(Scrapbook)

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Abstract

XXX obsolete/derecated/alive no more/get rid of this. XXX

Statements about the structure of the Cantor Set can illuminate relationships between hyperbolic geometry and the representation of real numbers by dyadic fractions and continued fractions. In particular, it is intersting to consider the symmetries of the Cantor Set, the operator representations of these symmetries, and the Hilbert spaces of functions on which these act. This is a scrapbook of statements about the Cantor Set, containing some old and well known statements, and some new research. The goal is to come to a better understanding of the structure of the real numbers are.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group, and fractals.

XXXX This paper is under construction; it is a rough, unfinished draft. XXX

1 The Cantor Set

The canonical construction of the Cantor set is to start with the topologically closed unit interval, and to progressively remove the open intervals that form the middle third of the interval. The result is a set with a variety curious topological properties; among these, that it is a closed set that is of measure zero, and yet is uncountable.

It is perhaps less commonly known that one can construct the so-called "fat Cantor sets" which are true cantor sets, but no longer have a measure of zero. The fact Cantor sets are topologically isomorphic to the traditional measure-zero Cantor set, and enjoy all of its properties. Let us review the construction of thin and fat Cantor sets.

1.1 Thin Cantor Set Construction

Rather than removing the middle third, consider removing an open segment of length q centered on the middle of the segment, with $0 < q < 1$. After removing the first segment, the remaining line segments have a length, or measure $\mu_1 = 1 - q$. At the next step, one must remove two segments. If we keep a geometric ratio, each of these two segments will have a length of $q(1-q)/2$. Thus, at the second step, there are four line segments, of total length $\mu_2 = 1 - q - 2q(1-q)/2 = (1-q)^2$. At the third step, one must remove four intervals, each of length $q(1-q)^2/4$. Repeating this process n times results in the remaining set consisting of 2^nclosed segments of total length

$\mu_n = (1-q)^n$. Understanding $\lim_{n\to\infty}\mu_n$ to be the measure of the resulting Cantor set, it is straightforward to see that the measure of this set is zero, for any value of $0 < q < 1$.

There is another, more analytic construction of this same set by means of the binary (dyadic) expansion of the real numbers. Given a real number $0 \le x \le 1$, one may represent it by means of its expansion in binary digits:

$$x = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \tag{1}$$

where each $b_k$takes on the value of 0 or 1. For most real numbers, this expansion is unique; however, the dyadic rationals, that is, rationals of the form $m/2^n$for some integers m,n have two distinct expansions. Using the series $\{b_k\}$, one then writes the corresponding normalized z-function as

$$c_x(z) = \frac{1-z}{z} \sum_{k=1}^{\infty} b_k z^k \tag{2}$$

Fixing z but varying this as a function of x, one can consider the function $f_z(x) = c_x(z)$ with $f_z : [0,1] \to [0,1]$. One finds, without too much difficulty, that the image of $f_z$is a Cantor set, and specifically, is the Cantor set where we identify $z = (1-q)/2$ with q from the previous construction. The removed open intervals of the previous construction correspond to the gaps left behind by the two distinct representations of the dyadic rationals. To be more precise, the mapping is actually injective: every real number has a unique image point, although the dyadic rationals can be assigned to two distinct points in the image. This is worth a slight bit of discussion. Let $D = \{m/2^n : m,n \in \mathbb{Z}, m,n > 0\}$ be the set of dyadic rationals. Then $f_z$will map $[0,1]\backslash D$ to open intervals contained within the Cantor set. We may then ask how to map the points in D. By always pickinng one, or the other, binary expansion for the points of D, we see that f_z is a map from the unit interval to half-open intervals contained in the Cantor set. To obtain a map onto the Cantor set, one must consider the topology of the unit interval, with each dyadic rational doubled: that is, the set $[0,1]\cup D$. Thus, in this very certain sense, we conclude that the Cantor set is isomorphic to $([0,1]\backslash D)\cup D\cup D$ considered as point sets. In particular, this mapping demonstrates that the cardinality of the Cantor set is equal to the cardinality of the unit interval (plus a bit extra, one may say).

