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Quantitative wave-particle duality

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In two-way interferometers that are equipped with which-way detection devices one can acquire partial knowledge of the way at the cost of a reduction in fringe visibility. Since the interference pattern demonstrates wave properties, whereas distinguishable ways are typical of particles, it is natural to quantify the wave aspects by the visibility $\mathcal{V}$ of the fringes and the particle aspects by the distinguishability $\mathcal{D}$ of the ways. The rules of quantum mechanics imply the inequality $\mathcal{D}^2 + \mathcal{V}^2 \leq 1$. This duality relation constitutes a fundamental quantitative statement about wave-particle duality and is logically independent of Heisenberg's uncertainty relation. Recent experimental tests of the duality relation confirm the theoretical predictions both in photon interferometry and in atom interferometry.

Introduction: Wave-particle folklore

Wave-particle duality made its first appearance in Einstein's seminal paper [1] of 1905 about the photoelectric effect, in which he introduced the notion of light quanta — localized, indivisible lumps of electromagnetic energy. The boldness of this step can hardly be appreciated today; we have been living for decades in a world where the dual wave-particle nature of all atomic objects is an undisputed fact of live.

Einstein put forward his hypothesis in a different era. The physicists of the 19$^{\text{th}}$ century had firmly established that light is a wave phenomenon and had put an end, so it seemed, to all attempts at a corpuscular description of the properties of light, such as the one favored by Newton [2].[1] The cornerstones in this development were Young's double-slit experiment of 1801 [5], the observation of the so-called Poisson spot — the bright center in the shadow of a sphere, predicted by Poisson in 1818 using Fresnel's diffraction theory of 1816 and verified immediately by Arago (see Ref. [6] for details about this episode), the theoretical work that culminated in Maxwell's discovery of the equations that carry his name

[1] Incidentally, Newton knew about Grimaldi's discovery of diffraction [3] and also of Bartholin's discovery of birefringence [4]. For him the latter, in particular, was evidence *against* a wave theory of light (see Query 28 in the 2$^{\text{nd}}$ edition of the *Opticks* [2]).

(read to the Royal Society in 1864 and published in 1865), and the verification by Hertz in 1888 of the electromagnetic waves predicted by Maxwell's theory [7].

Of course, Einstein would not argue with the experimental demonstrations of light waves. Indeed, in the *annus mirabilis* 1905 he also published his first paper on the special theory of relativity, in which he made explicit use of the wave nature of light (without even mentioning the recently proposed particle properties). Einstein's light-quantum hypothesis was therefore tantamount to claiming a Janus nature of light: Under some experimental circumstances (those of Young's double-slit experiment, for example) light will behave as if it were a wave, but as if it were a particle under other conditions (such as those of the photoelectric effect).

The hypothetical light-quanta were not to everybody's liking. Planck, for one, thought for a long time that Einstein was reaching too far. Legend has it that the discovery of the Compton effect in 1923 [8] finally convinced Planck, and perhaps also other skeptical minds, of the reality of light quanta. The ultimate justification came, of course, with the inception of quantum electrodynamics in the late 1920s and the advent of its renormalized form in the late 1940s.

Einstein's bold crossing of the psychological threshold that keeps us from easily attributing particle properties to wave phenomena remained a singular event until de Broglie went the other way in 1923 [9,10]: He conjectured that electrons and other objects with well-established particle properties should also exhibit features that are characteristic of waves. In some sense, de Broglie's step was even bolder than Einstein's inasmuch as there was no experimental fact that suggested such a thing. But unlike Einstein's light-quanta, de Broglie's electron waves were not met by hostile skepticism, rather they were considered a curiosity worth exploring. Schrödinger's wave equation of 1926 [11] was obviously a consequence of de Broglie's suggestion, and ultimate triumph came in 1927 with the Davisson-Germer experiment [12].

Thus firmly established, wave-particle duality became folklore, and statements such as "Each experiment must either be described in terms of particles or in terms of waves." are oft-repeated textbook wisdom. The implication that a choice between the two is mandatory and that only the right choice will give a consistent description is, however, leading far astray. Indeed, rather than putting misplaced emphasis on the particle *or* wave alternative, one should stress that light, electrons, atoms, ... are both waves *and* particles. Only when naive classical pictures are invoked, must a choice been made, simply because such classical visualizations do not accommodate objects that are — by their nature — neither particle nor wave, but particle-and-wave.

In view of this Janus nature of all quantum objects, it is imprudent — if not in general, then certainly in the present context — to call photons, electrons, atoms, ... *particles*, as one ordinarily does. I'll therefore adopt a suggestion of Bunge's (as reported by Lévy-Leblond [13]) and speak of *quantons*. Hopefully this helps to avoid (or to create?) confusion.

