Symmetries of the Riemann and Hurwitz Zetas

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Abstract

The Takagi curve is a simple, exactly solvable fractal curve exhibiting the symmetry of the Modular Group $SL(2,\mathbb{Z})$. By constructing a similar curve using a Dirichlet Series instead of the MacLaurin series, one gets a visually similar curve which obeys a more complex set of self-similarities. These can be expressed as a set of relations on the Riemann Zeta and Hurwitz Zeta.

This paper assumes familiarity with the development of the symmetries of the Takagi Curve and the Minkowski Question Mark exhibited in companion papers. In other respects, the presentation is kept as simple as possible, avoiding appeals to abstract mathematical concepts as much as possible.

This is a draft; this paper is still in the process of being written.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

1 Symmetries of the Riemann and Hurwitz Zetas

Introduction to be written. The moral of this section might be "if you fool with Dirichlet series, don't be surprised to find the Riemann Zeta", which is what we'll do and find.

The general course of investigation revolves around the realization that when one considers an iterated function, the iterates form an arithmetic series. This arithmetic series can then be studied using any one of dozens of tools evolved over the centuries for studying series, from number theory to analytic functions. In earlier chapters, we considered the iterated tent map $\tau(x)$. For a fixed x, iterating on this function generates a series of values $\tau^n(x)$. The iterates can then be used to define a Maclaurin series

$$\sum_{n=1}^{\infty} \tau^n(x)\, w^{n-1} \tag{1}$$

and this series can be studied as an analytic function of w. Conversely, we can also fix a value for w, and consider the series as a function of x. In the later case, we saw that the the resulting curve, as a function of x, is called the Takagi curve. This curve is fractal, and it exhibits self similarity, with the self-similarity given by the modular group $SL(2,\mathbb{Z})$.

In the following, we will consider instead the Dirichlet series

$$\sum_{n=1}^{\infty} \tau^n(x)\, n^{-s} \tag{2}$$

This series is not trivially self-similar the way that the Takagi curve was (although, visually, it is nearly indistinguishable). However, it does offer up many interesting symmetries none-the-less, due in part to the relatively simple structure of the iterated tent map. For example, any rational number $x = n/(3 \cdot 2^m)$ will eventually iterate to the fixed point $\tau(2/3) = 2/3$ and thus the series differs from the Riemann zeta only by finite number of terms. Attempting to find modular group group elements to this sum then give relationships on the Riemann zeta. Other rational numbers, when iterated under the tent map, eventually settle down to periodic orbits with period two or with longer recurrence times. The pre-periodic terms again give rise to harmonic-number-like terms, while the periodic parts give rise to relations on the Hurwitz zeta. Some of these relationships; and possibly all of them, are known classically. The point of interest here is perhaps the novel derivation of these symmetry relationships.

The periodic orbits of the iterated tent map are a set of measure zero on the real number line. Of some curiosity is what sort of statements one might be able to make for almost-periodic orbits, or whether one might be able to make meaningful statements for larger sets of points. These last questions are (currently) outside the scope of this paper.

1.1 The Analog of the Takagi Curve as a Dirichlet Series

We saw in an earlier chapter how to build the Takagi Curve as a MacLaurin sum over the Tent Map, and how this curve transformed under the three-dimensional representation of the Modular Group. One can consider other series built out of the Tent Map, and ask what thier transformation properties are. In this section, lets consider the Dirichlet series. These are related to the Takagi Curve through analytic transformations. Thus, we define

$$\zeta_s(x) \equiv \sum_{n=1}^{\infty} \tau^n(x)\, n^{-s} \tag{3}$$

where $\tau(x)$ is the tent map as defined before:

$$\tau(x) = \begin{cases} 2(x - \lfloor x \rfloor) & \text{for} \quad 0 \le x - \lfloor x \rfloor < 1/2 \\ 2 - 2(x - \lfloor x \rfloor) & \text{for} \quad 1 > x - \lfloor x \rfloor \ge 1/2 \end{cases}$$

