From: Philippe.Flajolet@inria.fr
To: Linas Vepstas (linas@lians.org)
Subject: Differences of Zeta values
Date: January 22, 2006

Dear Linas,

Here are a few more comments on your sequence. Nothing shattering here. I am just playing with the best way of organizing the material.

Also, I think I didn't answer your earlier question about a twinned submission to a journal. On general grounds, I don't think that's a good idea since journals are not prepared for this and this is likely to make your chances at best an inf of the chances of each paper if treated separately. Also, I prefer to give you my opinion: my impression is that your first part on GKW is perhaps hard to publish, as it is, since it seems to me to be largely a presentation of material that is known and already published. Of couse, I may be wrong and might underestimate what you have on these aspects. (Can you state a clear-cut theorem?) Also, what I had time to look at is only your PDF note, not the surrounding web pages that seem to contain a lot of material. Whatever you decide eventually, I'll be happy to try and provide advice and references, if I can.

Back to our business :-). In what follows I write things that you know. This is just meant to fix ideas plus a few things I checked yesterday with Maple. Also you may notice fluctuating notations. I try to reserve S,T,U,V,..., for sequencs related to yours. By giving them momentarily different names, I think it will make the last grand unification easier. I hope this won't be too confusing to read.

With very best wishes,

Philippe

DIFFERENCES OF ZETA VALUES

by P.F. and L.V.,
Sun Jan 22 15:01:27 CET 2006

Introduction

Let $\zeta(s) = \sum 1/n^s$ be the Riemann zeta function and

$$Z(s) := \zeta(s) - \frac{1}{s-1}$$

be its "regularized" version. For reasons related to the dynamics of continued fractions and the Gauss-Kuzmin-Wirsing operator, Linas Vepstas was led to consider the finite differences[1]

$$(1) \qquad T_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k Z(k+1) \equiv \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left(\zeta(k+1) - \frac{1}{k} \right).$$

(The continued fraction connection is not discussed here: see Linas' web pages for some of its aspects.)

[1] Should we perhaps decide to define T_n as $\sum (-1)^{k-1} \cdots$??

The sums T_n are easily evaluated with the help of any system that implements multiprecision arithmetics—we have used MAPLE. In the course of his numerical investigations, Linas discovered in 2004 the startling near-identity,

$$\begin{aligned} -T(1000) &= 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,10421\,59335\,9399\mathbf{9}\ \ldots \\ \gamma &= 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,10421\,59335\,9399\mathbf{2}\ \ldots, \end{aligned} \tag{2}$$

where γ is Euler's constant. The surprise is that the discrepancy is found at the 50th digit only (the error is about $7 \cdot 10^{-50}$). Such a coincidence is rather remarkable. Being made publicly available on the world-wide web, it caught Philippe Flajolet's attention in early 2006. The present note describes our ensuing exchanges, which led to a very precise explanation of what lies behind this near-identity.

As we shall prove, the real numbers $-T_n$ and γ agree to a number of digits that is approximately equal

$$\log_{10}\left(e^{-2\sqrt{\pi n}}\right) = \frac{2\sqrt{\pi n}}{\log 10} \approx 1.539\sqrt{n}, \tag{3}$$

a formula that instantiates nicely to 48.68 for $n = 1,000$.

The fact that $T_n + \gamma$ decrease very fast implies in particular that the Newton series representation of regularized zeta converges throughout the complex plane. We thus have, as a consequence of our approximations,

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{m \geq 1} (-1)^m (T_m + \gamma) \frac{(s-1)\cdots(s-m)}{m!},$$

which is valid for all $s \in \mathbb{C}$. To the best of our knowledge, this representation has not been investigated before.

In what follows, we first digress a bit and discuss briefly the topic of near-identities, of which Eq. (2) is typical, as we feel it has perhaps not received much attention in the recent literature. We then proceed with the asymptotic analysis of T_n and conclude with a few remarks on additional properties of this sequence.

