THE GAUSS-KUZMIN-WIRSING OPERATOR

LINAS VEPŠTAS <LINASVEPSTAS@GMAIL.COM>

ABSTRACT. This chapter presents an incomplete review of the Gauss-Kuzmin-Wirsing (GKW) operator. The GKW operator is the transfer operator of the Gauss map, and thus has connections to the theory of continued fractions.

The operator has a complicated structure, and has not been fully solved yet. Eigenvalues and eigenfunctions can be obtained numerically, but little else is known in the mathematical literature.

The review given here is incomplete; it is a diary of research results.

A few connections to the Minkowski Question Mark Function are probed. In particular, the Question Mark is used to define a transfer operator which is "topologically equivalent" to the GKW. This "equivalent" operator is solvable, and can be shown to have fractal eigenfunctions. However, the spectrum of this operator is not at all the same as that of the GKW. This is because the Jacobian of the transformation relating the two is given by $(?' \circ ?^{-1})(x)$, which is well-known as the prototypical "multi-fractal measure".

The general presentation attempts to assume very little math background. This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

1. THE GAUSS-KUZMIN-WIRSING OPERATOR

This text is a diary of ongoing research results. As such, it is disorganized. It is only sporadically updated.

The general layout is:

- Present the Gauss-Kuzmin-Wirsing operator, including bvasic,facts, theorems, raltionships.
- Show that the Minkowksi Question Mark converts the GKW into a sawtooth.
- Solve the two sawtooth transfer operators (these are exactly solvable).
- Review the Farey Map

2. THE GAUSS-KUZMIN-WIRSING OPERATOR

The map that truncates continued fractions is

$$h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \tag{2.1}$$

and is often called the Gauss Map. The Ruelle-Frobenius-Perron operator associated with the Gauss map is known as the Gauss-Kuzmin-Wirsing (GKW)[**?**, **?**] operator $\mathscr{L}_h$. It is the pushback of h, and as such, is a linear map between spaces of functions on the unit interval (Banach spaces). That is, given the vector space of functions from the closed unit interval to the real numbers

$$\mathscr{F} = \{f \,|\, f : [0,1] \to \mathbb{R}\}$$

then $\mathscr{L}_h$ is a linear operator mapping $\mathscr{F}$ to $\mathscr{F}$. Given $f \in \mathscr{F}$, it is represented by

Date: 2 January 2004 (revised Oct 2008).

$$[\mathscr{L}_h f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right) \tag{2.2}$$

This operator has not been "solved", in the sense that there is no known closed-form solution expressing its all of its eigenfunctions and eigenvectors. There is one classically known eigenvector, $f(x) = 1/(1+x)$, which corresponds to the unit eigenvalue; this solution was given by Gauss.

Aside from this, there is a large class of fractal, discontinuous-everywhere functions associated with eiegenvalue 1. The prototypical such solution is the derivative of the Minkowski Question Mark function $?(x)$. That is,

$$\left[\mathscr{L}_h ?'\right](x) = ?'(x)$$

A proper construction for the everywhere-discontinuous function $?'$, and the derivation of the above idetity, is given in [**?**], together with a construction of a class of other similar solutions.

2.1. **Relation to the Riemann Zeta.**

The Gauss map is connected to the Riemann Zeta by a Mellin Transform:

$$\zeta(s) = \frac{1}{s-1} - s\int_0^1 h(x)x^{s-1}dx \tag{2.3}$$

The Riemann zeta can be written under a change of variable as

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x\left[\mathscr{L}_h x^{s-1}\right] \tag{2.4}$$

and thus it seems that a better understanding of GKW may shed light on the Riemann Hypothesis. Most immediately, simple manipulations lead to a representation for the Riemann zeta in terms of binomial coefficeints. This relation is explored in detail in [**?**, **?**]; it leads to criteria that are shown to be equivalent to the Riemann Hypothesis. The same development can be used to obtain Dirichlet L-functions, and generalizations to other number-theoretic series, such as the totient series, or the Liouville series [**?**].

Todo: Give the Riemann Hypothesis can be understood as a vector equation (using the polunomial form for the GKW, below).

2.2. **Lack of simple solutions.**

Aside from the classical solution, $1/(1+x)$, there do not seem to be any "easy" polynomial series solutions to the operator, where a "solution" would be a closed-form spcification of the eigenvectors.

The author performed a combinatorial search of simple combinations and summations of rational functions and various classical functions, such as the exponential, the gamma, the digamma, the dilogarithm, Bessel functions and the exponential integral. No eigenvalues were found in this way, although, of course, various close approximations can be so obtained.

The naivest approaches to solving the GKW operator, which is suggested by the eqn 2.2, is blocked by the next two theorems.

Theorem 2.1. *There are no polynomial solutions to the equation*

$$f\left(\frac{1}{\tau+n}\right) = \lambda_n (\tau+n)^2 f(\tau)$$

Proof. Assume that there does exist such a solution. Write

$$f(\tau) = \sum_{k=0}^{\infty} a_k \tau^k$$

Inserting this into the hypothetical form leads to the equation

$$\lambda_n \sum_{k=0}^{\infty} \tau^k \left[a_{k-2} + 2na_{k-1} + n^2 a_k\right] =$$
$$\sum_{k=0}^{\infty} \tau^k \frac{(-1)^k}{n^k} \sum_{j=0}^{\infty} a_j \binom{j+k-1}{j-1} \frac{1}{n^j}$$

Setting $\tau = 1$ in the above allows it to be re-written as

$$\lambda_n (n-1)^2 \sum_{k=0}^{\infty} (-1)^k a_k = a_0 + \sum_{k=1}^{\infty} n^{-k} \sum_{j=0}^{k} \binom{k}{j} a_{j+1}$$

Since the a_k are independent of n by assumption, one must then have $a_0 = 0$, and for each term in the series on the right hand side, one must have, individually, that

$$0 = \sum_{j=0}^{k} \binom{k}{j} a_{j+1}$$

or $a_k = 0$ for all k. Thus the theorem is proved. □

Were it not for this theorem, a solution would have been provided by looking for quasi-modular-form-like functions f. Another naive avenue is also blocked:

Theorem 2.2. *There is only one series solution to*

$$f\left(\frac{1}{\tau+n}\right) = \frac{(\tau+n)(\tau+1)}{(\tau+n+1)} \lambda f(\tau)$$

and it is $\lambda = 1$ *and* $f(\tau) = a_0/(1+\tau)$ *for any constant* a_0.

Proof. As in the previous proof, assuume a series solution. Substituting this into the above, and performing a straightforward but tedious expansion in powers of τ, n, and then comparing terms, reveals that $\lambda = 1$, and that $a_k = (-1)^k a_0$. □

If the above had allowed solutions for something other than $\lambda = 1$, then one would have also had that $\mathscr{L}_h f = \lambda f$.

2.3. **Assorted Algebraic Identities.** This section lists an assortment of random algebraic results, none particularly deep; some are vaguely suggestive of deeper relations. These are listed here mostly for the sake of completeness. These are the sorts of identities one obtains by means of knuckle-headed persistence in the hope that that maybe one little algebraic twist will yeild a closed-form solution. These were obtained by the author long before he knew that $\mathscr{L}_h$ had a name or had been previously studied.

First, we notice that adjacent terms in the series can be made to cancel by shifting the series by one:

$$[\mathscr{L}_h f](x) - [\mathscr{L}_h f](x+1) = \frac{1}{(1+x)^2} f\left(\frac{1}{1+x}\right) \tag{2.5}$$

which holds for any function $f(x)$. Thus, if $\rho(x)$ is an eigenvector, so that $\mathcal{L}_h\rho = \lambda\rho$, then it would also solve

$$\frac{1}{(1+x)^2}\rho\left(\frac{1}{1+x}\right) = \lambda\left(\rho(x)-\rho(x+1)\right) \tag{2.6}$$

This can be solved easily to get the zeroth eigenvector

$$\rho_0(x) = \frac{1}{\ln 2}\,\frac{1}{1+x} \tag{2.7}$$

which satisfies $[\mathcal{L}_h\rho_0](x) = \rho_0(x)$ and the normalization is given by requiring

$$\int_0^1 \rho_0(x)\,dx = 1 \tag{2.8}$$

One can see one hint of the relationship between period-doubling and the GKW in the identity

$$\frac{1}{1+x} = \sum_{n=1}^{\infty}\frac{1}{2^n}\left[\frac{2}{x+n}-\frac{1}{x+n+1}\right] \tag{2.9}$$

A reflection identity: $f(x) = 1-(1+x)^{-2}$ satisfies $\mathcal{L}_h f = 1-f$.

