A Foolish Proof of RH

December 3, 2014

Abstract

What's wrong with this picture? Is there some glib handwaving masking the failing part of the proof?

1 Introduction

Wherein we provide the overall review of what is to come.

2 Musings

Outline form.

2.1 Finite summations

The divisor operator L_D and the mobius operator M_D, written in matrix form, are clearly inverses of one another. All summations are finite, the operators are upper-triangular by construction, all summations are textbook material.

2.1.1 Non-Singularity of the operators

Because they are invertible, we conclude that they are not singular. But in what space? Answer: in l_1. Certainly in the space of finite-length (bounded-length) vectors. How about infinite-length vectors?

- Proof: suppose there exists an arithmetic function aka vector $a = \{a_n\}$ such that $L_D a = 0$ But we really do have $M = L^{-1}$ and so the diagonal elements of ML really are all ones. which we can prove by finite means. So there are no such a. The space in question is simply the set of all bounded sequences $l_\infty = \{\{a_n\} : |a_n| < \infty\}$ i.e. elements can be any finite number.

Careful: for the proof above, it was sufficient to assume that $a \in l_\infty$ because, by assumption, we started with $La = 0$ and so there were from the get-go no issues about summability. However, L itself is not defined on l_∞ , since, just looking at the very first row, which are all-ones, its clear that this will be absolutely summable if and only if we pick $a \in l_1$ where as always, $l_1 = \{\{a_n\} : \sum_{n=1}^{\infty} |a_n| < \infty\}$. So both L amd M are absolutely defined on l_1. They're also defined for conditionally convergent sums. But we have a proof about the kernel that holds for l_∞ and certainly its the case that conditionally convergent series are a subset of l_∞.

For the next step, we are interested in conditionally convergent sums that are in $l_2 = \left\{ \{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$ because this is where Fourier series are. Everyone knows that of course $l_1 \subset l_2 \subset l_\infty$.

2.1.2 Summability class

Next, observe that neither L nor M change the summability class. This follows from the fact that the matrix elements of L and M are always +1, 0 or -1 and from the fact that they are upper-triangular. What do we mean by this statement? And where do we apply it? We claim that, if $a \in l_1$ then $|\sum_{n=1}^{\infty} L_{mn} a_n| < \sum_{n=1}^{\infty} |L_{mn} a_n| \leq \sum_{n=1}^{\infty} |a_n|$ because $|L_{mn}| \leq 1$ and likewise $|M_{mn}| \leq 1$. Similarly, we have the identity $\sum_{n=1}^{\infty} |L_{mn} a_n|^2 \leq \sum_{n=1}^{\infty} |a_n|^2$ and the analogous one for M, although these do not seem to be needed just yet.

2.2 Analytic sums and analytic continuation

Wherein we establish that the formal algebraic identity $m^{-s}\zeta(s) = \sum_{n=1}^{\infty} L_{mn} n^{-s}$ and its mate, $m^{-s}/\zeta(s) = \sum_{n=1}^{\infty} M_{mn} n^{-s}$, obtained by purely algebraic means, also hold as analytic summations on the entire complex s-plane. These require analytic continuation, in a purely classical sense, to work around the obvious poles.

2.2.1 Absolute convergence

OK, take a step back. We start by observing that $m^{-s}\zeta(s) = \sum_{n=1}^{\infty} L_{mn} n^{-s}$. This holds as a purely formal, algebraic manipulation. Clearly, it also holds as a numerical summation, for $\Re s > 1$. Clearly, it cannot possibly hold for $s = 1$ since $L_{1n} = 1$ for all n, and $\sum_{n=1}^{\infty} n^{-1} = \infty$ by which we mean the sum diverges. No surprise here, $\zeta(s)$ has a pole at $s = 1$. This pole will dog the rest of the arguments.

2.2.2 Analytic continuation

However, treated as a conditionally-convergent sum, we claim that numerical identity holds for $m^{-s}\zeta(s) = \sum_{n=1}^{\infty} L_{mn} n^{-s}$ when $\Re s > \frac{1}{2}$. Can this really hold true? What is meant by this? Well, let's do this row by row. For $m = 1$, we have that $\zeta(s) = \sum_{n=1}^{\infty} L_{1n} n^{-s} = \sum_{n=1}^{\infty} n^{-s}$ which is just the standard definition. We can do the standard manipulations of analytic continuation to define $\zeta(s)$ for the entirecomplex s-plane, as long as we avoid the pole at $s = 1$. Nothing exciting happens here, because $L_{1n} = 1$ for all n. How about $m = 2$? Here, we have that $L_{2n} = 0$ for n odd and $L_{2n} = 1$ for n even. So $\sum_{n=1}^{\infty} L_{2n} n^{-s} = \sum_{n=1}^{\infty} (2n)^{-s} = 2^{-s} \sum_{n=1}^{\infty} n^{-s} = 2^{-s}\zeta(s)$. There do not seem to be any dubious manipulations here, either. The summation $\sum_{n=1}^{\infty} (2n)^{-s}$ has an obvious pole at $s = 1$, but it can be analalytically continued past it. Thus, by the usual rules of analytic continuation, pulling out the factor 2^{-s} out of the summation is a valid and justified operation. The same argument obviously repeats for $m = 3, 4, 5, \cdots$ and so we conclude that the equation $m^{-s}\zeta(s) = \sum_{n=1}^{\infty} L_{mn} n^{-s}$ must hold on the entire complex plane, for any m. Well, we have not demonstrated that the strength of the pole at $s = 1$ is the same on the right and left hand sides of the equation, but this seems both immaterial, and not challenging to prove.