1. Near-identities

Let us says that α and β satisfy a near-identity, written

$$\alpha \doteq \beta,$$

if α and β agree to an unusual number of digits. There is of course no proper definition that can capture this informal notion, but, for reasonably natural constants, some exceptionally small error term is likely to be an indication of a mathematical truth, possibly of some depth.

Ramanujan had a keen interest in such questions: see especially his work on modular equations and approximations to π. In this orbit of ideas, one has for instance the remarkable near-indentity

$$e^{\pi\sqrt{163}} \doteq 262537412640768744, \tag{4}$$

since $e^{\pi\sqrt{163}} = 262537412640768743.99999999999\mathbf{2}\cdots$, which reflects several highly non-trivial properties of the modular invariant $j(q)$.

In recent times, John and Peter Borwein have dedicated a mind-boggling study to "Strange Series and High Precision Fraud" [*Monthly*, 1992]. Their most spectacular near-identity is

$$\sum_{n=1}^{\infty} \frac{\lfloor n e^{\pi\sqrt{163/9}} \rfloor}{2^n} \doteq 1280640, \tag{5}$$

where the agreement is now to at least half a billion digits(!). Such an exceptional coincidence results from properties of certain interesting lattice sums combined with approximation properties of $\exp(\pi\sqrt{n})$, like in (4).

Yet some other near-identities are related to various functional relations, when these involve exponentially small terms. An instance given by the Borweins and close to several others of Ramanujan is

$$\sum_{n=-\infty}^{\infty} \frac{1}{10^{(n/100)^2}} \doteq 100\sqrt{\frac{\pi}{\log 10}}, \tag{6}$$

which is correct to some 10,000 digits. This near-identity is a reflection of the transformation formula of theta functions.

There are many other, less known, near-identities that are related to Mellin transforms. For instance, one has the near-identities $\alpha \doteq -\frac{1}{2}$ and $\beta \doteq -\frac{1}{2}$, where

$$\begin{aligned} \alpha &= \sum_{n=1}^{\infty} (-1)^n \frac{(9/10)^n}{1+(9/10)^{2n}} = -0.24999\,99999\,99999\,99986 \cdots\cdots\cdots \\ \beta &= \sum_{n=1}^{\infty} (-1)^n \frac{(99/100)^n}{1+(99/100)^{2n}} - 0.24999 \cdots 99999\,9\mathbf{8}, \end{aligned} \tag{7}$$

where the digit 8 occurs in position 211 of β(!). These near-identities are due to Flajolet and Guillera (2000, unpublished) and surface in the study of q-analogues of polylogarithms. Beyond the teratology of sums, similar examples occur naturally in the analysis of algorithms: see works by Brent-Kung-Vallée[2], Kirschenhofer-Prodinger, etc. As a somewhat related example, the sum

$$\sum_{n=2}^{\infty} \frac{(-2\pi)^n}{(2n-2)(n-1)!\zeta(n)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)\zeta(2n+1)} = 3.14159\,26535\,\mathbf{1}, \tag{8}$$

differs from π by the tiny amount $7 \cdot 10^{-11}$. It belongs to a collection of near-identities empirically discovered by Guillera[3] around 1999. Such near-identities fail to be actual identities, in essence, because the zeta function has nontrivial zeros. They are related to several series considered by Ramanujan in his investigations surrounding the prime number theorem [see Hardy's *Lectures*].

Our near-identity, relative to differences of zeta values and expressed by Eq. (2), has a flavour of some of our previous examples. It will be established by means of Nörlund-Rice integrals [Flajolet-Sedgewick, 1995], a technology somewhat similar in spirit to Mellin transforms, which underlies (7) and (8). The asymptotic approximation obtained turns out to be exponentially small, like in the near-identity (6) What is new is the fact that the exponentially small error term in our main approximation comes from a saddle-point estimate of the Nörlund-Rice integrals. (See also [Flajolet-Gerhold-Salvy, in preparation] for a related analysis.)

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[2] Consider citing Gérard Maze at Zurich?

[3] See Jesus Guillera's web pages.

2. Nörlund and saddles

The original sum involves huge cancellations. For instance, in the expression of T_{1000}, the largest binomial coefficient is of the order of 10^{300}. Following Nörlund and later authors the best strategy consists in working with an integral representation.