Another: $\mathcal{L}_h[(1+?(x))/(1+x)^2] = 1-?(x)$ where $?(x)$ is the Minkowski Question Mark function.

Another: $\mathcal{L}_h[?(x)x^{-2}] = 2-?(x)$. One can construct a variety of identities of this sort, for example:

$$\mathcal{L}_h\left[?(x)\left(\frac{1}{(1+x)^2}-2\right)\right] = \frac{?(x)-2}{(1+x)^2} \tag{2.10}$$

but these types of exercises do not seem to lead to any sort of worthwhile recurrence relations.

Acting on the monomial, one gets

$$\left[\mathcal{L}_h x^k\right](x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^{k+2}} = \frac{(-)^{k+2}}{(k+1)!}\psi^{(k+1)}(1+x) \tag{2.11}$$

where $\psi^{(k)}(x)$ is the k'th derivative of the Gamma function. The true difficulty of finding the solution to GKW becomes clear when the search leads one to start discovering complicated identities, such as

$$\sum_{m=1}^{\infty}\frac{1}{m^2}\psi^{(1)}\left(1+\frac{1}{m}+x\right) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^2}\psi^{(1)}\left(\frac{1}{n+x}+1\right) \tag{2.12}$$

or to finding curiosities such as $f(x) = (1+ax)^2$ gives $\mathcal{L}_h f = \psi^{(1)}(1+x+a)$.

For $f(x) = (1+nx)^{-2}-1$ one gets $\mathcal{L}_h f = -\sum_{k=1}^{n}(x+k)^{-2}$

Acting on a general power, the map gives the Hurwitz Zeta:

$$[\mathcal{L}_h x^s](x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^{s+2}} = \zeta(s+2,x+1) \tag{2.13}$$

This allows eqn 2.4 to be written as

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x\zeta(s+1,x+1)$$

Further consideration leads to various conditionally convergent series:

$$\sum_{k=0}^{\infty}(-)^k\binom{k+m+1}{m}\zeta(k+m+2) = 1 \tag{2.14}$$

which holds for any integer m. We also have series such as

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k e^{-tk}=\frac{1}{2} \tag{2.15}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k (k+2)e^{-tk}=\frac{3}{4} \tag{2.16}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k (k+2)(k+3)e^{-tk}=\frac{7}{4} \tag{2.17}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k (k+2)(k+3)(k+4)e^{-tk}=\frac{45}{8} \tag{2.18}$$

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k (k+2)(k+3)(k+4)(k+5)e^{-tk}=\frac{93}{4} \tag{2.19}$$

Its not clear what the general expression for forms of the above type is. Similarly, if we let

$$S_m\equiv\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k\frac{(k+m+1)!}{(k+1)!}\left[\zeta(k+m+2)-1\right]e^{-tk} \tag{2.20}$$

then we find $S_0=1/2$, $S_1=1/4$, $S_2=1/4$, $S_3=3/8$ and $S_4=3/4$ but its again not clear what the general expression might be. The above sums are generated by considering

$$\psi(1+z)=\frac{-1}{1+z}+1-\gamma+\sum_{m=0}^{\infty}(-)^m\left[\zeta(m+2)-1\right]z^{m+1} \tag{2.21}$$

and then writing

$$z^{m+1}=(z+1-1)^{m+1}=\sum_{k=0}^{m}(-)^{m-k}\binom{m}{k}(z+1)^k$$

The generating function for the moments of the Minkowski Question Mark[**?**, **?**, **?**] participates in an curious identity. This generating function obeys the relation

$$\frac{1}{z^2}G\left(\frac{1}{z}\right)+\frac{1}{(z+1)^2}G\left(\frac{1}{z+1}\right)=\frac{1}{z(z+1)}$$

which holds for complex-valued z (such as, for example, $z\mapsto z+n$), and also

$$\frac{1}{z}+\frac{1}{z^2}G\left(\frac{1}{z}\right)=G(z)-2G(z+1)$$

From this, one has the curious shift-over-by-one relationship

$$\left[\mathscr{L}_h G\right](z)=G(1+z)+\left[\mathscr{L}_h K\right](z)$$

where we've defined $K(z)=G(1+z)$.

2.4. **Polynomial Representation.** One can attempt to solve GKW by working in the polynomial representation. One possible choice is to make one's Taylor expansion about $x=0$, but this turns out to be a very poor choice, as we shall soon see. Thus, if we write $\mathscr{L}_h f = g$ and substitute a Taylor's expansion for f and g, we get

$$\frac{g^{(m)}(0)}{m!} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (-)^m \frac{(k+m+1)!}{m!\,(k+1)!} \zeta(k+m+2) \tag{2.22}$$

or, adopting the bra-ket notation introduced earlier, we have

$$\langle m|\mathscr{L}_h|k\rangle = (-)^m \left(\begin{array}{c} k+m+1 \\ m \end{array} \right) \zeta(k+m+2) \tag{2.23}$$

where we've replaced the factorials by the binomial coefficient that they form. Unfortunately, this is clearly a very poorly conditioned matrix. One can make some progress, if one wishes, by applying a regulator and using Levin-type sequence acceleration techniques. One can thus find a number of curious identities, some of which we've listed previously. However, the difficulty of working with divergent sums seems to outweigh any advantages given by the relatively simple form of the matrix elements. Thus, we are lead to consider the matrix elements for a polynomial expansion about $x=1$. These are far more complex, but give a very well-conditioned matrix. These are:

$$G_{mn} = \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} k+m+1 \\ m \end{array} \right) [\zeta(k+m+2)-1] \tag{2.24}$$

satisfying

$$(-)^m \frac{g^{(m)}(1)}{m!} = \sum_{n=0}^{\infty} G_{mn} (-)^n \frac{f^{(n)}(1)}{n!} \tag{2.25}$$

Other authors have chosen to expand about $x=1/2$ [**?**] but it would appear that expansion about $x=1$ leads to the simplest tractable expansion. The general case is explored in the appendix.

The first eigenvalue is known as the GKW constant, and is about 0.3036.

The eigenvector equation to be solved is then

$$\sum_{n=0}^{\infty} G_{mn} v_n = \lambda v_m \tag{2.26}$$

where $v = \{v_n\}$ is an eigenvector with components v_n. This beast has a discrete spectrum (need reference for proof), and so we may label the eigenvalues and eigenvectors with a label k, so that the k'th eigen equation is

$$\sum_{n=0}^{\infty} G_{mn} v_{nk} = \lambda_k v_{mk} \tag{2.27}$$

The k'th polynomial eigenfunction of the GKW operator is then given by

$$\rho_k(x) = \sum_{n=0}^{\infty} v_{nk} (1-x)^k \tag{2.28}$$

The zeroth eigenfunction was given by Gauss as

$$\rho_0(x) = \frac{1}{1+x} \tag{2.29}$$

and corresponds to the eigenvector $v_{n0} = 2^{-n}$. Again, we note the curious appearance of powers of two.

Based on numerical explorations (reported below), the series appears to be easily convergent even for $x = 0$. In particular, it appears that $\lim_{n\to\infty} v_{nk} = 0$, and that furthermore, that $\lim_{n\to\infty} v_{nk}/v_{n+1,k} = 2$. One must not conclude that this implies that $v_{nk}\ \mathcal{O}(2^{-n})$, as this sort of asymptotic behaviour conflicts with the zeroth eigenfunction. Thus, one might guess at $v_{nk}\ \mathcal{O}(n^s 2^{-n})$ but numeric data suggests that $s = 0$. Numeric data also excludes the form $v_{nk}\ \mathcal{O}(2^{-n}\log n)$ although it is possible that the softer $v_{nk}\ \mathcal{O}(2^{-n}\log\log n)$ might hold.

