ON DIFFERENCES OF ZETA VALUES

PHILIPPE FLAJOLET AND LINAS VEPSTAS

ABSTRACT. Finite differences of values of the Riemann zeta function at the integers are explored. Such quantities, which occur as coefficients in Newton series representations, have surfaced in works of Bombieri–Lagarias, Maślanka, Coffey, Báez-Duarte, Voros and others. We apply the theory of Nörlund-Rice integrals in conjunction with the saddle point method and derive precise asymptotic estimates. The method extends to Dirichlet L-functions and our estimates appear to be partly related to earlier investigations surrounding Li's criterion for the Riemann hypothesis.

INTRODUCTION

In recent times, a variety of authors have, for a variety of reasons, been led to considering properties of representations of the Riemann zeta function $\zeta(s) = \sum 1/n^s$ as a *Newton interpolation series*. Amongst the many possible forms, we single out the one relative to a regularized version of Riemann zeta, namely,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} (-1)^n b_n \binom{s}{n}, \tag{1}$$

where $\binom{s}{n}$ is a binomial coefficient:

$$\binom{s}{n} := \frac{s(s-1)\cdots(s-n+1)}{n!}.$$

Corollary 1 in Section 5 establishes that the representation (1) is valid throughout the complex plane, its coefficients being determined by a general formula in the calculus of finite differences (see [13, 19, 20] and Section 1 below):

$$b_n = n(1-\gamma-H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} \binom{n}{k}(-1)^k \zeta(k), \tag{2}$$

Here, $H_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}$ is a harmonic number. Although the terms in the sum defining b_n become exponentially large (of order close to 2^n), the values of the b_n turn out to be exponentially small, while exhibiting a curious oscillatory behavior. We shall indeed prove the estimate (Theorem 1 in Section 4)

$$b_n = \left(\frac{2n}{\pi}\right)^{1/4} e^{-2\sqrt{\pi n}} \cos\left(2\sqrt{\pi n} - \frac{5\pi}{8}\right) + \mathcal{O}\left(n^{-1/4} e^{-2\sqrt{\pi n}}\right). \tag{3}$$

Our *first* motivation for investigating (1) and (2) was an attempt by one of us, Linas [2003, unpublished; available at `http://linas.org/math/poch-zeta.pdf`], to obtain alternative and tractable expressions for the Gauss-Kuzmin-Wirsing operator of continued fraction theory In particular, Linas' computations at that time revealed that the b_n

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tend rather fast to 0 and exhibit a surprising oscillatory pattern, both facts crying for explanation. The present paper, essentially elaborated in early 2006, represents the account of our joint attempts at understanding what goes on.

Second, the zeta function has received attention in physics, for its role in regularization and renormalization in quantum field theory. Motivated by such connections, Maślanka introduced in [16] what amounts to the Newton series representation of a regularized verson of $\zeta(s)$, namely, $(1-2s)\zeta(2s)$. (Further numerical observations relative to the corresponding coefficients are presented by this author in [17].) The growth of coefficients in Maślanka's expansion has been subsequently investigated by Báez-Duarte [2]. In particular, Báez-Duarte's analysis implies that the coefficients decrease to 0 faster than any power of $1/n$. The methods we develop to derive Equation (3) can be easily adapted to yield refinements of the estimates of Baez-Duarte and Maślanka.

A *third* reason for interest in the representation (1) and the companion coefficients (2) is *Li's criterion* [15] for the Riemann Hypothesis (RH). Let ρ range over the nontrivial zeros of $\zeta(s)$. Li's theorem asserts that RH is true if and only if all members of the sequence

$$\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right], \qquad n \geq 1,$$

are nonnegative. Bombieri and Lagarias [4] offer an insightful discussion of Li's criterion. Coffey [5], following Bombieri and Lagarias [4, Th. 2], has expressed the λ_n as a sum of two terms, one of which is an elementary variant b_n defined by

$$A_n(1,2) := \sum_{k=2}^{n} \binom{n}{k} (-1)^k (1 - 2^{-k}) \zeta(k).$$

Theorem 2 of [5] amounts to the property that the coefficients $A_n(1,2)$ decrease to 0. As we shall see in Section 6 and right after Equation (48) in Section 7, the methods originally developed for estimating b_n yield precise asymptotic information on $A_n(1,2)$ as well. Though the sums we deal with count amongst the far easier ones, our precise asymptotic estimates may contribute to bring some clarity in this range of problems.

In this essay, we approach the problem of asymptotically estimating differences of zeta values by means of a combination of two well established techniques. We start from a contour integral representation of these differences as defined by (2) (for this technique, see especially Nörlund's treatise [20] and the study [9]), then proceed to estimate the corresponding complex integral by means of the classical saddle point method of asymptotic analysis [7, 21]. Our approach parallels a recent paper of Voros [25] (motivated by Li's criterion), which our results supplement by providing a fairly detailed asymptotic analysis of differences of zeta values.

Section 1 presents the construction of a Newton series for the zeta function, and presents generating functions for its coefficients. This is followed, in Section 2, by a brief examination of numerical results. Section 3 gives the Nörlund integral representation for the coefficients, of which Section 4 provides a careful saddle point analysis. The convergence of the Newton series representation (1) is then discussed in Section 5, and Section 6 develops the corresponding analysis for Dirichlet L-functions. We end with a conclusion, Section 7, outlining other applications of Nörlund integrals in the realm of finite differences and zeta functions.

1. NEWTON SERIES AND ZETA VALUES

This section defines the Newton series for the Riemann zeta that is to be studied, demonstrates some of its basic properties, and gives some generating functions for its coefficients. In this paper, a Newton series is defined as

$$\Phi(s) = \sum_{n=0}^{\infty} (-1)^n c_n \binom{s}{n}. \tag{4}$$

Given a function $\phi(s)$, one may attempt to represent it in some region of the complex plane by means of such a series. Since the series $\Phi(s)$ terminates at $s = 0, 1, 2, \ldots$, the conditions $\phi(m) = \Phi(m)$ at the nonnegative integers imply that the candidate sequence $\{c_n\}$ is linearly related to the sequence of values $\{\phi(m)\}$ by

$$\phi(m) = \sum_{n=0}^{m} (-1)^n c_n \binom{m}{n}.$$

The triangular system can then be inverted to give (by the binomial transform [11, p. 192], or its Euler transform version [22, p. 43], or by direct elimination)

$$c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi(k), \qquad n = 0, 1, 2, \ldots . \tag{5}$$

This choice of coefficients for (4) determines the Newton series *associated* to ϕ. The coincidence of the function ϕ and its associated series Φ is, by construction, granted at least at all the nonnegative integers. The validity of $\Phi(s) = \phi(s)$ is often found to extend to large parts of the complex plane, but this fact requires specific properties beyond the mere convergence of the series in (4).