Cases in which a full account of all aspects of an experiment can be given by solely relying on the classical-particle analog or solely on the classical-wave analog, are rare exceptions. Much more typical are experiments that exhibit to some extent the particle nature of a quanton and also to some extent its wave nature. A particularly transparent investigation of these intermediate situations is afforded by two-way interferometers. The particle alternatives "along this way" and "along that way" can interfere as long as the actual way remains partly unknowable (not just unknown). Then the interference fringes demonstrate the reality of the wave aspect, and their visibility $\mathcal{V}$ is a natural quantitative measure for the waveness[2] of the situation. Similarly, the particleness[2] is naturally measured by the experimenter's ability to guess the way right and, as will be discussed below, we can quantify the particle aspect in terms of the betting odds for the optimal guessing strategy; the resulting number is the distinguishability $\mathcal{D}$ of the alternatives.

The duality relation $\mathcal{D}^2 + \mathcal{V}^2 \leq 1$, which is rederived below, then quantifies the notion of wave-particle duality. It encompasses the extreme situations of "$\mathcal{D} = 1$ while $\mathcal{V} = 0$" and "$\mathcal{D} = 0$ while $\mathcal{V} = 1$" in which the respective analogies of classical particles only or classical waves only can give a full account. And in all the intermediate situations it tells us which compromises are tolerated by Nature.

Heisenberg's particle picture, Schrödinger's wave picture

Schrödinger's wave equation (1926, [11]),

$$i\hbar\frac{\partial}{\partial t}\psi(x',t) = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x'}\right)^2 + V(x')\right]\psi(x',t)\,, \tag{1}$$

here written for a (nonrelativistic) quanton of mass m in one-dimensional motion under the influence of a force $F(x) = -\partial V(x)/\partial x$, is the basis for all appeals to classical-wave analogs.[3] Since all experiments can be analyzed with the aid of the Schrödinger equation, a wave description is always possible, if one insists on having one. Fine, but just as well we can proceed from Heisenberg's equations of motion (1925, [15]),

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{p(t)}{m}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p(t) = F(x(t))\,, \tag{2}$$

and have a particle description instead.

[2] Sit venia verbo.

[3] Note that Huygens' principle [14] is immediately valid for Schrödinger waves, but not for classical electromagnetic waves because they obey a second-order (in time) wave equation. I owe this remark to R. Glauber.

One could object that the symbols $x(t)$ and $p(t)$ do not possess the same significance as in Hamilton's equations of motion and that therefore the object described by (2) is not really analogous to a point particle of Newtonian mechanics. True, but the probability amplitude $\psi(x',t)$ has also very little in common with classical waves (of sound, say). Both analogies are just that: analogies. Sometimes they are helpful, sometimes they are not, sometimes they are misleading.

There is, of course, absolutely no need for an a priori choice between Heisenberg's particle picture and Schrödinger's wave picture, and from this point of view the textbook dictum "Each experiment must either be described in terms of particles or in terms of waves." is simply incorrect. Indeed, Eqs. (1) and (2) are two sides of the same coin because

$$\psi(x',t) = \langle x',t|\ \rangle \quad \text{with} \quad \langle x',t|x(t) = x'\langle x',t| \tag{3}$$

relates Schrödinger's wave function $\psi(x',t)$ directly to the eigenstates of Heisenberg's position operator $x(t)$ (and to the state vector $|\ \rangle$ that specifies the actual physical situation). As a consequence, Eqs. (1) and (2) are not even two different descriptions, they are really the same description in different mathematical representations.

Whereas Heisenberg's particle-type equations of motion and Schrödinger's wave equation quite obviously appeal to the classical notions of particles and waves (and to the classical intuition resulting from them), the contrast between Eqs. (1) and (2) is just one facet of wave-particle duality. In view of (3), it is perhaps not even a very important facet. To do justice to wave-particle duality, we must consider simple, yet instructive situations in which the experimenter can choose between emphasizing the wave aspects or the particle aspects at his discretion.

Two-way interferometers

Consider a quanton that is prepared in either one of two alternative states $|\mathsf{Q}_1\rangle$ and $|\mathsf{Q}_2\rangle$ or in an arbitrary superposition of them, as is the situation at the center stage of any two-way interferometer. For instance, in the case of Young's double-slit interferometer the alternatives Q_1, Q_2 are "through this slit" or "through that slit," in a Mach-Zehnder interferometer they stand for "reflected at the entry port" or "transmitted there," and "in the lower state" or "in the upper state" are the alternatives that are available in the two-state–atom Ramsey interferometers of Refs. [16,17].

The latter example should make it clear that the term *way* is not at all restricted to the meaning of a trajectory in real space, or a trajectory in phase space (perhaps smeared out a bit). Quite generally, *ways* are well-defined *alternatives* that are truly different and can be kept apart if a distinction should be asked for.

In formal terms, the property "truly different" is the orthogonality $\langle \mathsf{Q}_1 | \mathsf{Q}_2 \rangle = 0$ of the Hilbert space vectors associated with the alternatives Q_1 and Q_2.

Interference

We shall not take for granted that the quanton is in a pure state and therefore employ the statistical operator

$$\rho_{\mathrm{Q}} = |\mathsf{Q}_1\rangle w_1 \langle \mathsf{Q}_1| + |\mathsf{Q}_2\rangle w_2 \langle \mathsf{Q}_2| + |\mathsf{Q}_2\rangle \epsilon^* \sqrt{w_2 w_1} \langle \mathsf{Q}_1| + |\mathsf{Q}_1\rangle \epsilon \sqrt{w_1 w_2} \langle \mathsf{Q}_2| \tag{4}$$

to specify the quanton's state. The normalization to unit trace and the positivity of ρ_{Q} impose the restrictions

$$w_1 \geq 0\,, \quad w_2 \geq 0\,, \quad w_1 + w_2 = 1\,, \quad \bigl|\epsilon\bigr| \leq 1 \tag{5}$$

on the real parameters w_1, w_2 and the complex parameter ϵ.