The coefficients are oscillatory, with k half-oscillations in the k'th eigenvector. That is, for $k = 0$, all of the v_{n0} can be taken to be of the same sign. For $k = 1$, the v_{n1} change sign, once, between $n = 0$ and $n = 1$. For $k = 2$, the coefficients change sign twice, and so on. This is shown in the graph below.

The left eigenvectors are given by

$$\sum_{m=0}^{\infty} w_{km} G_{mn} = \lambda_k w_{kn} \tag{2.30}$$

and correspond to left eigenfunctions

$$\ell_k(x) = \sum_{n=0}^{\infty} w_{kn} (-1)^n \delta^{(n)}(1-x) \tag{2.31}$$

where $\delta^{(n)}(x)$ is the n'th derivative of the Dirac delta function. The zeroth left eigenvector is given by $w_{0n} = 1/(n+1)$. Very curiously, this is the harmonic series. Thus, true to form, it appears that yet again, we are in the presence of another manifestation of the duality between the dyadic rationals and rationals, the duality between the Stern-Brocot tree and the dyadic tree, the duality captured in the Minkowski Question Mark function.

In analogy to the right eigenvectors, the series again appears to be not only convergent in that $\lim_{n\to\infty} w_{kn} = 0$, but also that a strict ratio is maintained in the limit: $\lim_{n\to\infty}(n+2)\,w_{kn}/(n+1)\,w_{k,n+1} = 1$, with strict equality holding for all n, and not just in the limit, when $k = 0$. A similar oscillatory behaviour is shown as well.

2.5. **Identity.**

We have the identity

$$\sum_{n=0}^{\infty} G_{mn} p^{-n} = p\left[\zeta(m+1) - 1 - \zeta\left(m+1, 2+\frac{1}{p-1}\right)\right] \tag{2.32}$$

In particular, for $p = 2$, the right hand side equals p^{-m}; this corresponds to the known eigenvector. Note that for any value of p, the leading term on the right is $2^{-(m+1)}$. A simple way to arrive at this is to note that

$$\sum_{n=0}^{\infty} p^{-n}(1-x)^n = \frac{p}{p-1+x} \tag{2.33}$$

and then evaluate this simple expression under the action of the GKW operator.

2.6. **The Kernel.**

We would like to know what the kernel of the GKW operator is. Consider

$$k(x) = \frac{1}{x^2}\exp(2k+1)\frac{\pi}{x} \tag{2.34}$$

TABLE 1. GKW Eigenvalues

N	Eigenvalue λ_n	Ratio λ_n/λ_{n+1}
0	1	-3.29312425436788
1	-0.303663002898733	-3.0100062440358
2	0.100884509293104	-2.842124671335
3	-0.03549615904	-2.763682528286
4	0.01284379036244	-2.72242392332
5	-0.00471777751158	-2.69791537939
6	0.0017486751243	-2.6819312645
7	-0.0006520208580	-2.67077775
8	0.00024413145	-2.662606
9	-9.16889×10^{-5}	-2.65651
10	3.45147×10^{-5}	-2.6539
11	-1.3005×10^{-5}	
12	4.860×10^{-6}	
13	-1.7×10^{-6}	

This table lists te first dozen eigenvalues of the polynomial representation GKW operator. The numbers are certain to about about the last figure or two quoted. They were obtained by nmerically inverting a 55 by 55 matrix.

Then

$$\begin{aligned}\left[\mathscr{L}_h k\right](x) &= \exp\left((2k+1)\pi x\right)\sum_{n=1}^{\infty}\cos(2k+1)n\pi \\ &= \exp\left((2k+1)\pi x\right)\sum_{n=1}^{\infty}(-)^n \end{aligned} \tag{2.35}$$

The value of the sum is ambiguous. One is tempted to cancel terms pairwise, and thus declare $k(x)$ to belong to the kernel. On the other hand, the regularized sum is not zero:

$$\lim_{t\to 0}\sum_{k=0}^{\infty}(-)^k e^{-tk} = \frac{1}{2} \tag{2.36}$$

and so one is left in a bit of a quandary.

Some progress might be made by applying the theory of the Henstock-Kurzweil integral.

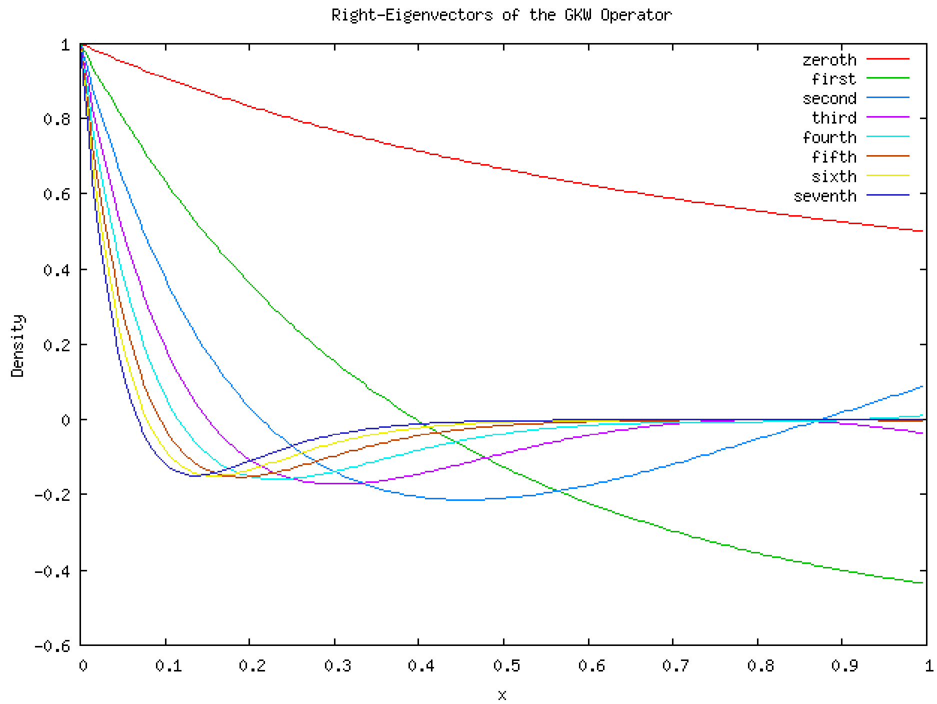
2.7. **Numeric Attacks.** One can mount numeric attacks on GKW. One can find, for instance, that

$$\rho_1(x) \approx \frac{-3}{4} + \frac{7}{4}\frac{1}{(1+x)^{5/2}} \tag{2.37}$$

is accurate to about one or two percent over the domain $x \in [0,1]$. It is associated with an eigenvalue $\lambda_1 \approx 0.303663$.

The matrix elements in 2.24 are easily numerically ... etc. Using the LAPACK DGEEV eigenvalue-finding routine. Can be numerically compared to previously published values (need ref).

FIGURE 2.1. GKW Right Eigenfunctions



This graph shows the right eigenfunctions of the GKW operator. These were computed numerically, by truncating the GKW operator to a 50 by 50 matrix and then solving for the eigenvectors of the matrix. The elements of the eigenvectors appear to be well-behaved, being oscillatory for small values, and then converging to zero rapidly. That is, if

3. The Singular Sawtooth of the First Kind

The Gauss Map $h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ is used in the construction of continued fractions. In this section, we will study a model for this function, where we replace the curve by a set of straight lines arranged between values of $1/n$ for integer n. This forms a singular sawtooth, with a singularity at $x = 0$.

