Skyrme->Fermions

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Abstract

Lets use the skyrme effective lagrangian and plug fermions into it, and then use resonant interaction techniques to factor it into free modes and bound modes. What do they look like?

Idea that surely others have explored, and surely won't work. But lets give it a whirl.

1 Introduction

Here's the basic idea: recent developments in resonant interactions show how to factor a non-linear system, e.g. one with solitons in it, into free modes and bound modes. The bound modes have no independent dynamics of thier own, but must evolve with the fundamental mode that is carrying the soliton. The free modes, by contrast, are free to interact, and, when they do, the interaction is given by the resonant interaction.

The Skyrme model for the nucleon is a soliton model. Can we perform this factorization into free and bound modes, and, when we do, what do those modes look like? Things can become even more interessting if we write the effective Lagrangian in terms of fermions. What do the free/bound modes look like then? Is there a snowballs chance that the algebra accidenally looks QCD-like, or is that just too unlikely? Well, I won't know until I try it.

As usual, In diving into this unlettered, unread. I suppose someone else knows this already. Probably Dan Freed. Oh well. From here on, mostly a rough sketch with few explanations.

Other notes:

- There is no quantum in here; this is a series of manipulations applied to purely classical fields.
- The fermions will be Weyl fermions, and will not be presumed to have any dynamics of thier own; they inherit from whatever the Skyrme effective lagrangian forces upon them. If they happen to self-organize into a Dirac form, that would be great, but it is not assumed a priori.
- The free/bound factorization requires recoding eveything into a Hamiltonian form, with conjugate variables, and thence into ladder operators. This follows the broad outlines set forth in the Janssen paper[3], a la Zakharov.

1.1 Review

Apparently, I am rediscovering old ground. A bit of work shows that the sigma model can be generically formulated on symmtric spaces, see e.g. [7] which then promptly gives eight references to sigma models as symmetric spaces. Yikes! However, all eight references are to supergravity theories; it is no wonder that I've been ignorant of these.

2 Skyrme Effective Lagrangian

From Wikipedia:

$$\mathscr{L} = \frac{-f_\pi^2}{4}\,\mathrm{tr} L_\mu L^\mu + \frac{1}{32e^2}\,\mathrm{tr}\left[L_\mu, L_\nu\right]^2 + \frac{f_\pi^2 m_\pi^2}{4}\mathrm{tr}\left(U + U^\dagger - 2\right)$$

where f_π is the pion-nuclean coupling strengh, g is the vector-pion coupling (rho meson) and

$$L_\mu = U^\dagger \partial_\mu U$$

and

$$U = \exp i\vec{\tau}\cdot\vec{\theta}$$

where $\vec{\tau}$ are the Pauli matrices, and $\vec{\theta}$ is the non-linear sigma field.

This is the Lagrangian describing the non-linear sigma model with a quartic term in Minkowski space. Its got a soliton, the Skyrmion.

To avoid confusion about the dimensionful constants, here they are:

$$f_\pi = 93 \text{ MeV} \qquad g_A = 1.29 \qquad m_\pi = 137 \text{ MeV} \qquad M_N = 939 \text{ MeV}$$

Let ϕ be the conventional pseudoscalar meson field; the normalization is such that $U = \exp i\phi\sqrt{2}/f_\pi$where f_π is the pion decay constant, g_A is the axial coupling, i.e. the coupling of the axial current to the nucleon, and m_π the pion mass, and M_N the nucleon mass. From phenomenology, one should have $e \approx 5$.[4]

I'd like to rephrase it so that, instead, it lives on some generic spin manifold, but this is probably not important(?) and not urgent, since I think what I'm looking for doesn't need that. It would be nice to use generic language, though. Somewhat easier said than done, since proof of the existance of the topological soliton requires the boundary of the manifold to be S_3 and so if it is going to be some spin manifold M it has to be one with $\pi_3(M) \neq 0$.

I do want to factor $\vec{\sigma}$ into a pair of spinors, however, i.e. so that it is manifestly spinorial, instead of a 3-vector. The pion is isospin-1, so the two fermions will be u and d. There will be messy algebra with error-prone indexes all over the place. I don't look forward to this.

