Lattice models and fractal measures

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Abstract

This paper reviews connections between the Bernoulli map, the baker's map and the Ising model. The Bernoulli map is possibly the simplest exactly solvable model of deterministic chaos. Its Frobenius-Perron operator or transfer operator has a set of well-known polynomial eigenvectors, given by the Bernoulli polynomials. Its also has a set of smooth but non-integrable eigenvectors given by the Hurwitz zeta function. Alternately, it has a set of fractal eigenvectors, of which the Blancmange or Takagi curve is one.

An even larger set of functions that are eigenvectors of the Bernoulli operator may be obtained by considering a one dimensional lattice model, such as the Ising model. Functions that are translationally invariant on a one-dimensional lattice are nothing more or less than eigenvectors of the Bernoulli operator with eigenvalue one. These functions, when graphed on the real number line, are in general not even continuous.

Functins that are continuous-nowhere are traditionally considered to be intractible, and usually, very little can be said about them. By identifying them with a set of simple, tractable operations on a one-dimensional lattice model, one obtains a set of tools with which to manipulate and analyze such functions.

A notable example of a discontinuouis-everywhere function that is not traditionally integrable, yet, when properly defined, can be integrated, is derivative of the Minkowski question mark function. It, and its generalizations are known as "multi-fractal measures". Thus, this paper explores what it means to "take the derivative of the question mark function". As a side-effect, all of the various functions that occur have a particular fractal self-symmetry, which can be made explicitly manifest; it is the symmetry of the dyadic monoid, that is, the period-doubling subset of the modular group $SL(2,\mathbb{Z})$.

This document started as a research diary noting various results, and is haphazardly structured.

AT THIS TIME, THIS PAPER MAKES SEVERAL INCORRECT STATEMENTS, AND SEVERAL MORE MISLEADING REMARKS. PLEASE BE PATIENT WHILE THE AUTHOR FIGURES OUT HOW TO BEST FIX THEM. CAVEAT EMPTOR.

This paper is part of a set of chapters that explore the relationship between the real numbers, the modular group $SL(2,\mathbb{Z})$, and fractals.

1 Intro

THIS IS A DRAFT WORK IN PROGRESS. The intro hasn't been written yet, but if it was, it would work like this:

Exactly solvable chaotic dynamical systems are interesting because they provide insight into the behaviour of chaotic dynamical systems "in the large". The simplest exactly solvable model of deterministic chaos is the iterated Bernoulli map[**?**]. It is a map of the unit interval onto itself, given by

$$b(x) = 2x \bmod 1 \tag{1}$$

Picking some "random" irrational x and then iterating this map, one sees $b^n(x)$ bounce all over the unit interval: the map is ergodic. One may compute correlation functions for it, and other statistical measures.

One method for studying a chaotic map is in terms of its Frobenius-Perron or transfer operator. The utility of the transfer operator can be introduced by an appeal to physics, and the statistical mechanics of continuum systems. The argument goes as follows: while iterated maps have a point dynamics that bounces all over the place, in nature, physical observables are smooth. Therefore, one should study the dynamics of a smooth distrubution of particles under the action of chaotic operators. The transfer operator gives exactly such an opportunity. It is defined as a certain map $L : \mathcal{F} \to \mathcal{F}$ where $\mathcal{F}$ is some collection of functions on the unit interval. For example one may take $\mathcal{F} = \mathcal{P}(x)$ the set of polynomials in one variable, or $\mathcal{F} = L^2([0,1])$ the set of square-integral functions on the unit interval. Given some iterated function (for example, $b(x)$), it is defined as the map

$$\mathcal{L}f = g \tag{2}$$

with $f, g \in \mathcal{F}$ such that, on a pointwise basis, one has

$$g(x) = \sum_{y \in b^{-1}(x)} \frac{f(y)}{b'(y)} \tag{3}$$

where $b'(y)$ is the derivative of b at y. When the pre-image $b^{-1}(x)$ is not a countable collection of points, the sum may be replaced by an integral. The transfer operator can be defined with varying degrees of sophistication and rigor, and can be found in many places in the literature; simple, workable definitions can be found in [**?**, **?**]. In short, while iterating a function such as $b(x)$ gives the time evolution of a point, iterating the operator L gives the time evolution of a smooth distribution. A general feature that is physically appealing is that the transfer operator typically has only one eigenstate associated with an eigenvalue of 1: this is the ground state or equilibrium state. All other eigenstates are associated with real eigenvalues that are less than one: upon iteration, the associated eigenstates must necessarily die away geometrically. This behaviour is physically appealing: it describes the approach to equilibrium of a system that may initially be far from equilibrium, and it provides a set of time scales at which that approach is made.

The transfer operator of the Bernoulli map is called the Bernoulli operator $\mathcal{L}_B$ and is given by

$$[\mathcal{L}_B f](x) = g(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{1+x}{2}\right) \right) \qquad (4)$$

When $\mathcal{F}$ is the space of polynomials, the eigenfunctions of $\mathcal{L}_B$ are the Bernoulli polynomials $B_n(x)$, and are associated with the eigenvalues 2^{-n}[?, ?, ?]. The constant function $B_0(x) = 1$ is associated with the eigenvalue 1; it is the only non-decaying eigenfunction associated with the iteration of $\mathcal{L}_B$: the equilibrium state is a constant distribution. The spectrum of $\mathcal{L}_B$ is discrete.

If one considers a larger space of functions $\mathcal{F}$, then one finds that $\mathcal{L}_B$ has a larger set of eigenvalues. For example, if one considers the collection $C^\infty(0,1)$ of infinitly-differentiable functions on the *open* unit interval (*i.e.* not containing the endpoints), then the spectrum of $\mathcal{L}_B$ is continuous, and takes on all values in the unit disk $|\lambda| < 1$ of the complex plane. The eigenstates are given by the Hurwitz zeta function[?]. The spectrum is degenerate, in that there are multiple eigenstates associated with each eigenvalue. Curiously, linear combinations of the Hurwitz zeta function may be taken, constructing the fractal Blancmange or Takagi curve. That is, for every eigenvalue on the complex unit disk, there is a corresponding fractal curve that is an eigenvector of $\mathcal{L}_B$, with that fractal having a dyadic self-symmetry. The Takagi curve is continuous but differentiable nowhere; however, it is bounded, and it is integrable. In this larger space, there is still only one unique eigenvector with eigenvalue one: this is still the constant function.

There is yet a larger space of functions $\mathcal{F}$ that one may consider, and this is the space of functions that are continuous nowhere on the unit interval and are unbounded in magnitude, but are, in a certain sense, integrable. In classical analysis, such functions are outcasts, seemingly intractible and uncoercible. However, they may in fact be constructed in a very natural fashion: these are functions on the state space of a one-dimensional lattice model. They may be mapped to the unit interval with the dyadic mapping, and when mapped in this way, are plainly discontinuous and generally beastly. However, the lattice posses an alternate topology, which makes these functions not only manipulable but well-studied. Examples of functions on the lattice include the energy of the Ising model, or the Boltzmann probability distribution of the set of states of the Ising model. In the lattice model, the Bernoulli map corresponds to nothing more than the shift of the lattice by one: a translation of the lattice. The eigenvectors of $\mathcal{L}_B$ are simply functions whose magnitude changes geometrically as the lattice is shifted by one: $\mathcal{L}_B$ is just the shift operator. Curiously, there is now a very large class of functions associated with the eigenvalue one: this is the class of translation-invariant functions.

The remainder of this paper is devoted to an exposition of the lattice-model construction of eigenstates of the transfer operator. An exploration is made not only of the Bernoulli map, which corresponds to a one-sided lattice, but also to the baker's map, which corresponds to a two-sided lattice. Since such continuous-nowhere functions seem outrageous, a motivation for this study is provided in the next section. It is a "multi-fractal measure", the derivative of the Minkowski question mark function, which is continuous nowhere, and unbounded, and yet is integrable. As multi-fractals

occur naturally in physical systems, one cannot simply dismiss such crazy eignevectors of $\mathcal{L}_B$ as unphysical: they are not only physical, but actually observed.

2 Distribution of the Rational numbers in the Farey tree

One way to enumerate all of the rationals is by placing them on the Farey tree (or the Stern-Brocot tree). It can be shown that the Farey tree enumerates all of the rationals in the unit interval exactly once[**?**]. However, the distribution of the rationals on the Farey tree is highly non-uniform, if the tree is walked breadth-first. The figure 1 shows that distribution. This distribution is highly non-uniform, and by fooling with the number of bins in the histogram, it becomes clear that the figure is highly discontinuous. As the number of bins is increased, it becomes clear that the distribution has a peculiar structure: it always has a humped form, even though the individual spikes vary strongly according to the number of bins.

The shape is suggestive, and the numerical evidence makes it clear that it is somehow the "derivative" of the Minkowski question mark function $?'(x) = d?(x)/dx$. Performing a simple numeric integral of the above, and comparing it to the exact expression shows a perfect match, as witnessed in the figure 2.

The derivative of the Minkowski question mark function cannot be defined using the classical analytic techniques. More precisely, traditional analytic tools applied to the "natural" topology of open sets on the real number line are insufficient to construct and describe the shape of this curve. This motivates the study of alternative topologies for the real number line, which constitutes one of the research programs of this paper.

When one does attempt to use classical techniques applied to the classical topology, the result is a number of perposterous-sounding statements, a sampling of which are given below. The question mark is highly singular. It is infinitely differentiable at every rational, and all derivatives vanish at every rational; yet the mapping is strictly monotonically increasing. Just examining the graph visually suggests that the derivatives must be non-zero "somewhere", although how to define how to value these derivatives is utterly unclear. Naively, they seem to be "infinite" at every irrational; but this fails to explain how to integrate.

Clearly, the classical approach is sorely lacking and inapplicable; something different is needed. Two possibilities arise: throw away traditional ideas of integration and differentiation, and replace them with something new, something sutiable for fractals, or keep these tools, but throw away the "natural" topology on the real number line. The latter approach seems to be far more promising. There are several popssible alternative topologies. Most immediately obvious is the topology of the Cantor set, which is readily mapped to the real number line. (discussed elsewhere in this cycle of papers). Less obvious, but more appropriate, is the topology of "cylinder sets" that is the natural topology for studying shifts, symbolic dynamics, and one-dimensional lattice models (discussed later).

