Square-root of Spacetime

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Abstract

What if fermions are just defects in spacetime? This is a discursive inquiry into the structure of spacetime, when spinor notation is used to represent spacetime itself, rather than the conventional use of spinors to define a spin structure. This is purely hypothetical; its not clear that this can be made to actually work. If it does work, it will probably turn out to be some crude but distorted reinvention of the simplest parts of string theory, arrived at by a different route. There's very little depth, here, yet; mostly just some hypothesizing.

1 Introduction

This paper is a set of notes pertaining to a crazy idea: what if the Dirac sea is just the square-root of spacetime itself? That is, instead of writing spacetime as the 4-vector $x = (\vec{x}, t)$ we instead write it as a 2-spinor $x = x_{\bar{\sigma}\sigma}$ where σ and $\bar{\sigma}$ are conjugate spinor indexes, each taking values of 1,2 so that the following holds as a matrix relation:

$$x_{\bar{\sigma}\sigma} = \left[\vec{x}\cdot\vec{\sigma} + t\mathbb{I}\right]_{\bar{\sigma}\sigma}$$

where $\vec{\sigma}$ are the Pauli matrices, as usual, and $\mathbb{I}$ is the unit 2x2 matrix. The fancy way of saying this is that the 2-spinor $x_{\bar{\sigma}\sigma}$ transforms according to the adjoint representation of $sl_2(\mathbb{C})$, which is, of course, the Lorentz group.[1]

Now imagine Minkowski spacetime, or, more generally, a differentiable pseudo-Riemannian manifold, with a single hole punched out of it. In that hole, place a single spinor: either $\psi = \psi_\sigma$ or $\bar{\psi} = \bar{\psi}_{\bar{\sigma}}$; these two transform according to the conjugate reps. of $sl_2(\mathbb{C})$. The intended question to be asked here is this: is there a way to consistently define this half-punctured manifold as being continuous and smooth in one index, but punctured in the other? The next question is then, for the punctured form, can one write something that is the spinor-analog to a closed form, analogous to the way one would write a closed form on an ordinary punctured differentiable manifold? That is, can one reinvent differential geometry and algebraic topology, but for spinors? In particular, could it be the case that the differential of the closed form is just the Dirac equation (or some suitable variant thereof, e.g. Majorana)?

[1] Note that, although the above is explicitly tied to 4-dimensional spacetime, it can be generalized to arbitrary Riemannian manifolds; see Jurgen Jost, "Riemannian Manifolds".

If one is interested in the physics of this, rather than the math, then the single-point hole needs to be replaced by a smooth curve, if it is to represent a fermion. That is, in this model, a fermion is a defect in an ordinary manifold; its an ordinary manifold with a missing (anti-)spinor. That is very much in contrast to the textbook-standard definition of fermions, which are fields defined "on top of" the manifold, or a field on an associated fiber bundle having the manifold as the base space.

Why would one ever want to define fermions this way? What's wrong with the ordinary textbook definition? There are two reasons for this: one is that maybe, just maybe, this provides a framework for thinking about non-locality and entanglement in quantum mechanics; the other is that this is a potential model for resolving the ER=EPR hypothesis. That is, by having half-punctured manifolds, perhaps we can perform surgery on them, and sew them back up again at the edges, so as to have a (geodesically-)complete manifold, without "holes" or half-boundaries. The hope is that this wacky half-surgery can "explain" entanglement, so that entanglement, or "spooky action at a distance", is just the consistency of the boundary conditions at the joint. It also resolves ER=EPR by making the half-surgery into an actual wormhole. For this to work, though, the wormhole cannot be the usual wormhole of GR, but rather, a half-wormhole, involving only one spinor component, but not the other. That is, the 4-manifold itself has to be dynamically flat (it must be Minkowski spacetime), while one of the two spinor components must be wildly curved, forming an ER bridge between two different locations of the flat spacetime. This, at least, is the dream. Can it be made to work?