In the case of the Newton series for $\zeta(s) - 1/(s-1)$, the general relation (5) provides the coefficients in the form

$$b_n = s_0 - ns_1 + \sum_{k=2}^{n} \binom{n}{k} (-1)^k \left[\zeta(k) - \frac{1}{k-1}\right], \tag{6}$$

where

$$s_0 = \left[\zeta(s) - \frac{1}{s-1}\right]_{s=0} = \frac{1}{2}, \qquad s_1 = \lim_{s\to 1} \left[\zeta(s) - \frac{1}{s-1}\right] = \gamma. \tag{7}$$

The harmonic numbers appear as[1]

$$\sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{k-1} = 1 - n + nH_{n-1}. \tag{8}$$

Equations (6), (7), (8) then entail that the b_n, as defined by (2), are indeed the coefficients of the Newton series associated to $\zeta(s) - 1/(s-1)$. A proof that the equality $\Phi(s) = \zeta(s) - 1/(s-1)$ holds for all complex s is given in Section 5, following the asymptotic analysis of the coefficients b_n and based on a theorem of Carlson.

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[1] Identity (8) is easily deduced from the classical partial fraction decomposition (see, e.g., [11, p. 188]),

$$\frac{n!}{x(x+1)\cdots(x+n)} = \sum_k \binom{n}{k} \frac{(-1)^k}{x+k},$$

upon letting $x \to -1$.

Before engaging in a detailed study of the b_n, we note a few simple facts regarding their elementary properties. Consider the quantities

$$\delta_n := \sum_{k=2}^{n} \binom{n}{k} (-1)^k \zeta(k), \tag{9}$$

which represent the nontrivial sums in the definition (2) of b_n and are, up to minor adjustments, differences of zeta values at the integers. Expanding the zeta function according to its definition and exchanging the order of summations in the resulting double sum yields

$$\delta_n = \sum_{\ell \geq 1} \left[\left(1 - \frac{1}{\ell}\right)^n - 1 + \frac{n}{\ell} \right]. \tag{10}$$

This rather simple sum shows a remarkably complex behavior; elucidating its behavior is one of the principal topics of this paper.

The ordinary generating function for the sequence $\{\delta_n\}$ is also of interest. Given the classical expansion [1, §6.3] of the logarithmic derivative $\psi(z) = \Gamma'(z)/\Gamma(z)$ of the Gamma function,

$$\psi(1+z) + \gamma = \zeta(2)z - \zeta(3)z^2 + \zeta(4)z^3 - \cdots,$$

one finds, by the usual generating function translation of the Euler transform [19, p. 311] or by an immediate verification based on the binomial theorem:

$$\sum_{n \geq 2} \delta_n z^n = \frac{z}{(1-z)^2} \left[\psi\left(\frac{1}{1-z}\right) + \gamma \right]. \tag{11}$$

The exponential generating function for the sequence $\{\delta_n\}$ reflects (10) and is even simpler:

$$\sum_{n \geq 2} \delta_n \frac{z^n}{n!} = e^z \sum_{n \geq 2} \zeta(n) \frac{(-z)^n}{n!} = e^z \sum_{\ell \geq 1} \left[e^{-z/\ell} - 1 + \frac{z}{\ell} \right]. \tag{12}$$

(The first equality is easily verified by expanding e^z and expressing the coefficient of $z^n/n!$ in the product as a convolution, itself seen to coincide with (9).)

2. Experimental analysis

Detailed experiments on the coefficients b_n conducted by one of us are at the origin of the present paper and we briefly discuss these since they illustrate some concrete numerical aspects of the sequence (b_n) while being potentially useful for similar problems. As it is usual when dealing with finite differences, the alternating binomial sums giving the b_n involve exponential cancellation since the binomial coefficients get almost as large as 2^n. We started by conducting evaluations of the b_n up to $n \approx 5000$, which requires determining zeta values up to several thousand digits of precision. Note that the zeta values can be computed rapidly to extremely high precision using several efficient algorithms (e.g., [6]), which are available in symbolic computation packages (MAPLE) and numerical libraries (PARI/GP).

A quick inspection of numerical data immediately reveals two features of the constants b_n: they are oscillatory with a slowly increasing (pseudo)period and their absolute values are rapidly decreasing. For instance[2]:

$$\begin{array}{llllll} b_1 & \doteq -7.72156 \cdot 10^{-2}, & b_2 & \doteq -9.49726 \cdot 10^{-3}, & b_5 & \doteq +7.15059 \cdot 10^{-4}, \\ b_{10} & \doteq -2.83697 \cdot 10^{-5}, & b_{20} & \doteq +2.15965 \cdot 10^{-9}, & b_{50} & \doteq -1.08802 \cdot 10^{-11}. \end{array}$$

A numeric fit of the oscillatory behavior of the function was first made. There are sign changes in the sequence $\{b_n\}$ at

$$n = 3, 7, 13, 21, 29, 40, 52, 65, 80, 97, 115, 135, 157, 180, \ldots,$$

the values growing roughly quadratically. A good fit for the kth sign-change was found experimentally to be of the form $q(k) = \frac{\pi}{4}k^2 + \mathcal{O}(k)$. The quadratic polynomial is then easily inverted to give an approximate oscillatory behavior of the b_n. Once the oscillatory behavior had been disposed of, the task of quantifying the general trend in the overall decrease of the sequence became easier. These observations then led us to conjecture

$$\beta(n) := \cos \pi \left(2\sqrt{\frac{n}{\pi}} + L \right) e^{-K\sqrt{n}}, \qquad K = 3.6 \pm 0.1, \tag{13}$$

as a rough approximation to b_n, where L is some real constant. This numerical fit then greatly helped us find the main estimate (3) above, which has the exact value $K = \sqrt{2\pi} \doteq 3.54490$ and an additional $n^{1/4}$ factor modulating the exponential.

3. The Nörlund integral representation

Our approach to the asymptotic estimation of the b_n relies on a complex integral representation of finite differences of an analytic function, to be found in Nörlund's classic treatise [20, §VIII.5] first published in 1924. In computer science, this representation was popularized by Knuth [14, p. 138], who attributed it to S.O. Rice, so that it also came to be known as "Rice's method"; see [9] for a review.

Lemma 1. *Let $\phi(s)$ be holomorphic in the half-plane $\Re(s) \geq n_0 - \frac{1}{2}$. Then the finite differences of the sequence $(\phi(k))$ admit the integral representation*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = \frac{(-1)^n}{2\pi i} \int_C \phi(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds, \tag{14}$$

where the contour of integration C encircles the integers $\{n_0, \ldots, n\}$ in a positive direction and is contained in $\Re(s) \geq n_0 - \frac{1}{2}$.

