Salut Philippe,

Here's the L-function version. The mechanism for deriving the result is nearly identical to the Riemann case. Doing this helped clarify, for me, the mechanism; several remarks about this. First, instead of considering an approx for cos, its clearer to keep that part exact, and instead consider two integrals. Each integral now has one saddle point, each the complex conjugate of the other. Slightly cleaner, it seems. Also, the saddle-point theorem is used naively, instead of being re-derived. Just plug in the values, and voila. For checking correctness, the Riemann case can be obtained by taking k=m=1 fairly naively throughout.

I also seem to be missing an overall factor of k somewhere fairly far in. I haven't proof-read this. Whether or not to fold this with the other paper or not is up to you. This is an interesting generalization, but I was unable to say anything wise by the time the end rolled around.

Finally note that a similar derivation may be made for the Dirichlet beta, for which the RH doesn't hold; so this is a more general feature, and not something limited to the Selberg class. It may be instructive to repeat the derivation for anything that has a functional equation, so as to understand how broad the result is.

–linas

Newton Series of L-functions

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11 February 2006

Abstract

The Newton series, or finite differences of the Dedekind L-functions are explored. As with similar sums for the Riemann zeta function, these sums are found to consist of a simple $\mathcal{O}(1/n)$ term plus an exponentially vanishing term of $\mathcal{O}(\exp(-K\sqrt{n}))$. The finite differences may be evaluated by re-expressing them as a Norlund-Rice integral, and then using saddle-point techniques to evaluate the integral. This provides both the ordinary term plus an asymptotic expansion for the exponentially vanishing term.

1 Introduction

Given the previous exploration of the finite differences of the Riemann zeta function and their remarkable properties, it seems worthwhile to explore if a similar set of relations hold for the Dirichlet L-functions. In fact, they do. This section reviews the the definition of finite differences, their relation to Newton series, and the definition of the L-functions. The next section proceeds directly to the evaluation of the finite differences of the L-functions.

Given a function $f(x)$, one defines its forward differences at point $x = a$ by

$$\Delta^n [f](a) = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} f(p+a) \tag{1}$$

where $\binom{n}{p}$ is the binomial coefficient. When one has only an arithmetic function or sequence of values $f_p = f(p)$, rather than a function of a continuous variable x, the above is referred to as the binomial transform of the sequence. If the function $f(x)$ is polynomially bounded (??), then the forward differences may be used to construct the umbral calculus analog of the Taylor's series for f:

$$f(z+a) = \sum_{n=0}^{\infty} \Delta^n [f](a) \, \frac{(z)_n}{n!} \tag{2}$$

where $(z)_n = z(z-1)\ldots(z-n+1)$ is the Pochhammer symbol or falling factorial. This series is known as the Newton series for f.

If the function $f(x)$ is analytic, then the forward differences may be expressed in terms of the Norlund-Rice integral

$$\Delta^n [f](a) = \frac{1}{2\pi i} \oint_C f(z+a) \frac{n!}{z(z-1)\cdots(z-n)} \, dz \tag{3}$$

with the contour C arranged so that it encircles the poles in the denominator, but not the poles of the function f. The utility of the Norlund-Rice integral is that it may be evaluated using saddle-point methods, a technique which will be applied at length below.

The Dirichlet L-functions are defined in terms of the Dirichlet characters, which are group representation characters of the cyclic group. They play an important role in number theory, and the generalized Riemann hypothesis is applied to the L-functions. The Dirichlet characters are multiplicative functions, and are periodic modulo k. That is, a character $\chi(n)$ is an arithmetic function of an integer n, with period k, so that $\chi(n+k) = \chi(n)$. A character is multiplicative, in that $\chi(mn) = \chi(m)\chi(n)$ for all integers m, n. Furthermore, one has that $\chi(1) = 1$ and $\chi(n) = 0$ whenever $\gcd(n,k) \neq 1$. The L-function associated with the character $\chi$is defined as

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{4}$$

All such L-functions may be re-expressed in terms of the Hurwitz zeta function as

$$L(\chi, s) = \frac{1}{k^s} \sum_{m=1}^{k} \chi(m) \zeta\left(s, \frac{m}{k}\right) \tag{5}$$

where k is the period of χ and $\zeta(s,q)$ is the Hurwitz zeta function, given by

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \tag{6}$$

Thus, the study of the analytic properties of the L-functions can be partially unified through the study of the Hurwitz zeta function.

