rademacher functions $r_{n_1} r_{n_2} r_{n_3} \cdots r_{n_k}$. It is not hard to see that these are orthonormal on the unit interval; indeed, the r_n themselves are orthonormal, in that

$$\int_0^1 r_m(x)\, r_n(x)\, dx = \delta_{nm}$$

These may be parametrized by means of an integer. Let m be an integer with the binary expansion

$$m = \sum_{k=1}^{\infty} \beta_k 2^{k-1}$$

FIGURE 4.6. Walsh Functions



The above figures show the Walsh functions s_m, in red, and the self-similar fractals ρ_m (defined in eqn. 4.8), in green, for $m = 1,2,3,7$. The value of $w = 0.6$ was used in the sum for the fractal curves.

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with each $\beta_k \in \{0,1\}$. Implicit is that there is some upper bound k above which all the β_k are zero. Then define the m'th Walsh function s_m as

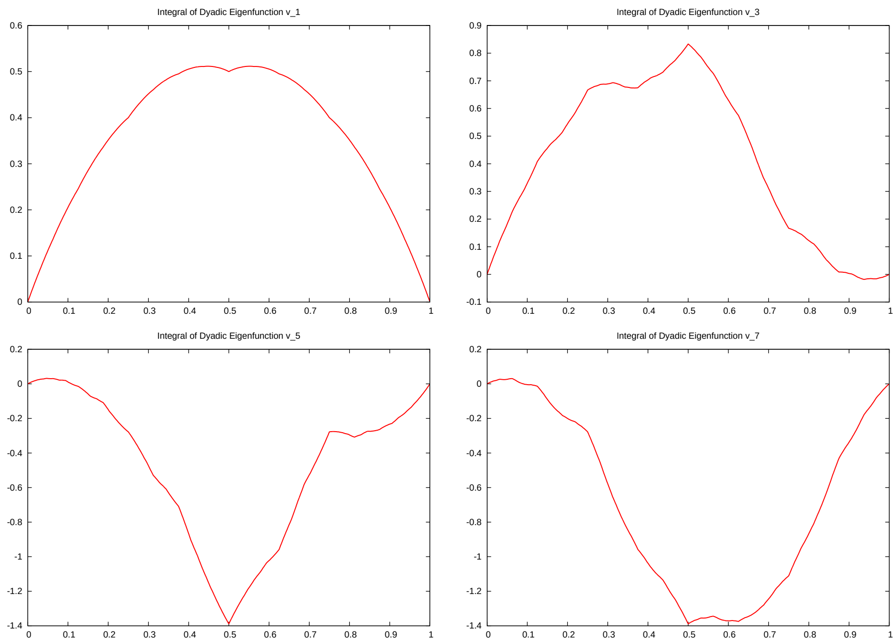
$$s_m(x) = \prod_{k=0}^{\infty} r_k^{\beta_k}(x)$$

In essence, if $\beta_k = 1$, then r_k appears in the product; if $\beta_k = 0$, then r_k does not. These are orthonormal, in that $s_m^2 = 1$, and

$$\int_0^1 s_m(x)\, s_n(x)\, dx = \delta_{nm}$$

and these form a complete basis for the space of square-integrable functions on the unit interval[7]. The Walsh functions have many curious properties; among those relevant here are that $s_m(2x) = s_{2m}(x)$ and that $s_m s_n = s_{m\oplus n}$ where $m \oplus n$ denotes the bit-wise XOR of the binary expansion of the two integers m and n, and $s_0(x) = 1$ for all x. That is, the non-negative integers form an abelian group under bit-wise XOR, and the Walsh functions are isomorphic, as a group. Treated as a product operation, it means that the Walsh functions can form not just a Banach space, but also a Banach algebra; but this algebra is just the usual ring of functions on the unit interval. The scaling property commutes with the XOR operation as well: $2(m \oplus n) = 2m \oplus 2n$. A sampling of these functions are shown in figure 4.6.

FIGURE 4.7. Integrals of Eigenfunctions



Figures showing integrals of the dyadic sawtooth eigenfunctions 4.9, for $w = 0.6$. Specifically, these figures show $\int_0^x v_k(y)\,dy$ for $k = 1, 3, 5, 7$. These integrals are easily computed, as the integrals of the Walsh functions are simply a sequence of upward or downward-pointing triangles or tents. Thus, the first figure is just an iterated triangle, leading to the blancmange or Takagi curve (corresponding to that shown in figure 4.4, but for $w = 0.3$; a factor of 1/2 resulting from integration). In general one has $\int_0^1 v_k(y)\,dy = 0$, which follows since the integral of any Walsh function is zero over the unit interval, and the v_k are simply linear combinations of the Walsh functions.

A set of fractal functions, corresponding 1-1 to the Walsh functions, and transforming under the dyadic monoid, are built as usual. Let

$$\rho_m(x) = \sum_{j=0}^{\infty} w^j s_m\left(2^j x\right) \tag{4.8}$$

It is not hard to see that

$$\rho_m\left(\frac{x}{2}\right) = s_{\left\lfloor \frac{m}{2} \right\rfloor}(x) + w\rho_m(x)$$

where $\lfloor v \rfloor$ denotes the floor of v (largest integer less than or equal to v). Under the action of reflection $r(x) = 1 - x$, the ρ_m is even or odd, depending on whether the number of 1 bits in m is even or odd. Let N be the smallest integer such that $m < 2^N$ (N is the length, in bits, of m, ignoring the infinite string of zero bits to the left; $N = \lfloor \log_2 m \rfloor + 1$). Then ρ_m transforms under an $N+1$ dimensional representation of the dyadic monoid, given by the basis

$$\begin{aligned} 1 &\to e_1 \\ s_{\left\lfloor \frac{m}{2^{N-1}} \right\rfloor} &\to e_2 \\ s_{\left\lfloor \frac{m}{2^{N-2}} \right\rfloor} &\to e_3 \\ &\cdots \\ s_{\left\lfloor \frac{m}{2} \right\rfloor} &\to e_N \\ \rho_m &\to e_{N+1} \end{aligned}$$

In this basis, the two monoid generators are

$$g = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & w \end{bmatrix} \quad \text{and} \quad r = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_N \end{bmatrix}$$

where $\sigma_k = \pm 1$ depending on the number of 1 bits in the binary expansion for $\left\lfloor m/2^{N-k} \right\rfloor$. To be precise, let $c_k = \sum_{j=N-k+1}^{N} \beta_j$ count the number of 1 bits in $\left\lfloor m/2^{N-k} \right\rfloor$; then $\sigma_k = (-1)^{c_k}$. Note that $\sigma_1 = -1$ always. This even/odd sign-changing behaviour is vaguely reminiscent of the Mobius function.

