Towards RH

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Abstract

Notes about the Riemann hypothesis, specifically, connection to Baker's map. "pattern recognition", a bunch of things that look similar.

1 Overview

Consider the Bernouli map $b(x) = 2x - \lfloor 2x \rfloor$, which is a function of the unit interval onto the unit interval. Its transfer operator is

$$[\mathscr{L}_B f](x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right) \tag{1}$$

which maps functions on the unit interval to other functions on the unit interval. This operator has eigenvectors given by the Hurwitz zeta function, which is easily expressed in terms of a closely related function, a variant of the polylogarithm:

$$\beta(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n^s} = \text{Li}_s\left(e^{2\pi ix}\right) \tag{2}$$

so that

$$[\mathscr{L}_B \beta](x,s) = 2^{-s}\beta(x,s) \tag{3}$$

That is, every possibly value of s corresponds to an eigenvalue. In the above, and in what follows, x is taken to lie on the unit interval.

There are several remarkable aspects. First, one has that $\beta(0,s) = \beta(1,s) = 0$ if and only if s is a zero of the Riemann zeta function. This seems important somehow.

XXX wtf. ?? To prove RH, one would need to show that some variant of $\mathscr{L}_B$ is unitary, and that this vanishing is somehow important for the space of functions to be considered. That is, one wants to somehow show that $\mathscr{L}_B = \sqrt{2}\exp iH$. where H is the Hilbert-Polya operator.

The eigenvectors are countably degenerate, in that s and $s' = s + 2\pi in/\ln 2$ for $n \in \mathbb{Z}$. Not sure, but I beleive no two zeros on the critical line correspond to degenerate eigenvalues; this would need proof.

Next curiosity: the Bernoulli operator can be interpreted as a one-sided shift operator on a one-dimensional lattice. The is suggestive of M.V. Berry's hypothesis that the Hilbert-Polya operator is somehow px for p the momentum operator, and x the position operator.

2 P-adics in general

The general p-adic operator is

$$[\mathscr{L}_p f](x) = \frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right) \tag{4}$$

and one has

$$\mathscr{L}_p \beta(x;s) = \frac{1}{p^s}\beta(x;s) \tag{5}$$

Note that $\beta(x;s)$ are the only eignefunctions that work for any value of p. One may construct fracal eigenfunctions that work for a given p, but those fractals are not eigenfuncions of the other values of p.

This allows the syntetic operator to be constructed:

$$\mathscr{Z} = \sum_{p=1}^{\infty} \mathscr{L}_p$$

having eigenvalues given by the Rieman zeta:

$$[\mathscr{Z}]\beta(x;s) = \left[\sum_{p=1}^{\infty} \mathscr{L}_p\right]\beta(x;s) = \sum_{p=1}^{\infty}\frac{1}{p^s}\beta(x;s) = \zeta(s)\beta(x;s) \tag{6}$$

Note that the Hurwitz zeta is the only eigenfunction of this, and that the fractal eigenfunctions are not. Similarly,

$$\mathscr{Z}_\chi = \sum_{p=1}^{\infty} \chi(p)\mathscr{L}_p$$

where χ is a Dirichlet character clearly has $\beta(x;s)$ as an eigenvector, and $L(s,\chi)$ as an eigenvalue.

The noteworthy observation here is that the zeros of the L-functions describe a set of eigenvectors that lie in the kernel of $\mathscr{Z}_\chi$.

Is there a short exact sequence of any sort?

Is there a Fredholm kernel we can apply here?

3 Integer sequences

Four examples of a discrete spectrum, taking values at the integers, embedded in a continuous spectrum.

3.1 The Simple Harmonic Oscillator

The quantized Hamiltonian $H = \frac{1}{2}\left(p^2 + x^2\right)$, with $H\psi = \lambda\psi$. Has a continuous spectrum $\lambda \in \mathbb{C}$. The continuous-spectrum eigenfunctions are not square integrable. The cannot be formed into (two-sided) coherent states (but one-sided is possible). There is also another discrete spectrum with decreasing energies ("the inverted harmonic oscillator"). Discrete eigenfiunctions are polynomials times an overall factor $\psi_0(x) = e^{-x^2/2}$.

3.2 Polynomial eigenfunctions of Bernoulli map

Defined above, $\mathscr{L}_2 B_n(x) = 2^{-n} B_n(x)$. Eigenfunctions are polynomials, and thus C^∞. The continuous spectrum is the periodic zeta $\beta(s;x)$ as defined above.