The figure 3 shows what could be called the Jacobian $(?' \circ ?^{-1})(x)$ of the question mark function. This figure simple relabels the x-axis by passing it through $?(x)$. This

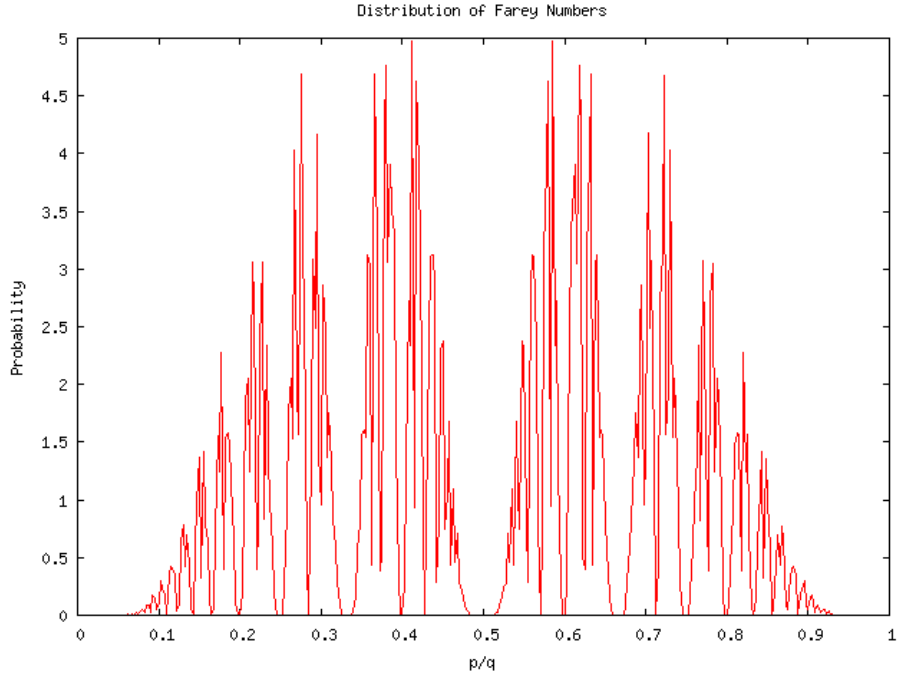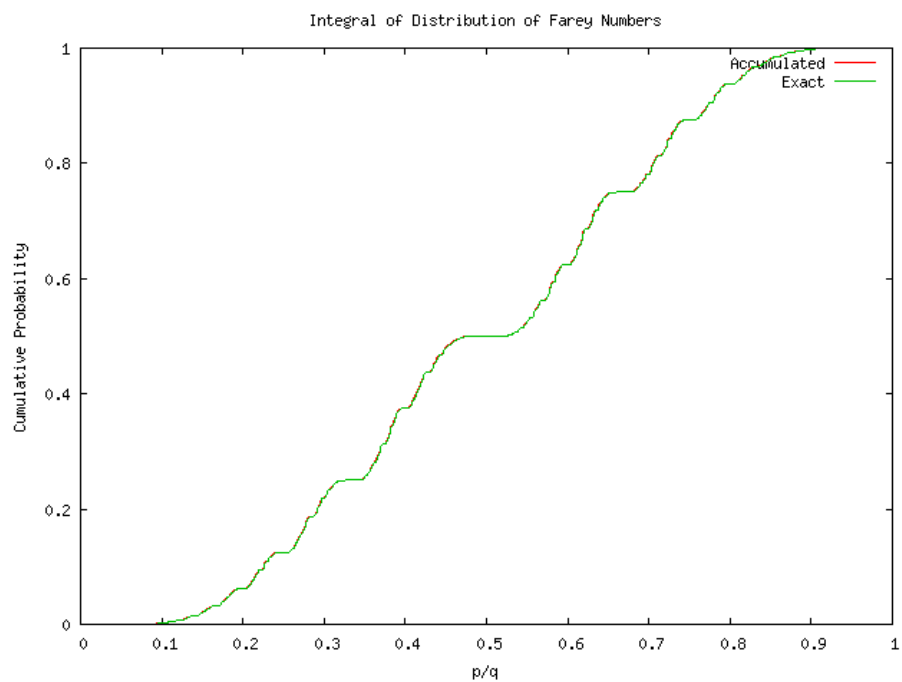Figure 1: Distribution of Farey Fractions

This figure shows a histogram of the first M=65536 Farey fractions according to their location on the unit interval. To create this figure, the unit interval is divided up into N=600 bins, and the bin count c_n for each bin n is incremented whenever a fraction p/q falls within the bin: $(n-1)/N \leq p/q < n/N$. At the end of binning, the count is normalized by multiplying each bin count c_n by N/M. The normalization ensures that the histogram is of measure one on the unit interval: that $\sum_{n=0}^{N} c_n = 1$. In a certain sense, the "shape" of this histogram is independent of the number of bins and number of fractions counted, although on a highly detailed level, the graph clearly varies with the number of bins N. In a proper formal treatment, we would need to show that the limit $\lim_{M \to \infty} c_n$ exists and is finite, for all n and N.

Figure 2: Sum of Farey Histogram

Integral of Distribution of Farey Numbers

This figure shows the sum of the bin counts of the Farey fractions, together with a graph of the Minkowski question mark function. The two graphs can hardly be told apart.

function appears to vanish only at the dyadics. Note the superficial resemblance of the profile to the Takagi curve. By visually comparing this function and the Takagi curve (also called the *blancmange curve*), one can intuitively sense that there must be a more formal relation between the two, as if the values of the one function could somehow be exponentiated to give the other. The trick is to write down this relation.

This is really a very curious circumstance. The above graphs show only the distribution of a finite number of points (they were, after all, computer generated). A finite set of points on the real number line is zero; yet we are using it to induce a measure that can be used for integration. How is this? The statistical properties of a finite sampling provide us with information about the limit. Its as if we were working with an anti-Cantor-set construction: we start with a dust, and average over intervals to discover analytic properties.

2.1 Notes

Some slightly formal but not very fruitful remarks follow. To prove that the measure vanishes on rationals, one must show two things: first, that the measure vanishes near 0 (or 1), in that the the bin count for bin c_0 (and c_1 and a few neighbors) is always bounded above by $N^{-\alpha}$ for some $\alpha > 1$. In fact, α is probably quite large, and the bound can be made very strong. Next, one needs to show a modular group symmetry for the histogram: thus for example, that $c_{pN/q}$ obeys a similar bound, but de-rated by a factor given by the modular group element that carries $0/1$ to p/q.

3 Spin lattice models

In ergodic theory, there is a well-known set of isomorphisms and correspondences between one dimensional spin lattice models (such as the Ising model or Potts model), sub-shifts of finite type, and motion on hyperbolic surfaces. See, for example, [**?**] for a review. This section reviews a small portion of this theory in a simplified manner, applying it in a concrete fashion to the question mark function.

The importance of lattice models to the study of dynamical systems arises from the fact that the natural topology of the one-dimensional lattices, the so-called "cylinder set topology" is a refinement of the natural topology of the real number line. As such, it allows for the construction and consistent discussion of a much larger class of functions than the natural topology of the real number line allows. In particular, functions which are well-behaved in various ways (such as being smooth, or being symmetric) in the lattice topology are non-differentiable and non-integrable when mapped to the real number line. Thus, the cylinder set topology allows for a natural setting for the discussion of objects that otherwise appear to be difficult to manipulate.

Possibly the very simplest exactly solvable model of deterministic chaos is the Bernoulli map[**?**]. The Bernoulli map can be understood to be the (one-sided) shift operator on the set of (infinite) string strings in two letters. The Bernoulli map, like all discrete dynamical systems, can be studied in terms of its Frobenius-Perron operator or transfer operator. The transfer operator is a curious construction. Consider, for example, the set of all possible functions from the set of strings in two letters, to the real

Figure 3: Jacobian

Distribution of Farey Numbers
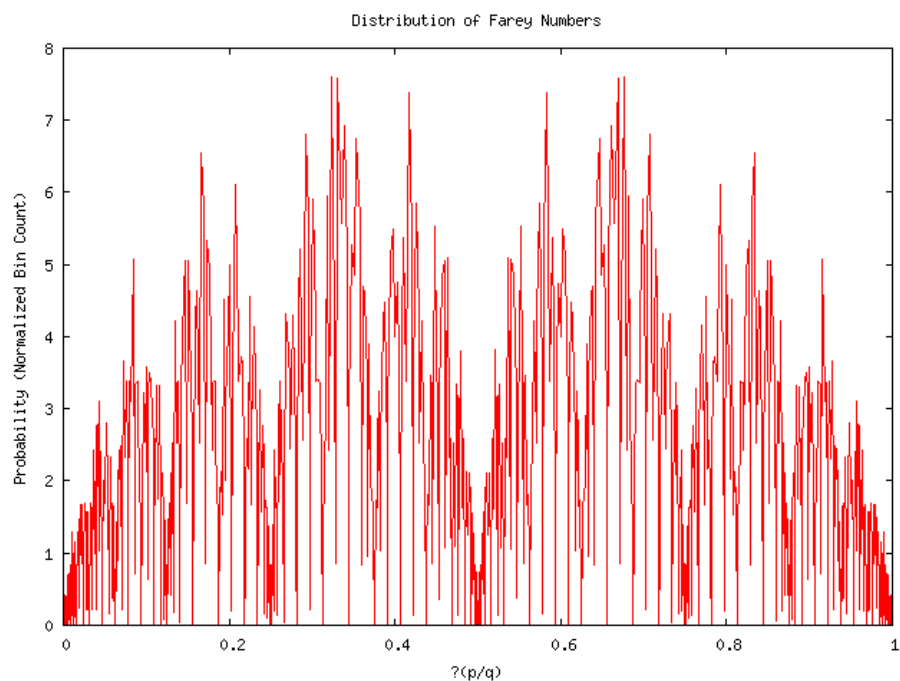
This figure shows the same Farey fraction distribution as in a previous figure, except that the x-axis is relabelled by passing it through the Minkowski question mark. In a certain sense, this can be taken to be the Jacobian $(?' \circ ?^{-1})(x)$ of the question mark.

Integral of ?'(?^-1 (x))
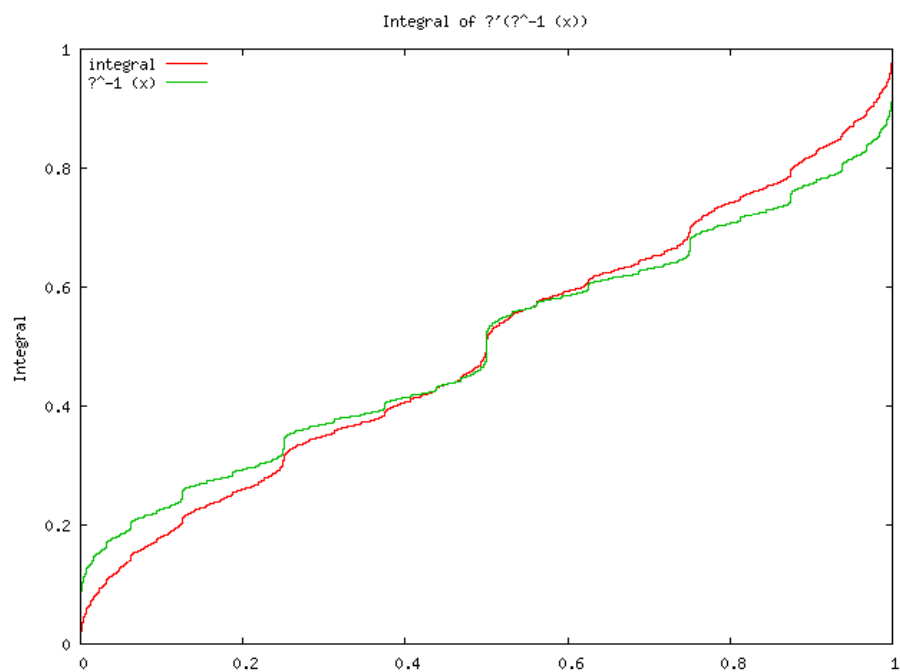
This figures shows the integral $\int_0^x (?' \circ ?^{-1})(y)\,dy$, and, by comparison, the value of $?^{-1}(x)$ which it resembles.