Proof. The integral on the right of (14) is the sum of its residues at $s = n_0, \ldots, n$, which precisely equals the sum on the left. □

An immediate consequence is the following representation for the differences of zeta values (δ_n is as in (9)):

$$\delta_n \equiv \sum_{k=2}^{n} \binom{n}{k} (-1)^k \zeta(k) = \frac{(-1)^{n-1}}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \zeta(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds. \tag{15}$$

(Choose $T > n$ and consider the finite contour (negatively oriented) consisting of the line from $\frac{3}{2} - iT$ to $\frac{3}{2} + iT$, followed by the clockwise arc of $|s| = \sqrt{T^2 + 9/4}$ that lies to the right of the given line. The contribution of the circular arc is $\mathcal{O}(T^{-n})$ and thus vanishes as $T \to +\infty$.)

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[2]The notation $x \doteq y$ designates a numerical approximation of x by y to the last decimal digit stated.

Since b_n is δ_n plus a correction term (see Equation (2)) and δ_n admits the integral representation (9), the first step of our analysis is to move the line of integration further to the left. It is well known that the Riemann zeta function is of finite order in any right half-plane [24, §5.1], that is, $|\zeta(s)| = \mathcal{O}(|s|^A)$ uniformly as $|s| \to \infty$, for some A depending on the half-plane under consideration. As a consequence, the integral of (15) remains convergent, when taken along any vertical line left of 0, as soon as n is large enough. Under these conditions, it is possible to replace the line of integration $\Re(s) = \frac{3}{2}$ by the line $\Re(s) = -\frac{1}{2}$, upon taking into account the residues of a double pole at $s = 1$ and a simple pole at $s = 0$. We find in this way

$$\delta_n = (-1)^{n-1}(R_1 + R_0) + \frac{(-1)^{n-1}}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \zeta(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds,$$

where, as shown by a routine calculation:

$$(-1)^n R_0 = -\frac{1}{2}, \qquad (-1)^n R_1 = -n(1 - \gamma - H_{n-1}).$$

The residues thus compensate exactly for the difference between δ_n and b_n, so that

$$b_n = \frac{(-1)^{n-1}}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \zeta(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds. \tag{16}$$

4. Saddle point analysis of zeta differences

The integrand in (16) is free of singularities on the left-hand side, and is thus a good candidate for estimation by the saddle point (or steepest-descent) method. This evaluation is the main topic of this section, and culminates with the derivation of one of the principal results of this paper, Theorem 1.

We make use of the functional equation of the Riemann zeta function under the form

$$\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1} \sin \frac{\pi s}{2} \zeta(1-s). \tag{17}$$

From this relation, upon performing the change of variables $s \mapsto -s$, we obtain

$$b_n = -\frac{1}{\pi i} \int_{1/2-i\infty}^{1/2+i\infty} (2\pi)^{-s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1+s) \frac{n!\,\Gamma(1+s)}{s(s+1)\cdots(s+n)}\, ds, \tag{18}$$

which is the starting point of our asymptotic analysis.

The integral representation (18) has several noticeable features. First, the integrand has no singularity at all in $\Re(s) \geq \frac{1}{2}$ and it appears to decay exponentially fast towards $\pm i\infty$ (see Equations (20), (21), and (21) below). This means that one can freely choose the abscissa c (with $c \geq \frac{1}{2}$) in the representation

$$b_n = -\frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi)^{-s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1+s) \frac{n!\,\Gamma(s+1)}{s(s+1)\cdots(s+n)}\, ds. \tag{19}$$

The very absence of singularities calls for an application of the saddle point method.

The factor $\zeta(1+s)$ remains bounded in modulus by a constant, and is in fact barely distinguishable from 1, as $\Re(s)$ increases, since

$$\zeta(s) = 1 + \mathcal{O}\left(2^{-\Re(s)}\right), \qquad \Re(s) \geq \frac{3}{2}. \tag{20}$$

Also, for large $|s|$, the complex version of Stirling's formula applies:

$$\Gamma(1+s) = s^s e^{-s} \sqrt{2\pi s} \left(1 + \mathcal{O}(|s|^{-1})\right), \qquad \Re(s) \geq 0. \tag{21}$$

Finally, the sine factor increases exponentially along vertical lines: one has

$$2i\sin\frac{\pi s}{2} = -\exp\left(-i\frac{\pi s}{2}\right) + \mathcal{O}\left(e^{-\pi\Im(s)/2}\right), \qquad \Im(s)\geq 0, \tag{22}$$

with a conjugate approximation holding for $\Im(s) < 0$.

In anticipation of applying saddle point methods, the approximations (20), (21), and (22) then suggest the function $e^{\omega(s)}$ as a simplified model of the integrand in the upper half-plane, where

$$\omega(s) = -s\log(2\pi) - i\frac{\pi s}{2} + \log\frac{n!\Gamma(s)^2}{\Gamma(s+n)}. \tag{23}$$

We shall demonstrate shortly that the location of the appropriate saddle points in the complex plane scale as $\sqrt{n}$, which may be confirmed by numerical experiments. Therefore, in performing an asymptotic analysis, it is appropriate to perform a change of variable $s = x\sqrt{n}$, and expand in descending powers of n, presuming x to be approximately constant. We find, uniformly for x in any compact region of $\Re(x) > 0$, $\Im(x) > 0$:

$$\left\{\begin{array}{lcl} \omega(x\sqrt{n}) & = & x\sqrt{n}\left[2\log x - 2 - \log(2\pi) - \frac{1}{2}i\pi\right] + \frac{1}{2}\log n \\ & & \qquad - \log x + \log(2\pi) - \frac{1}{2}x^2 + \mathcal{O}(n^{-1/2}) \\ \omega'(x\sqrt{n}) & = & \left[-\log(2\pi) - i\frac{\pi}{2} + 2\log x\right] - \left(x + \frac{1}{x}\right)\frac{1}{\sqrt{n}} + \mathcal{O}(n^{-1}). \\ \omega''(x\sqrt{n}) & = & \frac{2}{x\sqrt{n}} + \mathcal{O}(n^{-1}). \end{array}\right. \tag{24}$$

(The symbolic manipulation system MAPLE is a great help in such computations.)

From the second line of (24), an approximate root of $\omega'(s)$ is obtained by choosing the particular value x_0 of x that cancels ω' to main asymptotic order:

$$x_0 = e^{i\pi/4}\sqrt{2\pi}. \tag{25}$$

This corresponds to the following value for s,

$$\sigma \equiv \sigma(n) = x_0\sqrt{n} = (1+i)\sqrt{\pi n}, \tag{26}$$

which thus is also an approximate saddle point for $e^{\omega(s)}$. The substitution of this value given the first line of (24) then leads to

$$\exp\left(\omega(\sigma(n))\right) = \exp\left(2i\sqrt{\pi n}\right)\cdot\exp\left(-2\sqrt{\pi n}\right)\cdot\Pi(n), \tag{27}$$

where Π is an unspecified factor of at most polynomial growth. By using a suitable contour that passes though $\sigma(n)$, we thus expect the quantity in (27) to be an approximation (up to polynomial factors again) of b_n. This back-of-the-envelope calculation does predict the exponential decay of b_n as $\exp\left(-2\sqrt{\pi n}\right)$, in a way consistent with numerical data, while the fluctuations, $\sin\left(2\sqrt{\pi n} + \mathcal{O}(1)\right)$, are seen to be in stunning agreement with the empirically obtained formula (13).

