ON THE BERRY CONJECTURE (AND PERCHANCE A PROOF OF THE RH)

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ABSTRACT. A set of working notes for proving the RH, maybe, by route of Berry conjecture.

1. INTRO

The attack on the Riemann Hypothesis via the Berry conjecture requires the definition of position and momentum operators obeying the Heisenberg uncertainty principle $[p,x] = px - xp = \hbar$. In quantum mechanics, when formulated as an exercise in Fourier analysis, the standard identification of the momentum operator is that $p \to -id/dx$ (there are, of course, many other formulations as well). The Berry conjecture is the observation that x^s is an eigenstate of xp, that is,

$$x\frac{d}{dx}x^s = sx^s$$

and that, perhaps there exists some instantiation of the abstract operator xp that gives the zeroes of the Riemann zeta function as its eigenvalues. The primary conceit of this paper is that the author has found this operator. What can be made of this remains open.

2. BASIC DEFINITIONS

What follows are some basic notation and definitions required to explore the space of measurable fuctions on an infinite product topology.

Let $2 = \{0,1\}$ the set with two elements. Let $\Omega = 2^\omega$ be the set of all infinite-length strings in two letters. Let $\sigma \in \Omega$ be written as

$$\sigma = [\sigma_0, \sigma_1, \sigma_2, \cdots]$$

with $\sigma_k \in \{0,1\}$. Define the projection operators $\pi_k : \Omega \to 2$ as $\pi_k(\sigma) = \sigma_k$. On certain rare occasions, we'll have the need to talk about the p-adic generalization of this space, where 2 is replaced by the set of p items; it will be made clear when this generalization will need to be made. On other occasions, we'll have the need to discuss the two-sided version of the above, i.e. the set of strings which trail off to infinity on either the left or right. This will be explicitly constructed and discussed at a later stage.

The natural topology on Ω is the product topology. A topological sub-base for this topology is given by the cylinder sets $\pi_k^{-1}(0)$ and $\pi_k^{-1}(1)$. These two sets are, respectively, the set of all strings which have a 0 in the k'th position, and a 1 in the k'th position. These are open sets in the product toplogy. All other open sets of this toplogy are formed by finite intersections and countable unions of these two sets. Note that

$$\pi_k^{-1}(0) \cap \pi_k^{-1}(1) = \varnothing$$

for all k, and that

$$\pi_j^{-1}(0) \cup \pi_k^{-1}(1) = \Omega$$

Date: 14 October 2008.

for all j,k. In what follows, we will regularly make an "abuse of notation", and let Ω stand for both the space of strings, and for the product topology on Ω, and also, for the Borel sigma-algebra induced by the topology. Which is intended will hopefully be clear from the context.

We will be interested in exploring the spaces of measurable real and complex-valued functions on Ω. We'll be concerned with either one of two different measures μ, the Bernoulli measure μ_B or the Minkowski measure μ_M. The Bernoulli measure is well-known[1]; it assigns a measure of 1/2 to each of the basic cylinder sets; that is,

$$\mu_B\left(\pi_k^{-1}(0)\right)=\mu_B\left(\pi_k^{-1}(1)\right)=\frac{1}{2}$$

The Minkowski measure is far more subtle; it is defined in [2]; its properties will be reviewed in a later section, as needed. All measures assign 1 to the whole space: $\mu(\Omega)=\mu_B(\Omega)=\mu_M(\Omega)=1$. In what follows, the subscript B or M will be used to denote the Bernoulli or the Minkowski measures, and any related quantities as they occur. A lack of subscript is meant to denote relations that hold in general, for all measures.

The space of measureable functions on Ω will be denoted as $\mathscr{F}(\Omega)$. Its elements are maps $f:\Omega\to\mathbb{R}$ to the real numbers, or sometimes to the complex numbers: $f:\Omega\to\mathbb{C}$; which of these cases apply at any moment should be clear from context. These functions are (almost) always taken to be measurable, by which it is meant that the following identity *always* holds, for any two open sets A,B such that $A\cap B=\varnothing$:

$$\mu(A\cup B)f(A\cup B)=\mu(A)f(A)+\mu(B)f(B) \tag{2.1}$$

This identity will be frequently invoked.(XXX And therefore, perhaps, it should be justified in greater detail, by touching on more standard presentations of measure theory?)