References and background for this paper:

- Chapter 2 and parts of Chapter 3 of the book "Superspace, or One Thousand and One Lessons in Supersymmetry", (1983) S.J. Gates, Jr, M.T Grisaru, M Rocek and W. Siegel. This is a rather old book, but it provides some precise, crisp notation for writing down spinors, and defining derivatives of spinors. The primary difficulty with this book (and indeed, any book of supersymmetry) is that it assumes, *a priori*, as a given, the existence of "Grassman variables", anti-commuting variables with somewhat paradoxical and occasionally disconcerting properties. This defect can be cured, starting from first principles, by working with a universal enveloping algebra of a Clifford algebra. Or rather, by understanding that the origin of the Grassman variables, where they "come from", is from the quotient of the tensor algebra by the Clifford algebra. The mechanics of this is explained in the next book.

- "Riemannian Geometry and Geometric Analysis", (2002) Jurgen Jost. Chapter 1 reviews Clifford algebras and spin structures, thus "explaining" the origin of the anti-commuting Grassman variables,

- "Differential Forms in Algebraic Topology", (1982) Raoul Bott and Loring W. Tu. The first few chapters review general concepts for gluing differential forms across atlases, so as to obtain manifolds with non-trivial cohomologies. The intended program here is to do the same, but with spinors, defining half-cohomologies.

- Moshe Carmeli "..." where is this book? It reformulates all of GR, using spinor notation. Basically, its just ordinary GR, but with unfamiliar notation. I think this notation would be useful here.

- "Affine Lie Algebras and Quantum Groups", Jurgen Fuchs. The first chapter provides a lightning review of Lie algebras and their representation theory, and of the fairly central position of $sl_2(\mathbb{C})$ in that theory. Chapter 3 provides a review of bosonic and fermionic strings. Chapter 4 shows how tensor algebras can be constructed. which is sort-of a per-requiste for working with universal enveloping algebras.

As noted, its mostly just chapters 1 and 2 of these books that seem relevant to the problem at hand. Mostly, these just provide notation, formalities and general background to the subsequent development.

2 How might this work?

Before attempting to write down equations that capture the above ideas, perhaps a list of guiding principles could be made. First, a set of principles that would guide the interpretation of this model as a natural field-free Dirac sea.

- Positrons are electrons travelling backwards in time. Normally, this is obtained by charge conjugation and CT invariance. In the proposed model, the positron would be a manifold, with a ψ-hole punched in it; an electron would be a manifold with a $\bar{\psi}$-hole punched into it. It still seems reasonable that duality will naturally "fall out" of the new formalism, since ψ and $\bar{\psi}$ are already conjugate, while space-time is adjoint. There doesn't seem to be a challenge. Note that space-time itself becomes the Dirac sea; the "holes" in the sea are literally holes.

- Positrons anihilate electrons. This too seems natural in the proposed model: when two tracks meet up, one track being a ψ_σ-hole, the other a $\bar{\psi}_{\bar{\sigma}}$-hole, it seems reasonable that they should surgically join to form a single $x_{\bar{\sigma}\sigma}$ point, and the resulting manifold is smooth after the joint. Similarly for pair-creation: pair-creation is the rending-apart of the space-time manifold, with two half-tears propagrating along. Here, the model resembles the standard conceptual model of the Dirac sea: each side provides what the other is missing.

- Conservation of angular momentum. Presumably, the equations for the merger of ψ-holes and $\bar{\psi}$-holes do not require the violation of angular momentum conservation. If they do, then there is trouble, as normally, a pair of photons are required for conservation of angular momentum. It would be weird and interesting if this model somehow forced the use of an auxilliary $U(1)$-bundle to restore angular momentum. If this happened to be required, then it would be interesting, as it might explain why photons don't experience proper time.

