Report on the paper

"Efficient algorithm for computing the polylogarithm and Hurwitz zeta-functions"

by L. Vepstas

In the paper an algorithm for the evaluation of the polylogarithm $\mathrm{Li}_s(z)$ with further application to the Hurwitz zeta-function $\zeta(s,\alpha)$ is proposed. The result allows to compute $\mathrm{Li}_s(z)$ for general complex values of s and for complex z from a particular region, whereas the existing approaches for computing $\mathrm{Li}_s(z)$ are designed for integer s (Crandall) and for real s (Wood). The technique is based on Borwein's ideas for computation of the Riemann zeta-function which were extended to Dirichlet L-functions in [6]. Also, there are several results in the literature devoted to the numerical evaluation of the Hurwitz zeta-function, which, in my opinion, should be cited in the present paper. In the work of Balanzario the generalization of the Euler-Maclaurin formula is obtained and can be used for computations of $\zeta(s,\alpha)$ for $0 < \mathrm{Re}\, s < 1$. An algorithm (based on the so-called FEE method) for the fast evaluation of $\zeta(s,\alpha)$ for integer s and algebraic α was considered by Karatsuba.

The paper under review differs very much from the previously submitted version. However, in my opinion, the main result of the paper is acceptable for publication in "Numerical Algorithms" after certain changes suggested in the report.

Other remarks:

In the definition of the Hurwitz zeta-function (Page 1) the half-plane of absolute convergence should be specified and also the type of analytic continuation to the whole complex plane. The definition of the polylogarithm is also not precise as well as in other formulas (e.g. (1.3)).

The meaning of the sign $\gtrsim$ is not clear (Page 1 and further in the text).

The author should express himself more precisely concerning the citation [6] in the paper.

In the last paragraph of Section 1 it should be "... a certain specific integral..." and "... may be applied to express the integral...".

The last sentence in the first paragraph of Section 2 could be deleted.

The representation for the polylogarithm given in Lemma 2.1 is known. It is sufficient to give a quotation.

The last sentence before formula (3.1) should be rewritten as "The first is to rewrite the above integral formula as follows". In the next sentence it should be "... can be shown to be bounded...".

It should be clarified how formula (4.1) is obtained. Moreover, the step from formula (3.8) to formula (4.2) via estimate (4.1) should be explained more detailed.

In the last paragraph of Page 4 it should be specified about which polynomial the author is speaking.

The first sentence below formula (4.5) and the last paragraph of Section 4 should be deleted.

The left-hand sides in estimates (5.1), (5.2), (5.4) and in the estimate between (5.2) and (5.4) should be taken in absolute value.

It is not clear, why in estimate (5.4) the multiplier $n^{-\frac{1}{2}}$ appears (in the former version of the paper it was 4^{-n} !). The author should provide more details.

In the last sentence on Page 5 better write "... where $y_0(1 - y_0)|\log y_0| = 0.260345491\ldots$".

Moreover, the numbering of formulas on Page 5 is unusual (better delete (5.2) and (5.4)).

The phenomenon described at the end of Section 5 is well-known from the Riemann zeta-function theory.

In the formula in the first sentence of Section 6 it is better to use the sign "$:=$" to stress that it is a definition.

The duplication formula for the so-called periodic zeta-function is well-known in analytic number theory. Therefore the last paragraph of Page 6 and the reference [7] should be deleted.

Section 8 should be presented in a proper way! It would be nice to have a discussion of optimality of the presented algorithm.

In the list of references there are misprints (for example, it should be "Riemann zeta", it is not clear what the question mark in reference 5 means etc.).

On Page 8 it is not clear what means "... being values on the order of one, ... on the order of two..." also "(need ref.)" and what kind of zeroes (line -6) are meant (the numerical values of the first zeroes of the Riemann zeta-function differ from those which are listed. They are not rational, also the second one is approximately $\frac{1}{2} + i21.02$).

The title of Figure 3 should be specified.

In the formulas below Figure 3 and Figure 4 the imaginary unit i is missing.

The paper should be revised and after the corrections returned to the referee for the final decision about its publication.

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