

# Geometry of Space, Time and Other Things

## The Mathematics of Fiber Bundles

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## Fiber Bundles

- Central to physics: classical mechanics, electrodynamics, quantum field theory, gravitation, superconductivity.
- It was not always that way!
- Unified (pseudo-)Riemannian Geometry (i.e. Gravitation) with Symplectic Geometry (classical mechanics) with Electrodynamics with Yang-Mills theory with Superconductivity with Fermions (QFT)
- A single, unified framework for (almost) all of the fundamental theories of physics.
- And that is the topic today.

## Zen Koans

- There will be a lot of equations today
- Several semesters worth ...
- What do they MEAN? Intuitively ??
- Interpretation of poetry, jokes of Zen koans
- Intuition alone is FAULTY. Must trust formulas!
- Equations are tie-breakers for intuitive ideas
- Creativity and imagination are KEY

# Tee-shirt Equations

Before fiber bundles:

- Classical mechanics was Hamilton's equations

$$\dot{p} = -\frac{dH}{dq} \quad \dot{q} = \frac{dH}{dp}$$

- Electrodynamics was Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = 0$$

- Gravitation was Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Superconductivity was the Ginzberg–Landau equations

$$\mathcal{L} = \alpha |\phi|^2 + \beta |\phi|^4 + \frac{1}{2m} \left| \left( -i\hbar\vec{\nabla} - 2e\vec{A} \right) \phi \right|^2 + \frac{|\vec{B}|^2}{2}$$

- Standard Model = Yang-Mills + Higgs + Fermions

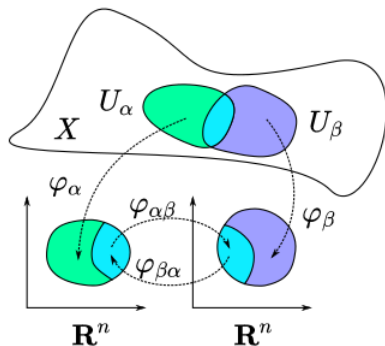
## Outline

- Manifold  $M$  as gluing of  $\mathbb{R}^n$  - coordinate charts
- (Integrable) vector fields as hair/fur that can be combed
- Tangent vector space  $T_p M$
- Back to basics: Vector spaces; notation:  $e_n$  as basis vector
- A frame field as  $e_n(p)$  varying from point to point  $p$ .
- Frame fields can twist around, rotate, swirl.
- The rotation matrix  $A$ . The connection  $A_i = \Gamma_{ij}^k$  aka Christoffel symbol
- Rotations & rotation matrices in 3D
- Curvature as total rotation after walking a loop.
- Parallel transport
- Geodesics

# Charts and Manifolds

An atlas is:

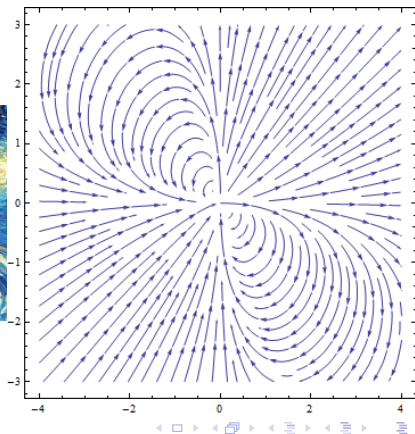
- A collection of regions  $U_\alpha$
- A collection of charts  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$
- A collection of “transition functions”  $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$



# Vector Fields

A vector field is:

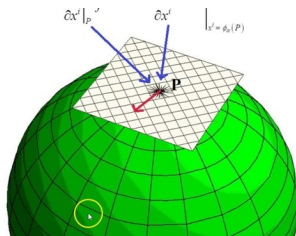
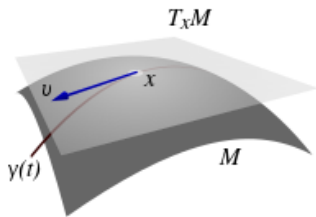
- A collection of vectors  $\vec{v}_p$
- One for each point  $p \in U_\alpha$
- Smooth, differentiable, integrable