It is expedient to have a more compact notation at hand that appeals to the analogy between any two-state degree of freedom and one of spin-$\frac{1}{2}$ type. Upon introducing the ladder operators

$$\tau = |\mathsf{Q}_2\rangle\langle \mathsf{Q}_1|\,, \qquad \tau^\dagger = |\mathsf{Q}_1\rangle\langle \mathsf{Q}_2|\,, \tag{6}$$

we have

$$\rho_{\mathrm{Q}} = w_1 \tau^\dagger \tau + w_2 \tau \tau^\dagger + \epsilon^* \sqrt{w_1 w_2}\, \tau + \epsilon \sqrt{w_1 w_2}\, \tau^\dagger\,, \tag{7}$$

and

$$\rho_{\mathrm{Q}} \mathrel{\widehat{=}} \begin{pmatrix} w_1 & \epsilon\sqrt{w_1 w_2} \\ \epsilon^* \sqrt{w_1 w_2} & w_2 \end{pmatrix} \tag{8}$$

is a useful 2×2-matrix representation.

The interferometer's main purpose is to probe whether there is any coherence between the alternatives Q_1 and Q_2 or, put differently, whether the alternatives can interfere. To this end, the probability p_ϕ for finding the quanton in the superposition state

$$\langle \phi | = 2^{-1/2} \left(\langle \mathsf{Q}_1 | + \mathrm{e}^{i\phi} \langle \mathsf{Q}_2 | \right) \tag{9}$$

is determined at the interferometer's output. Equivalent expressions are

$$p_\phi = \langle \phi | \rho_{\mathrm{Q}} | \phi \rangle = \operatorname{tr}_Q \left\{ O(\phi) \rho_{\mathrm{Q}} \right\} = \langle O(\phi) \rangle_Q\,, \tag{10}$$

where

$$O(\phi) = |\phi\rangle\langle\phi| = \frac{1}{2} \left(\tau^\dagger \tau + \tau \tau^\dagger + \mathrm{e}^{-i\phi} \tau + \mathrm{e}^{i\phi} \tau^\dagger \right) \tag{11}$$

is the observable in question, and $\operatorname{tr}_Q \{\ldots\}$, $\langle \ldots \rangle_Q$ are the respective injunctions to trace over the quanton degree of freedom and to evaluate an expectation value in the given quanton state.

We find the interference pattern

$$p_\phi = \frac{1}{2} + \sqrt{w_1 w_2}\,\mathrm{Re}\left(\mathrm{e}^{-i\phi}\epsilon\right) , \tag{12}$$

which has the extremal values

$$\left.\begin{matrix}\max\\ \min\end{matrix}\right\} = \frac{1}{2} \pm \bigl|\epsilon\bigr|\sqrt{w_1 w_2} , \tag{13}$$

so that the fringe visibility $\mathcal{V} = (\max - \min)/(\max + \min)$ is given by

$$\mathcal{V} = 2\bigl|\epsilon\bigr|\sqrt{w_1 w_2} = 2\bigl|\left\langle\tau\right\rangle_Q\bigr| . \tag{14}$$

Maximal fringe visibility, that is: $\mathcal{V} = 1$, requires $w_1 = w_2 = \frac{1}{2}$ and $\bigl|\epsilon\bigr| = 1$.

Which-way predictability

The statistical operator ρ_Q of (4) and (7) refers to single quantons, traversing the interferometer one by one under the reproducible conditions that specify ρ_Q. What can we say about the way that the next quanton will take? If $0 < w_1 = 1 - w_2 < 1$ holds both ways are possible and we cannot be sure, but if one way is more probable than the other we know which one to bet on. The likelihood $\mathcal{L}$ for guessing the way right is then the larger one of the two numbers w_1 and w_2,

$$\mathcal{L} = \mathrm{Max}\,\{w_1, w_2\} = \frac{1}{2} + \frac{1}{2}\bigl|w_1 - w_2\bigr| . \tag{15}$$

The $\frac{1}{2}$ term is, of course, just the betting odds obtained by flipping a coin and does not represent genuine which-way knowledge, and since $\mathcal{L}$ cannot exceed unity, the actual which-way knowledge is given by the modulus of $w_1 - w_2$.