- Conservation of energy. There are no overt, explicit masses in the proposed model, and so presumably a conservation of momentum does not offer any challanges. Maybe. The model proposes that the holes are accompanied by closed spinor-forms, i.e. obey a differential equation $d\psi = 0$ (where I've suppresed spinor indecies; presumably the correct equation will turn out to resemble $\partial\!\!\!/ \psi = 0$ in some way). Differential equations always have an implicit momentum; will pair annihilation result in a violation of the conservation of this implicit momentum? This is unclear. If it does (and it seems reasonable that maybe it should), then, once again, it would seem to force the use of an auxilliary $U(1)$-bundle to balance things out. That would be an interesting side-effect!

- There is also a different way of combining a ψ_σ-hole and a $\bar{\psi}_{\bar{\sigma}}$-hole: both spinor components are now missing, resulting in a normal hole in Minkowski space. This is catastrophic, and presumably forces a Schwarzschild solution. That seems unreasonable, given that the base assumption was that the untorn manifold was flat. The only plausible resolution here is that the formation of ordinary holes is somehow forbidden, or that the holes carry implicit energy-momentum (and thus explcitly curve space-time?) This really complicates things.

- Conservation of fermion number. The holes/tears are fundamentally topological, and thus cannot "decay". This explains the stability of fermions. That is, fermions correspond to cohomology classes, presumably of the form $H(X) = \mathbb{Z}_2 \times \mathbb{Z}_2$ or something like that. I wrote two copies of $\mathbb{Z}_2$, one for each conjugate spinor. Of course, the $H(X)$ can't be the normal de Rahm cohomology H; it has to be the $\partial\!\!\!/ \psi$ variant of it. Can we write $H\!\!\!/ (X)$? How does that work?

- Another topological analogy to fermions has long been noted in the field of materials science: screw dislocations in crystalline latices are fermion-like. They are also topologically stable. They are also highly mobile, extruding themselves, leading to "whiskers" on Galena crystals. Such whiskers are strong and stable, because there are no shear stresses parallel to its Burgers vector. Perhaps something similar could happen here? The point is we want the fermion-pair to be highly mobile, but stable, not subject to decay.

- Spin-statistics. Spin-statistics, in the sense of anti-commuting Grassman variables, can already be explained via Clifford algebras and spin structures. However, the proposed model also gives an alternative explanation: one cannot make two ψ-holes; the index is already missing, one can't remove it one more time.

A different, unrelated hope of this model is that it provides insight into entanglement. How could this work out?

- In normal entanglement, one has a pair of entangled qubits. Spinors are naturally qubits, because there is a natural ambiguity arising from the coordinate frame of the spinor, when composing to form a scalar (or vector). The ambiguity of the reference frame for a spinor is naturally a 2-sphere; that is, it is naturally a qubit. Thus, we have the curious situation that this model "explains" or "predicts" the existence of quantum mechanics! Is this claim is suspicious? Spin-1/2 does not imply quantization in and of itself: one can certainly devise a "classical" (non-quantized) spin-1/2 field as an associated fiber bundle. See, for example, chapter 6 of Bleecker.[1]

