Inertias

1 Solid Cylinder About Central Axis

The formula for the moment of inertia of a solid cylinder about the central axis is derived from the moment of inertia of thin cylindrical shells, which are all summed using integration. A solid cylinder with a single thin shell highlighted is shown by figure 1.

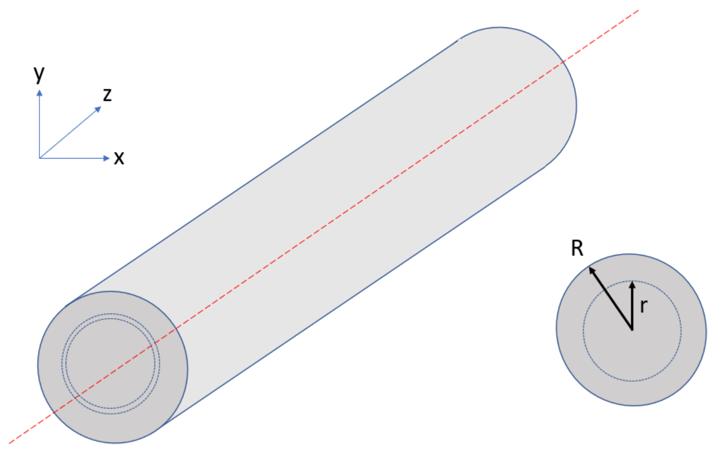

Figure 1: *A solid cylinder with a thin shell indicated*

The cylinder has the following properties:

1. Length = L
2. Mass = m
3. Density $\rho = \frac{m}{V} = \frac{m}{\pi R^2 L}$

Consider the cross-section of the cylinder and the thin shell, shown by Figure 2.

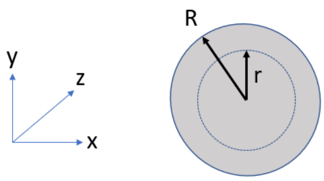

Figure 2: *Solid cylinder Cross Section*

The cylinder Cross-Section has the following properties:

1. Cylinder Radius = R
2. Shell Radius = r
3. Shell Thickness = dr

4. Shell Area $= 2\pi r dr$

The mass of the shell is given by Equation 1

$$dm = \rho dv = \rho L 2\pi r dr \tag{1}$$

The general definition for the Moment of Inertia is given by the expression shown by Equation 2.

$$I = \int_0^m r^2 dm \tag{2}$$

If we substitute the expression shown by Equation 1 for dm, we may instead perform the integral over the radius of the cylinder. This means that we are essentially finding the inertia of each shell and adding up all the infinitesimally thin shells across the whole cylinder, as shown by Equation 3.

$$I = \int_0^R r^2 \rho L 2\pi r dr \tag{3}$$

$$\rightarrow I = 2\pi\rho L \int_0^R r^3 dr \tag{4}$$

$$\rightarrow I = 2\pi\rho L \left[\frac{r^4}{4}\right]_0^R \tag{5}$$

$$\rightarrow I = 2\pi\rho L \frac{R^4}{4} \tag{6}$$

$$\rightarrow I = 2\pi \frac{m}{\pi R^2 L} L \frac{R^4}{4} \tag{7}$$

Finally producing the expression for the Moment of Inertia about the central axis of a solid cylinder, shown by Equation 8.

$$\boldsymbol{I = \frac{1}{2} m R^2} \tag{8}$$

2 Hollow Cylinder About Central Axis

The derivation follows the exact same procedure as that of the solid cylinder. Again, we will be finding the moment of inertia of infinitesimally thin cylindrical shells and summing them up using integration. Consider the cross-section of a hollow cylinder as shown by Figure 3.

