ELLIPSES AND POLYNOMIALS

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Abstract. Averaging a complex polynomial over an ellipse reveals a conservation law: the result is the same on all confocal ellipses. Starting with this classical observation we will single out one more time the ellipse as the unique shape among among all planar curves, with respect to some properties carrying a constructive mathematics flavor: numerical quadrature, orthogonal polynomials, boundary value problems and quantization.

Ellipses enchanted mathematicians for millennia. Think of the conic sections of Appolonius, Pascal's Theorem referring to six points on an ellipse or Kepler's laws in the dynamics of celestial bodies. The computation of the arc length of an ellipse has dominated the golden age of infinitesimal calculus, ultimately leading to a new discipline, nowadays called algebraic geometry, with elliptic integrals as principal characters. Even in our times only highly trained mathematicians operate with ease with elliptic integrals, elliptic curves and all derived concepts from them. It is fascinating to witness how the progress in number theory is intimately linked to exactly these objects. The present note goes on a different path, showing in the spirit of Archimedes that some averaging processes along ellipses are accessible by elementary techniques and they provide novel characterizations of this basic curve; second, and not unrelated, we will indicate that quantizing the ellipse, in a precise meaning to be explained below, generates interest nowadays.

1. Averaging complex polynomials

In modern terminology an ellipse is an *algebraic curve*, that is a collection Γ of pairs of real numbers (x, y) subject to a quadratic equation, such as the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1.1}$$

where a, b are positive real numbers. We can make a choice of coordinates and assume $a \geq b$. Geometrically, the ellipse is the locus of points (x, y) in the plane whose sum of distances to the foci $\pm c$ is constant. With the above selection of parameters $c = \sqrt{a^2 - b^2}$. We distinguish below between the curve Γ and the solid ellipse, that is the interior of Γ which we denote by E.

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One can compute in close form with basic calculus methods the length of Γ and the area of E. If we interpret these geometric quantities in mechanical terms as averages of a uniform point distribution along the ellipse or inside its interior, it is quite natural to relax the uniformity and ask for instance what happens if we average other physically relevant distribution of points? Imagine for instance a fluid flowing in the plane in the simplest possible way, without whirling and without changing the density on small areas. Technically this is called an irrotational and incompressible fluid flow in two dimensions (but never mind the sophisticated terminology). What happens if we average the velocity distribution of such a fluid flow over an ellipse?

To answer this question we adopt a rather bold stand, going back at least two centuries to Gauss and Green, and reformulate the problem in terms of the complex coordinate $z = x + iy$, with the imaginary quantity i subject to $i^2 = -1$. This gigantic step in mathematics turned out to be very beneficial for field theory, where our questions naturally belong, but also for geometry, number theory, algebra and the whole corpus of XIX-th century mathematics. The reason our problem is illuminated in this way is that irrotational and incompressible fields (like the velocity distribution of our fluid) are on regions without holes precisely complex polynomials or limits of them, called complex analytic functions. So, let's try to average on the solid ellipse E a polynomial

$$p(z) = c_0 + c_1 z + \ldots + c_d z^d,$$

where the coefficients $c_0.c_1, \ldots, c_d$ are complex numbers. Specifically, we want to compute the integral

$$\int_E p(z)dxdy.$$

This looks challenging, or at least uninviting. Instead, we do what all students of calculus learn: try a change of variable. For instance Joukowski's map, named after one of the pioneers of modern fluid mechanics, yields:

$$z = x + iy = \frac{1}{2}(w + \frac{1}{w}),$$

and via Euler's famous formula providing polar coordinates for w:

$$w = u + iv = re^{it} = r\cos t + ir\sin t,$$

we infer

$$x = \frac{1}{2}(r + \frac{1}{r})\cos t, \quad y = \frac{1}{2}(r - \frac{1}{r})\sin t.$$

Consequently, if w runs on the circle $r = \rho$ constant, the image of it is the ellipse with semi-axes

$$a = \frac{1}{2}(\rho + \frac{1}{\rho}), \quad b = \frac{1}{2}(\rho - \frac{1}{\rho}).$$

To see this just remember that $\cos^2 t + \sin^2 t = 1$. Note that in this case the foci are at ± 1, independent of the value of $\rho > 1$.