Figure 4: Takagi Curve
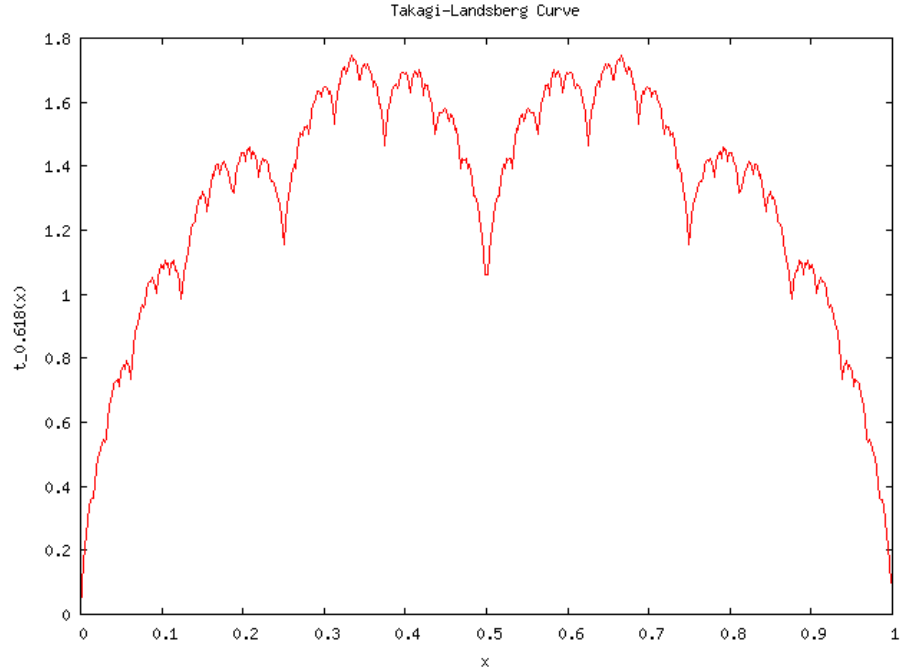
A graph of the Takagi-Landsberg curve $t_w(x)$ as a function of x for a fixed value of $w = 0.618$. Note the curious visual resemblance to the Jacobian of the Question mark function. The Takagi curve, explored in another chapter, is given by

$$t_w(x) = \sum_{k=0}^{\infty} w^k \tau\left(2^k x\right) \tag{5}$$

where $\tau(x)$ is the triangle function:

$$\tau(x) = \begin{cases} 2(x - \lfloor x \rfloor) & \text{for} \quad 0 \leq x - \lfloor x \rfloor < 1/2 \\ 2 - 2(x - \lfloor x \rfloor) & \text{for} \quad 1 > x - \lfloor x \rfloor \geq 1/2 \end{cases} \tag{6}$$

The fractal symmetries of the Takagi curve are given by a three-dimensional representation of a sub semi-group of the modular group (see other chapter).

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numbers. The transfer operator is then a representation of the the shift operator on this function space. That is, it describes what happens to the function when the underlying string is shifted by one. Transfer operators, being operators, may be decomposed in terms of eigenvalues and eigenvectors. Often one is interested in the polynomial or holomorphic eigenvectors. One may also study the square-integrable eigenvectors. Sometimes, the polynomial eigenvectors are a subset of the square-integrable eigenvectors, sometimes not. The transfer operator may be unitary in one function basis, but not in another; it need not in general be either a compact operator or a trace-class operator.

If one considers a one-dimensional lattice where each lattice location can take one of two values, then the Bernoulli operator is nothing more and nothing less than the shift or translation operator on this lattice. Lattice models provide a very natural way to construct translation-invariant functions on them. But of course (and as will be shown), a translation-invariant function on a lattice is nothing more and nothing less than an eigenvector of the transfer operator, with an eigenvalue of one: indeed this is the definition of translation invariance. The set of translation-invariant functions on a one-dimensional lattice is much, much larger than the set of square integrable eigenvectors of the transfer operator, which is in turn much larger than the set of polynomial (or holomorphic) eigenvectors of this operator. Thus, the study of lattice models provides a broader context for the discussion of dynamical systems, and the lattice model topology allows for the discussion and manipulation of functions (such as the derivative of the Minkowski function) that are not otherwise easy to manipulate.

The Bernoulli map is a one-sided shift operator, and it does not have a time-reversible dynamics. It corresponds to the shift operator on a lattice with extends only in one direction. The baker's map provides a simple, solvable time-reversible chaotic dynamical system. The baker's map corresponds to a bi-infinite one-dimensional lattice, where the shift can be made to the right or to the left. The connection to the Ising model is, in one sense, shallow: the Ising model just happens to show how to construct translation-invariant functions on a lattice, and thus construct eigenvectors (with unit eigenvalue) of the transfer operator. In another sense, the connection is much deeper: by understanding the cylinder set topology, and the fractal self-similarities that it generates, one may gain a better understanding of the structure of functional integrals in quantum field theory. The ideas are not disjoint: rather the methods and theorems from one field of study can be imported to better understand the other.

3.1 Overview

Consider the one-dimensional, two-state lattice model, with the lattice having a countable infinity of lattice positions. The set of all possible states of the lattice is the set of all possible (infinitely-long) strings in two letters. This set of strings is a very large set: it has $\aleph_1 = 2^\omega$ elements in the set. The cardinality of this set is the cardinality of the continuum. The (classical) Hamiltonian is a function that assigns a real value to each possible element of this set. The Hamiltonian is not the only physically interesting function; in physics, one may also be interested in the partition function, the mean magnetization, ore correlation functions. In mathematics, there are yet more interesting aspects.

When discussing such lattice models, a problem arises: how should one label the points in this set? Each corresponds to an infinite string in two letters. This set is not enumerable, so one cannot label them with integers. Operations with respect to the label, such as summation or integration, are potentially troublesome. Summation requires a countable label; summation is defined only on countable sets. The analog of summation for uncountable sets is integration; however, integration cannot be simply or naively defined on uncountable point-sets, and requires the framework of measure theory to be mathematically rigorous.

The following sections will develop the two-state one-dimensional lattice model, primarily with the goal of elucidating the structure of Minkowski's question mark function. It is important to realize that lattice models can be generalized in a large variety of ways, and many of these generalizations have important implicaitons for physics, and are mathematically interesting for a large variety of reasons. Almost none of these generalizations or connections are mentioned below; they can be found in [**?**]. The focus here really is intended to be the question mark function.

The discussion will explore several directions, focusing on the natural topology of one-dimensional lattice models, and relating that topology to the Cantor set, and the natural topology on the real numbers. The full exposition requires the application of measure theory and sigma algrebras to the one-dimensional lattice models; these ideas are presented here as simply as possible, assuming little prior acquaintance. One can also gain considerable intuition by sticking to naive point-set topology, that is, by working with a lattice of finite size N and imagining the possibility of taking the limit of $N \to \infty$. The finite-sized lattice provides a simple, concrete system for discussion, and is thus presented first, in the following section.

3.2 Naive lattice theory

The naive approach to lattice theory is to work with a finite-sized lattice, one with N lattice points, and then attempt to perform the $N \to \infty$ limit. In the following, we work with a one-dimensional, two-state lattice model, where the two states are alternately denoted by A, B or $0, 1$ or L, R or $-1, +1$, as suitable for the given context.

Given a lattice of N sites, there are 2^N different possible lattice configurations, that is, 2^N different possible strings in two letters. One defines an interaction potential between neighboring spins in the lattice. For the traditional Ising model, the interaction potential is

$$V(\sigma) = J\sigma_0\sigma_1 + M\sigma_0 \tag{7}$$

Here, $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots) \in 2^N$ is one possible configuration, and $\sigma_k \in \{-1, +1\}$ is the spin at lattice location k. The Ising potential consists of a nearest-neighbor interaction, with energy $J \in \mathbb{R}$ and the interaction of the spin with an external magnetic field M. The physical interpretation of the words "spin" and "magnetic field" or even "energy" is not important for the development here; these are merely the terms used by physicists to label these quantities.

The total energy of a given lattice configuration σ is given by

$$H(\sigma) = \sum_{k=0}^{N} V(\tau^k \sigma) \tag{8}$$

where τ is the shift operator: $\tau(\sigma_0, \sigma_1, \sigma_2, \ldots) = (\sigma_1, \sigma_2, \sigma_3, \ldots)$. That is, one takes the interaction potential to be translation invariant, and so sums over all nearest-neighbor interactions. In fact, the class of translation invariant interactions can be shown to be isomorphic to a certain class of measure-preserving dynamical systems, again, see [**?**] for details. The above summation glosses over boundary conditions, that is, it ignores what happens when $k = N$. This gloss can be hand-waved away by arguing that it makes no difference to the $N \to \infty$ limit. Similarly, the lattice is considered to extend only to the right, *i.e.* to exist for non-negative k values only; this is another gloss that makes little practical difference, and is useful to keep notation simple and unburdened.

Quantities, such as the energy, may be readily graphed by identifying lattice locations with bit positions in the bit-string representation of a dyadic rational. That is, one represents the lattice configuration σ with the rational $x(\sigma)$ given by

$$x(\sigma) = \sum_{k=0}^{N} \left(\frac{\sigma_k + 1}{2} \right) 2^{-(k+1)} = \sum_{k=0}^{N} b_k 2^{-(k+1)} \tag{9}$$

where $b_k \in \{0, 1\}$ is the lattice configuration at the k'th site. A graph of the energy, as a function of x, is shown in the figure 5. An obvious fractal-like shape is readily visible, and it is straight-forward to give an explicit representation of the action of the dyadic monoid on this fractal curve. This is given in a later section.

An important idea in statistical mechanics is not only the energy of each configuration σ, but also a probability or likely-hood that it occurs in the "canonical ensemble" of all possible configurations. The probability is usually given by the Boltzmann distribution

$$P(\sigma) = \frac{1}{Z(\Omega)} \exp\left(-\frac{H(\sigma)}{k_B T} \right) \tag{10}$$

where T is interpreted as the temperature and k_B is Boltzmann's constant. For this discussion, the physical interpretation of these values is not particularly important; what is important is that $P(\sigma)$ can be interpreted as a probability (more precisely, a *probability density function*): it is non-negative, and with an appropriate normalization factor $Z(\Omega)$, it sums to one. Here, $\Omega = 2^N = \{\sigma\}$ is used to denote the set of all possible configurations. A graph of this probability function for the Ising model is shown in figure 5.

At least a superficial resemblance of the probability to the distribution of the Farey numbers of figure 1 is readily apparent. This resemblance can be emphasized by graphing the the cumulative distribution as a function of $x(\sigma)$, that is, graphing

$$F(y) = \sum_{\sigma; x(\sigma) < y} P(x) \tag{11}$$

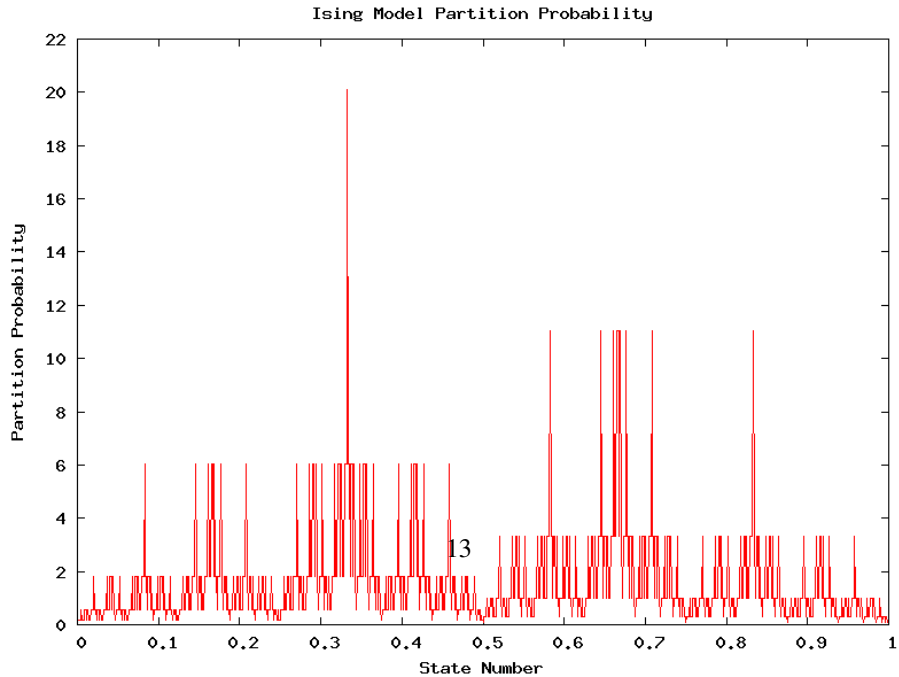A graph of the cumulative distribution is shown in the figure 6.