By substituting

$$n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} y^{s-1} e^{-ny} dy \tag{4}$$

and performing the sum, and making the change-of-variable $w = e^{-y}$ one finds that

$$\zeta_s(x) = \frac{1}{\Gamma(s)} \int_0^1 (-\ln w)^{s-1}\, t_w(x) dw \tag{5}$$

where $t_w(x)$ is the Takagi Curve as defined in previous sections. Note that a pole in $t_w(x)$ at $w = 1$ translates into a pole (with the same residue) at $s = 1$. Graphically, a picture of this Dirichlet series looks very similar to the Takagi curve itself, possibly looking a tad "noisier"; thus, figure **??** suffices to visualize this curve. XXX redo. Although visually similar, and thus "obviously" self-similar, the transformation properties of this curve are far, far messier. We get

$$(\zeta_s \circ g_D)(x) = \zeta_s(\frac{x}{2}) = x + \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-)^k \zeta_{s+k}(x) \tag{6}$$

for the simplest relation. This can be derived by using the identity

$$\left(\frac{n}{n+1}\right)^s = \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} \left(\frac{-1}{n}\right)^k \tag{7}$$

in the expansion for $\zeta_s(x/2)$.

Comparing equation 6 to the equivalent expression for the Takagi Curve, we see that the transformation law mixes curves with different order parameters s. Comparing this equation to the expression for the matrix elements for the Gauss-Kuzmin-Wirsing (GKW) Operator developed in earlier chapters, (xxx give ref) we see that this is essentially a summation over the columns of the GKW. Essentially, what we would like to be able to say is that if we could work with the diagonal form of the GKW, then we would get a curve that transforms under the simple 3D representation of the modular group, just as the Takagi Curve does. We develop idea this a bit further in a later section.

We can make an explict connection the the Riemann Zeta by considering specific values of x. For example, $t_w(1/3) = 2/3(1-w)$ and so we get

$$\zeta_s\left(\frac{1}{3}\right) = \frac{2}{3}\zeta(s) \tag{8}$$

where $\zeta(s)$ is the Riemann Zeta. Similarly, we have $\zeta_s(1/6) = (2\zeta(s)-1)/3$. Using this with equation 6 we find the following relation:

$$\zeta(s) = 1 + \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-)^k \zeta(s+k) \tag{9}$$

This sum is conditionally convergent, thus we prefer to make the analytic continuation $\zeta(s+k) = \zeta(s+k) - 1 + 1$ and perform the sum explicitly to get

$$\zeta(s) = 1 + 2^{-s} + \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-)^k \left[\zeta(s+k) - 1\right] \tag{10}$$

Although not strictly necessary for the above, we can repeat the analytic continuation trick to get

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-)^k \left[\zeta(s+k) - 1 - 2^{-(s+k)}\right] \tag{11}$$

or in general,

$$\zeta(s) = H_{m+1}(s) + \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-)^k \left[\zeta(s+k) - H_m(s+k)\right] \tag{12}$$

where we used $H_m(s)$ to stand for the harmonic number $H_m(s) = \sum_{k=1}^{m} k^{-s}$ and we require that $m \geq 1$ for the summation over zetas to be convergent.

This equation is interesting because, in a certain sense, it encapsulates the action of the generator g of the Modular Group on the Riemann Zeta. The equation itself is not "new", it can also be obtained using entirely classical methods: for example, starting with Abramowitz & Stegun equation 6.4.9, the classical series expansion for the polygamma function. What is novel is that it seems to show a certain funny type of fractal self-similarity in the Riemann Zeta. We can recursively apply this relation to find additional relations. The generator g^n is associated with

$$\zeta(s) = H_{m+n}(s) + \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) (-n)^k \left[\zeta(s+k) - H_m(s+k) \right] \tag{13}$$

where, for convergence of the right hand side, we should take $m \geq n$.

1.2 The Relation for p/q = 1/2

We would like to extend these relations to all group elements of the form $\gamma = g^{a_1} r g^{a_2} r \ldots r g^{a_N}$. This is easier said than done. The curve $\zeta_s(x)$ is even across $x = 1/2$ just like the Takagi Curve is, and so $(r_S \zeta_s)(x) = (\zeta_s \circ r_D)(x) = \zeta_s(x)$ and so we expect the polynomial parts of the transformation law to be identical to those for the Takagi curve. Thus, we can write the representation as

$$g_S = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & G \end{array} \right) \text{ and } r_S = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & I \end{array} \right) \tag{14}$$

which acts on the vector

$$\left(\begin{array}{c} 1 \\ x \\ \zeta_s(x) \end{array} \right)$$

just as in the Takagi representation. The difference is that here, G is understood to be the operator that mixes together different orders:

$$(G\zeta_s)(x) = \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) (-)^k \zeta_{s+k}(x) \tag{15}$$

Just to be crystal clear, we can now write equation 6 as

$$(g_S \zeta_s)(x) = (\zeta_s \circ g_D)(x) = x + (G\zeta_s)(x) \tag{16}$$

and then use matrix concatentation to construct general elements γ just as in the Takagi case.