• *Import material from first note, with adjustments.*

3. Miscellaneous formulæ

For the record, here are a few identities. Only a few should be kept in the final paper.

• We have the well-known identity

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = H_n,$$

with $H_n = 1 + \cdots + \frac{1}{n}$ a harmonic number. Thus, from (1), taking out the regularization term $1/k$, we get by exchange of summations and the binomial theorem

$$\begin{aligned} T_n &= H_n + \sum_{k=1}^{n} \binom{n}{k} (-1)^k \zeta(k+1) \\ &= H_n + \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left[\frac{1}{1^{k+1}} + \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} + \cdots \right] \\ &= H_n - \left[1 - (1 - \frac{1}{1})^n \right] - \frac{1}{2} \left[1 - (1 - \frac{1}{2})^n \right] - \frac{1}{3} \left[1 - (1 - \frac{1}{3})^n \right] - \cdots . \end{aligned}$$

Thus, defining

$$U_n = -T_n + H_n,$$

we have

$$U_n = \sum_{\ell \geq 1} \frac{1}{\ell} \left[1 - (1 - \frac{1}{\ell})^n \right], \tag{9}$$

with $U_0 = 0$. This simple calculation gives immediately[4] the ordinary generating function of the U_n:

$$\begin{aligned} U(z) &:= \sum_{n \geq 0} U_n z^n \\ &= \sum_{\ell \geq 1} \frac{1}{\ell} \left[\frac{1}{1-z} - \frac{1}{1 - z(1 - 1/\ell)} \right] \\ &= \frac{1}{1-z} \left(\gamma + \psi \left(\frac{1}{1-z} \right) \right). \end{aligned}$$

There, use has been made of the classical representation:

$$\psi(1+s) = \frac{d}{ds} \log \Gamma(1+s) = -\gamma + \sum_{\ell \geq 1} \left[\frac{1}{\ell} - \frac{1}{\ell + s} \right].$$

(This is not surprising since $\psi(1+s)$ is a generating function of zeta values, while differences are known to correspond to an Euler transformation.)

[4] Verified by Maple.

• The exponential generating function,

$$\widehat{U}(z) := \sum_{n\geq 0} U_n \frac{z^n}{n!},$$

comes out equally easily:

$$\widehat{U}(z) = e^z \sum_{\ell\geq 1} \frac{1}{\ell}\left[1 - e^{-z/\ell}\right].$$

Note that quantities similar to $U_n, U(z), \widehat{U}(z)$, but with the summation restricted to values of ℓ that are powers of 2, surface repeatedly in the analysis of digital trees. In that case, what plays the rôle of the zeta function is the simpler quantity $(1-2^s)^{-1}$.

• We have from (9)

$$U_n = \phi(n),$$

where

$$\phi(x) = \sum_{\ell\geq 1} \frac{1}{\ell}\left[1 - \left(1 - \frac{1}{\ell}\right)^x\right].$$

This provides an analytic lifting of the sequence U_n (hence of T_n) to complex values of the index. I don't think we can do much new with this observation.

• Similarly, the Mellin transform of $e^{-z}\widehat{U}(z)$ is

$$\int_0^\infty e^{-x}\widehat{U}(x)x^{s-1}\,dx = -\Gamma(s)\zeta(1-s), \qquad -1 < \Re(s) < 0.$$

This is also expected (by the Poisson-Mellin-Newton cycle). The Mellin transform of $U(z)$ is (essentially) obtained by replacing $\Gamma(s) \mapsto \pi/\sin\pi s$, as usual. Anyhow, we are moving around in circles.

Question [cryptic]: which formula results from the following sequence of actions: (i) start from Mellin of $\widehat{U}(z)$; (ii) express $\widehat{U}(z)$ as an inverse Mellin integral; (ii) insert the functional equation of zeta; (iii) perform the actual inverse Mellin transformation; (iv) expand. Do we get an interesting equivalent sum? The same can be directly tried on the Nörlund representation.