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A simple proof of Linas's theorem on Riemann zeta function

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Abstract: Linas Vepstas gives rapidly converging infinite representatives for values of Riemann zeta function at $(4m-1)$, where m is a natural number. In this paper, we give a new simple proof. Also, we obtain two equations of values of Bernoulli numbers ' generating function by applying a corollary given in this paper.
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1 Introduction

Linas[1] gave the following rapidly converging infinite representatives for values of $\zeta_{(4m-1)}$ using polylogarithm, where m is a natural number and B_k is the k'th Bernoulli number.

Theorem 1.1. *(Linas's theorem)*

$$\zeta_{(4m-1)} = -2\sum_{n=1}^{\infty} \frac{1}{n^{4m-1}\left(e^{2\pi n}-1\right)} - \frac{1}{2}\left(2\pi\right)^{4m-1}\sum_{j=0}^{2m}\left(-1\right)^{j}\frac{B_{2j}}{(2j)!}\frac{B_{4m-2j}}{(4m-2j)!}$$

In this paper, we give a simple proof of it, using Fourier series of $\cosh(x)$.

2 Preliminaries

Lemma 2.1.

$$L\coth(L) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{\left(\frac{\pi}{L}n\right)^2 + 1}$$

Proof. Using the Fourier series expansion method from $-L$ to L, where L is a positive real number, $\cosh(x)$ is expressed as following

$$\cosh(x) = \frac{\sinh(L)}{L} + \sum_{n=1}^{\infty} \frac{2}{L} \frac{(-1)^n}{\left(\frac{\pi}{L}n\right)^2 + 1} \sinh(L)\cos\left(\frac{\pi}{L}nx\right)$$

Substituting L for x, we obtained the following equation.

$$\cosh(L) = \frac{\sinh(L)}{L} + \frac{2}{L}\sinh(L)\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(\frac{\pi}{L}n\right)^2 + 1}(-1)^n$$

□

Corollary 2.1.1.

$$\sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^{4m-1}} = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^{4m}} + \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{4m-2}n^2 + k^{4m}}$$

Proof. From Lemma 2.1, the above equation can be obtained. □

Lemma 2.2.

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{4m} + k^{4m-2}n^2}$$

This series converges, where m is a positive integer.

Proof. From Corollary 2.1.1, the series above can be rewritten as follows,

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{4m} + k^{4m-2}n^2} = \frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{k^{4m}} + \frac{\pi\coth(k\pi)}{k^{4m-1}}\right)$$

Since the right-hand side expression above is always positive and monotonically decreases, Cauchy Condensation Test can be applied. Therefore, below is necessary and sufficient condition for it to converge.

$$\sum_{k=0}^{\infty} 2^k\left(-\frac{1}{\left(2^k\right)^{4m}} + \frac{\pi\coth\left(2^k\pi\right)}{\left(2^k\right)^{4m-1}}\right)$$

This series can be divided as follows,

$$(-1 + \pi\coth\pi) + \sum_{k=1}^{\infty} 2^k\left(-\frac{1}{\left(2^k\right)^{4m}} + \frac{\pi\coth\left(2^k\pi\right)}{\left(2^k\right)^{4m-1}}\right)$$

The previous parentheses are constants and by applying the ratio test, we can see that the part of the infinite series converges if the following conditions are satisfied.

$$\lim_{k\to\infty}\left|\frac{2^{k+1}\left(-\frac{1}{\left(2^{k+1}\right)^{4m}}+\frac{\pi\coth\left(2^{k+1}\pi\right)}{\left(2^{k+1}\right)^{4m-1}}\right)}{2^{k}\left(-\frac{1}{\left(2^{k}\right)^{4m}}+\frac{\pi\coth\left(2^{k}\pi\right)}{\left(2^{k}\right)^{4m-1}}\right)}\right|<1$$

The value of the left-hand side expression is 4^{1-2m}, which is less than 1. Thus, the series converges. □

Lemma 2.3.

