Uniqueness of Electrical Conductivity Through an Open Bounded Domain

Adam Miller

June 26, 2024

Abstract

The Calderón problem asks if it is possible to determine conductivity through a domain given measurements of voltage and current on the boundary of the domain. This result gives rise to an imaging method called Electrical Impedance Tomography, which has applications in medical imaging, exploration geophysics, and nondestructive testing. We show that on an open bounded subset Ω of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary, the Dirichlet-to-Neumann map, which measures the electrical current flowing through the boundary $\partial\Omega$, uniquely determines the electrical conductivity through Ω.

1 Introduction

In 1980, Alberto Calderón published "On an inverse boundary value problem" [Calderón, 1980]. An inverse problem is one of recovering unknown parameters from external observations on a system. The Calderón problem asks if it is possible to determine the electrical conductivity throughout a medium by making measurements of voltage and current on the boundary. We aim to give a proof that the electrical conductivity is determined *uniquely* by the map which measures the current through the boundary of the medium.

Calderón was motivated by exploration geophysics: the idea of being able to take measurements along the Earth's surface, and gaining information about what's happening in the interior. He was initially inspired by finding oil deposits underground, but his work has many geophysical applications. This includes finding ore deposits and water reservoirs through taking measurements of voltage and current on the Earth's surface.

Beyond this initial motivation, the Calderón problem led to development of a medical imaging technology called Electrical Impedance Tomography, which is noninvasive. By placing electrodes and taking measurements of voltage on the body, we are able to measure the conductivity throughout the body's tissues. From this, there have been exciting developments in the early detection of breast cancer. Since cancerous breast tissue has a slightly higher electrical conductivity than normal, benign tissue [Uhlmann, 2014], imaging the body using EIT allows us to detect any potential cancerous tissue without doing any invasive procedures, like taking a biopsy, or exposing the patient to radiation, like in a mammogram.

Unfortunately, many of the applications of Calderón's problem require taking voltage and current measurements on only part of the boundary. If we want to detect oil underground, it isn't feasible to gather data of electrical current along the entire Earth's surface. Similarly, if we want to detect any potential cancerous breast tissue, we would prefer to only place electrodes on the boundary of the breast instead of along the whole body. Proving the uniqueness result for partial data requires techniques in microlocal, semiclassical, and harmonic analysis, which are beyond the scope of this paper, so we instead only focus on proving the result given full data on the entire boundary. To see the case of partial data, look at Isakov's "On uniqueness in the inverse conductivity problem with local data" [Isakov, 2007].

In this paper, we aim to prove that given the map which measures the electrical current flowing through the boundary of the domain, the function which measures the electrical conductivity of the entire domain is determined uniquely. To do so, we will first define these maps, as well as discuss the partial differential equation for the conductivity of the domain. We are interested in finding the solutions to this conductivity equation, and we will focus mainly on looking at solutions called Complex Geometrical Optics solutions. These specific solutions are the primary tool that we will use to prove the uniqueness result.

2 Definitions and Preliminaries

We consider our domain $\Omega \subseteq \mathbb{R}^n$ for $n \geq 3$ to be open, bounded, and with smooth boundary. We denote $\overline{\Omega}$ to be the closure of Ω and $\partial\Omega$ to be the boundary of Ω. In practice, Ω could be a variety of different domains – for example, Ω could be the Earth's surface or a human's body.

Our work in the later sections relies on working in L^p spaces, so we recall the following definitions from measure theory [Folland, 1999];

Definition 1. Let $(X, \mathcal{M}, \mu)$ be a measure space. If f is a measurable function and $0 < p < \infty$, define the norm

$$||f||_{L^p(X)} = \left[\int_X |f|^p \, d\mu \right]^{1/p}.$$

Define the L^p *space* as

$$L^p(X) = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_{L^p(X)} < \infty\}.$$

For $p = \infty$, define

$$||f||_{L^\infty(\Omega)} = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}$$

and

$$L^\infty(X) = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_\infty < \infty\}.$$

A *Sobolev space* H^s is the function space of functions in $L^2(\mathbb{R})$ such that f and all weak derivatives of f up to order s have finite L^2 norm.

