Integral representation of differences of zeta values

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Abstract

[Part of my notes.] We present exact representations of the nth divided differences of certain values of the Riemann and Hurwitz zeta functions. The dominant part of these sums is given by a digamma function contribution and we discuss how further asymptotic results may be obtained.

Key words and phrases

Riemann and Hurwitz zeta functions, binomial coefficient, asymptotic form, alternating binomial sum, Pochhammer symbol

Integral representation of differences of zeta values

Let $\zeta(s)$ and $\zeta(s,a)$ be the Riemann and Hurwitz zeta functions, respectively. Let for $\kappa \neq 0$ and Re $b > 1/2$

$$S_n(\kappa,b) \equiv \sum_{j=2}^n (-1)^j \binom{n}{j} \kappa^j \zeta(bj), \tag{1}$$

and for Re $a > 0$

$$S_n(\kappa,b,a) \equiv \sum_{j=2}^n (-1)^j \binom{n}{j} \kappa^j \zeta(bj,a). \tag{2}$$

Let ${}_pF_q$ be the generalized hypergeometric function. We show and then discuss

Proposition 1. (a) For $b \geq 1$ an integer we have

$$S_n(\kappa,b) = \frac{\kappa n}{(b-1)!} \int_0^\infty \frac{e^{-t} t^{b-1}}{(1-e^{-t})} \left[1 - {}_1F_b\left(1-n; \frac{b+1}{b}, \frac{b+2}{b}, \ldots, \frac{2b-1}{b}, 2; \frac{\kappa}{b^b} t^b\right)\right] dt, \tag{3}$$

and (b)

$$S_n(\kappa,b,a) = \frac{\kappa n}{(b-1)!} \int_0^\infty \frac{e^{-at} t^{b-1}}{(1-e^{-t})} \left[1 - {}_1F_b\left(1-n; \frac{b+1}{b}, \frac{b+2}{b}, \ldots, \frac{2b-1}{b}, 2; \frac{\kappa}{b^b} t^b\right)\right] dt. \tag{4}$$

Let L_n^α be the associated Laguerre polynomial of degree n. We then obtain

Corollary 1. We have the special cases (a)

$$S_n(\kappa,1) = \kappa \int_0^\infty [n - L_{n-1}^1(\kappa t)] \frac{dt}{e^t - 1}, \tag{5}$$

and (b)

$$S_n(\kappa,1,a) = \kappa \int_0^\infty \frac{e^{-at}}{1-e^{-t}} [n - L_{n-1}^1(\kappa t)] dt. \tag{6}$$

The proof of Proposition 1 uses

Lemma 1. For $b \geq 1$ an integer we have

$$\sum_{j=2}^{n}(-1)^j \binom{n}{j} \frac{\kappa^j}{\Gamma(bj)} t^{bj-1} = \frac{\kappa n}{(b-1)!} t^{b-1} \left[1 - {}_1F_b\left(1-n; \frac{b+1}{b}, \frac{b+2}{b}, \ldots, \frac{2b-1}{b}, 2; \frac{\kappa}{b^b} t^b\right)\right], \tag{7}$$

where Γ is the Gamma function.

Proof of Lemma 1. We first express the binomial coefficient as a Pochhammer symbol $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$. We have

$$\binom{n}{j+1} = (-1)^{j+1}\frac{(-n)_{j+1}}{(j+1)!} = \frac{(-1)^{j+1}}{j!}(1-n)_j \frac{n}{(j+1)}. \tag{8}$$

We recall that

$$(b+1)_{bj} = (b^b)^j \prod_{\ell=1}^{b} \left(\frac{b+\ell}{b}\right)_j \equiv (b^b)^j \left(\frac{b+1}{b}\right)_j \left(\frac{b+2}{b}\right)_j \cdots \left(\frac{2b-1}{b}\right)_j (2)_j, \tag{9}$$

and

$$(b+1)_{bj} \equiv \frac{\Gamma[b(j+1)+1]}{\Gamma(b+1)} = b(j+1)\frac{\Gamma[b(j+1)]}{b!}, \tag{10}$$

so that

$$\frac{1}{\Gamma[b(j+1)]} = \frac{b(j+1)}{b!} \frac{1}{(b^b)^j \prod_{\ell=1}^{b}\left(\frac{b+\ell}{b}\right)_j}. \tag{11}$$

By a shift of the summation index on the left side of Eq. (7), the use of Eqs. (8) and (11), and the application of the series definition of ${}_pF_q$ the Lemma follows.

Proof of Proposition 1. Part (a) follows from the definition of the series (1), a standard integral representation of $\zeta(s)$, and Lemma 1. Similarly, (b) follows from definition (2), Lemma 1, and the representation valid for Re $s > 1$ and Re $a > 0$

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(a-1)t}}{e^t - 1} dt. \tag{12}$$

At $b=1$, Corollary 1 follows since

$$n{}_1F_1(1-n,2;w) = ne^w\ {}_1F_1(n+1,1-n;w) = L_{n-1}^1(w), \tag{13}$$

where the middle expression here represents Kummer's first transformation and ${}_1F_1$ is the confluent hypergeometric function.

