Modern Cosmology

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Abstract

These lecture notes cover the Modern Cosmology fourth year optional module (PHYS4016) and Particles and Gravity M.Sc course. It begins with a review of Friedmann models including an introduction to the thermal history of the universe, freeze-out, relics, recombination, the epoch of last scattering and dark matter candidates. We then go into more details concentrating initially on the Inflationary scenario and explaining why it is required in the early Universe. It leads us into physics beyond the standard model as we are introduced to scalar fields. A number of inflation models are presented, along with their equations of motion. The slow roll conditions are obtained and we introduce the ideas behind reheating the Universe at the end of inflation. The most important aspect of inflation, the associated generation of primordial density fluctuations are derived along with the power spectra. Particular emphasis is given to the connection between the generation of the fluctuations, the slow roll parameters and the cosmological observables. Moving on from Inflation we enter the world of large scale structure formation. Initially we adopt a Newtonian approach neglecting pressure. This allows us to define and introduce perturbation modes; matter transfer functions; nonlinear effects and the spherical collapse model. The Lagrangian approach is developed leading to a description of N-body simulations, dark-matter haloes and mass functions, as well as the importance of gas cooling. This culminates in a brief overview of galaxy formation. Gravitational lensing is introduced, we describe what it is and how it can be used to detect dark matter. The mechanisms required for generating Cosmic Microwave Background anisotropies are described and linked to the inflationary perturbations. The associated Boltzmann equations, power spectrum, tensor modes and polarisation signatures are described for the case of ΛCDM models. Finally, we describe the evidence for the existence of dark energy, which is believed to be driving the current acceleration of the universe. Although it fits the data the best, there are theoretical issues associated with using a cosmological constant and these are described along with the results obtained from adopting one. Alternative models involving an evolving scalar field, Quintessence

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are introduced and the associated fine tunings required with them are described. The possibility that the current acceleration is a manifestation of modified gravity is briefly reviewed. Material that we did not cover directly in lectures but is still cool stuff, part of the course and worth knowing about, is in the framed boxes.

Useful resources

- D. Baumann, *Cosmology*, (Cambridge University Press, 2022) This will be the main book we follow. Well written, covering all the main topics you will need (and more) with lots of problems.
- S. Dodelson, *Modern Cosmology*, (Academic Press, 2003) A very good book, also covering all the main topics you will need in this module.
- D.H. Lyth and A.R. Liddle, *The Primordial Density Perturbation*, (Cambridge University Press, Cambridge, 2009) This is especially strong on the inflation section and as the title says on how to generate primordial perturbations from inflation – written by two of the experts in Inflation. We make great use of it in Chapters (5) and (6).
- P.J.E. Peebles, *Principles of Physical Cosmology*, (Princeton University Press, Princeton, 1993). A classic, written by a master of the field. Tough going though, not for the light hearted.
- J.A. Peacock *Cosmological Physics*, (Cambridge University Press, Cambridge, 1999). A superb book describing the physics of the Big Bang and which is also very up-to-date.
- V. Mukhanov *Physical Foundations of Cosmology*, (Cambridge University Press, Cambridge, 2005). A wonderful book written by a pioneer in the field of cosmological perturbations.
- E. W. Kolb and M. S. Turner, *The Early Universe*, (Frontiers in Physics, Addison-Wesley Publishing Company, 1990). If there is one vintage book in cosmology then this is it. Nicely written and all material are still relevant.
- R. Durrer, *The Cosmic Microwave Background*, (Cambridge University Press, Cambridge, 2008). Only for the brave. Very mathematical, covers in great depth everything related to CMB theory and beyond. If you can master this book you can become master of the Universe.

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"The cosmos is all that is or ever was or ever will be. Our feeblest contemplations of the Cosmos stir us – there is a tingling in the spine, a catch in the voice, a faint sensation, as if a distant memory, or falling from a height. We know we are approaching the greatest of mysteries."

Carl Sagan, *Cosmos*

1 Review of Friedmann Models of Cosmology

We begin this course with a few lectures dedicated to reviewing the subject to the level we would expect a third year undergraduate to have reached. For completeness this is a fairly detailed set of notes and the lectures will not be going into as much detail. The extra material that we don't cover directly in the class have been beautifully framed for you. Our belief is you should have access to as much material as possible, even though we can not possibly cover it all in such a short period of time. For those not feeling so familiar with the background material, we would recommend you get to grips with this section, possibly with the aid of some of the recommended background literature.

1.1 Observational features of our Universe

Over the past few decades a number of features of our Universe appear to have been established.

- The observable patch has a radius of around 3000 Mpc (1 Mpc $\simeq 3.26 \times 10^6$ years $\simeq 3.08 \times 10^{24}$ cm).
- It is homogeneous and isotropic on scales larger than 100 Mpc with well developed inhomogeneous structure (i.e. clusters of galaxies, galaxies, stars ... us) on smaller scales.
- It is expanding according to Hubble's Law, although it is accelerating at present.
- The Universe is full of thermal microwave background radiation with temperature $T \simeq 2.725$K
- It contains baryonic matter, roughly one baryon per 10^9 photons, and very little anti-matter.
- Of the baryonic matter, 75% of it is in hydrogen, 25% helium and trace amounts of heavier elements.
- Baryons contribute 5% of the total energy density, the rest is dark and appears to be composed of cold dark mater with negligible pressure (~ 27 %) and dark energy with negative pressure (~ 68%).
- Observations of the fluctuations in the cosmic microwave background radiation indicate that when the universe was a thousand times smaller than it is today, the fluctuations in the energy density distribution were as small as 10^{-5} – very small indeed.

1.2 Metrics – a brief resume of some key results of General Relativity

We need to introduce the concept of a metric tensor in order to fully exploit the cosmological solutions we will be obtaining and in particular to allow us to discuss perturbations around those background solutions. Let's start with the **metric** of space-time which describes the physical distance between points. It is the metric which allows us to interpret the geometry of the Universe, ideas of luminosities and distances in cosmology.

To get an idea of the space-time metric, first consider the usual Euclidean distance between two points on a piece of flat paper with coordinate axis x_1 and x_2. This of course is given by Pythagoras

$$\Delta s^2 = \Delta x_1^2 + \Delta x_2^2,$$

where Δx_1 and Δx_2 are the separation in the x_1 and x_2 coordinates, and it is invariant in that it does not depend on what coordinate system we use (i.e. cartesian or polars). If the paper is replaced by a rubber sheet which expands, then the coordinate grid expands with the sheet and the physical distance between the points grows as well. If the expansion is uniform (i.e. same everywhere) we write

$$\Delta s^2 = a^2(t)[\Delta x_1^2 + \Delta x_2^2],$$

with $a(t)$ giving the rate of expansion, and coordinates x_1 and x_2 are comoving coordinates. In GR, we replace the spatial coordinates by space-time coordinates, and look for the distance between points in four dimensional space-time. In other words there is noting special about time here it is one of the coordinates. We also allow for the possibility that the spatial sections may be curved. The infinitesimal separation, ds is written in terms of the infinitesimal coordinate separation dx^μ as

$$ds^2 = \sum_{\mu,\,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is the **metric**, μ and ν are Greek indices taking values 0,1,2,3, x^0 is the time coordinate and x^1, x^2 and x^3 are the three spatial coordinates. In general $g_{\mu\nu}$ is a function of the coordinates, and this allows spacetime to be curved. From now on we will drop the explicit summation sign in these expressions, but it is implied that repeated indices are to be summed over i.e.

$$a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3. \tag{1.1}$$

A general vector $A^\mu = (A^0,\, A^i)$ has A^0 being the timelike component and A^i the three spatial components. Importantly, in relativity upper and lower indices are distinct, the former are associated with vectors and the latter with 1-forms. Going back and forth between them is done via the metric tensor,

$$A_\mu = g_{\mu\nu} A^\nu \;\;;\;\; A^\mu = g^{\mu\nu} A_\nu \tag{1.2}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and will be defined shortly. A vector and a 1-form can be contracted to produce an invariant, a scalar, i.e. the four-momentum squared of a massless particle must vanish

$$P^2 \equiv P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = 0.$$

In fact the metric raises and lowers indices on tensors in general, not just vectors. For example consider raising the indices on the metric tensor itself

$$g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta} \tag{1.3}$$

Note, that if the index $\alpha = \nu$, then the first term on the right is equal to the term on the left, which means that in that case we are forced to have (from the two extra factors of g on the RHS of Eqn. (1.3))

$$g^{\nu\beta} g_{\alpha\beta} = \delta^\nu_\alpha, \tag{1.4}$$

where δ^ν_α is the Kronecker delta equal to zero unless $\nu = \alpha$ in which case it is equal to unity. This is why we say $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$.
The metric tensor $g_{\mu\nu}$ is necessarily symmetric (it should not matter whether we write $dx^\mu dx^\nu$ or $dx^\nu dx^\mu$, so in principle it has four diagonal and six off-diagonal components. As stated above

it provides the link between the values of the coordinates and the more physical measure of the interval ds^2, also known as the *proper time*. Special Relativity is described by Minkowski space-time with coordinates $x^\mu = (ct, x^i)$, $i = 1, 2, 3$ and metric $g_{\mu\nu} = \eta_{\mu\nu}$ where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

For the case of an expanding universe, the two grid points move apart so that their physical separation is proportional to the scale factor. If today the comoving distance is x_0, then the physical distance between the two points at some earlier time t was $a(t)x_0$ (which is based on the normalisation that $a(t_0) = 1$). It suggests that in a spatially flat expanding universe the metric is (recall the coordinates are $x^\mu = (ct, x^i)$, $i = 1, 2, 3$)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{1.6}$$

or

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1.7}$$

Let us make sure we are happy with the meaning of this metric tensor. It is representing the relationship between time intervals and comoving intervals through

$$(\text{invariant interval})^2 = -(\text{time interval})^2 + (\text{scale factor})^2(\text{comoving interval})^2$$

There exists a universal cosmological time which we would associate with a clock ticking at constant comoving coordinates on the sky. The spatial part of the metric expands with time, given by the universal scale factor $a(t)$. This implies that particles at constant coordinates recede from the origin and therefore must undergo a Doppler redshift due to the increasing scale factor. Eqn. (1.6) is the Friedmann-Robertson-Walker (FRW) metric for a spatially flat Universe.

Because the expansion is uniform we can write

$$\mathbf{r}(t) = a(t)\mathbf{x}(t)$$

where $\mathbf{r}$ is the real (physical) distance and $\mathbf{x}$ is the comoving distance. The comoving coordinates are therefore carried along with the expansion, any objects remaining at fixed coordinate values (if they are not moving relative to each other).