2.2.3 Mobius

By similar arguments to the above, we claim that likewise, the equation $m^{-s}/\zeta(s) = \sum_{n=1}^{\infty} M_{mn} n^{-s}$ holds for all m and on the entire complex plane s. The important ingredients here are that, for $m = 1$, the summation is again classically known, and that, for $m > 1$ we have, as before, just a Hilbert hotel with interspersed empty rooms.

In this case, there are far more poles to maneuver around, thus making practical calculation far more difficult. However, there's no question that the classic number theory sums involving the Mobius function are somehow suspect.

2.3 Fourier series

Wherein we define $\beta(x;s)$ and show that it is well-behaved, in all of the usual textbook senses. That is, we define

$$\beta(x;s) = \sum_{n=1}^{\infty} n^{-s} \exp(2\pi i n x)$$

and show that its analytically plausible on the entire complex s-plane. This is again, classical stuff, covered in Apostol [1990, 1976]Apostol (xx which one?), for example. However, despite its seemingly innocuous written form, do not underestimate the complexity of the above: it has essential singularities on the complex x-plane, and not one but multiple branch cuts. It is closely related to the polylogarithm, and a complete development is given in Vepstas [2008]. At any rate, it is well-behaved for real $0 < x < 1$, and we shall not need it for any other values of x.

The other problem here is to understand the formally derived equations $L\beta(x;s) = \zeta(s)\beta(x;s)$ and the corresponding $M\beta(x;s) = \zeta(s)^{-1}\beta(x;s)$, and quite exactly what they really imply in terms of summability and analyticity.

2.3.1 Fourier series, definition

Now add the Fourier bits back in. We write $\alpha(x) = \sum_{n=1}^{\infty} a_n \exp(2\pi i x n)$. If we demand absolute convergence, then l_1. For Fourier series, we only demand l_2. That is, we are willing to accept sequences $a \in l_2$ if we want $\alpha(x)$ to be well-defined. Again, this is all standard textbook material.

2.3.2 Integrability, l_p and $L^{(p)}$ spaces

But let's review it anyway. In order for Fourier series to work, we have to have that $\alpha \in L^{(2)}$ where $L^{(2)} = \left\{ f : \int_0^1 |f(x)|^2 dx < \infty \right\}$. Claim that $a \in l_2$ is sufficient to guarantee that $\alpha \in L^{(2)}$. Need to find citation for this. This is really a special case of a well-known result from Banach spaces that relates l_p to $L^{(q)}$ for blah blah. Need citation for this.

The above does NOT mean that the summation holds for all x. It is very much NOT the case that $|\alpha(x)| < \infty$ for all x. Recall that $L^{(\infty)} \subset L^{(2)} \subset L^{(1)}$. But that's OK, because we don't need $L^{(\infty)}$.

2.3.3 The beta function

For what values of s is $\beta(x;s)$ well-defined? Well, since $\beta(x;s) = \sum_{n=1}^{\infty} n^{-s} \exp(2\pi inx)$ its clear that β is absolutely summable iff $\Re s > 1$and that it is conditionally summable iff $\Re s > \frac{1}{2}$. Basically, β is well-behaved in this area, in the classic textbook sense of well-behaved. That is, for fixed x, we have that

$$|\beta(x;s)| = \left| \sum_{n=1}^{\infty} n^{-s} \exp(2\pi inx) \right| < \sum_{n=1}^{\infty} \left| n^{-s} \exp(2\pi inx) \right| = \sum_{n=1}^{\infty} n^{-\Re s}$$

and so that we have $\beta \in L^{(\infty)}$ whenever $\Re s > 1$. But this just follows from a well-known correllary of the theory of Lesbesgue spaces. Which we already cited up above. And, again, we observe that the series $\{n^{-s}\} \in l_2$ whenever $\Re s > \frac{1}{2}$. We can also write this in dullardly detail:

$$\sum_{n=1}^{\infty} \left| n^{-s} \exp(2\pi inx) \right|^2 = \sum_{n=1}^{\infty} n^{-2\Re s} < \infty$$

when $\Re s > \frac{1}{2}$.

In fact, lets just claim that $\beta(x;s)$ is analytic on the entire complex s-plane, except for the obvious pole at $s = 1$. This can be acheived because $\beta(x;s)$ is bounded by $\zeta(s)$. That is, consider any analytic continuation of $\zeta(s)$ into any region R of the complex plane, to give an absolutely convergent summation $\zeta_R(s)$ on that region. The function $\beta(x;s)$ can be continued in the same way, giving an absolutely convergent series representation $\beta_R(x;s)$ having the property that .. err. well. that its bounded. It ain't got no pole surprises in store for us. It ain't got not pole but the one at $s = 1$.

