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Abstract

Questionable ansatz regarding the Berry-Keating variant of the Hilbert-Polya conjecture. The Berry-Keating conjecture states that the zeros of the Riemann zeta result from some unknown quantization of $(1/2 - ixd/dx)$. Then Ansatz here is that the correct operator is similar, via the Minkowski Question mark as the similarity transform. Viz. it is $?^{-1} \circ (1/2 - ixd/dx) \circ ?$. The below are a random collection of notes on this operator.

1 Introduction

Consider the operator

$$H = \frac{1}{2} - ix\frac{d}{dx} \tag{1}$$

Lets call this the Berry-Keating operator. We are looking for solutions

$$H\psi = \omega\psi$$

First, consider the spectrum of smooth solutions, *i.e.* ψ a polynomial or even Laurent series:

$$\psi(x) = \sum_n a_n x^n$$

Plugging in, we find that $\omega a_n = a_n/2 - ina_n$ which is solved by $a_k = \text{const}$ and $a_m = 0$ for all $m \neq k$: i.e. the monomial x^k is an eigenfunction with eigenvalue $\omega = 1/2 - ik$. This is shallow but suggestive: eigenvalues line up as expected, at least for integer k. For non-integer k, anything goes. Berry-Keating suggest that some boundary condition on ψ imposes the result that the k's are the zeros of the Riemann zeta.

2 The Question Mark Ansatz

The Ansatz is as follows: consider the operator

$$Q = ?^{-1} H ?$$

or equivalently the equation

$$\left(\frac{1}{2} - ix\frac{d}{dx}\right)?(f(x)) = \lambda\,?(f(x))$$

Here, ? is the Minkowski Question Mark function.

As a differential equation, this is identical to the previous one, since we can take $\psi(x) = ?(f(x))$. The difference here is that, by writing it this way, we are forced to explicitly consider non-smooth eigenfunctions; eigenfunctions that might be differentiable-nowhere. This seems silly on three accounts: first, the above is in the form of a differential equation, so how can we define it to act on non-differentiable functions? Secondly, the "derivative" of ? vanishes on all rationals, and is infinite on "most" reals, and seemingly undefined on the rest, at least, measure-theoretically. Thirdly, its a similarity transform; and it is "well known" that similarity transforms cannot alter the spectrum.

All three of these superficial objections can be dealt with equally superficially. One can give a precise definition for the derivative $?'(x)$ and a collection of useful algebraic identities for it; its not un-manipulable. For the second objection, there is a sound measure-theoretic approach, as long as some care is taken. And the objection about similarity transforms is, of course, not really applicable, as, along with the similarity transform, there is also an implicit change of the space on which the operator acts. Different spaces do mean different spectra. All three of these objections have been previously dealth with. [ref Mink Q, Bernouli facts, exact exprs papers]. Nevertheless, the undertaking seems vaguely absurd. We continue undaunted.

Re-writing the eigenvalue equation as

$$-ix\frac{d}{dx}?(f(x)) = \left(\lambda - \frac{1}{2}\right)?(f(x))$$

impresses on one an immediate problem: what is the analytic extension of $?(x)$ from the real-number line to the upper half-plane? It seems that such an extension should exist, given how the question mark almost behaves like a modular form.

3 Some identities

Some pertinent factoids from earlier work that are useful to keep in mind.

There are many self-similarity results, of the form:

$$?\left(\frac{x}{x+1}\right) = \frac{1}{2}?(x)$$

which can be extended to general Mobius transforms $(ax+b)/(cx+d)$ in SL(2,Z), albeit having a more complicated expression. Viz:

$$?\left(\frac{ax+b}{cx+d}\right) = \frac{M}{2^N} + (-1)^Q\,\frac{?(x)}{2^K}$$

for integers M,N,Q,K derived in previous papers (state which).

The expression

$$?' \circ ?^{-1}$$

can be given a precise meaning (reference: mink-exact.pdf), where, roughly speaking, $?'$ is the first derivative of ?. Specifically, it can be interpreted as a measure, obeying the standard axioms of measure theory, and having the values:

$$\int_a^b ?'(x)dx = ?(b) - ?(a)$$

which obeys self-symmetry relations as well:

$$\frac{1}{(1+x)^2} ?'\left(\frac{x}{1+x}\right) = \frac{1}{2} ?'(x)$$

and more generally

$$\frac{1}{(cx+d)^2} ?'\left(\frac{ax+b}{cx+d}\right) = \frac{?'(x)}{2^K}$$

for the same K as above.