Differentiating $\mathbf{r}(t) = a(t)\mathbf{x}(t)$ we find

$$\mathbf{v(t)} = H(t)\mathbf{r}(t) + a(t)\dot{\mathbf{x}}(t)\,, \tag{1.8}$$

where

$$H(t) = \left(\frac{\dot{a}}{a}\right). \tag{1.9}$$

defines the Hubble parameter $H(t)$. The second term on the right hand side of Eqn. (1.8) is known as the *peculiar velocity* and it accounts for the local dynamics of the objects in question being affected (for instance the local velocities of neighbouring galaxies). The first term is the key term for cosmology, it tells how the expansion rate of the universe is directly affecting the velocity of recession between the two objects, even if they are not moving relative to one another (i.e. even if $\dot{\mathbf{x}}(t) = 0$). For that case, it gives us directly Hubble's Law $\mathbf{v}(\mathbf{t}) = H(t)\mathbf{r}(\mathbf{t})$. By convention we often state that $a(t_0) = 1$ today – recall when we refer to values today we attach a subscript '0' to the quantity. In that case the comoving distance is the actual distance today. Of course it implies $a < 1$ in the past. Setting $t = t_0$ in (1.8) we obtain Hubble's law with $H_0 = H(t_0)$. A word of caution though on simply setting $a(t_0) = 1$. It is not always possible to do that, in particular as we shall see, there are restrictions on whether that can be done in a curved space setting.

H_0 is becoming very well constrained. For example Planck's 2015 Constraints paper [arXiv:1502.01589 [astro-ph.CO]] indicates

$$H_0 = 100h\text{km s}^{-1}\text{ Mpc}^{-1}, \qquad h = 0.678 \pm 0.009. \tag{1.10}$$

where our uncertainty in H_0 is parameterised by the constant h. The most recent direct measurements are reported in Betoule et al. (2014), A&A, 568, A22, [arXiv:1401.4064] who obtain $h = 0.685 \pm 0.0127$ which should be compared with Riess et al, Astrophys.J. **730** (2011) 119, Erratum-ibid. 732 (2011) 129, who obtained $h = 0.738 \pm 0.024$. The uncertainty is becoming remarkably small, now around 1-2%. As an example if $h = 0.72$, then if $v_{exp} = 7200\text{km s}^{-1}$ we have a separation of 100 Mpc. Our uncertainty in h feeds into almost all of the cosmological parameters as we will see. The units of H are inverse time, and so we can use as an estimate for the cosmological expansion time H_0^{-1}, the Hubble time: $H_0 = 100h\text{ km s}^{-1}\text{ Mpc}^{-1} \longrightarrow H_0^{-1} \sim 9.77h^{-1} \times 10^9$ years.

The evolution of the scale factor depends on the density of matter in the universe. We will shortly introduce perturbations to the metric, essential if we are to understand the generation of structure in the universe. The perturbed part of the metric will be determined by the associated inhomogeneities in the matter and radiation. Equation (1.6) is the metric for a spatially flat homogeneous and isotropic universe. In the problem set, you are asked to derive the corresponding metric for the curved space generalisation. The Cosmological Principle (CP) makes life easier for us here. it means that at any time the universe should have no preferred positions. So, the spatial part of the metric must have a constant curvature (same everywhere!) which could of course be zero as in the flat metric. The most general form of the spatial metric (ds_3^2) of a three-dimensional space with constant curvature is (written in spherical polars)):

$$ds_3^2 = a^2\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{1.11}$$

where $a^2 > 0$ and the constant K measures the curvature of space. It is the same K we will use in the Friedmann equation, and describes spherical ($K > 0$), flat ($K = 0$) and hyperbolic ($K < 0$) geometries respectively. It is often normalised to be $K = +1, 0, -1$ for the three geometries. Given the form of the spatial metric which satisfies the cosmological principle, we write down the full space-time metric (i.e. include the infinitesimal change in time dt). As with moving from the paper sheet to the rubber band which expands, we can allow the space to grow or shrink in time. This gives the general curved space Robertson-Walker metric

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.12}$$

Consider the spatial sections for a few minutes. Returning to Eqn. (1.11), it often proves useful to replace the radial coordinate r with χ which is defined by

$$d\chi^2 = \frac{dr^2}{1 - Kr^2} \tag{1.13}$$

By integrating this, it follows that

$$\chi = \text{arcsinh}\, r, \quad K = -1 \tag{1.14}$$
$$\chi = r, \quad K = 0 \tag{1.15}$$
$$\chi = \arcsin r, \quad K = +1 \tag{1.16}$$

The coordinate χ varies between $0 \leq \chi \leq \infty$ for flat and hyperbolic spaces, and $0 \leq \chi \leq \pi$ for positively curved spaces. The metric Eqn. (1.11) now becomes in terms of χ,

$$ds_3^2 = a^2(d\chi^2 + S_K^2(\chi)d\Omega^2)$$

with

$$S_K(\chi) = \sinh\chi, \quad K = -1 \tag{1.17}$$
$$S_K(\chi) = \chi, \quad K = 0 \tag{1.18}$$
$$S_K(\chi) = \sin\chi, \quad K = +1, \tag{1.19}$$

where

$$d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$$

It is worth looking a bit closer at the case of these constant curvature spaces.

Three-dimensional sphere (K=+1) From ds_3^2, the distance element on the surface of the 2-sphere of radius χ is

$$dl^2 = a^2\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2).$$

You should be able to see that this is the same line element as a sphere of radius $R = a\sin\chi$ in flat three-dimensional space, which means that we can straight away write the total surface area :

$$S_{2d}(\chi) = 4\pi R^2 = 4\pi a^2\sin^2\chi.$$

The behaviour is at first strange. As the radius χ increases, the surface area grows to a maximum value at $\chi = \pi/2$, then decreases, vanishing at $\chi = \pi$. A lower dimensional analogy may be useful. The surface of the globe plays the role of three-dimensional space with constant curvature, and the two dimensional surfaces correspond to circles of constant lattitude on the globe. Starting from the north pole ($\theta = 0$), the circle circumference grows as we go south, reaching a max at the equator ($\theta = \pi/2$), then decreases as we go further, disappearing at the south pole ($\theta = \pi$). The circles cover the whole surface of the globe as θ runs from 0 to π. The same happens here with χ running from 0 to π, it covers the whole three-dimensional space of constant curvature. The area of the globe is finite, implying the volume of the three-dimensional space should also be with constant positive curvature. To show this recall that the physical width of an infinitesimal shell is $dl = ad\chi$, hence the volume element between two spheres with radii χ and $\chi + d\chi$ is

$$dV = S_{2d}(\chi) a d\chi = 4\pi a^3 \sin^2 \chi d\chi.$$

The volume within the sphere of radius χ_0 is

$$V(\chi_0) = 4\pi a^3 \int_0^{\chi_0} \sin^2 \chi d\chi = 2\pi a^3 (\chi_0 - \frac{1}{2}\sin 2\chi_0).$$

A strange looking volume, but note that for the case where the radius is small $\chi_0 \ll 1$, we recover the familiar Euclidean result

$$V(\chi_0) = \frac{4}{3}\pi (a\chi_0)^3 + \cdots$$

The total volume is for $\chi_0 = \pi$ which gives the finite result

$$V = 2\pi^2 a^3.$$

Three-dimensional pseudo-sphere (K=-1)- constant negative curvature. The metric on the surface of the corresponding 2-dimensional sphere of radius χ is

$$dl^2 = a^2 \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2),$$

which following the argument for the positively curved space above gives the area of the sphere as

$$S_{2d}(\chi) = 4\pi R^2 = 4\pi a^2 \sinh^2 \chi$$

which increases exponentially for $\chi \gg 1$. Recall that $0 \leq \chi \leq \infty$, it follows that the total volume of the hyperbolic space is infinite, being given by $V = \int_0^\infty S_{2d} a d\chi$.

1.3 Light propagation and redshifts

We know from special relativity that light follows trajectories with zero proper time (null geodesics). Considering Eqn. (1.12) it follows that the radial equations of motion integrate to give

$$\int \frac{dr}{\sqrt{1 - Kr^2}} = c \int \frac{dt}{a(t)} \tag{1.20}$$

The comoving distance is a constant, whereas the domain of integration in time extends from $t_{\rm emit}$ to $t_{\rm obs}$, the times of emission and detection of a photon. It therefore follows that

$$\frac{dt_{\rm emit}}{dt_{\rm obs}} = \frac{a_{\rm emit}}{a_{\rm obs}} \tag{1.21}$$

implying that events on distant galaxies time-dilate. Now this dilation also applies to frequency so

$$\frac{\nu_{\rm emit}}{\nu_{\rm obs}} \equiv 1 + z = \frac{a_{\rm obs}}{a_{\rm emit}} \tag{1.22}$$

In other words by observing shifts in spectral lines, we can determine the size of the universe at the time the light was emitted – this is the key result which enables the discipline of observational cosmology.

1.4 The Geodesic Equation

What directions do particles go in in curved space? We know that in Minkowski space they travel in straight lines unless acted on by an external force. In curved space, the straight line is generalised to a *geodesic*, the path of a particle in the absence of external forces. To obtain it we generalise Newton's Law in the absence of forces, $d^2\mathbf{x}/dt^2 = 0$, to the case of an expanding universe. This is a standard GR calculation and is done in all decent textbooks. For the case of relativity two key modifications need to be made. The first is to allow the indices to run from 0 to 3 thereby allowing time to be one of the coordinates. The second emerges because of the fact we now have time as a coordinate. It implies we can not use it as our evolution parameter. Instead we introduce a parameter λ which monotonically increases along the particles path. [**Insert figure here**]. The geodesic equation then becomes (see for example Schutz for a proof)

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{1.23}$$

where the Christoffel symbol is given by

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right], \tag{1.24}$$

where to remind you $x^\mu = (ct, x^i)$, $i = 1, 2, 3$. Note the use of the inverse metric $g^{\mu\nu}$ defined in Eqn. (1.4). From Eqn. (1.6) we see that in the flat ($K = 0$), FRW metric the inverse is identical to $g_{\mu\nu}$ except that its spatial elements are $1/a^2$ instead of a^2. Using Eqns. (1.6) and (1.24) we can now derive the Christoffel symbols in a spatially flat expanding homogeneous universe. First evaluate the components with the upper index being zero, $\Gamma^0_{\alpha\beta}$. The fact that the metric is diagonal implies that $g^{\nu 0} = 0$ unless $\nu = 0$ in which case $g^{00} = -1$. We then have

$$\Gamma^0_{\alpha\beta} = \frac{-1}{2}\left[\frac{\partial g_{\alpha 0}}{\partial x^\beta} + \frac{\partial g_{\beta 0}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right]. \tag{1.25}$$

The first two terms are just derivatives of g_{00} so vanish because $g_{00} = -1$. We are left with

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^0}. \tag{1.26}$$

For this to be non-zero, we require α and β to be spatial indices, which we identify with the Roman letters i, j running from 1 to 3. Now since $x^0 = ct$ we have

$$\begin{aligned}\Gamma^0_{00} &= 0 \\ \Gamma^0_{0i} &= \Gamma^0_{i0} = 0 \\ \Gamma^0_{ij} &= \delta_{ij}\, a' a\end{aligned} \tag{1.27}$$

where $a' \equiv \frac{da}{d(ct)} = \frac{1}{c}\frac{da}{dt} = \frac{1}{c}\dot{a}$. [1] It is also straightforward to show that $\Gamma^i_{\alpha\beta}$ vanishes unless one of the lower indices is zero and one is spatial giving

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\frac{a'}{a} \tag{1.28}$$

[1] In section 1.7 we will be considering the case where primes will denote derivatives with respect to conformal time. We will remind you in that case what the new a' means.