To obtain the action of $\mathscr{L}_c$ on ρ_m, one needs an expression for the matrix elements $g^{n-1} rg$. These are easily obtained, because of the simple, bi-diagonal, shift form of g. The most relevant one, for $n \geq N$ is

$$g^{n-1} rg = \begin{bmatrix} 1 & 0 & & & \vdots & & \\ 1 & 0 & & \cdots & 0 & \cdots & \\ 1 & 0 & & & \vdots & & \\ \vdots & \vdots & & & & & \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ p_{n-N}(w) & \sigma_2 w^{n-N+1} & \sigma_3 w^{n-N+2} & \cdots & \sigma_{N-1} w^{n-2} & \sigma_N w^{n-1} & \sigma_N w^n \end{bmatrix}$$

Here, p_k is a polynomial of degree k, given by $p_k(w) = \left(1 - 2w^k + w^{k+1}\right)/(1-w)$. Note that $p_1(w) = 1 - w$ and $p_0(w) = -1$. It is given by the recurrence relation $p_{k+1} = 1 + w p_k$. Making use of the property that

$$\sum_{n=1}^{\infty} \frac{p_{n-N}(w)}{2^n} = 0$$

for all N, one can find that

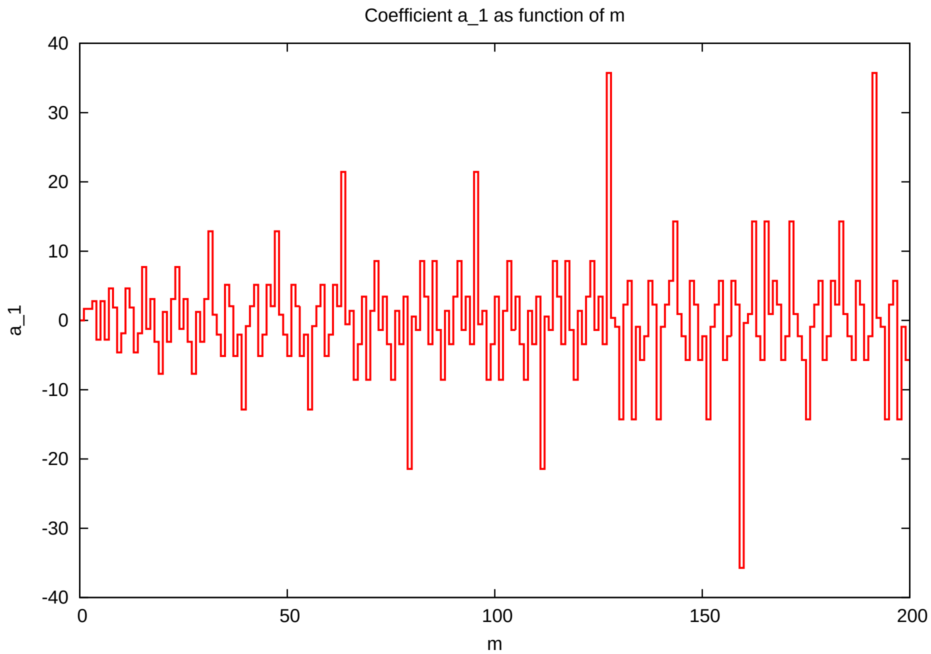
$$\mathscr{L}_c \rho_m = \frac{1}{2-w} \left(\frac{\sigma_2 s_1}{2^{N-2}} + \frac{\sigma_3 s_{m_2}}{2^{N-3}} + \frac{\sigma_4 s_{m_3}}{2^{N-4}} + \cdots + \frac{\sigma_{N-1} s_{m_{N-2}}}{2} + \sigma_N s_{m_{N-1}} + \sigma_N w \rho_m \right)$$

Here, we've made use of the notation

$$m_k = \left\lfloor \frac{m}{2^{N-k}} \right\rfloor = \sum_{j=N-k+1}^{N} \beta_j 2^{j-N+k-1} = m \gg (N-k)$$

where $\gg$ denotes the bit shift-right operator. Observe that $m_N = m$ and that $m_1 = 1$ always.

FIGURE 4.8. Coefficient a_1



This figure shows a graph of the coefficient a_1 as a function of m; that is a_1 for the m'th eigenvector v_m, at a value of $w = 0.6$. It is readily computed by using the recursive equation 4.10. Note that, for each power of 2, there is a pattern, repeated 4 times: twice right side up, and twice upside-down.

From the above, it is immediately clear that eigenfunctions of $\mathscr{L}_c$ can be constructed from linear combinations of ρ_m and the s_{m_k}, and that the eigenvalues will be $\sigma_N w/(2-w)$. To obtain these, one needs the action of $\mathscr{L}_c$ on s_m. This is given by

$$\mathscr{L}_c s_m = \sigma_N \frac{s_{m_{N-1}}}{2} + \sigma_{N-1} \frac{s_{m_{N-2}}}{4} + \cdots + \sigma_2 \frac{s_{m_1}}{2^{N-1}} = \sum_{k=1}^{N-1} \sigma_{k+1} \frac{s_{m_k}}{2^{N-k}}$$

The eigen-equation is then

$$\mathscr{L}_c v_m = \lambda_m v_m$$

with $\lambda_m = \sigma_N w/(2-w)$ and

$$v_m(x) = \rho_m(x) + \sum_{k=1}^{N-1} a_k s_{m_k}(x) \tag{4.9}$$

Here, the a_k are just numbers, and are given by a recurrence relation:

$$a_{N-k} = \frac{\sigma_N \sigma_{N-k+1}}{2^{k-1} w} \left[1 + (2-w) \sum_{j=1}^{k-1} 2^{j-1} a_{N-j} \right] \tag{4.10}$$

The first few of these are

$$\begin{aligned}
a_{N-1} &= \frac{1}{w} \\
a_{N-2} &= \frac{\sigma_N \sigma_{N-1}}{w^2} \\
a_{N-3} &= \frac{\sigma_{N-2}}{w^3} \left[\sigma_{N-1} + \frac{w}{2} \left(\sigma_N - \sigma_{N-1} \right) \right]
\end{aligned}$$

and, written in this "reduced" way, these expressions become progressively more complicated. The coefficient a_1, as a function of m, is shown in the graph 4.8. Putting this all together, the first few eigenfunctions are

$$\begin{aligned}
v_1(x) &= \rho_1(x) \\
v_2(x) &= \frac{s_1(x)}{w} + \rho_2(x) \\
v_3(x) &= \frac{s_1(x)}{w} + \rho_3(x) \\
v_4(x) &= \frac{s_1(x)}{w^2} + \frac{s_2(x)}{w} + \rho_4(x) \\
v_5(x) &= \frac{-s_1(x)}{w^2} + \frac{s_2(x)}{w} + \rho_5(x) \\
v_6(x) &= \frac{s_1(x)}{w^2} + \frac{s_3(x)}{w} + \rho_6(x) \\
v_7(x) &= \frac{-s_1(x)}{w^2} + \frac{s_3(x)}{w} + \rho_7(x)
\end{aligned}$$

Because all of these are constructed from step functions, they are easily integrated over: the integrals are composed of triangle waves.

Several properties of these functions should be noted. First, because the Walsh functions s_m formed a complete orthonormal set for $L_2[0,1]$, it follows that the ρ_m also span $L_2[0,1]$ (because each ρ_m contains s_m as a term). The ρ_m are also linearly independent (no ρ_m can be written as a sum of the others, because each contains a term s_m that cannot be canceled out). They are not, however, orthogonal: in general, one has $\int_0^1 \rho_m(x)\rho_n(x)\,dx \neq 0$ whenever $m = n2^k$ for some (positive or negative) integer k.