Via Joukowski map the unit circle $r = 1$ is collapsed onto the interval $[-1, 1]$ between the foci. Now our computation is becoming more transparent

$$\int_E p(z)dxdy = \int_{E\setminus[-1,1]} p(z)dxdy = \int_{1<r<\rho} p(z(w))\frac{1}{4}|1-\frac{1}{w^2}|^2 dudv, \quad (1.2)$$

and the latter integral is within reach, as we recall that on an annular region

$$\int_{1<r<\rho} w^k\overline{w}^\ell dudv = 0$$

whenever $k \neq \ell$. All in all, for the solid ellipse E with semi axes $a = \frac{1}{2}(\rho + \frac{1}{\rho}),\quad b = \frac{1}{2}(\rho - \frac{1}{\rho})$ and foci at ± 1 we find after simple algebra

$$\int_E p(z)dxdy = \frac{1}{2}(\rho^2 - \frac{1}{\rho^2})\int_{-1}^{1} p(x)\sqrt{1-x^2}dx. \quad (1.3)$$

Or, taking into account that $\mathrm{Area}(E) = \pi ab = \frac{\pi}{4}(\rho^2 - \frac{1}{\rho^2})$,

$$\int_E p(z)dxdy = \frac{2}{\pi}\mathrm{Area}(E)\int_{-1}^{1} p(x)\sqrt{1-x^2}dx,$$

or even better

$$\frac{1}{\mathrm{Area}(E)}\int_E p(z)dxdy = \frac{2}{\pi}\int_{-1}^{1} p(x)\sqrt{1-x^2}dx.$$

And this is true for all polynomials p regardless of their degree, and even for analytic functions, after passing to the limit. In other terms *the average on an ellipse of an incompressible, irrotational field depends only of its values on the interval between the two foci.* Moreover, this mean value is the same on all confocal ellipses.

We invite the reader to rewrite formula for arbitrary semi axes a, b and let a tend to b. The conclusion will be a famous quadrature rule discovered by Gauss: *The average of a complex polynomial on a disc coincides with its value at the center.*

We can also interpret (1.3) as a quadrature formula, which contrary to the standard terminology adopted in numerical mathematics, it applies to *all* polynomials. An inspiring text, with ample details for the topics touched in this section is Davis' booklet [2].

But this is not all, averaging an analytic function on the whole plane, denoted henceforth $\mathbb{C}$, minus the ellipse is even simpler:

Theorem. (Sakai) *Let $f(z)$ be a complex analytic function defined on the complement of an ellipse E, and integrable there. Then*

$$\int_{\mathbb{C}\setminus E} f(z)dxdy = 0.$$

For details and a classification of all such "null quadrature domains" see [8].

2. Orthogonal polynomials

The notion of quantization has reached a variety of interpretations. One of them, relevant for our note, concerns the passage from a geometric structure (in our case the ellipse) to a non-commutative entity (an infinite matrix) which mimics the passage from classical mechanics entities to quantum mechanical ones. Without entering into cumbersome details, the infinite matrix in question is in general a linear transformation acting on an infinite dimensional space. The basis with respect to which this matrix quantization is formed consists of *complex orthogonal polynomials* on the ellipse. To be more precise, we seek complex polynomials $p_k(z)$, each of degree $\deg p_k = k, \quad k \geq 0$, subject to the orthonormality conditions

$$\int_E p_k(z)\overline{p_\ell(z)}dxdy = \begin{cases} 1, & k = \ell, \\ 0 & k \neq \ell. \end{cases}$$

Joukowski's change of variable (1.2) helps one more time to construct these polynomials. Writing $z = \cos\zeta = \frac{1}{2}(e^{i\zeta} + e^{-i\zeta})$ and $w = e^{i\zeta}$ we identify an appropriate rectangle R in the ζ-plane which is mapped onto the ellipse E of semi axes $a = \frac{1}{2}(\rho + \frac{1}{\rho}), \quad b = \frac{1}{2}(\rho - \frac{1}{\rho})$. We see then that

$$U_k(z) = \frac{\sin((k+1)\zeta)}{\sin\zeta}$$

is a polynomial in $z = \cos\zeta$ of exact degree k. Moreover, the above change of variable formula yields

$$\int_E U_k(z)\overline{U_\ell(z)}dxdy = \int_R \frac{\sin((k+1)\zeta)}{\sin\zeta}\,\overline{\frac{\sin((\ell+1)\zeta)}{\sin\zeta}}|\sin\zeta|^2 dudv,$$

where $\zeta = u+iv$. Basic trigonometry implies then the orthogonality of these polynomials. Moreover, one computes without problems the normalization constant and finds that the polynomials

$$p_k(z) = 2\sqrt{\frac{k+1}{\pi}}(\rho^{2k+2} - \rho^{-2k-2})U_k(z), \quad k \geq 0,$$

are orthonormal on the ellipse E. The polynomials $U_k(z)$ are known as Tchebyschev polynomials of the second kind and they are remarkable in many respects.