Part two is to write the the Lagrangian as a Hamiltonian. This is required in order to get it into symplectic form, so that the ladder operators can be written down, so that canonical transformations can be applied to remove the non-interacting modes, so the resonant condition can be uncovered. This also promises to require a mind-boggling amount of algebra.

This should probably be done twice: once with pions, and once with fermions.
Questions:

1. How can the algebra be double-checked?
2. What algebra package can be used to expediate the calculations?

Next steps. We now want to:

- Write does a Lagrangian (well, we did already...)
- Write the equations of motion, I think we need these before we get to write the Hamiltonian.
- Write down a Hamiltonian. That means writing down a pair of conjugate fields.
- Write the ladder operators
- Linearize
- Perform canonical transformations.
- Factor into spinors. Oof dah.

But, why, exactly, am I expecting any of this to lead to anything useful, whatsoever? What would the first 5-6 steps yield, that aren't already implicitly visible? And the last step, with spinors, why would that be anything other than a morass? This whole program feels,stupid, unpromising...

How this compares to other things:

- 't Hooft shows that in the large N_C expansion i.e. $N_C \to \infty$ that the QCD Lagriangian looks like a weakly interacting meson lagrangian.[5, 6]
- The paper by Schechter and Weigel[4] provides an comprehensive review of low-energy particle phenomenology in the Skyrme model.

2.1 Conventions

The following sections will reconstruct the Skyrme model with N-dimensional flavor symmetry, where the symmetry group is

$$G = SU(N) \times SU(N)$$

However, almost all of the development will not require this specific form; it will be conducted in a generic fashion. When in doubt, the above can be used as a touchstone, especially so by setting $N = 2$.

2.1.1 Sigma model

From Wikipedia, the sigma model Lagrangain can be written as

$$\mathcal{L}(\phi_1, \cdots, \phi_n) = \frac{1}{2}\sum_{ij} g_{ij} d\phi^i \wedge * d\phi^j - V(\phi)$$

where g_{ij} is a Riemannian metric on the field space Φ. Equivalently,

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi) \left(\partial^\mu \phi^i\right) \left(\partial_\mu \phi^j\right) - V(\phi)$$

where $\partial_\mu \phi$ is a section of the jet bundle of $M \times \Phi$.

TODO: articulate the jet-bundle construction; show the equivalence of these two.

2.1.2 Matrix identities

Yilmaz[7] quotes a matrix identity which seems very useful: given a matrix C one has

$$de^C e^{-C} = dC + \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] + \cdots$$

and

$$e^{-C} de^C = dC - \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] - \cdots$$

which I guess follows from the Baker-Campbell-Hausdorff formula in some less-than-obvious way, to me. TODO: find a second authoritative more widely-known source for the above.

Writing $C = \theta^i T_i$ for Lie-algebra-valued T_i obeying $[T_i, T_j] = f_{ij}^{\ \ k} T_k$ one then obtains

$$\begin{aligned} de^{\theta^i T_i} e^{-\theta^i T_i} &= d\theta^i T_i + \frac{1}{2!}\theta^i d\theta^j f_{ij}^{\ \ k} T_k + \frac{1}{3!}\theta^i \theta^j d\theta^k f_{jk}^{\ \ l} f_{il}^{\ \ m} T_m + \cdots \\ &= T_i \Delta_{ij} d\theta^j \end{aligned}$$

with

$$\Delta = \sum_{n=0}^{\infty} \frac{M^n}{(n+1)!} = \left(e^M - I\right) M^{-1}$$

and $M_j^k = \theta^i f_{ij}^{\ \ k}$. Likewise,

$$\begin{aligned} e^{-\theta^i T_i} de^{\theta^i T_i} &= d\theta^i T_i - \frac{1}{2!}\theta^i d\theta^j f_{ij}^{\ \ k} T_k + \frac{1}{3!}\theta^i \theta^j d\theta^k f_{jk}^{\ \ l} f_{il}^{\ \ m} T_m - \cdots \\ &= T_i W_{ij} d\theta^j \end{aligned}$$

with

$$W = \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n+1)!} = \left(I - e^{-M}\right) M^{-1}$$