- By hypothesis, the pair-creation process creates the half-holes such that they are entangled, and more: they are still conjoined, but at different locations. That is, normal space-time is still given by points $x_{\bar{\sigma}\sigma}$. Pair-creation sunders this in such a way that there are now components, ψ_σ an $\bar{\psi}_{\bar{\sigma}}$, in such a way that ψ_σ is a tear or cut, at a 4-position y in Minkowski space, and $\bar{\psi}_{\bar{\sigma}}$ is a tear or cut at another 4-position w in Minkowski space, such that $\psi_\sigma \bar{\psi}_{\bar{\sigma}}$ is a bi-spinor, defining a single adjoint "location" that is a smooth continuous "sheet", without holes. That is, no actual holes were created; instead one should imagine that space-time is bi-sheeted, with σ labelling one sheet, and $\bar{\sigma}$ labelling the other, and these two sheets are normally glued together to obtain the normal 4D space-time coordinate $x_{\bar{\sigma}\sigma}$. Pair-creation is then the peeling apart of these two spinor-sheets, while somehow maintaining both continuity and smoothness of each sheet, and without actually creating any punctures at all. Or perhaps, the peeling apart can be imagined to be a half-puncturing, followed immediately by surgical re-attachement to restore smoothness; thus $\psi_\sigma(y)$ and $\bar{\psi}_{\bar{\sigma}}(w)$, although now located at two different spatial positions y and w, are still the same "sheet", so that $\psi_\sigma(y)\bar{\psi}_{\bar{\sigma}}(w)$ is an ordinary adjoint vector. That is, $\psi_\sigma(y)\bar{\psi}_{\bar{\sigma}}(w) = x_{\sigma\bar{\sigma}}(y,w)$ is still an ordinary 4-vector. It is still a kind-of 4-dimensional position, except that now it is in two places at once. Put differently, pair creation is the same thing as the explicit creation of an ER bridge, but that bridge is only between spinor-sheets, and not between space-time itself. Since, by hypothesis, the underlying manifold needs to remain flat, there clearly has to be lot of curvature or torsion or something like that, in the psinor sheets to acheive this. The mathematical challenge is to write down the above, together with some kind of spin-affine connection-thingy that makes sense of this peeling-apart surgery. Again, each peeled-apart component has to satisfy $\not{\partial}\psi = 0$ or some suitable variant thereof.
- In normal entanglement, one has a pair of entangled qubits. In this model, one has a pair of entangled fermions, ψ and ϕ, located in different regions of space-time (but once having been close together?), prepared in some product state. Because, in the multi-sheeted model, the fermions ψ and ϕ are glued up to thier partners $\bar{\psi}$ and $\bar{\phi}$, the process of entanglement of ψ and ϕ also requires the re-organization of the gluing onto $\bar{\psi}$ and $\bar{\phi}$, wherever they may be. Exactly how this works is unclear. Also, to be clear: there are no hiden variables anywhere in this picture; ψ should be visualized as a 2-sphere; the gluing of ψ to $\bar{\psi}$ is effectively just a sphere bundle, and the entanglement of ψ and ϕ is just a sphere bundle, with a different affince connection on it. The qubits are sphere-bundles.

The last few sentences suggest a fairly concrete starting point for this enterprise: construct an ordinary sphere bundle, or actually, two of them, and then glue the fibers together such that the gluing-together can be identified with a single point in the base-space.

3 Patching

So, start implementing the above program. XXX Draft.

3.1 Patching

The patching idea. Drill a pair of holes, removing all spacetime points, along $vt-x$ and $vt+x$, for non-negative x and t. Note for $v<1$ these are space-like holes. Widen them into empty cones (but narrow enough to still be space-like). Add another pair of concentric cones outside of them, so the outer set sit in Minkowski space. Note that the cones are of the form $B\times\mathbb{R}^{+}$ where B is a ball in 3D space (*i.e.* in the (x,y,z) coordinates), and $\mathbb{R}^{+}$ is the center of the ball, travelling forward over the time coordinate.

Take a single spinor, ψ_{σ} and patch it onto the $x_{\bar{\sigma}\sigma}$ on the left cone. Do this by creating charts, just like an ordinary coordinate-chart used to define an atlas for a topology, except that the coordinate space is $\mathbb{P}^{1}(\mathbb{C})$ and not the textbook-usual $\mathbb{R}^{n}$. Set up this patch so that its "flat". Create another patch on the right cone, using $\bar{\psi}_{\bar{\sigma}}$, again so its flat. This leaves one index on each cone that is not attached. As noted, the interior of the cone is a ball; the surface of the cone is a 2-sphere. Create a spinor-coordinate patch that glues these two remaining cones together. Give explict expressions for all of the charts and transition maps! FIXME.

Problem with this: The hollowed-out interiors of these cones no longer have a well-defined space-time. I guess that one could take the product of two spinor charts, and claim that this is space-time, but its not the full-degrees-of-freedom any more. But of course, we ripped apart spacetime, on purpose, so what can you expect? The cones can be shrunk back down so that they are very thin.

