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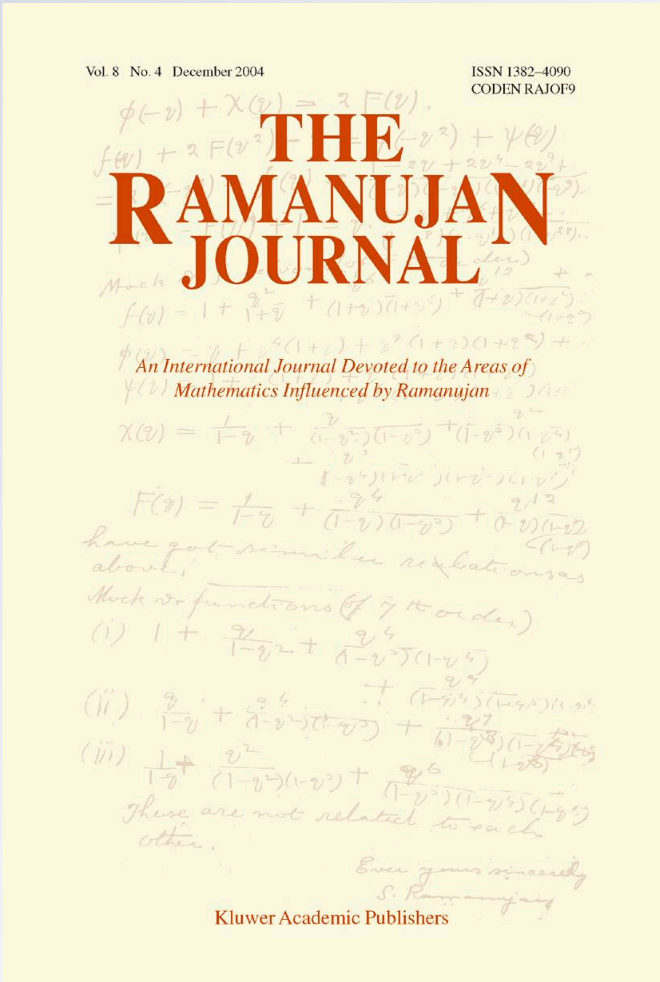

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On Plouffe's Ramanujan identities

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Abstract Recently, Simon Plouffe has discovered a number of identities for the Riemann zeta function at odd integer values. These identities are obtained numerically and are inspired by a prototypical series for Apéry's constant given by Ramanujan:

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}.$$

Such sums follow from a general relation given by Ramanujan, which is rediscovered and proved here using complex analytic techniques. The general relation is used to derive many of Plouffe's identities as corollaries. The resemblance of the general relation to the structure of theta functions and modular forms is briefly sketched.

Keywords Apéry's constant · Theta function · Modular form

Mathematics Subject Classification (2000) 11M06

1 Introduction

Inspired by an identity for $\zeta(3)$ given in Ramanujan's notebooks [4, Chap. 14, formulas 25.1 and 25.3],

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}.$$

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Plouffe describes a set of similar identities [8], [9] that were discovered numerically using arbitrary-precision software. For example, Plouffe gives an identity for $\zeta(7)$:

$$\zeta(7) = \frac{19\pi^7}{56700} - 2\sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n}-1)}.$$

This text provides an analytically derived formula for expressions of this type. The resulting general formula, valid for integer $m \geq 1$, is

$$\zeta(4m-1) = -2\sum_{n=1}^{\infty} \frac{1}{n^{4m-1}(e^{2\pi n}-1)} - \frac{1}{2}(2\pi)^{4m-1} \sum_{j=0}^{2m} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{4m-2j}}{(4m-2j)!}$$

where B_k is the kth Bernoulli number. The above is a special case of a yet more general formula, derived and presented in a later section, allowing pairs of such sums to be related. From this, one may obtain expressions such as

$$\zeta(3) = \frac{37\pi^3}{900} - \frac{2}{5}\sum_{n=1}^{\infty} \frac{1}{n^3}\left[\frac{4}{e^{\pi n}-1} + \frac{1}{e^{4\pi n}-1}\right].$$

There are an (uncountable) infinity of similar sums, each giving a different series summation for $\zeta(4m-1)$. Taking linear combinations of these, one may choose to cancel the zeta terms, to obtain summations for odd powers of π. Thus, for example, combining the above with Ramanujan's series for Apéry's constant, one gets

$$\frac{\pi^3}{180} = \sum_{n=1}^{\infty} \frac{1}{n^3}\left[\frac{4}{e^{\pi n}-1} - \frac{5}{e^{2\pi n}-1} + \frac{1}{e^{4\pi n}-1}\right].$$

Again, there are an uncountable infinity of such relations.

Plouffe also notes similar relations for the other odd integers; for example,

$$\zeta(5) = \frac{\pi^5}{294} - \frac{72}{35}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n}-1)} - \frac{2}{35}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n}+1)}.$$

The general form for this type of expression may be shown to be

$$\begin{aligned}&\left[1+(-4)^m - 2^{4m+1}\right]\zeta(4m+1)\\ &\quad = 2\sum_{n=1}^{\infty} \frac{1}{n^{4m+1}(e^{2\pi n}+1)} + 2\left[2^{4m+1} - (-4)^m\right]\sum_{n=1}^{\infty} \frac{1}{n^{4m+1}(e^{2\pi n}-1)}\end{aligned}$$

$$+ (2\pi)^{4m+1} \sum_{j=0}^{m} (-4)^{m+j} \frac{B_{4m-4j+2}}{(4m-4j+2)!} \frac{B_{4j}}{(4j)!}$$

$$+ \frac{1}{2}(2\pi)^{4m+1} \sum_{j=0}^{2m+1} (-4)^{j} \frac{B_{4m-2j+2}}{(4m-2j+2)!} \frac{B_{2j}}{(2j)!}.$$

The methods described in this text also allow for a large generalization of these types of sum. Defining

$$P_k(\tau) = \sum_{n=1}^{\infty} \frac{1}{n^k (e^{2\pi i n \tau} - 1)}$$

these generalizations follow from a modular equation relating $P_k(\tau)$ to $P_k(-1/\tau)$ for odd integers k, the derivation and proof of which is the one of the main topics of this note. The modular relation is not new; it appears in Ramanujan's Notebooks [4, Chap. 14, Entry 21] as

$$\begin{aligned} &\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k} - 1}\right\} \\ &= (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k} - 1}\right\} \\ &\quad - 2^{2n} \sum_{k=0}^{n+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2n+2-2k}}{(2n+2-2k)!} \alpha^{n+1-k} \beta^k \end{aligned}$$

where $\alpha > 0$, $\beta > 0$ with $\alpha\beta = \pi^2$ and n any positive integer. Berndt implies that this formula is the most studied of all the notebooks; it has been independently discovered perhaps a half-dozen times, and proven twice as often. It has been generalized to L-functions, and to rational values of k; Berndt provides a long list [4] of the various proofs and generalizations made.

