Mystery 3

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Twilight hangs over London. You make your way home from the pub. You reach your apartment, and, as exhaustion overtakes you, you sink into your chair. That instant, there is an urgent knock at the door. Inspector Lestrade comes barging in. "Watson," he pants. "There's been a murder. And we think Moriarty is involved."

Police have discovered the decomposing body of one of Moriarty's nemeses in a local park. According to the forensics, the person was killed yesterday between the hours of 7 and 8pm. They have arrested Moriarty, but can't find any witnesses that claim to have seen him in the park around that time.

Moriarty says: "Look – I'm a famous criminal. People would recognize me. If I killed the guy, what are the odds that nobody would have seen me at the park? Low! Therefore, you should conclude that I didn't do it."

Your goal in this mystery is to rigorously evaluate Moriarty's claim. Similar to Task 1, you will determine the posterior probability that Moriarty is guilty, given the fact that nobody has reported seeing him at the park. But now, there's an implicit third variable that we have to model: We don't know *how many people were at the park!*

So there is one known variable – the number of people who reported seeing Moriarty at the park (call it R, for "reports"). In this case, $R = 0$. And there are two unknowns: Whether Moriarty is guilty, and how many people were at the park. Call the first unknown G (if he's guilty, $G = 1$; if not, $G = 0$). Call the second unknown N (and assume that N can be between 0 and 1000). N is crucial to evaluating Moriarty's claim, but we only really care about G. What do we do?

The key is to evaluate the posterior probability of guilt given both R and N, and then marginalize out N to compute the probability of guilt given just R. In other words, we first compute $p(G \mid R, N)$, and then marginalize out N to compute $p(G \mid R)$.

Part A

Write a function that computes the posterior probability of guilt given a fixed R and N – i.e. $p(G \mid R, N)$. (If you're stuck, write out Bayes rule!) Assume that: (i) if Moriarty were actually there, each person at the park would have had a 90% chance of seeing him, (ii) if Moriarty were not actually there, each person at the park would have had a 20% chance of seeing him (because people sometimes make mistakes), and (iii) you're pretty damn sure *a priori* that Moriarty is guilty; the prior probability that $G = 1$ is 0.9.

Now, suppose that we magically know that there were exactly 7 people at the park that night. So $N = 7$. Compute and plot the posterior probability that Moriarty is guilty, for the following hypothetical values of R: 0, 1, 2, …, 7. In reality, $R = 0$ – in this case, what is the probability that Moriarty is guilty? This seems to support Moriarty's claim.

Part B

Of course, we have no reason to think that *N = 7*. In fact, we don't want to know the probability of guilt given some *N*. Instead, we want to know: Averaging across all *N*, what is the probability of guilt? To accomplish this, we use the following fact:

$$p(G \mid R) = \sum_{N=0}^{1000} p(G \mid R, N) * p(N)$$

In other words, if we average over the probability of guilt (conditioned on *N*), weighting each value by the prior probability of there being exactly *N* people in the park, then we get what we want: $p(G \mid R)$. (This is called *marginalizing out* N.)[1]

We already know $p(G \mid R, N)$; we computed it in Part A. What about $p(N)$? In words, this is the prior probability that there would have been *N* people in the park that night. The easiest way to model this probability is with a Poisson distribution. Basically, a Poisson distribution models the number of discrete events that occur in a given timeframe, assuming that the events are independent. (Google it if you want to learn more!) Using MATLAB syntax, you should assume that $p(N) = poisspdf(N, \lambda)$, where λ is the average number of people in the park. Assume that $\lambda = 7$ - i.e., on average, there are 7 people in the park.

Use these facts to write a function that computes the marginal probability that Moriarty is guilty, given the fact that there were zero reports (i.e. *R = 0*). (This function should not take *N* as an argument!) What is the marginal probability of guilt given *R = 0*? This number should be order of magnitudes higher than the number you found at the end of Part A. Why is the marginal probability of guilt (assuming that there's 7 people *on average* at the park) so much higher than the probability of guilt when there's exactly 7 people in the park? (Hint: Look up pictures of a Poisson distribution, or plot it yourself!)

Part C

You remember that there was actually a parade going on in a different part of town that night, and the park would have been quite empty. You now believe that the average number of people in the park was only 1, i.e. $\lambda = 1$. Redo Part B. What is the marginal probability of guilt now? Plot the marginal probability for $\lambda = 1$ to 7.

Part D

You then remember that Moriarty is also a master in disguise, and he could easily have escaped detection. Assume now that, if Moriarty were actually there, each person would only have a 40% chance of recognizing and reporting him. Redo Part B with this assumption. What is the marginal probability of guilt now? Note how dramatically your estimate has changed from Part A. In Bayesian modeling, your construction of the situation can strongly affect the resulting inference.

(If you want to see my solution to Mystery 3, check out "moriarty.m"; online here).

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[1] I've brushed over two technicalities here. First, the rightmost term in the equation should actually be $p(N \mid R)$, but for simplicity we assume that your prior over *N* does not depend on *R*, so it simplifies to $P(N)$. Second, the sum should actually go to $N = \infty$, but for simplicity we cap it at 1000.