Speculation: because they are punctured, they do allow an electric charge to be placed inside them, at no penalty (no actual singularity). That is, because they are punctured, they allow a non-trivial cohomology group around them, exactly the kind of cohomology needed for a naked classical electric charge. Cool. So the puncture is not useless. Oh, and in some strange way, the glueing creates a kind-of flux-tube-like thing between the two cones, so "of course", the charge on one cone has to be the opposite of the other: they are necessarily charge-conjugate.

3.2 Tangent bundles and sphere bundles

The starting point for this endevour is more-or-less an ordinary spin structure on flat Minkowski space-time. Details on how this is done can be found in Chapter 6 of Bleecker[1]. But first, some more hand-waving to paint the intuitive picture. A spinor ψ can be thought of as a point on the complex projective line, compactified as the Reimann sphere. In quantum mechanics, we normally expect to have our wave-functions to be normalized to unit-length. This can be acheived by delcaring them to be points in projective spaces, so that the unit-length normalization is a natural by-product of it being a projective space. A spinor ψ has two complex components: it is the complex projective line $\mathbb{P}^{1}(\mathbb{C})$ of all complex lines in $\mathbb{C}^{2}$. As a qubit, it is the same thing as the Bloch sphere, and is diffeomorphic to the 2-sphere. Wikipedia provides an adequate review of these relationships, and the different possible notations.

To make this work, start with a pair of 2-spinors, $\bar{\psi}$ and ϕ, organized as a direct sum into a 4-spinor $\rho=\phi\oplus\bar{\psi}$. These are understood to transform under the two-dimensional representations of $sl_{2}(\mathbb{C})$.[2] As usual, the bar denotes that the spinor

[2]The algebra $sl_{2}(\mathbb{C})$ is famously known two have a pair of complex-conjugate two-dimensional repre-

transforms according to the conjugate representation. Denote the spaces which these spinors inhabit as $\mathbb{P} = \mathbb{P}^1(\mathbb{C})$ and $\bar{\mathbb{P}}$, both being complex projective lines, but transforming under the conjugate representations. The spinors are just points in these two spaces.

There are two bundles that suggest themselves. One is the trivial bundle $M \times (\mathbb{P} \oplus \bar{\mathbb{P}})$ that is simply the Cartesian product of spaces, with M being Minkowski space. This can be understood as a space of Dirac-spinor-valued fields on M. The other is the trivial bundle $TM = M \times \mathbb{R}^4$ the tangent bundle on Minkowski space. These two can be made canonically isomorphic, in the usual sense. That is $\mathbb{R}^4 = \mathbb{P} \oplus \bar{\mathbb{P}}$ with the isomorphism being given by $x_{\bar{\sigma}\sigma} = \bar{\psi}_{\bar{\sigma}} \phi_\sigma$. Written as a matrix, this is

$$\begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} = \begin{bmatrix} \bar{\psi}_1\phi_1 & \bar{\psi}_1\phi_2 \\ \bar{\psi}_2\phi_1 & \bar{\psi}_2\phi_2 \end{bmatrix}$$

Argh. Fixme. Such a decomposition as above works only as coordinates on the light-cone. And the spinors are not fully free; they are limited and coupled to one-another, loosing a (complex) degree of freedom (two real degrees of freedom). Well, actually one degree of freedom; the other they never had cause they were already projective. But whatever. FIXME. Give explicit expressions.

4 Flat (Minkowski) space

How does one recognize flat space? One way is to examine the geodesics obtained from the Euler-Lagrange equations, and notice theier particularly trivial behavior. Lets take a look at what these look like, and what happens if we try to write them using spinor coordinates.

5 Now what?

There is no there, there. Maybe later.

Anyway, the current state of these ideas is completely broken and insane; its "obviously bad" in multiple ways. Back-burner.

References

[1] David Bleecker. *Gauge Theory and Variational Principles*. Addison-Wesley Publishing, 1981.

sentations. These are famously homeomorphic to one of the connected components of the Lorentz group. This is obviously a fundamental cornerstone to the thesis presented here. Do not pass go, do not collect $200 if you don't know this inside-out and upside-down.