We must now fix the contour of integration and provide final approximations. The contour adopted (Figure 1) goes through the saddle point $\sigma = \sigma(n)$ and symmetrically through its complex conjugate $\overline{\sigma} = \overline{\sigma(n)}$. In the upper half-plane, it traverses $\sigma(n)$ along a line of steepest descent whose direction, as determined from the argument of $\omega''(\sigma)$, is at an angle of $\frac{5\pi}{8}$ with the horizontal axis. The contour also includes parts of two vertical lines of respective abscissae $\Re(s) = c_1\sqrt{n}$ and $c_2\sqrt{n}$, where

$$0 < c_1 < \sqrt{\pi} < c_2 < 2\sqrt{\pi}.$$

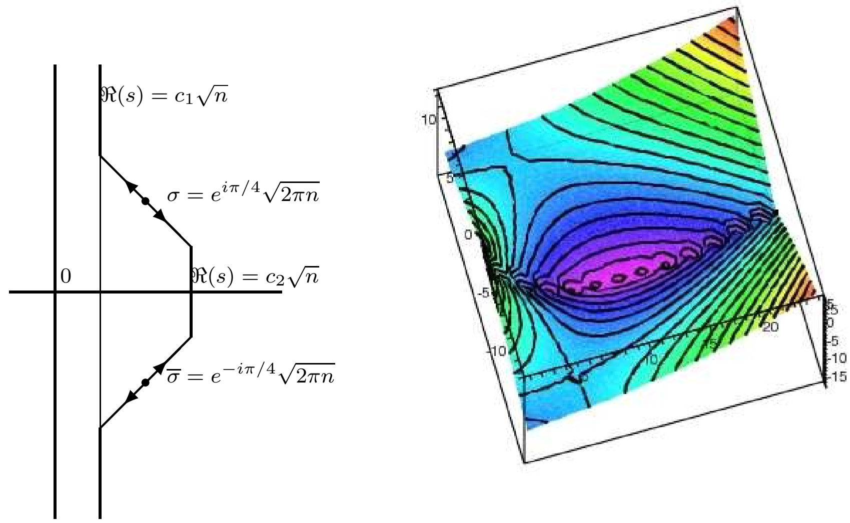

FIGURE 1. Left: The saddle point contour used for estimating b_n. The arrows point at the directions of steepest descent from the saddle points. Right: the landscape of the logarithm of the modulus of the integrand in the representation of b_n for $n = 10$.

The choice of the abscissae, c_1 and c_2, is not critical (it is even possible to adapt the analysis to $c_1 = c_2 = \sqrt{\pi}$). One verifies easily, from crude approximations, that the contributions arising from the vertical parts of the contour are $\mathcal{O}(e^{-L_0\sqrt{n}})$, for some $L_0 > 2\sqrt{\pi}$, i.e., they are exponentially small in the scale of the problem:

$$\int_{\text{vertical}} = \mathcal{O}\left(e^{-L_0\sqrt{n}}\right) \qquad L_0 > 2\sqrt{\pi}. \tag{28}$$

The slanted part of the contour is such that all the estimates of (24) apply. The scale of the problem is dictated by the value of $\omega''(\sigma)$, which is of order $\mathcal{O}(n^{-1/2})$. This indicates that the "second order" scaling to be adopted is $n^{1/4}$. Accordingly, we set

$$s = (1+i)\sqrt{\pi n} + e^{5i\pi/8} y n^{1/4}. \tag{29}$$

Define the *central region* of the slanted part of the contour by the condition that $|y| \leq \log^2 n$. Upon slightly varying the value of x around x_0, one verifies from (24) that, for large n, the quantity

$$\Re\left(\frac{1}{\sqrt{n}}\omega\left(x_0\sqrt{n} + e^{5i\pi/8}t\sqrt{n}\right)\right)$$

is an upward concave function of t near $t = 0$. There results, in the complement of the central part, $|y| \geq \log^2 n$, the approximation

$$\left|\exp\left(\omega\left(x_0\sqrt{n} + e^{5i\pi/8}yn^{1/4}\right)\right)\right| < e^{\omega(x_0\sqrt{n})} \cdot \exp\left(-L_1\log^2 n\right), \qquad L_1 > 0.$$

Figuratively:

$$\int_{\text{slanted}} = \int_{\text{central}} + \mathcal{O}\left(\exp\left(-L_1\log^2 n\right)\right). \tag{30}$$

Thus, from (28) and (30), only the central part of the slanted region matters asymptotically. This applies to $e^{\omega(s)}$ but also to the full integrand of the representation (19) of b_n, given the approximations (20)–(24).

We are finally ready to reap the crop. Take the integral representation of (19) with the contour deformed as indicated in Figure 1 and let b_n^+ be the contribution arising from the upper half-plane, to the effect that

$$b_n = 2\Re(b_n^+), \tag{31}$$

by conjugacy. In the central region,

$$s = x_0\sqrt{n} + e^{5i\pi/8} y n^{1/4},$$

the integrand of (19) becomes

(32)

$$\left(-\frac{1}{\pi i}\right) \cdot (2\pi)^{-1} \cdot \left(-\frac{1}{2i}\right) \cdot \left(1 + \mathcal{O}(2^{-\sqrt{\pi n}})\right) \cdot \frac{x_0}{\sqrt{n}} \cdot e^{\omega(x_0\sqrt{n})} \cdot e^{-y^2/\sqrt{2\pi}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right).$$

The various factors found there (compare (19) to $e^{\omega(s)}$ with $\omega(s)$ defined in (23)) are in sequence: the Cauchy integral prefactor; the correction $(2\pi)^{-1}$ to the functional equation of Riemann zeta; the factor $-1/(2i)$ relating the sine to its exponential approximation; the approximation of Riemann zeta; the correction $s/(s+n)$ of the Gamma factors; the main term $e^{\omega(\sigma)}$; the anticipated local Gaussian approximation; the errors resulting from approximations (20)–(24), which are of relative order $\mathcal{O}(n^{-1/2})$. Upon completing the tails of the integral and neglecting exponentially small corrections, we get

$$b_n^+ = K_0 e^{\omega(x_0\sqrt{n})} \frac{x_0}{\sqrt{n}} \int_{-\infty}^{+\infty} e^{-y^2/\sqrt{2\pi}}\, dy \cdot \left(e^{5i\pi/8} n^{1/4}\right) \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right), \tag{33}$$

where $K_0 = -1/(4\pi^2)$ is the constant factor of (32), while the factor following the integral translates the change of variables: $ds = e^{5i\pi/8} n^{1/4} dy$.