3.2.1 Fractal self-symmetry

In this section, we want to give explicit expression to the apparent fractal self-similarity seen in the Ising model. To simplify the discussion, this section assumes the limit $N \to \infty$, or at least that N is indeterminate when the $N \to \infty$ limit is ambiguous or ill-defined.

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Figure 5: Ising Model Configuration Energy
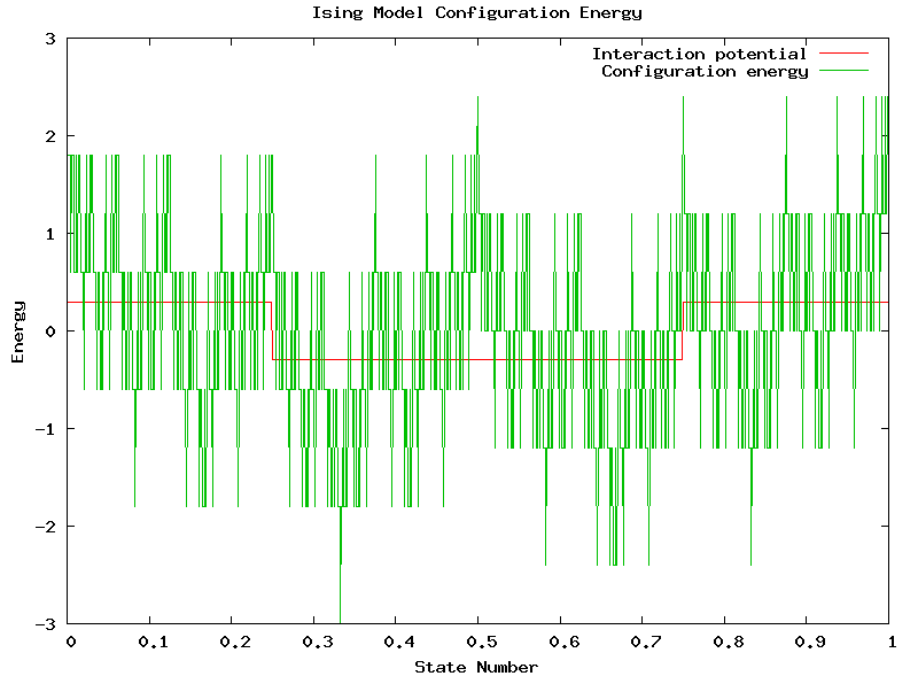
This figure shows the energy of various configurations of the Ising model. A lattice with ten lattice points is considered, so the total number of possible lattice states is $2^{10} = 1024$. The energy of each of these possible lattice configurations is shown in green, while the interaction energy is shown in red. The lattice states are labelled and ordered along the horizontal axis by the rational number $x = \sum_{k=1}^{10} b_k 2^{-k}$ where $b_k \in \{0, 1\}$ is the lattice configuration at the k'th location. The interaction energy, shown in red, is given by $V = 0.3(2b_0 - 1)(2b_1 - 1)$.

This figure shows the (un-normalized) partition probability $P(\sigma) = \exp(-H(\sigma)/k_B T)$ for the same lattice and interaction as above. The temperature and units are taken such that $k_B = T = 1$.

Figure 6: Ising Model Cumulative Distribution
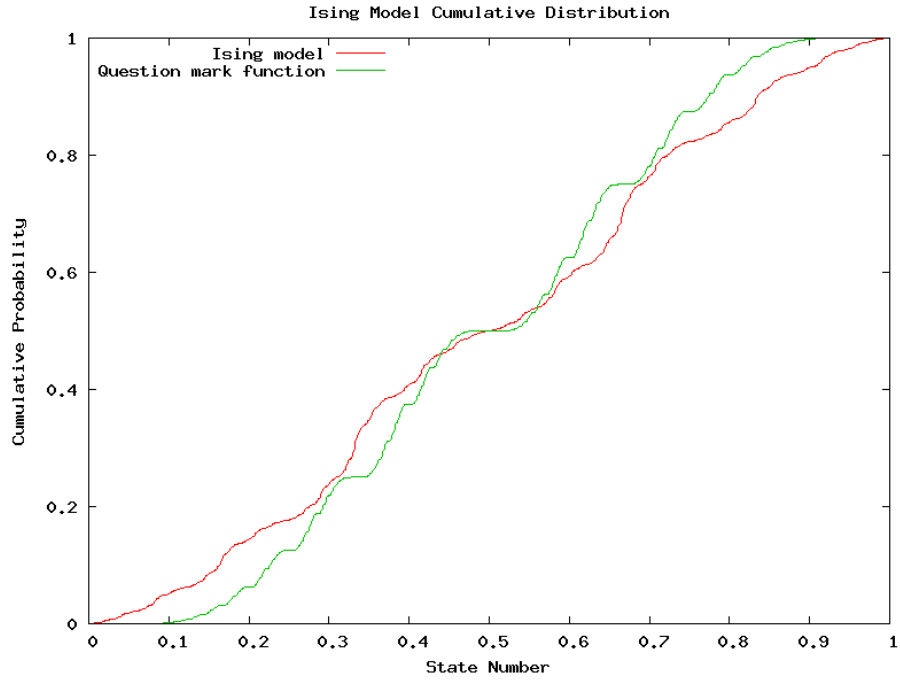
This figure shows the cumulative distribution $F(y)$ defined by equation 11, for the probability distribution 10 for the Ising model interaction 7. As in the other graphs, the lattice size is taken as $N = 10$, so that $2^N = 1024$ states are graphed. The Ising interaction is taken with $J = 0.3$ and $M = 0$. The cumulative distribution is fractal, and its fractal self-similarity can be given a precise expression. For comparison, the question mark function is also graphed. Although the shapes are clearly different, none-the-less, this begs the question: is there some interaction potential that can result in a perfect match?

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As before, the lattice is assumed to be one-sided: that is, it starts at 0 and stretches off to the right, so that lattice positions can be labelled by non-negative integers. The two-sided lattice is treated in a later section; while there are many similarities, there are also important differences between the one-sided and the two sided lattice.

Consider the behavior of the classical Hamiltonian 8 under the right shift of the lattice, with a new spin s introduced to occupy the zeroth position. One then has

$$H(s, \sigma_0, \sigma_1, \sigma_2, \cdots) = V(s, \sigma_0, \sigma_1, \sigma_2, \cdots) + H(\sigma_0, \sigma_1, \sigma_2, \cdots) \tag{12}$$

Taking $s = -1$ and using the dyadic mapping 9, the above may be written as

$$H\left(\frac{x}{2}\right) = V\left(\frac{x}{2}\right) + H(x) \tag{13}$$

while for $s = +1$ one has

$$H\left(\frac{1+x}{2}\right) = V\left(\frac{1+x}{2}\right) + H(x) \tag{14}$$

For the probability distribution, one has (after taking $k_B = T = 1$)

$$P\left(\frac{x}{2}\right) = e^{-V(x/2)} P(x) \tag{15}$$

and so on. As any true self-similarity, this can be iterated repeatedly, so that

$$P\left(\frac{x}{2^n}\right) = P(x) \prod_{k=1}^{n} e^{-V(x/2^k)} \tag{16}$$

For the Ising potential 7, the function $V(x)$ takes a particularly simple form:

$$V_{\text{Ising}}(x) = \begin{cases} J - M & \text{for } 0 \leq x < \frac{1}{4} \\ -J - M & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ -J + M & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\ J + M & \text{for } \frac{3}{4} \leq x \leq 1 \end{cases} \tag{17}$$

and so the probability distribution has a simple scaling:

$$P_{\text{Ising}}\left(\frac{x}{2^n}\right) = \lambda^n P_{\text{Ising}}(x) \qquad \text{for } 0 \leq x \leq \frac{1}{2} \tag{18}$$

where $\lambda = \exp(M - J)$.

Another natural symmetry operation of the lattice is the exchange of spin values $+\sigma_k \leftrightarrow -\sigma_k$. For the dyadic mapping, this corresponds to the exchange $x \leftrightarrow 1 - x$. Let the exchange of spin values be denoted by the operation r (for *reflection*); the dyadic representation of r is the function $r_D(x) = 1 - x$. Let the right shift operation, with an insertion of a spin of -1 at the zeroth lattice position be denoted by g. The dyadic representation of g is $g_D(x) = x/2$. The operations g and r, taken as a free monoid (that is a free group, but with non-negative powers only) generate the dyadic monoid. A synopsis of the properties of this monoid are given below, after which the analysis of self-similarity resumes.

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3.2.2 The dyadic monoid

Given operations g and r acting as representations of translation and reflection, one has in essence constructed everything that is needed to have a fractal with dyadic symmetry. That is, one need only to recognize that g and r generate the dyadic monoid of the modular group $SL(2,\mathbb{Z})$. The elements of this monoid are of the form

$$\gamma = g^a r g^b r g^c \cdots \tag{19}$$

where $a, b, c, \ldots$ are non-negative integers. This section provides a very brief synopsis of the general properties of monoid elements γ; these properties are presented in greater detail in [?, ?]. First, all such elements may be represented by the continued fraction

$$\gamma_C = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cdots}}} \tag{20}$$

Since every real number has an (essentially) unique representation in terms of continued fractions (up to a $\mathbb{Z}_2$ choice), this means the self-symmetries of the fractal may be labelled by the real numbers. The generators g and r also generate walks on the infinite binary tree: g may be taken to mean "always take the left branch" and r may be taken to mean "reverse the sense of left and right". Walks on the binary tree may also be represented as a sequence of left, right moves. An arbitrary string in two letters L and R may be taken to be a real number as well, which may be denoted by γ_D when the sequence of L, R moves take one to the same place as the series of g, r moves given by γ_C. That is, one has the representation as a binary fraction

$$\gamma_D = 0.\underbrace{000...0}_{a-1}\ \underbrace{11..1}_{b}\ \underbrace{00..0}_{c}\ \underbrace{11..1}_{d}\ \underbrace{00..0}_{e}1.. \tag{21}$$

where $L = 0$ and $R = 1$.

The Minkowski question mark function is nothing more and nothing less than the function that relates γ_C to γ_D. That is,

$$?(\gamma_C) = \gamma_D \tag{22}$$

In terms of the generators, on has $r(x) = 1 - r(1-x)$, which is satisfied by the question mark: $(? \circ r)(x) = ?(1-x) = 1 - ?(x) = (r \circ ?)(x)$. The generator g has inequivalent representations g_D and g_C for the dyadic and continued fraction representations, given by $g_D(x) = x/2$ and $g_C(x) = x/(x+1)$, so that one has

$$(g_D \circ ?)(x) = \frac{?(x)}{2} = ?\left(\frac{x}{x+1}\right) = (? \circ g_C)(x) \tag{23}$$

To summarize, a general monoid element $\gamma = g^a r g^b r g^c \cdots$ has two inequivalent representations as a real number.