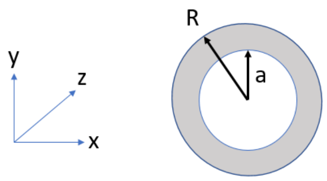

Figure 3: *A thin annulus*

Note for a hollow cylinder: Density $\rho = \frac{m}{V} = \frac{m}{\pi(R^2-a^2)L}$
Most of the derivation is the same as for a solid cylinder, and to avoid unnecessary repetition is omitted here. The only difference is that the integral must be performed over different limits. For the solid cylinder, the integration was performed from the central axis to the outer radius of the cylinder, as shown by the limits shown by Equation 3. For the case of the hollow cylinder, we must integrate from a to R, as shown by Equation 9.

$$I = \int_a^R r^2 \rho L 2\pi r dr \tag{9}$$

$$\rightarrow I = 2\pi\rho L \int_a^R r^3 dr \tag{10}$$

$$\rightarrow I = 2\pi\rho L \left[\frac{r^4}{4}\right]_a^R \tag{11}$$

$$\rightarrow I = 2\pi\rho L \left(\frac{R^4}{4} - \frac{a^4}{4}\right) \tag{12}$$

$$\rightarrow I = 2\pi \frac{m}{\pi(R^2-a^2)L} L \frac{1}{4} \left(R^2 - a^2\right)\left(R^2 + a^2\right) \tag{13}$$

Finally producing the expression for the Moment of Inertia about the central axis of a hollow cylinder, shown by Equation 14.

$$\boldsymbol{I = \frac{1}{2} m \left(R^2 + a^2\right)} \tag{14}$$

3 Solid Cylinder About Central Diameter

Consider a solid cylinder, shown by Figure 4.

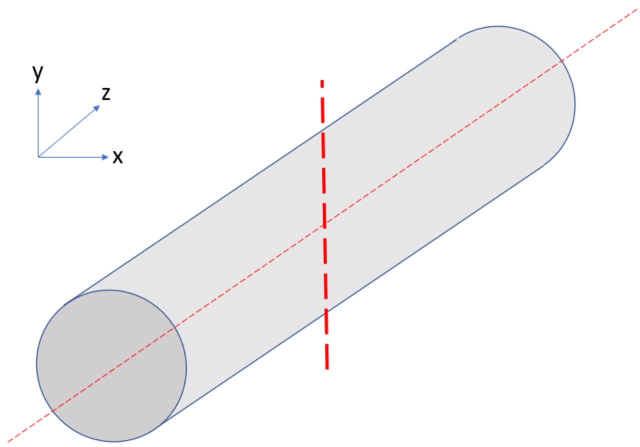

Figure 4: *A solid cylinder*

The cylinder has the following properties:

1. Length = L
2. Mass = m
3. Density $\rho = \frac{m}{V} = \frac{m}{\pi R^2 L}$

This cylinder is composed of many infinitely thin disks, shown by Figure 5.

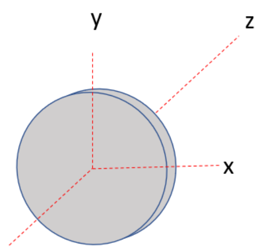

Figure 5: *An infinitely thin disk*

A thin disk can be considered as a cylinder with a small length, for which we have already derived the moment of inertia about the central axis:

$$I = \frac{1}{2} m R^2 \tag{15}$$

For an infinitely thin disk, with thickness dz:

$$dI_z = \frac{1}{2} dm R^2 \tag{16}$$

The small amount of mass of the infinitely thin disk is given by Equation 17.

$$dm = \rho dV = \frac{m}{V} A dz = \frac{m}{L} dz \tag{17}$$

By the perpendicular axis theorem:

$$dI_z = dI_x + dI_y \tag{18}$$

x and y are equal by symmetry, thus:

$$dI_x = \frac{1}{2}dI_z = \frac{1}{4}dmR^2 \tag{19}$$

Consider a disk which is distance z from the origin which coincides with the Centre of Mass. We may use the parallel axis theorem to find the Inertia:

$$I_{parallel} = I_{COM} + md^2 \tag{20}$$

where d is the distance from the centre of mass to the axis.

$$dI_x = \frac{1}{4}dmR^2 + dmz^2 \tag{21}$$

Substitute Equation 17 and integrate over the whole cylinder, shown by Equation 22:

$$I_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} dI_x \tag{22}$$

$$\rightarrow I_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{4}R^2\frac{m}{L}dz + \int_{-\frac{L}{2}}^{\frac{L}{2}} z^2\frac{m}{L}dz \tag{23}$$

$$\rightarrow I_x = \left[\frac{1}{4}R^2\frac{m}{L}z + \frac{1}{3}\frac{m}{L}z^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} \tag{24}$$

$$\rightarrow \frac{1}{4}\frac{m}{L}R^2\left(\frac{L}{2} - \left(-\frac{L}{2}\right)\right) + \frac{1}{3}\frac{m}{L}\left(\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right) \tag{25}$$

$$\tag{26}$$

Finally producing the expression for the Moment of Inertia about the central diameter of a solid cylinder, shown by Equation 27.

$$\boldsymbol{I_x = \frac{1}{4}mR^2 + \frac{1}{12}mL^2} \tag{27}$$

4 Solid Cylinder About Endpoint Diameter

Consider a solid cylinder, shown by Figure 6.

