From: Philippe.Flajolet@inria.fr
To: Linas Vepstas (linas@lians.org)
Subject: Differences of Zeta values
Date: February 27, 2006

Dear Linas,

For convergence acceleration of our joint paper, I include some half-baked thoughts on the Baez-Duarte criterion. My impression is that it might not fit in that well.

While glancing through your note regarding ''Norlund-Rice candidates'', I was under the impression that you were trying to apply saddle point. As a matter of fact what I had in mind was taking residues of poles (arising from 1/zeta) into account.

I write below what I think is a simple direct proof of criteria in the style of Baez-Duarte. The method is quite accomodating with respect to variations in the denominator.

This write up makes me doubt now that this topic is mainstream with respect to our general purpose = Nörlund + zeta + saddle point.

Best, Philippe

ON BAEZ-DUARTE'S CRITERION

Define

$$B_n := \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{\zeta(k)}.$$

Given the absolutely convergent representation,

$$\zeta(s) = \sum_{\ell=1}^{\infty} \frac{\mu(\ell)}{\ell^s},$$

at $s = 2, \ldots, n$, it is possible to reorganize the terms in B_n, to the effect that

$$B_n = \sum_{2\le k\le n;\ 1\le\ell} \mu(\ell) \binom{n}{k} \frac{(-1)^k}{\ell^k},$$

that is,

$$B_n = \sum_{\ell\ge1} \mu(\ell) \left[\left(1 - \frac{1}{l}\right)^n - 1 + \frac{n}{\ell} \right].$$

Note that the general term in the last expression is, for large ℓ, of order $n\ell^{-2}$, which is summable.

We propose to compare B_n with $C(n)$, where the function $C(x)$ is defined for $x \ge 0$ by

$$C(x) := \sum_{\ell\ge1} \mu(\ell) \left[e^{-x/\ell} - 1 + \frac{x}{\ell} \right].$$

I claim that

$$B_n - C(n) = O(\log n).$$

Indeed set $\ell_0(x) = x/\log x$. We write $B_n - C(n) = S_< + S_\geq$, where

$$S_< := \sum_{\ell < \ell_0(n)} \mu(\ell) \left[\left(1 - \frac{1}{\ell}\right)^n - e^{-n/\ell} \right]$$

This sum comprises $\ell_0(n)$ terms each of order $1/n$, so that

$$S_< = O\left(\frac{1}{\log n}\right).$$

The complementary sum $S_\geq$ satisfies

$$\begin{aligned} S_\geq &= \sum_{\ell \geq \ell_0} \mu(\ell) e^{-n/\ell} \left[\exp\left(n \log(1 - 1/\ell) + n/\ell\right) - 1\right] \\ &= \sum_{\ell \geq \ell_0(n)} O\left(\frac{n}{\ell^2}\right) = O\left(\frac{n}{\ell_0(n)}\right) = O(\log n). \end{aligned}$$

Thus, by an elementary calculation, it suffices to investigate the behaviour of the real function $C(x)$.

Now the function $C(x)$ is $O(x)$ when $x \to \infty$ (split the sum according to $\ell_1(x) = x$) and $O(x^2)$ when $x \to 0$. Thus, its Mellin transform,

$$C^*(s) = \int_0^\infty C(x) x^{s-1}\, dx,$$

is defined in the strip $-2 < \Re(s) < -1$ (at least), and its value is easily computed:

$$C^*(s) = \frac{\Gamma(s)}{\zeta(-s)}, \qquad -2 < \Re(s) < -1.$$

Lemma 1. *Under the Riemann hypothesis (RH), one has, for any $\epsilon > 0$, the estimate*

$$C(x) = O\left(x^{1/2+\epsilon}\right), \qquad x \to +\infty.$$

Proof. Start from the usual integral representation of the inverse Mellin transform,

$$C(x) = \frac{1}{2i\pi} \int_{-c-i\infty}^{-c+i\infty} \frac{\Gamma(s)}{\zeta(-s)} x^{-s}\, ds,$$

with $c = -3/2$ say. Under RH, it is known that

$$\frac{1}{\zeta(s)} = O(t^\delta),$$

for any $\delta > 0$, with $\frac{1}{2} < \Re(s) < 1$ constant and $\Im(s) = t$. (See [Titchmarsh], §14.2, p. 337.) Thus, we can safely move the line of integration in the inverse Mellin integral to $c = -\frac{1}{2} - \epsilon$. The inverse Mellin integral is absolutely convergent at this value of c, and a factor of the form $O(x^{1/2+\epsilon})$ comes out of the integral. QED

Lemma 2. *For any fixed $\epsilon > 0$, the estimate*

$$C(x) = O\left(x^{1/2+\epsilon}\right), \qquad x \to +\infty$$

implies that $\zeta(s)$ has no zero in $\Re(s) > \frac{1}{2} + \epsilon$.

Proof. Write $C^*(s) = C_0^*(s) + C_\infty^*(s)$, where

$$C_\infty(s) = \int_1^\infty C(x) x^{s-1}\, dx.$$

The function $C_0^*(s)$ is analytic in the right half-plane $\Re(s) > -2$. The function $C_\infty^*(s)$ is analytic in the left half-plane $\Re(s) < -\frac{1}{2} - \epsilon$, given the growth assumption on $C(x)$. In that case, $\zeta(-s)$ can have no zero in $\Re(s) < -\frac{1}{2} - \epsilon$.

Note. I have decided to go through the standard Mellin transform after an approximation (of B_n by $C(n)$). It should be possible to operate directly with the Nörlund-Rice integral representation of the original sum B_n.

Also, since there are areas where $1/\zeta$ is guaranteed not to be too small, we should have

$$C(x) = \sum \frac{\Gamma(-\rho)}{\zeta'(\rho)} x^\rho + \text{small}\,.$$

See the similar discussion in [Titchmarsh], $9.8, p. 219 (which is *unconditional*—no RH needed!).

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