

A simple proof of Linas's theorem on Riemann zeta function

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Received: DD Month YYYY **Revised:** DD Month YYYY **Accepted:** DD Month YYYY

Abstract: Linas Vepstas gives rapidly converging infinite representatives for values of Riemann zeta function at $(4m - 1)$, where m is a natural number. In this paper, we give a new simple proof. Also, we obtain two equation of values of Bernoulli numbers ' generating function by applying a corollary given in this paper.

Keywords: Analysis, Riemann zeta function, Fourier series, hyperbolic function

2010 Mathematics Subject Classification: 11M06.

1 Introduction

Linas[1] gave the following rapidly converging infinite representatives for values of $\zeta_{(4m-1)}$ using polylogarithm, where m is a natural number and B_k is the k 'th Bernoulli number.

Theorem 1.1. (*Linas's theorem*)

$$\zeta_{(4m-1)} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{4m-1} (e^{2\pi n} - 1)} - \frac{1}{2} (2\pi)^{4m-1} \sum_{j=0}^{2m} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{4m-2j}}{(4m-2j)!}$$

In this paper, we give a simple proof of it, using Fourier series of $\cosh(x)$.

2 Preliminaries

Lemma 2.1.

$$L \coth(L) = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{\left(\frac{\pi}{L}n\right)^2 + 1}$$

Proof. Using the Fourier series expansion method from $-L$ to L , where L is a positive real number, $\cosh(x)$ is expressed as following

$$\cosh(x) = \frac{\sinh(L)}{L} + \sum_{n=1}^{\infty} \frac{2}{L} \frac{(-1)^n}{\left(\frac{\pi}{L}n\right)^2 + 1} \sinh(L) \cos\left(\frac{\pi}{L}nx\right)$$

Substituting L for x , we obtained the following equation.

$$\cosh(L) = \frac{\sinh(L)}{L} + \frac{2}{L} \sinh(L) \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(\frac{\pi}{L}n\right)^2 + 1} (-1)^n$$

□

$$\sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^{4m-1}} = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^{4m}} + \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{4m-2}n^2 + k^{4m}}$$

Proof. From Lemma 2.1, the above equation can be obtained.

□

Lemma 2.2.

$$\frac{1}{k^{4m} + k^{4m-2}n^2} = \frac{1}{2} \sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

Proof.

$$\sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}} = \sum_{s=1}^{2m-1} \frac{1}{k^{4m}} \left((-1)^{s+1} \frac{k^{2s}}{n^{2s}} \right)$$

Calculate the equation above and the series can be expressed as follows.

$$\frac{1}{k^{4m} + k^{4m-2}n^2} + \frac{1}{k^2n^{4m-2} + n^{4m}} = \sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

Considering the symmetry of the left-hand side, we can obtain the equation.

□

3 A Proof of Main Theorem

Following relation is well known.

$$\zeta_{(2n)} = (-1)^{n+1} \frac{B_{2n} (2\pi)^{2n}}{2 (2n)!}$$

We can rewrite the Theorem 1.1 by this equation as follows.

$$\zeta_{(4m-1)} + 2 \sum_{k=1}^{\infty} \frac{1}{k^{4m-1} (e^{2\pi k} - 1)} = \frac{1}{\pi} \sum_{s=0}^{2m} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)}$$

The equation above can be proved by transforming the equation from Corollary 2.1 using Lemma 2.2 by the following process.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\coth(\pi k)}{k^{4m-1}} &= \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^{4m}} + \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{4m-2} n^2 + n^{4m}} \\ &= \frac{1}{\pi} \left(\sum_{k=1}^{\infty} \frac{1}{k^{4m}} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s} n^{2s}} \right) \\ &= \frac{1}{\pi} \left(\zeta_{(4m)} + \sum_{s=1}^{2m-1} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)} \right) \\ &= \frac{1}{\pi} \sum_{s=0}^{2m} (-1)^{s+1} \zeta_{(4m-2s)} \zeta_{(2s)} \end{aligned}$$

4 Appendix

This appendix gives following equations.

$$\begin{aligned} \frac{1}{e^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2 + 1} \\ \frac{\pi}{e^{\pi} - 1} &= 2 \sum_{n=1}^{\infty} \left(\frac{1}{4n^2 + 1} + \frac{(-1)^n}{4n^2 - 1} \right) \end{aligned}$$

These equations come from following corollary.

$$\frac{x}{e^x - 1} = -\frac{x}{2} + 1 + 2 \sum_{n=1}^{\infty} \frac{1}{4 \left(\frac{\pi}{x} n \right)^2 + 1}$$

Proof. Substituting $x/2$ for L in Lemma 2.1, we get the infinite representative above. □

Substituting $2, \pi$ for x in Corollary 4, we get the following equations.

$$\begin{aligned} \frac{1}{e^2 - 1} &= \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2 + 1} \\ \frac{\pi}{e^{\pi} - 1} &= -\frac{\pi}{2} + 1 + 2 \sum_{n=1}^{\infty} \frac{1}{4n^2 + 1} \end{aligned}$$

Since the following equation can be obtained by following calculation,

$$\begin{aligned}
2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} &= 2 \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{(-1)^{n+1}}{4n^2 - 1} \\
&= 2 \lim_{k \rightarrow \infty} \sum_{n=1}^k \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \cdot \frac{(-1)^{n+1}}{2} \\
&= \lim_{k \rightarrow \infty} \left(\left(\frac{1}{1} - \frac{1}{3} \right) - \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) - \dots \pm \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \right) \\
&= \lim_{k \rightarrow \infty} \left(2 \cdot \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \pm \frac{1}{2k-1} \right) - 1 \mp \frac{1}{2k+1} \right) \\
&= 2 \lim_{k \rightarrow \infty} \left(\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \pm \frac{1}{2k-1} \right) \mp \frac{1}{2k+1} \right) - 1 \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

replacing them gives that equation.

References

- [1] Linas Vepštas *On Plouffe's Ramanujan identities*, The Ramanujan Journal. **27**(4) (2012), 387–408.