1.2 Fat Cantor Set Construction

A fat cantor set may be constructed similarly, but removing less and less at each step. AKA Smith-Volterra-Cantor Set ...

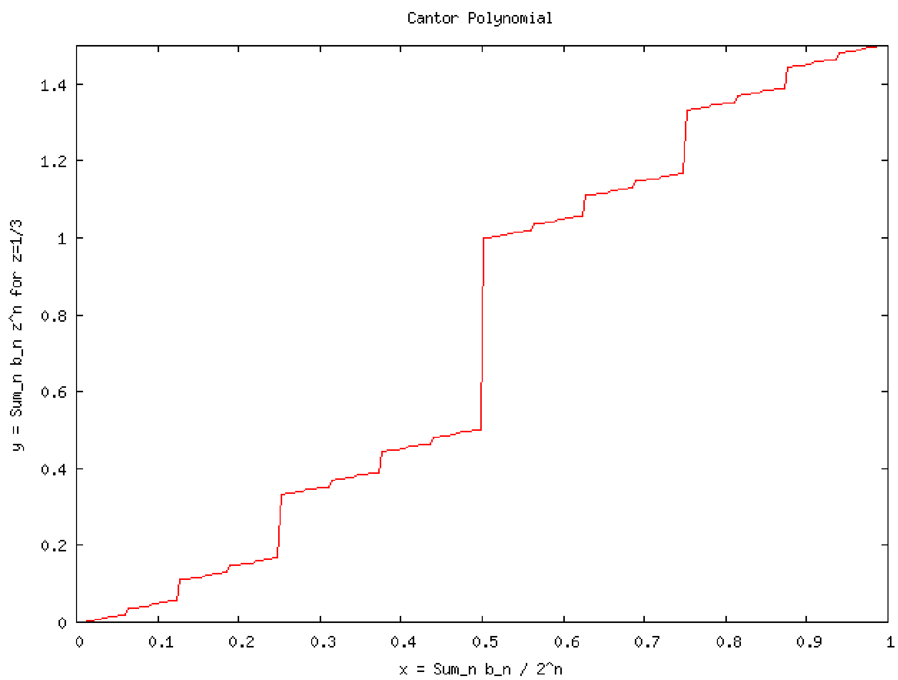
ToDo/Homework: Review topology, review the isomorphism.

by removal of open middle thi

2 Hyperbolic Rotations of Binary Trees

Note that by imposing the modular group symmetry on the real number line, we've essentially introduced a hyperbolic manifold that is homomorphic to the real-number

Figure 1: Cantor polynomial



Graph of $f_z(x)$ for $z = 1/3$. XXX this is improperly scaled on the vertical axis.

line. The existence of this hyperbolic manifold and its negative curvature essentially 'explains' why trajectories of iterated functions have positive Lyapunov exponents. Of course they do, since their 'true' trajectories should be considered to live on the hyperbolic manifold rather than on the real-number line. We try to make this clear below.

One can get a much better sense of the hyperbolic nature induced by this symmetry group by looking at the discrete 'rotations' of the binary tree. Rotations are frequently used in computer science algorithms to rebalance finite binary trees while at the same time preserving the order of the elements in the tree. We can think of a rotation as kind of like draping the flexible, droopy tree over a peg, letting gravity do its job, and declaring the node on the peg as the new root of the tree. In fact, we have to do a bit of minor surgery to get this right; we have to cut a branch and re-attach it at the free spot where the old root used to be.