Much of this paper is devoted to (yet another! independently discovered) proof of this relation, followed by a series of lemmas that provide the connection to Plouffe's results. In searching for curious and interesting special cases of this relation, one senses that only the tip of the iceberg has been seen. Unexplored possibilities include, for example, considering $\tau \in \mathbb{Q}[i]$, the field of Gaussian rationals, or from considering Diophantine roots of quadratics.

The rest of this paper is roughly laid out as follows: The second section provides a review of previous related results. The third section gives a relationship between the sums and the polylogarithm, and thence to an integral on the complex plane. The fourth section examines the related contour integral, which is easily integrated via Cauchy's residue theorem to give a finite sum involving the Bernoulli numbers. The fifth section relates the contour integral to the polylogarithm integral, thus resulting in a functional equation for $P_k(\tau)$. The sixth section applies the functional equation, providing a variety of lemmas, many of which explain Plouffe's discoveries. The seventh section give a pair of relationships on the Bernoulli numbers that arise naturally in this context. The eighth section explores the modular nature of the relations on $P_k(\tau)$, followed by a conclusion. An appendix gives a derivation of an integral representation of the polylogarithm, which is central to the analysis.

2 Related sums

A large number of similar sums have been explored before; this section reviews some of these. Perhaps the most forthright is a sum given by Ramanujan in a famous letter to Hardy [6], stating that

$$\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\cdots=\frac{1}{24}.$$

A generalization of this sum,

$$\sum_{n=1}^{\infty}\frac{n^{4k+1}}{e^{2\pi n}-1}=\frac{B_{4k+2}}{4(2k+1)}$$

is proved by Berndt [2], and attributed to Glaisher [5]. This and many related results are derived by Zucker [12], based on the theory of Jacobian elliptic functions. A similar result is stated by Apostol in the form of an exercise [1, see exercise 15 at end of Chap. 1]:

$$\sum_{n=1;n\text{odd}}^{\infty}\frac{n^{4k+1}}{1+e^{n\pi}}=\frac{2^{4k+1}-1}{8k+4}B_{4k+2}.$$

Many sums resembling those in this note are given by Zucker [11]. Some of these are

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi nx}-1)}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{\coth(\pi mx)-1}{m},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{2\pi nx}-1)}=\frac{1}{2}\sum_{m=1}^{\infty}(-1)^m\frac{\coth(\pi mx)-1}{m},$$

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi nx}+1)}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1-\tanh(\pi mx)}{m},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(e^{2\pi nx}+1)}=\frac{1}{2}\sum_{m=1}^{\infty}(-1)^m\frac{1-\tanh(\pi mx)}{m}.$$

The fount of inspiration for such sums is Ramanujan. Sums given in Chap. 14 of Part II of the Ramanujan's Notebooks [4] include entry 8:

$$\alpha\sum_{n=1}^{\infty}\frac{\sinh(2\alpha nk)}{e^{2\alpha^2 n}-1}+\beta\sum_{n=1}^{\infty}\frac{\sin(2\beta nk)}{e^{2\beta^2 n}-1}=\frac{\alpha}{4}\coth(\alpha k)-\frac{\beta}{4}\cot(\beta k)-\frac{k}{2}$$

and another similar one relating cos and cosh. Above, k is any positive integer, $\alpha\beta = \pi$ and $0 < \beta k < \pi$.

Entry 13 generalizes the sums mentioned previously,

$$\alpha^k \sum_{n=1}^{\infty} \frac{n^{2k-1}}{e^{2\alpha n} - 1} - (-\beta)^k \sum_{n=1}^{\infty} \frac{n^{2k-1}}{e^{2\beta n} - 1} = \left[\alpha^k - (-\beta)^k\right] \frac{B_{2k}}{4k}.$$

This time, one takes $\alpha\beta = \pi^2$and $k > 1$ an integer. The above follows from sums on the divisor function, as is frequently noted.

Sums involving pairs of Bernoulli numbers also appear in the analysis of the Dedekind eta function. Thus, Sho Iseki's transformation formula, as described by Apostol [1, see Theorem 3.5], is

$$\Lambda(\alpha, \beta, z) = \Lambda\left(1 - \beta, \alpha, \frac{1}{z}\right) - \pi z \sum_{n=0}^{2} \binom{2}{n} \frac{B_{2-n}(\alpha) B_n(\beta)}{(iz)^n}$$

where Λ is given by

$$\Lambda(\alpha, \beta, z) = \sum_{r=0}^{\infty} \left[\lambda\big(z(r + \alpha) - i\beta\big) + \lambda\big(z(r + 1 - \alpha) + i\beta\big)\right]$$

and

$$\lambda(x) = \sum_{m=1}^{\infty} \frac{e^{-2\pi m x}}{m}.$$

A sum linking the Bernoulli and Euler numbers is given by Berndt [3]:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\operatorname{sech}[(2n-1)\pi\sqrt{3}/2]}{(2n-1)^{6k+1}}$$
$$= \frac{1}{2}(-1)^{k+1} \pi^{6k+1} \sum_{m=0}^{3k} \frac{E_{2m+1}}{(2m+1)!} \frac{B_{6k-2m}}{(6k-2m)!} \cos\left[(2m+1)\frac{\pi}{3}\right].$$

Perhaps the results that are closest to those presented in this paper are those noted by Borwein et al. [7, Sect. 5], and in particular, giving a very similar result involving $\zeta(4m-1)$ and $\zeta(4m+1)$.

3 The polylogarithm

The recurring theme in Plouffe's identities is the sum

$$S_s(x) = \sum_{n=1}^{\infty} \frac{1}{n^s (e^{xn} - 1)}$$

with s usually an odd positive integer and $x = \pi$ or $x = 2\pi$ or possibly other interesting values, such as $x = \pi\sqrt{m}$ for some integer m. This sum may be converted into a sum over polylogarithms, and subsequently into an integral. The integral may, after some difficulties, be converted into a contour integral, whereupon it may be evaluated by Cauchy's residue theorem. The result is a finite sum whose general structure resembles those of Plouffe's and Ramanujan's identities. This section develops the first part of this analysis.