These are apparently the vielbeins, because these can be plugged directly into the sigma model Lagrangian. That is, let $L = U^\dagger dU$ and $U = \exp \vec{T} \cdot \vec{\theta}$

$$\mathrm{tr} L \wedge * L = W_i^m W_j^n \; d\theta^i \wedge * d\theta^j \; \mathrm{tr} T_m T_n$$

which is a fairly slick trick. To be clear: the Riemannian metric g_{ij} appearing in the sigma model lagrangian is being written as

$$g_{ij} = W_i^{\ m} W_j^{\ n} \kappa_{mn}$$

where $\kappa_{mn} = \mathrm{tr} T_m T_n$ is the Killing form, which can be explicitly written in terms of the structure constants as

$$\kappa_{mn} = f_{im}^{\ \ j} f_{jn}^{\ \ i}$$

The Killing form can be taken as the "flat" metric on $\mathfrak{g}$. Whenever a curved matric can be written in terms of a "flat" metric in this way, the factors $W_i^{\ m}$ are called the vielbeins. Cool!

I have not double-checked, but for the special case of SU(2) I think this boils down to the old tried-and-true identities

$$e^{i\vec{\tau}\cdot\vec{\theta}} = \cos|\theta| + i\vec{\tau}\cdot\vec{\theta}\frac{\sin|\theta|}{|\theta|}$$

and the usual L_μ follows from that. Huh. TODO:

$$L_\mu = ...$$

2.1.3 Sigma model redux

From what I can tell, Yilmaz is saying that the above can even work for the symmetric space, in that one can split, with some care, the basis T_i for $\mathfrak{g}$ into those that are in $\mathfrak{m}$ and $\mathfrak{h}$ where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. That is, one can write a parameterization for G/H as

$$\phi = e^{\theta^i m_i}$$

with m_i the basis vectors of $\mathfrak{m}$.

Write # as the "generalized transpose" , induced by the Cartan involution α as $X^\# = -\alpha(X)$ for any $X \in \mathfrak{g}$. Depending on $\mathfrak{g}$, this generalized transpose is either the the conjugate transpose, or the ordinary matrix transpose, depending on whether the algebra is complex, or not, has a real representation or not, etc.

Write $\mathscr{M} = \phi^\# \phi$, where we are using $\mathscr{M}$ because this is the "internal metric" on the sigma space. The sigma model Lagrangian is then (up to a constant c)

$$\mathscr{L} = -\frac{c}{4}\mathrm{tr}\left(d\mathscr{M} \wedge * d\mathscr{M}^{-1}\right)$$

which can be written as

$$\mathscr{L} = \frac{c}{2}\mathrm{tr}\left(\left(\mathscr{G} + \mathscr{G}^\#\right) \wedge * \mathscr{G}\right)$$

where $\mathscr{G} = \phi^{-1} d\phi$ is the pullback of the Maurer-Cartan form. Yilmaz provides an explicit derivation of the above, its not hard.

2.1.4 Other terms in the Lagrangian

Apparently, in the supergravity world, it is not uncommon[7] to add an additional term

$$\mathscr{L} = \frac{c_2}{2}\mathscr{M}_{ij}F^i \wedge *F^j$$

where $F^i = dA^i$ for some A^i and and c_2 some constant. I guess the whole thing is somehow gauge invariant again, although some work would be needed to show this.

2.2 Symmetric spaces

This and the next two sections flesh out the details of an idea presented by Harland[2].

Let G be a compact connected Lie group. Let $\alpha : G \to G$ satisfy $\alpha(gg') = \alpha(g)\alpha(g')$ and also $\alpha^2(g) = \alpha(\alpha(g)) = g$. This is the definition of an involution; it is called "the Cartan involution". The fixed points $h \in G$ of α are by definition $\alpha(h) = h$. These form a subgroup $H \subset G$ and the quotient space G/H is a symmetric space – by defintion, symmetric spaces are quotients of a group by it's involution-invariant subgroup. It's assumed that α is non-trivial, *i.e.* that H is a proper subgroup that H is not the trivial subgroup.

This function α can be interpreted as a permutation of two "parts" of G; the subgroup H is then the diagonal of a product of these two parts. Later on, these will be identified as the left and right chiral pieces of the sigma model; there is no need to do this, yet, and so the following proceeds in a generic fashion. (There are actually three possibilities: G is a real simple Lie group, or G is the product of two compact simple Lie groups (this is the conventional sigma model) or G is the complexification of a compact simple Lie group, in which case G/H is non-compact, and H is the maximal compact subgroup of G.)