Another important property is that both g_D and g_C can also be understood to be maps from the unit interval to a specific subinterval. This, for example,

$$\gamma_D(x) = (g_D^{a_1} r_D g_D^{a_2} r_D ... r_D g_D^{a_N})(x) = \frac{1}{2^{a_1}} - \frac{1}{2^{a_1+a_2}} + \frac{1}{2^{a_1+a_2+a_3}} - ... + (-)^{N+1} \frac{x}{2^{a_1+a_2+a_3+...+a_N}}$$

(24)

maps the unit interval into a dyadic sub-interval. Use of continued fractions gives another, different interval map for γ_C.

3.2.3 Self-similarity again

Returning to the self-symmetries of the classical lattice models, one can write transformation properties of the probability distribution as

$$(P \circ g_D)(x) = P\left(\frac{x}{2}\right) = e^{-V(x/2)}P(x) = e^{-(V \circ g_D)(x)}P(x) \tag{25}$$

and

$$(P \circ r)(x) = P(1-x) \tag{26}$$

One may now ask for, and provide, the self-similarity transformation that corresponds to the interval map 24. The iterated application of the above two equations then provides a formula that relates the probability distribution on a sub-interval to the probability distribution on the entire interval.

The simpler case, treated first, occurs when the potential is symmetric under the exchange of $+\sigma_k \leftrightarrow -\sigma_k$; that is, when $V(x) = V(1-x)$. One may then easily show that $P(x) = P(1-x)$.

Consider, for example, the interval given by $grg^3(x) = \frac{1}{2} - \frac{x}{8}$. Then, repeatedly applying the above, one obtains

$$
\begin{aligned}
P\left(\frac{1}{2} - \frac{x}{8}\right) &= P\left(grg^3\right) \\
&= e^{-V(grg^3)}P\left(rg^3\right) \\
&= e^{-V(grg^3)}P\left(g^3\right) \\
&= e^{-V(grg^3)}e^{-V(g^3)}e^{-V(g^2)}e^{-V(g)}P(x)
\end{aligned}
\tag{27}
$$

where the argument x was dropped to keep the notation simple. This formula relates in a precise (and computable) way the probability distribution on a subinterval to the whole interval. The process can be repeated to obtain a relation joining any dyadic subinterval to the whole interval.

For potentials that are not symmetric, the development has some additional complications. Write the potential as in terms of even and odd parts $V(x) = V_e(x) + V_o(x)$, with $V_e(x) = V_e(1-x)$ and $V_o(x) = -V_o(1-x)$. It follows from equation 8 that the energy can be decomposed in the same way. The probability is multiplicative, so one has $P(x) = P_e(x)P_o(x)$ with $P_e(x) = \exp(-H_e(x))$ and similarly for P_o. Under inversion , one then has $P(1-x) = P_e(x)/P_o(x)$.

The point here is that the probability distribution has a self-similar structure, one might say a "covariant" structure, under the action of the dyadic monoid. The generators of the monoid correspond to a right-shift on the one-sided lattice, and the inversion of spins.

The Ising model interaction potential has the general form

$$V(\sigma_0, \sigma_1, \sigma_2, \cdots) = \sigma_0 f(\sigma_1, \sigma_2, \cdots) \tag{28}$$

for some general function f of the lattice spins $\sigma_1, \sigma_2, \cdots$. Using the dyadic mapping , this condition implies that

$$V\left(\frac{1+x}{2}\right) = -V\left(\frac{x}{2}\right) \tag{29}$$

In addition, for $M = 0$, the Ising model potential has symmetry under the exchange of $+\sigma_k \leftrightarrow -\sigma_k$.

3.2.4 An exact solution

The graph6 is remarkably suggestive, hinting that there may be some interaction that gives a cumulative probability distribution that is exactly the question mark function. There is not just one, but several interactions that give such a cumulative distribution. One may obtained by a simple, explicit manipulation. The other is hinted at numerically, under the name of the Kac model, given below.

Suppose one wants to have

$$P(x) = \left(?' \circ ?^{-1}\right)(x) \tag{30}$$

exactly. One may then work backwards to find the potential, as follows. One must have $P(?(x)) = ?'(x)$, and, applying the symmetries of the question mark function,

$$\frac{?(x)}{2} = ?\left(\frac{x}{1+x}\right) \tag{31}$$

one has simply that

$$P\left(\frac{?(x)}{2}\right) = P\left(?\left(\frac{x}{1+x}\right)\right) = ?'\left(\frac{x}{1+x}\right) \tag{32}$$

Applying the symmetry of the lattice model under the action of g as discussed above, one gets an expression involving the potential

$$P\left(\frac{?(x)}{2}\right) = \exp\left(-V\left(\frac{?(x)}{2}\right)\right) P(?(x)) \tag{33}$$

from which one deduces

$$?'\left(\frac{x}{1+x}\right) = ?'(x) \exp\left(-V\left(\frac{?(x)}{2}\right)\right) \tag{34}$$

On the other hand, one has

$$\frac{?'(x)}{2} = \frac{1}{(1+x)^2} ?'\left(\frac{x}{1+x}\right) \tag{35}$$

which may be obtained by naively differentiating the relation 31. From this one deduces that

$$\frac{(1+x)^2}{2} = \exp\left(-V\left(\frac{?(x)}{2}\right)\right) \tag{36}$$

which is easily solved for V to obtain

$$V(y) = \log 2 - 2\log\left(1 + ?^{-1}(2y)\right) \qquad \text{for } 0 \le y \le \frac{1}{2} \tag{37}$$

and $V(y) = V(1-y)$ for $1/2 < x \le 1$.

It is not clear how to write this interaction function as sum of interactions between lattice spins, and that makes this function less appealing from a physical viewpoint. One may learn something about the dyadic mapping by attempting to express this function as a set of spin interactions. XXX do this exercise. A different potential, the Kac model, discussed below, has a simple pair-wise interaction between spins, falling off geometrically as the distance. The Kac model also seems to converge to the same probability distribution, although it clearly is a very different interaction.

3.2.5 The Kac model and the question mark function

The description of the Kac model is similar to that for the Ising model, with the exception that the interaction potential is not limited to a nearest neighbor interaction, but instead extends between distant sites, with a geometrically decreasing strength. That is, the potential is given by

$$V(\sigma) = -J(1-\lambda)\sigma_0 \sum_{k=1}^{N} \sigma_k \lambda^k \tag{38}$$

with λ a real number determining the strength of the interaction.

The energy or classical Hamiltonian of a given configuration is given by equation 8. As before, the boundary conditions are glossed over, but with a difference: the interaction 38 may be truncated at the edge, or one may choose to use periodic boundary conditions $\sigma_{N+1} = \sigma_0$. A graph of the configuration energy is shown in figure 7 for the case of $\lambda = 1/2$, $J = -1$. For this value of λ, the interaction potential as a function of x is easily seen to be the tent map

$$V(x) = \begin{cases} 2x - 1/2 & \text{for } 0 \le x \le 1/2 \\ -2x + 3/2 & \text{for } 1/2 \le x \le 1 \end{cases} \tag{39}$$

Readily apparent in the figure is the general resemblance of the configuration energy to the shape of the curve in figure 3. The resemblance can be made more precise.

Does the sum over partition probabilities equal the question mark function, as these figures all suggest? The numerical evidence suggests that the answer is yes, as shown in figure 9.

To be explicit, the numerical evidence from finite-size lattices suggests that

$$\left(?' \circ ?^{-1}\right)(y) = \lim_{N \to \infty} \frac{1}{Z(N)} \sum_{\substack{\sigma \\ x(\sigma) < y}} \exp H(\sigma) \tag{40}$$

Figure 7: Kac Model Configuration Energy
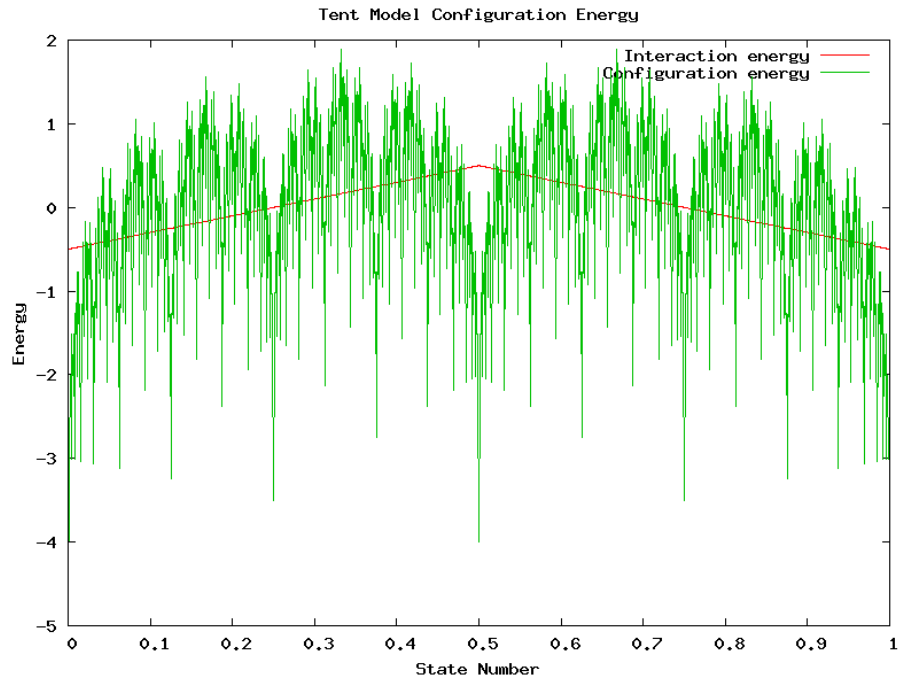
This figure shows the configuration energy for the tent interaction, that is, for the Kac model with $\lambda = 1/2$. The red tent-shaped curve (labelled "Interaction Energy") is simply a graph of the interaction potential $V(\sigma)$ of equation 38 as a function of x. The green curve, labelled "Configuration Energy", is a graph of the classical Hamiltonian of equation 8 for the Kac potential. A general resemblance to the figure 3 should be immediately apparent.