Thus for example, lets rotate the dyadic tree so that the node at 1/4 becomes the new root. We do this by chopping off the tree rooted at 3/8'ths and re-attaching at as the left subtree of the old root at 1/2. (xxx we desperately need a diagram here) Denoting this rotation with the symbol θ, we have an isomorphism of trees: that is,

$$\theta\left(\frac{1}{2}\right)=\frac{1}{4}$$

$$\theta\left(\frac{1}{4}\right)=\frac{1}{8} \qquad \theta\left(\frac{3}{4}\right)=\frac{1}{2}$$

$$\theta\left(\frac{1}{8}\right)=\frac{1}{16} \quad \theta\left(\frac{3}{8}\right)=\frac{3}{16} \quad \theta\left(\frac{5}{8}\right)=\frac{3}{8} \quad \theta\left(\frac{7}{8}\right)=\frac{3}{4}$$

and the rest of the tree hanging as normal under these nodes. Its not hard to see that this rotation is order-preserving, that is, $\theta(x)<\theta(y)$ whenever $x<y$ and thus monotonic, one-to-one and onto. This is the stretch-and-shrink map

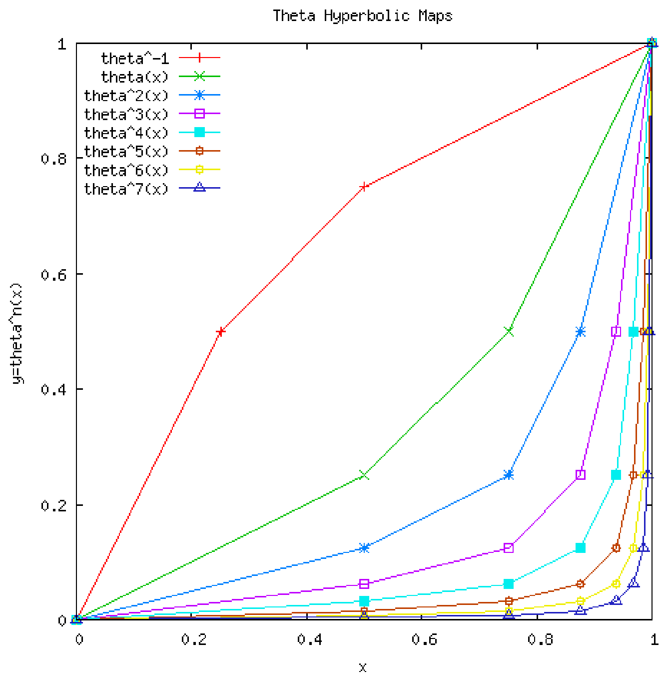
$$\theta(x)=\begin{cases} x/2 & \text{for} \quad 0\leq x\leq 1/2 \\ x-1/4 & \text{for} \quad 1/2\leq x\leq 3/4 \\ 2x-1 & \text{for} \quad 3/4\leq x\leq 1 \end{cases}$$

which is stretching in one interval and shrinking in another. Since we've seen that the Modular Group maps intervals to intervals, some stretching and some shrinking, we immediately recognize that three group elements were used to construct this map. Since we've already enumerated the modular group in terms of maps of intervals, we see that the three different group elements making up this map are the one that re-parented the 1/8 tree at 1/4, the 3/8 tree at 5/8 and the 3/4 tree at 7/8. In terms of tree surgery, it is enough to specify these three remapings, as the rest of the binary trees hanging below are structurally unaltered. Note that the 3/8'ths to 5/8'ths mapping does not change the denominator: this map neither stretches nor shrinks, its a lateral translation.

Note that the map is clearly invertible, with its inverse being

$$\theta^{-1}(x)=xxx$$

Figure 2: Theta Maps



This figure shows $\theta^n(x)$ for $n = -1$ and positive n up to 7. Marker-points highlight the endpoints of the straight line segments that make up each of these maps. Note that these maps are symmetric around the $y = 1 - x$ line, and that the endpoints of each segment lie on the hyperbola $xy = 2^{-n-2}$, with the exception of the endpoints at 0 and 1. Since each map contains more segments, each map becomes a better approximation to the hyperbola.

Notice that the shrinking portion of the map is just the inverse of the stretching portion of the map. Iterating these maps just makes them even more hyperbolic. Figure 2 shows some of these.