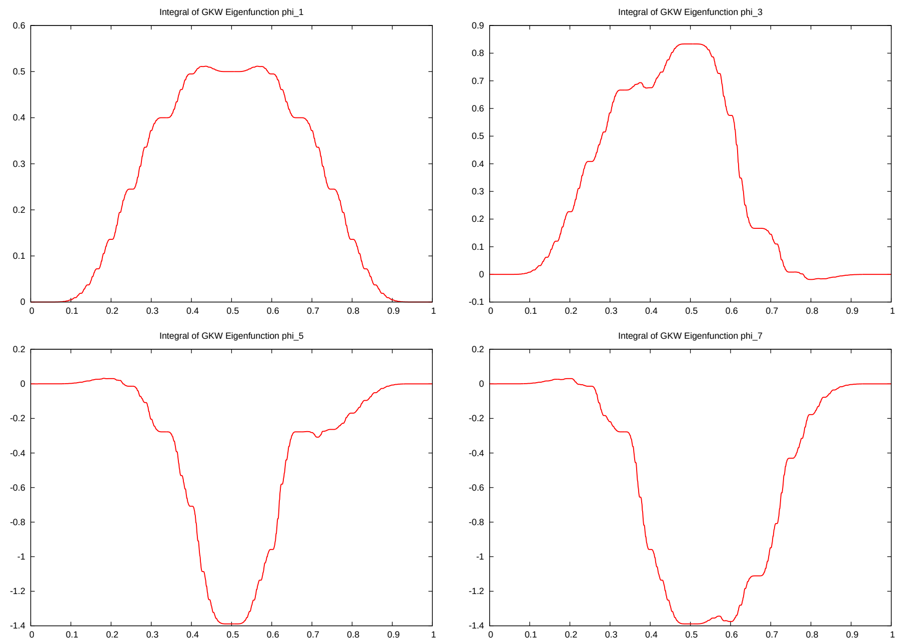
By contrast, the v_m are not linearly independent. Indeed, one readily verifies that $v_2 = v_1/w$, that $v_4 = v_1/w^2$, and so on. Likewise, $v_6 = v_3/w$. In general, one has that $v_{2^k m} = v_m/w^k$ so that the v_m are linearly independent of one-another if and only if m is odd. Since $\mathscr{L}_c$ has a non-trivial kernel, it follows that the v_m cannot span $L_2[0,1]$.

4.11. **Fractal Eigenfunctions of the GKW Operator.** To recap: the motivation for all of the preceding mechanics is to find solutions to the GKW operator, by means of the similarity transformation described in section 4.3. By construction, the v_k are, by similarity transform, the desired eigenfunctions. Unfortunately, they are not even continuous under the natural topology on the reals (they are discontinuous at all dyadic rationals). However, since they do span (XXX span what, precisely?), and one may assume that certain linear combinations of these may be continuous, or even differentiable.

That is, for any sequence of real numbers c_k, the sum

$$f(x) = \sum_{k=1}^{\infty} c_k v_{2k+1}(x)$$

FIGURE 4.9. Integrals of GKW eigenfunctions



These figures show the integrals of GKW eigenfunctions $\phi_k = ?' v_k \circ ?$ for $w = 0.6$. Specifically, these figures show $\int_0^x \phi_k(y)\,dy = \int_0^{?(x)} v_k(z)\,dz$ for $k = 1,3,5,7$. These integrals are easily computed, as the integrals of the v_k are easily computed, as noted in the caption to the previous set of graphs, and the computation of the Question Mark is equally straight-forward.

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is an eigenfunction of the dyadic sawtooth $\mathscr{L}_c f = \lambda f$ and likewise, $\phi = ?' f \circ ?$ is an eigenfunction of the GKW, $\mathscr{L}_h \phi = \lambda \phi$ with the same eigenvalue λ. There must exist certain sequences of $\{c_k\}$ for which ϕ is continuous, and, of these, some for which ϕ is differentiable, or even a polynomial series, although, clearly, the last is not possible for just any value λ, as the spectrum of the GKW is discrete when the eigenfunctions are infinitely differentiable.

The continuous ones are the ones for which the filters are bounded:

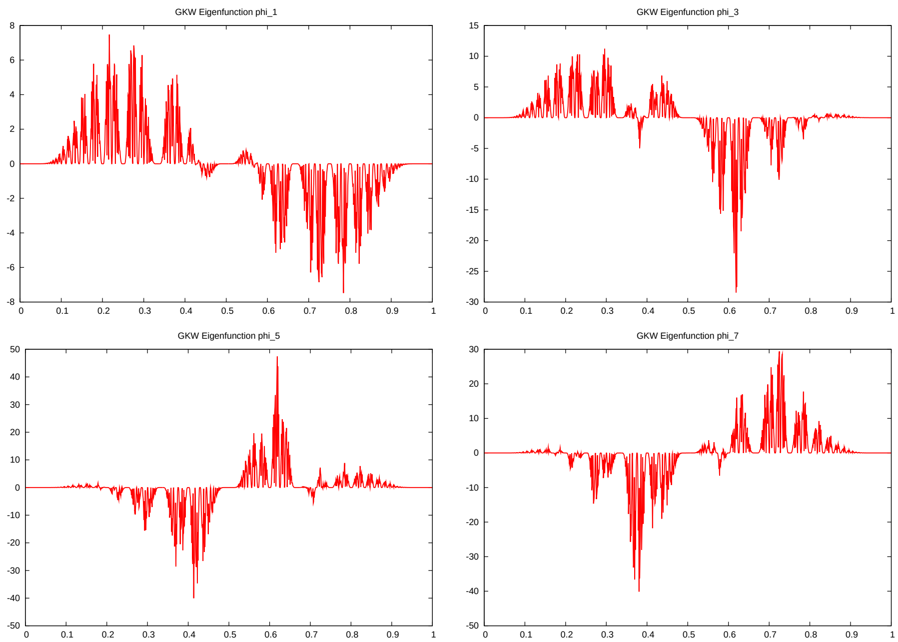
$$\lim_{a\to b} \frac{1}{b-a} \int_a^b \phi(x)\,dx = \lim_{a\to b} \frac{1}{b-a} \int_{?(a)}^{?(b)} f(x)\,dx < \infty$$

This condition, derived in section 4.3, is required of continuous solutions to the GKW, and is explored in the next sections. A sampling of some of the eigenfunctions, and their integrals, is shown in figures 4.7, 4.9 and 4.10.

5. FUNCTIONS ON THE CANTOR SET

The above developments present an insight that leads to a question. Basically, we've seen an operator that has a discrete spectrum when one considers only functions that are infinitely differentiable. It has a continuous spectrum when one considers functions on the Cantor set. Somewhere between these two extremes one imagines a transition or some twist

FIGURE 4.10. Fractal Eigenfunctions of GKW



These figures show the fractal GKW eigenfunctions $\phi_k = ?' v_k \circ ?$ for $w = 0.6$ and $k = 1,3,5,7$. Although these functions are discontinuous at all rationals, they are none-the-less easily approximated by taking numerical differences of $\Phi_k(x) = \int_0^x \phi_k(y)\,dy = \int_0^{?(x)} v_k(z)\,dz$. As discussed in the captions to the previous graphs, these integrals are easily computed, and taking numerical differences is straight-forward. Presumably, one may find certain linear combinations of these that are continuous everywhere, or even differentiable everywhere.

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or constraint that forces the once spectrum into the other. Where, exactly, does this happen? Perhaps when one forces differentiability? Can the transition from continuous to discrete be somehow "smooth"? There are various phenomena on physics where continuous spectra break up onto discrete bands; might something similar happen here? The discussion that follows aims to review a more general setting, and consider different approaches to findings answers to these questions.