To find the polylogarithm, one expands

$$S_s(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{e^{-xn(m+1)}}{n^s}$$
$$= \sum_{m=1}^{\infty} \mathrm{Li}_s\left(e^{-xm}\right)$$

which is generally valid for $\Re x > 0$. The above is easily obtained by applying the expansion

$$\frac{1}{1-z} = \sum_{m=0}^{\infty} z^m$$

and the series definition of the polylogarithm:

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

The polylogarithm may be expressed in terms of an integral as

$$\mathrm{Li}_s\left(e^{-u}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\zeta(z+s)u^{-z}\,dz.$$

A derivation of this is given in the appendix. Here, $\Gamma(z) = (z-1)!$ is the classical Gamma function. The line of integration is taken to be to the right of all of the poles in the integrand, namely, $c > 1$. Using this in the summation, one obtains

$$S_s(x) = \frac{1}{2\pi i}\sum_{m=1}^{\infty}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\zeta(z+s)(xm)^{-z}\,dz$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)}{x^z}\zeta(z+s)\zeta(z)\,dz.$$

The exchange of the order of summation and integration is justified precisely when one has $c > 1$. The last integral has poles at $z = 1$ and $z + s = 1$ coming from the zeta functions and poles at all of the non-positive integers coming from the Gamma function. The last integral shows that the series is the inverse Mellin transform of $\Gamma(z)\zeta(z+s)\zeta(z)$.

If the integral can somehow be converted into a closed contour on the left, then it may be evaluated in a straight-forward way by means of Cauchy's residue theorem. Performing this closure is in fact harder than one might hope, as there are non-zero contributions to the contour from its closure. The next section evaluates the Cauchy integral, assuming that the contour can be closed. The section after that computes the contributions from closing the contour integral. Upon doing this, Plouffe's identities, and many more, become available.

4 The contour integral

Define the contour integral as

$$I_s(x) = \frac{1}{2\pi i} \oint_\gamma \frac{\Gamma(z)}{x^z} \zeta(z+s)\zeta(z)\, dz$$

where the contour γ encircles the poles at $z=1$, $z+s=1$ and $z=0,-1,-2,\ldots$ in the usual, right-handed fashion. Then, one uses Cauchy's theorem, which states that

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz$$

for simple poles, and that

$$f'(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^2}\, dz$$

for double poles. For the pole at $z=1$, one obtains the residue

$$\text{Res}(z=1) = \frac{\zeta(s+1)}{x}.$$

For the poles at $z=-n$, one obtains the residue

$$\text{Res}(z=-n) = \frac{(-x)^n}{n!}\zeta(s-n)\zeta(-n)$$

and so one has

$$I_s(x) = \text{Res}(z=1-s) + \frac{\zeta(s+1)}{x} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\zeta(s-n)\zeta(-n).$$

For s not an integer, one has

$$\text{Res}(z=1-s) = \frac{\Gamma(1-s)}{x^{1-s}}\zeta(1-s).$$

However, the interesting case is for $s=k$ a positive integer. In this case, the pole overlays another pole from the Gamma, and one has a double pole. This is just a little trickier to evaluate:

$$\frac{1}{2\pi i}\oint_{z=1-k}\frac{\Gamma(z)}{x^z}\zeta(z+s)\zeta(z)\,dz=\frac{1}{2\pi i}\oint_{z=1-k}\frac{f(z)}{(z+k-1)^2}\,dz$$
$$=\frac{d}{dz}\left[(z+k-1)^2\Gamma(z)\zeta(z+k)\frac{\zeta(z)}{x^z}\right]\bigg|_{z=1-k}.$$

To perform the derivative, one will need to use the identities

$$\frac{d}{ds}(s-1)\zeta(s)\bigg|_{s=1}=\gamma$$

where $\gamma=0.577\ldots$ is the Euler–Mascheroni constant, and

$$\frac{d}{dz}(z+n)\Gamma(z)\bigg|_{z=-n}=(-1)^n\frac{\psi(n+1)}{\Gamma(n+1)}=(-1)^n\frac{H_n-\gamma}{n!}$$

where $\psi(z)$ is the digamma function, and H_n is the nth harmonic number. Putting these together, one obtains

$$\frac{1}{2\pi i}\oint_{z=1-k}\frac{\Gamma(z)}{x^z}\zeta(z+s)\zeta(z)\,dz=\frac{(-x)^{k-1}}{(k-1)!}\big[\zeta'(1-k)+(H_{k-1}-\ln 2\pi)\zeta(1-k)\big].$$

Adding this to the other contributions, one gets

$$I_k(x)=\frac{\zeta(k+1)}{x}+\sum_{\substack{n=0\\ n\neq k-1}}^{\infty}\frac{(-x)^n}{n!}\zeta(k-n)\zeta(-n)$$
$$+\frac{(-x)^{k-1}}{(k-1)!}\big[\zeta'(1-k)+(H_{k-1}-\ln 2\pi)\zeta(1-k)\big].$$

When k is an odd integer, the above simplifies in two ways. First, $\zeta(1-k)$ vanishes, because the zeta function vanishes at all negative even integers. Similarly, the infinite sum becomes finite: when k is an odd integer, one has either $\zeta(k-n)=0$ or $\zeta(-n)=0$ for all $n>k$. Thus, for k an odd integer, one has

$$I_k(x)=\frac{\zeta(k+1)}{x}+\sum_{\substack{n=0\\ n\neq k-1}}^{k}\frac{(-x)^n}{n!}\zeta(k-n)\zeta(-n)+\frac{(-x)^{k-1}}{(k-1)!}\zeta'(1-k).$$

For the remainder of the paper, it is assumed that, in this context, $k>1$ is an odd integer, unless explicitly stated otherwise. The above evaluation of the contour integral is the main result of this section. To see that this is a key result, one may substitute $k=3$ and $x=2\pi$ to obtain

$$I_3(2\pi)=-\zeta(3)+\frac{7\pi^3}{180}$$

which should be recognizable as a portion of Ramanujan's identity. For $k = 7$, one has

$$I_7(2\pi) = -\zeta(7) + \frac{19\pi^7}{56700}$$

which resembles one of the results given by Plouffe. To complete the connection, one must relate the contour integral $I_k(x)$ to the sum $S_k(x)$. This is done in the next section.