Figure 8: Partition Probabilities


Tent Model Partition Probability
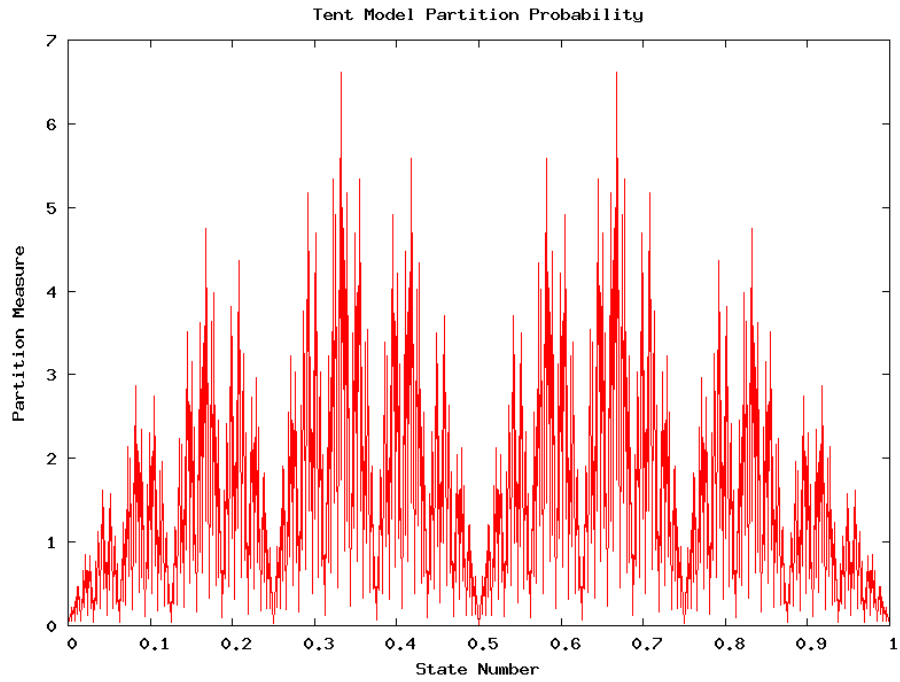
The above graph shows $P(x) = \exp H(x)$ for the Kac model configuration energy as a function of $x = x(\sigma)$. A minus sign that might normally appear here has been absorbed into the constant J. This figure is for $N = 10$, that is, a lattice with $2^{10} = 1024$ possible configurations. An even stronger resemblance to figure 3, which graphs $(?' \circ ?^{-1})(x)$, should be noted.

Weighted Tent Model Partition Probability
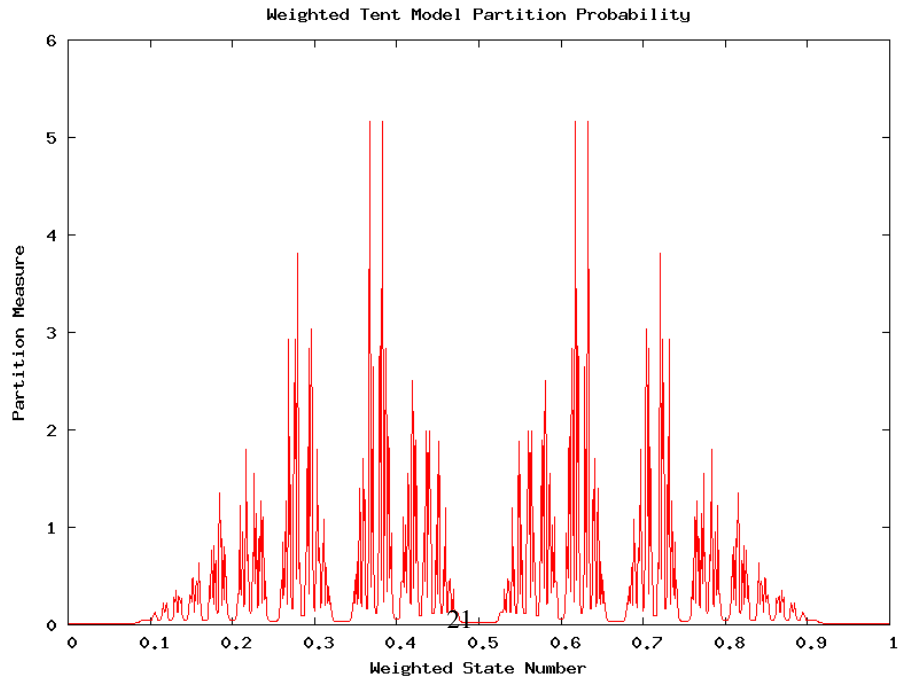
This figure shows a graph of $\exp H(?(x))$. The partition probability is graphed in this way to favor resemblance to figure 1, which shows $?'(x)$. Again, $N = 10$.

Figure 9: Cumulative Partition Probability
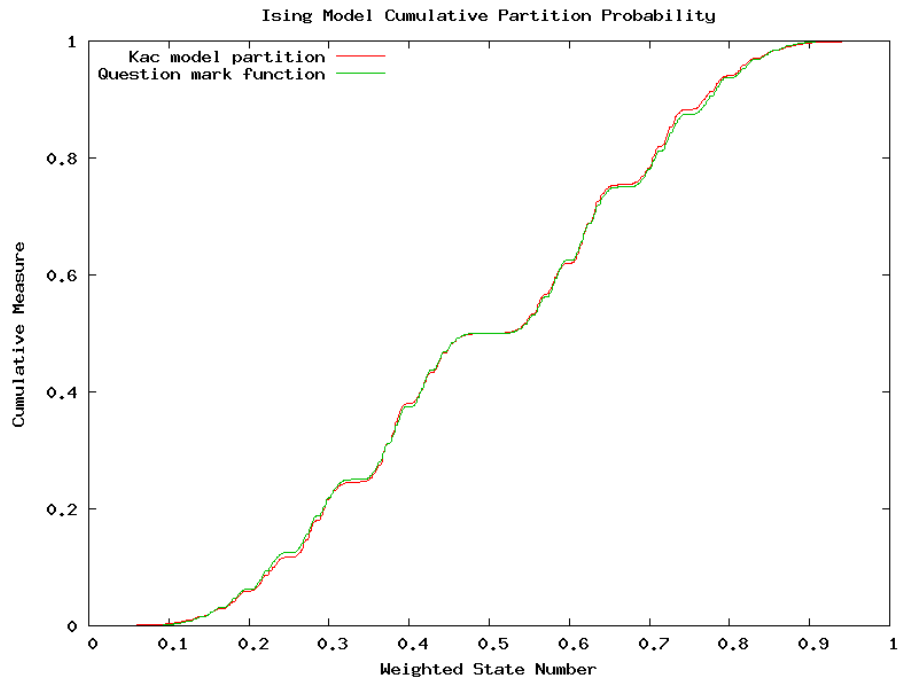
This figure shows a graph of the Minkowski question mark function $?(y)$ compared to a graph of the cumulative probability function $\sum_{\sigma \text{ s.t. } x(\sigma)<y} \exp H(?(x(\sigma)))$. As for other cases, this graph is for $N = 10$. As the value of N is increased, the graphs may be seen to numerically approach one-another.

where $Z(N)$ is a normalization dependent on N:

$$Z(N) = \sum_{\sigma} \exp H(\sigma) \tag{41}$$

(The use of N instead of $\Omega = 2^N$ here is merely a notational convenience). In physics, this normalization is known as the "partition function": it is a function of not only the temperature (which we have taken as one) but also of the various constants appearing in the interaction potential, such as λ and J.

3.2.6 Physical interpretation

A physical interpretation can be given to the appearance of the question mark function. In the limit of large N, any lattice configurations with long runs of repeated spins will have a very large energy (when J is taken positive), and thus the exponential give such cases an exponentially small probability. In the dyadic mapping, long runs of repeated spins correspond to dyadic rationals: that is, numbers whose first few bits may vary, but whose binary expansion eventually terminates. Thus, for example, $1/2$ has the binary expansion $0.10000\ldots$ and the long string of zeros leads to a large energy and a small probability; in the limit of large N this becomes an infinite energy and a zero probability. This effect can be clearly seen in the graphs. Of course, strings such as $0.100\ldots001$ are also disfavored; in the dyadic mapping, such strings are located "near" $1/2$, and account for the pronounced dip in that area. Similarly, strings near zero and one begin with (rather than end with) long runs of zeros and ones, and are also seen to be dis-favored. (There is a possibility in principle, which does not occur here, of Anderson localized states, which are states having only a finite extent but still having finite energy. Viewed as a subshift of finite type, localized states would be identified with instantons. The point is that the binary expansion for 1/2 need not have an *a priori* probability of zero.)

Now, the question mark function maps the rational numbers to the dyadic rationals; in the Kac model, the dyadic rationals have a zero thermodynamic probability. In this sense, we may understand why the derivative of the question mark function is necessarily zero at all rationals. Conversely, one may see that the derivative of the question mark function is necessarily non-zero in those places where the strings of letter alternate, ad infinitum. A special case of alternating strings are periodic strings: strings that may begin with some random pattern, but eventually settle down to a periodic pattern. Periodic strings are interesting because their energy can be precisely computed even in the limit of large N. Periodic binary strings correspond to those rationals that are not dyadic; for example, $2/3 = 0.10101\ldots$ These are the only such periodic strings, as there are no irrationals that have a periodic binary expansion. For the question mark function, periodic strings also have a significance: they correspond to the quadratic irrationals. This result was noted hundreds of years ago (by Gauss??), in the study of Diophantine equations: the solutions to the Diophantine equations are the quadratic irrationals, which may be specified as continued fractions with a periodic component in the continued fraction. The question mark function maps such periodic continued fractions to periodic binary strings: thus the quadratic irrationals and the rational numbers are in perfect one-to-one correspondence.

3.2.7 The two-sided lattice

The two-sided lattice is the lattice that stretches off to both the left and the right. The state of the two-sided lattice may be represented by two numbers $0 \leq x, y \leq 1$ with x given by equation 9, and y likewise, but instead being a sum over the negative indexes, starting at $k = -1$. The energy is a simple sum of the energy of the half-lattices to the left and to the right, plus an interaction term involving spins on both the left and right. The probability density is a product of the probability densities on the left and right, adjusted by the interaction between the left and right sides.

The shift operator τ can be understood to be an operator that pops a bit off the right-hand half lattice, and pushes it onto the left half-lattice. In the dyadic notation, it can be recognized as the so-called *Baker's map*, a map from the unit square onto itself that cuts the square into two pieces, and stacks and squashes them. That is, given a pair of numbers (x, y) in the unit square, one has

$$\tau(x, y) = \left(\frac{x + \lfloor 2y \rfloor}{2}, 2y - \lfloor 2y \rfloor \right) \tag{42}$$

The Baker's map can be studied in its own right as a discrete-time dynamical system, in that τ can be taken to define the time evolution of a unit square over a single time-step. The time evolution is invertible, in that the inverse map τ^{-1} is uniquely defined, being

$$\tau^{-1}(x, y) = \left(2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2} \right) \tag{43}$$

The inverse map can be seen to be identical to the forward map with x and y interchanged: that is, translation to the left on the lattice is identical to translation to the right, with left and right exchanged. One may say the map is PT-symmetric, with P standing for the "parity exchange" operation $x \leftrightarrow y$ and T being the "time inversion" operator $\tau \leftrightarrow \tau^{-1}$.

The two-sided lattice may be studied either as a lattice model, or as a dynamical system with time evolutino given by τ. It is interesting to juxtapose these two viewpoints. As a lattice model, one is typically only interested in lattices for which the energy and the probability density are invariant under the action of the translation operator. That is, one is interested only in the classical Hamiltonians of eqn 8 which are translation-invariant:

$$H(\tau^n \sigma) = H(\sigma) \tag{44}$$

for all $n \in \mathbb{Z}$.