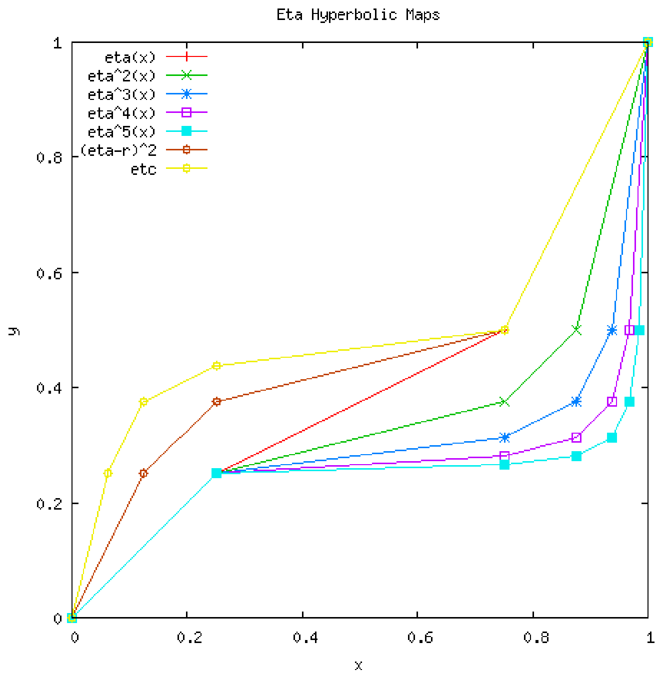
The reflection map $r(x) = 1 - x$ conjugates θ to its inverse, that is, $r\theta^n r = \theta^{-n}$.

We demonstrate a more complex mapping by rebalancing the binary tree so that the 3/8 node becomes root. This requires two tree movements: switching the 7/16 node so that it lives under 1/2, and switching the 5/16 node so that it lives under the now empty slot at 1/4. This is shown in the diagram (xxx need to create the pretty-picture of the rotated tree xxx). This rotation is given by the mapping

$$\eta(x) = \begin{cases} x & \text{for} \quad 0 \le x \le 1/4 \\ x/2 + 1/8 & \text{for} \quad 1/4 \le x \le 3/4 \\ 2x - 1 & \text{for} \quad 3/4 \le x \le 1 \end{cases}$$

A graph of this this map and its first few iterates are shown in figure 3.

Figure 3: The Eta Map



This figure shows two different hyperbolic sequences, one for the iterated eta, and also $r\eta r\eta$ and that sequence $\sqrt[4]{z}$

Looking at these, it becomes clear that one can insert hyperbolas wherever one wishes, as long as the resulting map is monotonically increasing.

And so we can see the negative curvature. we also have an explicit metric on this space. Note that this in turn induces a metric on the Stern-Brocot Tree. We can consider the path of 'geodesics' under this map, say of iterating θover and over. clearly these 'geodesics' separate.

Note that this is essentially a model of two-dimensional space-time, where the Lorentz transformations are given by elements of the modular group.

We of course are now begging to ask about 3+1 "spacetime" generated by the complex numbers, (xxx this is actually called the Picard group see Fricke and Klien, circa 1897.) which is generated by $SL(2, ZxZ)$ which is a subgroup of $GL(2, C)$. Studying how rotations work on this manifold would be interesting, as well as defining precisely its relation to Minkowski spacetime.xxx move this to the todo-list. Higher dimensional manifolds seem to be generated by quaternions and the octonians. We also can't help but take a general pot-shot and exclaim 'of course quantum mechanics is chaotic: Hamiltonian evolution takes place on a fundamentally hyperbolic manifold, viz. the Minkowski spacetime'. The wave functions will of course be fractal. In some freaky way, this is also why we get quantization: we don't truly have the general symmetries, we have instead interval-mapping symmetries that are limited to a discrete (but infinite) set of intervals.

3 Conclusions

To be written. $\{xxx\}$

References

[Man88] Benoit Mandelbrot, in *The Science of Fractal Images, ed. Heinz-Otto Peitgen, Dietmar Saupe,* (Springer-Verlag, 1988) p. 246

[deR57] Georges de Rham, On Some Curves Defined by Functional Equations (1957), reprinted in Classics on Fractals, ed. Gerald A. Edgar, (Addison-Wesley, 1993) pp. 285-298