Let Δ denote the Cantor set. Consider the space of functions $\mathscr{F}(\Delta \to \mathbb{R})$ from the Cantor set to the real numbers. Since the Cantor set $\Delta = 2^\omega$ has the cardinality of the continuum, this space is clearly very large, a bit too large to work with, and doesn't match the characteristics of the systems studied above. Most notably, we've constructed $?'$ so that it is explicitly integrable; thus, we should consider the space of integrable functions on the Cantor set. The Cantor set has a topological basis that is countable. One may consider real-valued functions on this basis; a subset of these are integrable, as shown below. Eventually, we want to limit these further, and consider the further subsets of continuous and then differentiable functions.

5.1. **Integrability.** Begin by defining a basis $\mathscr{B}$ for a topology on the Cantor set. The basis can be visualized as being given by the infinite dyadic tree. The topmost node in the tree stands for the the unit interval $I_1 = [0,1]$. The two child nodes on either side should be understood to stand for $I_2 = \left[0, \frac{1}{2}\right]$ and $I_3 = \left[\frac{1}{2}, 1\right]$. Continuing in this fashion, one has $I_4 = \left[0, \frac{1}{4}\right]$ and $I_5 = \left[\frac{1}{4}, \frac{1}{2}\right]$ and so on for the next row. The basis is countable: the nodes in the infinite dyadic tree are countable. There is a natural way of numbering the nodes: the top node may be called 1, the left and right children are 2 and 3, respectively, the next row being 4,5,6,7, and so on. The intervals are numbered likewise, and so one has $\mathscr{B} = \{I_k | k \in \mathbb{N}\}$. The full topology $\mathscr{T}$ on the Cantor set is given by the finite intersections and countable unions of the elements of this basis. This topology is one and the same as the standard product topology of cylinder sets on Cartesian products; in this case, the Cartesian product is simply $2 \times 2 \times 2 \times \cdots = 2^\omega$.

Consider the vector space of functions $\mathscr{F}(\mathscr{B} \to \mathbb{R})$ from this basis to the real numbers. Because $\mathscr{B}$ is countable, we can equate this with the space of sequences from $\mathbb{N}$ to $\mathbb{R}$. We are interested in a subspace of these functions that could be considered to be "integrable". Loosely speaking, we are interested in those sequences $\{a_k\}$ which can be interpreted as integrals over the corresponding $\{I_k\}$, so that

$$a_k = \int_{I_k} f(x)\, dx \tag{5.1}$$

for some real-valued function $f : [0,1] \to \mathbb{R}$ on the unit interval. Thus motivated, define the subspace of integrable functions $\mathscr{I} \subset \mathscr{F}(\mathscr{B} \to \mathbb{R})$ as the set

$$\mathscr{I} = \{a_k \in \mathbb{R} | k \in \mathbb{N}, a_1 = a_2 + a_3, a_2 = a_4 + a_5, \cdots, a_n = a_{2n} + a_{2n+1}, \cdots\}$$

The summation constraint is meant to be natural: so that, for example, the constraint $a_1 = a_2 + a_3$ is meant to be understood to be

$$\int_0^1 f = \int_0^{\frac{1}{2}} f + \int_{\frac{1}{2}}^1 f$$

and so on, for each of the basis elements in $\mathscr{B}$. Any integrable function on the unit interval gives rise to a sequence in $\mathscr{I}$. This is easy to see: given the definite integral $F(x) = \int_0^x f(y)\, dy$, the corresponding sequence is just

$$a_k = F\left(\frac{k+1-2^N}{2^N}\right) - F\left(\frac{k-2^N}{2^N}\right) \tag{5.2}$$

with $N = \lfloor \log_2 k \rfloor$, so that, for example, $a_3 = \int_{\frac{1}{2}}^1 f(y) dy$.

Now, the lemma to establish here is that the space of such sequences is isomorphic to the space of integrable functions on the unit interval; that the two may be taken to be the same thing. We've already established one direction above. For the other direction, given a sequence $\{a_k\}$, one must show that there exists a unique, corresponding f. But this follows primarily from the fact that the intersections and unions of elements of $\mathscr{B}$ are very nearly trivial, and from the fact that the dyadic rationals are dense in the reals. We make no effort to be more rigorous, as this is essentially an old theorem from Banach Theory: the Walsh functions provide a basis for the Banach space L_1 [7]. A bit of addition and subtraction provides the lemma above.

Note that, as a sequence space, $\mathscr{I}$ is not bounded: the values a_k may get arbitrarily large (positive or negative) for large k, as long as the constraint $a_k = a_{2k} + a_{2k+1}$ is obeyed. This means that $\mathscr{I}$ cannot be a subspace of the classical Banach space l_∞ of bounded sequences. A more interesting case is the subspace$\mathscr{M} \subset \mathscr{I}$ of measures. This is given by those sequences $\{a_k\}$ for which $a_k \geq 0$, and $a_1 = 1$. Thus, it follows that $a_k < 1$ for all $k > 1$ and so $\mathscr{M} \subset l_\infty$. In general, measures may be delta-function-like; for example $f(x) = 0.2 + 0.8\delta(x - 1/3)$ yields a sequence $\{a_k\}$ such that for any $n \in \mathbb{N}$ there exists a $k > n$ such that $a_k > 0.75$. This example shows that nothing smaller than the full space l_∞ can contain such sequences. More generally, even for $f \in L_p$ with $p < \infty$, one can still find sequences (derived from p'th roots of the delta function) that are not convergent, and thus properly live only in l_∞. However, if f is bounded, *i.e.* if $f \in L_\infty$, then obviously one has that the sequence $\{a_k\} \in l_2$. One can do slightly better.

Recall the standard definition of the Banach space l_p: it is the set of sequences $\{a_k\} \in l_p$ which satisfy $\sum_{k=1}^\infty |a_k|^p < \infty$.

Lemma. *If $f \in L_\infty$, then the sequence $\{a_k\} \in l_p$ for any $p > 1$.*

Proof. If $f \in L_\infty$, then $\|f\| \leq C$ for some constant $C < \infty$. That is, $|f|$ is bounded on the unit interval. As a result, one has that $|a_k| \leq C2^{-N}$ where $N = \lfloor \log_2 k \rfloor$. One can use this to bound the sum:

$$\begin{aligned}\sum_{k=1}^\infty |a_k|^p &= \sum_{N=0}^\infty \sum_{k=2^N}^{2^{N+1}-1} |a_k|^p \\ &\leq \sum_{N=0}^\infty \sum_{k=2^N}^{2^{N+1}-1} \left|\frac{C}{2^N}\right|^p \\ &= C^p \sum_{N=0}^\infty \left(\frac{1}{2^{p-1}}\right)^N \\ &< C^p \qquad \forall p > 1\end{aligned}$$

The last line showing the desired result. □

The bound is strict:

Lemma. *There do not exist any sequences $a \in \mathscr{I}$ which are also in l_1, other than the trivial sequence $\{a_k | a_k = 0\}$.*

Proof. One has $a_1 = \sum_{k=2^N}^{2^{N+1}-1} a_k$ for all N. If the sequence isn't trivial, then there must exist some M for which $\sum_{k=2^M}^{2^{M+1}-1} |a_k| = C > 0$. Since $a_k = a_{2k} + a_{2k+1}$, one must have $|a_k| \leq |a_{2k}| + |a_{2k+1}|$. It follows that $\sum_{k=1}^{2^N-1} |a_k| \geq (N - M) C$ and so as $N \to \infty$, the series is bouned below by an ever-larger number. That is, the sequence is summable only if $C = 0$, that is, if $a_k = 0$ for all k. □

In practice, none of the sequences we will deal with are in L_∞, nor are they obviously in any L_p for any $p > 1$. Yet they all appear to be in $\mathscr{I} \cap l_p$ for $p > 1$. The function $?'$ is a measure; an explicit proof of this is given in [30]. All of the eigenfunctions constructed above are integrable; these are shown in figures 4.7 and 4.9. As these integrals are bounded, we may conclude the corresponding sequences are elements of $\mathscr{I} \cap l_p$ for $p > 1$.