First, however, to drive the point home, one must observe that most of the terms in the above expression are rational multiples of powers of π. This follows from the zeta function being related to the Bernoulli numbers B_n at even integers:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

for integer $n \geq 0$. At the negative values, one has

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

while the derivative is

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1)$$

for integer $n > 1$. Using these in the above expression for $I_k(x)$, and rearranging terms a bit, one gets

$$\begin{aligned} -2I_k(x) = \zeta(k)\left[1 - \left(\frac{x}{2\pi i}\right)^{k-1}\right] \\ + \frac{1}{x(k+1)!} \sum_{j=0}^{(k+1)/2} \binom{k+1}{2j} x^{2j} (2\pi i)^{k+1-2j} B_{2j} B_{k+1-2j}. \end{aligned}$$

The above introduces the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Although the imaginary number $i = \sqrt{-1}$ appears in the above, it is always squared, and thus is just a sign-keeping device. Every term in the sum is purely real.

When the contour integral is written in this form, it may now be seen that for x being any rational multiple of π, that is, $x = p\pi/q$ for any integers p, q, that the coefficient of $\zeta(k)$ is a rational number, and that the second term is another rational times π^k.

A further curiosity in this regard is noted by Plouffe: if one takes $x = \pi\sqrt{p/q}$ for integers p and q, one also gets simple expressions: because k is odd, the coefficient of $\zeta(k)$ is still a rational, and the coefficient of π^k is $\sqrt{p/q}$ times some rational. For rational x, one still sees that the sum is a rational polynomial in π^2, and for

$x = \sqrt{\pi p/q}$, one still finds that the sum is a rational polynomial in π. The ocean-full of rationals here suggest that some sort of p-adic analysis might be interesting. The appearance of the square root suggests that there is a relation to complex multiplication, or that one may have interesting results on the field of Gaussian integers.

5 Evaluating the contour integral

The goal of this section is to relate the sum $S_k(x)$ to the contour integral $I_k(x)$.

Theorem 1 *For odd integers k, one has*

$$S_k(x) = I_k(x) + (-1)^{(k-1)/2}\left(\frac{x}{2\pi}\right)^{k-1} S_k\left(\frac{4\pi^2}{x}\right).$$

For the remainder of this text, this will be referred to as the "functional equation for S_k".

Proof To prove this result, consider evaluating the contour integral $I_s(x)$ for a tall rectangular contour surrounding the poles at $z = 1, 0, -1, \ldots, 1-s$. Thus, write $I_s(x) = A + B + C + D$ with A being the integral from $c - ih$ to $c + ih$ for a constant $c > 1$ and the height h large, eventually taking the limit $h \to \infty$. That is, A forms the right hand side of the rectangular contour. In the limit of $h \to \infty$, one has by definition

$$A = S_s(x).$$

Let B and C be the top and bottom of the contour, so that for B, the integral runs from $c + ih$ to $ih - s - \varepsilon$ leftwards. For C, the integral runs rightwards from $-ih - s - \varepsilon$ to $c - ih$; here we take $\varepsilon > 0$. The integral D on the left hand side of the rectangle closes the contour, running downwards, from $ih - s - \varepsilon$ to $-ih - s - \varepsilon$.

The integrals B and C will vanish in the limit of the height $h \to \infty$. This can be easily seen after a simple change of variable:

$$\begin{aligned} B &= \frac{1}{2\pi i}\int_{ih+c}^{ih-s-\varepsilon} \frac{\Gamma(z)}{x^z}\zeta(s+z)\zeta(z)\,dz \\ &= \frac{1}{2\pi i}\int_{c}^{-s-\varepsilon} \frac{\Gamma(u+ih)}{x^{u+ih}}\zeta(s+u+ih)\zeta(u+ih)\,du. \end{aligned}$$

Much of the integrand is $\mathcal{O}(1)$ in h, or polynomially thereabouts. The integrand is dominated by the Gamma function, which, from Stirling's approximation, may be seen to be

$$\Gamma(u+ih) = \mathcal{O}\big(e^{-\pi h/2}\big).$$

The complex conjugate argument applies to C, and thus B and C vanish in the limit of $h \to \infty$.

To evaluate D, begin by writing

$$D = -\frac{1}{2\pi i}\int_{-ih-s-\varepsilon}^{ih-s-\varepsilon} \frac{\Gamma(z)}{x^z}\zeta(s+z)\zeta(z)\,dz$$
$$= -\frac{1}{2\pi i}x^{s+\varepsilon}\int_{-ih}^{ih} x^u \Gamma(-u-s-\varepsilon)\zeta(-u-\varepsilon)\zeta(-u-s-\varepsilon)\,du$$

after a change of variable $z = -u - s - \varepsilon$. One then applies the functional equations for Gamma:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

and for zeta:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$$

to obtain

$$D = \frac{1}{2\pi^2 i}\left(\frac{x}{2\pi}\right)^{s+\varepsilon}\left(\frac{1}{2\pi}\right)^{\varepsilon}\int_{-ih}^{ih}\left(\frac{x}{4\pi^2}\right)^u \frac{\sin\frac{u+\varepsilon}{2}\pi\,\sin\frac{u+s+\varepsilon}{2}\pi}{\sin(u+s+\varepsilon)\pi}$$
$$\times\,\Gamma(1+u+\varepsilon)\zeta(1+u+\varepsilon)\zeta(1+s+u+\varepsilon)\,du.$$

Another change of variable, this time as $w = 1 + u + \varepsilon$, takes the integral to a slightly more recognizable form:

$$D = -\frac{1}{\pi i}\left(\frac{x}{2\pi}\right)^{s-1}\int_{1+\varepsilon-ih}^{1+\varepsilon+ih}\left(\frac{x}{4\pi^2}\right)^w \frac{\cos\frac{w}{2}\pi\,\cos\frac{w+s}{2}\pi}{\sin(w+s)\pi}\Gamma(w)\zeta(w)\zeta(w+s)\,dw.$$

Next, by taking $s = k$ to be an odd integer, the trigonometric piece simplifies and looses its w dependence:

$$\frac{\cos\frac{w}{2}\pi\,\cos\frac{w+k}{2}\pi}{\sin(w+k)\pi} = \frac{1}{2}(-1)^{(k+1)/2}.$$

Pulling out this piece, one regains a recognizable integral, so that, in the limit $h \to \infty$, one finally reaches the claimed result:

$$D = (-1)^{(k+1)/2}\left(\frac{x}{2\pi}\right)^{k-1} S_k\left(\frac{4\pi^2}{x}\right)$$

that is,

$$I_k(x) = A + D = S_k(x) - (-1)^{(k-1)/2}\left(\frac{x}{2\pi}\right)^{k-1} S_k\left(\frac{4\pi^2}{x}\right)$$

for odd integer k. □

The next section will review the application of this and the preceding sections to specific, simple values of x, thus regaining many of Plouffe's sums.

