31 Dec 2007
Austin TX

Salut Philippe,

I just received a note from I.Kaporin, who points us at two references, that overlap substantially with our paper. They are located at http://riemannzeta.narod.ru/zhvmzeta.htm; both authored by Eremin, A.Yu., Kaporin, I.E., Kerimov, and published in Russian, in 1985 and 1988.

In the first paper ("O vichuslenii zeta-funktsi Riemana v complexnoi oblasti", "On the computation of the Riemann zeta function for complex argument" in my poor translation), they develop the series expansion

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} (-1)^n \kappa_n \binom{s-1}{n} \tag{1}$$

where

$$\kappa_n = \gamma + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left[\zeta(k+1) - \frac{1}{k}\right]$$

which is to be compared to our

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} (-1)^n b_n \binom{s}{n}$$

and similar but slightly more verbose expression for the b_n. That paper finds, as we do that

$$|\kappa_n| \leq 2\exp -2\sqrt{\pi n}$$

via a formula reminiscent of our steepest descent formula; see the integral representation for the κ_n just after formula (2.4), in Thm. 1. I don't quite understand why, but it seems to have to location of the saddle-points built-in to the formula. Also, similarly, they give an exact expression for the series $\sum \kappa_n t^n$ that is similar to our results. I suppose that some of the other expressions in our paper might be slightly simplified by expanding about $s = 1$ instead of $s = 0$.

After that, they wander off into some linear equations whose purpose is a bit hard to follow, but I think they are meant to be an algorithm for efficiently computing the κ_n ("Method vichuslenuja kozfutsuentov ryada Newtona"). This ends with a table of values for the κ_n to 9 digits ("Rezultati chuslenik experimentov") and a table of $\zeta(s)$ values for assorted complex-valued s, to 15 digits. From this, I gather that they are proposing that (1) is a convenient numerical technique for computing $\zeta(s)$. (I cannot speak Russian, although I can barely read the Cyrillic alphabet and a few stray words.)

The second paper, from 1988, is "O vichuslenii prouzvoduik zeta-funktsii Riemana v komplexnoi oblasti", which I think translates to "The computiation of derivatives of the Riemann zeta for complex argument". This recaps the starting point of the first paper, but then seems to promptly head off into an Euler-Maclaurin approximation (equation 2.1) and then some recurrence relations, followed immediately by tables for $\zeta'(s)/\zeta(s)$ for real s. This is followed by what appears to be an appendix, something about the "interpolation formula of Newton for the Riemann zeta" which evaluates in detail the integral formula for the κ_n. This time, they find not only the exponential decay, but also the leading factor $(\pi/n)^{1/4}$. There are some higher-order factors given, although I can't see that lower-order factors weren't neglected in obtaining these. Although they numerically observed the oscillation of the κ_n in the 1985 paper, they do not seem to discuss the oscillations in either the first or the second paper.

–linas

p.s. I am cc'ing serebro53@mail.ru which is the email address given on that page; I do not know who that email belongs to, but perhaps they can clarify or amplify what I've said above.