SCRAPBOOK

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ABSTRACT. Collection of random observations pertaining to number theory topics.

SIGMA ALGEBRAS

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Sigma algebra on $\mathbb{N}$. The sigma algebra for functions on the integers is given by $2^{\mathbb{N}}$, and, specifically, by individual *points* in $2^{\mathbb{N}}$. Each point should be interpreted as a Dirac-delta/membership function. This is in contrast to the standard sigma algebra for the Cantor set, which is the set of all finite-length strings.

Non-standard sigma algebras on $2^{\mathbb{N}}$. The standard sigma algebra for the Cantor set $2^{\mathbb{N}}$ is given by the set of all finite-length strings. Why? Because this fits well with the standard topology on the reals. However, we can also consider cylinder sets which are not finite-length. To do this, define a cylinder set as a point (an infinitely long string) in $\{0,1,*\}^{\mathbb{N}} = 3^{\mathbb{N}}$ So, for example, the string $101*1***\cdots$ is the collection of all infiintely long strings whose 1st, 2nd and 3rd bits are 101, whose 5th bit is 1 and all other bits are 'don't care'. Clearly, we can consider strings in $3^{\mathbb{N}}$with non-* bits stretching off to infinity: these form a valid sigma algebra.

Using the standard Bernoulli measure, these sets have measure zero. Why? The standard measure gives a binary string of length n the measure of 2^{-n} (or, more generally p and $1-p$ multiplied n times). Thus, for the standard Bernoulli measure, we conider only those strings in $3^{\mathbb{N}}$ which have a *finite* number of 0's an 1's in them, and thus end with an infinite number of trailing *'s.

Are there measures on the sigma algebra $(\Omega,\mathscr{B}) = (2^{\mathbb{N}},3^{\mathbb{N}})$ which are not Bernoulli measures? i.e. assign non-zero measures to infiintely long strings in $3^{\mathbb{N}}$? Well, there are some, which are "trivial" extensions of the Bernoulli measure: so: fix any one, given string in $3^{\mathbb{N}}$and then assign a Bernoulli measure to all strings that differ by a finite number of positions. We can extend this to any finite number of strings in $3^{\mathbb{N}}$, partition up the total measure between these, and then sub-partition these up using the standard Bernoulli measure. We can 'trivially' go one step further: pick a countable number of infinite strings in $3^{\mathbb{N}}$, partition up the total measure between those, and then sub-partition each of these with Bernoulli measures. So, with this trivial extension, the best we can do is to pick out a countable subset from $3^{\mathbb{N}}$ and distribute the measure across this countable subset.

Theorem. *For standard measure theory, this is the best that one can do. That is, one cannot use standard measure theory, and at the same time contemplate a function $\mu : 3^{\mathbb{N}} \to \mathbb{R}^+$ that assigns a non-zero value to uncountably many points in $3^{\mathbb{N}}$.*

Proof. The reason for this is that, for standard measure theory, we always always ask that μ be sigma-additive. The sigma-additivity condition is for countable disjoint unions. To do what we want above, we would need to extend the notion of sigma-additivity to an uncountable class of sets. That is, we would have to have some mechanism for assigning a non-zero, but infinitessimal amount of the measure to various sets. We'd have to extend the usual sigma-additivity summation into an integral. This is beyond the scope of standard measure theory. □

I hope above wasn't too confusing.