6 Lemmas and applications

The first corollary demonstrates Plouffe's simplest sums for $\zeta(4m-1)$.

Corollary 2 *For m integer, one has*

$$I_{4m-1}(2\pi) = 2S_{4m-1}(2\pi).$$

Proof Substitute $x = 2\pi$ in the functional equation. □

This corollary provides the first concrete result of this exposition, namely that

$$2\sum_{n=1}^{\infty} \frac{1}{n^{4m-1}(e^{2\pi n}-1)} = -\zeta(4m-1) + \frac{(2\pi)^{4m-1}}{(4m)!} \sum_{j=0}^{2m} \binom{4m}{2j} (-1)^j B_{2j} B_{4m-2j}$$

which completely resolves one set of relationships given by Plouffe. The functional equation opens additional possibilities. By substituting $x = 2\pi p/q$, one obtains, for $k = 4m-1$,

$$q^{k-1} \sum_{n=1}^{\infty} \frac{1}{n^k(e^{2\pi pn/q}-1)} + p^{k-1} \sum_{n=1}^{\infty} \frac{1}{n^k(e^{2\pi qn/p}-1)} = q^{k-1} I_k\left(\frac{2\pi p}{q}\right).$$

Thus, for example, by choosing $p = 2$, $q = 1$ and $k = 3$, on obtains

$$\zeta(3) = \frac{37\pi^3}{900} - \frac{2}{5} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[\frac{4}{e^{\pi n}-1} + \frac{1}{e^{4\pi n}-1}\right]$$

and similarly, for $k = 7$,

$$\zeta(7) = \frac{409\pi^7}{94500} - \frac{2}{5} \sum_{n=1}^{\infty} \frac{1}{n^7} \left[\frac{4}{e^{\pi n}-1} + \frac{1}{e^{4\pi n}-1}\right]$$

and one may proceed in a similar manner. There are an uncountable infinity of such relations (since p/q need not be rational). One may take arbitrary linear combinations of these; or if one desires, one may subtract to cancel out zeta terms, leaving behind an (uncountable) infinity of relations for powers of π.

The next corollary shows that something more is needed to obtain identities for $\zeta(5)$, $\zeta(9)$, and so on, since the most direct approach does not give any information for such sums.

Corollary 3 *For k an odd integer, one has*

$$I_k(x) = (-1)^{(k+1)/2} \left(\frac{4\pi^2}{x}\right)^{k-1} I_k\left(\frac{4\pi^2}{x}\right).$$

Proof This may be proved by applying the functional equation twice in a row. That is, it may be proved by substituting $x \to 4\pi^2/x$ in the functional equation and then employing the result. □

Rather than encoding identities for $\zeta(4m+1)$, the above leads to a well-known identity on the Bernoulli numbers. Taking $x = 2\pi$, one obtains $I_k(2\pi) = 0$, from which one may find:

$$0 = \sum_{j=0}^{2m+1} (-1)^j \frac{B_{4m-2j+2}}{(4m-2j+2)!} \frac{B_{2j}}{(2j)!}.$$

The functional equation for S_k does not provide any statements about S_k when $k = 4m+1$. To obtain results on sums involving $k = 4m+1$, one must introduce

$$T_s(x) = \sum_{n=1}^{\infty} \frac{1}{n^s (e^{xn}+1)}.$$

Theorem 4 *One has*

$$T_s(x) = S_s(x) - 2S_s(2x).$$

Proof This may be proved by re-writing in terms of the polylogarithm, along the lines of the earlier development:

$$T_s(x) = -\sum_{m=1}^{\infty} (-1)^m \operatorname{Li}_s\left(e^{-xm}\right).$$

The even and odd terms are regrouped, as

$$T_s(x) = \sum_{m=1}^{\infty} \operatorname{Li}_s\left(e^{-xm}\right) - 2\operatorname{Li}_s\left(e^{-2xm}\right)$$

which is seen to be a sum of S_s's. □

The results for $\zeta(5)$, etc. follow from a critical observation: that

$$S_s(x + 2\pi i) = S_s(x)$$

is a periodic function. This periodicity is employed directly in the next theorem.

Theorem 5 *For positive integer m, one has*

$$S_{4m+1}(2\pi) = I_{4m+1}\left(2\pi(1+i)\right) + (-1)^m \left[\frac{T_{4m+1}(2\pi)}{4^m} + 2\cdot 4^m S_{4m+1}(2\pi) - 4^m I_{4m+1}(\pi)\right].$$

Proof Using periodicity, one writes, for $k = 4m + 1$,

$$S_k(2\pi + 2\pi i) = S_k(2\pi) = I_k\big(2\pi(1+i)\big) + 2^{(k-1)/2} e^{3i\pi(k-1)/4} S_k\big(\pi(1-i)\big).$$

The series $S_k(\pi(1-i))$ doesn't have an imaginary part; rather, it is an alternating series, which may be expanded and written as

$$S_k\big(\pi(1-i)\big) = -T_k(\pi) + \frac{T_k(2\pi) + S_k(2\pi)}{2^k}.$$

The $T_k(\pi)$ term may be eliminated by writing

$$T_k(\pi) = S_k(\pi) - 2S_k(2\pi)$$

and the $S_k(\pi)$ term may be eliminated by

$$\begin{aligned} S_k(\pi) &= I_k(\pi) + 2^{1-k} S_k(4\pi) \\ &= I_k(\pi) + 2^{-k}\big[S_k(2\pi) - T_k(2\pi)\big]. \end{aligned}$$

Performing the various substitutions suggested above proves the theorem. □

As an example of the application of the above theorem, take $m = 1$, that is, $k = 5$. One easily finds

$$I_5(\pi) = -\frac{15}{32}\zeta(5) + \frac{\pi^5}{9 \cdot 64}$$

and

$$I_5\big(2\pi(1+i)\big) = -\frac{5}{2}\zeta(5) + \frac{\pi^5}{9 \cdot 15}.$$

Combining these, one gets

$$\zeta(5) = \frac{\pi^5}{294} - \frac{2}{35}\big[T_5(2\pi) + 36 S_5(2\pi)\big]$$

which is given by Plouffe. The theorem may be used to generate similar expressions for all $\zeta(4m+1)$.