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{4m}+k^{4m-2}n^2}=\frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{2m-1}\frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

Proof.

$$\sum_{s=1}^{2m-1}\frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}=\sum_{s=1}^{2m-1}\frac{1}{k^{4m}}\left((-1)^{s+1}\frac{k^{2s}}{n^{2s}}\right)$$

It is well-known that for integer p, the following equation is valid

$$\begin{aligned}\frac{\alpha^p-\beta^p}{\alpha-\beta} &= \alpha^{p-1}+\alpha^{p-2}\beta+\alpha^{p-3}\beta^2+\cdots+\beta^{p-1}\\ &= \sum_{s=1}^{p}\alpha^{p-2}\beta^{s-1}\\ &= \frac{\alpha^p}{\beta}\sum_{s=1}^{p}\left(\frac{\beta}{\alpha}\right)^s\end{aligned}$$

One can divide both sides by $1/\alpha^p\beta^p$ to get

$$\frac{1}{\alpha-\beta}\left[\frac{1}{\beta^p}-\frac{1}{\alpha^p}\right]=\frac{1}{\beta^{p+1}}\sum_{s=1}^{p}\left(\frac{\beta}{\alpha}\right)^s$$

Now, let $\alpha=n^2$ and $\beta=-k^2$ and $p=2m-1$. This gives equation below

$$\frac{1}{k^{4m}+k^{4m-2}n^2}+\frac{1}{k^2n^{4m-2}+n^{4m}}=\sum_{s=1}^{2m-1}\frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

From the result of Lemma 2.2, we can say that the infinite sum of the left-hand side and each term of the left-hand side converge, so we obtain the following equality.

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{4m-2}n^2+k^{4m}}+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{4m}+k^2n^{4m-2}}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{2m-1}\frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

Since these dual series are positive term series and converge, we can rewrite them as follows.

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{k^{4m-2}n^2+k^{4m}}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{2m-1}\frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

□

3 A Proof of Main Theorem

The following relation is well known.

$$\zeta_{(2n)} = (-1)^{n+1} \frac{B_{2n} (2\pi)^{2n}}{2 (2n)!}$$

We can rewrite the Theorem 1.1 by this equation as follows.

$$\zeta_{(4m-1)} + 2 \sum_{k=1}^{\infty} \frac{1}{k^{4m-1} (e^{2\pi k} - 1)} = \frac{1}{\pi} \sum_{s=0}^{2m} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)}$$

The equation above can be proved by transforming the equation from Corollary 2.1 using Lemma 2.3 by the following process.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^{4m-1}} &= \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^{4m}} + \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{4m-2} n^2 + k^{4m}} \\
&= \frac{1}{\pi} \left(\sum_{k=1}^{\infty} \frac{1}{k^{4m}} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s} n^{2s}} \right) \\
&= \frac{1}{\pi} \left(\zeta_{(4m)} + \sum_{s=1}^{2m-1} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)} \right) \\
&= \frac{1}{\pi} \sum_{s=0}^{2m} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)}
\end{aligned}$$

4 Appendix

This appendix gives following equations.

$$\begin{aligned}
\frac{1}{e^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2 + 1} \\
\frac{\pi}{e^{\pi} - 1} &= 2 \sum_{n=1}^{\infty} \left(\frac{1}{4n^2 + 1} + \frac{(-1)^n}{4n^2 - 1} \right)
\end{aligned}$$

These equations come from following corollary.

Corollary 4.0.1.

$$\frac{x}{e^x - 1} = -\frac{x}{2} + 1 + 2 \sum_{n=1}^{\infty} \frac{1}{4 \left(\frac{\pi}{x} n \right)^2 + 1}$$

Proof. Substituting $x/2$ for L in Lemma 2.1, we get the infinite representative above. □

Substituting 2,π for x in Corollary 4.0.1, we get the following equations.

$$\begin{aligned}
\frac{1}{e^2-1} &= \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2+1} \\
\frac{\pi}{e^\pi-1} &= -\frac{\pi}{2}+1+2\sum_{n=1}^{\infty} \frac{1}{4n^2+1}
\end{aligned}$$

Since the following equation can be obtained by following calculation,

$$\begin{aligned}
2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} &= 2\lim_{k\to\infty}\sum_{n=1}^{k} \frac{(-1)^{n+1}}{4n^2-1} \\
&= 2\lim_{k\to\infty}\sum_{n=1}^{k} \left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\cdot\frac{(-1)^{n+1}}{2} \\
&= \lim_{k\to\infty}\left(\left(\frac{1}{1}-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)-\cdots\pm\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)\right) \\
&= \lim_{k\to\infty}\left(2\cdot\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\pm\frac{1}{2k-1}\right)-1\mp\frac{1}{2k+1}\right) \\
&= 2\lim_{k\to\infty}\left(\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\pm\frac{1}{2k-1}\right)\mp\frac{1}{2k+1}\right)-1 \\
&= \frac{\pi}{2}-1
\end{aligned}$$

replacing them gives that equation.

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References

[1] Linas Vepštas *On Plouffe's Ramanujan identities*, The Ramanujan Journal. **27**(4) (2012), 387–408.