Some of the above identities are most easily obtained by considering continued fractions, as in [ref chap-minkowski.pdf], and the others by considering the shift operator on a 1D lattice.

Then there's this:

$$?'(y) = \prod_{k=0}^{\infty} \frac{A' \circ A_k(y)}{2}$$

where

$$A(y) = \begin{cases} \frac{y}{1-y} & \text{for } 0 \le y \le \frac{1}{2} \\ \frac{2y-1}{y} & \text{for } \frac{1}{2} \le y \le 1 \end{cases}$$

and the iterated function A_k is defined so that

$$A_{k+1}(y) = A_k \circ A(y)$$

with $A_0(y) = y$ and $A_1(y) = A(y)$.

The $A_k(y)$ are piece-wise smooth, and have a differentiable extension $\widetilde{A}_k$ such that $A_k(y) = \left\lfloor \widetilde{A}_k(y) \right\rfloor$ i.e. such that they are the fractional part.

A handy sum:

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{2}{x+n} - \frac{1}{x+n+1}\right]$$

4 Analytic extension

Some attempts at an analytic extension of $?(x)$ to the upper-half plane. Summary: in this and the next section, some progress can be made, but it appears to not lead to anything useful.

Lets try this. Let

$$x = \sum_n a_n 2^{-n} \quad \text{and} \quad y = \sum_n b_n 2^{-n}$$

be the binary expansions for two real numbers x and y. Consider then an extension to the complex plane $a_n \to a_n(u)$ and likewise $b_n \to b_n(w)$ for u, w complex, and $a_n(0) = a_n$ and likewise $b_n(0) = b_n$. Thus, $x \to X(u)$ and $y \to Y(w)$ become complex-analytic functions with some non-zero radius of convergence. The upper-case is used to distinguish the function from its value at zero. We are fishing for a consistent expansion, in that, for some point z in the (upper-half) complex plane, we have that

$$X(z-x) = Y(z-y) \tag{2}$$

for some finite radius of convergence around z. At a minimum, we hope for convergence for $|z| < \Im z$ i.e. the region of convergence extends to the real line, and better yet, for $|z| \leq |x-y|$ i.e. the region of convergence extends to encompass both x and y. Can this be done? Lets try. Write the analytic extension of a_n as

$$a_n(u) = \sum_{k=0}^{\infty} a_{nk} u^k \tag{3}$$

so that

$$X(u) = \sum_{n=1}^{\infty} 2^{-n} \sum_{k=0}^{\infty} a_{nk} u^k$$

and likewise for $Y(w)$. Then, expanding,

$$X(z-x) = \sum_{m=0}^{\infty} z^m \sum_{k=m}^{\infty} \binom{k}{m} (-x)^{k-m} \sum_{n=1}^{\infty} a_{nk} 2^{-n}$$

and consistency requires that this match, z-term by z-term, with the analogous expansion for $Y(z-y)$. We being by noting that $a_{n0} = a_n$ is fixed. For convenience, let

$$A_k = \sum_{n=1}^{\infty} a_{nk} 2^{-n} \tag{4}$$

so that

$$X(u) = \sum_{k=0}^{\infty} A_k u^k$$

For balance, i.e. eqn 2, we need to solve the equations

$$\sum_{k=m}^{\infty} \binom{k}{m} (-x)^{k-m} A_k = \sum_{k=m}^{\infty} \binom{k}{m} (-y)^{k-m} B_k$$

for each value of m. This is solvable, as its upper-triangular. Write

$$C_{mk}^{(x)} = \begin{cases} \binom{k}{m} x^{k-m} & \text{for } k \geq m \\ 0 & \text{for } k < m \end{cases}$$

so that we are solving

$$\sum_{k=0}^{\infty} C_{mk}^{(-x)} A_k = \sum_{k=0}^{\infty} C_{mk}^{(-y)} B_k \tag{5}$$