Now, we must define the physical quantities that we're measuring:

Definition 2. *Conductivity* is the ability of a material to conduct electrical current. *Potential* is the amount of work energy needed per unit of electric charge to move the charge from a given point to another fixed point. *Voltage* is the difference in electrical potential between two points. *Sources* and *sinks* in current are points through which electrical current enters and exits the domain.

Let $\gamma(x)$ be a positive C^2 (twice continuously differentiable) function on $\overline{\Omega}$ which describes the conductivity in Ω. Let $f(x) \in H^{1/2}(\partial\Omega)$ be the voltage potential at the boundary $\partial\Omega$. Such an f can be obtained through taking measurements using a voltmeter and placing electrodes along $\partial\Omega$. We require that $f \in H^{1/2}(\partial\Omega)$ due to a technique using Fourier series which is beyond the scope of this paper. Now, given that there are no sources or sinks in current in Ω, we have that there is a function $u \in H^1(\Omega)$ which solves the conductivity equation

$$\nabla \cdot \gamma \nabla u = 0$$

in Ω [Calderón, 1980]. Our main goal is to find the values $\gamma(x)$ for the conductivity at each $x \in \Omega$ given knowledge of the voltage potential along the boundary $\partial\Omega$. For a given partial differential equation, a *Dirichlet Problem* is the problem of finding a function which solves the equation in the interior of Ω, given that we know a function which solves it on $\partial\Omega$. So, we have that the Dirichlet problem for the conductivity equation is given by

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}, \tag{1}$$

and we know that there is a function $u \in H^1(\Omega)$ which solves it [Calderón, 1980]. When discussing these Dirichlet problems, we want to only consider those which will give us useful information. We call such problems well-posed.

Definition 3. A problem is *well-posed* if the problem has a solution, the solution is unique, and the behavior of the solution changes continuously respect to the initial conditions.

So, the Dirichlet problem (1) for the conductivity equation is well-posed if for every $f \in H^{1/2}(\Omega)$, there is a unique solution $u \in H^1(\Omega)$ such that $||u||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}$. In general, for any voltage potential $f \in H^{1/2}(\partial\Omega)$, the above Dirichlet problem is not well-posed. To illustrate this, we consider the following example as constructed in [Uhlmann, 2014].:

Example 2.1. Let $\lambda > 0$ be an eigenvalue of the Laplacian Δ in Ω and let $f = -\lambda$. Then there exists a nonzero $u \in H^1(\Omega)$ such that $\Delta u = \lambda u$ in Ω, but $u|_{\partial\Omega} = 0$, violating the condition that $||u||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}$, since the solution is nonzero on Ω and is identically zero on the boundary. So, the Dirichlet Problem for the conductivity equation is not well posed.

In order to avoid any counterexamples as the one illustrated above, we aim to reduce to a case in which the problems are always well-posed.

Now, our function f on $\partial\Omega$ describes the voltage potential along $\partial\Omega$, but we would like to recover values for the current flowing through $\partial\Omega$. The map which provides these values is called the Dirichlet-to-Neumann map.

Definition 4. Let ν be the unit outer normal vector to the boundary $\partial\Omega$. The *Dirichlet-to-Neumann map*, or voltage-to-current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Bigg|_{d\Omega}.$$

With the inner product on $\partial\Omega$ with respect to the surface measure

$$\langle f, g \rangle_{\partial\Omega} = \int_{\partial\Omega} fg \, dS,$$

we obtain that if u is the solution to the Dirichlet problem (1) and $v \in H^1(\Omega)$ is any function which is equal to g on the boundary $\partial\Omega$, the Dirichlet-to-Neumann map is defined weakly as

$$\langle \Lambda_\gamma f, g \rangle_{\partial\Omega} = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx. \tag{2}$$

It can also be shown that the Dirichlet-to-Neumann map is self-adjoint [Uhlmann, 2014]. Now, we can see that our main goal is showing that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.

3 Complex Geometrical Optics Solutions

In order to prove that the conductivity γ is determined uniquely by the Dirichlet-to-Neumann map, we rely on a technique using complex geometrical optics solutions.