Which-way knowledge of this elementary kind originates in an asymmetry of the set-up; it is (usually) available at the outset. We call it the *predictability* of the ways and denote it by the symbol $\mathcal{P}$. If there is no additional which-way knowledge at our disposal, as is the situation to which the likelihood of (15) applies, then

$$\mathcal{L} = \frac{1}{2} + \frac{1}{2}\mathcal{P} \tag{16}$$

holds, and thus the predictability is given by

$$\mathcal{P} = \bigl|w_1 - w_2\bigr| = \Bigl|\left\langle\tau^\dagger\tau - \tau\tau^\dagger\right\rangle\Bigr| . \tag{17}$$

The inequality

$$\mathcal{P}^2 + \mathcal{V}^2 \leq 1 \tag{18}$$

is an immediate implication of the restrictions (5), that is: of the positivity and normalization of ρ_Q. The equal sign holds in (18) if ρ_Q is a pure state or projector, that is: $\rho_Q^2 = \rho_Q$, which is the case whenever $|\epsilon| = 1$ or, much less interestingly, $w_1 w_2 = 0$.

Equation (18) can be found implicitly or explicitly in a number of publications; I am aware of Refs. [18–26] but presumably there are others. In the situations considered in Refs. [18–20,22–25] predictability comes about either because the beam splitter at the entry port is asymmetric[4] or because one of two partial beams is attenuated behind a symmetric beam splitter.[5] Markedly different are the scenarios studied by Glauber [21] and by Jaeger, Shimony, and Vaidman [26].

Glauber lets a single photon pass through a double-slit and then amplifies the radiation by so much that the limit of classical electromagnetism is reached. Owing to the noise that unavoidably accompanies the amplification, the resulting classical amplitudes are, as a rule, not symmetric although the quantum input into the amplifiers is. The degree of asymmetry is, of course, not predictable and each repetition of the experiment will have a random value of the asymmetry. Each repetition will also yield a full interference pattern produced by the abundance of photons present after the amplification. The fringe visibility $\mathcal{V}$ is larger if the asymmetry is less pronounced. Upon measuring the asymmetry by a number that is the analog of the predictability $\mathcal{P}$ of (17), Glauber finds that the equal sign holds in (18) for each shot, while the values of $\mathcal{P}$ and $\mathcal{V}$ vary from shot to shot in an unpredictable fashion.

Jaeger, Shimony, and Vaidman[6] assume that the impurity of ρ_Q, for which $|\epsilon| < 1$ is the evidence, originates in an entanglement of the quanton with some other degree(s) of freedom, whereby the whole system is in a pure state. In formal terms, this amounts to the statement

$$\rho_Q = \operatorname{tr}_{\text{aux}} \{ |\Psi\rangle\langle\Psi| \} \tag{19}$$

with

$$\langle\Psi| = \sqrt{w_1}\,\langle \mathsf{Q}_1, \mathsf{A}| + \sqrt{w_2}\,\langle \mathsf{Q}_2, \mathsf{B}| \,, \tag{20}$$

where A and B stand for the respective quantum numbers of the imagined auxiliary system that is traced over in (19). The quanton state ρ_Q of (4) is reproduced, provided that $\epsilon = \langle \mathsf{B}|\mathsf{A}\rangle$ is ensured. Now, if this is more than imagination,[7] one can make a better prediction about the way taken by the next quanton by exploiting the entanglement of (20). As we shall see below, the improved predictability $\mathcal{P}_{\text{aux}}$ obtained in this manner is bounded by

$$\mathcal{P} \le \mathcal{P}_{\text{aux}} \le \sqrt{1 - \mathcal{V}^2} \,, \tag{21}$$

and the inequality (18) follows.

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[4] This includes the example of [18], where a Young interferometer has slits of different width.

[5] The experiments reported in Refs. [19,20,23] employed this method.

[6] These authors actually speak of *distinguishability* rather than *predictability*. We reserve the term distinguishability for the numerical measure introduced below in (34).

[7] For an explicit example of a realistic experimental situation of this kind see Fig. 3 in [27], which builds on problem 9-6 in [28].

Which-way marking

Rather than predicting the way for the next quanton, one can try to find out which way was taken by the last one that went through the interferometer. To go beyond what can be predicted in the first place, the atom must leave a trace on its way through the interferometer. This is achieved by an interaction with another degree of freedom that serves as a which-way *marker*. Owing to the resulting entanglement of the quanton's two-way degree of freedom with the marker degree(s) of freedom, a suitable measurement on the marker can supply information about the actual way, about which we'll say more in the next section.

Prior to their interaction the quanton and the marker are not entangled and their joint state P is just a (tensorial) product

$$\text{before:} \qquad \mathrm{P} = \rho_{\mathrm{Q}}\rho_{\mathrm{M}} \tag{22}$$

of the quanton state ρ_{Q} and the initial marker state ρ_{M}. The net effect of the interaction is a unitary transformation,

$$\text{after:} \qquad \mathrm{P} = S^{\dagger}\rho_{\mathrm{Q}}\rho_{\mathrm{M}}S \qquad \text{with} \qquad SS^{\dagger} = S^{\dagger}S = 1\,. \tag{23}$$

