From: Philippe.Flajolet@inria.fr
To: lbaezd@cantv.net
Cc: Linas Vepstas (linas@austin.ibm.com, linas@linas.org)
Subject: Differences of Zeta values
Date: Thu May 11 10:45:47 CEST 2006

Dear Luis,

Thanks for your kind and informative message dated Thu 11 May 2006 00:18:53 CEDT. As a matter of fact, for reasons that should soon become clear, I was proposing to write to you within the next couple of weeks—this for several reasons.

`I am taking the liberty of copying below my recent letter to Prof Vepstas.`

You did well as I hadn't been aware of your exchange(s). In case you'd have experienced difficulties in communicating with Linas, beware that the email address that is on his web page is only semi-functional, while the IBM address seems to work fine.

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At 22:42 -0400 04/05/06, Luis Baez-Duarte wrote:
Dear Professor Vepstas,

I have  been  made aware of  your  very  interesting paper   "A series
representation      for the   Riemann       Zeta  derived   from  the
Gauss-Kuzmin-Wirsing  Operator" of which   I received a preprint dated
2004 and  revised in 2005.  I would like  to be permitted to draw your
attention to  the  fact that  an   expansion of the zeta  function  in
Pochhammer polynomials, very similar to (1) in your paper, it involves
a  change in scale and  multiplication by $s - 1$  to obtain an entire
function, was  discovered earlier   by  K.  Maslanka. It   was   first
published as

The Beauty of Nothingness: Essay on the Zeta Function of Riemann,
Acta Cosmologica XXIII-1, p. 13-18, 1998;

and later as

A hypergeometric-like Representation of Zeta function of Riemann,
posted at arXiv:math-ph/0105007 v1 4 May 2001;
see also http://functions.wolfram.com, citation index: 10.01.06.0012.01 and
10.01.17.0003.01.

Unfortunately these  original publications contained incomplete proofs
of the convergence  of the expansion. Later I  found a  rigorous proof
and posted it in ArXiv:

L. Baez-Duarte, On   Maslanka's Representation  for the  Riemann  Zeta
Function, arXiv:math.NT/0307214v1, 16 July 2003.

The result is also mentioned in my recently published paper

L.  Baez-Duarte,  A Sequential  Riesz-Like Criterion  for  the Riemann
Hypothesis,  International  Journal   of Mathematics and  Mathematical
Sciences 21 (2005) 3527-3537.
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It may also be of interest to note that the book "Polynomial
Expansions of Analytic Functions" by R. P. Boas and R. C. Buck,
Erbegnisse der Math. Wiss, Band 19, contains a rather complete theory
of expansions in Pochhammer polynomials which they call Newton
polynomials. However, and this is important, if you look at pages 34
and 35 you'll see the formula that gives both Maslanka's and Vepsta's
coefficients, but the function $f(z)=(z-1)\zeta(z)$ does not satisfy
the conditions under which it is derived there because although $f(z)$
is of order 1 it is of infinite type.
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OK. Let me start by recounting my side of the story. I became aware of Linas' site while googling around with the search term "*continued fractions*", an old pet subject of mine. I noticed Linas' startling numerical observations regarding differences of zeta values. I should say that I also have a recurring interest in special functions and the calculus of finite differences. As a matter of fact, a few years back, Sedgewick and I published[1]

> [FlSe95] Mellin transforms and asymptotics: finite differences and Rice's integrals. Philippe Flajolet and Robert Sedgewick (1995).

Given this, I could envision a possible plan for attacking Linas' conjectures. In early January 2006, Linas and I started working together via email. We soon realized that, indeed, with a bit of work, differences of zeta values could be rather precisely estimated, and the Newton series representation for zeta could be established. We have a whole collection of separate notes that we plan to unify into an ArXiv preprint, then submit to a journal. With luck, we should have something readable to send to you by the end of May 2006.

While working on our joint project, Linas and I did our homework as regards bibliography. Thanks to you, Coffey, and Maślanka carefully posting your findings on ArXiv, we became aware of the ongoing debate regarding zeta, Newton, R.H., and differences. In particular I could appreciate the clarity and perceptiveness of your own mathematical works. (There are also, as I came to realize only recently, relevant interesting studies by Voros.)