The following notation for cylinder sets will be convenient. Let $C_0=\pi_0^{-1}(0)$ and $C_1=\pi_0^{-1}(1)$. For a general string of binary digits $a,b,c,\ldots$ define

$$C_{abc\ldots}=\pi_0^{-1}(a)\cap\pi_1^{-1}(a)\cap\pi_2^{-1}(c)\cap\cdots$$

Notice that

$$\begin{aligned}\Omega=&C_1\cup C_0=C_1\cup C_{01}\cup C_{00}\\=&C_1\cup C_{01}\cup C_{001}\cup C_{000}\\=&C_1\cup C_{01}\cup C_{001}\cup C_{0001}\cup\cdots\end{aligned}$$

For convenience, write d_k for the $C_{0\cdots01}$ where the 1 occurs in the k'th position; and so $d_0=C_1$ and $d_1=C_{01}$ and $d_2=C_{001}$ and so on. The d_k are clearly pairwise-disjoint: $d_j\cap d_k=\varnothing$ for $j\neq k$ and the previous shows that, by construction, they span the entire space:

$$\Omega=\bigcup_{k=0}^{\infty}d_k \tag{2.2}$$

Combining this with the eqn 2.1 leads to a lemma that demonstrates that, in order to completely define a function f, it is sufficient to define it on in terms of the d_k:

Lemma 2.1. *One has that, for any $f\in\mathscr{F}(\Omega)$, that*

$$f(\Omega)=\sum_{k=0}^{\infty}\mu(d_k)f(d_k)$$

and that, more generally, for an open set $A \subset \Omega$, that

$$\mu(A) f(A) = \sum_{k=0}^{\infty} \mu(A \cap d_k) f(A \cap d_k)$$

Proof. This follows from the observation that $d_j \cap d_k = \varnothing$ for any $j \neq k$, and from the fact that the measure μ is meant to be sigma-additive; that is, the relation 2.1 can be safely applied a countable number of times. □

This lemma may be thought of as an "extension theorem": it shows how to define a function f on the whole space, given its values only on a set of parts $A \cap d_k$.

2.1. **The Shift Operator.** A final, critical device will be the shift operator $\tau : \Omega \to \Omega$, which truncates one digit off of an infinite string:

$$\tau : [\sigma_0, \sigma_1, \sigma_2, \cdots] \mapsto [\sigma_1, \sigma_2, \cdots]$$

Considered as a map of cylinder sets, it acts as $\tau : \pi_{k+1}^{-1}(a) \mapsto \pi_k^{-1}(a)$ while $\tau\left(\pi_0^{-1}(a)\right) = \Omega$. In terms of the sets d_k defined above, one has that $\tau(d_{k+1}) = d_k$ while $\tau(d_0) = \Omega$. Of course, τ is not one-to-one; to remove the ambiguity, we will *define* τ^{-1} as $\tau^{-1}\left(\pi_k^{-1}(a)\right) = \pi_{k+1}^{-1}(a)$. Perhaps it would have been more correct to define τ^{-1} first, and then define τ as the inverse of τ^{-1}.

The shift operator is distributive over set union and intersection, so that, for any sets $A, B \subset \Omega$, one has

$$\tau(A \cap B) = \tau(A) \cap \tau(B)$$

and also

$$\tau(A \cup B) = \tau(A) \cup \tau(B)$$

and similarly for τ^{-1}.

The shift operator induces a map $\tau_* : \mathscr{F}(\Omega) \to \mathscr{F}(\Omega)$, the push-forward, defined as $\tau_* f = f \circ \tau^{-1}$. The push-forward is a bounded linear operator, and it can be identified with the Ruelle-Frobenius-Perron operator[2]; this correspondance will be be much explored in this article. In anticipation of this, the notation $\mathscr{L} f = \tau_* f = f \circ \tau^{-1}$ will be used for the push-forward of τ. The choice of $\mathscr{L}$ emphasizes that the operator is linear:

$$\mathscr{L}(af + bg) = a\mathscr{L} f + b\mathscr{L} g$$

for $a, b \in \mathbb{C}$ and $f, g \in \mathscr{F}(\Omega)$.