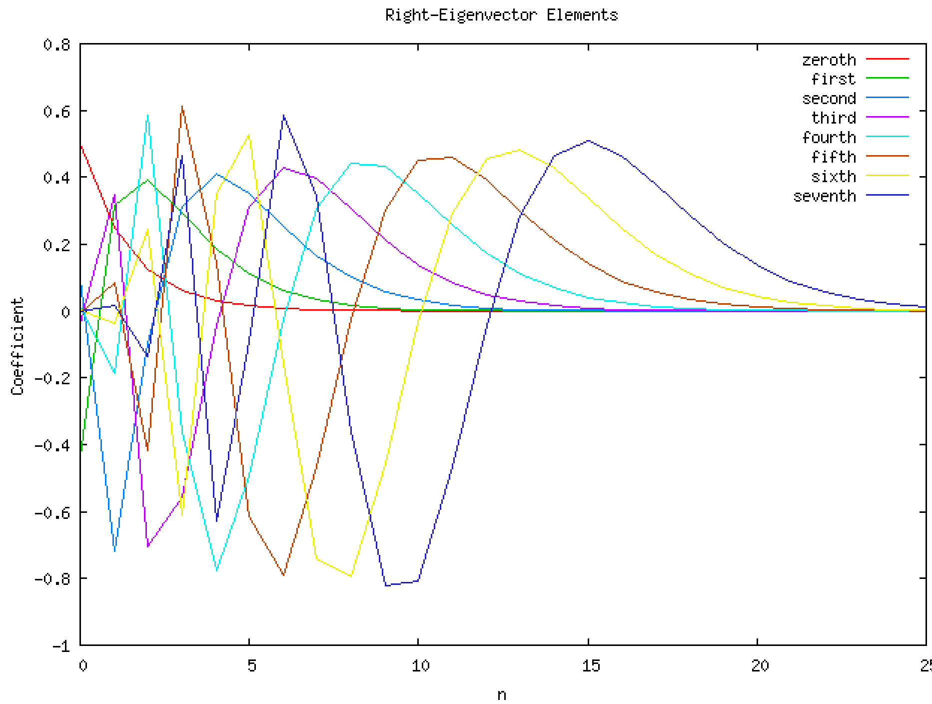
$$w(x) = \begin{cases} 2-2x & \text{for} \quad \frac{1}{2} < x \leq 1 \\ 3-6x & \text{for} \quad \frac{1}{3} < x \leq \frac{1}{2} \\ 4-12x & \text{for} \quad \frac{1}{4} < x \leq \frac{1}{3} \\ n+1-n(n+1)x & \text{for} \quad \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases} \tag{3.1}$$

This function is pictured in figure 3.1.

The Frobenius-Perron operator for this function is exactly solvable, and provides a toy model of the Gauss-Kuzmin-Wirsing operator. The Frobenius-Perron operator of this sawtooth, acting on a general function $f(x)$, is given by

$$\left[\mathscr{L}_w f\right](x) = \sum_{x':w(x')=x} \frac{f(x')}{|dw(x')/dx'|} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} f\left(\frac{n+1-x}{n(n+1)}\right) \tag{3.2}$$

FIGURE 2.2. Right Eigenvector Coefficients



This figure shows a graph of the coefficients of the first eight right-eigenvectors of the GKW operator. The red line corresponds to the zeroth eigenvector with components $v_{n0} = 2^{-(n+1)}$. All of these were obtained numerically. The normalization used here is to require that $\sum_{n=0}^{\infty} v_{nk} = 1$.

We develop a representation of this operator in the monomial-basis Hilbert space below.

3.1. **The Polynomial Eigenfunctions.** We will want to consider the action of this operator on polynomials of $y = 1 - x$, so that we can express $f(x)$ as a Taylor's expansion about $y = 0$. Lets make this change-of-variable now, and write

$$f(y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} y^k \equiv \sum_{k=0}^{\infty} a_k y^k \tag{3.3}$$

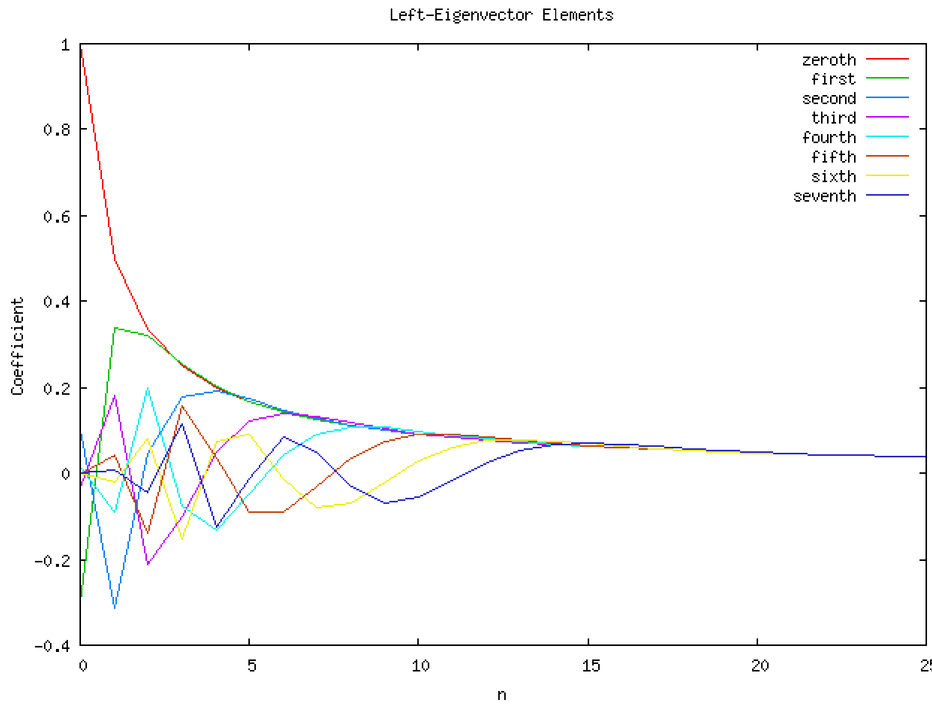
so that

$$[\mathscr{L}_w f](y) = \sum_{k=0}^{\infty} b_k y^k = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{k=0}^{\infty} a_k \left(\frac{n+y}{n(n+1)} \right)^k \tag{3.4}$$

Rearranging the sums, and equating terms with the same power of y, we define the matrix elements W_{mk} so that

$$b_m = \sum_{k=0}^{\infty} W_{mk} a_k \tag{3.5}$$

FIGURE 2.3. Left Eigenvector Coefficients



This figure shows a graph of the coefficients of the first eight left-eigenvectors of the GKW operator. The red line corresponds to the zeroth eigenvector with components $w_{n0} = 1/(n+1)$. All of these were obtained numerically. The normalization used here is to require that w_{nk} $1/(n+1)$ for large values of n.

and find that

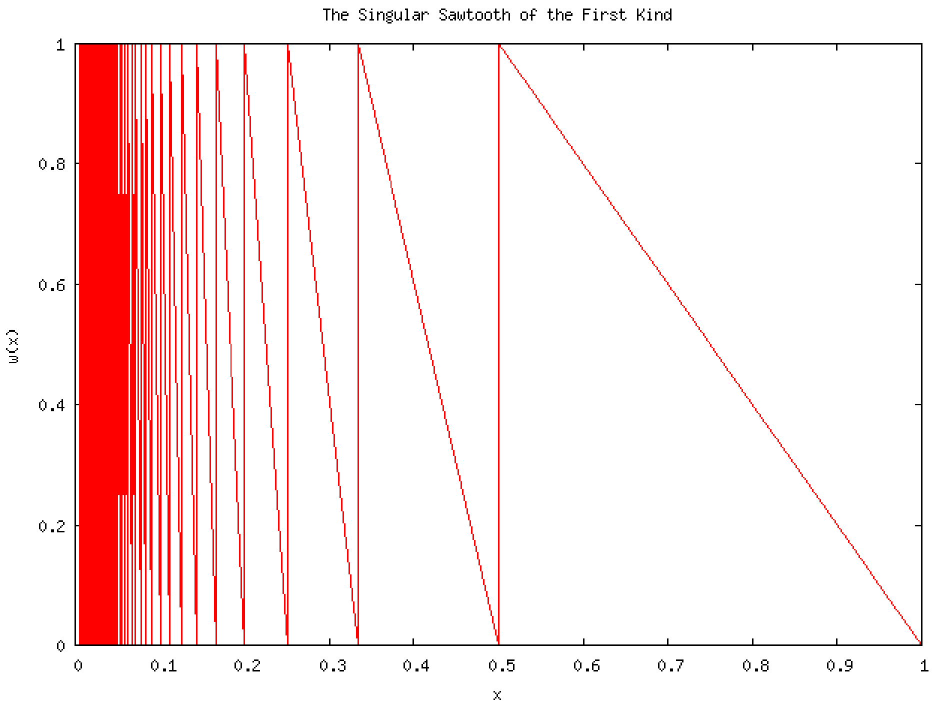
$$W_{mk} = \begin{cases} \begin{pmatrix} k \\ m \end{pmatrix} \sum_{n=1}^{\infty} n^{-m-1}(n+1)^{-k-1} & \text{for} \quad k \geq m \\ 0 & \text{for} \quad k < m \end{cases} \tag{3.6}$$