Considered as a dynamical system, one is typically interested in the time evolution of densities on the unit square, that is, of real-valued maps $\rho : [0, 1] \times [0, 1] \to \mathbb{R}$. The time evolution of a density can be simply understood as a physical model, where the density is a local density of some "dust" of points (x, y), with the time evolution of each point given by τ. Thus, the time evolution of this "dust" or density is given by the transfer operator $\mathcal{L}$ as

$$[\mathcal{L}\rho](x, y) = \left(\rho \circ \tau^{-1} \right)(x, y) = \rho \left(2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2} \right) \tag{45}$$

Figure 10: Ising Tartan
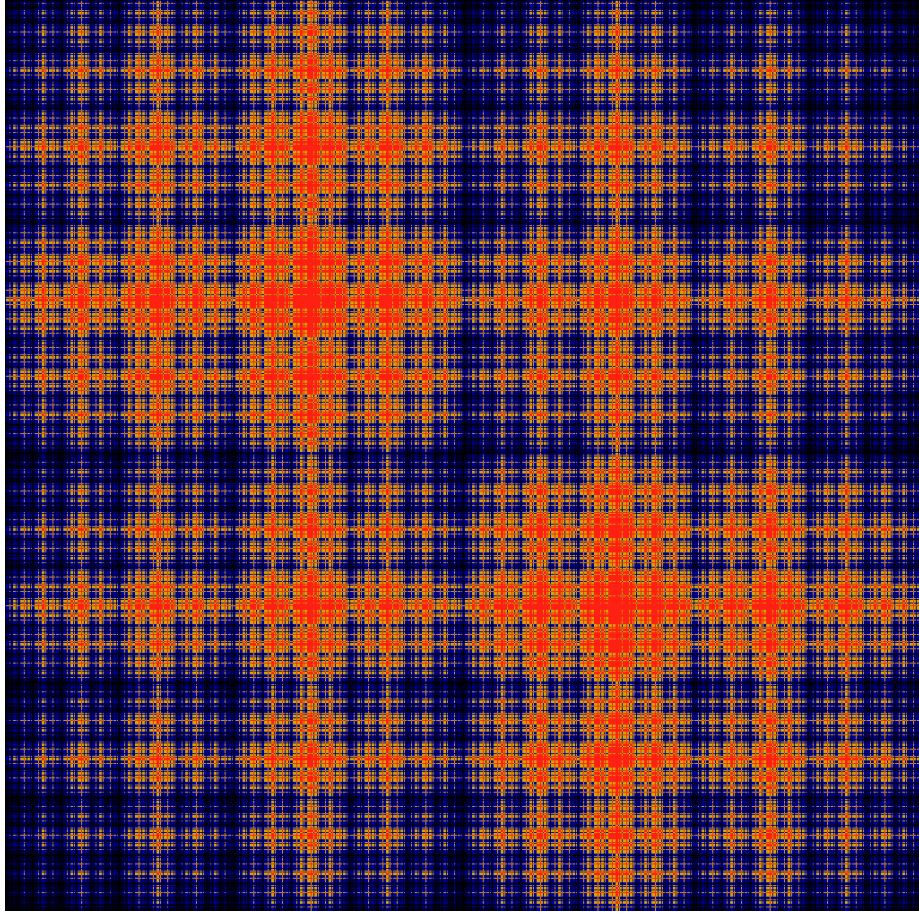
This tartan-like graph shows the Ising model probability density $P(\sigma)$ for the two-sided lattice using the dyadic mapping. That is, the lattice configuration $\sigma = (\sigma_{-N-1}, \cdots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_N)$ is represented by two numbers $0 \leq x, y \leq 1$ with

$$x(\sigma) = \sum_{k=0}^{N} \left(\frac{\sigma_k + 1}{2} \right) 2^{-(k+1)}$$

and

$$y(\sigma) = \sum_{k=0}^{N} \left(\frac{\sigma_{-k-1} + 1}{2} \right) 2^{-(k+1)}$$

The energy of a given configuration σ is computed using 8, with the sum running from $-N$ to $+N$, of course. The probability density $P(\sigma)$ is given by equation 10.

The graph here assumes the Ising potential, with $J = 0.3$ and $M = 0$ for a finite sized lattice with $N = 10$. The color choices here are such that black represents values where $P(\sigma) = P(x,y)$ are zero, blue are small values, with yellow and red being progressively larger values. This fractal tartan is invariant under the Baker's map.

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The transfer operator is thus a map $L : \mathcal{F} \to \mathcal{F}$ where $\mathcal{F}$ is the set of functions on the unit square. Clearly L is a linear operator. One is interested in characterizing the eigenfunctions and eigenvalues of L. The general idea behind equation 8 can be immediately appealed to, to construct some of the eigenstates of L. That is, on considers functions $f(x, y)$ of the form

$$f = \sum_{k=-\infty}^{\infty} \lambda^k g \circ \tau^k \tag{46}$$

given some function $g(x, y)$ and number λ. Formally, such an f is an eigenstate of L with eigenvalue λ: that is, $Lf = \lambda f$. In practial terms, it is evident that not all possible λ can make the sum convergent, although one might expect that λ on the unit circle $|\lambda| = 1$ of the complex plane might lead to a convergent sum. Thus immediately, one deduces that L must surely be a unitary operator. The unitarity of L makes intuitive sense in a certain way: the map τ is invertible, and one expects time-reversible evolution to be described by unitary operators. The figure 10 shows a tartan-like distribution corresponding to $\lambda = 1$ and $g(x, y) = V_{\text{Ising}}(x)$ the Ising model potential. Some additional tartans, for other values of λ, are shown in figure xx.

The proper analysis of the transfer operator L requires that the function space $\mathcal{F}$ be pinned down more precisely. Classic results on time-symmetry breaking [**?**, **?**, **?**] indicate that whether or not L is unitary depends on the function space $\mathcal{F}$.

XXX develop more fully. Talk about Gibbs states, which are the stationary states of the transfer operator. Also talk about the decyaing states i.e. the eignenfunctions of the transfer operator that have an eignevalue of less than one. Possibly review time-symmetry breaking in the spectrum of the decaying states (i.e. the time-reversed decaying state is a growing state).

3.2.8 Gibbs states and fractal self-similarity

The physics of the Kac model provides insight to why the question mark has flat spots at every rational, and why its derivative is non-vanishing on the irrationals: only the irrationals are mapped to strings that don't terminate. In physics, such states are called "Gibbs states": they are the states that are associated with a finite thermodynamic energy. From the above presentation, we can deduce that the Gibbs states are in some sense "dense" in the space of all configurations; conversely, so are the non-Gibbs states. A rigorous notion of "dense", however, requires a topology; a topology for the lattice models is constructed in a later section.

The Gibbs states may also be used to "explain" the fractal self-similarity of the question mark function. Both the Ising and the Kac potentials are translationally invariant, in that the total energy does not change as the shift operator is applied to the lattice. The translation invariance is an explicit part of the construction: it is used in equation 8 to construct the total energy. In a certain sense, the Gibbs states are precisely those states that are translationally invariant. That is, if one chooses some Gibbs state, and then chops off the leading binary digit, one must necessarily have another Gibbs state.

At first blush, the matching of real numbers to lattice model configurations given by the dyadic map of equation 9 may seem arbitrary: one might imagine a large class

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of maps that could connect these spaces. The seeming fractal nature of the problem may be a trick of the mind, wrapped up in the mapping of equation 9 instead of being intrinsic to the physics of the configuration space. This concern raises two important questions: does there exist a non-fractal mapping from the space of configuration states to the real number line, and how should the concept of "fractal" be expressed on spaces, such as the space of states, whose topology is not naturally that of the real number line?

A reasonable requirement on a map from the configuration space to the reals would be that it somehow preserves the translation invariance. Translation invariance is an important physical requirement for a lattice model, and one would like to have this requirement reflected in the mapping. That is, if $f : \Omega \to \mathbb{R}$ is some map that assigns a real number to each point $\sigma \in \Omega$, the set of all possible states, then one might require that there exists a non-trivial function $g : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ \tau)(\sigma) = (g \circ f)(\sigma) \tag{47}$$

where τ is the shift operator. This requirement that the mapping preserve translation invariance is simply a requirement that there exists a commuting diagram:

$$
\begin{array}{ccc}
\Omega & \xrightarrow{f} & \mathbb{R} \\
\tau \downarrow & & \downarrow g \\
\Omega & \xrightarrow[f]{} & \mathbb{R}
\end{array}
\tag{48}
$$

It should now be clear that the representation of translation invariance on the real number line is a requirement that the mapping have a fixed point at which it is self-similar: that is, one must have

$$(f \circ \tau^n)(\sigma) = (g^n \circ f)(\sigma) \tag{49}$$

for all positive integers n. Translation invariance implies iteration; it is well understood that, in general, the iteration of functions of the real number line results in fractals. The above result is general; it applies to all one-dimensional lattice models, and is not dependent one the form of the interaction, or even on the number of states at a given lattice location.

For the two-state model, there is another symmetry that one might expect to preserve: this is the symmetry under the interchange of the two states. In the absence of a magnetic-field-type interaction (so $M = 0$ in equation 7), one might expect an absolute symmetry. Let $\rho : \Omega \to \Omega$ be this reflection operator, the operator that, given a string of letters σ, exchanges letter A (or state -1) for the letter B (or state +1), and *vice-versa*. Corresponding to this, one expects a function $r : \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

$$
\begin{array}{ccc}
\Omega & \xrightarrow{f} & \mathbb{R} \\
\rho \downarrow & & \downarrow r \\
\Omega & \xrightarrow[f]{} & \mathbb{R}
\end{array}
\tag{50}
$$

Clearly, one has $\rho^2(\sigma) = \sigma$ the identity function, and so $r^2(x) = x$ is the identity as well.

Given such a translation-invariant homomorphism f, that has functions g and r acting as representations of translation and reflection, one has in essence constructed

everything that is needed to have a fractal with dyadic symmetry. That is, one need only to recognize that g and r generate the dyadic monoid of the modular group $SL(2,\mathbb{Z})$. The elements of this monoid are of the form

$$\gamma = g^a \circ r \circ g^b \circ r \circ g^c \circ \cdots \tag{51}$$

where $a,b,c,\ldots$ are non-negative integers.

Thus, we have demonstrated a corollary, of sorts: no matter what the form of the function f, if it preserves the translation invariance of the lattice, then the Minkowski question mark will inevitably appear in the theory.

This construction holds for essentially all two-state lattice models. That is, let f be the translation-invariant homomorphism as above, and let $h : \Omega \to X$ be just about any function to some space X which commutes with the reflection operator ρ. Then $(h \circ f^{-1}) : \mathbb{R} \to X$ is necessarily a fractal, having elements γ of the dyadic monoid giving the set of self-symmetries of the fractal. (The details on why anything having dyadic monoid symmetry is a fractal are voluminously and excruciatingly detailed in my other writings). In particular, the result here is independent of the details of the interaction between spins (barring the magnetic interaction, which can also be taken care of). We conclude: ***all such lattice models have a fractal self-symmetry***. (xxx ToDo: cavalierly used f^{-1} in the above, without justifying its existence. Baaad. Fix that.)

3.2.9 Free monoids and the self-symmetries of lattices

The set of strings generated by g,r is the free monoid $\mathbb{N} \times \mathbb{Z}_2$, the monoid analog of a free group. The notation here is that the exponent on g is any element of the natural numbers $\mathbb{N}$ while $\mathbb{Z}_2$ indicates that exponent on r is 0 or 1 because $r^2 = e$. The Cayley graph of this monoid is the infinite binary tree, and, as mentioned above, it is shared by the monoid $\mathbb{N} \times \mathbb{N}$ when interpreted as sequences of left-right moves walking the tree. That is, the monoid generated by $\mathbb{N} \times \mathbb{N}$ is the set of all strings in two letters.

It should be clear from the above construction that the K-state lattice model, of which the Potts model is an example, would have a set of symmetries that are the free monoid of $\mathbb{N} \times \mathbb{Z}_K$. (Properly, the symmetries are isometries, although the K-adic metric has not yet been defined here.) The point here is that the "ordinary" Euclidean translations are not the only symmetries that occur naturally in the studies of lattice models.