Let $\mathfrak{g}$ be the corresponding Lie algbera of G. Then p descends to $\mathfrak{g}$ and since $\alpha^2 = 1$ the identity, it's eigenvalues are ± 1. It splits $\mathfrak{g}$ into odd and even subspaces $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. The odd-parity subspace $\mathfrak{m}$ can be identified as the tangent space of G/H at the identity and the even-parity subspace $\mathfrak{h}$ is the Lie algebra of H. Note that

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h} \quad \text{and} \quad [\mathfrak{m},\mathfrak{h}] \subseteq \mathfrak{m} \quad \text{and} \quad [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$$

The easiest way to see this is to note that $\alpha[X,Y] = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{g}$. Write $X = X_\mathfrak{m} \oplus X_\mathfrak{h}$ and note the $\alpha\left(X_\mathfrak{m} \oplus X_\mathfrak{h}\right) = (-X_\mathfrak{m}) \oplus X_\mathfrak{h}$. Then clearly the parity of $\left[X_\mathfrak{m} \oplus 0, 0 \oplus Y_\mathfrak{h}\right]$ is negative, while the parity of $\left[0 \oplus X_\mathfrak{h}, 0 \oplus Y_\mathfrak{h}\right]$ and $[X_\mathfrak{m} \oplus 0, Y_\mathfrak{m} \oplus 0]$ is positive.

Two more identities will be useful. The adjoint action $\mathrm{Ad}_g X = gXg^{-1}$ of $g \in G$ on $X \in \mathfrak{g}$ behaves as

$$\alpha(\mathrm{Ad}_g X) = \alpha\left(gXg^{-1}\right) = \alpha(g)\,\alpha(X)\,\alpha\left(g^{-1}\right) = \mathrm{Ad}_{\alpha(g)}\alpha(X)$$

Given a curve $\gamma : \mathbb{R} \to G$ on the group manifold, the tangent vector $X = \gamma^{-1}\partial_t\gamma$ to the curve transforms as

$$\alpha\left(\gamma^{-1}\partial_t\gamma\right) = \alpha\left(\gamma^{-1}\right)\partial_t\alpha(\gamma)$$

Let's belabor the obvious; this will be useful a bit later. Since G is a Lie group, the tangent space TG has a natural coordinate system given by the adjoint map. That is, given a point $g \in G$ and a basis vector $e_i \in \mathfrak{g}$, the corresponding basis at point g is given by $\mathrm{Ad}_g e_i = g e_i g^{-1}$. This basis splits into two parts, given by lifting the projections the projections $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ to all of TG.

2.2.1 Maurer-Cartan form

The Cartan form ω is a one-form on G given by the differential $L_{g^{-1}}$ and specifically, $\omega_g = L_{g^{-1}*}$ The definition of the differential is given in Bleeker section 0.2.5 page 8. If G is embeded in $GL(n)$ then $\omega_g = g^{-1}dg$ *i.e.* the "left logarithmic derivative of the identity map of G". Taking G to be a principle bundle over a single point, ω is the principle connection on that bundle.

Given a homogenous space G/H and a section $s : G/H \to G$ then the pullback $\theta = s^*\omega$ transforms in the appropriate gauge-like way, and has a vanishing curvature. See Wikipedia "Maurer–Cartan form" for more.

This pullback can be decomposed as $\theta = \pi_{\mathfrak{m}}(\theta) \oplus \pi_{\mathfrak{h}}(\theta)$.

There is apparently an equivalent characterization where a metric is provided on G/H, which is then decomposed into vielbeins[7]. The rest of that article appears to deal with the case where G is non-compact, which appears to be the case of interest for supergravity.

2.3 Vector and Gauge fields

The goal of this section is to set up the basic ingredients for the sigma model as the dynamics of a field ϕ, the sigma field, valued in the symmetric space G/H. This is not the conventional construction but close to it. The end goal of the construction is to define

$$L_\mu = \pi_{\mathfrak{m}} \circ \left(g^{-1}\partial_\mu g\right)$$

where $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ is the projection above, and then use this in the Skyrme Lagrangian. This is well and fine, but practical calculations require the idea to be articulated. I'll try to do this carefully, to avoid later confusion due to glibness. This necessitates a dive back to first principles and basic (textbook) definitions (The textbook being Bleecker).