This result remains abstract until we plug in specific values of x to get back relations on the Riemann Zeta. This is where the true adventure begins. We could try, for example, $x = (grg)(1/3) = 5/12$ but this can lead to some confusion. To unclutter the issue, pick $x = 1/5$ and ignore connections to g and r for a moment. We then have $\tau(1/5) = 2/5$ and $\tau^2(1/5) = 4/5$ and $\tau^3(1/5) = 2/5$ and so we see that after one step, the iteration immediately settles down to an orbit of period 2. We then have

$$\begin{aligned} \zeta_s\left(\frac{1}{5}\right) &= \sum_{n=1}^{\infty} \tau^n\left(\frac{1}{5}\right) n^{-s} \\ &= \frac{2}{5}\left(1+\frac{1}{3^s}+\frac{1}{5^s}+\ldots\right)+\frac{4}{5}\left(\frac{1}{2^s}+\frac{1}{4^s}+\ldots\right) \\ &= \frac{2}{5}\left(1-2^{-s}\right)\zeta(s)+\frac{4}{5}2^{-s}\zeta(s) \\ &= \frac{2}{5}\zeta(s)\left(1+\frac{1}{2^s}\right) \end{aligned} \tag{17}$$

Inserting this into equation 6 gives

$$\zeta(s)\left(1-\frac{1}{2^s}\right) = 1+\frac{1}{2^s}\sum_{k=0}^{\infty}\left(\begin{array}{c} s+k-1 \\ s-1 \end{array}\right)\left(\frac{-1}{2}\right)^k \zeta(s+k) \tag{18}$$

What happens if we pick an x giving an orbit of period three? For instance, $x = 1/7$ has an orbit (2/7, 4/7, 6/7). Unfortunately, this gives rise to a sum that is not directly solvable in terms of the Riemann zeta, as one prominently must wrestle with the series

$$1+\frac{1}{4^s}+\frac{1}{7^s}+\frac{1}{10^s}+\ldots \tag{19}$$

This series can be expressed as the Hurwitz Zeta, as can sums involving orbits with higher periods. We come back to these in a later section; for now we will pursue the period-2 case.

1.3 Relations for Half-Integer Values

We can consider equations 13 and 18 to be the generators of a free semigroup of symmetries on the Riemann Zeta. It thus becomes interesting to ask what some of the generated expressions look like; this section presents some of these.

We can simplify the presentation by developing a shorthand for the recurring elements in the sums. Define

$$Z_m(s,x) = \sum_{k=0}^{\infty} (-)^k \left(\begin{array}{c} s+k-1 \\ s-1 \end{array}\right) \left[\zeta(s+k) - H_m(s+k)\right] x^k \tag{20}$$

For fixed, real s this is a monotonically decreasing function of x. In this notation, we can re-write equation 18 as

$$\zeta(s)\left(1-\frac{1}{2^s}\right) = 1+\frac{1}{2^s}Z_0\left(s,\frac{1}{2}\right) \tag{21}$$

This relation can be iterated, in that the left hand side can be brought over the the right and expanded, giving a double sum. One of teh two sums is then readily performed. This iteration is a bit tedius, but when brought to the end gives the series

$$\begin{aligned}
\zeta(s)\left(1-\frac{1}{2^s}\right) &= 1+\frac{1}{3^s}+\frac{1}{4^s}\left[Z_0\left(s,\frac{1}{4}\right)+Z_0\left(s,\frac{3}{4}\right)\right] \\
&= 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+ \\
&\quad +\frac{1}{8^s}\left[Z_0\left(s,\frac{1}{8}\right)+Z_0\left(s,\frac{3}{8}\right)+Z_0\left(s,\frac{5}{8}\right)+Z_0\left(s,\frac{7}{8}\right)\right] \\
&= \sum_{k=0}^{2^{m-1}-1}\frac{1}{(2k+1)^s}+\frac{1}{2^m}Z_0\left(s,\frac{2k+1}{2^m}\right) \qquad (22)
\end{aligned}$$

There does not seem to be any simple way to further simplify the above expression, and, in particular, to obtain an expression involving only (for example) $Z_0(s,1/4)$. We will, however, be able to find this in a later section, where we consider points with orbits of period four.