2.2. **The Bernoulli Measure.** We note some properties of the Bernoulli measure that will prove useful. The Bernoulli measure μ_B is invariant under the inverse shift; that is,

$$\mu_B\left(\tau^{-1}(A)\right) = \mu_B(A)$$

this invariance being a defining property of a class of measures to which the Bernoulli measure belongs (the subscript B here stands for "Bernoulli" and not something else). In general, most other measures will not have this shift-invariance property; in particular, the Minkowski measure does not.

The Bernoulli measure is, by its nature, a product measure, and so one has, for example, that

$$\mu_B(d_k) = \frac{1}{2^{k+1}}$$

3. The Position Operator

We focus now on obtaining a workable expression for the position operator. Given an arbitrary sequence of real or complex numbers $\{a_k\}$, consider the linear operator $\mathscr{A}$: $\mathscr{F}(\Omega) \to \mathscr{F}(\Omega)$ defined by

$$[\mathscr{A}f](A) = \frac{1}{\mu(A)} \sum_{k=0}^{\infty} a_k \mu(A \cap d_k) f(A \cap d_k)$$

where, of course, $f \in \mathscr{F}(\Omega)$ and $A \subset \Omega$.

Lemma 3.1. *The codomain of $\mathscr{A}$ is $\mathscr{F}(\Omega)$, that is, $\mathscr{A}f \in \mathscr{F}(\Omega)$.*

Proof. One need only verify that $\mathscr{A}f$ obeys formula 2.1, which is

$$\mu(A \cup B)[\mathscr{A}f](A \cup B) = \mu(A)[\mathscr{A}f](A) + \mu(B)[\mathscr{A}f](B)$$

for any two sets $A, B \subset \Omega$ with $A \cap B = \varnothing$. But this follows directly from the definition of $\mathscr{A}$ and the application of 2.1 under the sum. □

The operator $\mathscr{A}$ will now be used to construct the position operator for the Bernoulli measure.

Consider the operator $\mathscr{A} = \varphi$ defined by the series $a_k = k$. When coupled with the Bernoulli measure (thus gaining a subscript), the operator φ_B will be refered to as the the position operator. The following theorem gives a glimmer as to why it deserves this name.

Theorem 3.2. *For the Bernoulli measure, one has that*

$$\mathscr{L}_B \varphi_B - \varphi_B \mathscr{L}_B = \mathscr{L}_B$$

Proof. Recall that the generic definition for the pushforward $\mathscr{L}$ is

$$[\mathscr{L}f](A) = \left(f \circ \tau^{-1}\right)(A)$$

One then has that

$$[\mathscr{A}\mathscr{L}f](A) = \frac{1}{\mu(A)} \sum_{k=0}^{\infty} a_k \mu(A \cap d_k) \left(f \circ \tau^{-1}\right)(A \cap d_k)$$

while

$$\begin{aligned}
[\mathscr{L}\mathscr{A}f](A) =& \frac{1}{\mu(\tau^{-1}(A))} \sum_{k=0}^{\infty} a_k \mu\left(\tau^{-1}(A) \cap d_k\right) f\left(\tau^{-1}(A) \cap d_k\right) \\
=& \frac{a_0}{\mu(\tau^{-1}(A))} \mu\left(\tau^{-1}(A) \cap d_0\right) f\left(\tau^{-1}(A) \cap d_0\right) \\
&+ \frac{1}{\mu(\tau^{-1}(A))} \sum_{k=1}^{\infty} a_k \mu\left(\tau^{-1}(A \cap d_{k-1})\right) f\left(\tau^{-1}(A \cap d_{k-1})\right)
\end{aligned}$$