Let $P \to M$ be a smooth principal fiber bundle (PFB) over a smooth manifold M with structure group G. Let $p \in P$ be a point above x. By definition, P_x is diffeomorphic to G for all $x \in M$. Given $p \in P$, we have G acting on the right, so that $p \mapsto pg$. Write $R_g : P \to P$ for this right-action.

Given a subset $U \subset M$, a connection is a $\mathfrak{g}$-valued 1-form ω on U that is covariant under gauge transformations. For the present purposes, a gauge transformation is a map $g : U \to G$ and the connection transforms as

$$\omega \mapsto g^{-1}\omega g + g^{-1}dg$$

Earlier, G was split into $G/H \times H$ and correspondingly, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and this split can be applied to the connection. Since ω is $\mathfrak{g}$-valued, write $\omega = \pi_{\mathfrak{m}}(\omega) \oplus \pi_{\mathfrak{h}}(\omega) =$

$\omega_{\mathfrak{m}} \oplus \omega_{\mathfrak{h}}$. Under gauge transformations $h \in H$, one has that

$$\pi_{\mathfrak{m}}(\omega) \mapsto \pi_{\mathfrak{m}}\left(h^{-1}\omega h + h^{-1}dh\right) = h^{-1}\pi_{\mathfrak{m}}(\omega)\,h$$

because $h^{-1}dh$ is an $\mathfrak{h}$-valued 1-form, and thus projected out (that is, $\pi_{\mathfrak{m}}(\mathfrak{h}) = 0$). The other projection picks up this term:

$$\pi_{\mathfrak{h}}(\omega) \mapsto \pi_{\mathfrak{h}}\left(h^{-1}\omega h + h^{-1}dh\right) = h^{-1}\pi_{\mathfrak{h}}(\omega)\,h + h^{-1}dh$$

The ingredients to the sigma model are obtained by setting the connection ω to zero globally; that is, $\omega = 0$ which in turn implies that P is globally trivial: $P = M \times G$. We can still perform gauge transformations. These are

$$\omega = 0 \mapsto g^{-1}dg$$

so that the connection is "pure gauge". The component $\pi_{\mathfrak{m}}\left(g^{-1}dg\right)$ is non-zero, as is $\pi_{\mathfrak{h}}\left(g^{-1}dg\right)$. Writing $F = G/H$ as the space of the sigma field, the component $\pi_{\mathfrak{m}}\left(g^{-1}dg\right)$ is to be re-interpreted as a 1-form valued in the tangent space $T_f(F)$ where $f = \pi(g)$ and $\pi : G \to F$ is the projection.

What happens if we do *not* set ω to zero? Then I guess that, if we add the dynamical term Yilmaz mentions, then we get a new gauge-invariant Lagrangian.

2.3.1 Associated bundle

Let F be some manifold (the "field" manifold), such that G acts on F on the left, *i.e.* so that for $f \in F$ we have $f \mapsto gf$. The associated fiber bundle is

$$P \times_G F \to M$$

The space of all sections of this bundle is isomorphic to the space of all maps $\phi : P \to F$ such that $\phi(pg) = g^{-1}\phi(p)$. A specific choice of ϕ is a "particle field".[1] That this terminology is correct and appropriate can be most easily inuited by considering a flat bundle $P = M \times G$ and then considering a section (that is, ignoring G) to get a map $M \to F$. That map to be extended back over all of P with the desired properties.

Conventionally, a (bosonic) particle field corresponds to an F being a vector space V, and G being a gauge group, acting on it by means of a representation in $GL(V)$. This won't be done here, because the goal is to construct the sigma model so that $F = G/H$. Another difference is that the group G is "too big"; the field F is invariant under H and not G. Thus, some factorization is required.

TODO: Why is saying any of this interesting? I was hoping it would make it easier to describe the 1-jet, but no luck so far.

2.3.2 Physics notation

The function $\phi : M \to G/H$ is to be interpreted as the physical field of the sigma model, the sigma-field.

[1] Bleeker, section 3.1.1, page 43.

