Deformations of Shifts

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Abstract

A research diary looking into the deformation of tensor algebras of shift spaces and shift operators. This is an exploration of what happens when ideas are borrowed from representation theory, including ideas from the construction of Clifford algebras, and applies them to shift spaces.

1 Introduction

Talking about deformations is hard without first establishing notation; and so a lightning review.

1.1 The Bernoulli Shift

A lightning review of the Bernoulli shift.

1.1.1 Bit sequence

A countably infinite sequence of bits $\mathbb{B} = \{b_k : k \in \mathbb{N}, b_k \in \mathbb{Z}_2\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots = \mathbb{Z}_2^\omega$. This set is the Cantor set, discussed below.

1.1.2 Representing Reals

Given a specific sequence of bits $\beta = (b_0, b_1, \cdots)$ there exists a corresponding real number in the interval $[0,1] = \{x : x \in \mathbb{R}, 0 \le x \le 1\}$, given by $x = x(\beta) = \sum_{k=0}^\infty b_k 2^{-k-1}$. This map is a surjection onto the unit reals: every dyadic rational $n/2^{-m}$ has two distinct bit-sequence representations, one ending in an infinite sequence of 0's, and the other ending in an infinite sequence of 1's. It is often convenient to think of this map as a projection from $\mathbb{B}$ to the unit interval $[0,1] \subset \mathbb{R}$.

1.1.3 Formal Power Series

More generally, one can consider the "analytic" series $f(\beta; z) = \sum_{k=0}^\infty b_k z^k$ for $z \in \mathbb{C}$. As a general rule, such series are not differentiable on the real axis (i.e. not differentiable when holding β fixed, and allowing z to be a varying real number). Such functions do poses a fractal (self-similar) structure in z (while holding β fixed). The self-similarity arises by means of the application of the shift operator. Such series can be defined for $|z| > 1$ but have a variety of associated differentiability issues.

1.1.4 Shift Operator

The left-shift operator $L : \mathbb{B} \to \mathbb{B}$ acts on sequences of bits as $L : (b_0, b_1, b_2, \cdots) = (b_1, b_2, \cdots)$. There is a right-shift $K : \mathbb{B} \to \mathbb{B}$, commonly called the Koopman operator, given by $K : (b_0, b_1, \cdots) = (0, b_0, b_1, \cdots)$. Note that $LK = I$ the identity, but that $KL \neq I$ as the shifted bit represents a loss of information. It is often convenient to visualize these shift operators as matrix operators, with a string of all-ones just above, or just below the diagonal.

Creating a deformation that restores KL as the identity is one of the primary goals of studying how deformations work with shifts.

1.1.5 Bernoulli map

The action of the left shift on $\mathbb{B}$ projects naturally to an action on the unit interval $[0,1] \subset \mathbb{R}$. This action is termed the Bernoulli map, and can be obtained as follows. Given $x = x(\beta) = \sum_{k=0}^{\infty} b_k 2^{-k-1}$, define $Lx = [Lx](\beta) = x(L\beta) = \sum_{k=0}^{\infty} b_{k+1} 2^{-k-1}$ and it is straight-forward to verify that

$$Lx = \lfloor 2x \rfloor = \begin{cases} 2x & \text{for } 0 \leq x < 1/2 \\ 2x - 1 & \text{for } 1/2 < x \leq 1 \end{cases}$$

This is just the Bernoulli map.

1.1.6 Pushforward

The Bernoulli map also pushes forward on the space of functions $f : [0,1] \to \mathbb{R}$ (or $\mathbb{C}$, as desired, or other additive field). The pushforward is given by the inverse image, that is, $Lf = f \circ L^{-1}$. But, as noted, the inverse L^{-1} doesn't properly exist, but is instead played by the two Koopman operators K_0 or K_1 which insert either a 0 or a 1 in front of a string of bits. Thier action on $\mathbb{B}$ projects naturally to an action on the unit interval as $K_0 x = [K_0 x](\beta) = x(K_0 \beta) = x/2$ and similarly, $K_1 x = (x+1)/2$. Thus, the pushforward is given by

$$[Lf](x) = \frac{1}{\mu_0} f\left(\frac{x}{2}\right) + \frac{1}{\mu_1} f\left(\frac{x+1}{2}\right)$$

where μ is a measure on $\mathbb{B}$, defined in an upcoming section. But convention, it is taken as $1/2$ and so one has

$$[Lf](x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$$

In this form, the L is a transfer operator or the Ruelle-Frobenius-Perron operator: it is an operator action on a space of functions. For the Bernoulli shift, this can be called the Bernoulli operator.