where we use $\begin{pmatrix} k \\ m \end{pmatrix}$ to denote the binomial coefficient. This matrix is upper-triangular, and thus has its eigenvalues along the diagonal. These are

$$\lambda_k = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}(n+1)^{k+1}} \tag{3.7}$$

so that $\lambda_0 = 1$ and $\lambda_1 = 2\zeta(2) - 3$ where $\zeta(x)$ is the Riemann zeta. Numerically, we can see that the first few eigenvalues are $\lambda_1 = 0.289868...$ and $\lambda_2 = 0.130396...$ and $\lambda_3 = 0.0633278...$ and $\lambda_4 = 0.031383...$ and $\lambda_5 = 0.0156468...$ We can trivially see that the ratio of eigenvalues settles down to $\lambda_k/\lambda_{k+1} = 2$ for large k, since the first term of the sum will dominate for large k.

FIGURE 3.1. The Singular Sawtooth of the First Kind



This sawtooth function joins values of $1/n$ with straight lines.

We can solve the operator through recursion on the matrix elements of a related operator, by observing that

$$Z_{mk} \equiv \sum_{n=1}^{\infty} \frac{1}{n^m (n+1)^k} \left[\frac{1}{n} - \frac{1}{n+1} \right] = Z_{m,k-1} - Z_{m-1,k} \tag{3.8}$$

These recurrence relations are bounded on the edges by $Z_{00} = 1$, $Z_{01} = 2 - \zeta(2)$ and thus

$$Z_{0k} \quad = Z_{0,k-1} - (\zeta(k+1) - 1) = 1 - \sum_{j=1}^{k} [\zeta(j+1) - 1] \tag{3.9}$$

and $Z_{10} = \zeta(2) - 1$ so that

$$Z_{m0} = \zeta(m+1) - Z_{m-1,0} = (-)^m \left[1 + \sum_{j=1}^{m} (-)^j \zeta(j+1) \right] \tag{3.10}$$

and we have $Z_{mk} = W_{mk}$ for $m \leq k$. The first few eigenfunctions are

$$e_0(y) = 1$$

$$e_1(y) = 1 - 2y$$

$$\begin{aligned} e_2(y) \quad &= \quad \frac{15 - 13\zeta(2) - 9\zeta(3) + 2\zeta(2)[\zeta(2) + 3\zeta(3)]}{3(13\zeta(2) - 8\zeta(3))(3 - 2\zeta(2))} + \\ &\quad + \frac{6\zeta(2) + 2\zeta(3) - 12}{13 - 8\zeta(2)} y + y^2 \end{aligned} \tag{3.11}$$

We see that although the eigenfunctions are polynomials and are exactly solvable, they quickly spiral out of control.

XXX To Do: Double-check e_2 Provide the closed-form finite-sum matrix elements. Provide graphs of the first dozen polynomials. Discuss the similarity transform that takes $w(x)$ to $h(x)$ and discuss why this fails to preserve the eigenvalues. What are the shift-states of this operator? What are the continuous-eigenvalue (square-integrable) eigenfunctions? Graph these eigenfunctions, see what kind of fractals they look like.

4. Singular Sawtooth of the Second Kind

The singular sawtooth of the second kind is given by the dyadic-space conjugate of the continued-fraction shift function $h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$, that is,

$$c(x) = ?\left(\frac{1}{?^{-1}(x)} - \left\lfloor \frac{1}{?^{-1}(x)} \right\rfloor\right) = (? \circ h \circ ?^{-1})(x) \tag{4.1}$$

where $?(x)$ is the Minkowski Question Mark, presented in earlier chapters. This map consists of straight-line segments between values of $1/2^k$, as pictured in figure 4.1, and can be written as

$$c(x) = 2 - 2^n x \text{ for } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \tag{4.2}$$

Just as the Gauss Map is able to lop off the leading term of the continued fraction expansion for x, so this map is able to lop off all of the leading zeros of the binary expansion for x. The downward slope of the sawtooth just reflects the binary expansion, exchanging 1's for 0's, so that the next iteration can chop of the next contiguous chunk. Thus, the orbits of points under this map are completely isomorphic to the orbits of points under the Gauss Map. This is indeed the very idea of a "conjugate map".

The Frobenius-Perron operator of this function provides a second model of the Gauss-Kuzmin-Wirsing operator. It can be solved exactly; unfortunately, while one might think that there is a similarity transform to take it back to GKW, it turns out this similarity transform is dastardly singular, being just the Jacobian of the Minkowski Question Mark $(?' \circ ?^{-1})(x)$, which we examine in detail in a related paper of this series. Thus, although the point dynamics of this sawtooth map are completely isomorphic to the point dynamics of the Gauss Map, the spectra of the associated transfer operators are *not* identical. This is perhaps the most surprising result of this paper. The transfer operator is

$$\left[\mathcal{L}_C f\right](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\frac{2-x}{2^n}\right) \tag{4.3}$$

The following sections develop this operator in different function spaces.

4.1. The Polynomial Basis Eigenfunctions.

As before, we change variables to $y = 1 - x$, expand both sides in terms of y, and match terms to find the matrix elements

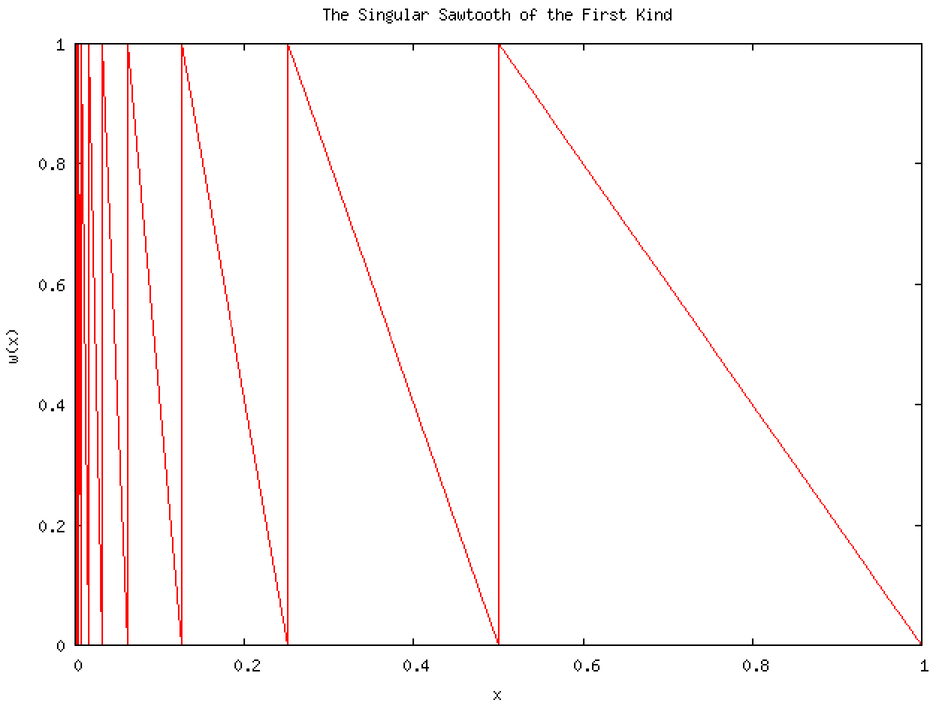
$$C_{mk} = \left(\begin{array}{c} k \\ m \end{array}\right) \frac{1}{2^{k+1} - 1} \tag{4.4}$$

which is upper triangular (we take the binomial coefficients to be vanishing when $k < m$). The eigenvalues lie along the diagonal. The first few are $\lambda_0 = 1$, $\lambda_1 = 1/3$, $\lambda_2 = 1/7$, *etc.* with the ratio of successive eigenvalues tending to 2. The first few eigenvectors are

$$e_0 = 1$$

$$e_1 = 1 - 2y = 2x - 1$$

FIGURE 4.1. Singular Sawtooth, Second Kind



Picture of the second kind of sawtooth.