Anyhow, Linas and I still decided to go on with our joint project of publishing something [attempting to publish, at least] on the following counts: (*i*) we have a complete asymptotic expansion (in descending powers of n) with fluctuating terms for differences of zeta values; (*ii*) the method applies equally well to all sorts of variants of zeta, like $\zeta(s)-1/(s-1)$, $s\zeta(s+1)$, $\zeta(2s)$, L-functions, etc; (*iii*) we have what we feel to be a less problem-specific approach to establishing the convergence of the corresponding Newton series (based on Carlson's beautiful theorem relative to holomorphic functions that are not too large in right half-planes and vanish at the integers). For the record, the main asymptotic term of the nth coefficient in the Newton series of $\zeta(s)-1/(s-1)$ is found to be

$$2^{3/4}\pi^{-1/4}e^{-2\sqrt{\pi n}}\cos\left(2\sqrt{\pi n}+\frac{3\pi}{8}\right).$$

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> It may also be of interest to note that the book "Polynomial
> Expansions of Analytic Functions" by R. P. Boas and R. C. Buck,
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[1]Almost everything recent is on my web page. Most of the rest, including some scans, is at the private URL `algo.inria.fr/flajolet/Reprints`.

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> Erbegnisse  der Math. Wiss, Band  19, contains a rather complete
> theory of expansions in Pochhammer polynomials which they call
> Newton polynomials.
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Like, I guess, most of the world that tends to speak of "Newton series". `:-):-)` More seriously, my pet reference on this is Nörlund's book *Differenzenrechnung* (1923), but you need at least a superficial knowledge of German[2]. I tend to call nowadays the integral representation of coefficients of Newton series "Nörlund-Rice integrals" (see [FlSe95] for the Rice side of the terminology).

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> However,  and this is important,  if you look at
> pages  34 and 35  you'll see the formula  that gives both Maslanka's
> and  Vepsta's coefficients,  but   the function $f(z)=(z-1)\zeta(z)$
> does  not satisfy  the conditions under  which it  is derived  there
> because although $f(z)$ is of order 1 it is of infinite type.
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Will need to look that up. I will need to check whether it's much different from what Nörlund had to say.

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> L.  Baez-Duarte,  A Sequential Riesz-Like  Criterion for the Riemann
> Hypothesis, International  Journal of Mathematics  and  Mathematical
> Sciences 21 (2005) 3527-3537.
```

That's another reason I wanted to write to you. Refer to (on my web site)

> [FlVa00]. Continued Fractions, Comparison Algorithms, and Fine Structure Constants. Philippe Flajolet and Brigitte Vallée (2000).

There, Brigitte and I accidentally bumped into a version of Baez-Duarte's criterion!!! To be a bit more precise, we considered a sort of difference of inverse zeta values in the form

$$D_n = \sum_{k\geq 1} \binom{n}{k} (-1)^k \frac{A(k)}{\zeta(2k)},$$

with $A(s)$ a certain well-controlled Dirichlet series, and observed[3] that

$$R.H \qquad \iff \qquad Q(n) = O\left(n^{1/4+\epsilon}\right),$$

where

$$D_n = K_0 n \log n + K_1 n + Q(n) + \text{smaller order terms.}$$

What is of interest in this context is that the Flajolet-Vallée sequence arise naturally from a problem in continued fraction theory: *"Given n random real numbers of* $[0,1]$*, how many continued fraction digits do you need on average to distinguish them (e.g., sort them)?"* We actually didn't think of this as a criterion: It is perhaps of interest that, as we noted in that paper (as well as in an earlier study of Clément-Flajolet-Vallée [ClFlVa01], published in 2001), there is a sort of exponential "dampening" of the effect of the nontrivial zeta zeros that renders such criteria

[2] If French is more congenial to you (as it should! :-)), then you can get some of the original papers by Nörlund freely from the NUMDAM site `http://www.numdam.org:80/numdam-bin/recherche`.

[3] In the published paper, we actually made a slightly stronger assertion, which needs to be amended.

not very effective for testing R.H. numerically. (I have seen since that similar remarks have been made on ArXiv concerning the numerical (im)practicability of some of the other recent criteria.)

Well, I guess that's all for the moment,

With best wishes,

Philippe