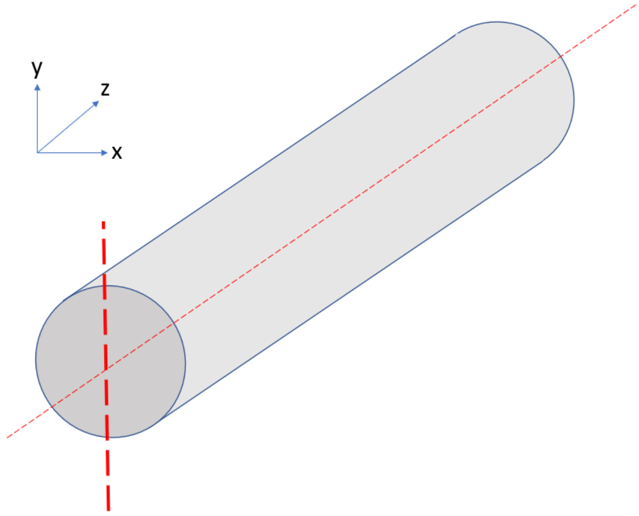

Figure 6: *A solid cylinder*

The process is exactly the same as deriving the Moment of Inertia about the central diameter. The only difference is the application of the limits of integration over the cylinder. Previously, the limits were as described by Equation 22. Starting again from that point:

$$I_x = \int_0^L dI_x \tag{28}$$

$$\rightarrow I_x = \int_0^L \frac{1}{4}R^2 \frac{m}{L} dz + \int_0^L z^2 \frac{m}{L} dz \tag{29}$$

$$\rightarrow I_x = \left[\frac{1}{4}R^2 \frac{m}{L} z + \frac{1}{3}\frac{m}{L} z^3 \right]_0^L \tag{30}$$

Finally producing the expression for the Moment of Inertia about the endpoint diameter of a solid cylinder, shown by Equation 31.

$$\boldsymbol{I_x = \frac{1}{4}mR^2 + \frac{1}{3}mL^2} \tag{31}$$

5 Solid Cylinder About Arbitrary Diameter

Consider a solid cylinder, shown by Figure 7.

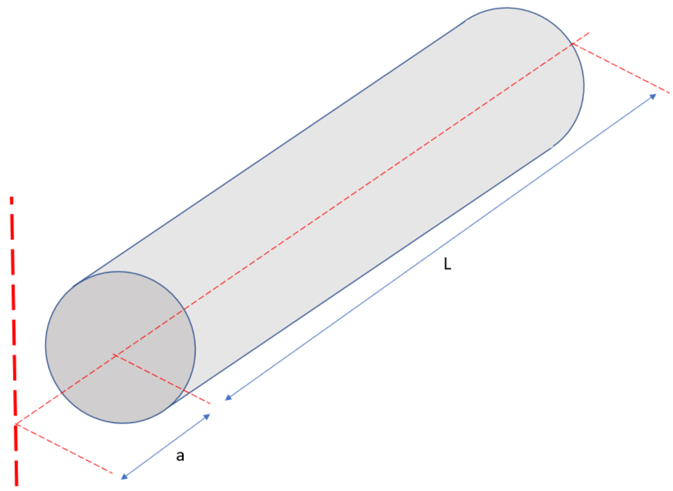

Figure 7: *A solid cylinder*

The process is exactly the same as deriving the Moment of Inertia about the central diameter and endpoint Diameter. The only difference is the application of the limits of integration over the cylinder. Previously, the limits were as described by Equation 22. Starting again from that point:

$$I_x = \int_a^{L+a} dI_x \tag{32}$$

$$\rightarrow I_x = \int_a^{L+a} \frac{1}{4}R^2\frac{m}{L}dz + \int_a^{L+a} z^2\frac{m}{L}dz \tag{33}$$

$$\rightarrow I_x = \left[\frac{1}{4}R^2\frac{m}{L}z + \frac{1}{3}\frac{m}{L}z^3\right]_a^{L+a} \tag{34}$$

Finally producing the expression for the Moment of Inertia about an arbitrary diameter of a solid cylinder, shown by Equation 35.

$$\boldsymbol{I_x = \frac{1}{4}mR^2 + \frac{1}{3}\left(3a^2 + 3aL + L^2\right)} \tag{35}$$

6 Hollow Cylinder About Central Diameter

Consider a hollow cylinder, shown by Figure 8.