$$e_2 = 1 - \frac{18}{5}y + \frac{12}{5}y^2 = \frac{-1}{5}\left(1+6x-12x^2\right)$$

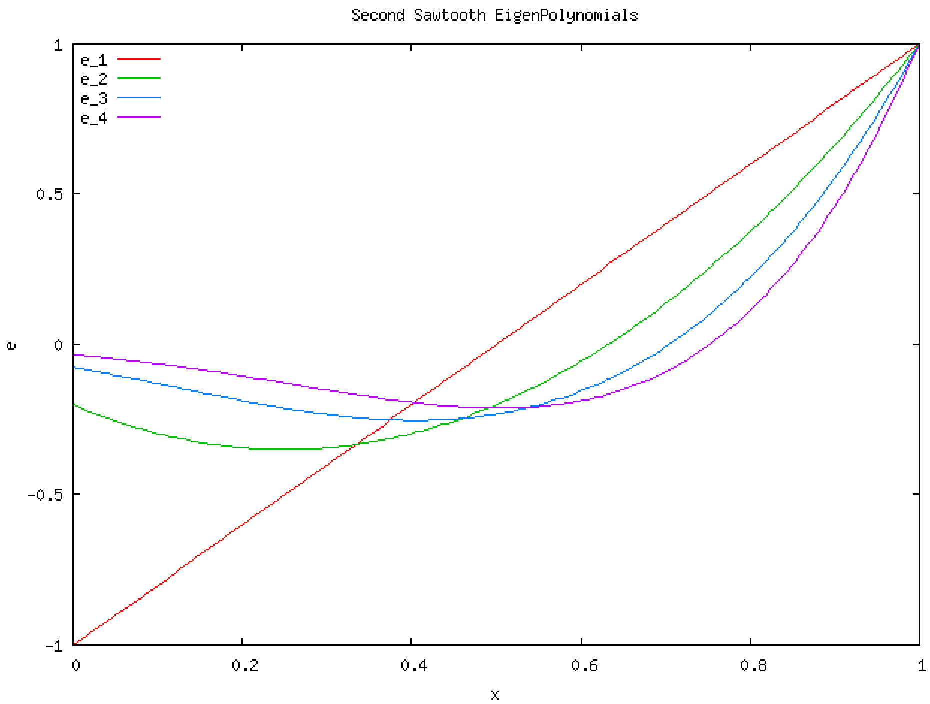
$$e_3 = 1 - \frac{66}{13}y + \frac{84}{13}y^2 - \frac{32}{13}y^3 = \frac{-1}{13}\left(1+6x+12x^2-32x^3\right)$$

$$\begin{aligned} e_4 &= 1 - \frac{36450}{5597}y + \frac{67620}{5597}y^2 - \frac{50400}{5597}y^3 + \frac{13440}{5597}y^4 \\ &= \frac{-1}{29\cdot 193}\left(193+1350x+2940x^2+3360x^3-13440x^4\right) \end{aligned} \tag{4.5}$$

As we can see, the complexity of individual eigenvectors spirals out of control; there's no obvious simple closed form expression for higher eigenvectors. Indeed, it seems that the simplest algorithm is to directly invert the matrix. This is curious, because the matrix itself is so curiously simple, and superficially similar to that for the Bernoulli map.

4.2. **The Failure of the Similarity Transform for the Polynomial Basis.** Under normal circumstances, whenever one has a pair of maps $\alpha(x)$ and $\beta(x)$ that are conjugate to each other through an invertible function $\phi(x)$ such that $\alpha(x) = (\phi\circ\beta\circ\phi^{-1})(x)$, then there exists a similarity transform S_ϕ such that the Frobenius-Perron operators are also conjugate; that is, $\mathscr{L}_\alpha = S_\phi \mathscr{L}_\beta S_\phi^{-1}$ where $S_\phi^{-1} = S_{\phi^{-1}}$. Formally, one finds that $S_\phi = 1/(\phi'\circ\phi^{-1})$ where the prime denotes differentiation: $\phi'(x) = d\phi(x)/dx$. Since the continued-fraction shift function is conjugate to the sawtooth, one might hope that GKW would be conjugate to $\mathscr{L}_C$, that is, $\mathscr{L}_h = S_? \mathscr{L}_C S_?^{-1}$. Unfortunately, the Minkowski Question Mark is highly singular and is not traditionally differentiable, and so we cannot build such a similarity

FIGURE 4.2. Second Sawtooth Polynomials



Eigenvectors of the second sawtooth.

transform using the polynomial function basis. Another way to deduce this is to note that the similarity transform S_ϕ, working as a traditional, ordinary operator, normally preserves the eigenvalues; that is, the eigenvalues of $\mathscr{L}_\alpha$ equal those of $\mathscr{L}_\beta$. In the current case, we see trouble in that the eigenvalues of $\mathscr{L}_C$ are not those of GKW. They are not even 'close', in that the ratio of tends to $\lambda_k/\lambda_{k+1} = 2$ whereas for the GKW the ratio is 2.61803... (the square of golden mean, see [**?**].

However, there are suggestive elements. For example, the function argument $(2-x)/2^n$ is just the dyadic polynomial $(g_D^{n-1} r_D g_D)(x)$. Tantalizingly, the corresponding Moebius transform is $(g_C^{n-1} r_C g_C)(x) = 1/(n+x)$ which is the function argument to the GKW operator. This suggests the tantalizing re-write of the terms of GKW as

$$f\left(\frac{1}{n+x}\right) = f \circ ?^{-1}\left(\frac{2-?(x)}{2^n}\right) \tag{4.6}$$

Also, one can do strange things such as xxx but why do we want to do that?

The point is to not give up hope on the operator relationships, even though the polynomial basis breaks the relationship. Thus, we are motivated to explore other bases, and not just the polynomial basis. Fortunately, we can find some of these.

4.3. **Fractal Eigenfunctions of the Second Sawtooth.** The Takagi curve can be used to build an alternate set of eigenfunctions for the second sawtooth, possessing continuous-spectrum eigenvalues. These eigenfunctions are not differentiable, and thus cannot be obtained through polynomials, and thus are not visible when working with the operator in a polynomial-basis Hilbert Space. They can be used to build an alternate function space, in which the Second Sawtooth remains exactly solvable.

We recognize from the studying of the dyadic representation of the Modular Group $SL(2,\mathbb{Z})$ that the second sawtooth is expressed in terms of the group element

$$\left[g_D^{k-1} r_D g_D\right](x) = \frac{1}{2^{k-1}} - \frac{x}{2^k} \tag{4.7}$$

that is,

$$\left[\mathscr{L}_C f\right](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\left[g_D^{n-1} r_D g_D\right](x)\right) \tag{4.8}$$

and thus functions $f(x)$ possessing the modular group symmetry are candidates for solving the operator $\mathscr{L}_C$. The candidates we have in mind are of course the family of Takagi Curves.