1.5 Einstein Equations

Einstein's equations relate the components of the Einstein tensor describing the geometry of space-time to the energy-momentum tensor describing the energy. The equation (including a cosmological constant Λ) is given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{1.29}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor (which depends on the metric and its derivatives), R is the Ricci scalar and is given by the contraction of the Ricci tensor ($R = g^{\mu\nu} R_{\mu\nu}$). G is Newton's constant, and $T_{\mu\nu}$ is the energy-momentum tensor. It is a symmetric tensor which describes the constituents of matter in the Universe. Note that whereas the left hand side of Eqn. (1.29) is a function of the metric, the right hand side is a function of the energy – Einstein's equation relates geometry and matter !

The Ricci tensor is given by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}, \tag{1.30}$$

where $\Gamma^{\alpha}_{\mu\nu,\alpha} \equiv \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}$ etc... For the case of the FRW universe we have already obtained the Christoffel symbols Eqns. (1.27) and (1.28) with all the others being zero. Using this we see that there are only two sets of nonvanishing components of the Ricci tensor; one with $\mu = \nu = 0$ and the other with $\mu = \nu = i$. For the case of R_{00} we have

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{00} - \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha}, \tag{1.31}$$

which then simplifies to

$$R_{00} = -\Gamma^{i}_{0i,0} - \Gamma^{i}_{j0}\Gamma^{j}_{0i}, \tag{1.32}$$

and finally using Eqn. (1.28) we end up with

$$\begin{aligned} R_{00} &= -\delta_{ii}\left(\frac{a'}{a}\right)' - \left(\frac{a'}{a}\right)^2 \delta_{ij}\delta_{ij} \\ &= -3\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right] - 3\left(\frac{a'}{a}\right)^2 \\ &= -3\frac{a''}{a} \end{aligned} \tag{1.33}$$

The factors of three on the second line are from the Kronecker δ functions – recall δ_{ii} means summing over all three spatial indices counting one for each. The space-space component is given by

$$R_{ij} = \Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{ij} - \Gamma^{\alpha}_{\beta i}\Gamma^{\beta}_{j\alpha}, \tag{1.34}$$

which for our FRW Universe becomes

$$R_{ij} = \Gamma^{0}_{ij,0} + \Gamma^{k}_{0k}\Gamma^{0}_{ij} - \Gamma^{0}_{ki}\Gamma^{k}_{j0} - \Gamma^{k}_{0i}\Gamma^{0}_{jk}, \tag{1.35}$$

which in turn becomes

$$R_{ij} = \delta_{ij}\,[2a'^2 + aa'']. \tag{1.36}$$

The Ricci scalar follows,

$$\begin{aligned} R &= g^{\mu\nu} R_{\mu\nu} \\ &= -R_{00} + \frac{1}{a^2} R_{ii} \\ &= 6\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right], \end{aligned} \tag{1.37}$$

where again remember the sum over i leads to a factor of three in R_{ii}. The Friedmann equation comes from the considering only the time-time coordinate of the Einstein equations:

$$R_{00} - \frac{1}{2} g_{00} R = \frac{8\pi G}{c^4} T_{00} - \Lambda g_{00} \tag{1.38}$$

For the case of a perfect isotropic fluid, the energy momentum tensor is given by

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{1.39}$$

where $\rho(t)$ is the energy density and $p(t)$ is the pressure of the fluid. Hence using Eqn. (1.39) in Eqn. (1.38) with Eqns. (1.33) and (1.37) we obtain Einstein's equation in a spatially flat ($K = 0$) FRW universe[2]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) + \frac{\Lambda c^2}{3} \tag{1.40}$$

This is the **Friedmann equation** in the $K = 0$ universe. The general curvature case with a cosmological constant follows from considering the metric Eqn. (1.12) in Eqn. (1.29)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) - \frac{K c^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{1.41}$$

A second equation follows when we consider the space-space component of Einstein's equation

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij} \tag{1.42}$$

Of course the only non-trivial components are when $i = j$ and we obtain

$$2\dot{a}^2 + a\ddot{a} - \frac{1}{2} a^2\, 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{8\pi G}{c^2} a^2 p(t) - a^2 \Lambda c^2 \tag{1.43}$$

which simplifies to give

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{c^2} p(t) + \Lambda c^2 \tag{1.44}$$

The general curvature case follows as before

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{c^2} p(t) - \frac{K c^2}{a^2} + \Lambda c^2 \tag{1.45}$$

[2]Note we have gone from a' to $\dot{a}$ by multiplying through by c^2. Recall $a' = \frac{\dot{a}}{c}$

Equations (1.41) and (1.45) are the basis of the standard big bang cosmological model including the current ΛCDM model. Note that they can be combined to give an **acceleration** equation where the curvature term drops out,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho(t) + \frac{3p(t)}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{1.46}$$

Note the cosmological constant term can be omitted if we make the following replacement in Eqn. (1.46)

$$\begin{aligned} \rho &\rightarrow \rho + \frac{\Lambda c^2}{8\pi G} \\ p &\rightarrow p - \frac{\Lambda c^4}{8\pi G} \end{aligned}$$

Therefore the cosmological constant can be interpreted as arising from a form of energy which has negative pressure, equal in magnitude to its (positive) energy density:

$$p = -\rho c^2 \tag{1.47}$$

which of course is consistent with $\dot{\rho} = 0$ in Eqn. (1.56) below. Such a form of energy is a generalization of the notion of a cosmological constant and is known as dark energy. In fact, in order to get a term which causes an acceleration of the universe expansion, it is enough to have a source of unusual matter which satisfies

$$p(t) < -\frac{\rho(t)c^2}{3}. \tag{1.48}$$

Such a source is sometimes known as quintessence, and is usually associated with a fluid comprised of a scalar field. It is of course unusual to have a fluid with a negative pressure, yet this is required if we are to have a universe accelerating.

Returning to the Friedmann equation (1.41) where we have re-absorbed the explicit cosmological constant into the general energy density ρ we have

$$H^2 = \frac{8\pi G}{3}\rho(t) - \frac{Kc^2}{a^2}. \tag{1.49}$$

It allows us to define the **critical density** which is the density of matter required to yield a flat universe

$$\rho_c \equiv \frac{3H^2}{8\pi G}. \tag{1.50}$$

Another quantity that can be defined is the dimensionless **density parameter** as the ratio of the denisty to the critical density:

$$\Omega \equiv \frac{\rho(t)}{\rho_c} = \frac{8\pi G\rho}{3H^2} \tag{1.51}$$

Todays values of these parameters are usually given a zero subscript, i.e. H_0, ρ_0, Ω_0. Recall Eqn. (1.10) for the Hubble parameter today, then the current density of the universe is

$$\begin{aligned} \rho_0 &= 1.878 \times 10^{-26}\Omega_0 h^2 \text{ kg m}^{-3} \\ &= 2.775 \times 10^{11}\Omega_0 h^2 M_\circ \text{Mpc}^{-3} \end{aligned} \tag{1.52}$$

Of course the critical density ρ_c corresponds to the case $\Omega_0 = 1$ or a spatially flat universe.

1.6 Evolution of energy

Let's end this section by deriving the fluid equation. The key starting point is the idea of energy momentum conservation which in the case of an expanding universe implies the *covariant* derivative of the energy momentum tensor vanishes. We are not going to derive the result here, but simply state it, although it is not difficult to derive:

$$T^{\mu}_{\nu;\mu} \equiv T^{\mu}_{\nu,\mu} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu} T^{\mu}_{\alpha} = 0 \tag{1.53}$$

Eqn. (1.53) actually corresponds to four separate equations because ν can take on four values. Considering the case $\nu = 0$ then we have

$$T^{\mu}_{0,\mu} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}_{0} - \Gamma^{\alpha}_{0\mu} T^{\mu}_{\alpha} = 0 \tag{1.54}$$

However, from Eqn. (1.39) we see that assuming isotropy implies T^0_i vanishes, hence the dummy indices μ in the first term and α in the second must be equal to zero leaving us with

$$-\frac{\partial(\rho(t)c^2)}{\partial(ct)} - \Gamma^{\mu}_{0\mu}\rho(t)c^2 - \Gamma^{\alpha}_{0\mu} T^{\mu}_{\alpha} = 0 \tag{1.55}$$

Now we can simplify further, since we know that $\Gamma^{\alpha}_{0\mu}$ vanishes unless μ and α are spatial indices and are equal to each other, as seen in Eqn. (1.28). This then leaves us with the result for the well known fluid equation in an expanding universe

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{1.56}$$

Now Eqn. (1.56) is not the end of the story, we need to relate the pressure (p) and energy density (ρ) of the fluid. This is done by assuming there is a unique **Equation of state** of the form $p = p(\rho)$, for each fluid. For the types of fluid we will be considering (no torsion) it is thought to be a simple linear relation which is written generically in one of two equivalent ways:

$$p = w\rho \qquad \text{or} \qquad p = (\gamma - 1)\rho$$

where of course $w = \gamma - 1$, and both w and γ being known as the equation of state parameter. There are some particularly important cases which crop up in cosmology: Matter – $w = 0$, includes non-relativistic particles such as baryons as well as cold dark matter and is sometimes called dust. It is pressureless and satisfies $p = 0$, which is a good approximation for atoms which seldom interact in a cooled universe. Galaxies also obey $p = 0$, as they mainly interact gravitationally. Radiation – $w = 1/3$, describes any massless (and very light) particles which move with speed approaching c. From your electromagnetic wave courses you will recall that light exerts a radiation pressure with equation of state $p = \frac{\rho c^2}{3}$. Cosmological Constant – $w = -1$ is the energy density associated with quantum fluctuations. The corresponding equation of state satisfies $p = -\rho c^2$, in other words the pressure is negative! It is vital for the *Inflationary Universe Scenario* as we shall see later.

