NEWTON SERIES FROM MELLIN TRANSFORMS

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10 JANUARY 2007

1. Development

Starting with any Mellin transform:

$$\mathbf{M}_f(s) = \int_0^\infty f(x)x^{s-1}dx \tag{1.0.1}$$

Then using the Laguerre series formula:

$$x^{s-1} = \Gamma(s+\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\alpha+1)}\frac{\Gamma(s)}{\Gamma(s-n)}L_n^\alpha(x) \tag{1.0.2}$$

Where 1.0.2 is valid for $0 < x < \infty$, $s > -\frac{1}{2}(\alpha+1)+1$, and $\alpha > -1$.
We substitute 1.0.2 into 1.0.1, choosing α to agree with the area of convergence.
We now write the Laguerre polynomial $L_n^\alpha(x)$ as a contour integral:

$$L_n^\alpha(x) = \frac{1}{2\pi i}\int_{\gamma_0}\frac{e^{-x\frac{\tau}{1-\tau}}}{(1-\tau)^{\alpha+1}\tau^{n+1}}d\tau \tag{1.0.3}$$

Where γ_0 is any contour that encircles the origin.
On moving the summation to the outside and interchanging the order of integration we have:

(1.0.4)
$$\mathbf{M}_f(s) = \Gamma(s+\alpha)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\alpha+1)}\frac{\Gamma(s)}{\Gamma(s-n)}\frac{1}{2\pi i}\int_{\gamma_0}\int_0^\infty f(x)\frac{e^{-x\frac{\tau}{1-\tau}}}{(1-\tau)^{\alpha+1}\tau^{n+1}}dxd\tau$$

Let us denote the quantity for the double integral as b_n.

Also note, the inner integral is just the Laplace transform of $f(x)$ with

Date: 10 January 2007.

respect to the parameter $\tau/(1-\tau)$, $\mathbf{L}_f(\tau/(1-\tau))$.
Thus the final formula for the Newton series coefficients is therefore:

$$b_n = \frac{1}{n!}\frac{d^n}{d\tau^n}[(1-\tau)^{-\alpha-1}\mathbf{L}_f(\tau/(1-\tau))]_{\tau=0} \tag{1.0.5}$$

2. Examples

The simplest example is the Mellin transform of e^{-x}.

Here using $\alpha = 0$ we see $b_0 = 1$ and all other $b_n = 0$, giving as expected from 1.0.4:

$$\mathbf{M}_{e^{-x}}(s) = \Gamma(s)$$

As another example we take the integral form for

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-x}}{1-e^{-x}}x^{s-1}dx$$

, valid for $\sigma > 1$.
Rewriting the above as

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{1}{\Gamma(s)}\int_0^\infty e^{-x}(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2})x^{s-1}dx$$

and placing into the form of 1.0.4, we choose $\alpha = 1$ and get for b_n:

$$b_n = -\frac{1}{n!}\frac{d^n}{d\tau^n}[\frac{1}{2(1-\tau)} + \frac{log(1-\tau) + \Psi(\frac{1}{1-\tau})}{(1-\tau)^2}]_{\tau=0}$$

(After adding and subtracting $exp(-x)/x$ and using the integral forms for $log(s)$ and $\Psi(s)$.)

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References

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