The asymptotic form of b_n is now completely determined by (31) and (33). We have obtained:

Theorem 1. *The Newton coefficient* b_n *of* $\zeta(s) - 1/(s-1)$ *defined in* (2) *satisfies*

$$b_n = \left(\frac{2n}{\pi}\right)^{1/4} e^{-2\sqrt{\pi n}} \cos\left(2\sqrt{\pi n} - \frac{5\pi}{8}\right) + \mathcal{O}\left(e^{-2\sqrt{\pi n}} n^{-1/4}\right). \tag{34}$$

The agreement between asymptotic and exact values is quite good, even for small values of n (Figure 2).

In summary, the foregoing developments justify the validity of applying the saddle point formula to the Nörlund Rice integral representation (19) of zeta value differences. Under its general form, this formula (which, without further assumptions, remains a heuristic) reads

$$\int e^{-Nf(x)}\, dx = \sqrt{\frac{2\pi}{Nf''(x_0)}} e^{-Nf(x_0)} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{35}$$

Here the analytic function $f(x)$ must have a (simple) saddle point at x_0, that is, $f'(x_0) = 0$ and $f''(x_0)$ is the second derivative of f at the saddle point. In the case of differences of zeta values, the appropriate scaling parameter is $s = x\sqrt{n}$ corresponding to $N = \sqrt{n}$, and the function f is

$$f(x) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \omega\left(x\sqrt{n}\right),$$

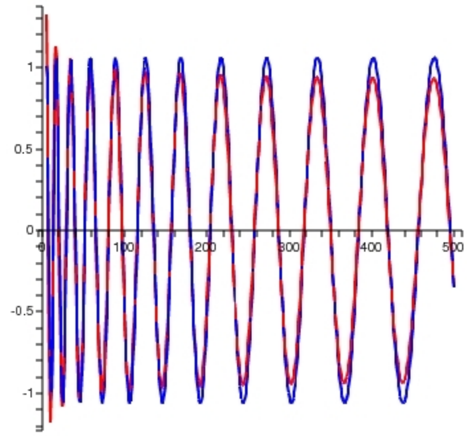

FIGURE 2. A comparative plot of b_n and the main term of its approximation (34), both multiplied by $e^{2\sqrt{\pi n}} n^{-1/4}$, for $n = 5 \,..\, 500$.

up to smaller order corrections that we could treat as constants in the range of the saddle point. As we shall see in Section 6, this paradigm adapts to sums involving Dirichlet L-functions.

5. CONVERGENCE OF THE NEWTON SERIES OF ZETA

The fact that the coefficients b_n decay to zero faster than any polynomial in $1/n$ implies that the Newton series

$$\Phi(s) = \sum_{n=0}^{\infty} (-1)^n b_n \binom{s}{n}, \tag{36}$$

with b_n given by (2), converges throughout the complex plane, and consequently defines an entire function. Set $Z(s) := \zeta(s) - 1/(s-1)$ with $Z(1) = \gamma$. We have, by construction $\Phi(s) = Z(s)$ at $s = 0, 1, 2 \ldots$, but the relation between Φ and Z at other points is still unclear.

Corollary 1. *The Newton series of* (36) *is a convergent representation of the function* $\zeta(s) - 1/(s-1)$ *valid at all points* $s \in \mathbb{C}$.

Proof. Here is our favorite proof. A classic theorem of Carlson (for a discussion and a proof, see, e.g., Hardy's Lectures [12, pp. 188-191] or Titchmarsh's treatise [23, §5.81]) says the following: *Assume that* (i) $g(s)$ *is analytic and such that*

$$|g(s)| < Ce^{A|s|},$$

where $A < \pi$, *in the right half-plane of complex values of* s, *and* (ii) $g(0) = g(1) = \ldots = 0$. *Then* $g(s)$ *vanishes identically.*

To complete the proof, it suffices to apply Carlson's theorem to the difference $g(s) = \Phi(s+2) - Z(s+2)$. Condition (ii) is satisfied by construction of the Newton series. Condition (i) results from the fact that $Z(s+2)$ is $\mathcal{O}(1)$ while a general bound due to

Nörlund (Equation (58) of [20, p. 228]) and valid for all convergent Newton series asserts that $|\Phi(s+2)|$ is of growth at most $e^{\frac{\pi}{2}|s|}$, throughout $\Re(s) > -\frac{1}{2}$. □

An alternative proof can be given starting from a contour integral representation for the remainder of a general Newton series given [20, p. 223]. Yet another proof derives from a turnkey theorem of Nörlund, quoted in [19, p. 311]: *In order that a function $F(x)$ should admit a Newton series development, it is necessary and sufficient that $F(x)$ should be holomorphic in a certain half-plane $\Re(x) > \alpha$ and should there satisfy the inequality $|F(x)| < C2^{|x|}$, where C is a fixed positive number.* In a short note, Báez-Duarte [2] justified a similar looking Newton series representation of the zeta function due to Maślanka—however his bounds on the Newton coefficients are less precise than ours and his arguments (based on a doubly indexed sequence of polynomials) seem to be somewhat problem-specific.

6. Dirichlet L-functions

The methods employed to deal with differences of zeta values have a more general scope, and we may reasonably expect them to be applicable to other kinds of Dirichlet series. Such is indeed the case for any Dirichlet L-function,

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a multiplicative character of some period k, that is, for all integers m, n, one has: $\chi(n+k) = \chi(n)$, $\chi(mn) = \chi(m)\chi(n)$, $\chi(1) = 1$, and $\chi(n) = 0$ whenever $\gcd(n,k) \neq 1$.

Let $\zeta(s,q)$ be the Hurwitz zeta function defined by

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \tag{37}$$

Any Dirichlet L-function may be represented as a combination of Hurwitz zeta functions,

$$L(\chi, s) = \frac{1}{k^s} \sum_{m=1}^{k} \chi(m) \zeta\left(s, \frac{m}{k}\right), \tag{38}$$

where k is the period of χ. In particular, the coefficients of the Newton series for $L(\chi, s)$ are simple linear combinations of the quantities

$$A_n(m,k) = \sum_{\ell=2}^{n} \binom{n}{\ell} (-1)^{\ell} \frac{\zeta\left(\ell, \frac{m}{k}\right)}{k^{\ell}}, \tag{39}$$

which we adopt as our fundamental object of study.