We can solve for more general equations of state of the form p=$w\rho c^2$. The fluid equation becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0 \longrightarrow \frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}(1+w) = 0 \longrightarrow \rho \propto a^{-3(1+w)} \tag{1.57}$$

The Friedmann equation then becomes (for $k = 0$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^{-3(1+w)} \longrightarrow \dot{a} \propto a^{-(1+3w)/2} \longrightarrow a^{3(1+w)/2} \propto t \longrightarrow a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{1.58}$$

You can check it with the known cases of matter and radiation. For example for **matter** $w = 0$ we obtain the $\Omega_m = 1$ Einstein de Sitter solution

$$\rho_m(t) = \rho_{m0} \left(\frac{a_0}{a}\right)^3 \text{ and } a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} \tag{1.59}$$

whereas for **radiation** $w = 1/3$ we obtain the radiation dominated or Tolman universe:

$$\rho_r(t) = \rho_{r0} \left(\frac{a_0}{a}\right)^4 \text{ and } a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} \tag{1.60}$$

The extra factor of a in the relative energy densities between matter and radiation is just a reflection of the fact that the number density of particles is diluted by the expansion, with photons also having their energy reduced by the redshift. For the case of a **cosmological constant** $w = -1$ we have

$$\rho_v(t) = \rho_{v0} \text{ and } a(t) = a_0 \exp(H(t - t_0)) \tag{1.61}$$

where $H^2 = \frac{8\pi G}{3}\rho_{v0}$ is a constant.

Recall that the total energy density is made up of contributions from matter ρ_m, radiation ρ_r, and something that resembles a cosmological constant type term today say ρ_v, with equation of state parameter w which we would usually set to $w = -1$. Then recall the scale factor-redshift relation given in Eqn. (1.22), (i.e. $1 + z = \frac{a_0}{a}$) and the critical density as defined in Eqn. (1.50), we can write

$$\begin{aligned} \frac{8\pi G}{3}\rho &= \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_v) \\ &= \frac{8\pi G}{3H_0^2} H_0^2 \left(\rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4 + \rho_{v0}(1+z)^{3(1+w)}\right) \\ &= H_0^2 \left(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}(1+z)^{3(1+w)}\right) \end{aligned} \tag{1.62}$$

Returning to the Friedmann equation (1.41) we see that it has the awkward K factor in it. It is not an observable as such and so we really need to eliminate if we are to make progress observationally. We do it by realising that it is a constant and that it can be written in terms of the observed parameters today. In particular from (1.41) applied today we have

$$\frac{Kc^2}{a_0^2} = H_0^2 \left(\Omega_0 - 1\right) \tag{1.63}$$

where $\Omega_0 = \Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0}$. It then follows upon substitution of Eqn. (1.63) into Eqn. (1.41) that

$$H^2(z) = H_0^2 \left(\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}(1+z)^{3(1+w)} - (\Omega_0 - 1)(1+z)^2\right). \tag{1.64}$$

This equation is in a very useful form as it can be integrated immediately to get t(z). The integrals are straightforward, at least numerically, and for the case of a flat universe ($\Omega_0 = 1$) they can be performed analytically allowing us to obtain $a(t)$. As can be seen from Eqn. (1.64), curvature can always be neglected at sufficiently early times ($z \gg 1$), as can vacuum density (except for when we consider the theory of inflation as it postulates that the vacuum density was very much higher in the very distant past).

1.7 Cosmological solutions

We can now begin to address the question of what the cosmological solutions to the Friedmann equation for different cases. We have already obtained the case of single fluid components in a flat universe Eqns. (1.59)-(1.61). Thats fine when one fluid completely dominates the dynamics but that is not generally the case, for instance we know the universe actually contains both matter and radiation, and so we need to solve for them together. Moreover, in the very early universe we may expect it to be dominated initially by radiation before moving onto a period of Λ domination corresponding to the onset of inflation.

Here we deal with these more general cases which requires the introduction of **conformal time** as a useful aid in obtaining these more complicated solutions. Conformal time η, is defined by $cdt = a(\eta)d\eta$, or

$$\eta \equiv \int \frac{cdt}{a(t)}. \tag{1.65}$$

In the following section 1.7.1 we set the speed of light to unity for convenience (i.e. we set $c = 1$).

1.7.1 Solutions in general curved space with one component of matter

The Friedmann equation and acceleration equation become in terms of the new time variable:

$$\begin{aligned} a'^2 &= \frac{8\pi G}{3}\rho a^4 - Ka^2 \\ a'' &= \frac{4\pi G}{3}(\rho - 3p)a^3 - Ka \end{aligned}$$

where $a' \equiv \frac{da}{d\eta}$ etc.... (Note the new meaning of a' as opposed to that in section 1.4 – it will keep this meaning from now on.) For the case of radiation $p = \rho/3$, the acceleration equation now becomes

$$a'' + Ka = 0,$$

which can be integrated and leads to

$$\begin{aligned} a(\eta) &= c_r \sinh\eta, \qquad K = -1, \quad 0 \le \eta \le \infty \\ a(\eta) &= c_r \eta, \qquad K = 0, \quad 0 \le \eta \le \infty \\ a(\eta) &= c_r \sin\eta, \qquad K = +1, \quad 0 \le \eta \le \pi \end{aligned}$$

where c_r is a constant of integration and the second one has been fixed by demanding $a(\eta = 0) = 0$. The physical time then follows from,

$$t = \int a(\eta)d\eta,$$

giving

$$\begin{aligned} t &= c_r(\cosh\eta - 1), \qquad K = -1 \\ t &= c_r\eta^2/2, \qquad K = 0 \\ t &= c_r(1 - \cos\eta), \qquad K = +1 \end{aligned}$$

These solutions are parametric in that although we have $a(\eta)$ and $t(\eta)$ it is not generally possible to obtain an analytic expression for $a(t)$ apart of course for the $K=0$ case. Indeed in the $K=0$, radiation dominated universe it immediately follows that $a(t) \propto t^{1/2}$ with $H = 1/2t$.

For the case of dust domination, $p_m = 0$ and the acceleration equation is solved to give

$$\begin{aligned}
a(\eta) &= c_m(\cosh\eta - 1), \quad && K=-1, \quad 0 \le \eta \le \infty \\
a(\eta) &= c_m\eta^2, && K=0, \quad 0 \le \eta \le \infty \\
a(\eta) &= c_m(1-\sin\eta), && K=+1, \quad 0 \le \eta \le 2\pi
\end{aligned}$$

where c_m is a constant of integration – why not have a go to show it?

1.7.2 Combined matter and radiation solutions – K=0 case

Consider just a mixture of matter and radiation in a spatially flat universe. The energy density of matter scales as a^{-3} and radiation as a^{-4}, allowing us to write the combination as

$$\rho = \rho_m + \rho_r = \frac{\rho_{eq}}{2}\left(\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4\right),$$

where a_{eq} is the scale factor when the two components have equal energy densities – an epoch known as matter-radiation equality. Note that in the acceleration equation, the contribution from radiation vanishes because $\rho_r - 3p_r = 0$, leaving us with the pressureless matter contribution,

$$a'' = \frac{2\pi G}{3}\rho_{eq}a_{eq}^3.$$

The RHS is constant, so the integral is trivial giving,

$$a(\eta) = \frac{\pi G}{3}\rho_{eq}a_{eq}^3\eta^2 + C\eta,$$

and one of the constants of integration has been fixed by demanding $a(\eta=0)=0$. We fix C by inserting the solution for $a(\eta)$ into the Friedmann equation

$$a'^2 = \frac{8\pi G}{3}\rho a^4$$

with ρ given above. This gives

$$C = (4\pi G\rho_{eq}a_{eq}^4/3)^{1/2}$$

hence

$$a(\eta) = a_{eq}\left(\left(\frac{\eta}{\eta_*}\right)^2 + 2\left(\frac{\eta}{\eta_*}\right)\right)$$

with

$$\eta_* = (\pi G\rho_{eq}a_{eq}^2/3)^{-1/2} = \eta_{eq}/(\sqrt{2}-1)$$

Note that we obtain the expected results in the appropriate limit. For $\eta \ll \eta_{eq}$, radiation dominates and we have $a \propto \eta$, whereas for $\eta \gg \eta$, matter has come to dominate and we have $a \propto \eta^2$. To see that this gives the usual proper time dependence, simply insert the scale factor into $t = \int a d\eta$.

1.7.3 Radiation - Λ solution – K=0 case

We can solve for a system with just radiation and a cosmological constant ($w = -1$) in a straightforward manner. From Eqn. (1.64) we have

$$H^2(z) = H_0^2 \left(\Omega_{r0} \left(\frac{a_0}{a} \right)^4 + \Omega_{\Lambda 0} \right). \tag{1.66}$$

This simplifies to

$$\frac{d}{dt} a^2 = 2H_0 \sqrt{\Omega_{r0}} a_0^2 \left(1 + \alpha^2 \left(\frac{a}{a_0} \right)^4 \right)^{\frac{1}{2}}, \tag{1.67}$$

where $\alpha^2 \equiv \frac{\Omega_{\Lambda 0}}{\Omega_{r0}}$. Defining $y = \frac{a^2}{a_0^2}$ we then have

$$\dot{y} = 2H_0 \sqrt{\Omega_{r0}} (1 + \alpha^2 y^2)^{\frac{1}{2}} \tag{1.68}$$

which can be integrated to give

$$a(t) = a_0 \left(\frac{\Omega_{r0}}{\Omega_{\Lambda 0}} \right)^{\frac{1}{4}} \left(\sinh(2H_0 \Omega_{v0}^{\frac{1}{2}} t) \right)^{\frac{1}{2}} \tag{1.69}$$

where the initial condition $a(0) = 0$ has been used. For early times or small t we obtain $a(t) \propto t^{\frac{1}{2}}$ corresponding to radiation domination, and for large t we obtain exponential expansion as found in vacuum dominated de-Sitter type evolution $a(t) \propto \exp(H_0 \Omega_{v0}^{\frac{1}{2}} t)$. This would be an appropriate model for the onset of a phase of inflation following a big-bang singularity.

