Identities inspired by Ramanujan Notebooks (part 2)

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These are new findings related to the ones found in 1998. I had the idea that maybe the sums in the first note I wrote were looking in the bad direction. They are good yes but maybe there is better. Actually there is, instead of considering powers of exp(2π) I should consider powers of exp(kπ).

After a few experiments with Maple (computer algebra system) and PSLQ (integer relations algorithm) here are the findings and remarks.

- I could go up to Zeta(13) and powers of exp(24π) after that the numbers are too close together for any computation, 1/exp(30π) is of the order of 10^{-40}.
- I used a precision of up to 1000 digits.
- The powers of exp(kπ) are related to the divisors of a number.
- I could find relations with ζ(2n) but not even powers of π. I have identities for π and π^3 alone.
- I have found no relations for other constants like the Catalan constant so far but I am working on it.
- Some formulas are with the exponential sums but with no powers of π.

$$\pi = 72\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)}$$

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}+1)} - \frac{2}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}+1)}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

$$\zeta(5) = \frac{-7}{1840}\pi^5 + \frac{328}{115}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{419}{460}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{9}{115}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{3\pi n}-1)}$$
$$+ \frac{261}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{6\pi n}-1)} - \frac{9}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{12\pi n}-1)}$$

$$\zeta(7) = \frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - \frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} - \frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

$$\zeta(9) = \frac{64}{3}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{\pi n}-1)} + \frac{441}{20}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{2\pi n}-1)} - 32\sum_{n=1}^{\infty}\frac{1}{n^9(e^{3\pi n}-1)}$$
$$- \frac{4763}{60}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{4\pi n}-1)} + \frac{529}{8}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{6\pi n}-1)} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{12\pi n}-1)}$$

$$\zeta(5) = \frac{-149}{43983}\pi^5 + \frac{785}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{12\pi n}-1)} - \frac{22765}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{6\pi n}-1)} + \frac{1570}{543}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{3\pi n}-1)}$$
$$- \frac{61}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20\pi n}-1)} + \frac{1769}{4344}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{10\pi n}-1)} - \frac{122}{543}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{5\pi n}-1)}$$

This last formula is interesting since the lowest term gives a rate of convergence of the order of $1/\exp(3\pi)$.

I take the occasion to thank Bruce Berndt for the remarkable piece of work he did with these collected works of S. Ramanujan.

References

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