We begin by defining the Takagi Curve as

$$t_w(x) = \sum_{k=0}^{\infty} w^k \tau\left(2^k x - \left\lfloor 2^k x \right\rfloor\right) \tag{4.9}$$

where $\tau(x)$ is the triangle wave:

$$\tau(x) = \begin{cases} 2x & \text{when} \quad 0 \le x \le 1/2 \\ 2(1-x) & \text{when} \quad 1/2 \le x \le 1 \end{cases} \tag{4.10}$$

This form of the Takagi curve transforms under the three-dimensional representation of the Modular Group. Specifically, we write

$$g_3^n = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1/2^n & 0 \\ 0 & q_n(w) & w \end{array} \right) \tag{4.11}$$

where $q_n(w)$ is the polynomial

$$q_n(w) = \frac{1}{2^{n-1}} \sum_{k-0}^{n-1} (2w)^k = \frac{1}{2^{n-1}} \left(\frac{1-(2w)^n}{1-2w} \right) \tag{4.12}$$

We write out the full matrix form for $g_3^{k-1} r_3 g_3$ and apply the group action isomorphism $t_w g_D^{k-1} r_D g_D = g_3^{k-1} r_3 g_3 t_w$ to obtain

$$t_w\left(\frac{1}{2^{k-1}} - \frac{x}{2^k}\right) = q_{k-1}(w) + x\left(w^{k-1} - q_{k-1}(w)/2\right) + w^k t_w(x) \tag{4.13}$$

Inserting the above back into the definition for the sawtooth operator, and performing the sum, we get

$$\left[\mathscr{L}_C t_w\right](x) = \frac{4}{3(2-w)} + \frac{x}{3(2-w)} + \frac{w t_w(x)}{2-w} \tag{4.14}$$

From this, we can immediately read off the eigenvalue as $w/(2-w)$. To get the eigenfunction, we need to complete the diagonalization by using $\left[\mathscr{L}_C 1\right](x) = 1$ and $\left[\mathscr{L}_C x\right](x) = (2-x)/3$ to get the eigenfunction

$$E_2(x) = \frac{2-w}{2(w+1)(w-1)} + \frac{x}{2(w+1)} + t_w(x) \tag{4.15}$$

It should be clear from this presentation that the higher and lower dimensional Takagi curves, of both even and odd parity, give eigenvectors and eigenvalues as well. We present a few more here:

XXX to do, present $E_1$and the odd-parity $E_2$and both parities for E_3 as well. Show graphs as well.

XXX Note a shallow relationship to the Gaussian Binomial: viz.

$$\left(\begin{array}{c} n \\ 1 \end{array}\right)_x \equiv \frac{1-x^n}{1-x}$$

Previously, we saw that the Takagi Curves served as basis vectors for a space of degenerate eigenfunctions of the Bernoulli Map, associated with arbitrary eigenvalue. We saw that this space could also be spanned by the Hurwitz Zeta, through a change of basis. Thus, we expect that we can extend these results to this map as well.

5. The Isola Map

Stefano Isola proposes studying a map of deceptive simplicity[**?**]. Given by

$$F(x) = \begin{cases} x/(1-x) & \text{if } 0 \leq x \leq 1/2 \\ (1-x)/x & \text{if } 1/2 \leq x \leq 1 \end{cases} \tag{5.1}$$

it is symmetric about $x = 1/2$: that is, $F(x) = F(1-x)$, and has a very simple tent-like shape, and this is the source of the deception. One wants to hastily conclude that it is topologically equivalent to the standard tent map

$$\tau(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2-2x & \text{if } 1/2 \leq x \leq 1 \end{cases} \tag{5.2}$$

and thus that the spectrum of its Frobenius-Perron Operator is identical to that of the Bernoulli Map, and that this map can be trivially brushed aside as belonging to that conjugacy class. Nothing could be farther from the truth. In fact, it is conjugate, but the conjugating function is the Minkowski Question Mark:

$$F(x) = (?^{-1} \circ \tau \circ ?)(x) \tag{5.3}$$

and so the relationship is anything but trivial. The easiest way to see this is to note that we can write F and τ are combinations of the modular group element g^{-1}:

$$F(x) = \begin{cases} g_C^{-1}(x) & \text{if } 0 \leq x \leq 1/2 \\ (g_C^{-1} \circ r)(x) & \text{if } 1/2 \leq x \leq 1 \end{cases} \tag{5.4}$$

following the notation of earlier chapters, and

$$\tau(x) = \begin{cases} g_D^{-1}(x) & \text{if } 0 \leq x \leq 1/2 \\ (g_D^{-1} \circ r)(x) & \text{if } 1/2 \leq x \leq 1 \end{cases} \tag{5.5}$$

Just as we saw with the second sawtooth, the point dynamics of the Isola Map and the Tent Map are isomorphic to each other, but the eigenvalue spectra are inequivalent. The Ruelle-Frobenius-Perron operator for the Isola Map is

$$[\mathscr{P} f](x) = \frac{1}{(1+x)^2}\left[f\left(\frac{x}{x+1}\right) + f\left(\frac{1}{x+1}\right)\right] \tag{5.6}$$

Isola shows how the Gauss-Kuzmin-Wirsing operator can be constructed through some simple operator relationships on $\mathscr{P}$ and so it is a worthwhile goal to attempt to solve $\mathscr{P}$. As we will see below, this seems to be an even harder task.

Closely related is a modular variant of the Bernoulli shift, given by

$$A(y) = ?^{-1}(\text{frac}(2?(y))) = \begin{cases} \frac{y}{1-y} & \text{for } 0 \leq y \leq \frac{1}{2} \\ \frac{2y-1}{y} & \text{for } \frac{1}{2} \leq y \leq 1 \end{cases} \tag{5.7}$$

The assoicated transfer operator is

$$[\mathscr{L}_A f](y) = \frac{1}{(1+y)^2} f\left(\frac{y}{1+y}\right) + \frac{1}{(2-y)^2} f\left(\frac{1}{2-y}\right)$$

which again has the curious relationship

$$\mathscr{L}_A ?' = ?'$$

as given in [**?**].

5.1. **The (Lack of a) Polynomial Basis.** Based on our previous luck, we attempt to define the operator $\mathscr{P}$ in the polynomial basis. First, we attempt an expansion at $x=0$. This leads to

$$\begin{aligned} \mathscr{P}_{nk} &\equiv \langle n|\mathscr{P}|k\rangle \\ &= (-)^n \left[\binom{k+n+1}{k+1} + \Theta_{k\leq n} (-)^k \binom{n+1}{k+1} \right] \end{aligned} \tag{5.8}$$

This matrix is not triangular, and is thus not directly solvable. It is also very ill-conditioned, making it not numerically tractable, at least, not in any simple fashion. As n gets large, the matrix elements grow exponentially on the diagonal. This is easily seen by applying Stirling's asymptotic formula for the factorial to the binomial; one easily gets

$$\binom{n}{k} \approx \frac{2^{n+1}}{\sqrt{2\pi n}} \exp\left(\frac{-(2k-n)^2}{2n}\right) \tag{5.9}$$

when n and k get large. Thus, along the diagonal, $\mathscr{P}_{nn} \approx 4^{n+1}/2\sqrt{\pi n}$, and the matrix is not tractable numerically, and would be painful to work with analytically, without defining some sort of regulator. Thus, we are motivated to look at the expansion at $x=1/2$. Here, however, the situation is not much better. Defining $y = x - 1/2$ so that

$$[\mathscr{Q} f](x) = \frac{1}{(y+3/2)^2} \left[f\left(\frac{y+1/2}{y+3/2}\right) + f\left(\frac{1}{y+3/2}\right) \right] \tag{5.10}$$

we work through the same set of steps to obtain

$$\begin{aligned} \mathscr{Q}_{nk} &\equiv \langle n|\mathscr{Q}|k\rangle \\ &= \left(\frac{-2}{3}\right)^{n+2} \left[\left(\frac{2}{3}\right)^k \binom{k+n+1}{k+1} + \left(\frac{1}{3}\right)^k \sum_{p=0}^{\min(n,k)} (-3)^p \binom{n+k-p+1}{k+1} \binom{k}{p} \right] \end{aligned} \tag{5.11}$$

which is far more complex, and only marginally less divergent: $\mathscr{Q}_{nn} \sim (16/9)^{n+1}$. There is hardly any hope that a Taylor's expansion around any other value of x will give a tractable result; the trick of using the Taylor's expansion to obtain polynomial eigenstates fails in this case. Indeed, it seems likely that the eigenstates will not be analytic, although it is not clear to me what theorem would establish or disprove this conjecture.