1.7.4 Matter - Λ solution – K=0 case

We follow the procedure as in the radiation case starting with

$$H^2(z) = H_0^2 \left(\Omega_{m0} \left(\frac{a_0}{a} \right)^3 + \Omega_{\Lambda 0} \right). \tag{1.70}$$

This time by introducing $y = \left(\frac{a}{a_0} \right)^{\frac{3}{2}}$ we obtain

$$a(t) = a_0 \left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}} \right)^{\frac{1}{3}} \left(\sinh \left(\frac{3}{2} H_0 \Omega_{v0}^{\frac{1}{2}} t \right) \right)^{\frac{2}{3}}. \tag{1.71}$$

Once again for small t we obtain the matter dominated regime $a(t) \propto t^{\frac{2}{3}}$ which then evolves into a vacuum dominated exponential expansion at late times, $a(t) \propto \exp(H_0 \Omega_{v0}^{\frac{1}{2}} t)$. It could well be this type of transition which has described our recent universe.

1.7.5 More general solutions – $K \neq 0$ case

It is possible to derive some more general solutions for non-flat spacetimes and this is set as an exercise in one of the problem sets . Of course observations of the large scale features of our Universe indicate that the curvature contribution to the energy budget is likely to be no more than around 1% today, but it does not rule out the possibility that we live in an open or closed universe. From a formal point of view, knowing this difference is vital, for example a commonly recited prediction of the Landscape of string theory is that the universe is likely to be open. We have from Eqn. (1.63) that

$$\frac{Kc^2}{a_0^2} = H_0^2\,(\Omega_0 - 1)$$

where $\Omega_0 = \Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0}$.

If we ignore radiation then it follows that the behaviour of $\Omega_{m0} + \Omega_{\Lambda 0}$ will determine the nature of the curvature. In particular in a plot of $\Omega_{m0} - \Omega_{\Lambda 0}$ the diagonal line $\Omega_{m0} + \Omega_{\Lambda 0} = 1$ is the crucial one, separating open and closed models. If $\Omega_{\Lambda 0} < 0$, the solution will always recollapse, whereas having $\Omega_{\Lambda 0} > 0$ does not guarantee expansion to infinity, especially if the matter density is high. Also, if $\Omega_{\Lambda 0}$ is large enough, then for closed models there was no big bang in the past, the universe must have emerged from a 'bounce' at some finite minimum radius. These features can be seen in Figure. (1.1).

1.8 Observational Parameters

The big bang does not fix many of the parameters we will encounter in these lectures, such as $H_0, \rho_{mat0}, \rho_{\rm rad0}\ldots$. We can determine a few of them from observations.

Expansion rate H_0 – Hubble's constant

$$H_0 = 100h\text{km s}^{-1}\text{ Mpc}^{-1}, \qquad\qquad h = 0.678 \pm 0.009.$$

The deceleration parameter : q_0

As the name suggests q_0 provides information about the acceleration of the universe. It is obtained from the scale factor by Taylor expanding it about $a(t_0)$:

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{a}(t_0) + \cdot$$

Dividing by $a(t_0)$ we write

$$\frac{a(t)}{a(t_0)} = 1 + (t - t_0)H_0 - \frac{q_0 H_0^2}{2}(t - t_0)^2 + \cdot$$

It follows by comparing the two series that q_0 is defined by

$$q_0 \equiv -\frac{\ddot{a}(t_0)}{a(t_0)H_0^2} = -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)}$$

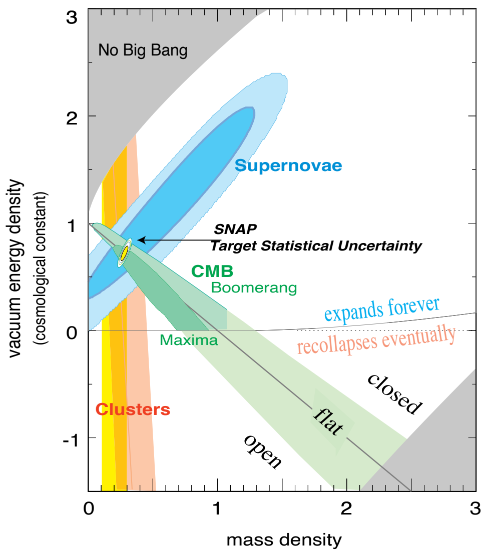

Figure 1.1: The Ω_{m0}-$\Omega_{\Lambda 0}$ confidence regions constrained from the observations of SN Ia, CMB and galaxy clustering. We also show the expected confidence region from a SNAP (Supernova project) satellite for a flat universe with $\Omega_{m0} = 0.28$.

The deceleration parameter is useful because we can measure it directly on large scales. The observations of Type Ia supernovae suggestions in fact that $q_0 \sim -0.6 < 0$ implying $\ddot{a} > 0$ i.e. that the universe is accelerating today.

1.9 Horizons and distances in cosmology

The cosmological horizon (particle horizon)

We need to think about measuring distance scales in cosmology, as its a vital skill we need to understand if we are to say anything about the make up of the universe. Lets start with a big question, how big is the observable universe, or how far has light travelled since the big bang? Recall the line element for the FRW universe given in Eqn. (1.12)

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

For a light ray travelling from $(r = 0, t = t_{\rm em})$ to $(r = r_0, t = t)$ it travels along a radial null direction hence $ds^2 = 0$ with $(d\phi = d\theta = 0)$.

A neat way of analysing problems associated with the propagation of light is to once again introduce the conformal-time η defined in terms of proper time t by

$$\eta \equiv \int \frac{cdt}{a(t)}.$$

Then using Eqns. (1.13)-(1.19) in Eqn. (1.12) and introducing the conformal time, then the metric takes the form

$$ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)) \tag{1.72}$$

where to remind you we have

$$\begin{aligned} S_K(\chi) &= \sinh\chi, \quad K = -1 \\ S_K(\chi) &= \chi, \quad\quad\ K = 0 \\ S_K(\chi) &= \sin\chi, \quad\ K = +1. \end{aligned}$$

Given that the radial trajectory has $d\phi = d\theta = 0$, we see from (1.72) that the function $\chi(\eta)$ along the light geodesic is completely determined by $ds^2 = 0$ or

$$d\eta^2 - d\chi^2 = 0.$$

The beautiful result that follows is that light geodesics satisfy

$$\chi(\eta) = \pm\eta + \text{const} \tag{1.73}$$

solutions that are straight lines at angles $\pm 45^\circ$ in the $\eta - \chi$ plane. In particular it is true of all geometries, whether it be $K = 0$ or $K = \pm 1$.

Now we can begin to talk about different **horizons**. Light in a universe of finite age can only have travelled a finite distance in that time, meaning that the volume of space within which we can have received signals is finite. The boundary of this volume is the **particle horizon**, and we would expect it to have a value of order 14 billion light years today, corresponding to the age of the universe. Given the solution Eqn (1.73), the maximum comoving distance light can propagate is

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{cdt}{a} \tag{1.74}$$

where η_i (or t_i) is the beginning of the universe. So, at time η, events at $\chi > \chi_p(\eta)$ are inaccessible to us located at $\chi = 0$. It is usually fine to choose $\eta_i = t_i = 0$, especially when there is an initial singularity (big bang), but in some cases of non-singular backgrounds that isn't possible and we take a non-zero value instead – a de Sitter universe is an example. To get the **physical size** of the particle horizon, mulitply χ_p by the scale factor today:

$$R(t) = a(t)\chi_p = a(t)\int_{t_i}^{t} \frac{cdt}{a(t)},$$

which is the **radius of the observable universe** at time t. This radius may be finite or infinite depending on how the scale factor evolves. We can easily see that for cosmological models which decelerate, $R(t)$ is always finite. To show this, consider $a(t) \propto t^\alpha$, $(0 < \alpha < 1)$. This gives

$$R(t) = \frac{ct}{1-\alpha}$$

which is finite at any given time t, but note that it grows linearly with t. For the Einstein-de Sitter Universe (K=0, matter dominated), we know $\alpha = \frac{2}{3}$, and in that case today we have

$$R(t_0) = 3ct_0.$$

The implication of this is that light from any galaxy that is now further away from us than $R(t_0)$ can not have reached us by today. The sphere of radius $R(t_0)$ centred on us is said to be our **cosmological horizon**, and is also known as the **particle horizon**. Note that $R(t_0) > ct_0$, the maximum distance light could travel in Minkowski space. How can that be? The reason is that the universe continues to expand as light makes its way across it. $R(t_0)$ is the distance as measured in the present universe. It was smaller earlier and easier to make progress across it.

Event horizon

This can be thought of as the complement of the particle horizon in that it encloses the set of points from which signals sent at a given moment in time t (η) will **never** be received by an observer in the future. In terms of the co-moving coordinates the points are at

$$\chi > \chi_{\rm ev}(t) = \eta_{\rm max} - \eta = \int_t^{t_{\rm max}} \frac{cdt'}{a(t')}$$

where $\eta_{\rm max}$ refers to the final moment of conformal time. The **physical size** of the event horizon at time t is

$$R_{\mathbf{ev}}(t) = ca(t) \int_t^{t_{\rm max}} \frac{dt'}{a(t')}.$$

Note if the universe expands forever then $t_{\rm max} \to \infty$. For the case of a $K = 0$ or $K = -1$ decelerating universe, then $\chi_{\rm ev}$ and $R_{\mathbf{ev}} \to \infty$. In that case there is no event horizon. However if the universe is accelerating, then we see that $R_{\mathbf{ev}}$ is finite even for $K = 0$ or -1, hence in that case there is an event horizon. To see this, consider the case of a flat de Sitter space universe given by $a(t) \propto e^{H_\Lambda t}$ where H_Λ is constant. Then we have

$$R_{\mathbf{ev}}(t) = ce^{H_\Lambda t} \int_t^{\infty} e^{-H_\Lambda t'} dt' = cH_\Lambda^{-1}$$

and is finite, having a size which is the curvature scale of the universe. It means that any event that occurs at a distance larger than cH_Λ^{-1} at a time t can **never** be seen by an observer. Because the space between the event and observer is expanding so rapidly, it can not influence her future. Note that for a closed ($K = 1$) universe which is decelerating, then the time available for future observations is finite because the universe will eventually re collapse. In that case there is both an event horizon and a particle horizon.