For the second part of the above equation, the identity $\tau(d_k) = d_{k-1}$ was made use of, together with the distributivity of τ^{-1} over set intersection. By employing the shift invariance of the Bernoulli measure μ_B, namely that

$$\mu_B\left(\tau^{-1}(A)\right) = \mu_B(A)$$

one then immediately obtains that

$$\begin{aligned}
[(\mathscr{L}_B \mathscr{A}_{\mathscr{B}} - \mathscr{A}_B \mathscr{L}_B) f](A) =& \frac{a_0}{\mu_B(A)} \mu_B\left(\tau^{-1}(A) \cap d_0\right) f\left(\tau^{-1}(A) \cap d_0\right) \\
&+ \frac{1}{\mu_B(A)} \sum_{k=0}^{\infty} (a_{k+1} - a_k) \mu_B(A \cap d_k) \left(f \circ \tau^{-1}\right)(A \cap d_k)
\end{aligned}$$

The theorem is completed by substituting $a_k = k$ into the above, and then observing that, in accordance with lemma 2.1, one has that

$$\frac{1}{\mu_B(A)} \sum_{k=0}^{\infty} \mu_B(A \cap d_k) \left(f \circ \tau^{-1}\right)(A \cap d_k) = [\mathscr{L}_B f](A)$$

As this holds for all sets $A \subset \Omega$, the theorem is proven. □

The position operator is not a bounded operator; its eigenvalues may be arbitrarily large. A discrete set of eigenfunctions showing this behaviour can be immediately demonstrated. Consider a set of maps $\theta_k : \Omega \to \mathbb{R}$ given by

$$\theta_k(A \cap d_j) = \begin{cases} 0 & \text{for } j \neq k \\ 0 & \text{for } j = k \text{ and } A \cap d_k = \varnothing \\ \frac{\mu_B(A)}{\mu_B(A \cap d_k)} & \text{for } j = k \text{ and } A \cap d_k \neq \varnothing \end{cases}$$

From this definition, it is clear that

$$\varphi_B \theta_k = k\theta_k$$

that is, the θ_k are eigenvectors with eigenvalue k, and thus, φ_B is unbounded (and, thus, is not a continuous operator).

The θ_k can be thought of as set-membership functions on Ω. Since the d_k span the entire space (eqn 2.2), the lemma 2.1 shows that the above is sufficient to define the θ_k for all sets in the product topology on Ω. Plugging in, one has the much simpler definition:

$$\theta_k(A) = \begin{cases} 0 & \text{for } A \cap d_k = \varnothing \\ 1 & \text{for } A \cap d_k \neq \varnothing \end{cases}$$

and so, for a general point $\sigma \in \Omega$, one has

$$\theta_k(\sigma) = 1_{d_k}(\sigma) = \begin{cases} 0 & \text{for } \sigma \notin d_k \\ 1 & \text{for } \sigma \in d_k \end{cases}$$

which emphasizes the set membership aspect.

Theorem 3.3. *The eigenvalue spectrum of φ_B is discrete; the eigenvalues are integers.*

Proof. The goal is to solve $\varphi_B f = \lambda f$ and see if perhaps non-integer-valued λ can be found. But

$$\begin{aligned} 0 &= (\varphi_B - \lambda) f \\ &= \sum_{k=0}^{\infty} (k - \lambda) \mu_B(A \cap d_k) f(A \cap d_k) \end{aligned}$$

As this must hold for any open set A, and, in particular, for those sets A which cause the sum to consist of a finite number of terms, one has only two alternatives: either $k - \lambda = 0$ or $f(A \cap d_k) = 0$. The first alternative gives the discrete eigenvalue spectrum; but if λ is not an integer, then the second alternative must hold for all k, thus, by lemma 2.1, implying that $f = 0$. Thus, the eigenvalue spectrum is discrete. □

The eigenspaces are highly degenerate: any function that has support only on one given d_k will also be an eigenvector with eigenvalue k. These eigenspaces may be given a basis consisting of Haar functions, in the usual manner.

whoopsy daisy.

REFERENCES

[1] Achim Klenke. *Probability Theory*. Springer, 2008. ISBN 978-1-84800-047-6.
[2] Linas Vepstas. On the minkowski measure. *ArXiv*, arXiv:0810.1265, 2008.

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