The polynomial-basis matrix elements for this operator are much better behaved. They are given by

$$M_{nk} \equiv \langle n|\mathscr{L}_A|k\rangle = \frac{1}{2^{2+n+k}} \binom{n+k+1}{k+1} + \Theta_{n\geq k} (-1)^{k+n} \binom{n+1}{k+1}$$

(Notice the direction of teh Heaviside function is reversed, is this correct, or an error? XXX Needs double-checking).

The leading factor of two in the above makes all the difference in the world for this operator. This time, applying Stirlings formula to evaluate the matrix elements on the diagonal gives

$$M_{nn} \approx 1 + \frac{0.76}{\sqrt{n}}$$

thus implying that this operator, at least, is not hopelessly badly behaved.

The difference between the two is, perhaps, due to the former not being diagonalizable except in Jordan block fom.

6. Conclusions

Apologies for the format of this paper. It's a veritable candy store of goodies; there are all these yummy toys to play with, which one first?

Appendix A. Expansion about arbitrary location

This appendix discusses, in greater detail, the polynomial matrix elements that may be obtained by expanding around other points within the unit interval.

Consider $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\,(x-a)^n/n!$ and $g(x)$ likewise expanded about $x = b$. With this expansion, the operator relation $\mathscr{L}_h f = g$ becomes

$$\frac{g^{(m)}(b)}{m!} = \sum_{n=0}^{\infty} \mathscr{L}_{mn}^{(b,a)} \frac{f^{(n)}(a)}{n!} \tag{A.1}$$

Without much difficulty, one discovers that the matrix elements are given by

$$\mathscr{L}_{mn}^{(b,a)} = (-1)^m \sum_{k=0}^{n} (-a)^{n-k} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k+m+1 \\ m \end{array}\right) \zeta_H(k+m+2, 1+b) \tag{A.2}$$

where $\zeta_H(s,q)$ is the Hurwitz zeta function:

$$\zeta_H(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \tag{A.3}$$

Substituting $a = b = 1/2$, one obtains the expansion of [**?**], which is

$$\mathscr{L}_{mn}^{(1/2,1/2)} = (-1)^m \sum_{k=0}^{n} \left(\frac{-1}{2}\right)^{n-k} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k+m+1 \\ m \end{array}\right) \times \tag{A.4}$$

$$\left[2^{m+k+2}\left(\zeta(k+m+2) - 1\right) - \zeta(k+m+2)\right] \tag{A.5}$$

All of these expressions for the matrix elements for the GKW operator have a common form. It consists of two summations: the outer summation, and the summation defining the Hurwitz zeta function. Pulling out this second summation, one finds terms consisting of a series of polynomials, which are most simply expressed in terms of Gauss' hypergeometric series:

$$\Gamma_{mn}(x) \equiv (m+1)\,{}_2F_1\left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array} ; x\right] = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k+m+1 \\ m \end{array}\right) (-x)^k \tag{A.6}$$

These have a curious superficial resemblance to the shifted Legendre polynomial

$$\widetilde{P}_n(x) \equiv \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} k+n \\ n \end{array}\right) (-x)^k \tag{A.7}$$

Switching the order of summation in equation A.2 gives the following:

$$\mathscr{L}_{mn}^{(b,a)} = (-1)^{m+n} a^n \sum_{j=0}^{\infty} \frac{1}{(j+1+b)^{m+2}} \, {}_2F_1 \left[\begin{array}{cc} -n & m+2 \\ & 2 \end{array} ; \frac{-1}{a(j+1+b)} \right] \tag{A.8}$$

Oddly, this appears to be in the form of a slightly generalized form of the GKW operator, an operator given by Dieter Mayer[**?**],

$$\left[\mathscr{L}^{(s)} f\right](x) \equiv \sum_{n=1}^{\infty} \frac{1}{(n+x)^s} f\left(\frac{1}{n+x}\right) \tag{A.9}$$

where s is taken to be $s = m+2$ and $f =_2 F_1$. The general appearance and the matrix elements of this generalized operator are only a slight variation of those for the GKW; for completeness, these are stated here:

$$\left[\mathscr{L}^{(s)}\right]_{mn}^{(b,a)} = (-1)^m \sum_{k=0}^{n} (-a)^{n-k} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} m+k+s-1 \\ m \end{array}\right) \zeta_H(m+k+s, 1+b) \tag{A.10}$$

The corresponding hypergeometric identity that comes into play is

$$\sum_{k=0}^{n} (-x)^k \left(\begin{array}{c} n \\ k \end{array}\right) \left(\begin{array}{c} m+k+s-1 \\ m \end{array}\right) = \left(\begin{array}{c} m+s-1 \\ m \end{array}\right) {}_2F_1 \left[\begin{array}{cc} -n & m+s \\ & s \end{array} ; x \right] \tag{A.11}$$

As a final note, recall that the Hurwitz zeta may be expressed as the polygamma function for integer arguments, where the polygamma functions are the chain of logarithmic derivatives of the gamma function. Thus, one may also expresses the matrix elements of $\mathscr{L}$ in the curious form

$$\mathscr{L}_{mn}^{(b,a)} = \frac{(-a)^{n+1}}{m!} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \left(\frac{1}{a}\right)^{k+1} \frac{1}{(k+1)!} \frac{d^{k+1}}{dx^{k+1}} \psi^{(m)}(1+b) \tag{A.12}$$

Here, the curious operator making an appearance is

$$\left[P_{n,y} f\right](x) = \sum_{k=0}^{n} (-y)^k \left(\begin{array}{c} n \\ k \end{array}\right) \frac{f^{(k)}(x)}{k!} \tag{A.13}$$

where $f^{(k)}(x)$ is the k'th derivative of f at x. The operator $P_{n,y}$is upper-triangular, with all eigenvalues equal to 1, and all eigenvectors being polynomials (or analytic series for n not an integer).

REFERENCES

[1] Giedrius Alkauskas. Generating and zeta functions, structure, spectral and analytic properties of the moments of minkowski question mark function authors:. *ArXiv*, 0801.0056, 2007.

[2] Giedrius Alkauskas. The moments of minkowski ?(x) function: dyadic period functions. *ArXiv*, 0801.0051, 2007.

[3] Giedrius Alkauskas. Minkowski question mark function and its generalizations, associated with p-continued fractions: fractals, explicit series for the dyadic period function and moments. *ArXiv*, 0805.1717, 2008.

[4] Keith Briggs. A precise computation of the gauss-kuzmin-wirsing constant. http://keithbriggs.info/documents/wirsing.pdf, 2003.

[5] Philippe Flajolet and Linas Vepstas. On differences of zeta values. *Journal of Computational and Applied Mathematics*, 220:58–73, November 2007. arxiv:math.CA/0611332v2.

[6] Phillipe Flajolet and Brigitte Vallée. On the gauss-kuzmin-wirsing constant. http://algo.inria.fr/flajolet/Publications/gauss-kuzmin.ps, Oct 1995.

[7] Stefano Isola. On the spectrum of farey and gauss maps. preprint, between 2000 and 2004.

[8] Kuzmin. xx?xx. *Atti del Congresso*, 1928.

[9] Dieter H. Mayer. Continued fractions and related transformations. In C. Series T. Bedford, M. Keane, editor, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, chapter 7, pages 175–222. Oxford University Press, 1991.
[10] Linas Vepstas. A series representation for the riemann zeta derived from the gauss-kuzmin-wirsing operator. http://linas.org/math/poch-zeta.pdf, 2004.
[11] Linas Vepstas. Notes relating to newton series for the riemann zeta function. http://linas.org/math/norlund-l-func.pdf, Nov 2006.
[12] Linas Vepstas. On the minkowski measure. *ArXiv*, arXiv:0810.1265, 2008.
[13] Eduard Wirsing. On the theorem of gauss-kusmin-lévy and a frobenius-type theorem for function spaces. *Acta Arithmetica*, 26:507–528, 1974.