1.1.7 Eigenfunctions

There are a variety of different spaces on which the Bernoulli operator can act. Three interesting cases are the spaces of polynomials (in one variable), the space of square-integrable functions, and the space of formal power series. Taken as an operator, it has an associated spectrum. On the polynomials, the spectrum is discrete, real and contained inside the unit circle. On the square integrable functions, the spectrum is continuous and consists of the entire unit disk. For the case of polynomials, the eigenfunctinos are the Bernoulli polynomials; this follows from the identity

$$\frac{1}{2}\left(B_n\left(\frac{x}{2}\right)+B_n\left(\frac{1+x}{2}\right)\right)=\frac{1}{2^n}B_n(x)$$

and and so one may write $LB_n=\lambda B_n$ with eigenvalue $\lambda=2^{-n}$.

Note that $B_0(x)=1$ is the Frobenius-Perron eigenvalue; it is the largest eigenvalue, and it is shift-invariant. Since it is shift-invariant, it can be taken as the measure on the unit interval. In this case, it corresponds with the usual measure on the reals; this is not always the case for other shifts. Here, the important conclusion is thaat, in general, thanks to the Frobenius-Peron theorem, eigenfunctions with unit eigenvalue exist, and these can be taken as shift-invariant measures on the shift space.

1.1.8 Cantor set

The space $\mathbb{B}$ of infinite strings in two symbols is the Cantor set.

1.1.9 Product topology; Cylinder sets

The one-sided-infinite bit-sequence space $\mathbb{B}=\prod\mathbb{Z}_2$ has a natural topology, the product topology. The topology is generated by the finite intersection and countable union from a basis of open sets C_0^k and C_1^k, termed "cylinder sets". These are given by

$$\begin{aligned}C_0^k&=\prod_{j=0}^{k-1}\mathbb{Z}_2\times\{0\}\times\prod_{j=k+1}^{\infty}\mathbb{Z}_2=(*,*,\cdots,*,0,*,\cdots)\\C_1^k&=\prod_{j=0}^{k-1}\mathbb{Z}_2\times\{1\}\times\prod_{j=k+1}^{\infty}\mathbb{Z}_2=(*,*,\cdots,*,1,*,\cdots)\end{aligned}$$

that is, sequences consisting of $k-1$ places of $*=\mathbb{Z}_2$ "don't care" or "wildcard" values, then either a 0 for C_0^k or a 1 for C_1^k in the k'th position, followed by more wildcard values. Clearly, $C_0^k\cap C_1^k=\varnothing$ and $C_0^k\cup C_1^k=\mathbb{B}$. The result of a finite intersection and arbitrary union of such sets can be written as a semi-infinite string γ of three symbols $\{0,1,*\}$ such that the first two symbols appear a finite number of times.

Note that the left and right shifts act naturally on the cylinder sets; for example, $L^mC_b^k=C_b^{k-m}$ for $m\leq k$.

1.1.10 Measure

The space $\mathbb{B}$ endowed with the product topology can be assigned a sigma-additive measure μ, usually called the Bernoulli distribution. The measure is sub-additive, in that $\mu(A\cup B) \leq \mu(A) + \mu(B)$ with equality holding whenever the open sets A and B are disjoint: $A\cap B = \varnothing$. The measure of the empty set $\varnothing$ is zero: $\mu(\varnothing) = 0$ and the measure of the full space is unity: $\mu(\mathbb{B}) = 1$. The measure of a single point $\beta = (b_0, b_1, \cdots)$ is zero: $\mu(\beta) = 0$.