Definition 5. A *Complex Geometrical Optics* (CGO) solution to a given partial differential equation is a solution of the form

$$u(x) = e^{i\zeta \cdot x}(1 + r), \tag{3}$$

where $\zeta \in \mathbb{C}^n$ and r is an approximation term which varies from equation to equation to make the solution exact.

We are interested in finding solutions of this form because they have very unique properties. As we will see later in Lemma 3.2, complex geometrical optics solutions allow us to recover explicit expressions for solutions along the asymptotic limit as we let $|\zeta| \to \infty$. In many cases with partial differential equations, we are unable to acquire general solutions. However, with complex geometrical optics solutions, we are able to look at the behavior as $|\zeta| \to \infty$ to gather information on what an explicit solution would look like. Later, in the proof of Lemma 4.1, we will see that looking at the behavior as $|\zeta| \to \infty$ is crucial.

As we mentioned earlier, the Dirichlet problem for the conductivity equation is not generally well-posed, and we would like to reduce to a case in which the problems are always well-posed. To do this, we look at the Schrödinger equation for an induced voltage potential $q \in L^\infty(\Omega)$:

Definition 6. If $u \in H^1(\Omega)$ solves the Dirichlet problem (1), then for a voltage potential $q \in L^\infty(\Omega)$, we have the Schrödinger equation

$$(\Delta + q)u = 0 \text{ in } \Omega. \tag{4}$$

As the main term in this equation is the Laplacian Δ, this equation is much easier to work with than the conductivity equation $\nabla \cdot \gamma \nabla u = 0$ in Ω. As discussed and proven in [Uhlmann, 2009], we have that this class of Dirichlet problems for Schrödinger equations of voltage potentials induced by conductivities is the reduction we are looking for:

Lemma 3.1. *For a voltage potential $q \in L^\infty(\Omega)$ which is induced by an electrical conductivity γ, the Dirichlet problem for the Schrödinger equation* (4) *is always well-posed.*

Now, our attention is focused on only these Dirichlet problems for Schrödinger equations of voltage potentials induced by an electrical conductivity. Since these problems are well-posed, we know that there is a unique solution. The following result from [Uhlmann, 2014] describes that under certain conditions, we are able to recover an explicit expression for a complex geometrical optics solution to these equations:

Lemma 3.2. *Let $q \in L^\infty(\Omega)$. There is a constant C depending on Ω and n such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C||q||_{L^\infty(\Omega)}, 1)$, for any function $a \in H^2(\Omega)$ satisfying*

$$\zeta \cdot \nabla a = 0 \text{ in } \Omega,$$

the equation $(\Delta + q)u = 0$ in Ω has a complex geometrical optics solution

$$u(x) = e^{i\zeta \cdot x}(a + r),$$

where r satisfies

$$||r||_{L^2(\Omega)} \leq \frac{C}{|\zeta|}||(\Delta + q)a||_{L^2(\Omega)}.$$

The proof for this lemma is given in [Uhlmann, 2014], and is beyond the scope of this paper. Writing $\zeta = \text{Re}(\zeta) + i\text{Im}(\zeta)$, we see that $\zeta \cdot \zeta = 0 \iff |\text{Re}(\zeta)| = |\text{Im}(\zeta)|$ and $\text{Re}(\zeta) \cdot \text{Im}(\zeta) = 0$. From this, we see that we are able to easily construct situations where we can get a complex geometrical optics solution, and we can also get a bound for the L^2 norm of r. Since the denominator of the bound is $|\zeta|$, we observe that the term with r will vanish if we let $|\zeta| \to \infty$.

4 Uniqueness of Electrical Conductivity in Ω

We begin by giving a proof of the uniqueness of the voltage potentials given that their Dirichlet-to-Neumann maps agree on the boundary. Then, we will show that the uniqueness result for any γ_1, γ_2 follows from this reduction to the Schrödinger equation.