Let let $h: M \to H$ be a gauge transformation. Define the even and odd parity vector and axial vector fields as

$$L_\mu = \pi_{\mathfrak{m}} \circ \left(g^{-1}\partial_\mu g\right) \quad \text{and} \quad A_\mu = \pi_{\mathfrak{h}} \circ \left(g^{-1}\partial_\mu g\right)$$

where $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$ are the projections $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$. Under gauge transformations $g \to gh$, these transform as

$$L_\mu \mapsto h^{-1}L_\mu h \quad \text{and} \quad A_\mu \mapsto h^{-1}A_\mu h + h^{-1}\partial_\mu h$$

Note that L_μ transforms as a differential 1-form, while A_μ transforms as a gauge field; that is, A_μ is the affine connection (on the tangent bundle of G/H).

TODO: The above uses physics notation. It's "obviously the same" as the earlier presentation phrased in terms of 1-forms, but its different. The lack of a bridge between the notations is irksome.

2.3.3 Darboux derivative

According to the Wikipedia article "Darboux derivative", if one has a map $f: M \to G$, then the "most natural" way to write the derivative of f is to write it as the Darboux derivative, which is done by using the Maurer–Cartan form on G. It seems like L_μ is the same thing as the Darboux derivative, right? Or not? If not, then what is the Darboux derivative?

I think this just means that the Maurer-Cartan form can be written as $g^{-1}dg$ and that the pullback of it, by f, gives L_μ.

2.4 Chiral vector fields

The standard chiral interpretation is to take the vector and axial-vector spaces as the diagonal and modulo spaces of a product of the "left" and "right" chiral fields. Write

$$G = H \times H$$

Then $\alpha(h_L, h_R) = (h_R, h_L)$ the permutation of the two elements satisfies $\alpha^2 = 1$ and thus can be used as a symmetrizer to split out the diagonal. The fixed points are $\alpha(h,h) = (h,h)$. Then $G/H = (H \times H)/H$ is diffeomorphic to H, the explicit diffeomorphism is

$$(h_L, h_R) \mapsto U = h_L h_R^{-1}$$

All products of compact connected Lie groups work like this – the diagonal is a symmetric space diffeomorphic to the group itself.

Given bases vectors e_i for $\mathfrak{h}$ we can write an explicit basis for $\mathfrak{m}$ as $(e_i, -e_i)$ which corresponds to the $\alpha = -1$ eigenspace, and (e_i, e_i) as the basis for the $\alpha = +1$ eigenspace. These two are the axial-vector and the vector subspaces.

The historical basis for this chiral split is anchored in beta decay; the weak interactions distinguish left from right. This implied that the flavor symmetries must also have chiral components.

3 Hamiltonian

The goal of this section is to re-write the Lagrangian as a Hamiltonian, and obtain an explicit expression for the conjugate momentum. If this is possible, the next step is to write the ladder operators in terms of the conjugate pair. With these in hand, one can/should be able to find canonical transformations that split the space into "free" and "bound" sectors. The intent is to emulate what is being done by Janssen in [3].

Some random notes:

To fourth order in the pion field $\vec{\theta}$, we should get this:

$$\mathscr{L} = \frac{1}{2}\partial_\mu \vec{\theta} \cdot \partial^\mu \vec{\theta} + \frac{1}{6f_\pi^2}\left\{ \left(\vec{\theta}\cdot\partial_\mu \vec{\theta}\right)^2 - \left|\vec{\theta}\right|^2 \partial_\mu \vec{\theta} \cdot \partial^\mu \vec{\theta} \right\} + \mathscr{O}\left(\left|\vec{\theta}\right|^6\right)$$

From phenomenology, $f_\pi \sim \sqrt{N_C}$ and $e = 1/\sqrt{N_C}$[4]

3.1 Hamiltons equations

The proper formulation would be to write the Lagrangian in terms of 1-jets, as is done in section 3.3.1 page 50 of Bleeker. Doing so just right now is too confusing, so I will punt on this.

Lets stupidly plow ahead. We need a base manifold M that is Lorentzian, i.e. pseudo-Riemannian with signature $(n-1,1)$, so that we can (unambiguously) split out a time component.