So $C_{mk}^{(-x)}$ is upper-triangular, non-degenerate, and thus invertible. It is quasi-singular, as it has all 1's on the diagonal. Setting $x = 0$ we get $C_{km}^{(0)} = \delta_{km}$ the identity operator. It would appear that we have almost complete freedom to choose B_k at this point; see below. We can get an easy, intuitive feel for C by writing it out:

$$C^{(-x)} = \begin{bmatrix} 1 & -x & x^2 & -x^3 & x^4 & \cdots \\ 0 & 1 & -2x & 3x^2 & -4x^3 & \cdots \\ \vdots & 0 & 1 & -3x & 6x^2 & \\ & \vdots & 0 & 1 & -4x & \\ & & & 0 & 1 & \\ & & & & & \ddots \end{bmatrix}$$

which has both a left and right inverse given by

$$D_{mk}^{(x)} = C_{mk}^{(-x)}$$

In fact, it forms a nice Abelian group, viz:

$$\sum_{k=0}^{\infty} C_{mk}^{(a)} C_{kn}^{(b)} = C_{mn}^{(a+b)} \tag{6}$$

for any real or complex a, b, which follows from the binomial theorem in general.

First attempt; it fails

Thus, the desired solution is then

$$A_m = \sum_{k=0}^{\infty} C_{mk}^{(x-y)} B_k$$

Things seem to get funky here. Some care needs to be taken, here; we have the prior constraint that $X(0) = x$, that is,

$$A_0 = \sum_{n=1}^{\infty} a_{no} 2^{-n} = x$$

and likewise $B_0 = y$. Thus, the first row ($m = 0$) gives:

$$x = y + B_1(x-y) + B_2(x-y)^2 + B_3(x-y)^3 + \cdots$$

Recall, both x and y are fixed, here, and the goal is to provide some reasonable values for B_k such that this sum converges, as well as the sums for all the other rows. Due to the presence of the binomial coefficient, these rows grow large, and it would be best to provide an a set of B_k's that shrink as fast as the factorial. Thus, a reasonable ansatz is $B_k \sim 1/k!$ from which we obtain a solution

$$B_k = \frac{x-y}{k!(e^{x-y}-1)}$$

which is flawed ... technically, this is a solution to the problem as originally stated, but... think of it this way: Fix y=0. The B_k then depend on x but we want a solution that doesn't ... That is, this choice of B means that the expansion $Y(w)$ depends on x. That is, $Y(w) = Y_x(w)$

Can we fix this with a sheaf of some kind? We've got a sheaf of analytic functions at x and they were constructed to be consistent with the functions $X_y(u)$. Can we do anything useful with this sheaf? What did we intend to do with it, in the first place?

Second attempt; it fails

Go back to the symmetric version, in eqn 5. Make the same Ansatz, that $B_k \sim \frac{1}{k!}$. Then, for $m = 0$ one has

$$y - B_1 y + B_2 y^2 - B_3 y^3 + \cdots = \alpha$$

when

$$B_k = \frac{\alpha - y}{k!(e^{-y}-1)}$$

where α is a freely chosen constant. Making the same choice for the A_k results in a balance at $m = 0$. However, for $m \neq 0$ we get:

$$\sum_{k=0}^{\infty} C_{mk}^{(y)} B_k = \frac{(\alpha - y)e^{-y}}{(e^{-y}-1)m!}$$

which cannot balance the corresponding A-side in 5 unless we pick α =y. ... which sets all $B_k = 0$ (and breaks the $m = 0$ equations). WTF. So, clearly, the Ansatz cannot work.

Third attempt; success!

The way out of this is to solve a variant of 5, namely

$$\sum_{k=0}^{\infty} C_{mk}^{(-y)} B_k = \alpha_m$$

for some set (any set!?) of constants α_m. The solution is trivial, on account of eqn 6, namely:

$$B_k = \sum_{p=0}^{\infty} C_{kp}^{(y)} \alpha_p$$

for completely free constants α_p. Hmmm.

There seem to be several directions to go here. First, lets try $\alpha_p = 0$ for all p. This gives $A_k = B_k = 0$. Going back to the definition of these, in eqn 4, we have

$$0 = \sum_{n=1}^{\infty} a_{nk} 2^{-n}$$

for all k. We can certainly find many non-zero values for the a_{nk}; however, in the end this fails, because $X(u)$ is no longer dependent on u at this point, which defeats the purpose.