5.1.1. *Incidence algebra.* The constraint of working only with integrable functions on $\mathscr{B}$ leads to the curious relation that the elements $a \in \mathscr{I}$ satisfy an operator equation: $Wa = a$ with

$$W = \left[\begin{array}{cccccccccc} 0 & 1 & 1 & 0 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \cdots & \\ & & & & \cdots & & & & & \end{array}\right]$$

or, equivalently, that the elements of $\mathscr{I}$ lie in the kernel of $M = I - W$. Clearly W is bounded and thus continuous. However, it is not a projection; $W^2 \neq W$; it seems to be 'weakly nilpotent' (nilpotent, but of infinite order); and M is unipotent. Neither $\mathscr{I}$ nor $\mathscr{M}$ are complemented; that is, given a sequence x, there is no unique way of writing $x = a + y$ with $a \in \mathscr{I}$. Indeed, any a with $a_1 = x_1$ will do. This means that there is no projection operator from the space of sequences to either $\mathscr{I}$ or $\mathscr{M}$.

The operator $M = I - W$ corresponds to the Möbius function of the incidence algebra of the infinite dyadic tree taken as a poset; the corresponding zeta function is $Z = 1 + W + W^2 + W^3 + \cdots$. To see this, recall the definition of an incidence algebra on a poset, and specifically, of the zeta and Möbius functions of the incidence algebra. Let S be a partially ordered set, and let $Z : S \times S \to \mathbb{N}$ be the function

$$Z(a,b) = \begin{cases} 1 & \text{if } a \geq b \\ 0 & \text{otherwise} \end{cases}$$

For convenience, one orders all of the elements of $S = \{s_1, s_2, \cdots\}$ such that if $s_i > s_j$, then $i < j$. This ordering is easily done by choosing s_1 to be a maximal element of S, and s_2 to be a maximal element of $S \backslash s_1$, etc. Then writing $Z_{ij} = Z(s_i, s_j)$, one sees that Z_{ij} is upper-triangular, with all $1's$ on the diagonal, and thus unipotent. Define $N = I - Z$ and define $M = I + N + N^2 + \cdots$. For a finite set S (so that M, N and Z are all finite dimensional matrices), one then has $MZ = ZM = I$. This M is called the Möbius function of the incidence algebra of the poset. To see that this is exactly the M as given above for the infinite dyadic tree, one needs only to make the association that the s_i are just the intervals I_i given at the beginning of this section.

The relationship between M and Z in the infinite-dimensional case, however, is more subtle, and depends on the space of sequences.

Theorem. *On l_1, the operators are well-behaved and invertible: $MZ = ZM = I$.*

Proof. From 5.1, we conclude that ker M is trivial on l_1; thus $ZM = I$. For the other case, note that the matrix elements $Z_{1j} = 1$ for all j. Since l_1 consists of summable sequences, this implies that Z is bounded; indeed, $\|Z\| = 1$ on l_1. □

By contrast, the set $\mathscr{I}$ is defined as ker M, thus $ZM \neq I$ on $\mathscr{I}$. The lemma 5.1 establishes that Z cannot be bounded on sequences in $\mathscr{I}$ (since $\mathscr{I} \cap l_1$ is trivial.) . Thus, for the domain of interest, one must conclude $MZ \neq I$.

The correspondance here between the Mobius function on (semi-)lattices and its kernel being used to define a notion of an integrable function is new to this author. Is this correspondance accidental, or can it be used to define integrability in general?

5.2. **Continuity.** Given the space $\mathscr{I}$ of integrable functions (as defined above), how does one isolate the subspace of continuous functions? In part because we've chosen to work with a basis for the Cantor set topology, the mechanics of traditional limits and delta-epsilon proofs appear to be inappropriate and not (easily) modified for the current vocabulary. Thus, some alternative tools need to be developed. This is done here.

Given the interpretation of the a_k as integrals over the intervals I_k, we are then motivated to find a function $F(x)$ such that a_k are given by equations 5.1, 5.2. Then, very roughly speaking, we wish to have

$$f(x) = \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon) - F(x)}{\varepsilon}$$

as the function being integrated. We are interested in those functions $f(x)$ that are continuous in x. However, the above limit is not really well-defined or practical in the current situation; for as we saw, when $F = ?$ then this is not sufficient to rigorously define $?'$ on the real unit interval. This is indeed the whole reason why the setting and manipulations are performed on Cantor set, rather than the reals; why the tools use the language of intervals and cylinder sets.

Consider the problem of continuity at x. We wish to have the left and right limits be equal, in other words, for $\lim_{\varepsilon\to 0} f(x+\varepsilon) - f(x-\varepsilon) = 0$, or, writing $\varepsilon = 1/2^n$, for

$$\lim_{n\to\infty} 2^n \left(F\left(x+\frac{1}{2^n}\right) - F\left(x-\frac{1}{2^n}\right)\right) = 0$$

For $x = \frac{1}{2}$, one has that the interval $I_{3\cdot 2^n} = \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}}\right]$ and so continuity at $x = 1/2$ is obtained by admitting only those sequences for which

$$\lim_{n\to\infty} 2^{n+1} \left(a_{3\cdot 2^n} - a_{3\cdot 2^n - 1}\right) = 0$$

The above is both a necessary and sufficient condition for continuity. It may be replaced by a weaker, sufficient condition that

$$\left| \sum_{n=0}^{\infty} 2^{n+1} \left(a_{3\cdot 2^n} - a_{3\cdot 2^n - 1}\right) \right| < \infty \tag{5.3}$$

This may be strengthened by demanding that, for all $p > 0$, that

$$\sum_{n=0}^{\infty} 2^{n+1} \left|a_{3\cdot 2^n} - a_{3\cdot 2^n - 1}\right|^p < \infty$$

The above is readily generalized to a continuity condition at any dyadic rational. The midpoint for interval I_k is given by

$$m_k = \frac{k+1-2^N}{2^{N+1}}$$

where, as before, $N = \lfloor \log_2 k \rfloor$. Continuity at m_k is given by

$$\lim_{n\to\infty} 2^{n+N+1} \left(a_{(2k+1)2^n} - a_{(2k+1)2^n - 1}\right) = 0$$

The condition 5.3 may be generalized into an operator relation. Consider the operator B given by matrix elements B_{kj}

$$B_{kj} = 2^{n+N+1} \left(\delta_{j,(2k+1)2^n} - \delta_{j,(2k+1)2^n - 1}\right)$$

where δ is the Kronecker delta function. Then, in order for a series $a = \{a_k\}$ to represent a function continuous at m_k, it is sufficient to have $\left|\sum_{j=1}^{\infty} B_{kj} a_j\right| < \infty$. Define

$$\|Ba\| = \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} B_{kj} a_j \right|$$

Clearly, any a for which $\|Ba\| < \infty$ represents a function that is continuous at the dyadic rationals. In fact, this condition is sufficient to guarantee that it is continuous everywhere.