There are several alternative integral representations for $S_n(\kappa,b)$ and $S_n(\kappa,b,a)$ other than those obtainable from changes of variable in Proposition 1. As an example we have

Lemma 2. Let $\sigma(x)$ be the Stirling polynomials and B_n the Bernoulli numbers. Then we have (a)

$$S_n(\kappa,b) = \frac{\kappa n}{(b-1)!}\left\{\sum_{j=0}^{\infty}\sigma_j(1)\int_0^{2\pi} t^{b+j-2}e^{-t}\left[1-\ {}_1F_b\left(1-n;\frac{b+1}{b},\frac{b+2}{b},\ldots,\frac{2b-1}{b},2;\frac{\kappa}{b^b}t^b\right)\right]dt\right.$$
$$\left.+\int_{2\pi}^{\infty}\frac{e^{-t}t^{b-1}}{(1-e^{-t})}\left[1-\ {}_1F_b\left(1-n;\frac{b+1}{b},\frac{b+2}{b},\ldots,\frac{2b-1}{b},2;\frac{\kappa}{b^b}t^b\right)\right]dt\right\}, \tag{14}$$

and (b)

$$S_n(\kappa,b) = \frac{\kappa n}{(b-1)!}\left\{\sum_{j=0}^{\infty}\frac{B_j}{j!}\int_0^{2\pi} t^{b+j-2}\left[1-\ {}_1F_b\left(1-n;\frac{b+1}{b},\frac{b+2}{b},\ldots,\frac{2b-1}{b},2;\frac{\kappa}{b^b}t^b\right)\right]dt\right.$$
$$\left.+\int_{2\pi}^{\infty}\frac{t^{b-1}}{(e^{t}-1)}\left[1-\ {}_1F_b\left(1-n;\frac{b+1}{b},\frac{b+2}{b},\ldots,\frac{2b-1}{b},2;\frac{\kappa}{b^b}t^b\right)\right]dt\right\}. \tag{15}$$

Proof of Lemma 2. We apply the generating functions for the functions $1/(1-e^{-t})$ and $1/(e^t-1)$ that are valid for $|t|<2\pi$.

We previously showed:

Proposition 2. Let

$$d_n(p) = \frac{1}{n!}\frac{d^n}{dz^n}\ln\Gamma\left[\frac{1}{p(1-z)}\right]_{z=0}, \qquad p\neq 0. \tag{16}$$

Then

$$d_n(p) = \frac{1}{p}\psi\left(\frac{1}{p}\right) - \frac{1}{pn}\int_0^\infty \frac{e^{(1-1/p)t}}{(e^t-1)}\left[L_{n-1}^1\left(\frac{t}{p}\right) - n\right]dt, \qquad n \geq 1, \tag{17}$$

and $d_1(p) = \psi(1/p)/p$ and $\psi = \Gamma'/\Gamma$ is the digamma function. In particular,

$$d_n(1) = -\gamma - \frac{1}{n}\int_0^\infty \frac{\left[L_{n-1}^1(t) - n\right]}{(e^t-1)}dt, \qquad n \geq 1, \tag{18}$$

and $d_1(1) = -\gamma$, where γ is the Euler constant. Moreover,

$$d_n(p) = \frac{1}{p}\psi\left(\frac{1}{p}\right) + \frac{1}{n}\sum_{j=0}^\infty\left[\frac{n}{pj+1} - 1 + \left(\frac{jp}{jp+1}\right)^n\right] \tag{19a}$$

$$\geq \frac{1}{p}\psi\left(\frac{1}{p}\right) + \frac{1}{p}[\psi(n) + \gamma - 1] + \frac{1}{np}. \tag{19b}$$

Remark. In the case of positive integer argument, digamma function values are immediately expressible as harmonic numbers H_n: $H_{\ell-1} = \psi(\ell) + \gamma$.

Therefore, we have the relation

$$d_n(1/\kappa) = \kappa\psi(\kappa) + \frac{1}{n}\left[S_n(\kappa,1) + \kappa\sum_{j=1}^\infty \frac{(1-\kappa)^j}{j!}\int_0^\infty \frac{t^j}{e^t-1}[n - L_{n-1}^1(\kappa t)]dt\right]. \tag{20}$$

In particular, we have $d_n(1) = -\gamma + S_n(1,1)/n$ and

$$d_n(2) = \frac{1}{2}\psi\left(\frac{1}{2}\right) + \frac{1}{n}\left[S_n(1/2,1) + \frac{1}{2}\sum_{j=1}^\infty \frac{1}{2^j j!}\int_0^\infty \frac{t^j}{e^t-1}[n - L_{n-1}^1(t/2)]dt\right]. \tag{21}$$

Asymptotic forms are known for the confluent hypergeometric function, the associated Laguerre polynomials, and many instances of the generalized hypergeometric functions and these may be applied to the above exact representations. For example, we have for real α and $n \to \infty$

$$L_n^\alpha(x) = n^{-\alpha/2}e^{x/2}x^{-\alpha/2}J_\alpha(2\sqrt{nx}) + O(n^{-3/4}), \tag{22a}$$

and

$$L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} x^{-\alpha/2-1/4} e^{x/2} n^{\alpha/2-1/4} \cos\left(2\sqrt{nx} - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) + O(n^{\alpha/2-3/4}), \qquad x > 0. \tag{22b}$$

Equation (22b) is Fejér's formula and the bound for the remainder holds uniformly in a fixed interval of the positive x axis. Equation (22b) is consistent with Eq. (22a) due to the asymptotic form of the Bessel function J_α. In particular we have for $n \to \infty$

$$L_{n-1}^1(x) = \frac{1}{\sqrt{\pi}} e^{x/2} x^{-3/4} n^{1/4} \cos(2\sqrt{(n-1)x} - 3\pi/4) + O(n^{-1/4}), \qquad x > 0. \tag{23}$$

Further asymptotic relations for L_{n-1}^1 are developed and applied according to whether one is in the oscillatory region of its zeros or to the right in the exponentially growing region. We therefore finish with some remarks on the location of the real and simple zeros of this polynomial. Its first zero x_1 is at $O(1/n)$ and in more detail $2/(n-1) < x_1 < 6/(n+1)$. The largest zero x_n is at $O(n)$ and for instance $2(n-1) < x_n < 2n + \sqrt{4n^2 - 3/4} \simeq 4n$.