To convert a Lagrangian into a Hamiltonian, we need to pick a coordinate q and define the generalized momeum as $p = \partial\mathscr{L}/\partial\dot{q}$ and then finally define the tautolgical 1-form $\dot{q}p$ and then write $\mathscr{H} = \dot{q}p - \mathscr{L}$. What should we pick as q? Should it be the field $\phi \in G/H$ or should it be the paramterized field $\vec{\theta}$ which comes from writing $\phi = \exp\vec{\theta}\cdot\vec{m}$ where the m_i are the Lie algebra vectors spanning the subspace $\mathfrak{m}$? The later are nice, because they feel "flat", in that we have an explicit expression for the vielbeins and can work with the Killing form as the flat metric. We know how to perform computations with them. On the other hand, why not use ϕ directly as the coordinate, and insert the metric tensor everywhere, as needed? Perhaps try both?

It seems we also have several choices for notation: index notation, or a more abstract notation in terms of forms and pullbacks. Or both. Probably both, because looking at one style can rescue the other when the expressions get muddled. OK, so that's a plan.

The total differential is

$$d\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\vec{\theta}}\cdot d\vec{\theta} + \frac{\partial\mathscr{L}}{\partial\dot{\vec{\theta}}}\cdot d\dot{\vec{\theta}} + \frac{\partial\mathscr{L}}{\partial t}dt$$

and so, for the generalized momenta $\vec{\eta}$, write

$$\vec{\eta} = \frac{\partial\mathscr{L}}{\partial\dot{\vec{\theta}}}$$

so that per the usual

$$\mathscr{H} = \dot{\vec{\theta}}\cdot\vec{\eta} - \mathscr{L}$$

It seems easiest to take the metric signature +++- and to write $\partial_\mu = (\partial_x, \partial_t)$ with the understanding that ∂_x runs over all of the spatial dimensions of the base manifold. Starting from

$$\mathscr{L} = \frac{-f_\pi^2}{4} \operatorname{tr} L_\mu L^\mu + \frac{1}{32e^2} \operatorname{tr}\left[L_\mu, L_\nu\right]^2 + \frac{f_\pi^2 m_\pi^2}{4} \operatorname{tr}\left(U + U^\dagger - 2\right)$$

we have

$$\operatorname{tr} L_\mu L^\mu = \operatorname{tr} L_x L_x - \operatorname{tr} L_t L_t$$

where $L_\mu = \pi_\mathfrak{m} \circ \left(g^{-1}\partial_\mu g\right)$.

Let's try to do this properly, instead of winging it. We've got four manifolds to deal with. There is the base space M which we just split into space-like and time-like parts. The global structure of M is uninteresting (for now), so may as well take it to be flat. Then there's G as a manifold, G/H as a manifold, and the fiber H taken as a manifold. The first step is to establish

Let $g \in G$ be a point on the group manifold G. Given a basis e_i for the Lie algebra $\mathfrak{g}$, it is extended to the tangent field

$$\partial_i = xxxwtf \frac{d}{dt} g^{-1}$$

There are two ways to approach this; one is to write $\gamma : \mathbb{R} \to G$ be a curve that passes through the identity at $t = 0$, and then take ... I forget ... $g^{-1}\gamma g$? The tangent vector is then $\dot{\gamma}(0) = \dot{\vec{\theta}}$ at time $t = 0$ and so $L_t = \pi_\mathfrak{m} \circ \left(g^{-1}\dot{\vec{\theta}} g\right)$ ...

The other way is to use the BCH formula and infinitessimal variations. Write the group element $g \in G$ as $g = \exp \vec{e} \cdot \vec{\theta}$ where $\vec{e}_i \in \mathfrak{g}$ are a set of basis vectors for the Lie algebra $\mathfrak{g}$. Write the tangent as $\vec{\theta} \mapsto \vec{\theta} + \delta\theta$ so that

$$g^{-1}\partial_t g = \exp -\vec{e} \cdot \vec{\theta} \, \exp \vec{e} \cdot \left(\vec{\theta} + \delta\theta\right) = \exp\left(\vec{e} \cdot \delta\theta - \frac{1}{2}\left[\vec{e} \cdot \vec{\theta}, \vec{e} \cdot \delta\theta\right]\right)$$

Err, try again. Let $\gamma : \mathbb{R} \to G$ be a curve that passes through the identity at $t = 0$.

Yuck, this is a confusing mess. We have to backtrack and fix things up above.

4 Weyl Spinors

Lets recall how spinors are constructed. Given a vector space V and a bilinear product ... write .... Basically we want to write the Lie group as a spin group. This is possible for any Lie group. See ref. [1] and also Jost.

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