Theorem. *If $a \in \mathscr{I}$ and $\|Ba\| < \infty$, then the function represented by a is continuous everywhere.*

Proof. To see this, consider a function $f(x)$ that is discontinuous at a point x, and define $a_k = \int_{I_k} f(y)dy$ as usual. If x is equal to the dyadic rational m_k, then the arguments above have already shown that $\left|\sum_{j=1}^{\infty} B_{kj}a_j\right|$ would be unbounded. Consider x that is not a dyadic rational. If m_k is a midpoint that is near x (and $f(x)$ is continuous at m_k), then the sum $c_k = \left|\sum_{j=1}^{\infty} B_{kj}a_j\right|$ is bounded. However, for any finite constant C, one can always find a value of k such that $c_k > C$. One does so simply by considering values of k such that m_k is ever closer to x. Thus, the sum $\sum_k c_k$ cannot be finite. Thus, it is impossible to find a discontinuous function $f(x)$ for which $\|Ba\| < \infty$ and so the theorem is proved. $\square$

This motivates the definition of a space $\mathscr{C}$ of sequences that contains only continuous functions. This is the space where B is a bounded operator:

$$\mathscr{C} = \{a \in \mathscr{I} \,|\, \|Ba\| < \infty\}$$

Many, but perhaps not all, continuous functions are contained in $\mathscr{C}$; it is not entirely clear if one can construct continuous functions f for which the corresponding sequence a fails to satisfy the above.

6. Eigenfunctions of the GKW, redux

The goal of this section is to examine the measure $?'$ in greater detail. In order to avoid the difficulties associated with examining $?'$ at a point, it is far more convenient to study the integral

$$I(a,b) = \frac{1}{b-a}\int_a^b ?'(x)\,dx = \frac{?(b) - ?(a)}{b-a} \tag{6.1}$$

in the limit of $a \to b$. It is straightforward to verify that, when b is a rational number, that in this limit, I vanishes: the Minkowski question mark has zero derivative on the rationals. In fact it is very, very flat, exponentially so, with all derivatives vanishing, on the rationals: this is easily seen by recalling the definition 2.4 of the question mark: given a continued fraction expansion of a rational number $[a_1, a_2, \cdots, a_N]$, that one has that the flat part goes as 2^{-a_N} as $a_N \to \infty$. By contrast, the derivative is "infinite" on the quadratic irrationals; the rate of divergence is explored in figure 6.1.

7. The Isola Map

Stefano Isola proposes studying a map of deceptive simplicity[14]. Given by

$$F(x) = \begin{cases} x/(1-x) & \text{if } 0 \le x \le 1/2 \\ (1-x)/x & \text{if } 1/2 \le x \le 1 \end{cases}$$

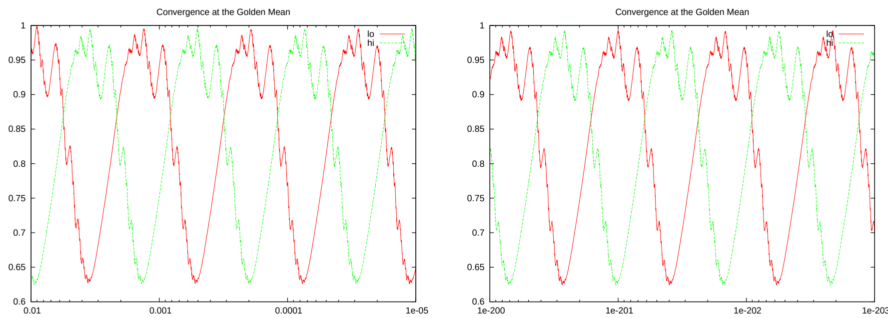
it is symmetric about $x = 1/2$: that is, $F(x) = F(1-x)$, and has a very simple tent-like shape, and this is the source of the deception. One wants to hastily conclude that it is topologically equivalent to the standard tent map

$$\tau(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2-2x & \text{if } 1/2 \le x \le 1 \end{cases}$$

and thus that the spectrum of its Frobenius-Perron Operator is identical to that of the Bernoulli Map, and that this map can be trivially brushed aside as belonging to that conjugacy class. Nothing could be farther from the truth. In fact, it is conjugate, but the conjugating function is the Minkowski Question Mark:

$$F(x) = (?^{-1} \circ \tau \circ ?)(x)$$

FIGURE 6.1. Convergence at the Golden Mean



The above figures show rescaled graphs of the integral 6.1 at the Golden Mean $\phi = \left(\sqrt{5}-1\right)/2$, for which $?(\phi) = 2/3$. Specifically, the graphs show

$$\mathrm{hi}(\varepsilon) = \varepsilon^{0.2798} I(\phi, \phi+\varepsilon) \qquad \text{and} \qquad \mathrm{lo}(\varepsilon) = \varepsilon^{0.2798} I(\phi-\varepsilon, \phi)$$

for two different ranges: $10^{-5} < \varepsilon < 10^{-2}$ and $10^{-203} < \varepsilon < 10^{-200}$. Note the logarithmic scale. The oscillatory behaviour is given by $\cos(1.040\pi \log \varepsilon)$ to a good approximation. That the oscillations are very regular should be apparent by comparing the left and right figures. Qualitatively similar behaviors can be seen at other quadratic irrationals, although with different growth amplitudes and periods of oscillation.

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and so the relationship is anything but trivial. The easiest way to see this is to note that we can write F and τ are combinations of the modular group element g^{-1}:

$$F(x) = \begin{cases} g_C^{-1}(x) & \text{if } 0 \le x \le 1/2 \\ (g_C^{-1} \circ r)(x) & \text{if } 1/2 \le x \le 1 \end{cases}$$

following the notation of earlier chapters, and

$$\tau(x) = \begin{cases} g_D^{-1}(x) & \text{if } 0 \le x \le 1/2 \\ (g_D^{-1} \circ r)(x) & \text{if } 1/2 \le x \le 1 \end{cases}$$

Just as we saw with the second sawtooth, the point dynamics of the Isola Map and the Tent Map are isomorphic to each other, but the eigenvalue spectra are inequivalent. The Ruelle-Frobenius-Perron operator for the Isola Map is

$$[\mathscr{P} f](x) = \frac{1}{(1+x)^2}\left[f\left(\frac{x}{x+1}\right) + f\left(\frac{1}{x+1}\right)\right]$$

Isola shows how the Gauss-Kuzmin-Wirsing operator can be constructed through some simple operator relationships on $\mathscr{P}$ and so it is a worthwhile goal to attempt to solve $\mathscr{P}$. As we will see below, this seems to be an even harder task.

