Geometry of Space, Time and Other Things

The Mathematics of Fiber Bundles

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Fiber Bundles

- Central to physics: classical mechanics, electrodynamics, quantum field theory, gravitation, superconductivity.
- It was not always that way!
- Unified (pseudo-)Riemannian Geometry (i.e. Gravitation) with Symplectic Geometry (classical mechanics) with Electrodynamcis with Yang-Mills theory with Superconductivity with Fermions (QFT)
- A single, unified framework for (almost) all of the fundamental theories of physics.
- And that is the topic today.

Forumlas and Intuition

Zen Koans

- There will will be a lot of equations today
- Several semesters worth ...
- Get familiar with commonly used widespread notation
- What does those formulas MEAN? Intuitively ??
- Interpretation of poetry, jokes of Zen koans
- Inutition alone is FAULTY. Formulas are PRECISE!
- Equations are tie-breakers for intuitive ideas
- Creativity and imagination are KEY

Tee-shirt Equations

Before fiber bundles, it was a hot mess:

- Classical mechanics was Hamilton's equations

$$\dot{p} = -\frac{dH}{dq} \qquad \dot{q} = \frac{dH}{dp}$$

- Electrodynamics was Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = 0$$

- Gravitation was Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Superconductivity was the Ginzberg–Landau equations

$$\mathscr{L} = \alpha \left|\phi\right|^2 + \beta \left|\phi\right|^4 + \frac{1}{2m} \left| \left(-i\hbar\vec{\nabla} - 2e\vec{A} \right) \phi \right|^2 + \frac{\left|\vec{B}\right|^2}{2}$$

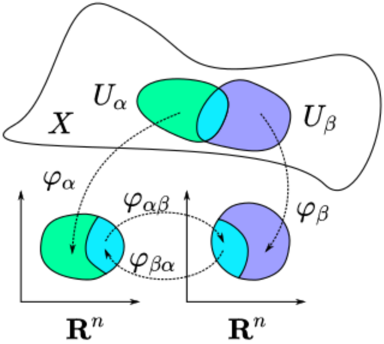
- Standard Model = Yang-Mills + Higgs + Fermions

Outline

- Manifold M as gluing of $\mathbb{R}^n$ - coordinate charts
- (Integrable) vector fields as hair/fur that can be combed
- Tangent vector space T_pM
- Back to basics: Vector spaces; notation: e_n as basis vector
- A frame field as $e_n(p)$ varying from point to point p.
- Frame fields can twist around, rotate, swirl.
- The rotation matrix A. The connection $A_i = \Gamma_{ij}^{\ k}$ aka Christoffel symbol
- Rotations & rotation matrices in 3D
- Curvature as total rotation after walking a loop.
- Parallel transport
- Geodesics

An atlas is:

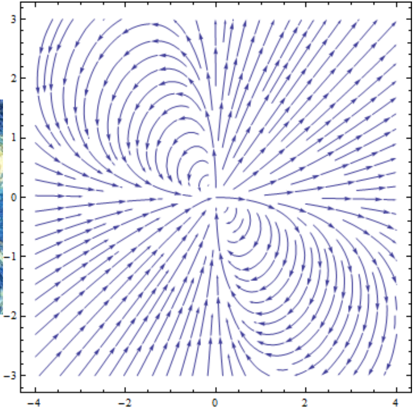
- A collection of regions U_α
- A collection of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$
- A collection of "transition functions" $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$



Vector Fields

A vector field is:

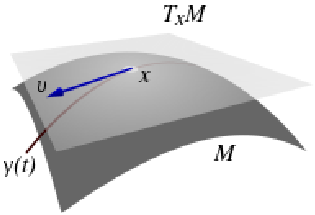
- A collection of vectors $\vec{v}_p$
- One for each point $p \in U_\alpha$
- Smooth, differentiable, integrable

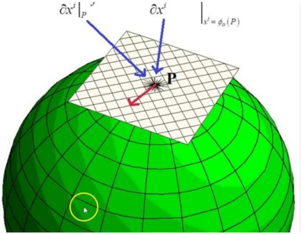


Tangent vector spaces

The tangent vector space T_pM is:

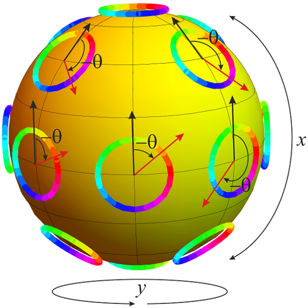
- A point $p \in U_\alpha$(that is, a point in $p \in M$)
- The collection of ALL possible vectors $\vec{v}_p \in T_pM$

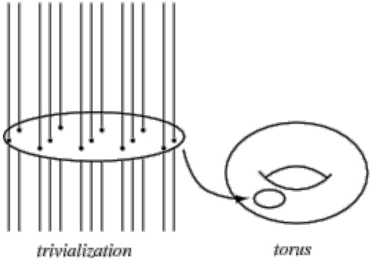




Tangent bundles - Fiber bundles

- The tangent bundle TM is the set of all T_pM for all $p \in M$
- The sphere bundle SM is a set of spheres S_pM, one for each $p \in M$
- The circle bundle is a set of circles, one for one for each $p \in M$
- The fiber bundle E is a set of fibers F, one for one for each $p \in M$