This transformation may also change the values of w_1, w_2, and ϵ, because it can contain a contribution that affects only the quanton but not the marker. Therefore we can incorporate a diagonalizing quanton transformation into the unitary operator S, and this will be expedient. All of this amounts to writing the 2×2 matrix representation of the final P in the form[8]

$$\mathrm{P} \mathrel{\widehat{=}} \begin{pmatrix} U_1^{\dagger} & U_+^{\dagger} \\ U_-^{\dagger} & U_2^{\dagger} \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \rho_{\mathrm{M}} \begin{pmatrix} U_1 & U_- \\ U_+ & U_2 \end{pmatrix}, \tag{24}$$

where $\cos^2\theta$ and $\sin^2\theta$ are the eigenvalues of ρ_{Q} before the interaction. The operators U_1, U_2, U_+, and U_- act only on the marker; as a rule they are not unitary themselves. Unitarity is only required of the whole transformation, not of individual pieces. Thus we need equalities such as

$$U_1U_1^{\dagger} + U_-U_-^{\dagger} = 1\,, \qquad U_2U_2^{\dagger} + U_+U_+^{\dagger} = 1 \tag{25}$$

and six more to hold. No other restrictions will be applied to the U's; and the initial marker state ρ_{M} is also quite arbitrary. It is not even necessary to state what constitutes the marker — in some sense, anything that we want to count as part of the quanton's environment can be regarded as part of the marker. In other words, the quanton has to leave a trace in its environment if we want to be able to find out the way it took.

[8]This is the most general form. The simplest, ideal case is dealt with in [29]; the $\theta = 0$ situation is treated in [30], and a special case thereof in [31].

Upon introducing the convenient abbreviations

$$X_1 = \sqrt{\rho_{\mathrm{M}}}\, U_1 \,, \quad Y_1 = \sqrt{\rho_{\mathrm{M}}}\, U_- \,, \qquad X_2 = \sqrt{\rho_{\mathrm{M}}}\, U_2 \,, \quad Y_2 = \sqrt{\rho_{\mathrm{M}}}\, U_+ \,, \tag{26}$$

we get

$$\mathrm{P} \mathrel{\widehat{=}} \cos^2\theta \begin{pmatrix} X_1^\dagger X_1 & X_1^\dagger Y_1 \\ Y_1^\dagger X_1 & Y_1^\dagger Y_1 \end{pmatrix} + \sin^2\theta \begin{pmatrix} Y_2^\dagger Y_2 & Y_2^\dagger X_2 \\ X_2^\dagger Y_2 & X_2^\dagger X_2 \end{pmatrix} . \tag{27}$$

The "after" state of the quanton is obtained by tracing over the marker's degree(s) of freedom; of course, it is of the form (8) with

$$\begin{aligned} w_1 &= \mathrm{tr}_M \left\{ \cos^2\theta\, X_1^\dagger X_1 + \sin^2\theta\, Y_2^\dagger Y_2 \right\} , \\ w_2 &= \mathrm{tr}_M \left\{ \cos^2\theta\, Y_1^\dagger Y_1 + \sin^2\theta\, X_2^\dagger X_2 \right\} , \\ \epsilon\sqrt{w_1 w_2} &= \mathrm{tr}_M \left\{ \cos^2\theta\, X_1^\dagger Y_1 + \sin^2\theta\, Y_2^\dagger X_2 \right\} . \end{aligned} \tag{28}$$

In accordance with (14) and (17), these parameter values now determine the visibility $\mathcal{V}$ of the interference fringes and the predictability $\mathcal{P}$ of the way along which the final, probing part of the interferometer is approached. In general, all three may differ from the values that one would have without the quanton-marker interaction. It is, however, possible to arrange matter such that only the value of ϵ is affected. This is the case in the ideal scenario studied theoretically in [29] and realized recently in the experiments reported in Refs. [32] and [33,34].

Which-way distinguishability

The final state $\rho_{\mathrm{M}}^{(\mathrm{fin})}$ of the marker is obtained by tracing P of (24) over the quanton's two-way degree of freedom. We find that

$$\rho_{\mathrm{M}}^{(\mathrm{fin})} = w_1 \rho_{\mathrm{M}}^{(1)} + w_2 \rho_{\mathrm{M}}^{(2)} \tag{29}$$

contains two contributions, one for each way, that are given by

$$\begin{aligned} w_1 \rho_{\mathrm{M}}^{(1)} &= \cos^2\theta\, X_1^\dagger X_1 + \sin^2\theta\, Y_2^\dagger Y_2 \,, \\ w_2 \rho_{\mathrm{M}}^{(2)} &= \cos^2\theta\, Y_1^\dagger Y_1 + \sin^2\theta\, X_2^\dagger X_2 \,. \end{aligned} \tag{30}$$

To learn something about the actual way, we must extract the which-way information stored in $\rho_{\mathrm{M}}^{(\mathrm{fin})}$, which requires the measurement of a suitably chosen marker observable M. The measurement result is one of the eigenvalues m_1, m_2, m_3, ..., assumed to be discrete and nondegenerate. If $|m_k\rangle$ denotes the marker ket to eigenvalue m_k, then

$$\langle m_k|\rho_{\mathrm{M}}^{(\mathrm{fin})}|m_k\rangle = \underbrace{w_1\langle m_k|\rho_{\mathrm{M}}^{(1)}|m_k\rangle}_{\text{first way}} + \underbrace{w_2\langle m_k|\rho_{\mathrm{M}}^{(2)}|m_k\rangle}_{\text{second way}} \tag{31}$$