As before, we try the Ansatz $\alpha_p = 1/p!$ subject to the constraint $B_0 = y$. Proceeding as before, we get $B_0 = y = e^y$ which is clearly wrong. We start to get an inkling: the constants α_p, whatever they are, cannot be freely chosen. They must be constants, i.e. they must be independent of y (and of x). So, from the first row, we have

$$B_0 = y = \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \cdots$$

so if these really are constants, we must have $\alpha_0 = 0$ and $\alpha_1 = 1$ and all other $\alpha_p = 0$. This gives $B_0 = y$ and $B_1 = 1$ and $B_k = 0$ for all other k.

From this, we conclude that the only possible extension is $Y(w) = y + w$ and likewise $X(u) = x + u$ so that

$$X(z-x) = z = Y(z-y)$$

so balance holds, the way we want it to. Moving to 4, we already know $a_{n0} = a_n$ are constrained as binary digits. After this, we have

$$1 = \sum_{n=1}^{\infty} a_{n1} 2^{-n}$$

for $k = 1$ and

$$0 = \sum_{n=1}^{\infty} a_{nk} 2^{-n}$$

for $k > 1$. Recall that the point of this is to obtain extensions of the a_n as given in 3. It would appear that we have a huge freedom of choice in these extensions, given the above constraints. So, for example, $a_{n1} = 1$ solves the first constraint, and $a_{1k} = -1$ with $a_{nk} = 1$ solves the second, for $n > 1$, $k > 1$. Plugging in, this gives

$$a_1(u) = a_1 + u\left(\frac{1-2u}{1-u}\right)$$

and for $n > 1$the corresponding

$$a_n(u) = a_n + \frac{u}{1-u}$$

Note these are just one possible choice, out of an infinite number of choices. It would be interesting to know if there is a choice that gives the same form for all $a_n(u)$. The correctness of the above is easily checked: one has

$$\begin{aligned}
x(u) = &\sum_{n=1}^{\infty} a_n(u) 2^{-n} \\
=& x + \frac{1}{2}\left[u\left(\frac{1-2u}{1-u}\right) + \frac{u}{1-u}\right] \\
=& x + u
\end{aligned}$$

so this form gives exctly what ws expected.

Recall that here, u is the "local" coordinate system, centered at x; the global coordinate system is z with $u = z - x$.

Recap

What have we done here? Suppose we have a real number $0 \le x \le 1$ with a binary-digit expansion

$$x = \sum_{n=1}^{\infty} a_n 2^{-n}$$

so that each $a_n \in \{0, 1\}$. (Note that for rational x, this expansion is not unique; there are two equivalent expansions). Then we have found an infinite set of analytic continuations $a_n^c(z)$ of the a_n to the complex plane z. This gives the sum the extension

$$x(z) = \sum_{n=1}^{\infty} a_n^c(z) 2^{-n}$$

In order to make this extension self-consistent for different x, it turns out that the condition $x(z) = z$ is forced. Although one might have imagined that there were other possibilities, in retrospect, this condition is "obvious": it even sounds absurdly trivial: it just says that the real number line has a unique extension to the complex plane, which *is* the complex plane.

One possible example of this extension (out of an infinity of possibilities) is given by

$$a_1^c(z) = a_1 + (z - x)\left(\frac{1 - 2(z - x)}{1 - (z - x)}\right)$$

and for $n > 1$ the corresponding

$$a_n^c(z) = a_n + \frac{z - x}{1 - (z - x)}$$

Strange but true. Its convenient to call a given choice of a's a "gauge choice". Hmm. Now, what do we do with this?

Now what?