Conventionally, the measure is real-valued, although in the following it will sometimes be convenient to allow a complex-valued measure, and/or to loosen the constraint that the measure of the full space is unity.

The Bernoulli measure is specifically that of an infinite sequence of coin-flips. Specifically, define $\mu_B\left(C_0^k\right) = p$ for some real value $0 \leq p \leq 1$, so that $\mu_B\left(C_1^k\right) = 1 - p$ and so $\mu_B\left(C_0^k \cup C_1^k\right) = \mu_B\left(C_0^k\right) + \mu_B\left(C_1^k\right) = \mu_B(\mathbb{B}) = 1$. The Bernoulli measure is translation-invariant, in that $\mu_B\left(L^m C_b^k\right) = \mu_B\left(C_b^{k-m}\right) = p$ for $m \leq k$.

One is certainly free to consider non-translation-invariant measures. Certainly, eigenfunctions to the shift operator with non-unit eigenvalues correspond to such measures, but even more are possible.

1.1.11 Binary tree

The binary tree extends naturally to PSL(2,Z) and is rampant in the theory of elliptic functions, modular forms and number theory in general.

1.2 Continued fractions

- Representation of reals
- Baire space
- Gauss map is the shift
- Farey fractions generate a binary tree
- 1-1 map of farey fractions (all rationals) to dyadic rationals
- 1-1 of quadratic irrationals to rationals
- 1-1 map of rationals to perdiodic orbits in SL(2,Z)/Gamma
- Riemann surfaces

1.3 Two-sided shifts

- Bakers map
- Commutative

1.4 Non-commutative shifts

- x, d/dx on polynomials
- harmonic osc raising and lowering operators.
- Quantization via PBW therem, start product, Moyal product, Weyl algebra
- Motivation: lets look at the simpler cases, more exhaustively.

2 Definitions

With that review and motivations, lets define the tensor product of shifts. For convenience, we focus on the Bernoulli shift.

2.1 Tensor product

Let $U = \left\{\gamma \in \{0,1,*\}^{\omega} : 0 \text{ and } 1 \text{ appear a finite number of times}\right\}$ be the topology of open sets (cylinder sets) γ, each having a finite measure. These sets will correspond to the idea of "vectors" in the normal sense.

To define the tensor product, one wants to consiser a scalar-valued (real-valued) function bilinear in the product components, namely, $f : U \otimes U \to \mathbb{R}$ such that

$$f\left(\left(a\gamma + b\gamma'\right) \otimes \gamma''\right) = af\left(\gamma \otimes \gamma''\right) + bf\left(\gamma' \otimes \gamma''\right)$$

This can be accomplished by allowing a measure μ play the role of a basis vector carrying index $\gamma \in U$. That is, define a vector space V, the basis vectors for which are the vectors $e_\gamma = \mu_\gamma = \mu(\gamma)$. The basis for the tensor space $V \otimes V$ is then clearly $e_\gamma \otimes e_{\gamma'}$ which can clearly be identified with the number $\mu(\gamma)\mu(\gamma')$. This is clearly bilinear, as, with some abuse of notation,

$$\begin{aligned}\left(ae_\gamma + be_{\gamma'}\right) \otimes e_{\gamma''} &= \left(a\mu(\gamma) + b\mu(\gamma')\right)\mu(\gamma'') \\ &= ae_\gamma \otimes e_{\gamma''} + be_{\gamma\prime} \otimes e_{\gamma''}\end{aligned}$$

All tensors can be written as such linear combinations. The conceptual description of "basis vectors" presented here is rather awkward; a superior way of talking about the vector space will be provided shortly.

Note that U is countable: each string γ has only a finite number of symbols that are not the wildcard $*$, and these can be placed in lexicographic order.

2.2 Markov Property

Armed with a vector space V and a suitable definition of a tensor product, one can proceed to build the tensor algebra $TV = 1 \oplus V \oplus (V \otimes V) \oplus \cdots$ in the canonical fashion. The canonical construction endows the space not only with a product but also a codeproduct, unit and counit and the antipode, thus rendering the tensor algebra a Hopf algebra.