is the probability for getting this eigenvalue. Having found m_k in the measurement of M, we won't know for sure which way was traversed, unless one of the two contributions to $\langle m_k|\rho_{\rm M}^{(\rm fin)}|m_k\rangle$ happens to vanish. If both terms on the right-hand side of (31) are positive, the best guess we can make is to bet on the way that contributes most to $\langle m_k|\rho_{\rm M}^{(\rm fin)}|m_k\rangle$ because that is the more likely way under the given circumstances.[9] In many repetitions of the experiment, the resulting likelihood $\mathcal{L}$ for guessing the way right is then

$$\mathcal{L}_M = \sum_k \mathrm{Max}\left\{w_1\langle m_k|\rho_{\rm M}^{(1)}|m_k\rangle, w_2\langle m_k|\rho_{\rm M}^{(2)}|m_k\rangle\right\} = \frac{1}{2}+\frac{1}{2}\mathcal{K}_M\,. \tag{32}$$

The number $\mathcal{K}_M$ quantifies the which-way *knowledge* gained by measuring the marker observable M [recall Eqs. (15) and (16)]. It is given by

$$\mathcal{K}_M = \sum_k \left|\langle m_k|(w_1\rho_{\rm M}^{(1)} - w_2\rho_{\rm M}^{(2)})|m_k\rangle\right| \tag{33}$$

and depends, of course, on the choice for M. The inequality $\mathcal{K}_M \geq \mathcal{P}$ is an immediate consequence of the familiar triangle inequality; it states that we don't lose which-way knowledge when taking a look at the marker.

Some observables M will yield more which-way knowledge than others. The largest value of $\mathcal{K}_M$ is the *distinguishability* $\mathcal{D}$ of the ways. It obtains when the $|m_k\rangle$'s are eigenkets of the difference $w_1\rho_{\rm M}^{(1)} - w_2\rho_{\rm M}^{(2)}$,[10] and is explicitly given by[11]

$$\mathcal{D} = \underset{M}{\mathrm{Max}}\left\{\mathcal{K}_M\right\} = \mathrm{tr}_M\left\{\left|w_1\rho_{\rm M}^{(1)} - w_2\rho_{\rm M}^{(2)}\right|\right\}. \tag{34}$$

The modulus $\left|\cdots\right|$ of an operator that appears here can be defined by $\left|O\right| = (O^\dagger O)^{1/2}$, but in the relevant case of a selfadjoint O the meaning of $\mathrm{tr}\left\{\left|O\right|\right\}$ is just the injunction to sum the moduli of the eigenvalues of O.

The inequalities

$$\mathcal{P} \leq \mathcal{K}_M \leq \mathcal{D} \tag{35}$$

state an obvious hierarchy: The distinguishability $\mathcal{D}$ represents Nature's information about the way taken; the knowledge $\mathcal{K}_M$ is what Man can learn from a measurement of the marker observable M; and the predictability $\mathcal{P}$ is Man's knowledge before measuring M. It is worth emphasizing that the numbers $\mathcal{P}$, $\mathcal{K}_M$, $\mathcal{D}$, and $\mathcal{L}_M$ quantify information or knowledge without invoking an entropic concept of some kind.

In any practical experiment, the experimenter is limited by constraints of various sorts and may not be able to extract all the which-way information that is potentially available. Some further details concerning these issues can be found in the recent paper by Björk and Karlsson [36], in whose terminology the knowledge $\mathcal{K}$ is the *measured distinguishability.*

[9]This betting strategy was introduced by Wootters and Zurek [18].

[10]This is a consequence of Peierls' inequality; see [35], for example.

[11]Mathematically speaking, $\mathcal{D}$ is the distance between $w_1\rho_{\rm M}^{(1)}$ and $w_2\rho_{\rm M}^{(2)}$ in the trace-class norm.

Duality relation

The visibility given by (14) and (28) is

$$\begin{aligned}\mathcal{V} &= 2\Big|\,\mathrm{tr}_M\left\{\cos^2\theta\, X_1^\dagger Y_1 \,+\sin^2\theta\, Y_2^\dagger X_2\right\}\Big| \\ &= \begin{cases} 2\Big|\,\mathrm{tr}_M\left\{X_1^\dagger Y_1\right\}\Big| \equiv \mathcal{V}_1 \text{ for } \cos^2\theta = 1\,, \\ 2\Big|\,\mathrm{tr}_M\left\{X_2^\dagger Y_2\right\}\Big| \equiv \mathcal{V}_2 \text{ for } \sin^2\theta = 1\,, \end{cases}\end{aligned} \tag{36}$$

and we have the distinguishability

$$\begin{aligned}\mathcal{D} &= \mathrm{tr}_M\left\{\Big|\cos^2\theta\left(X_1^\dagger X_1 - Y_1^\dagger Y_1\right) - \sin^2\theta\left(X_2^\dagger X_2 - Y_2^\dagger Y_2\right)\Big|\right\} \\ &= \begin{cases} \mathrm{tr}_M\left\{\left|X_1^\dagger X_1 - Y_1^\dagger Y_1\right|\right\} \equiv \mathcal{D}_1 \text{ for } \cos^2\theta = 1\,, \\ \mathrm{tr}_M\left\{\left|X_2^\dagger X_2 - Y_2^\dagger Y_2\right|\right\} \equiv \mathcal{D}_2 \text{ for } \sin^2\theta = 1\,, \end{cases}\end{aligned} \tag{37}$$

from (30) and (34). They obey the *duality relation*

$$\mathcal{D}^2 + \mathcal{V}^2 \le 1\,, \tag{38}$$

which is more stringent than (18), since $\mathcal{D}$ cannot be less than $\mathcal{P}$.