Closely related is a modular variant of the Bernoulli shift, given by

$$A(y) = ?^{-1}(\mathrm{frac}(2?(y))) = \begin{cases} \frac{y}{1-y} & \text{for } 0 \le y \le \frac{1}{2} \\ \frac{2y-1}{y} & \text{for } \frac{1}{2} \le y \le 1 \end{cases} \tag{7.1}$$

The associated transfer operator is

$$[\mathscr{L}_A f](y) = \frac{1}{(1+y)^2} f\left(\frac{y}{1+y}\right) + \frac{1}{(2-y)^2} f\left(\frac{1}{2-y}\right)$$

which again has the curious relationship

$$\mathcal{L}_A ?' = ?'$$

as given in [30].

7.1. **The (Lack of a) Polynomial Basis.** Based on our previous luck, we attempt to define the operator $\mathscr{P}$ in the polynomial basis. First, we attempt an expansion at $x=0$. This leads to

$$\begin{aligned}\mathscr{P}_{nk} \equiv & \langle n|\mathscr{P}|k\rangle \\ = & (-)^n \left[\left(\begin{array}{c} k+n+1 \\ k+1 \end{array} \right) + \Theta_{k\leq n} (-)^k \left(\begin{array}{c} n+1 \\ k+1 \end{array} \right) \right]\end{aligned}$$

This matrix is not triangular, and is thus not directly solvable. It is also very ill-conditioned, making it not numerically tractable, at least, not in any simple fashion. As n gets large, the matrix elements grow exponentially on the diagonal. This is easily seen by applying Stirling's asymptotic formula for the factorial to the binomial; one easily gets

$$\left(\begin{array}{c} n \\ k \end{array} \right) \approx \frac{2^{n+1}}{\sqrt{2\pi n}} \exp\left(\frac{-(2k-n)^2}{2n} \right)$$

when n and k get large. Thus, along the diagonal, $\mathscr{P}_{nn} \approx 4^{n+1}/2\sqrt{\pi n}$, and the matrix is not tractable numerically, and would be painful to work with analytically, without defining some sort of regulator. Thus, we are motivated to look at the expansion at $x=1/2$. Here, however, the situation is not much better. Defining $y = x - 1/2$ so that

$$[\mathscr{Q}f](x) = \frac{1}{(y+3/2)^2} \left[f\left(\frac{y+1/2}{y+3/2} \right) + f\left(\frac{1}{y+3/2} \right) \right]$$

we work through the same set of steps to obtain

$$\begin{aligned}\mathscr{Q}_{nk} \equiv & \langle n|\mathscr{Q}|k\rangle \\ = & \left(\frac{-2}{3} \right)^{n+2} \left[\left(\frac{2}{3} \right)^k \left(\begin{array}{c} k+n+1 \\ k+1 \end{array} \right) + \left(\frac{1}{3} \right)^k \sum_{p=0}^{\min(n,k)} (-3)^p \left(\begin{array}{c} n+k-p+1 \\ k+1 \end{array} \right) \left(\begin{array}{c} k \\ p \end{array} \right) \right]\end{aligned}$$

which is far more complex, and only marginally less divergent: $\mathscr{Q}_{nn} \sim (16/9)^{n+1}$. There is hardly any hope that a Taylor's expansion around any other value of x will give a tractable result; the trick of using the Taylor's expansion to obtain polynomial eigenstates fails in this case. Indeed, it seems likely that the eigenstates will not be analytic, although it is not clear to me what theorem would establish or disprove this conjecture.

The polynomial-basis matrix elements for this operator are much better behaved. They are given by

$$M_{nk} \equiv \langle n|\mathcal{L}_A|k\rangle = \frac{1}{2^{2+n+k}} \binom{n+k+1}{k+1} + \Theta_{n\geq k} (-1)^{k+n} \binom{n+1}{k+1}$$

(Notice the direction of the Heaviside function is reversed, is this correct, or an error? XXX Needs double-checking).

The leading factor of two in the above makes all the difference in the world for this operator. This time, applying Stirling's formula to evaluate the matrix elements on the diagonal gives

$$M_{nn} \approx 1 + \frac{0.76}{\sqrt{n}}$$

thus implying that this operator, at least, is not hopelessly badly behaved.

The difference between the two is, perhaps, due to the former not being diagonalizable except in Jordan block form.

8. Conclusions

Apologies for the format of this paper. It's a veritable candy store of goodies; there are all these yummy toys to play with, which one first?

ToDo list:

- show why Kolmogorov entropy is in trouble, desired for the Ornstein isomorphism theorem. Need Koopman to do this.
- show koopman operator, desired for Wold decomposition. Clarify kernel

Appendix A. Expansion about arbitrary location

This appendix provides expressions for the polynomial-basis matrix elements for the GKW and Mayer-Ruelle operators. Specifically, they provide expressions for the Taylor's expansions about points other than 0 or 1. Several relations to the hypergeometric series are also provided.

Consider $f(x)=\sum_{n=0}^{\infty}f^{(n)}(a)\,(x-a)^n/n!$ and $g(x)$ likewise expanded about $x=b$. With this expansion, the operator relation $\mathscr{L}_h f=g$ becomes

$$\frac{g^{(m)}(b)}{m!}=\sum_{n=0}^{\infty}\mathscr{L}_{mn}^{(b,a)}\,\frac{f^{(n)}(a)}{n!}$$

which is taken to define the meaning of $\mathscr{L}_{mn}^{(b,a)}$. Without much difficulty, one discovers that the matrix elements are given by

$$\mathscr{L}_{mn}^{(b,a)}=(-1)^m\sum_{k=0}^{n}(-a)^{n-k}\binom{n}{k}\binom{k+m+1}{m}\zeta_H(k+m+2,1+b) \tag{A.1}$$

where $\zeta_H(s,q)$ is the Hurwitz zeta function:

$$\zeta_H(s,q)=\sum_{n=0}^{\infty}\frac{1}{(n+q)^s}$$

Substituting $a=b=1/2$, one obtains the expansion of [6], which is

$$\begin{aligned}\mathscr{L}_{mn}^{(1/2,1/2)}=&(-1)^m\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^{n-k}\binom{n}{k}\binom{k+m+1}{m}\times\\&\left[2^{m+k+2}\left(\zeta(k+m+2)-1\right)-\zeta(k+m+2)\right]\end{aligned}$$

All of these expressions for the matrix elements for the GKW operator have a common form. It consists of two summations: the outer summation, and the summation defining the Hurwitz zeta function. Exchanging the order of summation, one finds terms consisting of a series of polynomials, which are most simply expressed in terms of Gauss's hypergeometric series:

$$\Gamma_{mn}(x)\equiv(m+1)\,{}_2F_1\left[\begin{matrix}-n & m+2\\ & 2\end{matrix};x\right]=\sum_{k=0}^{n}\binom{n}{k}\binom{k+m+1}{m}(-x)^k$$

These have a curious superficial resemblance to the shifted Legendre polynomial

$$\widetilde{P}_n(x)\equiv\sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{n}(-x)^k$$

Switching the order of summation in equation A.1 gives the following:

$$\mathscr{L}_{mn}^{(b,a)} = (-1)^{m+n} a^n \sum_{j=1}^{\infty} \frac{1}{(j+b)^{m+2}} \, {}_2F_1 \left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array} ; \frac{-1}{a(j+b)} \right]$$

Curiously, the above is in the form of the Mayer-Ruelle operator. This operator is a slightly generalized form of the GKW operator given by Dieter Mayer[19],

$$\left[\mathscr{M}^{(s)} f\right](x) \equiv \sum_{n=1}^{\infty} \frac{1}{(n+x)^s} f\left(\frac{1}{n+x}\right)$$

Then, taking $s = m+2$, $x = b$ and $f(x) = {}_2F_1 \left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array} ; \frac{-x}{a} \right]$, one has that the GKW matrix elements are just specific values resulting from the application of the Mayer-Ruelle operator to the hypergeometric series:

$$\mathscr{L}_{mn}^{(b,a)} = (-1)^{m+n} a^n \left[\mathscr{M}^{(m+2)} f\right](b)$$

The simplicity of this form can be further reinforces by setting $a = b = 1$ so that

$$G_{mn} = \left[\mathscr{M}^{(m+2)} f\right](1)$$

where, of course, G_{mn} are the matrix elements of the GKW operator as defined in eqn 2.6.