Curiously, the theorem yields results for $m = 0$ as well. In this case, one finds

$$S_1(2\pi) + T_1(2\pi) = \frac{\pi}{6} - \frac{3}{4}\log 2.$$

Many identities for π are possible by taking two different expressions for a given zeta, and subtracting them, leaving behind a rational combination of the sums and π. Thus, for example, the following theorem for Apéry's constant:

Theorem 6 *A series expression for* π^3 *is given by*

$$\pi^3 = 720 \cdot S_3(\pi) - 900 \cdot S_3(2\pi) + 180 \cdot S_3(4\pi).$$

Proof This follows by taking the general expression for $k = 4m - 1$:

$$\begin{aligned} 16^m S_{4m-1}(\pi) + 4S_{4m-1}(4\pi) = 16^m I_{4m-1}(\pi) \\ &= -\frac{1}{2}\zeta(4m-1)\left[16^m + 4\right] \\ &\quad - (2\pi)^{4m-1} \sum_{j=0}^{2m} (-4)^j \frac{B_{4m-2j}}{(4m-2j)!} \frac{B_{2j}}{(2j)!} \end{aligned}$$

solving for $\zeta(4m-1)$ and then using

$$2S_{4m-1}(2\pi) = I_{4m-1}(2\pi) = -\zeta(4m-1) + (2\pi)^{4m-1} \sum_{j=0}^{2m} (-1)^j \frac{B_{4m-2j}}{(4m-2j)!} \frac{B_{2j}}{(2j)!}$$

to eliminate the appearance of the zeta. The resulting expression may then be solved for π^{4m-1}. □

Relationships involving square roots also arise naturally.

Theorem 7 *One has*

$$\zeta(3) = \frac{5\pi^3}{72} - \frac{1}{2} S_3(2\pi\sqrt{3}) - \frac{3}{2} S_3\left(\frac{2\pi\sqrt{3}}{3}\right).$$

Proof It follows from the general expression given earlier that

$$S_k\left(\frac{2\pi p}{q}\right) + \left(\frac{p}{q}\right)^{k-1} S_k\left(\frac{2\pi q}{p}\right) = I_k\left(\frac{2\pi p}{q}\right).$$

Here, making the substitution $q = \sqrt{p}$ one obtains

$$\begin{aligned} S_k(2\pi\sqrt{p}) + p^{(k-1)/2} S_k\left(\frac{2\pi\sqrt{p}}{p}\right) &= -\frac{1}{2}\zeta(k)\left[1 - (-p)^{(k-1)/2}\right] \\ &\quad + \frac{(-1)^{(k-1)/2}}{2\sqrt{p}}(2\pi)^k \\ &\quad \times \sum_{j=0}^{(k+1)/2} (-p)^j \frac{B_{k+1-2j}}{(k+1-2j)!} \frac{B_{2j}}{(2j)!}. \end{aligned}$$

The specific result follows after choosing $k = 3$ and $p = 3$. □

7 Some Bernoulli number identities

In addition to the previously noted identity

$$0 = \sum_{j=0}^{2m+1} (-1)^j \frac{B_{4m-2j+2}}{(4m-2j+2)!} \frac{B_{2j}}{(2j)!}$$

there are several other identities on sums of Bernoulli numbers that result from the previous developments. These are briefly stated here.

Theorem 8 *For integer m, one has*

$$0 = \sum_{j=0}^{m} (-4)^j \left[\frac{B_{4m-4j+2}}{(4m-4j+2)!} \frac{B_{4j}}{(4j)!} + 2 \frac{B_{4m-4j}}{(4m-4j)!} \frac{B_{4j+2}}{(4j+2)!} \right].$$

Proof Consider the sums resulting from $0 = I_{4m+1}(2\pi(1+i)) - I_{4m+1}(2\pi(1-i))$. □

Theorem 9 *For integer m, one has*

$$\sum_{k=0}^{2m} (-1)^k \frac{B_{4m-2k}}{(4m-2k)!} \frac{B_{2k}}{(2k)!} = \sum_{j=0}^{m} (-4)^j \frac{B_{4m-4j}}{(4m-4j)!} \frac{B_{4j}}{(4j)!} - 2 \sum_{j=0}^{m-1} (-4)^j \frac{B_{4m-4j-2}}{(4m-4j-2)!} \frac{B_{4j+2}}{(4j+2)!}.$$

Proof Consider the sums resulting from the identity

$$I_{4m-1}(2\pi) = I_{4m-1}\big(2\pi(1+i)\big) + I_{4m-1}\big(2\pi(1-i)\big).$$ □

8 Modular relations

The various sums and quantities above can be seen to be quasi-modular by making a simple change of variable, namely by making the substitution $x = 2\pi i\tau$. By "quasi-modular", it is meant that the various terms almost have simple behaviors under the Möbius transformation $\tau \to (a\tau + b)/(c\tau + d)$ for integer a, b, c and d. In this respect, the sums bear close resemblance to theta functions, which obey similar relations. These relationships are brought to focus here.

First, define $K_k(\tau) = I_k(2\pi i\tau)$. This change of variable results in a definition which seems simpler than that for I_k. An expansion in τ is often referred to as a "Fourier series" in the context of hyperbolic geometry:

$$K_k(\tau) = \frac{\tau^{k-1} - 1}{2} \zeta(k) - \frac{(2\pi i)^k}{2\tau} \sum_{j=0}^{(k+1)/2} \tau^{2j} \frac{B_{2j}}{(2j)!} \frac{B_{k+1-2j}}{(k+1-2j)!}.$$

This quantity is almost a modular form of weight $k-1$, in that

$$K_k\left(\frac{-1}{\tau}\right) = -\tau^{1-k} K_k(\tau).$$

Its only "almost" a modular form, because it is not periodic in τ, that is,

$$K_k(\tau+1) \neq K_k(\tau).$$

By contrast, $P_k(\tau) = S_k(2\pi i\tau)$ is periodic:

$$P_k(\tau+1) = P_k(\tau)$$

but is not quite modular under inversion:

$$P_k\left(\frac{-1}{\tau}\right) = \tau^{1-k} P_k(\tau) + K_k\left(\frac{-1}{\tau}\right).$$

One may define a simple variant of the sums that does have a simple transformation under inversion, namely

$$M_k(\tau) = P_k(\tau) - \frac{1}{2} K_k(\tau)$$

which transforms as

$$M_k\left(\frac{-1}{\tau}\right) = -\tau^{1-k} M_k(\tau).$$

However, M_k is then not periodic.