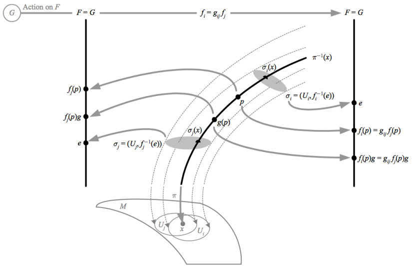


trivialization of a line bundle on a torus

Fiber Bundles

Properties of Fiber bundles

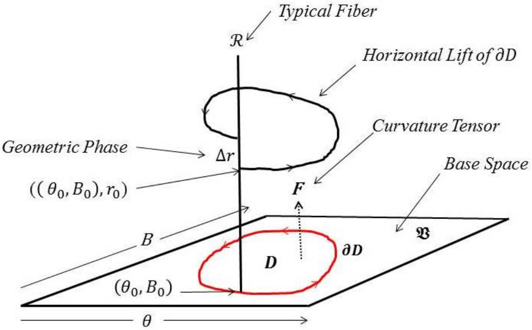
- Locally, they are trivial products $U_\alpha \times F$ of a chart U_α and a fiber F
- Neighboring fibers need to be glued (soldered) together; the connection!
- Works best when fibers have some natural symmetry
- A group G that moves you up and down a fiber F



$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}
$$

Curvature

The result of glueing is curvature



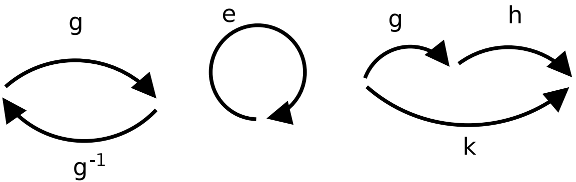
Groups and actions

Examples of Groups:

- Rotation group
- Translation group
- Permutation group

A Group G is a set where:

- Inverses: for all $g \in G \;\; \exists g^{-1} \in G$ s.t. $gg^{-1} = e$
- Identity element: $e \in G$ s.t. $\forall g \in G \;\; e \cdot g = g$
- Closure: For all $g, h \in G \, \exists k \in G$ s.t. $gh = k$



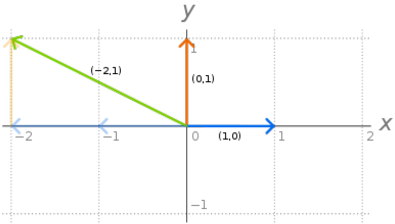
Vectors and Bases

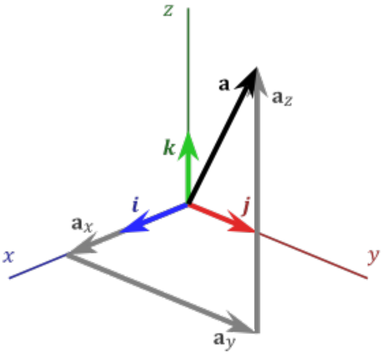
A Vector $\vec{v} \in \mathbb{R}^n$ in n-dimensional space is:

- A collection of n real numbers: $\vec{v} = (v_1, v_2, v_3, \cdots, v_n)$

A vector space basis for $\mathbb{R}^n$ is a collection of n vectors $\{\vec{e}_k\}$:

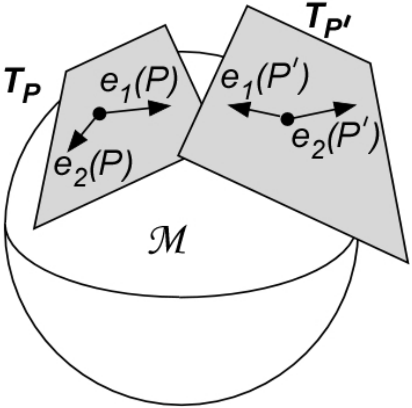
- Where $e_1 = (1, 0, 0, \cdots, 0)$ and $e_2 = (0, 1, 0, \cdots, 0)$ and $e_3 = (0, 0, 1, 0, \cdots, 0)$ and ...





A frame field is:

- A collection of basis vectors $\{\vec{e}_k(p)\}$
- One for each point $p \in U_\alpha$



Maaadf

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Geometry with Formulas

The hardest part with formulas is (1) there are so many (2) there are many different ways of writing down the *same* equations, using wildly different notation.

- Introduce Lie derivative $L_X f$
- introduce covariant derivative D = d+A - rosetta stone of different notations
- geodesics as solutions of Hamilton's equations i.e. as linear, first-order diffeq NOT second order!

$$\dot{p} = -\frac{dH}{dq} \qquad \dot{q} = \frac{dH}{dp}$$

  where H=squared-length-of-curve
- exp as the map that moves along geodesics
- geodesic completeness
- introduce metric as inner product of frame fields $g_{\mu\nu} = e_\mu \cdot e_\nu = e_\mu^a e_\nu^b \eta_{ab}$
- point out that metric was NOT needed to define curvature, geodesics, parallel transport

- (metric is almost kind-of useless except that its a standard touch-stone for GR)
- Repeat Einstein eqns.
- replace frame field by generic fiber bundle
- e.g. U(1) for electromagnetism, SU(n) for yang-mills
- Maxwell's equations are nothing more than Hamilton's eqns on U(1) + Bianchi identites

$$F = dA \quad d*F = 0$$

- Maxwell's eqn's have singularities called "electric charges" and geodesics go "splat" on an electric charge
- Swarzschild BH's are just like electric charges: geodesics go splat when they get there.
- Yang-Mills/Einstein

$$F = dA + A \wedge A \qquad D*F = 0$$

is the same as

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X,Y]$$

provide a rosetta-stone correspondance