Uniqueness of Electrical Conductivity Through an Open Bounded Domain

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Inverse Problems

Inverse Problem

An *Inverse Problem* is a problem of recovering unknown parameters from external observations (measurements, data) on a system.

On an inverse boundary value problem

The Calderón Problem

Is it possible to determine the electrical conductivity throughout a domain Ω by making measurements of voltage and current on the boundary?

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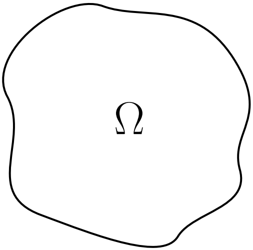
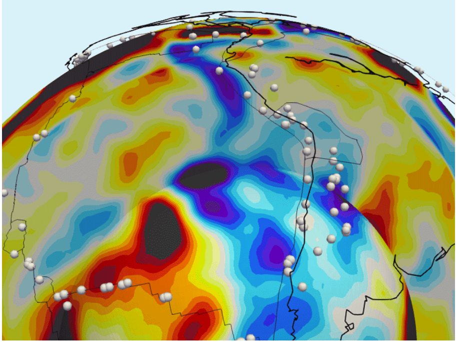


Figure: $\Omega \subseteq \mathbb{R}^n, n \geq 2$

Motivation

Motivation

- Exploration Geophysics



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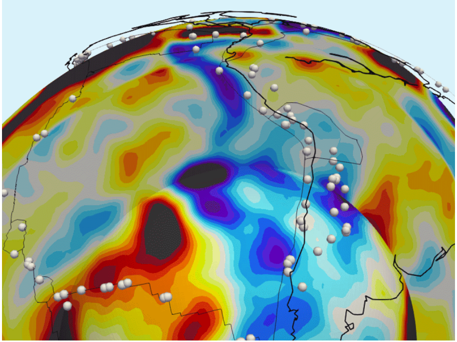

  - Oil
  - Ore deposits
  - Water reservoirs

Dirichlet Problem

Dirichlet Problem

A *Dirichlet Problem* for a particular partial differential equation (PDE) is finding a function which solves the PDE on the interior of Ω given one that solves it on the boundary $\partial\Omega$.

Conductivity Problem

- Conductivity function $\gamma(x)$

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- Voltage Potential function $f(x)$ on $\partial\Omega$

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- Conductivity function $\gamma(x)$
- Voltage Potential function $f(x)$ on $\partial\Omega$
- u solves the Dirichlet problem for the conductivity equation:

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}.$$

Dirichlet-to-Neumann Map

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$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}$$

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$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}$$

- Describes the current flowing through the boundary of Ω.

The Main Result

Theorem

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ on $\partial\Omega$, then $\gamma_1 = \gamma_2$ on Ω.

- Recall that Λ_γ is the Dirichlet-to-Neumann map for the positive conductivity function γ.
- If two conductivity functions have the same Dirichlet-to-Neumann map on $\partial\Omega$, then they must be the same.

Complex Geometrical Optics solutions

A Complex Geometrical Optics (CGO) solution to a PDE is of the form

$$u(x) = e^{i\zeta\cdot x}(1 + r(x))$$

The Schrödinger Equation

CGOs of the Schrödinger Equation

For a voltage potential $q \in L^\infty(\Omega)$, the Schrödinger equation

$$(\Delta + q)u = 0 \text{ in } \Omega$$

has a complex geometrical optics solution of the form

$$u(x) = e^{\zeta \cdot x} \gamma^{-\frac{1}{2}}(x) \left(1 + O\left(\frac{1}{|\zeta|} \right) \right), \ |\zeta| \to \infty.$$

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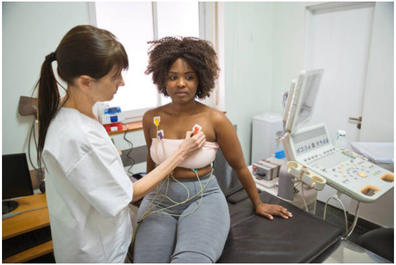
- This CGO solution is the key to proving the uniqueness result.

Detection of Corrosion

- Detection of cracks, corrosion, and leaks
- Nondestructive testing of materials

Electrical Impedance Tomography



- Early Detection of Breast Cancer
- Monitoring Lung Health

Thank you!