The analytic structure of P_k is curious: it has a pole at $\tau = i\infty$ and, by the inversion formula and periodicity, at every rational value of τ. This is clearly visible in Figs. 1 and 2, which show P_{-1} and P_{-5} on the q-series or "punctured disk" coordinates $q = e^{2\pi i\tau}$:

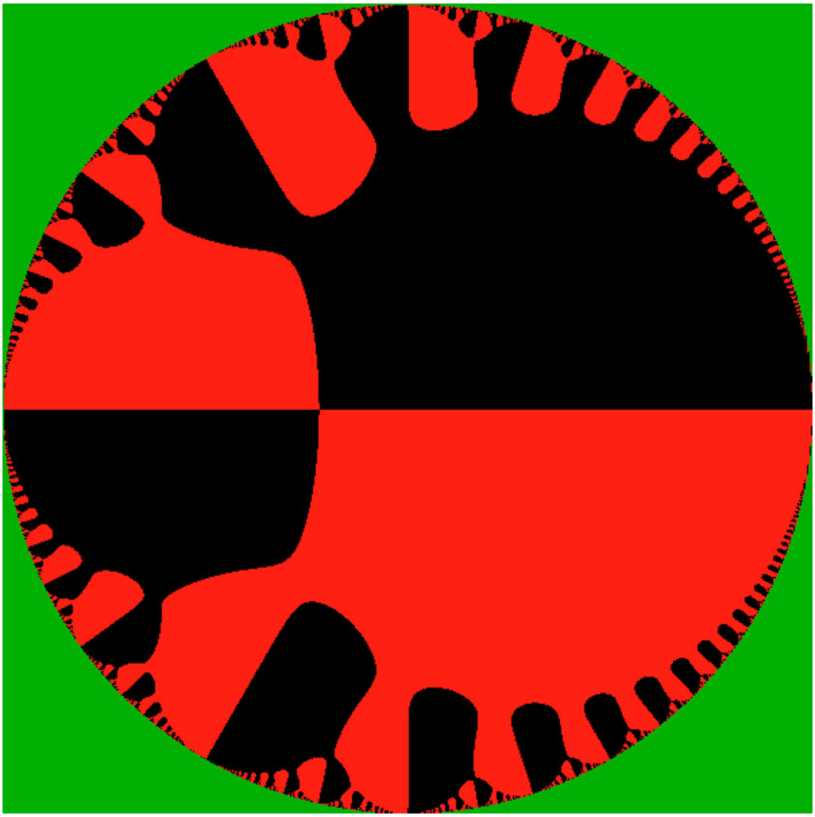
$$P_s(q) = \sum_{n=1}^{\infty} \frac{1}{n^s(q^n-1)}.$$

In this form, the relation to modular forms becomes the most immediate. Consider the Lambert series:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \frac{q^n}{1-q^n} = -\zeta(s) - P_s(q) = \sum_{m=1}^{\infty} \sigma_{-s}(m) q^m.$$

Here, $\sigma_s(m)$ is the divisor function:

$$\sigma_s(m) = \sum_{n|m} n^s$$



This graphic shows the phase

$$\arg P_{-1}(q) = \arg \sum_{n=1}^{\infty} \frac{n}{q^n - 1}$$

where $\arg f(z) = \Im \log f(z)$ is the usual arg of a function. The color scheme is such that black represents areas where arg > 0 and red represents areas where arg < 0. The absence of other colors indicates that the phase is rather closely confined to the vicinity of 0 for most all of the disk. Numerically, the absolute value of the phase is smaller than 10^{-3} for much of the disk. In particular, this indicates that there are no zeros at all in the interior of the disk, as a zero would be surrounded by a region where the phase wraps around by 2π. The function $P_s(q)$ does have poles at $q = e^{2\pi i m/n}$ for all rationals m/n; these are visible at the edges of the disk. The fractal nature of this image is the characteristic signature of a modular form of weight 2; the self-similar regions are just copies of the fundamental region of the modular group $SL(2, \mathbb{Z})$.

Fig. 1 Phase of $P_{-1}(q)$ on the unit disk

with the notation $n|m$ denoting that the sum extends over all divisors n of m. This should be compared to the Eisenstein series [1, see Sect. 3.10]

$$G_{2k}(q) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m.$$

This graphic shows the phase

$$\arg P_{-5}(q) = \arg \sum_{n=1}^{\infty} \frac{n^5}{q^n - 1}.$$

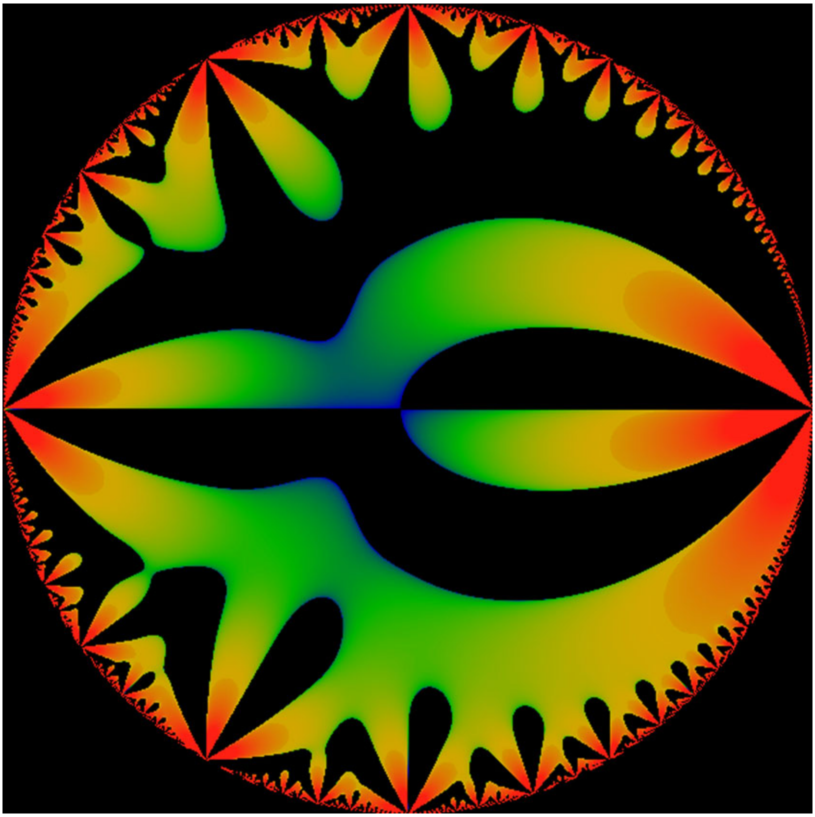
The color scheme is as in Fig. 1, and similar conclusions apply: there are no zeros at all in the interior of the disk. The absolute value of the phase is tiny: numerically, it is within 10^{-9} of zero for much of the disk. The fractal nature of this image is the characteristic signature of a modular form of weight 6; it should be compared to the image of the modular invariant g_3 shown in Fig. 3.