Luminosity distance

We now turn our attention to some of the most pressing aspects of observational cosmology, determining the distances to cosmological objects. It is through this that we can talk about determining the cosmological parameters, and yet it is not easy, we cant simply use a tape measure! The Lumnosity distance has a simple definition in Minkowski space (with no expansion to cause us any trouble). If we consider a source emitting light with absolute luminosity L_s, then the flux of light we receive $\mathcal{F}$ at a distance d is given by the inverse square law

$$\mathcal{F} = \frac{L_s}{4\pi d^2}$$

in other words the flux is the luminosity per unit area of the sphere of radius d. We dont know what the 'true' distance is in an expanding universe, so we use the Minkowski result and turn it into a definition of a new distance scale called the luminosity distance d_L:

$$d_L^2 \equiv \frac{L_s}{4\pi\mathcal{F}}. \tag{1.75}$$

Let us consider an object with absolute luminosity L_s located at a coordinate distance χ_s from an observer at $\chi = 0$. It proves convenient to adopt the metric given in Eqn. (1.72), namely

$$ds^2 = -c^2dt^2 + a^2(t)\left[d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.76}$$

Now the energy of light emitted from the object with time interval Δt_1 is denoted as ΔE_1, whereas the energy which reaches us on the sphere with radius χ_s is written as ΔE_0. We note that ΔE_1 and ΔE_0 are proportional to the frequencies of light at $\chi = \chi_s$ and $\chi = 0$, respectively, i.e., $\Delta E_1 \propto \nu_1$ and $\Delta E_0 \propto \nu_0$. The luminosities L_s and L_0 are given by

$$L_s = \frac{\Delta E_1}{\Delta t_1}, \quad L_0 = \frac{\Delta E_0}{\Delta t_0}. \tag{1.77}$$

The speed of light is given by $c = \nu_1\lambda_1 = \nu_0\lambda_0$, where λ_1 and λ_0 are the wavelengths at $\chi = \chi_s$ and $\chi = 0$. Then from $1+z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a}$ we find

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = \frac{\Delta t_0}{\Delta t_1} = \frac{\Delta E_1}{\Delta E_0} = 1+z\,, \tag{1.78}$$

where we have also used $\nu_0\Delta t_0 = \nu_1\Delta t_1$. Combining Eq. (1.77) with Eq. (1.78), we obtain

$$L_s = L_0(1+z)^2\,. \tag{1.79}$$

The two factors of (1+z) have arisen from the fact that each photon loses energy as it travels from the source to us, and that the number of photons arriving per second decreases over time as the universe expands. The light traveling along the χ direction satisfies the geodesic equation $ds^2 = -c^2dt^2 + a^2(t)d\chi^2 = 0$. We then obtain

$$\chi_s = \int_0^{\chi_s} d\chi = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \frac{c}{a_0}\int_0^z \frac{dz'}{H(z')}\,. \tag{1.80}$$

Note that we have used the relation $\dot{z} = -H(1+z)$ coming from the relation $1+z = \frac{a_0}{a}$. From the metric (1.76) we find that the area of the sphere at $t = t_0$ is given by $S = 4\pi(a_0S_K(\chi_s))^2$. Hence the observed energy flux is

$$\mathcal{F} = \frac{L_0}{4\pi(a_0S_K(\chi_s))^2}\,. \tag{1.81}$$

Substituting Eqs. (1.79) and (1.81) into Eq. (1.75), we obtain the luminosity distance in an expanding universe:

$$d_L = a_0S_K(\chi_s)(1+z)\,. \tag{1.82}$$

For the case of a flat FRW background with $S_K(\chi) = \chi$ we then find

$$d_L = a_0\chi_s(1+z).$$

A consequence of this relation is that distant objects appear to be further away than they really are, again because the redshift decreases their apparent luminosity L_0. We can compare the luminosity distance to the proper or physical distance which is defined at an instant of time. A radial ray of light travels a proper distance given by $ds = a(t)d\chi$. The physical distance to the source is therefore given by integrating this at a fixed time

$$d_{\rm phy} = a(t)\int_0^{\chi_s} d\chi = a(t_0)\chi_s \tag{1.83}$$

for today. Note that if $z \ll 1$ we have

$$d_L = a_0\chi_s = d_{\rm phy} \tag{1.84}$$

in all the curvature cases since from Eqns. (1.17)-(1.19) we have $S_K(\chi_s)) \sim \chi_s$ for $\chi_s \ll 1$, which means that objects are really as far away as they look.
Now, the lumnosity distance depends upon the cosmological model, hence we can use it to say which model best fits the data. In other words we can plot d_L v z for different cosmologies and compare it to the actual data points. This is what we turn our attention to now. Lets concentrate on the spatially flat case where

$$d_L = a_0\chi_s(1+z)\,.$$

Using Eqn. (1.80) we have

$$d_L = c(1+z)\int_0^z \frac{dz'}{H(z')}\,, \tag{1.85}$$

and the Hubble rate $H(z)$ can be expressed in terms of $d_L(z)$:

$$H(z) = \left\{\frac{d}{dz}\left(\frac{d_L(z)}{c(1+z)}\right)\right\}^{-1}\,. \tag{1.86}$$

If we measure the luminosity distance observationally, we can determine the expansion rate of the universe! On the other hand substituting for $H(z)$ from Eqn. (1.64) into Eqn. (1.85) we can predict the form of d_L for any given FRW cosmology.

In Fig. 1.2 we plot the luminosity distance (1.85) for a two component flat universe (non-relativistic fluid with $w_m = 0$ and cosmological constant with $w_v = -1$) satisfying $\Omega_{m0}+\Omega_{\Lambda 0} = 1$. Notice that $d_L \simeq z/H_0$ for small values of z. The luminosity distance becomes larger when the cosmological constant is present. We can prove that it should be like this by expanding out the integral. If we take $cH_0^{-1} = 3000h^{-1}$Mpc, then for this particular case we have

$$d_L = 3000h^{-1}\ \text{Mpc}\ (1+z)\int_0^z \frac{dz'}{[1-\Omega_{m0}+\Omega_{m0}(1+z')^3]^{\frac{1}{2}}} \tag{1.87}$$

Solving this numerically gives Figure 1 for different values of $\Omega_{\Lambda 0}$ (i.e. todays value). However we can obtain the $z \ll 1$ expansion. After a little bit of algebra we obtain

$$d_L \simeq 3000h^{-1}\ \text{Mpc}\ \left[z + z^2\left(1-\frac{3}{4}\Omega_{m0}\right) + O(z^3)\right],$$

confirming the linear expansion for small z and showing the rather weak dependence on the background cosmology.

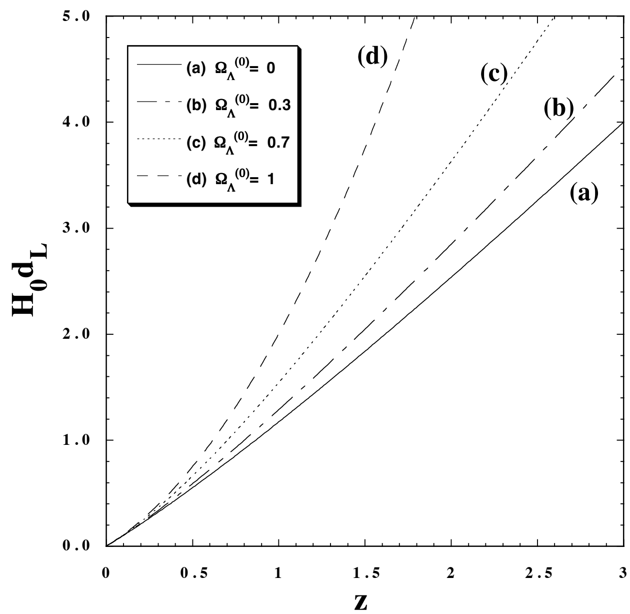

Figure 1.2: Luminosity distance d_L in the units of H_0^{-1} for a two component flat universe with a non-relativistic fluid ($w_m = 0$) and a cosmological constant ($w_v = -1$). We plot $H_0 d_L$ for various values of $\Omega_{\Lambda 0}$.

Constraints from Supernovae Type Ia – or how to win a Nobel prize in Physics

The direct evidence for the current acceleration of the universe is related to the Nobel prize winning observations of luminosity distances of high redshift supernovae. The apparent magnitude m of the source with an absolute magnitude M is related to the luminosity distance d_L via the relation

$$m - M = 5 \log_{10} \left(\frac{d_L}{\text{Mpc}} \right) + 25 \,. \tag{1.88}$$

This comes from taking the logarithm of Eq. (1.75) by noting that m and M are related to the logarithms of $\mathcal{F}$ and L_s, respectively. The numerical factors arise because of conventional definitions of m and M in astronomy. The Type Ia supernova (SN Ia) can be observed when white dwarf stars exceed the mass of the Chandrasekhar limit and explode. The belief is that SN Ia are formed in the same way irrespective of where they are in the universe, which means that they have a common absolute magnitude M independent of the redshift z. Thus they can be treated as an ideal standard candle. We can measure the apparent magnitude m and the redshift z observationally, which of course depends upon the objects we observe.

