Comments on NNTDM submission Paper id: 2021 / 269 Revision 1

Review date: 28 October July 2021

Review summary: The result is quite remarkable! That such an algebraic derivation is possible is unexpected. The paper should be published.

There does appear to be a minor typo in section 3 that does not change any results. I found the first step of the proof of Lemma 2.3 to be opaque; the rest of this note provides an easier-to-read presentatation of the first step.

A typographical error

Top of page 4, first formula, right hand side currently reads

$$\frac{1}{k^{4m-2}n^2+n^{4m}}$$

It should read

$$\frac{1}{k^{4m-2}n^2+k^{4m}}$$

A simpler derivation of Lemma 2.3

I found the first step of the proof of Lemma 2.3 to be opaque and painful to verify. A more transparent derivation might go as follows.

Famously, for integer p, one has

$$\begin{aligned}\frac{\alpha^p-\beta^p}{\alpha-\beta} &= \alpha^{p-1}+\beta\alpha^{p-2}+\beta^2\alpha^{p-3}+\cdots+\beta^{p-1}\\ &= \sum_{s=1}^{p}\alpha^{p-s}\beta^{s-1}\\ &= \frac{\alpha^p}{\beta}\sum_{s=1}^{p}\left(\frac{\beta}{\alpha}\right)^s\end{aligned}$$

Let $\alpha=n^2$ and $\beta=-k^2$ and $p=2m-1$. One can substitute straight away or perhaps first divide both sides by $1/\alpha^p\beta^p$ to get

$$\frac{1}{\alpha-\beta}\left[\frac{1}{\beta^p}-\frac{1}{\alpha^p}\right]=\frac{1}{\beta^{p+1}}\sum_{s=1}^{p}\left(\frac{\beta}{\alpha}\right)^s$$

Now, when one makes the proposed substitution, this gives the important identity in the proof of Lemma 2.3

$$\frac{-1}{k^{4m}}\sum_{s=1}^{2m-1}\left(\frac{-k^2}{n^2}\right)^s=\frac{1}{n^{4m-2}\left(k^2+n^2\right)}+\frac{1}{k^{4m-2}\left(k^2+n^2\right)}$$

The rest of the proof of Lemma 2.3 follows.