The distinct aspect here is that the shift space, as studied, is not studied as a completely "free" vector space, but rather, with special focus on the shift operator. Of particular interest is the subspace of TV that is compatible with the shifts, that is, that part where the shifts act as homomorphisms preserving the Hopf algebra structure. This is easy to say, but needs explicit verification.

The most notable aspect of the shift is the sigma-additivity of the measures placed on the shift space. This appears to extend to a Markov-like property on the tensor product. Specifically, if $A, B, \cdots \in U$ are cylinder sets that are a partition of unity, then $A \cap B \cap \cdots = \varnothing$ and $\mu(A) + \mu(B) + \cdots = 1$. Given some element $\gamma \in U$, one may tensor to construct $A \otimes \gamma, B \otimes \gamma, \cdots \in U \otimes U$ having the property that $\mu(A \otimes \gamma) + \mu(B \otimes \gamma) + \cdots = \mu(\gamma)$. This last is termed "Markov-like": conventionally, a square matrix has the Markov property if all of the elements in a fixed colum sum to one. The Markov property is important, as it is a homomorphism of probability vectors: a probability vector times a Markov matrix is still a probability vector. The goal here is have a similar property for the measure. It is no longer appropriate to talk about "columns" and "rows", but it is appropriate to insist on the preservation of sigma-additivity.

By homomorphism, it would seem that sigma-additivity can be extended to the entire tensor algebra. Whether it also extends to the coalgebra and the antipode is not verified, as these aspects of the algebra are not important to the current pursuit.

2.3 Dual Space

The next question is how to extend the shift operator over the tensor product in a homomorphic way. Rather than thinking of V in some unstructured way, as some unstructured vector space, it is more appropriate to choose for V one of the possible spaces on which the Ruelle-Frobenius-Perron operator is defined.

To make this concrete, a breif detour to construct the dual space to V is needed. So, consider the space of eigenvectors of the transfer operator. For the Bernoulli shift acting on the polynomials, the eigenvectors were the Bernoulli polynomials B_n. For a given $\gamma \in U$, the corresponding elements in V are the integrals, over the invariant measure, of the B_n. These are

$$\int_\gamma B_n d\mu = \int_\gamma B_n(x)\, dx$$

where $\mu(x) = B_0(x) = 1$ was the shift-invariant measure. The integral is real-valued, and can be taken literally and concretely when working in the canonical projection from $\mathbb{B}$ to $[0,1]$: so, for $\gamma = C_0^0$ one has

$$\int_\gamma B_n d\mu = \int_0^{1/2} B_n(x)\, dx$$

since C_0^0 is the set of all binary strings begining with 0, i.e. the set $0 \le x \le 1/2$. For $\gamma = C_0^1$, it is

$$\int_\gamma B_n d\mu = \int_0^{1/4} B_n(x)\, dx + \int_{1/2}^{3/4} B_n(x)\, dx$$

since $C_0^1 \mapsto [0,1/4] \cup [1/2,3/4]$ and so on.

Since the integral is real-valued, we have effectively constructed the dual space to V. This helps emphasize that the elements of V are measures, and the elements of the dual space V^* are (real or complex-valued) functions (on the Cantor set $\mathbb{B}$, or on it's projection to $[0,1]$, as convenient).

Thus, a general element $v \in V$ can be written as

$$v = \int_\gamma d\mu$$

while a dual element $v^* \in V^*$ is a series

$$v^* = \sum_{n=0}^{\infty} \alpha_n B_n(x)$$

for some sequence $\{\alpha_n\}$ (of real or complex numbers). Thus, the dual v^* is a map $v^* : V \to \mathbb{R}$ (or to $\mathbb{C}$) that takes

$$v^* : v \mapsto \int_\gamma \sum_{n=0}^{\infty} \alpha_n B_n(x)\, d\mu$$

On.

2.4 Shift operator

Shift invaiance...