Comments on NNTDM submission Paper id: 2021 / 269

Review date: 5 July 2021

Review summary: A central lemma that the ressult depends on is incorrect. Fixing the lemma gives a different kind of result, one that is interesting and peculiar in it's own right.

Lemma 2.2 is incorrect

The lemma is not right. I will provide to simple demonstrations of this, and a variant that is perhaps desirable.

Direct substitution in Lemma 2.2

One way of quickly checking results is to try direct subsitution. The lemma claims

$$\frac{1}{k^{4m}+n^2k^{4m-2}} \quad \overset{?}{=} \quad \frac{1}{2}\sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}}$$

If this is to hold for all m then it must hold for $m=1$. By direct substitution, one gets

$$\frac{1}{k^4+n^2k^2} \quad \overset{?}{=} \quad \frac{1}{2}\sum_{s=1}^{1} \frac{(-1)^{s+1}}{k^{4-2s}n^{2s}} = \frac{1}{2}\frac{1}{k^2n^2}$$

which is clearly false. Attempting $m=2$ shows that it won't work there, either.

Factorization

A different way of seeing the result is to factorize terms more carefully. That is, write

$$\frac{1}{k^{4m}} \cdot \frac{1}{1+n^2k^{-2}} = \frac{1}{k^{4m}+n^2k^{4m-2}} \quad \overset{?}{=} \quad \frac{1}{2}\sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{4m-2s}n^{2s}} = \frac{1}{2k^{4m}}\sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{-2s}n^{2s}}$$

dropping the leading factor on both sides, one gets

$$\frac{1}{1+n^2k^{-2}} \quad \overset{?}{=} \quad \frac{1}{2}\sum_{s=1}^{2m-1} \frac{(-1)^{s+1}}{k^{-2s}n^{2s}}$$

The left hand side does not involve m in any way. The right hand side does – it is a sum of m terms. None of these terms are zero, and the leading terms are always identical; therefor this cannot possibly be right.

Desired result

Below follows a simple derivation of what perhaps might have been hoped for. Consider the expansion

$$\frac{1}{1-x} = \sum_{p=0}^{\infty} x^p$$

This holds for all real $-1 < x < 1$. Setting $x = -k^2/n^2$ and requiring that $k < n$, one gets

$$\frac{1}{1+k^2/n^2} = \frac{n^2}{n^2+k^2} = \sum_{p=0}^{\infty} (-1)^p \frac{k^{2p}}{n^{2p}}$$

Multiplying both sides by $1/n^2 k^{4m-2}$ gives

$$\frac{1}{k^{4m}+n^2 k^{4m-2}} = \sum_{p=0}^{\infty} (-1)^p \frac{1}{k^{4m-2-2p} n^{2p+2}}$$

Here, the left hand side is that of Lemma 2.2. The right hand side is clearly something different.

Note that this holds only for $k < n$. It is not a well-conditioned sum otherwise, as, for $k > n$, each term is becomes progressively larger. To btain a convergent sum, one can play some games with analytic continuation, but that is outside of the scope of the current effforts.

Main theorem

Sadly, this ruins the main theorem. The expansion for the hyperbolic cotangent (Lemma 2.1) appears to be correct; I verified this, but did not double-check. Unfortunately, the revised identity holds only for $k < n$ and therefore cannot be plugged in directly. Despite this, it can sometimes be useful to play some games, and treat the problem as a formal series, hoping that a re-arrangement of terms by the end can rescue the situation. So lets try this. Started from the hyperbolic cotangent,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\coth \pi k}{k^{4m-1}} =& \frac{1}{\pi}\left(\sum_{k=1}^{\infty} \frac{1}{k^{4m}} + 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{k^{4m-2}n^2 + k^{4m}}\right) \\
=& \frac{1}{\pi}\left(\zeta(4m) + 2\sum_{k=1}^{\infty} \frac{1}{k^{4m-2}} \sum_{n=1}^{\infty} \frac{1}{k^2+n^2}\right) \\
=& \frac{1}{\pi}\left(\zeta(4m) + 2\sum_{k=1}^{\infty} \frac{1}{k^{4m-2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{p=0}^{\infty} (-1)^p \frac{k^{2p}}{n^{2p}}\right) \\
=& \frac{1}{\pi}\left(\zeta(4m) + 2\sum_{p=0}^{\infty} (-1)^p \sum_{k=1}^{\infty} \frac{1}{k^{4m-2-2p}} \sum_{n=1}^{\infty} \frac{1}{n^{2p+2}}\right) \\
=& \frac{1}{\pi}\left(\zeta(4m) + 2\sum_{p=0}^{\infty} (-1)^p \zeta(4m-2-2p)\,\zeta(2p+2)\right)
\end{aligned}$$

Well, this is certainly a peculiar kind of beast. The middle bits rrequired working with a sum that is only formally defined, but cannot converge numerically, because the $k < n$ condition is violated. Despite this, the final sum does appear to be convergent.

To bring it closer to the form that Plouffe and Ramanujan provide, the hyperbolic cotangent should be substituted:

$$\sum_{k=1}^{\infty} \frac{\coth \pi k}{k^{4m-1}} = \sum_{k=1}^{\infty} \frac{e^{2\pi k}+1}{k^{4m-1}\left(e^{2\pi k}-1\right)}$$

So, this is similar, but curiously different. It lacks the rapid convergence properties, which is what Plouffe was interested in: Plouffe has some arbitrary-precision software, and so he can (easily) evaluate these sums to thousands of digits. In this way, he was able to discover the indicated identites numerically, but he had no algebraic, formal proof that they held. As it happened, a variant of them, in a non-obvious form, can be found in Ramanujan's diaries (without proof).

The above form lacks the rapid convergence poroperties, because of the factor of $e^{2\pi k}$ in the numerator. The infinite sum over the double zeta is also numerically discouraging.

However, the fact that this shows up as a kind of hidden or cryptic Fourier transform is notable. A revised paper with the above corrections seems publishable, I guess. Explicitly strengthening and tying together that this is a "secret" or "hidden" Fourier transform can certainly make things more interesting. It seems worthwhile to make this as explicit as possible.