Iterating on equation 13 gives corresponding equations for general half-integer and dyadic values. Turning the crank once gives a relation for general half-integer values:

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=H_{2m+2n+1}(s)-\frac{H_{n+m}(s)}{2^s}+\frac{1}{2^s}Z_m\left(s,n+\frac{1}{2}\right) \qquad (23)$$

This can be better understood by noting that

$$H_{2m+2n+1}(s)-\frac{H_{n+m}(s)}{2^s}=\sum_{k=0}^{m+n}\frac{1}{(2k+1)^s} \qquad (24)$$

and of course that

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^s} \qquad (25)$$

and so we see that we have a general resemblance between equation 13 and 23. As before, we pick a value for m large enough to make sure that the right hand side of equation 23 remains convergent ($m>n$ is enough), and it reappears on the left-hand side in a corresponding fashion. Turning the crank a second time produces

$$\begin{aligned}
\zeta(s)\left(1-\frac{1}{2^s}\right) &= H_{4m+2n+3}(s)-\frac{H_{2m+n+1}(s)}{2^s}+ \qquad (26) \\
&\quad \frac{1}{4^s}\left[Z_m\left(s,\frac{2n+1}{4}\right)+Z_m\left(s,\frac{2n+3}{4}\right)\right] \qquad (27)
\end{aligned}$$

with additional turns generating more relations in this series.

1.4 Hurwitz Zeta Relations

Orbits with longer periods give relations on the Hurwitz Zeta. First,we start by describing the equivalence classes of all values of x with longer periods (XXX finish this section). For example, $x = 1/5$ iterates to the orbit (2/5, 4/5) of period two. Iterates of $x = 1/7$ converge to the orbit (2/7, 4/7, 6/7) of period three. Iterates of $x = 1/9$ converge to (2/9, 4/9, 8/9) of period 3. Next, we have (2/11, 4/11, 8/11, 6/11, 10/11) of period 5, and, in general, for p prime, $x = 1/p$ will iterate to an orbit of length $(p-1)/2$. By contrast, $x = 1/15$ will iterate to the orbit (2/15, 4/15, 8/15, 14/15) of period four. XXX ToDo: Give a general analysis, based on expansion in terms of binary digits.

The pre-periodic part gives rise to a harmonic-number-like term, a finite sum. The periodic part gives the Hurwitz zeta.

Define the Hurwitz zeta as

$$\zeta_H(s,q) = \sum_{k=0}^{\infty} (k+q)^{-s} \tag{28}$$

It is then relatively straightforward to see that if we pick x such that it has an orbit of length n, then $\zeta_S(x)$ will involve a sum over $\zeta_H(s, m/n)$ with m ranging from 1 to n. For example, consider $x = 1/7$, with an orbit of period 3. Then we have

$$\zeta_s\left(\frac{1}{7}\right) = \frac{2}{7} 3^{-s} \left[\zeta_H\left(s, \frac{1}{3}\right) + 2\zeta_H\left(s, \frac{2}{3}\right) + 3\zeta_H(s,1)\right] \tag{29}$$

Using this, together with equation 6 gives a sum relation of the form

$$\zeta_H\left(s, \frac{2}{3}\right) + 2\zeta_H\left(s, \frac{3}{3}\right) + 3\zeta_H\left(s, \frac{1}{3}\right) = 3^{s+1} +$$

$$\sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{3}\right)^k \left[\zeta_H\left(s+k, \frac{1}{3}\right) + 2\zeta_H\left(s+k, \frac{2}{3}\right) + 3\zeta_H(s+k,1)\right] \tag{30}$$

Iterating on $x = 2/7$ gives another relation, as does $x = 4/7$, but with the coefficients permuted. We can write these three permutations in matrix form, to make the permutations clearer:

$$\left[\begin{array}{cccc} 3 & 1 & 2 & -3 \\ 1 & 2 & 3 & -1 \\ 2 & 3 & 1 & -2 \end{array} \right] \left[\begin{array}{c} \zeta_H\left(s, \frac{1}{3}\right) \\ \zeta_H\left(s, \frac{2}{3}\right) \\ \zeta_H\left(s, \frac{3}{3}\right) \\ 3^s \end{array} \right] = \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{3}\right)^k \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \right] \left[\begin{array}{c} \zeta_H\left(s+k, \frac{1}{3}\right) \\ \zeta_H\left(s+k, \frac{2}{3}\right) \\ \zeta_H\left(s+k, \frac{3}{3}\right) \end{array} \right] \tag{31}$$