Now, in general, when some function on a topological space has some self-resemblance after some action, one says that this function is self-similar. When there are a countable number of such self-resemblances, one might say that the function is "translationally invariant", or so, or show that a fixed point results. When the number of such self-resemblances is dense in some way, or tends to the uncountable, or is characterized by a free group or free monoid, one says the function is a fractal. This motivates the following definition of a fractal:

Definition Let X and Y be topological spaces, and $f : X \to Y$ be some map between them. Let $\rho : X \to X$ and $\tau : X \to X$ be two isomorphisms of the spaces. The map f is said to preserves these isomorphisms, if there exist maps $r : Y \to Y$ and

$g : Y \to Y$ such that $f \circ \rho = r \circ f$ and $f \circ \tau = g \circ f$. If such an f exists, and if ρ and τ together generate some free monid, then we say that f has the **structure of a fractal**.

This definition seems almost silly, because after a bit of reflection, one realizes that there are many, many systems in physics and mathematics that would have the structure of a fractal according to this definition. For example, every Lie group has a discrete lattice that acts freely on the Lie group, and thus one may find maps that have the structure of a fractal. On the other hand, this is perhaps not unexpected: the Anosov flow on the tangent space of a Lie group typically has expanding and contracting directions, making the tangent manifold hyperbolic, and the flow ergodic. Ergodicity is typically associated with fractal behaviour, and hyperbolic structures typically admit ergodic flows, and so the above definition of a fractal structure is perhaps not ludicrous.

The requirement that ρ and τ generate some free monoid can be loosened to a monoid with a presentation, as long as the monoid is hyperbolic. Clearly, if ρ and τ had a presentation $\rho\tau = \tau\rho$ then the generated monoid would be flat Euclidean space, which is clearly not fractal.

3.3 Measure Theory

ToDo This is what needs to be done for the following sections.

Several questions arise. First, can the relation 40 be formally derived? Can the left-hand side of this equation be given a precise meaning? A strict derivation should provide a topology on the configuration space 2^N in the limit of $N \to \infty$, and then clarify how the topology of that space maps to the natural topology of the real numbers, as well as its relationship to the topology of the Cantor set. In particular, the derivation should make clear what aspects of the fractal behavior can be attributed to the mapping of the topologies, and what aspects depend on the choice of some particular translation-invariant long-range interaction.

Notable is the appearance and importance of the exponential function. In the normal study of number theory, rationals, modular forms, the question mark function, and the like, the exponential function makes almost no appearance ever. Thus, it's appearance here should be more than just curious; it begs for a deeper explanation.

It may be immediately intuited that this function gives a preferential weight to configurations with aligned spins, but why this was not inherently incorporated into the Hamiltonian deserves a review.

The next section begins a more formal development, starting with a a review of a topology for lattice models, and the sigma algebra that may be built on it.

3.3.1 Measure theory and sigma-algebras on a lattice

From the above discussion, it is clear that taking the $N \to \infty$ limit of a one-dimensional lattice model will require a way to promote the sums over configuration space into integrals. As pointed out in the introduction, this is not a trivial matter, because the cardinality 2^N of the configuration space changes from finite to uncountable in the limit. In particular, it is known from measure theory that it is impossible to define integration

on a point set with the cardinality of the continuum. To properly define integration, one must define a topology and a sigma-algebra first. A general presentation of this idea is given here.

The machinery of measure theory forces a radical shift in the language, notation and concepts used to discuss the problem. First of all, one can no longer employ the idea that the space consists of a set of points. There is a reason for this: not every subset of of an uncountably-infinite set is "measurable". In fact, almost all subsets of an uncountably-infinite set are not measurable. So the very first step of measure theory is to throw away almost all subsets of the total space. One keeps only the measurable sets. The collection of measurable sets is countable, and to each such set one assigns a (non-negative) size or "measure". This collection forms a topology, in the sense that the intersection or union of any two measurable sets is also a measurable set, and has a size. Such a collection of sets, equipped with a size, and obeying an additional axiom, (that of sigma-additivity, to be defined), is called a sigma algebra.

In this new language, the Hamiltonian takes on a new and very different appearance. It is no longer a function from points to the reals, but a function from elements of the sigma algebra to the reals. Intuitively, this "new" Hamilton can be visualized or intuited as the "average" of the old, classical Hamiltonian, the average taken over a certain subset of points. However, this intuition is dangerous, and can lead to errors: in particular, one has the chicken-and-egg problem of how to define the "average" of the old, classical Hamiltonian. The new Hamiltonian is different, and cannot be constructed from the old Hamiltonian. However, there is a way to prove that the new Hamiltonian is a faithful representation of the old one. If one has chosen one's sigma algebra wisely, then for every possible point in the old set, there is a sequence or filter or net of elements of the sigma algebra that contain the old point, and the measure of the elements in this sequence decreases to zero.

Suppose that the elements of this net are denoted by a_n for integer n, and have a measure $\mu(a_n)$ The net is ordered so that $\mu(a_n) < \mu(a_k)$ whenever $n > k$. The "new" Hamiltonian is some function that assigns a real value to each a_n. We can then say that the new and the old Hamiltonians are equal or equivalent when, for the "classical" point p, the following are satisfied:

$$p \in a_n \quad \forall n \in \mathbb{Z}^+$$

and

$$\lim_{n \to \infty} \mu(a_n) = 0$$

and

$$\lim_{n \to \infty} \frac{H(a_n)}{\mu(a_n)} = E_p$$

where E_p is the classical energy of the point p which the net is converging to. Again: to be clear: the point p is a possible configuration of the lattice model; each and every point p corresponds to an infinite string in two letters, and vice-versa, in one-to-one correspondence. If a net can be found for all points p, and the above relationship holds for all points p, and it holds for all possible nets to the point p, then one may honestly and truthfully insists that the "new" and the "old" Hamiltonians "are the same".

More generally, this sort of a measure-theoretic reformulation of integration over an uncountable set of points is referred to as "second quantization" in physics, and the integration is referred to as "functional integration". A word of caution: standard treatments of second quantization in physics texts do not follow the above construction. This approach only becomes apparent in mathematical treatments of second quantization (need ref), where the aim is to construct a topological space over which integration can be performed. In the following, the "old" Hamiltonian shall be called the "classical Hamiltonian", as it deals with points. The "new" Hamiltonian will be called the "quantum Hamiltonian", because the sigma algebra may be used to define functional integrals.

3.3.2   The cylinder set topology

The natural topology for the configuration space of one-dimensional lattice models has a sub-base that consists of cylinder sets. It is "natural" in the sense that there is a simple and natural action of the shift operator on the cylinder sets, and that there is a natural way to assign a size to the cylinder sets.

The sub-base of a topology is a collection of sets such that any finite intersection or union of such sets belongs to the topology, and that furthermore, every set in the topology can be written as a finite intersection or union of sets in the sub-base.

A sigma algebra then consists of the topology, together with a function, the measure, that assigns a non-negative, real value to every element of the topology. The measure must satisfy a property called "sigma-additivity": the measure of the union of a number of disjoint sets must equal the sum of the measures of each set. Notationally, sigma-additivity requires that:

$$\mu\left(\bigcup_{i\in I} a_i\right) = \sum_{i\in I} \mu(a_i) \tag{52}$$

where each of the sets a_i are pairwise-disjoint from the other sets, and I is a (countable) collection of indecies of the sets being considered.

The cylinder sets may be defined as follows. Let the positions on the lattice be labelled by an integer n. The elements of the sub-base of the sigma algebra can be completely enumerated by ordered pairs (n,s) where s is a string in two letters of finite length k, for integer $k < \infty$. The pair (n,s) is to be visualized as the set of all states where the lattice values between location n and $n+k-1$ are equal to the string s. Let $C_{(n,s)}$ denote an element of this sub-base.

The sub-base of a topology is a collection of sets, which, by intersection and union, generate the rest of the topology. A brief review of intersection and union is in order. The intersection $C_{(n,s)} \cap C_{(m,t)}$ is the set of all configurations that match s at n *AND* match t at m. The union $C_{(n,s)} \cup C_{(m,t)}$ is as above, with *OR* taking the place of *AND*.

Let the two letters that occur in the string be A and B. Then, for example,

$$C_{(n,A)} \cap C_{(n,B)} = \varnothing$$

and

$$C_{(n,AAA)} \cap C_{(n,AAB)} = \varnothing$$

and

$$C_{(n,A)} \cup C_{(n,B)} = \Omega$$

with Ω representing the entire space. One noteworthy aspect of this topology is the somewhat intuitively "backwards" relation between the string labels and the sets, in that when two strings overlap, its is likely that the intersection of the corresponding sets will be empty, whereas when the strings don't overlap, the intersection will never be empty. Thus, for example,

$$C_{(n,s)} \cap C_{(m,t)} \neq \varnothing$$

whenever $m > n + \text{len}(s)$ where $\text{len}(s)$ is the length of the string.

3.3.3 Measures on cylinder sets

The measure is a function μ that assigns to each element $C_{(n,s)}$ a real, non-negative value. It is convenient to normalize the measure such that $\mu(\varnothing) = 0$ and $\mu(\Omega) = 1$. The requirement of sigma-additivity implies that all other values will be less than one. A measure will be said to be *translation invariant* if $\mu(C_{(n,s)}) = \mu(C_{(m,s)})$ for all m, n.

ToDo XXX finish this section.

We can use sigma additivity to construct a collection of translation invariant measures as follows. Let $\mu(C_{(n,A)}) = x$ for some $0 \leq x \leq 1$. Then sigma-additivity requires that $\mu(C_{(n,B)}) = 1 - x$ which follows from the fact that the union of these two disjoint sets is the entire space. One may readily deduce that

$$\mu(C_{(n,s)}) = x^{\#A(s)}(1-x)^{\#B(s)} \tag{53}$$

where $\#A(s)$ is the number of times the letter 'A' occurs in the string "s", and likewise for $\#B(s)$. In most physics applications, the canonical and symmetric choice would be $x = 1/2$, but it should be clear that this is not mathematically constrained. One might even be able to make physical arguments to have "x" be something other than one-half, if, for example (and maybe a bad example), some external force is causing there to be more spins pointing in one direction than the other, on average.

Lets pursue this idea just a bit further. Let $\sigma_k = \{s : \text{len}(s) = k\}$ be the set of all strings of length "k". Then, $\forall s, t \in \sigma_k, s \neq t$ one has $C_{(n,s)} \cap C_{(n,t)} = \varnothing$, that is, this set of strings defines sets that are pair-wise disjoint. One has

$$\bigcup_{s \in \sigma_k} C_{(n,s)} = \Omega$$

and, from sigma additivity, we deduce

$$1 = \mu\left(\bigcup_{s \in \sigma_k} C_{(n,s)}\right) = \sum_{s \in \sigma_k} \mu\left(C_{(n,s)}\right) = \sum_{s \in \sigma_k} x^{\#A(s)}(1-x)^{\#B(s)} = \sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} = 1$$

3.3.4 ToDo

To-do list:

– Define the measure on the cylinder sets that the Kac model induces (to be copied from wikipedia page)

– review: the "natural" measure on the real-number line maps to the "natural" measure on the cylinder sets via the question mark.

– use the transfer matrix to move out of the measure into to the potential: the Ruelle-Frobenius-Perron operator to compute the partition function.

– lather rinse repeat .

4 Conclusion

In conclusion, we conclude ???

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