GKW Lattice

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Abstract

The Minkowski Question Mark function is a highly singular function that has, among other properties, is able to completely alter the spectrum of an operator when it is used as a similarity transform. The goal of this paper is to formulate the tools to understand how this is possible. Specifically, it requires passing from operators defined on the Hardy space of functions defined on the real unit interval, to a space of functions defined on the Cantor set. This paper explores this formulation for the Gauss-Kuzmin-Wirsing (GKW) operator, which is the transfer operator (Ruelle operator) for the shift operator on continued fractions.

This is a diary of partial results.

1 Introduction

The observation that motivates this paper is that while the GKW operator has one spectrum in its ordinary formulation, a similarity transform based on the Minkowski Question Mark function mutates it into a different operator with a different spectrum entirely. The goal of this paper is to understand how this happens, and to provide the machinery for dealing with it.

The next observation is that the machinery of operators defined on Banach spaces over functions on the real-valued unit interval (Hardy spaces) is insufficient for the task. This is because the the singular nature of the Minkwoski Question Mark. However, the Question Mark does seem to have a natural representation on the Cantor set. Thus, the basic chore here is to examine both the GKW operator, and related operators, acting on the set of functions defined on the Cantor set, and clarifying how this space is related to the usual Hardy space setting.

To be concrete, the motivation stems from the following. Consider the Gauss Map, which is defined as the function $h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ acting on the unit interval. It drew Gauss's attention because it is the shift operator in the space of continued fractions: if one writes out the continued-fraction expansion for $x \in [0,1]$ as

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \cdots]$$

then one has that

$$h(x) = [a_2, a_3, \cdots]$$

whence the name “shift operator”. One may then consider the transfer operator formed from the Gauss Map; this is called the Gauss-Kuzmin-Wirsing or GKW operator, and is given by

$$\left[\mathscr{L}_h f\right](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right) \tag{1}$$

where f is just an ordinary function on the unit interval: $f:[0,1]\to\mathbb{R}$. This operator has not been solved in closed form, although Gauss did provide the eigenvector for the largest eigenvalue (which is 1). The next eigenvalue is known as the GKW constant, and is roughly $\lambda_1 = 0.30366300\cdots$.

Consider now a closely-related function, the Gauss Map transformed by a similarity transformation, with the Minkowski Question Mark providing the similarity transformation:

$$c(x) = ?\left(\frac{1}{?^{-1}(x)} - \left\lfloor \frac{1}{?^{-1}(x)} \right\rfloor\right) = (?\circ h \circ ?^{-1})(x)$$

This map is very simple: it consists of straight-line segments between values of $1/2^k$, and can be written as

$$c(x) = 2 - 2^n x \ \text{ for } \ \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$$

The transfer operator corresponding to this function is

$$\left[\mathscr{L}_c f\right](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\frac{2-x}{2^n}\right) \tag{2}$$

and is very different from the GKW operator: first, it is exactly solvable, in closed form, and its spectrum is different from that of the GKW operator: it is very simply $\lambda_k = 1/\left(2^{k+1}-1\right)$ *i.e.* $\lambda_0 = 1$, $\lambda_1 = 1/3$, $\lambda_2 = 1/7$, *etc.*

The surprise in all of the above is that the spectra for these two operators differ. This is a surprise because, ordinarily, a similarity transformation is unable to alter the spectrum of an operator. But the Question Mark is no ordinary function: it is quite singular; its derivative, while tricky to define correctly, is properly discontinuous-everywhere. Thus the question is posed: What is going on here, that these spectra differ, and how should the machinery be properly formed so that this strange situation can be better understood?

The answer appears to be to formulate on the Cantor set. Blah.

2 Conclusion

Work in Progress

References