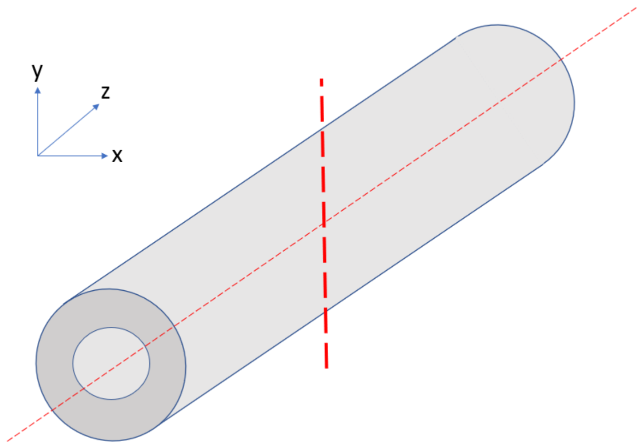

Figure 8: *A hollow cylinder*

The cylinder has the following properties:

1. Length = L
2. Mass = m
3. Density $\rho = \frac{m}{V} = \frac{m}{\pi(R^2-a^2)L}$

This cylinder is composed of many infinitely annuli, shown by Figure 9.

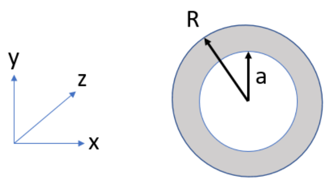

Figure 9: *A thin annulus*

We have already derived the moment of inertia of a hollow cylinder about its central axis. The moment of inertia of a thin annulus is given by Equation 36.

$$dI_z = \frac{1}{2}dm\left(R^2 + a^2\right) \tag{36}$$

The small amount of mass of the infinitely thin annulus is given by Equation 37.

$$dm = \rho dV = \frac{m}{V}Adz = \frac{m}{L}dz \tag{37}$$

By the perpendicular axis theorem:

$$dI_z = dI_x + dI_y \tag{38}$$

x and y are equal by symmetry, thus:

$$dI_x = \frac{1}{2}dI_z = \frac{1}{4}dm\left(R^2 + a^2\right) \tag{39}$$

Consider an annulus which is distance z from the origin which coincides with the Centre of Mass. We may use the parallel axis theorem to find the Inertia:

$$I_{parallel} = I_{COM} + md^2 \tag{40}$$

where d is the distance from the centre of mass to the axis.

$$dI_x = \frac{1}{4}dm\left(R^2 + a^2\right) + dmz^2 \tag{41}$$

Substitute Equation 37 and integrate over the whole cylinder, shown by Equation 42:

$$I_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} dI_x \tag{42}$$

$$\rightarrow I_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}dz + \int_{-\frac{L}{2}}^{\frac{L}{2}} z^2\frac{m}{L}dz \tag{43}$$

$$\rightarrow I_x = \left[\frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}z + \frac{1}{3}\frac{m}{L}z^3\right]_{-\frac{L}{2}}^{\frac{L}{2}} \tag{44}$$

$$\rightarrow \frac{1}{4}\frac{m}{L}\left(R^2 + a^2\right)\left(\frac{L}{2} - \left(-\frac{L}{2}\right)\right) + \frac{1}{3}\frac{m}{L}\left(\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right) \tag{45}$$

$$\tag{46}$$

Finally producing the expression for the Moment of Inertia about the central diameter of a hollow cylinder, shown by Equation 47.

$$\boldsymbol{I_x = \frac{1}{4}m\left(R^2 + a^2\right) + \frac{1}{12}mL^2} \tag{47}$$

7 Hollow Cylinder About Endpoint Diameter

Consider a solid cylinder, shown by Figure 10.