Lemma 4.1. *Let $q_1, q_2 \in L^\infty(\Omega)$ such that the Dirichlet problems for the Schrödinger equations $(\Delta + q_1)u_1 = 0$ and $(\Delta + q_2)u_2 = 0$ in Ω are well-posed. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$ in Ω.*

Proof. For $j = 1, 2$, let f_j be such that u_j is equal to f_j on the boundary of Ω. Then, we claim that

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2 \rangle_{\partial\Omega} = \int_\Omega (q_1 - q_2)u_1 u_2 \, dx.$$

By the weak definition of the Dirichlet-to-Neumann map (2), we have

$$\langle \Lambda_{q_1} f_1, f_2 \rangle_{\partial\Omega} = \int_\Omega (\nabla u_1 \cdot \nabla u_2 + q_1 u_1 u_2) \, dx,$$

and similarly since the Dirichlet-to-Neumann map is self-adjoint, we have

$$\langle \Lambda_{q_2} f_1, f_2 \rangle_{\partial\Omega} = \langle f_1, \Lambda_{q_2} f_2 \rangle_{\partial\Omega} = \langle \Lambda_{q_2} f_2, f_1 \rangle_{\partial\Omega} = \int_\Omega (\nabla u_2 \cdot \nabla u_1 + q_2 u_2 u_1) \, dx.$$

Thus

$$\langle(\Lambda_{q_1}-\Lambda_{q_2})f_1,f_2\rangle_{\partial\Omega}=\langle\Lambda_{q_1}f_1,f_2\rangle_{\partial\Omega}-\langle\Lambda_{q_2}f_1,f_2\rangle_{\partial\Omega}=\int_\Omega(q_1-q_2)u_1u_2\,dx.$$

Now, since $\Lambda_{q_1}=\Lambda_{q_2}$, we know that

$$\int_\Omega(q_1-q_2)u_1u_2\,dx=0 \tag{5}$$

for any solutions u_1,u_2 to the equations $(\Delta+q_1)u_1=0$ and $(\Delta+q_2)u_2=0$ in Ω.
Now, our goal is to apply Lemma 3.2 in such a way which implies that $q_1=q_2$. Fix $\xi\in\mathbb{R}^n$, and choose two unit vectors $\vec{v_1},\vec{v_2}\in\mathbb{R}^n$ such that the set $\{\xi,\vec{v_1},\vec{v_2}\}$ is orthogonal. Then set

$$\zeta=s(\vec{v_1}+i\vec{v_2})\in\mathbb{C}^n$$

for some $s>0$. Notice that since $\vec{v_1}$ and $\vec{v_2}$ are unit vectors, we have that $s=|\zeta|$. By orthogonality of v_1 and v_2, we have that $\zeta\cdot\zeta=0$ for any s we choose, since $|\mathrm{Re}(\zeta)|=|\mathrm{Im}(\zeta)|$ and $\mathrm{Re}(\zeta)\cdot\mathrm{Im}(\zeta)=0$. Additionally, since $\{\xi,\vec{v_1},\vec{v_2}\}$ is orthogonal, we have that ζ and ξ are orthogonal. Now, we begin constructing equations for which Lemma 3.2 will provide us solutions. First, set $a=e^{ix\cdot\xi}$. Then

$$\zeta\cdot\nabla a=\zeta\cdot\nabla(e^{ix\cdot\xi})=(\zeta\cdot\xi)e^{ix\cdot\xi}=0,$$

since ζ and ξ are orthogonal. So, choosing $s=|\zeta|\geq\max\{C_1||q||_{L^\infty(\Omega)},1\}$, we have by Lemma 3.2 that there is a constant C_1 such that the equation $(\Delta+q)u=0$ in Ω has a complex geometrical optics solution

$$u_1(x)=e^{i\zeta\cdot x}(e^{ix\cdot\xi}+r_1),$$

where $||r_1||_{L^2(\Omega)}\leq\frac{C_1}{s}||(\Delta+q)e^{ix\cdot\xi}||_{L^2(\Omega)}$. Similarly, we construct a second equation taking $-\zeta$ and $a\equiv1$. We still have that $(-\zeta)\cdot(-\zeta)=0$, and since a is a constant, $\zeta\cdot\nabla a=0$. Then, choosing $s=|\zeta|\geq\max\{C_2||q||_{L^\infty(\Omega)},1\}$, we have by Lemma 3.2 that there is a constant c_2 such that the equation $(\Delta+q)u=0$ in Ω has a complex geometrical optics solution