So we extended the binary bit-sum to the complex plane. Now what? Conversion to continued fractions requires bit-counting. So, consider the binary expansion

$$x = \quad 0.\underbrace{000...0}_{c_1}\ \underbrace{11..1}_{c_2}\ \underbrace{00..0}_{c_3}\ \underbrace{11..1}_{c_4}\ \underbrace{00..0}_{c_5}1..$$

Consider first c_k for even $k > 2$. Its the count of the number of 1-bits, and so has a plausible analytic extension equal to

$$c_k \to c_k(u) = c_k \cdot \left(\frac{u}{1-u}\right)$$

if we follow the "gauge choice" given previously. For odd k, which counts zeros, there are two plausible extensions: one that leaves c_k a constant; another which counts 1=1-0; that is,

$$c_k \to c_k(u) = c_k \cdot \left(1 - \frac{u}{1-u}\right) = c_k \left(\frac{1-2u}{1-u}\right)$$

Applying similar reasoning, we can get $c_1(u)$ and $c_2(u)$, special-casing for a zero count:

$$c_1 \to c_1(u) = \begin{cases} u\left(\frac{1-2u+2u^2}{1-u}\right) + (c_1 - 1)\left(\frac{1-2u}{1-u}\right) & \text{for } c_1 > 0 \\ 0 & \text{for } c_1 = 0 \end{cases}$$

and

$$c_2 \to c_2(u) = \begin{cases} c_2\left(\frac{u}{1-u}\right) & \text{for } c_1 \neq 0 \\ u\left(\frac{1-2u}{1-u}\right) + (c_2 - 1)\left(\frac{u}{1-u}\right) & \text{for } c_1 = 0 \end{cases}$$

These formulas are pretty whacked. Its even crazier to ponder the summation

$$x^s(u) = \sum_{k=1}^{\infty} (-1)^{k+1}\, 2^{-(c_1+c_2+\cdots+c_k)}$$

which obeys $x(0) = x$ but otherwise has a crazy analytic structure. Not quite clear why it's interesting... similarly, we can contemplate the extended continued fraction

$$\begin{aligned} x^f(u) =& [c_1+1, c_2, c_3, \cdots] \\ =& \frac{1}{c_1 + 1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots}}} \end{aligned}$$

but, again, other than the fact that we can do this, what's the point? Note that $x^f(0) = ?^{-1}(x)$ so this is kind-of a continuation of that ...

A numeric exploration of this is in misc/frontal and neither the $c_k(u)$ nor the $x^s(u)$ nor the $x^f(u)$ are very interesting. They do stuff .. but neither is it trivial, nor does it suggest anything modular. Seems like a dead-end.

5 The Question Mark as a kind-of metric

Well, we already know that ? can be constructed as a measure [ref]. It can also serve as a metric on the real-number line, and that metric can be taken as the limit of a metric on the upper half-plane. Specificically, consider a point z in the fundamental domain (i.e. $-\frac{1}{2} \leq \Re z \leq \frac{1}{2}$ and $|z| \geq 1$). Then, the action of $g^{a_1} r g^{a_2} r \cdots r g^{a_n}$ maps z to a different domain. The cusp of that domain is given precisely by the continued fraction $[a_1+1, a_2, \cdots, a_n]$, viz, exactly the definition used to construct the question mark. The cusps are in one-to-one correspondance with the rationals, in just the way needed to construct the question mark. Thus the ordinary Eucliden metric on the real number line can be considered to be the limit of the Poincare metric on the upper-half plane. Hand calculations needed to get the asymptotic expansion.

6 Realization & Spaces

Some words/formulas clarifying how to think of the Berry-Keating operator in eqn 1 and the conjecture. Consider the simple harmonic oscillator (SHO); it has the standard, well-know raising and lowering operators a and a^* with $[a, a^*] = 1$ and the number operator $N = aa^*$ and eigenstates $|n\rangle$ so that $N|n\rangle = n|n\rangle$. The egenstates can be taken to be the basis vectors of a Hilbert space; it comes with a norm; it's l_2; its complete; it has a complex coonjugate. The Berry-Keating operator 1 is worded in terms of the (Weyl?) realization of the shift operators as x and d/dx so that $N = xd/dx$ and $|n\rangle = x^n$. However, there is no particular reason (yet) to stick to this realization; and its somewhat convientient to explore an SHO realization, or other more abstract formulations.

A standard construction is to build 'coherent states', labelled as $|\alpha\rangle$, as a superposition, such that $a|\alpha\rangle = \alpha|\alpha\rangle$ for any complex number α. The construction is

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n |n\rangle$$

and since $a|n\rangle = |n-1\rangle$ the relation $a|\alpha\rangle = \alpha|\alpha\rangle$ follows. In the Weyl realization, $|\alpha\rangle = (1-\alpha x)^{-1}$. Note that the Weyl rep and the SHO rep have competing and conflicting notions of completeness and convergence. We gloss over these for now, but these should really really be clarified.