In order to get a feeling of the phenomenon let us consider two supernovae 1992P at low-redshift $z = 0.026$ with $m = 16.08$ and 1997ap at high-redshift redshift $z = 0.83$ with $m = 24.32$. As we have already mentioned, the luminosity distance is approximately given by $d_L(z) \simeq z/H_0$ for $z \ll 1$. Using 1992P, we find that the absolute magnitude is estimated by $M = -19.09$ from Eq. (1.88). Here we adopted the value $cH_0^{-1} = 2998h^{-1}\,\text{Mpc}$ with $h = 0.72$. Then the luminosity distance of 1997ap is obtained by substituting $m = 24.32$ and $M = -19.09$ for Eq. (1.88):

$$H_0 d_L \simeq 1.16\,, \quad \text{for} \quad z = 0.83\,. \tag{1.89}$$

From Eq. (1.85) the theoretical estimate for the luminosity distance in a two component flat universe is

$$H_0 d_L \simeq 0.95, \qquad \Omega_{m0} \simeq 1\,, \tag{1.90}$$

$$H_0 d_L \simeq 1.23, \qquad \Omega_{m0} \simeq 0.3,\ \Omega_{\Lambda 0} \simeq 0.7\,. \tag{1.91}$$

This estimation is clearly consistent with that required for a dark energy dominated universe as can be seen also in Fig. 1.2. Of course, from a statistical point of view, one can not strongly claim that that our universe is really accelerating by just picking up a single data set. Up to 1998 Perlmutter *et al.* [supernova cosmology project (SCP)] had discovered 42 SN Ia in the redshift range $z = 0.18$-0.83, whereas Riess *et al.* [high-z supernova team (HSST)] had found 14 SN Ia in the range $z = 0.16$-0.62 and 34 nearby SN Ia. Assuming a flat universe ($\Omega_{m0} + \Omega_{\Lambda 0} = 1$), Perlmutter *et al.* found $\Omega_{m0} = 0.28^{+0.09}_{-0.08}$ (1σ statistical) $^{+0.05}_{-0.04}$ (identified systematics), thus showing that about 70 % of the energy density of the present universe consists of dark energy. In 2004 Riess *et al.* reported the measurement of 16 high-redshift SN Ia with redshift $z > 1.25$ with the Hubble Space Telescope (HST). By including 170 previously known SN Ia data points, they showed that the universe exhibited a transition from deceleration to acceleration at > 99 % confidence level. A best-fit value of Ω_{m0} was found to be $\Omega_{m0} = 0.29^{+0.05}_{-0.03}$ (the error bar is 1σ). Figure 1.3 illustrates the observational values of the luminosity distance d_L versus redshift z together with the theoretical curves derived from Eq. (1.85). This shows that a matter dominated universe without a cosmological constant ($\Omega_0 = 1$) does not fit to the data. A best-fit value of Ω_{m0} obtained in a joint analysis is $\Omega_{m0} = 0.31^{+0.08}_{-0.08}$, which is consistent with the result by Riess *et al.*. In 2011, Saul Perlmutter, Brian Schmidt and Adam Riess deservedly shared the Nobel prize for these remarkable observations.

Angular diameter distance : d_{diam}

What follows is based on the book by Mukhanov, *Physical Foundations of Cosmology*, including the maths as well. In particular we will be deriving some of the key results in chapters 1 and 2 of the book.

Objects of a given physical size l are assumed to be perpendicular to our line of sight. If it subtends an angle $\Delta\theta$ (which is always small in astronomy because the distance scales are so large) then we define d_{diam} through

$$d_{\text{diam}} \equiv \frac{l}{\Delta\theta} \tag{1.92}$$

We begin by deriving a few useful results. Recall that in terms of conformal time η (defined through $cdt = a(\eta)d\eta$) we can write the metric as

$$ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)) \tag{1.93}$$

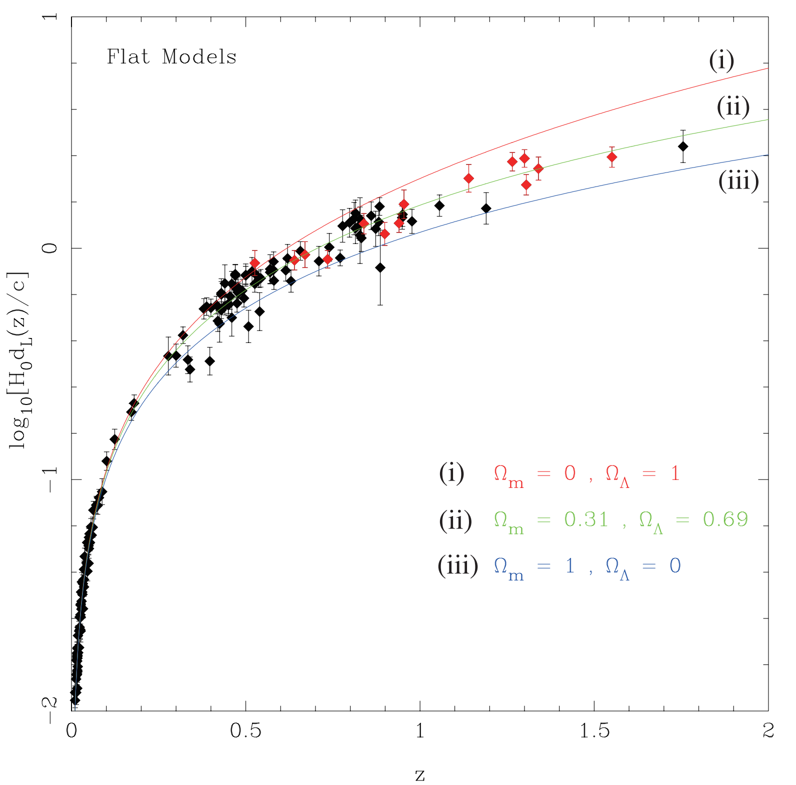

Figure 1.3: The luminosity distance $H_0 d_L$ (log plot) versus the redshift z for a flat cosmological model. The black points come from the "Gold" data sets by Riess *et al.*, whereas the red points show the recent data from HST. Three curves show the theoretical values of $H_0 d_L$ for (i) $\Omega_{m0} = 0$, $\Omega_{\Lambda 0} = 1$, (ii) $\Omega_{m0} = 0.31$, $\Omega_{\Lambda 0} = 0.69$ and (iii) $\Omega_{m0} = 1$, $\Omega_{\Lambda 0} = 0$.

where χ and $S_K(\chi)$ are defined in Eqns. (1.13)-(1.19). We are going to concentrate on the case of a dust dominated universe, but we could do any cosmology, the technique applies equally well. There is a neat result for the size of the particle horizon in a dust dominated universe. It turns out that for any value of Ω_0 the following result holds

$$S_K(\chi_p) = \frac{2}{a_0 H_0 \Omega_0}. \tag{1.94}$$

Lets prove it – it will bring together many of the results of the course to date.

Case 1: $K = -1$ – dust dominated universe

We quoted the result for the scale factor earlier in section 1.7.1, but lets derive it here. For dust domination we know $\rho_m \propto a^{-3}$, hence we define the constant

$$M = \frac{4\pi G}{3}\rho_m a^3. \tag{1.95}$$

The Friedmann and acceleration equations are:

$$\begin{aligned} a'^2 &= \frac{8\pi G}{3}\rho a^4 - K a^2 & (1.96)\\ a'' &= \frac{4\pi G}{3}(\rho - 3p)a^3 - Ka & (1.97) \end{aligned}$$

and with $K = -1$ the acceleration equation becomes

$$a'' = M + a.$$

The general solution is

$$a = A\sinh\eta + B\cosh\eta - M,$$

which simplifies with initial condition $a(\eta = 0) = 0$ to

$$a = A\sinh\eta + M(\cosh\eta - 1).$$

The integration constant A is determined by substitution into (1.96) yielding the final solution

$$a(\eta) = M(\cosh\eta - 1). \tag{1.98}$$

Now we want our results in terms of observables such as H_0 and Ω_0, not in terms of M. We can easily do this, by recalling that in a dust dominated universe (where we ignore other contributions such as radiation and a cosmological constant)

$$\Omega_0 \equiv \frac{\rho_0}{\rho_{\rm cr0}} = \frac{\rho_{m0}}{\rho_{\rm cr0}} = \frac{3M}{4\pi G a_0^3} \times \frac{8\pi G}{3H_0^2} = \frac{2M}{a_0^3 H_0^2},$$

hence

$$\frac{a_0}{M} = \frac{2}{\Omega_0 a_0^2 H_0^2}. \tag{1.99}$$

Making use of the Friedmann equation we have

$$H^2 = \frac{8\pi G}{3}\rho + \frac{1}{a^2}$$

Replacing ρ_m with (1.95) and (1.99), this simplifies to give

$$1 + \Omega_0 a_0^2 H_0^2 = H_0^2 a_0^2 \tag{1.100}$$

which will prove useful shortly. Now onto $S_K(\chi_p)$. From (1.17) and (1.74) with $\eta_i = 0$ because we are looking for the particle horizon, we have

$$S_K(\chi_p) = \sinh \chi_p = \sinh \eta \tag{1.101}$$

We can rewrite this in terms of the scale factor using (1.98) to give

$$S_K^2(\chi_p) = \left(\frac{a_0}{M}\right)\left(2 + \frac{a_0}{M}\right) \tag{1.102}$$

Finally using (1.99) and (1.100) we obtain the desired result (1.94):

$$S_K(\chi_p) = \frac{2}{a_0 H_0 \Omega_0}.$$

Case 2: $K = 0$ – dust dominated universe

We adopt the same approach as before, but hopefully it will be a bit quicker. The key equations are (1.18) and (1.74). The solution to the acceleration equation (1.97) for $K = 0$, which satisfies the initial condition $a(\eta = 0) = 0$ and the Friedmann equation (1.96) is

$$a = \frac{M}{2}\eta^2. \tag{1.103}$$

Hence

$$S_K^2(\chi_p) = \frac{2a_0}{M} = \frac{4}{\Omega_0 a_0^2 H_0^2}$$

using (1.99). Recall that $K = 0$ implies by definition that $\Omega_0 = 1$, hence we can include an extra factor of Ω_0 to give the desired result

$$S_K(\chi_p) = \frac{2}{a_0 H_0 \Omega_0}.$$

Case 3: $K = +1$ – dust dominated universe

I leave this one to you my friends. Try it !

We have just obtained the particle horizon for the case of dust dominated cosmologies in the three different curved scenarios. When it comes to measuring the angular diameter distance, we are generally not looking all the way back to the beginning of the universe, rather we are looking back to a redshift z where a galaxy is emitting light that we are detecting today. We need to evaluate $S_K(\chi_{em}(z))$ and so turn our attention to this. Of course, the limit $z \to \infty$ should recover our result $S_K(\chi_p)$. We will do it again for a matter dominated scenario, and consider the case of an open $K = -1$ universe. The same result actually applies to the closed universe, and you are encouraged to try and show it. The technique is the same.