$$u_2(x)=e^{-i\zeta\cdot x}(1+r_2),$$

where $||r_2||_{L^2(\Omega)}\leq\frac{C_2}{|\zeta|}||(\Delta+q)1||_{L^2(\Omega)}$. Now, we would like to make the bounds for the L^2 norms of r_1 and r_2 a bit nicer. Choosing $C=\max\{C_1||(\Delta+q)e^{ix\cdot\xi}||_{L^2(\Omega)},C_2||(\Delta+q)||_{L^2(\Omega)}\}$ and $s\geq\max\{C_1||q||_{L^\infty(\Omega)},C_2||q||_{L^\infty(\Omega)},1\}$, then we have that

$$||r_j||_{L^2(\Omega)}\leq\frac{C}{s}\text{ for }j=1,2.$$

Now, we can plug in our solutions u_1 and u_2 into (5). We get

$$\int_\Omega(q_1-q_2)e^{i\zeta\cdot x}(e^{ix\cdot\xi}+r_1)e^{-i\zeta\cdot x}(1+r_2)=\int_\Omega(q_1-q_2)(e^{ix\cdot\xi}+r_1)(1+r_2)=0, \tag{6}$$

where r_1,r_2 are dependent only on s, with $||r_j||_{L^2(\Omega)}\leq\frac{C}{s}$. So, we can take the limit as $s\to\infty$, and r_1 and r_2 will vanish. Taking the limit as $s\to\infty$ in (6), we obtain

$$\int_\Omega(q_1-q_2)e^{ix\cdot\xi}=0.$$

But, $\xi\in\mathbb{R}^n$ was arbitrary, so this holds for all $\xi\in\mathbb{R}^n$. This implies that $q_1=q_2$, so our proof is done. $\square$

Before we can prove the main result, we first give the following lemma; if we have a voltage potential q which comes from the conductivity function γ, then the Dirichlet Problem is always well-posed and there is a relation between the Dirichlet-to-Neumann maps Λ_γ and Λ_q.

Lemma 4.2. *If γ is a positive function in $C^2(\overline{\Omega})$ and $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$, then $q \in L^\infty(\Omega)$, the Dirichlet problem for $\Delta + q$ is well-posed, and*

$$\Lambda_q f = \frac{1}{\sqrt{\gamma}}\Lambda_\gamma\left(\frac{f}{\sqrt{\gamma}}\right) + \frac{1}{2\gamma}\frac{\partial\gamma}{\partial\nu}f\bigg|_{\partial\Omega} \tag{7}$$

for every $f \in H^{1/2}(\Omega)$.

This lemma is proven in [Sylvester and Uhlmann, 1987], and we accept it as true. Now that we have a relation between Λ_γ and Λ_q, we are ready to prove that that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \gamma_1 = \gamma_2$.

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n\,(n \geq 3)$ be an open, bounded set with smooth boundary. Let $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ be two positive functions. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

Proof. Let $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ be two positive functions and assume that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. For $j = 1, 2$, set

$$q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}.$$

By Lemma 4.2, for $j = 1, 2$, we have $q_j \in L^\infty(\Omega)$, the Dirichlet problems for $\Delta + q_j$ are well-posed, and $\gamma_1 \mid_{\partial\Omega} = \gamma_2 \mid_{\partial\Omega}$. At this point, we have that γ_1 and γ_2 agree on the boundary of Ω, and we assumed that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then by (7), for any $f \in H^{1/2}(\partial\Omega)$, we have

$$\begin{aligned}
\Lambda_{q_1} f &= \frac{1}{\gamma_1}\Lambda_{\gamma_1}\left(\frac{f}{\sqrt{\gamma_1}}\right) + \frac{1}{2\gamma_1}\frac{\partial\gamma_1}{\partial\nu}f\bigg|_{\partial\Omega} \\
&= \frac{1}{\gamma_2}\Lambda_{\gamma_2}\left(\frac{f}{\sqrt{\gamma_2}}\right) + \frac{1}{2\gamma_2}\frac{\partial\gamma_2}{\partial\nu}f\bigg|_{\partial\Omega} \\
&= \Lambda_{\gamma_2} f.
\end{aligned}$$