Polylogarithm

There are several related constructions that can be made at this point. One is a state

$$|s\rangle = \sum_{n=1}^{\infty} n^{-s} |n\rangle$$

which has the property that $N|s\rangle = |s+1\rangle$. That is, what was previously the 'number operator' has become a 'shift operator'. How strange! In the Weyl rep, we have

$$|s\rangle = \mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

This begs a question: if N behaves like a raising operator, then what is the corresponding lowering operator? Can we build coherent states for the lowering operator? What happens if we repeat this construction a second time, a third time, etc? Exactly what convergence and completeness properties apply in these cases? How are the various spaces related to the SHO Hilbert space?

Some other questions: what is the subspace orthogonal to $|s\rangle$? What is the nature of the space that one gets if s is restricted to non-negative integers? That is, how is it relatedd to the original SHO Hilbert space? What kind of a space do we get if we consider the extension where each $|s\rangle$ is considered to be a unique vector, so that the space has an uncountable basis?

Wait – there's more confusion. Consider the conjugate coherent state

$$\langle z| = \sum_{n=0}^{\infty} z^n \langle n|$$

Then one has that $\langle z|s\rangle = \mathrm{Li}_s(z)$ which begs the question: what, exactly, is the correspondance between the conjugate states, and the Weyl rep?

What is the length of $|s\rangle$? Well, since it is a meber of Hilbert space, we are allowed to write

$$\langle s| = \sum_{n=0}^{\infty} n^{-s^*} \langle n|$$

where $s^* = \sigma - i\tau$ when $s = \sigma + i\tau$. The length is then

$$\langle s|s\rangle = \sum_{n=0}^{\infty} n^{-(s^*+s)} = \sum_{n=0}^{\infty} n^{-2\sigma}$$

which is finite for $\sigma > 1/2$. From this, we conclude that $|s\rangle$ is a member of the Hilbert space whenever $\sigma > 1/2$. RH-violating zeros, if any, then correspond to non-zero vectors in this Hilbert space.

Coherent states

Restricting s to integers, we can then write a coherent state as

$$\begin{aligned}
|w\rangle &= \sum_{s=0}^{\infty} w^s |s\rangle \\
&= \sum_{s=0}^{\infty} w^s \sum_{n=1}^{\infty} n^{-s} |n\rangle \\
&= \sum_{n=1}^{\infty} |n\rangle \sum_{s=0}^{\infty} \left(\frac{w}{n}\right)^s \\
&= \sum_{n=1}^{\infty} \frac{1}{1-w/n} |n\rangle
\end{aligned}$$

Another construction

Another construction of particular interest is the construction of an operator N_s and a set of coherent states $|n;s\rangle$ such that $N_s|n;s\rangle = n^{-s}|n;s\rangle$ for complex-valued s. Thus, what we wish to do is to write an equation such as this:

$$\mathrm{Tr}\, N_s = \zeta(s)$$

when N_s is expresed as

$$N_s = \sum_{n=1}^{\infty} |n;s\rangle\, n^{-s}\, \langle n;s|$$

We are making several misleading leaps here: an implicit assumption that such an N is a nuclear operator, which would be required to have a well-defined trace. That the states $|n;s\rangle$ span some kind of space is clear; what is not clear is how this space might vary as a function of s, or quite even what this means, exactly. That is, the standard SHO states are constructed to be the orthonormal basis vectors of a Hilbert space. The definition of N_s as done above makes a (faulty) assumption of the orthonormality of the $|n;s\rangle$, which is not the case. Should we consider N_s to act on that Hilbert space, or perhaps it is better considered to be acting on some other space altogether? So the above is hand-waving about possible directions, rather than any concrete claims...

The dual space is spanned by

$$\langle\alpha;\omega| = \frac{1}{1-\alpha\omega}\sum_{n=0}^{\infty}\omega^n\langle n|$$

having the property that $\langle\alpha;\omega|\,\alpha\rangle = 1$. WTF this is already off the tracks.

The problem is that "normal" coherent states do not have well-defeined orthogonal vectors $\langle\beta|\,\alpha\rangle = 0$. However, for N_s we can build this when s is a zero ... wtf.