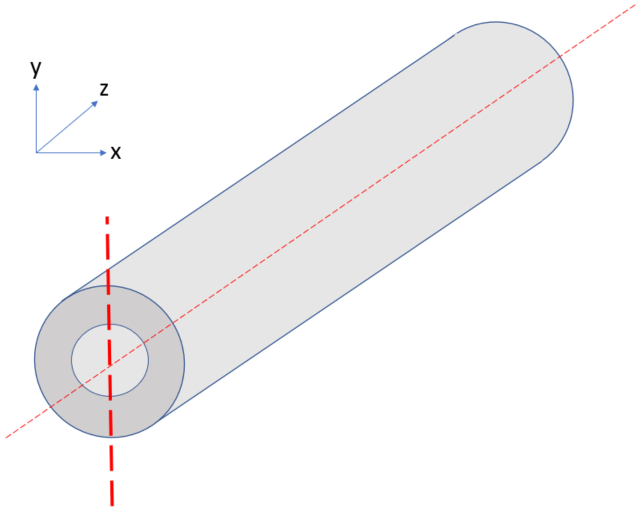

Figure 10: *A Hollow cylinder*

The process is exactly the same as deriving the Moment of Inertia about the central diameter. The only difference is the application of the limits of integration over the cylinder. Previously, the limits were as described by Equation 42. Starting again from that point:

$$I_x = \int_0^L dI_x \tag{48}$$

$$\rightarrow I_x = \int_0^L \frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}dz + \int_0^L z^2\frac{m}{L}dz \tag{49}$$

$$\rightarrow I_x = \left[\frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}z + \frac{1}{3}\frac{m}{L}z^3\right]_0^L \tag{50}$$

Finally producing the expression for the Moment of Inertia about the endpoint diameter of a solid cylinder, shown by Equation 51.

$$\boldsymbol{I_x = \frac{1}{4}m\left(R^2 + a^2\right) + \frac{1}{3}mL^2} \tag{51}$$

8 Hollow Cylinder About Arbitrary Diameter

Consider a hollow cylinder, shown by Figure 11.

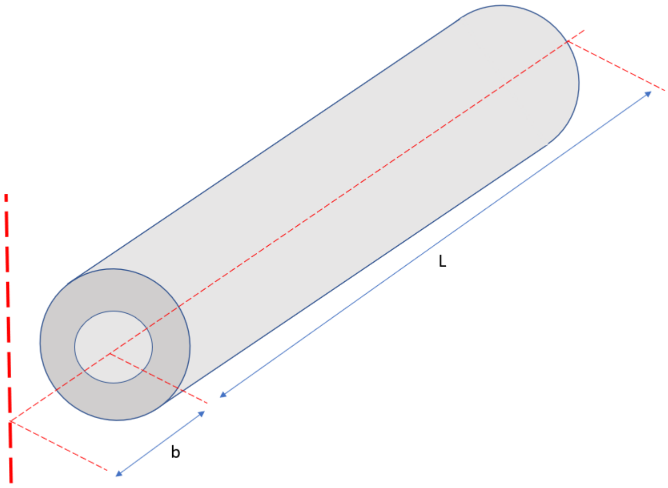

Figure 11: *A Hollow cylinder*

The process is exactly the same as deriving the Moment of Inertia about the central diameter and endpoint Diameter. The only difference is the application of the limits of integration over the cylinder. Previously, the limits were as described by Equation 42. Starting again from that point:

$$I_x = \int_b^{L+b} dI_x \tag{52}$$

$$\rightarrow I_x = \int_b^{L+b} \frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}dz + \int_b^{L+b} z^2 \frac{m}{L}dz \tag{53}$$

$$\rightarrow I_x = \left[\frac{1}{4}\left(R^2 + a^2\right)\frac{m}{L}z + \frac{1}{3}\frac{m}{L}z^3\right]_b^{L+b} \tag{54}$$

Finally producing the expression for the Moment of Inertia about an arbitrary diameter of a solid cylinder, shown by Equation 55.

$$\boldsymbol{I_x = \frac{1}{4}m\left(R^2 + a^2\right) + \frac{1}{3}m\left(3b^2 + 3bL + L^2\right)} \tag{55}$$