The physical inequality (38) is demonstrated with the aid of the mathematical statement

$$\left(\mathrm{tr}\left\{\left|X^\dagger X - Y^\dagger Y\right|\right\}\right)^2 + 4\,\mathrm{tr}\left\{X^\dagger Y\right\}\mathrm{tr}\left\{Y^\dagger X\right\} \le \left(\mathrm{tr}\left\{X^\dagger X + Y^\dagger Y\right\}\right)^2\,, \tag{39}$$

which is valid if all these traces are finite and can be proven in various manners. One simple proof is given in [30]. It is not necessary to repeat this piece of mathematics here, but it is worth emphasizing that the proofs of (39) make no use whatsoever of any uncertainty relation of the Heisenberg-Robertson kind [37,38],

$$\delta A\,\delta B \ge \frac{1}{2}\left|\langle i(AB - BA)\rangle\right|\,, \tag{40}$$

for the spreads of two observables A and B and the expectation value of their commutator.

To close the gap between (39) and (38), we first note that Eqs. (25) and (26) establish

$$\mathrm{tr}\left\{X_1^\dagger X_1 + Y_1^\dagger Y_1\right\} = 1\,, \qquad \mathrm{tr}\left\{X_2^\dagger X_2 + Y_2^\dagger Y_2\right\} = 1\,. \tag{41}$$

In conjunction with (36), (37), and (39) the $\cos^2\theta = 1$ and $\sin^2\theta = 1$ versions of (38) then follow immediately,

$$\mathcal{D}_1^2 + \mathcal{V}_1^2 \le 1\,, \qquad \mathcal{D}_2^2 + \mathcal{V}_2^2 \le 1\,. \tag{42}$$

Now, triangle inequalities (for complex numbers and the trace-class norm, respectively) yield

$$\mathcal{V} \le \mathcal{V}_1 \cos^2\theta + \mathcal{V}_2 \sin^2\theta\,, \qquad \mathcal{D} \le \mathcal{D}_1 \cos^2\theta + \mathcal{D}_2 \sin^2\theta\,, \tag{43}$$

which we combine with

$$\mathcal{D}_1\mathcal{D}_2 + \mathcal{V}_1\mathcal{V}_2 \le \sqrt{\mathcal{D}_1^2 + \mathcal{V}_1^2}\,\sqrt{\mathcal{D}_2^2 + \mathcal{V}_2^2} \tag{44}$$

to arrive at

$$\mathcal{D}^2 + \mathcal{V}^2 \le \left(\cos^2\theta\,\sqrt{\mathcal{D}_1^2 + \mathcal{V}_1^2} + \sin^2\theta\,\sqrt{\mathcal{D}_2^2 + \mathcal{V}_2^2}\right)^2 . \tag{45}$$

In view of (42) the right-hand side does not exceed unity, and therefore we have succeeded in deriving (38), indeed.

If both ρ_{Q} and ρ_{M} are pure states before the quanton-marker interaction, it is an easy matter to show that the equality sign holds in the duality relation (38). The situation is then mathematically congruent with the one of (19)–(21) whereby the auxiliary degrees of freedom play the role of the marker and $\mathcal{P}_{\mathrm{aux}}$ is the analog of $\mathcal{K}$. Equation (21) then obtains as a consequence of (35) and (38).

Einsteins's recoiling slit

The derivation of the duality relation (38) does not rely on the uncertainty relation (40), and one doesn't use (38) to establish (40). This is summarized in the statement

The duality relation (38) and
the uncertainty relation (40)
are logically independent. (46)

Arguments showing that there is really no room for (40) in any derivation of (38) have been given elsewhere (see [29], for instance), and rather than repeating them, it is perhaps more useful to consider in some detail what might appear to be a counter example: Einstein's recoiling slit, a thought experiment that was refuted by Bohr in whose reasoning the position-momentum uncertainty relation was a crucial ingredient [39].

We recall: Einstein proposed to use the recoil of a single slit that illuminates the double-slit of a Young interferometer to determine the way for each quanton sent through. All quantons together would build up the interference pattern, and the recorded recoils would reveal the way of each quanton. The analysis is somewhat simplified if we assume that the double-slit is halfway between the single slit and the screen. Then, an easy geometrical consideration shows that the amount of recoil in question, that is: $\pm\frac{1}{2}\hbar k$, is simply related to the expected fringe spacing $2\pi/k$.[12]

[12] These and other details are spelled out in [40]; the present treatment focuses on the bare essentials.