A notable conclusion of the above computation is that Tchebyshev polynomials $U_k(z)$ are simultaneously orthogonal with respect to *all* confocal solid ellipses. And in particular they are orthogonal on elliptic shells (that is set theoretic differences of two confocal ellipses), and in the limit they are orthogonal with respect to the arc length, along all confocal ellipses, this time regarded as curves. The converse is striking, roughly speaking stating the following:

Theorem. (Szegö) *If two closed curves in the plane share the same sequence of orthogonal polynomials with respect to arc length, then they are* *confocal ellipses.*

For the technical assumptions and a classification of all possible cases, even for weighted line measures, see [9].

The "position operator" which quantizes the ellipse is simply the multiplication by the variable z on an appropriate space. Represented in the basis given by the orthonormal polynomials $p_k(z)$ this operator becomes a banded tri-diagonal matrix. This is due to the fact that Tchebyshev polynomials satisfy a simple *three term recurrence relation*:

$$2zU_n(z) = U_{n-1}(z) + U_{n+1}(z), \quad n \geq 1.$$

For instance, given $s > 0$, the infinite matrix

$$M_z = \begin{pmatrix} 0 & s & 0 & \dots & 0 & 0 \\ 1 & 0 & s & \dots & 0 & 0 \\ 0 & 1 & 0 & s & \ddots & 0 \\ \vdots & & 1 & \ddots & s & \vdots \\ 0 & 0 & 0 & \dots & 0 & \ddots \\ 0 & 0 & \dots & & \ddots & \end{pmatrix}$$

represents the position operator on the ellipse with semi axes $1 \pm s$.

A converse was recently proved, characterizing the ellipses by the above matricidal structure:

Theorem. [6] *Let Ω be a bounded planar domain with smooth boundary and let $(q_n)_{n=0}^{\infty}$ denote the associated orthogonal polynomials with respect to area measure on Ω. If these polynomials satisfy a finite term relation, that is there exists an integer k such that $zq_n(z)$ is a linear combination of $q_{n-k}(z), q_{n-k+1}(z), q_n(z), q_{n+1}(z)$, for all $n \geq k$, then Ω is an ellipse.*

3. The Dirichlet problem

Although we did not touch the proof of the last theorem, we mention that it is derived from yet another characterization of ellipses, this time related to a classical boundary value problem for harmonic functions.

Remember our irrotational and incompressible field. One of the leading threads in XIX-th and XX-th century mathematical physics, known as the Dirichlet problem, asks to prove that the potential of such a field is determined by an arbitrary data on the boundary of a given domain. Without dwelling into the technical aspects, this amounts to solve in a domain $\Omega \subset \mathbb{C}$ the partial differential equation $\Delta u = 0$ with imposed boundary condition $u|_{\partial\Omega} = f$. Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ stands for Laplace's operator. The devil is in the details, as rough boundaries $\partial\Omega$ and irregular data f require a heavy arsenal of methods to solve this problem. We remain on the easy side, as we only deal with ellipses and polynomials.

The first notable fact goes back to a 1877 article by Ferres, see [3] for a reference in the proper context:

Theorem. (Ferres) *Dirichlet's problem on an ellipse with polynomial data has a polynomial solution.*

The modern proof is elementary and we reproduce it below, for the reader equipped with some preparation in linear algebra. Suppose that the boundary Γ of the ellipse is given by a quadratic equation $q(x, y) = 0$, as in (1.1). Let f be a polynomial of degree d. The vector space V of polynomials of degree less than or equal to d is finite dimensional, and carries the linear transform $T : V \longrightarrow V$ defined as

$$T(h) = \Delta(qh), \quad h \in V.$$

A degree count shows that the range of T is indeed contained in V. The transformation T is injective, because $T(h) = 0$ would imply that the polynomial qh is harmonic. But qh vanishes on the boundary Γ and the maximum principle for harmonic functions then implies $qh = 0$ everywhere, hence $h = 0$. Since T is injective, it is also surjective, by the rank-nulity theorem. Which means that there exists $h \in V$ so that $\Delta(qh) = \Delta(f)$. Then $u = f - qg$ is the harmonic polynomial solving our Dirichlet problem.