Theorem 2. *The differences of Hurwitz zeta values, $A_n(m,k)$ defined by* (39)*, satisfy the estimate*

$$A_n(m,k) = \left(\frac{m}{k} - \frac{1}{2}\right) - \frac{n}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_{n-1}\right] + a_n(m,k) \tag{40}$$

where the $a_n(m,k)$ are exponentially small:

$$\begin{aligned} a_n(m,k) \quad &= \quad \frac{1}{k}\left(\frac{2n}{\pi k}\right)^{1/4} \exp\left(-\sqrt{\frac{4\pi n}{k}}\right) \cos\left(\sqrt{\frac{4\pi n}{k}} - \frac{5\pi}{8} - \frac{2\pi m}{k}\right) \\ &\quad + \mathcal{O}\left(n^{-1/4} e^{-2\sqrt{\pi n/k}}\right). \end{aligned} \tag{41}$$

Here $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Gamma function.

The previous results for Riemann zeta may be regained by setting $m = k = 1$, so that $\delta_n = A_n(1,1)$ and $b_n = a_n(1,1)$.

Proof. Converting the sum to the Nörlund-Rice integral, and extending the contour to infinity, like before, one obtains

$$A_n(m,k) = \frac{(-1)^n}{2\pi i}\, n! \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\zeta\left(s, \frac{m}{k}\right)}{k^s s(s-1)\cdots(s-n)}\, ds \tag{42}$$

Moving the contour to the left, one encounters a single pole at $s = 0$ and a double pole at $s = 1$. The residue of the pole at $s = 0$ is

$$\text{Res}(s=0) = \zeta\left(0, \frac{m}{k}\right) = \frac{1}{2} - \frac{m}{k}.$$

(See [26, p. 271] for this evaluation.) The double pole at $s = 1$ evaluates to

$$\text{Res}(s=1) = \frac{n}{k}\left[\psi\left(\frac{m}{k}\right) + \ln k + 1 - H_{n-1}\right]$$

Combining these, one obtains (40) where the a_n are given by

$$a_n(m,k) = \frac{(-1)^n}{2\pi i}\, n! \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\zeta\left(s, \frac{m}{k}\right)}{k^s s(s-1)\cdots(s-n)}\, ds. \tag{43}$$

As before, the $a_n(m,k)$ have the remarkable property of being exponentially small; that is, $a_n(m,k) = \mathcal{O}\left(e^{-K\sqrt{n}}\right)$, for a constant K that only depends on k. The precise behavior of the exponentially small term may be obtained by using a saddle point analysis parallel to the one given in the previous sections. Its application here is abbreviated, as there are no major differences in the course of the derivations.

The term $a_n(m,k)$ is represented by the integral of (43). At this point, the functional equation for the Hurwitz zeta may be applied. This equation is

$$\zeta\left(1-s, \frac{m}{k}\right) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{p=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi p m}{k}\right) \zeta\left(s, \frac{p}{k}\right), \tag{44}$$

and it either follows from adapting the "second proof" of Riemann for the common zeta function [24, §2.4] or from the transformation formula of Lerch's transcendent ϕ found in [26, p. 280]. This allows the integral in (43) to be expressed as a sum:

$$a_n(m,k) = -\frac{2n!}{k\pi i} \sum_{p=1}^{k} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} \cos\left(\frac{\pi s}{2} - \frac{2\pi p m}{k}\right) \zeta\left(s, \frac{p}{k}\right)\, ds$$

It proves convenient to pull the phase factor out of the cosine part and write the integral as

$$\begin{aligned} a_n(m,k) \quad = \quad & -\frac{n!}{k\pi i} \sum_{p=1}^{k} \exp\left(i\frac{2\pi p m}{k}\right) \\ & \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{1}{(2\pi)^s} \frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} \exp\left(-i\frac{\pi s}{2}\right) \zeta\left(s, \frac{p}{k}\right)\, ds + \mathbf{c.c.}, \end{aligned}$$

where $\mathbf{c.c.}$ ("*complex conjugate*") means that i should be replaced by $-i$ in the two exp parts.

To recast the equation above into the form needed for the saddle point method, an asymptotic expansion of the integrands needs to be made for large n. As before, the appropriate scaling parameter is $x = s/\sqrt{n}$. The asymptotic expansion is then performed by holding x constant, and taking n large. Thus, one writes

$$a_n(m,k) = -\frac{1}{k\pi i}\sum_{p=1}^{k}\left[e^{i2\pi pm/k}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\omega(x\sqrt{n})}dx \ +e^{-i2\pi pm/k}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\overline{\omega}(x\sqrt{n})}dx\right]. \tag{45}$$

Proceeding, one finds

$$\omega(s) = \log n! + \frac{1}{2}\log n - s\log\left(\frac{2\pi p}{k}\right) - i\frac{\pi s}{2} + \log\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)} + \mathcal{O}\left(\left(\frac{p}{k+p}\right)^s\right)$$

where the approximation $\log\zeta\left(s,p/k\right) = (k/p)^s + \mathcal{O}\left((p/(k+p))^s\right)$, for large $\Re(s)$, has been made. Expanding to $\mathcal{O}(1/\sqrt{n})$ and collecting terms, one obtains

$$\begin{aligned}\omega(x\sqrt{n}) \quad &= \quad \frac{1}{2}\log n - x\sqrt{n}\left[\log\frac{2\pi p}{k} + i\frac{\pi}{2} + 2 - 2\log x\right] \\ &\qquad + \log 2\pi - 2\log x - \frac{x^2}{2} + \mathcal{O}\left(n^{-1/2}\right)\end{aligned} \tag{46}$$

The saddle point is obtained by solving $\omega'(x\sqrt{n}) = 0$. To lowest order, one has $x_0 = (1+i)\sqrt{\pi p/k}$. Also, $\omega''(x\sqrt{n}) = 2/x\sqrt{n} + \mathcal{O}(n^{-1})$. Substituting in the saddle point formula (35), one directly finds

$$\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\omega(x\sqrt{n})}dx = \left(\frac{2\pi^3 pn}{k}\right)^{1/4} e^{i\pi/8}\exp\left(-(1+i)\sqrt{\frac{4\pi pn}{k}}\right) + \mathcal{O}\left(n^{-1/4}e^{-2\sqrt{\pi pn/k}}\right) \tag{47}$$

while the integral for $\overline{\omega}$ is the complex conjugate quantity (having a saddle point at the complex conjugate location). Inserting this into equation (45) gives a sum of contributions for $p = 1, \ldots, k$, of which, for large n, only the $p = 1$ term is seen to contribute significantly. So, one has the estimation (41) of the statement. $\square$

7. PERSPECTIVE

The previous methods serve to unify and make precise estimates carried out in the literature by a diversity of approaches. For instance, the study of quantities arising in connection with Li's criterion calls for estimating, in the notations of (39),

$$A_n(1,2) = \sum_{\ell=2}^{n}\binom{n}{\ell}(-1)^\ell(1-2^{-\ell})\zeta(\ell). \tag{48}$$