3.3 Fractal eigenfunctions of Bernoulli map

The square wave aka Haar wavelet

$$h(x) = \begin{cases} +1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \end{cases}$$

is in the kernel of $\mathscr{L}_2 h(x) = 0$ and thus can be used to create a fractal eigenfunction

$$\psi(x) = \sum_{n=0}^{\infty} \lambda^n h(2^n x)$$

so that $\mathscr{L}_2 \psi = \lambda \psi$ for any $|\lambda| < 1$ in the unit disk. The fractal symmetry of this fractal eigenfunction transforms as a 2D representation of the dyadic monoid that is the subset of the modular group.

Integrating the Haar wavelet gives the triangle wave, which transforms as the 3D representation. Intrgrating again to get a periodic parabola gives something that transforms under the 4D rep, *etc*. By contrast, using sine-waves to construct the eigenfunctions result in eigenfunctions that transform in an infinite-dimensional kind-of-way; thus, the general fourier-analysis input into the thing will be inf dimensional,with the square, triangle, etc. cases forming a discrete spectrum.

3.4 Monodromy of the Polylogarithm

The monodromy group of the polylogarithm $\mathrm{Li}_s(z)$, for general s, has an inf-dimensional representation, with one of the two generators of the monodromy acting as a shift operator(!). The exceptions occur for $s = n$ a positive integer. For $n = 2$, the monodromy group is nothing other than the integer Heisenberg group, generated by p, x with $[p, x] = 1$. Here, p is the monodromy generator of winding around the branch point at $z = 0$, while x is the monodromy generator of winding around the branch point at $z = 1$. The integer Heisenberg group has the usual representation by 3x3 matrices. For $n = 3$, there is a 4D representation (but not the Heisenberg group), and so on.

Of course, the Riemann zeta lies on the branch point of the polylogarithm, in that $\mathrm{Li}_s(1) = \zeta(s)$. However, its the $z = 0$ monodromy that is the shift state, while the $z = 1$ monodromy "injects" into the shift. Hmmm.

4 Dualities

The evenly-spaced spectrum is curiously dual to the zeros of RH. For example, the sums:

$$\begin{aligned}
\delta_n &= \sum_{k=2}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^k \zeta(k) \\
&= \sum_{k=1}^{\infty} \left[\left(1-\frac{1}{k}\right)^n - 1 + \frac{n}{k} \right]
\end{aligned}$$

and the sums in Li's criterion (per Bombieri):

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n \right]$$

Uhh XXX to do:

Write $\rho = \frac{1}{2} + i\tau$ then is $\sum_\tau \left[\left(1-\frac{1}{\tau}\right)^n - 1 \right]$ oscillatory, the way that the δ_n are? is this true for all such sequences, per Bombieri? Also, what is $\sum_n \lambda_n z^n$ and $\sum_n \lambda_n z^n / n!$?

5 Singular in some way

The Rieman zeta is tangled up with a singularity in several ways. Two examples follow.

5.1 Polylog branch point

For the polylogarithm $\mathrm{Li}_s(z)$, one has $\mathrm{Li}_s(1) = \zeta(s)$. But the polylogarithm has a branch point at $z = 1$, so this relation asks for the value at a branch point! Performing the same analysis for the Dirichlet L-functions shows that these are also at the branch point of the polylogarithm. One the other hand, it can't be that bad, since, from the Dirichlet eta function, one has

$$\zeta(s) = -\frac{1}{1-2^{1-s}} \mathrm{Li}_s(-1)$$

and the polylog has no branch point at $z = -1$.

5.2 Theta function

Starting with the classical theta function $\vartheta(z;\tau)$, defined as

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2 + 2\pi i n z}$$

one has the Mellin transform

$$\int_0^\infty [\vartheta(z,it) - 1]\, t^{s/2} \frac{dt}{t} = \pi^{(s-1)/2} \Gamma\left(\frac{1-s}{2}\right) [\zeta(1-s;z) + \zeta(1-s;1-z)]$$

for z not an integer. For $z = n$ an integer, the relation takes a different form:

$$\int_0^\infty [\vartheta(n,it) - 1]\, t^{s/2} \frac{dt}{t} = 2\pi^{(s-1)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(s)$$

the trick here is that th $z \to n$ limit is non-trivial, because the theta function has a non-trivial limit

$$\lim_{t\to 0} \vartheta(z;it) = \sum_{n=-\infty}^{\infty} \delta(z-n)$$

So it seems the Riemann zeta is associated with the singular points.