These permutation matrices are not singular. We can bring the one on the right into diagonal form:

$$\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{array} \right] \left[\begin{array}{c} \zeta_H\left(s, \frac{1}{3}\right) \\ \zeta_H\left(s, \frac{2}{3}\right) \\ \zeta_H\left(s, \frac{3}{3}\right) \\ 3^s \end{array} \right] = \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{3}\right)^k \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} \zeta_H\left(s+k, \frac{1}{3}\right) \\ \zeta_H\left(s+k, \frac{2}{3}\right) \\ \zeta_H\left(s+k, \frac{3}{3}\right) \end{array} \right] \tag{32}$$

and so we can immediately read off three relations.

It is now readily appearent how to generalize this structure to an orbit of arbitrary length. Consider an orbit of period n, and let $a_1, a_2, \ldots, a_n$ be the cyclic sequence such that $\tau(a_1) = a_2$, and so on: $\tau^k(a_j) = a_{j+k \bmod n}$ giving the cyclic group. Let A be the matrix of the permuted sequence sitting inside the sum:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_2 & a_3 & a_4 & \ldots & a_1 \\ a_3 & a_4 & a_5 & \ldots & a_2 \\ \vdots & \vdots & & & \vdots \\ a_n & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix} \tag{33}$$

and let B be the matrix sitting on the left hand side, outside of the sum:

$$B = \begin{bmatrix} a_n & a_1 & a_2 & \ldots & a_{n-1} & -a_n \\ a_1 & a_2 & a_3 & \ldots & a_n & -a_1 \\ a_2 & a_3 & a_4 & \ldots & a_1 & -a_2 \\ \vdots & \vdots & & & \vdots & \\ a_{n-1} & a_n & a_1 & \ldots & a_{n-2} & -a_{n-1} \end{bmatrix} \tag{34}$$

Clearly A is invertible and B is just shifted by one, so that

$$A^{-1}B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix} \tag{35}$$

and thus we can read off two relations. The bottom row gives

$$\zeta_H\left(s, \frac{1}{n}\right) = n^s + \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{n}\right)^k \zeta(s+k) \tag{36}$$

whereas the other rows give

$$\zeta_H\left(s, \frac{m+1}{n}\right) = \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{n}\right)^k \zeta_H\left(s+k, \frac{m}{n}\right) \tag{37}$$

This last relation is formal, in that the sum on right-hand side is divergent, due to the first term of the Hurwitz zeta. However, this term is very easily removed and resummed, leaving the perfectly convergent formula below

$$\zeta_H\left(s, \frac{m+1}{n}\right) = \left(\frac{n}{m+1}\right)^s + \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-1}{n}\right)^k \left[\zeta_H\left(s+k, \frac{m}{n}\right) - \left(\frac{n}{m}\right)^{s+k}\right] \tag{38}$$

which is finite and valid for all $m,n \geq 1$. Note that the derivation of this result does not depend on actual generators of the orbit, but only on thier periods. Thus, for example, both $x = 2/7$ and $x = 1/9$ had a period of three; both would give rise to the same relations, as both are isomorphic to the cyclic group of period three, which is essentially unique.

We can go a few steps farther. Previously, we gave a relation involving 1/4 and 3/4'ths. We can now pull this apart into its components, and give a distinct 3/4'ths relationship:

$$= \sum_{k=0}^{\infty} \left(\begin{array}{c} s+k-1 \\ s-1 \end{array} \right) \left(\frac{-3}{4} \right)^k \zeta(s+k)$$

1.5 Reflection Formula

Note that although we do not have self-similarity in the same sense as the Takagi curve, one still has a reflection symmetry on sub-segments. For example:

$$\begin{aligned} (\zeta_s \circ gr)(x) = \zeta_s \left(\frac{1-x}{2} \right) &= \\ &= (\zeta_s \circ g)(x) - 1 + 2x \\ &= \zeta_s \left(\frac{x}{2} \right) - 1 + 2x \end{aligned} \tag{39}$$

and we can generalize this for arbitrary g^n...

1.6 Conclusions

This section to be written.

Note similarity to Hurwitz Zeta functional equation.

Note similarity to relation given by Gauss (see mathworld Hurwitz Zeta page)

References

I can't figure out how to make bibtex work !!!! and worse, they don't show in html! See the DVI/PDF version for refs.

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