Bombieri and Lagarias encountered this quantity in [4, Th. 2] and Coffey (see his $S_1(n)$ in [5]) proved, by means of series rearrangements akin to (10) used in conjunction with Euler-Maclaurin summation the inequality

$$A_n(1,2) \geq \frac{n}{2}\log n + (\gamma-1)\frac{n}{2} + \frac{1}{2}. \tag{49}$$

Our analysis quantifies $A_n(1,2)$ to be

$$A_n(1,2) = \frac{n}{2}\psi(n) + n(\gamma - \frac{1}{2} + \frac{1}{2}\log 2) + o(1),$$

where the $o(1)$ error term above is $a_n(1,2)$, which is exponentially small and oscillating:

$$a_n(1,2) = \frac{1}{2}\left(\frac{n}{\pi}\right)^{1/4} \exp\left(-\sqrt{2\pi n}\right) \cos\left(\sqrt{2\pi n} - \frac{5\pi}{8}\right) + \mathcal{O}\left(n^{-1/4} e^{-\sqrt{2\pi n}}\right) \tag{50}$$

Another observation is that the combination of Nörlund-Rice integrals and saddle point estimates applies to many "desingularized" versions of the Riemann zeta function, like

$$(1-2^{1-s})\zeta(s), \quad (s-1)\zeta(s), \quad \zeta(2s) - \frac{1}{2s-1}, \quad (2s-1)\zeta(2s).$$

The first one is directly amenable to Theorem 2. The Newton series involving $\zeta(2s)$ include Maślanka's expansion [16] (relative to $(2s-1)\zeta(2s)$) and have a striking feature—their Newton coefficients are polynomials in π with rational coefficients. In addition, the exponential smallness of error terms in asymptotic expansions of finite differences of this sort has the peculiar feature of inducing near-identities that relate rational combinations of zeta values and Euler's constant. For instance, defining the following elementary variant of b_n,

$$C_n := -\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{\zeta(k+1)}{k+1},$$

we find to more than 35 digits of accuracy,

$$\left\{ \begin{array}{rcl} C_{499} - H_{499} + 1 & = & 0.57\underline{\mathbf{8}}21\,56649\,01532\,86060\,65120\,90082\,40243\ldots \\ \gamma & = & 0.57\underline{\mathbf{7}}21\,56649\,01532\,86060\,65120\,90082\,40243\ldots, \end{array} \right.$$

where the sole discrepancy observed is in the third decimal digit.

The Nörlund integrals are also of interest in the context of differences of inverse zeta values, for which curious relations with the Riemann hypothesis have been noticed by Flajolet and Vallée [10], and independently by Báez-Duarte [3]. Consider the typical quantity

$$d_n = \sum_{k=2}^{n} \binom{n}{k} (-1)^k \frac{1}{\zeta(k)}, \tag{51}$$

which arises as coefficient in the Newton series representation of $1/\zeta(s)$. Its asymptotic analysis can be approached by means of a Nörlund-Rice representation as noted in related contexts by the authors of [10] and more recently by Maślanka in [18]. The following developments provide a rigorous basis for some of the observations made in [18], simplifies the criterion for RH that is implicit in [10], and points in the direction of easy generalizations relative to $1/\zeta(2s)$, $\zeta'(s)/\zeta(s)$, $\zeta(s-1)/\zeta(s)$, or other similar functions.

Theorem 3. *The differences of inverse zeta values d_n defined by* (51) *are such that the following two assertions are equivalent:*

> **FVBD Hypothesis (akin to** [3, 10]**).** *For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*
> $$|d_n| < C_\epsilon n^{1/2+\epsilon}.$$
>
> **RH (Riemann hypothesis).** *The Riemann zeta function $\zeta(s)$ is free of zeros in the half-plane $\Re(s) > \frac{1}{2}$.*

Proof. (i) Assume **RH**. Under RH, it is known that, given any $\sigma_0 > \frac{1}{2}$ and any $\epsilon > 0$, one has

$$\frac{1}{\zeta(s)} = \mathcal{O}(|t|^\epsilon), \qquad \text{for } \Re(s) = \sigma_0, \text{ where } t = \Im(s) \tag{52}$$

(see Equation (14.2.6) of [24, p. 337]). Then, start from the Nörlund integral representation (cf Lemma 1 and Equation (15)),

$$d_n = J_n(c), \quad \text{where } J_n(c) := \frac{(-1)^{n-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\zeta(s)} \frac{n!}{s(s-1)\cdots(s-n)}\, ds, \tag{53}$$

which is valid unconditionally for $c \in (1,2)$. Next, we propose to move the line of integration to $c = \sigma_0$. To this effect, observe that the integral $J_n(\sigma_0)$ defined in (53) converges and is $\mathcal{O}(n^{\sigma_0})$, since

$$\begin{aligned} |J_n(\sigma_0)| &\leq \frac{1}{2\pi\sigma_0}\binom{n-\sigma_0}{-\sigma_0}^{-1} \int_{-\infty}^{\infty} |\zeta(\sigma_0+it)|^{-1} \left| \frac{-\sigma_0\cdots(-\sigma_0+n)}{(-\sigma_0-it)\cdots(-\sigma_0-it+n)} \right| dt \\ &\leq \frac{1}{2\pi\sigma_0}\binom{n-\sigma_0}{-\sigma_0}^{-1} \int_{-\infty}^{\infty} |\zeta(\sigma_0+it)|^{-1} \left| \frac{-\sigma_0(-\sigma_0+1)}{(-\sigma_0-it)(-\sigma_0-it+1)} \right| dt \\ &= \mathcal{O}\left(n^{\sigma_0}\right). \end{aligned}$$

There, the second line results from the fact that, for x, t real, one has $|(x/(x-it)| \leq 1$; the third line summarizes the asymptotic estimate $\binom{n-\sigma_0}{-\sigma_0}^{-1} = \mathcal{O}(n^{\sigma_0})$ (by Stirling's formula) as well as the fact that the integral factor is convergent (since the integrand decays at least as fast as $\mathcal{O}(|t|^{-2+\epsilon})$ as $|t| \to +\infty$).

(ii) Assume **FVBD**. First, a reorganization similar to the one leading to (10) but based on the expansion of $1/\zeta(s)$ shows that

$$d_n = \sum_{\ell=1}^{\infty} \mu(\ell) \left[\left(1 - \frac{1}{\ell}\right)^n - 1 + \frac{n}{\ell} \right],$$

with $\mu(\ell)$ the Möbius function. The general term of the sum decreases like n/ℓ^2, which ensures absolute convergence. Next, introduce the function

$$D(x) = \sum_{\ell \geq 1} \mu(\ell) \left[e^{-x/\ell} - 1 + \frac{x}{\ell} \right],$$

whose general term decreases like x^2/ℓ^2.