Bohr's argument is summarized as follows. (i) To ensure good fringe visibility the initial position uncertainty δx of the recoiling slit must be small compared with the fringe spacing,

$$\delta x \ll 2\pi/k\,; \tag{47}$$

(ii) if one wants to distinguish recoil $+\frac{1}{2}\hbar k$ from recoil $-\frac{1}{2}\hbar k$, then the initial momentum uncertainty δp must be small compared with their difference,

$$\delta p \ll \hbar k\,; \tag{48}$$

(iii) the implied statement

$$\delta x\,\delta p\,(\ll)^2 2\pi\hbar \tag{49}$$

violates Heisenberg's uncertainty relation; (iv) therefore, one cannot have fringes [condition (47)] and also which-way knowledge [condition (48)], and we must conclude that Einstein's idea doesn't work.

Although this familiar argument appears to rely heavily on $\delta x\,\delta p \geq \hbar/2$, this is actually not the case because Bohr's requirement (48) is too strong. All that one really needs is that $\hbar k$ is large compared to the momentum-space coherence length $\hbar/\delta x$ (which itself can be much smaller than the momentum uncertainty δp)

$$\hbar k \gg \hbar/\delta x\,. \tag{50}$$

And this immediately contradicts condition (47) without any reference to the position-momentum uncertainty relation. Whereas Bohr's conclusion, namely that Einstein's idea doesn't work, was undoubtedly correct, his argument was not as convincing as it looks at first glance.

Here is a simple example illustrating these points. Suppose that the single slit is initially in a center-of-mass state that has a momentum-space wave function of the form

$$\psi(p') = \sqrt{\xi/2}\left(\mathrm{e}^{-\xi|p'-p_0|} + \mathrm{e}^{-\xi|p'+p_0|}\right) \tag{51}$$

with $\xi p_0 \gg 1$. The momentum density $\bigl|\psi(p')\bigr|^2$ has two peaks at $p' = \pm p_0$ whose separation $2p_0$ is much larger than their width $1/\xi$. Recoils of $\pm\frac{1}{2}\hbar k$ would change the momentum distribution to $\bigl|\psi_\pm(p')\bigr|^2 = \bigl|\psi(p' \mp \frac{1}{2}\hbar k)\bigr|^2$, and one can clearly tell them apart if $\hbar k$ is sufficiently larger than the peak width $1/\xi$. A relative recoil $\hbar k$ larger than the momentum uncertainty $\delta p = p_0$, as requested by Bohr's condition (48), is not needed for a reliable distinction of $\bigl|\psi_+(p')\bigr|^2$ from $\bigl|\psi_-(p')\bigr|^2$. Since $\delta x = \hbar\xi$ here, the requirement $\hbar k \gg 1/\xi$ amounts to condition (50), indeed.

Recent experiments

Quite a few experiments (see Refs. [19,20,23,41,42,17,33] and there are surely others) have demonstrated a systematic loss of fringe visibility when which-way information is made potentially available. But the actual extraction of which-way knowledge was realized only recently in experiments performed in Konstanz and Los Alamos.

The Konstanz experiment by Dürr, Nonn, and Rempe [34] used standing optical-light waves to split and reunite a beam of cold Rb atoms. Hyperfine states of the atoms were used for the which-way marking, and which-way knowledge was gained by a state-sensitive detection of the atoms. A large range of values for the visibility $\mathcal{V}$ and the acquired knowledge $\mathcal{K}$ could be realized, and good agreement was found with

$$\mathcal{K}^2 + \mathcal{V}^2 \leq 1\,, \tag{52}$$

which is the version of the duality relation (38) that is relevant for the comparison with actual experimental data.

The Los Alamos experiment by Schwindt and Kwiat [32] used a Mach-Zehnder interferometer for photons and the photon polarization for the which-way marking. A polarization-sensitive detection of the photons made which-way knowledge available. Again, the measured visibility and the extracted knowledge were in full accord with (51). It was possible to get very close to the upper limit, so that the experimental values of the knowledge $\mathcal{K}$ could not have been much short of the distinguishability $\mathcal{D}$.

In both experiments the orbital motion of an object (atom or photon) was used for the two-way degree of freedom of the quanton, and internal degrees of freedom (hyperfine levels or polarization) of the same object were employed for the marker degrees of freedom. This is also the case in a proposed neutron experiment [43]. Experiments in which the quanton and the marker are different physical objects, not just different degrees of freedom, would be very welcome. Perhaps the Ramsey interferometers of Refs. [16,17] are well suited for this purpose.

Summary

After a brief review of some of the historical aspects of wave-particle duality, the quantification of this notion in the context of two-way interferometry was recalled. The central statement is the duality relation (38). This inequality is a fundamental quantitative statement about wave-particle duality. It is not a variant of the uncertainty relation, but rather a logically independent restriction.

Two recent experimental tests of the duality relation were reported, one using an atom interferometer, the other a photon interferometer. Both found very good agreement with the theoretical expectations.

Further, the Bohr-Einstein debate about the acquisition of which-way knowledge by measuring the momentum of a recoiling slit was reconsidered. It was shown that the position-momentum uncertainty relation really plays no role in the argument that demonstrates that good which-way knowledge and well visible fringes are mutually exclusive.

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