2.3.4 The Eigen-Equation

We already know, by purely formal means, that $L\beta(x;s) = \zeta(s)\beta(x;s)$. For what values of s is this well defined? Well, how did we even obtain this equation? We started withthe equation $m^{-s}\zeta(s) = \sum_{n=1}^{\infty} L_{mn} n^{-s}$, which we already showed is analytic on the complex s-plane, in section 2.2. Now, sum over m to obtain $\sum_{m=1}^{\infty} \exp(2\pi imx) \sum_{n=1}^{\infty} L_{mn} n^{-s} = \sum_{m=1}^{\infty} \exp(2\pi imx)\, m^{-s}\zeta(s)$. The left-hand side converges if the right-hand side does. The right hand side is clearly just $\zeta(s)\beta(x;s)$. What is the left-hand side? Why, its just the definition of $[L\beta](x;s)$.

The notation here is perhaps confusing, and so is worth exploring. Suppose we have a vector (aka an arithmetic function) v_k. We define $v(x) \equiv \sum_{k=1}^{\infty} v_k \exp(2\pi ikx)$, and, as previously explored $v(x)$ is square-integrable if v_k is square-summable. That is, this definition of $v(x)$ is well-defined. Now we consider the product $w_n = \sum_{k=1}^{\infty} L_{nk} v_k$.

Let us assume that the summations are appropriately convergent. Then we have that $w(x) = [Lv](x)$. That is the only meaning of the square brackets, here. The are simply saying that the product of L and v should be taken first, and then the fourier transform to position-space happens second. This is the notation being used, when writing $[L\beta](x;s)$.

So, to recap, we have that $[L\beta](x;s) = \zeta(s)\beta(x;s)$ holds on the entire complex s-plane (excepting a pole at $s = 1$), and for $0 < x < 1$. In fact, we can even be a little bit stronger and claim that it holds for $0 \leq x \leq 1$ but this does not seem to have any material effect on the argument. For the $L^{(2)}$ integrability, the endpoints doing matter.

Persuing an identical argument, we get that $[M\beta](x;s) = \zeta(s)^{-1}\beta(x;s)$ is valid, as a Fourier equation, over the entire complex s-plance. Note the choice of language: We are *not* saying that this is an operator equation, because we have *not* discussed what happens under a change of basis. Likewise, we have avoided the connotations associated with the words "Hilbert space". Its not that we expect anything bad to happen under a change of basis; its rather a side topic that does not seem to have any bearing on the argument. In all cses, the operator L is always taken in the basis where its matrix elements are as initially defined.

We have taken the liberty of calling this an eigen-equation, despite the fact that the spectrum appears to be continuous, rather than discrete. Explorations of the spectrum are interesting in thier own right, but, again, seem to have no particular bearing on the argument at hand.

2.3.5 Integrability Redux

We would like to make the claim that, if $f \in L^{(2)}$ then $Lf \in L^{(2)}$also, where Lf is short-hand for $[Lf](x)$, and, as explained above, $[Lf](x)$ is, in turn, a short-hand for a Fourier sum perfomed on the product formed from L and the Fourier decomposition of f. However, for the moment, it does not seem that this claim is needed anywhere, so we leave it unjustified.

2.4 The critical strip

Suppose that the Riemann zeta function has a zero at $s = s_0 = \sigma + i\tau$ with $\sigma > \frac{1}{2}$. This implies that $[L\beta](x;s_0) = 0$. Likewise, we conclude that $[M\beta](x;s)$ has a pole at $s = s_0$.

So, lets multiply this by M. What happens? In particular, in what sense can we expect $M = L^{-1}$ to still hold in this new setting?

Well, therein lies the rub, dudn't it? We do have that $M = L^{-1}$ as ever before. Through the round-about manipulations, though, nothing really happens here. That is, we are just droping a pole onto a zero, and the two shall cancel; this should be no surprise. Right?

To get a proof of the RH, we need to make a stronger claim. We need to prove that L is not singular on $L^{(2)}$, that it is invertible on $L^{(2)}$ and that M is it's inverse on $L^{(2)}$. All that we have for certain, right now is that $M = L^{-1}$ holds on l_1, and that we can gingerly extend certain related equations to $L^{(2)}$ by carefully navigating around a pole located at $s = 1$.

By doing these same ginger manipulations, we can in fact obtain $M = L^{-1}$ as an analytic identity, but that is all. That is, the pole of one is co-located with the zero of the other, and they cancel each-other out precisely. This is NOT the same as saying that L does not have any zeros in $L^{(2)}$.

3 Conclusion

Say something here.

References

Tom M. Apostol. *Introduction to Analytic Number Theory*. Springer; New York, 1976.

Tom M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Springer, 2nd ed. edition, 1990.

Linas Vepstas. An efficient algorithm for accelerating the convergence of oscillatory series, useful for computing the polylogarithm and hurwitz zeta functions. *Numerical Algorithms*, 47(3):211–252, 2008. doi: 10.1007/s11075-007-9153-8. preprint: arXiv:math.CA/0702243 (February 2007).