Fix any small $\delta > 0$ ($\delta = \frac{1}{10}$ is suitable) and define $\ell_0 = \lfloor x^{1-\delta} \rfloor$. The difference $d_n - D(n)$ satisfies

$$\begin{aligned} d_n - D(n) &= \sum_{\ell=1}^{\infty} \mu(\ell) \left[\left(1 - \frac{1}{\ell}\right)^n - e^{-n/\ell} \right] \\ &= \left(\sum_{\ell < \ell_0} + \sum_{\ell \geq \ell_0} \right) \mu(\ell) e^{-n/\ell} \left[e^{n/\ell + n\log(1-1/\ell)} - 1 \right] \\ &= \mathcal{O}\left(\ell_0 e^{-n/\ell_0}\right) + \sum_{\ell \geq \ell_0} \mathcal{O}\left(\frac{n}{\ell^2}\right) \quad = \quad \mathcal{O}\left(n^{\delta}\right), \end{aligned} \tag{54}$$

by series reorganization, a split of the sum according to $\ell \gtreqless \ell_0$, and trivial majorizations.

Given (54), the FVBD Hypothesis implies that $D(x) = \mathcal{O}(x^{1/2+\epsilon})$, at least when x is a positive *integer*. To extend this estimate to real values of x, it suffices to note that $D(x)$ is differentiable on $\mathbb{R}_{>0}$, and

$$D'(x) = -\sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell} \left[e^{-x/\ell} - 1 \right],$$

is proved to be $\mathcal{O}(1)$ by bounding techniques similar to (54). Thus, assuming the FVBD Hypothesis, the estimate

$$D(x) = \mathcal{O}\left(x^{1/2+\epsilon}\right), \qquad x \to +\infty \tag{55}$$

holds for *real* values of x.

Regarding the behaviour of $D(x)$ at 0, the general term of $D(x)$ is asymptotic to $x^2/(2\ell^2)$, so that $D(x) = \mathcal{O}(x^2)$, as $x \to 0^+$. This, combined with the estimate of $D(x)$ at infinity expressed by (55), implies (under the FVBD Hypothesis, still) that the Mellin transform

$$D^\star(s) := \int_0^\infty D(x) x^{s-1}\, dx, \tag{56}$$

exists and is an analytic function of s for all s in the strip $-2 < \Re(s) < -\frac{1}{2} - \epsilon$. On the other hand, the usual properties of Mellin transforms (see, e.g., the survey [8]) imply that

$$D^\star(s) = \left(\sum_{\ell=1}^\infty \mu(\ell)\ell^s\right) \cdot \int_0^\infty \left[e^{-x} - 1 + x\right] x^{s-1}\, dx = \frac{\Gamma(s)}{\zeta(-s)}, \tag{57}$$

at least for s such that $-2 < \Re(s) < 1$, which ensures that the expansion of $1/\zeta(-s)$ is absolutely convergent. The comparison of the analytic character of (56) in $-2 < \Re(s) < -\frac{1}{2} - \epsilon$ (implied by the FVBD Hypothesis) and of the explicit form of (57) shows that the Riemann Hypothesis is a consequence of the FVBD Hypothesis. □

Numerically, for comparatively low values of n, it would seem that d_n tends slowly but steadily to $2 = -1/\zeta(0)$. For instance, we have $d_{20} \doteq 1.93$, $d_{50} \doteq 1.987$, $d_{100} = 1.996$, $d_{200} \doteq 1.9991$. However, it appears from our previous analysis and a residue calculation applied to (53) that there must be complicated oscillations due to the nontrivial zeta zeros—these oscillations in fact eventually *dominate*, though at a rather late stage, as we now explain following [10]. Indeed, assuming for notational convenience the simplicity of the nontrivial zeros of $\zeta(s)$, one has (unconditionally)

$$d_n = \sum_\rho{}^{\star} \frac{1}{\zeta'(\rho)} \frac{\Gamma(n+1)\Gamma(-\rho)}{\Gamma(n+1-\rho)} + 2 + o(1), \tag{58}$$

where the summation extends to all nontrivial zeros ρ of $\zeta(s)$ with $0 < \Re(\rho) < 1$, while the starred sum ($\sum^\star$) means that zeros should be suitably grouped, following the careful discussion in §9.8 of Titchmarsh's treatise [24, p. 219] in relation to a formula of Ramanujan. A simplified model of the sequence d_n then follows from the fact that, for large n, any individual term of the sum in (58) corresponding to a zeta zero $\rho = \sigma + i\tau$ is asymptotically

$$\frac{\Gamma(-\rho)}{\zeta'(\rho)} n^\sigma e^{i\tau \log n}. \tag{59}$$

Such a term involves a logarithmically oscillating component, a slowly growing component n^σ ($\sqrt{n}$ under RH), as well as a multiplier that is likely to be extremely small numerically, since it involves the quantity $\Gamma(-\rho) \asymp e^{-\pi|\tau|/2}$. For the first nontrivial zeta zero at $\rho \doteq \frac{1}{2} + i\,14.13$, the term (59) is very roughly

$$10^{-9}\sqrt{n}\cos\left(14.13\,\log n\right), \tag{60}$$

and for the next zero, at $\rho \doteq \frac{1}{2} + i\,21.022$, the numerical coefficient drops to about 10^{-14}. The corresponding oscillations then have the curious feature of being numerically detectable only for very large values of n: for instance, in order for the first term given by (60) to attain the value 1, one needs $n \approx 10^{18}$, while for the contribution of the second zero, one would need $n \approx 10^{28}$. Since it is known that all zeros of $\zeta(s)$ lie on the critical line till height $T_0 \approx 5 \cdot 10^8$, we can estimate in this way that the presence of a nontrivial zero (if any) off the critical line could only be detected in the asymptotic behaviour of d_n for values of n larger than $N_0 \approx e^{\pi T_0} \approx 10^{600,000,000}$. In summary, from a numerical point of view, a possible failure of RH, though in theory traceable through the asymptotic behavior of d_n, is in reality violently counterbalanced by the exponential decay of the Γ factor, hence it must remain totally undetectable in practice—analogous facts were observed in [10, 18]. It is finally of interest to note that such phenomena do occur in nature, specifically, in the determination by Flajolet and Vallée in [10] of *the expected number of continued fraction digits that are necessary to sort* n *real numbers drawn uniformly at random from the unit interval*.

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P.F.: ALGORITHMS PROJECT, INRIA-ROCQUENCOURT, F-78153 LE CHESNAY, FRANCE

L.V. 35 MIDDLE STREET W, AUSTIN TEXAS 57721 USA