2 Forward differences

In analogy to the study of the forward differences of the Riemann zeta function, the remainder of this paper will concern itself with the analysis of the series given by

$$L_n = \sum_{p=1}^{n} (-1)^p \binom{n}{p} \frac{L(\chi, p+1)}{p+1} \tag{7}$$

Because of the relation 5 connecting the Hurwitz zeta function to the L-function, it is sufficient to study sums of the form

$$A_n(m,k) = \sum_{p=1}^{n} (-1)^p \binom{n}{p} \frac{\zeta\left(p+1, \frac{m}{k}\right)}{k^{p+1}(p+1)} \tag{8}$$

since

$$L_n = \sum_{m=1}^{k} \chi(m) A_n(m,k) \tag{9}$$

Converting the sum to the Norlund-Rice integral, and extending the contour to the half-circle at positive infinity, and noting that the half-circle does not contribute to the integral, one obtains

$$A_n(m,k) = \frac{(-1)^n}{2\pi i}\, n! \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta\left(s+1, \frac{m}{k}\right)}{(s+1)k^{s+1} s(s-1)\cdots(s-n)}\, ds \tag{10}$$

Moving the integral to the left, one encounters a single pole at $s=-1$ and a double pole at $s=0$. The residue of the pole at $s=-1$ is

$$\mathrm{Res}(s=-1) = \frac{-1}{n+1} \zeta\left(0, \frac{m}{k}\right) \tag{11}$$

where one has the curious identity in the form of a multiplication theorem for the digamma function:

$$\zeta\left(0, \frac{m}{k}\right) = \frac{-1}{\pi k} \sum_{p=1}^{k} \sin\left(\frac{2\pi pm}{k}\right) \psi\left(\frac{p}{k}\right) = -B_1\left(\frac{m}{k}\right) = \frac{1}{2} - \frac{m}{k} \tag{12}$$

Here, ψ is the digamma function, the logarithmic derivative of the Gamma function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) \tag{13}$$

and $B_1$is the Bernoulli polynomial of order 1. The double pole at $s=0$ evaluates to

$$\mathrm{Res}(s=0) = \frac{-1}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_n\right] \tag{14}$$

where H_n are the harmonic numbers

$$H_n = \sum_{j=1}^{n} \frac{1}{j} \tag{15}$$

Combining these, one obtains

$$A_n(m,k) = \frac{1}{n+1}\left(\frac{m}{k} - \frac{1}{2}\right) - \frac{1}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_n\right] + a_n(m,k) \tag{16}$$

The remaining term has the remarkable property of being exponentially small; that is,

$$a_n(m,k) = \mathcal{O}\left(e^{-\sqrt{Kn}}\right) \tag{17}$$

for a constant K of order m/k. The next section develops an explicit asymptotic form for this term.

3 Saddle-point methods

The term $a_n(m,k)$ is represented by the integral

$$a_n(m,k) = \frac{(-1)^n}{2\pi i} n! \int_{\frac{-3}{2}-i\infty}^{\frac{-3}{2}+i\infty} \frac{\zeta\left(s+1,\frac{m}{k}\right)}{(s+1)k^{s+1}s(s-1)\cdots(s-n)}\, ds \tag{18}$$

which resulted from shifting the integration contour past the poles. At this point, the functional equation for the Hurwitz zeta may be applied. This equation is

$$\zeta\left(1-s,\frac{m}{k}\right) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{p=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi pm}{k}\right) \zeta\left(s,\frac{p}{k}\right) \tag{19}$$

This allows the integral to be expressed as

$$a_n(m,k) = \frac{n!}{k\pi i} \sum_{p=1}^{k} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n+1)} \cos\left(\frac{\pi s}{2} - \frac{2\pi pm}{k}\right) \zeta\left(s,\frac{p}{k}\right) ds \tag{20}$$

It will prove to be convenient to pull the phase factor out of the cosine part; we do this now, and write this integral as

$$\begin{aligned} a_n(m,k) &= \frac{n!}{k\pi i} \sum_{p=1}^{k} \exp\left(i\frac{2\pi pm}{k}\right) \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n+1)} \exp\left(-i\frac{\pi s}{2}\right) \zeta\left(s,\frac{p}{k}\right) ds \\ &+ c.c. \end{aligned} \tag{21}$$

where $c.c.$ means that i should be replaced by $-i$ in the two exp parts.