So, $\Lambda_{q_1} = \Lambda_{q_2}$ on Ω, and we apply Theorem 4.1 to give us that $q_1 = q_2$ on Ω. Therefore,

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \text{ on } \Omega. \tag{8}$$

From here, we would like to conclude that $\gamma_1 = \gamma_2$. To do this, let

$$q = \frac{\Delta\gamma_1}{\sqrt{\gamma_1}} = \frac{\Delta\gamma_2}{\sqrt{\gamma_2}},$$

and notice that both $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ solve the Schrödinger equation $(\Delta + q)u = 0$. Therefore, since the Schrödinger equation for a voltage potential $q \in L^\infty(\Omega)$ coming from a conductivity is well-posed (by Lemma 3.1), we have that the solutions are unique. Thus $\sqrt{\gamma_1} = \sqrt{\gamma_2}$, and we conclude that $\gamma_1 = \gamma_2$. □

5 Electrical Impedance Tomography

Now that we have proven the uniqueness of the electrical conductivity of the domain Ω given the Dirichlet-to-Neumann map on $\partial\Omega$, we return to the motivation which we provided in the introduction to illustrate an example of an application of this result. Electrical Impedance Tomography (EIT) is a technology in medical imaging developed from the Calderón Problem which allows us to determine the conductivity of tissue internal to the body without performing any sort of invasive procedure. In this example, Ω is the interior of the human body – this is good, since the human body exists in $\mathbb{R}^3$ and has a smooth boundary. We can think of the boundary of Ω to be the skin, and the interior to be the inside of the body, which includes the internal tissue.

Figure 1: Placing electrodes on the body.

Then, by placing electrodes along the body as in the above figure, we are able to take measurements of a voltage potential $q \in L^\infty(\Omega)$ which is induced by the electrical conductivity of the body. From this, we are able to define the Dirichlet-to-Neumann map Λ_q, which will give the values for the electrical current flowing through the boundary, or the skin. Then, Theorem 4.3 gives us that the conductivity of the interior of the body is determined uniquely by Λ_q. In other words, this means that if we take measurements using electrodes and get two voltage potentials q_1 and q_2 such that $\Lambda_{q_1} = \Lambda_{q_2}$, then the electrical conductivity of the body will be given uniquely.

As mentioned earlier, this has major applications in the early detection of breast cancer. Since the electrical conductivity of malignant tissue is higher than that of benign tissue, and we have shown that our measurements of electrical conductivity throughout the body is determined uniquely, we are able to conclude whether or not a patient's body contains cancerous tissue only by making measurements of the voltage potential along the surface of the skin. This is especially important, as many of our current forms of breast cancer detection leave harmful, long-lasting impacts on the patient. Mammograms, as a form of X-Ray, use radiation that can be harmful to the patient, especially if performed more than once in a short time period. Taking a biopsy, while having less of a long-lasting effect, still is an invasive procedure, and can be unsuccessful in detecting breast cancer if the cancerous tissue is concentrated in one spot. Utilizing Electrical Impedance Tomography for breast cancer detection, however, is entirely noninvasive.

References

[Calderón, 1980] Calderón, A. (1980). On an inverse boundary value problem. *Computational and Applied Mathematics - COMPUT APPL MATH*, 25.

[Folland, 1999] Folland (1999). *Real Analysis: Modern Techniques and Their Applications, 2nd Edition.* John Wiley and Sons, Inc.

[Isakov, 2007] Isakov, V. (2007). On uniqueness in the inverse conductivity problem with local data. *Inverse Problems and Imaging*, 1(1):95–105.

[Sylvester and Uhlmann, 1987] Sylvester, J. and Uhlmann, G. (1987). A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, 125(1):153–169.

[Uhlmann, 2009] Uhlmann, G. (2009). Electrical impedance tomography and calderón's problem. *Inverse Problems*, 25:123011.

[Uhlmann, 2014] Uhlmann, G. (2014). Inverse problems: seeing the unseen. *Bull. Math. Sci.*, 4:209–279.