Following the same procedures and definitions as above, the matrix elements of the Ruelle-Mayer operator $\mathscr{M}^{(s)}$ may be written as

$$\text{(A.2)} \quad \left[\mathscr{M}^{(s)}\right]_{mn}^{(b,a)} = (-1)^m \sum_{k=0}^{n} (-a)^{n-k} \binom{n}{k} \binom{m+k+s-1}{m} \zeta_H(m+k+s, 1+b)$$

The above gets a bit easier to read if one sets $a = b = 1$ to obtain

$$\left[\mathscr{M}^{(s)}\right]_{mn} = (-1)^{m+n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k+s-1}{m} \left[\zeta(m+k+s) - 1\right]$$

which obviously reduces to the GKW operator for $s = 2$:

$$G_{mn} = (-1)^{m+n} \left[\mathscr{M}^{(2)}\right]_{mn}$$

The corresponding hypergeometric identity that comes into play is

$$\sum_{k=0}^{n} (-x)^k \binom{n}{k} \binom{m+k+s-1}{m} = \binom{m+s-1}{m} \, {}_2F_1 \left[\begin{array}{cc} -n & m+s \\ & s \end{array} ; x \right]$$

As a final note, recall that the Hurwitz zeta may be expressed as the polygamma function for integer arguments, where the polygamma functions are the chain of logarithmic derivatives of the gamma function. Thus, one may also expresses the matrix elements of $\mathscr{L}$ in the curious form

$$\mathscr{L}_{mn}^{(b,a)} = \frac{(-a)^{n+1}}{m!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{a}\right)^{k+1} \frac{1}{(k+1)!} \frac{d^{k+1}}{dx^{k+1}} \psi^{(m)}(1+b)$$

Here, the curious operator making an appearance is

$$\left[P_{n,y} f\right](x) = \sum_{k=0}^{n} (-y)^k \binom{n}{k} \frac{f^{(k)}(x)}{k!}$$

where $f^{(k)}(x)$ is the k'th derivative of f at x. The operator $P_{n,y}$is upper-triangular, with all eigenvalues equal to 1, and all eigenvectors being polynomials (or analytic series for n not an integer).

Appendix B. Similarity Transforms

This appendix demonstrates the behaviour of the transfer operator under the action of a similarity transformation. The demonstration proceeds using the simplest, most basic manipulations possible, so as to be easy to verify.

Given a differentiable function $\alpha : X \to X$, the transfer operator (Perron-Frobenius-Ruelle operator) $\mathscr{L}_\alpha : \mathscr{F}(X) \to \mathscr{F}(X)$ is a bounded linear operator acting on a space of functions $\mathscr{F}(X)$ on X. One possible, simple definition for this operator is

$$[\mathscr{L}_\alpha f](x) = \sum_{y\in\alpha^{-1}(x)} \frac{1}{|\alpha'(y)|} f(y)$$

where $x \in X$ and $f \in \mathscr{F}(X)$, while α' denotes the derivative of α. The derivation of the similarity transform proceeds by substituting the similarity relation $\alpha = \phi^{-1} \circ \beta \circ \phi$, with $\phi : X \to V$ being one-to-one and onto (and so being uniquely invertible), and being differentiable, so that ϕ' can be defined. Thus, $\beta : V \to V$ is a function conjugate to α, and is differentiable as well. Substituting, one then has:

$$\begin{aligned}[\mathscr{L}_\alpha f](x) &= \sum_{y\in\left(\phi^{-1}\circ\beta\circ\phi\right)^{-1}(x)} \frac{1}{\left|(\phi^{-1}\circ\beta\circ\phi)'(y)\right|} f(y) \\ &= \sum_{\phi(y)\in(\beta\circ\phi)^{-1}(x)} \frac{1}{\left|(\phi^{-1}\circ\beta\circ\phi)'(y)\right|} f(y)\end{aligned}$$

Let $w = \phi(y)$ and $v = \phi(x)$ so that

$$\begin{aligned}[\mathscr{L}_\alpha f]\left(\phi^{-1}(v)\right) &= \sum_{w\in\beta^{-1}(v)} \frac{1}{\left|(\phi^{-1}\circ\beta\circ\phi)'(\phi^{-1}(w))\right|} f\left(\phi^{-1}(w)\right) \\ &= \sum_{w\in\beta^{-1}(v)} \frac{f\left(\phi^{-1}(w)\right)}{\left|(\phi^{-1\prime}\circ\beta)(w)\cdot\beta'(w)\cdot(\phi'\circ\phi^{-1})(w)\right|}\end{aligned}$$

Observe that, for all values of $w \in \beta^{-1}(v)$, one has that $\left(\phi^{-1\prime}\circ\beta\right)(w) = \phi^{-1\prime}(v)$ and so this term can be brought out of the summation. Next, one has that $1 = \left(\phi\circ\phi^{-1}\right)' = \left(\phi'\circ\phi^{-1}\right)\cdot\phi^{-1\prime}$ and so one has

$$\frac{1}{|(\phi'\circ\phi^{-1})(v)|}[\mathscr{L}_\alpha f]\left(\phi^{-1}(v)\right) = \sum_{w\in\beta^{-1}(v)} \frac{1}{|\beta'(w)|}\left(\frac{\left(f\circ\phi^{-1}\right)(w)}{|(\phi'\circ\phi^{-1})(w)|}\right)$$

Define an operator $S_\phi : \mathscr{F}(X) \to \mathscr{F}(V)$ that acts on $f \in \mathscr{F}(X)$ as

$$\left[S_\phi f\right](v) = \frac{\left(f\circ\phi^{-1}\right)(v)}{|(\phi'\circ\phi^{-1})(v)|}$$

Then the previous equation can be written as

$$\left[S_\phi\mathscr{L}_\alpha f\right](v) = \left[\mathscr{L}_\beta S_\phi f\right](v)$$

Since this holds for all f and v, one must have

$$\mathscr{L}_\beta = S_\phi\mathscr{L}_\alpha S_\phi^{-1}$$

where clearly, $S_\phi^{-1} = S_{\phi^{-1}}$. Thus, the two transfer operators are conjugate to one-another when the functions generating them are similar. The primary ingredient for the above derivation was that the similarity transform ϕ needed to be differentiable, in order to make the manipulations legitimate.

In fact, the conjugacy transformation leaves the spectrum of the operators unchanged. To see this, one must switch to slightly more abstract language: first, one must establish that the transfer operator is bounded (as, indeed, it is) and that it is expressible as a nuclear operator, so that it can be written as a countable sum of (bounded, summable) eigenvalues and basis vectors.

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