# Tangent vector spaces

The tangent vector space  $T_p M$  is:

- A point  $p \in U_\alpha$
- The collection of ALL possible vectors  $\vec{v}_p \in T_p M$





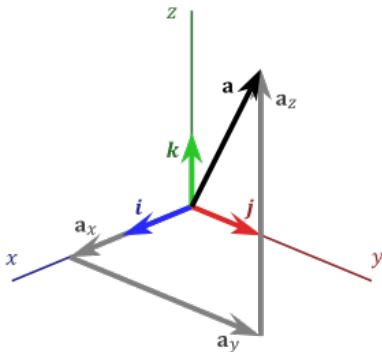
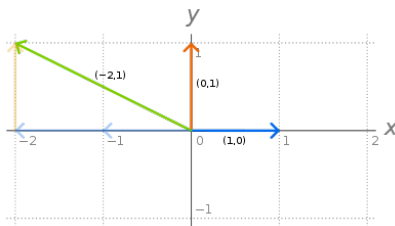
# Vectors and Bases

A Vector  $\vec{v} \in \mathbb{R}^n$  in  $n$ -dimensional space is:

- A collection of  $n$  real numbers:  $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$

A vector space basis for  $\mathbb{R}^n$  is a collection of  $n$  vectors  $\{\vec{e}_k\}$ :

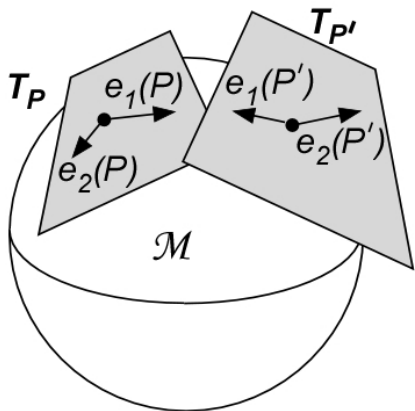
- Where  $e_1 = (1, 0, 0, \dots, 0)$  and  $e_2 = (0, 1, 0, \dots, 0)$  and  $e_3 = (0, 0, 1, 0, \dots, 0)$  and ...



# Frame fields

A frame field is:

- A collection of basis vectors  $\{\vec{e}_k(p)\}$
- One for each point  $p \in U_\alpha$







# Geometry with Formulas

The hardest part with formulas is (1) there are so many (2) there are many different ways of writing down the \*same\* equations, using wildly different notation.

- Introduce Lie derivative  $L_X f$
- introduce covariant derivative  $D = d+A$  - rosetta stone of different notations
- geodesics as solutions of Hamilton's equations i.e. as linear, first-order diffeq NOT second order!

$$\dot{p} = -\frac{dH}{dq} \quad \dot{q} = \frac{dH}{dp}$$

where  $H$ =squared-length-of-curve

- exp as the map that moves along geodesics
- geodesic completeness
- introduce metric as inner product of frame fields
$$g_{\mu\nu} = e_\mu \cdot e_\nu = e_\mu^a e_\nu^b \eta_{ab}$$
- point out that metric was NOT needed to define curvature, geodesics, parallel transport

- (metric is almost kind-of useless except that its a standard touch-stone for GR)
- Repeat Einstein eqns.
- replace frame field by generic fiber bundle
- e.g.  $U(1)$  for electromagnetism,  $SU(n)$  for yang-mills
- Maxwell's equations are nothing more than Hamilton's eqns on  $U(1)$  + Bianchi identities

$$F = dA \quad d * F = 0$$

- Maxwell's eqn's have singularities called "electric charges" and geodesics go "splat" on an electric charge
- Schwarzschild BH's are just like electric charges: geodesics go splat when they get there.
- Yang-Mills/Einstein

$$F = dA + A \wedge A \quad D * F = 0$$

is the same as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X, Y]$$

provide a rosetta-stone correspondance