It is a habit by now to try to prove that ellipses are the only domains which fulfill a certain stringent condition, as Ferres' theorem above. This is so, at least for planar domains with a good boundary:

Theorem. (Chamberland-Siegel) *Assume that a bounded planar domain Ω has a real algebraic boundary. If the Dirichlet problem on Ω has a polynomial solution for all polynomial boundary data, then Ω is an ellipse.*

It was conjectured by Khavinson and Shapiro that the above remains true under relaxed boundary assumptions, we refer for the precise statements to [1, 3].

4. Positive polynomials and quantization

An alternate and more elegant method of quantizing the ellipse is to perform a purely algebraic substitution of the complex variable z by a Hilbert space linear operator T. To this aim, we convert first the equation of the ellipse to complex coordinates, starting with the basic relations $\overline{z} = x - iy$ and

$$2x = z + \overline{z}, \quad 2iy = z - \overline{z}.$$

Since $z\overline{z} = x^2 + y^2$ and $z^2 + \overline{z}^2 = 2x^2 - 2y^2$ we immediately recognize that

$$z\overline{z} - c(z^2 + \overline{z}^2) = 1 \tag{4.1}$$

is the equation of an ellipse provided $0 \le c < 1/2$:

$$(1 - 2c)x^2 + (1 + 2c)y^2 = 1.$$

Assume T is a linear bounded operator acting on a complex Hilbert space H and assume that T satisfies the above equation of the ellipse:

$$T^*T - cT - cT^* = I,$$

where I is the identity transformation. The canonical position operators in quantum mechanics are hermitian, it is true mostly unbounded, meaning that they can be regarded as an array of *real* proper values arranged along a basis of independent vectors/states. This vague picture was made precise by molding about a century ago the concepts of spectral measure and spectral decomposition. The operator T above which quantizes the ellipse is not far from possessing a spectral decomposition.

Let us consider first the case $c = 0$, that is Γ is a circle. Then equation (4.1) becomes $T^*T = I$. This implies that T is an isometric transform, meaning that it preserves lengths and angles between vectors. It is known that isometries can be related to an array of *complex* proper values lying on the unit circle. In technical terms we say that T is the restriction of a unitary transformation to an invariant subspace. Well, the fact that unitary transformations can be "diagonalized" is the heart of all spectral decompositions, and it is a consequence of a purely algebraic observation [4]:

Theorem. (Fejér-F.Riesz) *Any non-negative polynomial on the unit circle is a hermitian square.*

In symbols this can be translated as follows: $g(e^{i\theta}, e^{-i\theta}) \geq 0$, for all $\theta \in [-\pi, \pi]$, implies that there is a complex polynomial $G(z)$ with the property $g(e^{i\theta}, e^{-i\theta}) = |G(e^{i\theta})|^2, \quad \theta \in [-\pi, \pi]$.

The case $c \neq 0$ is even more surprising. First, one finds by elementary algebra that not every positive polynomial along an ellipse which is not a circle can be represented as a sum of hermitian squares. In spite of this apparently negative warning, one proves that any linear bounded operator T satisfying the ellipse equation (4.1) is subnormal, that is T has the expected representation by an array of complex proper values spread on the ellipse. In its turn, this conclusion was reached via a deep observation of Quillen [7], again of purely algebraic nature, stating that *every positive polynomial on a sphere in* $\mathbb{C}^d$ *is a sum of hermitian squares.* Details can be found in [5].

To understand a little what it is at stake in the above remarks, we invite the reader with some training in linear algebra to prove that a complex matrix A satisfying $A^*A - cA - cA^* = I$ for some $c \in [0, 1/2)$ is normal, that is $A^*A = AA^*$.

We started by invoking a conservation law derived from computing averages of polynomials on ellipses. A fascinating new chapter of quantum physics and statistical mechanics build exactly on this observation is in full swing right now. The article [10] marks the beginning of this new adventure having ellipses in the front row. But this is a topics for another note.

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