Fig. 2 Phase of $P_{-5}(q)$ on the unit disk

That is, the sums P_s can be re-written in terms of the Eisenstein series G_{1-s}, and the behavior of one under the action of the modular group can be given in terms of the other.

9 Conclusions

Most of the sums discussed here are suggestive of linear algebra. So for example, one may write the sum S_k as the dot-product between the (infinite-dimensional) vector n^{-k} and the vector $(e^{2\pi n} - 1)^{-1}$. The evaluation of these sums forms a regu-



This figure shows the imaginary part of the modular invariant g_3, one of the invariants of an elliptic curve. Specifically, it shows the imaginary part of

$$g_3(q) = \frac{8\pi^6}{27}\left[1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}\right]$$

which is a modular form of weight 6. The colors are chosen such that black represents areas that are negative, blue and green represent areas with smaller values, and red those areas with the largest values.

Fig. 3 Graph of the modular invariant g_3

lar pattern in k, suggesting that, for example, n^{-k} could be taken to be the matrix elements of some linear operator. However, the significance of this operator (aside from assorted shallow results and relations) is completely unclear. The sums over the Bernoulli numbers are even reminiscent of some crazy Atiyah–Singer-like indexing; but the underlying operators are utterly unclear. Put another way, it is well-known in physics and mathematics that regular patterns are the result of symmetries; the sums discussed here form a regular pattern, but the nature of the symmetry that generates it is unclear. During manipulations, one gets the sense that there are plenty of other relations to be discovered; certainly, the mere heft of Ramanujan's tomes suggest as much. Yet, there is no picture of a generator that can be operated to generate the myriad of relations; some symmetry group presentation seems to be missing.

A similar problem exists in the theory of hypergeometric series, where there is an embarrassment of riches in terms of relations and identities, and yet a unifying theory is lacking. The ingredients to the sums discussed here include the Gamma function and the Bernoulli polynomials; among many other properties, these have a common set of relations in the p-adic "multiplication theorems"; the sums here vaguely resemble the multiplication theorems of characteristic zero. There are also similar phenomena and sums that occur in the theory of dynamical systems, and in particular, in symbolic dynamics; there, a group structure, or at least, a monoid structure, together with an explicit treatment in terms of linear operators, is more common. Any of these connections present intriguing avenues for future research; however, the overall problem, of discovering the underlying symmetry that leads to such relations, seems unattainably hard to solve.

Acknowledgement Thanks to Simon Plouffe for generating interest in such sums.

Appendix: Polylog integral

This appendix proves the following theorem:

Theorem 10 *The polylogarithm may be written as the integral*

$$\mathrm{Li}_s\left(e^{-u}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(r)\zeta(r+s)u^{-r}\,dr.$$

Proof The proof below is cribbed from the Wikipedia article on polylogarithms [10]. One begins by writing the Mellin transform of the polylog as

$$M_s(r) = \int_0^\infty \mathrm{Li}_s\left(ye^{-u}\right)u^r\,\frac{du}{u}.$$

Using an integral representation of the polylogarithm,

$$\mathrm{Li}_s(w) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{w^{-1}e^t - 1}\,dt$$

and substituting, one obtains

$$M_s(r) = \frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{u^{r-1}t^{s-1}}{y^{-1}e^{t+u} - 1}\,dt\,du.$$

A change of variable $t = ab$ and $u = a(1-b)$ with $dt\,du = a\,da\,db$ gives

$$\begin{aligned} M_s(r) &= \frac{1}{\Gamma(s)}\int_0^1 b^{s-1}(1-b)^{r-1}\,db\int_0^\infty \frac{a^{r+s-1}}{y^{-1}e^a - 1}\,da \\ &= \Gamma(r)\,\mathrm{Li}_{r+s}(y). \end{aligned}$$

The inverse Mellin transform may now be employed to write

$$\mathrm{Li}_s\left(ye^{-u}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{-r}\,\Gamma(r)\,\mathrm{Li}_{r+s}(y)\,dr.$$

By setting $y = 1$, one then uses $\mathrm{Li}_{s+r}(1) = \zeta(s+r)$ to obtain the desired result. □

References

1. Apostol, T.M.: Modular Functions and Dirichlet Series in Number Theory, 2nd edn. Springer, Berlin (1990)
2. Berndt, B.C.: Modular transformations and generalizations of several formulae of Ramanujan. Rocky Mt. J. Math. **7**, 147–189 (1977)
3. Berndt, B.C.: Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan. J. Reine Angew. Math. **303/304**, 332–365 (1978)
4. Berndt, B.C.: Ramanujan's Notebooks, Part II. Springer, Berlin (1989). ISBN 0-387-96794-X
5. Glaisher, J.W.L.: On the series which represent the twelve elliptic and the four zeta functions. Mess. Math. **18**, 1–84 (1889)
6. Hardy, G.H.: Obituary notice. In: Proceedings of the Royal Society, Series A, vol. 99, pp. xiii–xxix (1921)
7. Crandall, R.E., Borwein, J.M., Bradley, D.M.: Computational strategies for the Riemann zeta function. J. Comput. Appl. Math. **21**, 11 (2000). http://www.maths.ex.ac.uk/mwatkins/zeta/borwein1.pdf
8. Plouffe, S.: Identities Inspired from Ramanujan Notebooks II. http://www.lacim.uqam.ca/plouffe/identities.html, 21 July 1998
9. Plouffe, S.: Identities Inspired by Ramanujan Notebooks (Part 2). http://www.lacim.uqam.ca/plouffe/inspired2.pdf, April 2006
10. Wikipedia. Polylogarithm (2006). http://en.wikipedia.org/wiki/Polylogarithm
11. Zucker, I.J.: Some infinite series of exponential and hyperbolic functions. SIAM J. Math. Anal. **15**(2), 406–413 (1984)
12. Zucker, I.J.: The summation of series of hyperbolic functions. SIAM J. Math. Anal. **10**(1), 192–206 (1979)