For large values of n, this integral may be evaluated by means of the saddle-point method. The saddle-point method, or method of steepest descents, may be applied whenever the integrand can be approximated by a sharply peaked Gaussian, as the above can be for large n. More precisely, The saddle-point theorem states that

$$\int e^{-Nf(x)}dx \approx \sqrt{\frac{2\pi}{N\,|f''(x_0)|}} e^{-Nf(x_0)} \left[1 - \frac{f^{(4)}(x_0)}{8N\,|f''(x_0)|^2} + \cdots\right] \tag{22}$$

is an asymptotic expansion for large N. Here, the function f is taken to have a local minimum at $x = x_0$ and $f''(x_0)$ and $f^{(4)}(x_0)$ are the second and fourth derivatives at the local minimum.

To recast the equation 21 into the form needed for the method of steepest descents, an asymptotic expansion of the integrands will need to be made for large n. After such an expansion, it is seen that the saddle point occurs at large values of s, and so an asymptotic expansion in large s is warranted as well. As it is confusing and laborious to simultaneously expand in two parameters, it is better to seek out an order parameter to couple the two. This may be done as follows. One notes that the integrands have a minimum, on the real s axis, near $s = \sigma_0 = \sqrt{\pi kn/p}$ and so the appropriate scaling parameter is $z = s/\sqrt{n}$. One should then immediately perform a change of variable from s to z. The asymptotic expansion is then performed by holding z constant, and taking n large. Thus, one writes

$$a_n(m,k) = \frac{1}{k\pi i}\sum_{p=1}^{k}\left[\exp\left(i\frac{2\pi pm}{k}\right)\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\psi(z)}dz \right. \tag{23}$$

$$\left. +\exp\left(-i\frac{2\pi pm}{k}\right)\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\overline{\psi}(z)}dz\right] \tag{24}$$

where $\overline{\psi}$ is the complex conjugate of ψ.

Proceeding, one has

$$\psi(z) = \log n! + \frac{1}{2}\log n + \phi\left(z\sqrt{n}\right) \tag{25}$$

and

$$\phi(s) \approx -s\log\left(\frac{2\pi p}{k}\right) - i\frac{\pi s}{2} + \log\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n+1)} \tag{26}$$

where the approximation that $\zeta(s, p/k) \approx (k/p)^s$ for large s has been made. More generally, one has

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log n} \tag{27}$$

where $\Lambda(n)$ is the Mangoldt function. (What about Hurwitz?) The asymptotic expansion for the Gamma function is given by the Stirling expansion,

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi + \sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \tag{28}$$

and B_k are the Bernoulli numbers. Expanding to $O(1/n)$ and collecting terms, one obtains

$$\psi(z) \approx -\frac{1}{2}\log n - z\sqrt{n}\left[\log\frac{2\pi p}{k} + i\frac{\pi}{2} + 2 - 2\log z\right]$$

$$+\log 2\pi - 2\log z - \frac{z^2}{2} \tag{29}$$

$$+\frac{1}{\sqrt{n}}\left[\frac{7}{6z} - \frac{z}{2} + \frac{z^3}{3}\right] + \frac{1}{n}\left[\frac{73}{144z^2} + \frac{5z^2}{4}\right] + O\left(n^{-3/2}\right) \tag{30}$$

The saddle point may be obtained by solving $\psi'(z) = 0$. To lowest order, one obtains $z_0 = (1+i)\sqrt{\pi p/k}$. To use the saddle-point formula, one needs $\psi''(z_0) = 2\sqrt{n}/z + O(1)$. Substituting, one directly obtains

$$\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\psi(z)}dz \approx \left(\frac{2\pi^3 p}{n^3 k}\right)^{1/4} e^{i\pi/8}\exp\left(-(1+i)\sqrt{\frac{4\pi pn}{k}}\right) + O\,(??) \tag{31}$$

while the integral for $\overline{\psi}$ is the complex conjugate of this (having a saddle point at the complex conjugate location). Inserting this into equation 23 gives

$$a_n(m,k) \approx \frac{1}{k}\left(\frac{2}{\pi n^3}\right)^{1/4} \sum_{p=1}^{k} \left(\frac{p}{k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi pn}{k}}\right) \sin\left(\frac{2\pi pm}{k} + \frac{\pi}{8} - \sqrt{\frac{4\pi pn}{k}}\right) \tag{32}$$

For large n, only the $p=1$ term contributes significantly, and so one may write

$$a_n(m,k) \approx \frac{1}{k}\left(\frac{2}{\pi k n^3}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi n}{k}}\right) \sin\left(\frac{2\pi m}{k} + \frac{\pi}{8} - \sqrt{\frac{4\pi n}{k}}\right) \tag{33}$$

which demonstrates the desired result: the terms $a_n$are exponentially small.

4 Conclusion

We conclude by briefly returning to the structure of the Dirichlet L-functions. The L-function coefficients defined in equation 7 are now given by

$$L_n = \sum_{m=1}^{k} \chi(m) A_n(m,k) \tag{34}$$

Writing

$$A_n = B_n + a_n \tag{35}$$

so that $B_n(m,k)$represents the non-exponential part, one may state a few results. For the non-principal characters, one has $\sum_{m=1}^{k} \chi(m) = 0$ and thus, the first term simplifies to

$$\sum_{m=1}^{k} \chi(m) B_n(m,k) = \frac{1}{k} \sum_{m=1}^{k} \chi(m) \left[\frac{m}{n+1} - \psi\left(\frac{m}{k}\right)\right] \tag{36}$$

For the principal character χ_1, one has $\sum_{m=1}^{k} \chi_1(m) = \varphi(k)$ with $\varphi(k)$ the Euler totient function. Thus, for the principal character, one obtains

$$\sum_{m=1}^{k} \chi_1(m) B_n(m,k) = -\varphi(k)\left[\frac{1}{2(n+1)} + \frac{1}{k}(\ln k + 1 - H_n)\right] + \frac{1}{k} \sum_{m=1}^{k} \chi(m) \left[\frac{m}{n+1} - \psi\left(\frac{m}{k}\right)\right] \tag{37}$$

By contrast, the exponentially small term invokes a linear combination of Gauss sums. The Gauss sum associated with a character $\chi$is

$$G(n,\chi) = \sum_{m \bmod k} \chi(m) e^{2\pi i m n/k} \tag{38}$$

and so, to leading order

$$\sum_{m=1}^{k}\chi(m)a_n(m,k) \approx \frac{1}{2ik}\left(\frac{2}{\pi kn^3}\right)^{1/4}\exp\left(-\sqrt{\frac{4\pi n}{k}}\right)$$
$$\left[\exp i\left(\frac{\pi}{8}-\sqrt{\frac{4\pi n}{k}}\right)G(1,\chi)-\exp -i\left(\frac{\pi}{8}-\sqrt{\frac{4\pi n}{k}}\right)G(-1,\chi)\right] \quad (39)$$

The higher-order terms with $p > 1$ dropped from equation 32 correspond to terms involving $G(p,\chi)$.

That's all. Not sure what more to say at this point. – linas

5 Appendix

A related but simpler integral can be found in Tom M. Apostol, *Introduction to Analytic Number Theory*, Lemma 3, Chapter 13. The integral resembles equation **??**. Re-expressed so as to heighten the resemblance, it states

$$\begin{aligned} b_n(m) &= \frac{(-1)^n}{2\pi i}n!\int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty}\frac{1}{(s+1)m^{s+1}s(s-1)\cdots(s-n)}ds \\ &= \frac{(-1)^n}{2\pi i}n!\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{m^s}{s(s+1)\cdots(s+n+1)}ds \\ &= \begin{cases}\frac{1}{n+1}\left(1-\frac{1}{m}\right)^{n+1} & \text{for } m\geq 1\\ 0 & \text{for } m<1\end{cases} \end{aligned} \quad (40)$$

A naive application of this identity to the integrals in this paper leads to divergent formal sums.