We are wanting to evaluate (1.17) using (1.74) but with $\eta_i = \eta_{\rm em}$ corresponding to the finite conformal time when the light ray was emitted. Therefore we have

$$S_K(\chi_{\rm em}(z)) = S_K(\eta_0 - \eta_{\rm em}) = \sinh(\eta_0 - \eta_{\rm em}).$$

Expanding, we have

$$S_K(\chi_{\rm em}(z)) = \sinh\eta_0 \cosh\eta_{\rm em} - \cosh\eta_0 \sinh\eta_{\rm em}.$$

We can use the solution (1.98) to rewrite this as

$$S_K(\chi_{\rm em}(z)) = \left(\frac{a_{\rm em}}{M}+1\right)\sqrt{\left(\frac{a_0}{M}+1\right)^2-1} - \left(\frac{a_0}{M}+1\right)\sqrt{\left(\frac{a_{\rm em}}{M}+1\right)^2-1}.$$

Now recalling that $(1+z) = \frac{a_0}{a_{\rm em}}$ for the redshift of the emitting galaxy, and rewriting $\frac{a_{\rm em}}{M}$ as $\frac{a_{\rm em}}{a_0}\frac{a_0}{M}$, we can use (1.99) and (1.100) to obtain, after some manipulating:

$$S_K(\chi_{\rm em}(z)) = \frac{2}{\Omega_0^2 a_0 H_0(1+z)}\left(1+\Omega_0 z - \sqrt{1+\Omega_0 z} + \frac{1-\sqrt{1+\Omega_0 z}}{a_0^2 H_0^2}\right)$$

which using (1.100) to write $\frac{1}{a_0^2 H_0^2} = 1 - \Omega_0$ eventually leads to the desired result:

$$S_K(\chi_{\rm em}(z)) = \frac{2}{\Omega_0^2 a_0 H_0(1+z)}\left(\Omega_0 z + (\Omega_0 - 2)(\sqrt{1+\Omega_0 z} - 1)\right). \tag{1.104}$$

Although we have obtained it for the open universe, the result also holds true for the closed $K = 1$ universe, as well as a flat universe. Note, that as $\Omega_0 z \gg 1$ we recover the result for $S_K(\chi_p)$ of (1.94), corresponding to the case where we are going further back in time.

We now turn to the actual calculation of the Angular-diameter-redshift relation, as applied to general curved spacetime, and not just flat spatial sections. Usually we don't worry about an expanding universe, and think Euclidean with a static space in which case an object with a given fixed transverse size l on the sky subtends an angle which is inversely proportional to the distance to the object, i.e. $d = \frac{l}{\Delta\theta}$. In an expanding universe we have to be more careful. Start with an extended object of given transverse size l situated at a comoving distance $\chi_{\rm em}$ from us the observer. We are free to align the object as we want to, and so set $\phi = $ const. Photons emitted from the object at time $t_{\rm em}$ propagate along radial geodesics arriving today with an apparent angular separation $\Delta\theta$. The proper size of the object, l is given by the interval between the emission events at the endpoints. In this case $dt = d\chi = d\phi = 0$ and so in the full metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[d\chi^2 + S_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.105}$$

we have

$$l = \sqrt{|ds^2|} = a(t_{\rm em}) S_K(\chi_{\rm em}) \Delta\theta. \tag{1.106}$$

Comparing this with Eqn. (1.92) and Eqn. (1.82) we see that

$$d_{\rm diam} = a(t_{\rm em}) S_K(\chi_{\rm em}) = a(t_0)(1+z)^{-1} S_K(\chi_{\rm em}) = d_L (1+z)^{-2} \tag{1.107}$$

or

$$d_L = d_{\rm diam}(1+z)^2 \tag{1.108}$$

for all curved spaces.

Returning to $d_{\rm diam}$ we see that the angle subtended by the object is

$$\Delta\theta = \frac{l}{a(t_{\rm em}) S_K(\chi_{\rm em})} = \frac{l}{a(\eta_0 - \chi_{\rm em}) S_K(\chi_{\rm em})}, \tag{1.109}$$

the final term arising because the physical time $t_{\rm em}$ corresponds to the conformal $\eta_{\rm em} = \eta_0 - \chi_{\rm em}$. A **nearby object** satisfies $\chi_{\rm em} \ll \eta_0$, hence

$$a(\eta_0 - \chi_{\rm em}) \simeq a(\eta_0), \qquad S_K(\chi_{\rm em}) \simeq \chi_{\rm em},$$

where the final equality follows from (1.17)-(1.19) using the approximation $\chi_{\rm em} \ll 1$. It follows that

$$\Delta\theta \simeq \frac{l}{a(\eta_0)\chi_{\rm em}} = \frac{l}{d_{\rm phy}}, \tag{1.110}$$

where the physical distance is given by Eqn. (1.83), hence we see that for nearby objects, $\Delta\theta$ is inversely proportional to the physical distance as we would expect, which of course from Eqn. (1.84) is the same as the luminosity distance. What if the object is **far away**, say near the particle horizon? In that limit $\eta_0 - \chi_{\rm em} \ll \eta_0$, so

$$a(\eta_0 - \chi_{\rm em}) \ll a(\eta_0), \qquad S_K(\chi_{\rm em}) \to S_K(\chi_p) = \frac{2}{a_0 H_0 \Omega_0},$$

where $S_K(\chi_p)$ was derived in (1.94) and of importance here is the fact it is constant. The angular size of the object now becomes

$$\Delta\theta \propto \frac{l}{a(\eta_0 - \chi_{\rm em})}, \tag{1.111}$$

which increases with distance away from us. In fact as it approaches the horizon its image covers the whole sky ! So, the angular size of objects peak nearby as expected but also far away. Why then isn't the sky full of these images of distant objects? Its because the luminosity drops off rapidly with increasing distance, so the remote object do not outshine the nearby ones.

How can we imagine this behaviour? For an incomplete but useful analogy, lets go down a dimension and live at the north pole on the surface of a 2-sphere again. We are looking at the way the size of a given object (say a hug iceberg) varies as we change its distance form us. The object lies across lines of latitude, meaning that the light from it travels to us on lines of longitude (or meridians), because these are the geodesics on the earths surface. We find that if we are north of the equator, the angular size of the iceberg decrease as it goes further away towards the equator. However, once south of the equator, its angular size increases as we go further south, eventually covering the whole sky at the south pole. In actual fact, the angular size of a very remote object grows in a flat universe as well, because the scale factor is changing with time.

Angular diameter size is usually given as a function of redshift z. Using

$$1 + z = \frac{a_0}{a(t_{\rm em})}$$

equation (1.109) becomes

$$\Delta\theta = (1+z)\frac{l}{a_0 S_K(\chi_{\rm em}(z))}, \tag{1.112}$$

where $\chi_{\rm em}(z)$ is obtained from the usual definition in the following way: the comoving distance to an object that emitted a photon at $t_{\rm em}$ which arrives today is

$$\chi = \eta_0 - \eta_{\rm em} = \int_{t_{\rm em}}^{t_0} \frac{cdt}{a(t)}.$$

Changing variable to z using $(1+z) = \frac{a_0}{a(t)}$ we have

$$dz = -\frac{a_0}{a^2(t)}\dot{a}(t)dt = -(1+z)H(t)dt,$$

from which two neat things follow, the age of the universe at a redshift z is given by

$$t = \int_z^\infty \frac{dz}{(1+z)H(z)} \tag{1.113}$$

and we can write

$$\chi_{\rm em}(z) = \frac{1}{a_0}\int_0^z \frac{cdz}{H(z)}. \tag{1.114}$$

Ok, now we have to cut to the chase and start putting in some solutions.

Lets start with a flat universe filled with dust, $(K = 0, p = 0)$. From (1.18) we know that $S_K(\chi_{\rm em}) = \chi_{\rm em}$, hence we need $\chi_{\rm em}(z)$. For that we require $H(z)$ in (1.114). This is given by Eqn. (1.64) with $\Omega_{m0} = \Omega_0 = 1, \Omega_{\Lambda 0} = \Omega_{r0} = 0$.

We can easily solve the integrals in (1.113) and (1.114) to give

$$t(z) = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}} \tag{1.115}$$

$$\chi(z) = \frac{2c}{a_0 H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{1.116}$$

Substituting (1.116) into (1.112) we obtain

$$\Delta\theta = \frac{lH_0}{2c}\frac{(1+z)^{3/2}}{(1+z)^{1/2}-1}. \tag{1.117}$$

Notice the small and large z limits (recalling $H_0 = \frac{2}{3t}$):

$$\Delta\theta = \frac{l}{3ct_0}\frac{(1+\frac{3}{2}z+O(z^2))}{(1+\frac{1}{2}z-1+O(z^2))} \simeq \frac{2l}{3ct_0 z} \qquad z \ll 1,$$

$$\Delta\theta = \frac{z}{3ct_0} \qquad z \gg 1.$$

In both limits therefore the object appears large. In fact objects appear at their smallest when $\frac{d\Delta\theta}{dz} = 0$. It then follows by differentiating (1.117) that after a little bit of straightforward algebra, the corresponding redshift is given by

$$\frac{d\Delta\theta}{dz} = 0 \longrightarrow z = \frac{5}{4},$$

as can be seen in Figure. (1.4). The angular diameter distance can easily be obtained for more general cosmologies given the general result (1.112) and (1.104) for the case of dust dominated non-flat universe:

$$\Delta\theta = \frac{lH_0}{2}\frac{\Omega_0^2(1+z)^2}{\Omega_0 z + (\Omega_0 - 2)((1+\Omega_0 z)^{1/2}-1}. \tag{1.118}$$

We can look at the small and large z behaviour, which is as in the flat case. That means there is a minimum somewhere? Where is it as a function of Ω_0?

The use of angular diameter versus redshift to test cosmological models has met with limited success to date, mainly because of the lack of standard rulers. One exception though is the single standard ruler obtained from measurements of the CMB. It has been possible to measure temperatures in two random directions in the sky, and the temperature difference depends on the angular separation. Measuring the power spectrum associated with this temperature difference shows a series of peaks and troughs as the angular separation is varied from large to small scales. The ' first acoustic peak' is determined by the sound horizon at recombination, which corresponds to the maximum distance a sound wave in the baryon-radiation fluid can have propagated by recombination. This sound horizon acts as a standard ruler of length $l_s \sim H^{-1}(z_r)$. Recombination occurs at $z_r \simeq 1100$. Now since we are at such a large redshift, it implies $\Omega_0 z_r \gg 1$, so we can set $\chi_{\rm em}(z_r) = \chi_p$. This then means we can use $S_K(\chi_p)$ which we have evaluated for a dust dominated universe in (1.94). Substituting into (1.112) we obtain

$$\Delta\theta_r \simeq \frac{z_r H_0 \Omega_0}{2H(z_r)} \simeq \frac{1}{2} z_r^{-1/2}\Omega_0^{1/2} \simeq 0.87^\circ \Omega_0^{1/2}, \tag{1.119}$$

having used $H_0/H(z_r) \simeq (\Omega_0 z_r^3)^{-1/2}$ from (1.64). The beauty of this result is that it only depends on Ω_0, so the first doppler peak determines the spatial curvature of the universe ! The results to date suggest everything is consistent with a flat $\Omega_0 = 1$ universe.

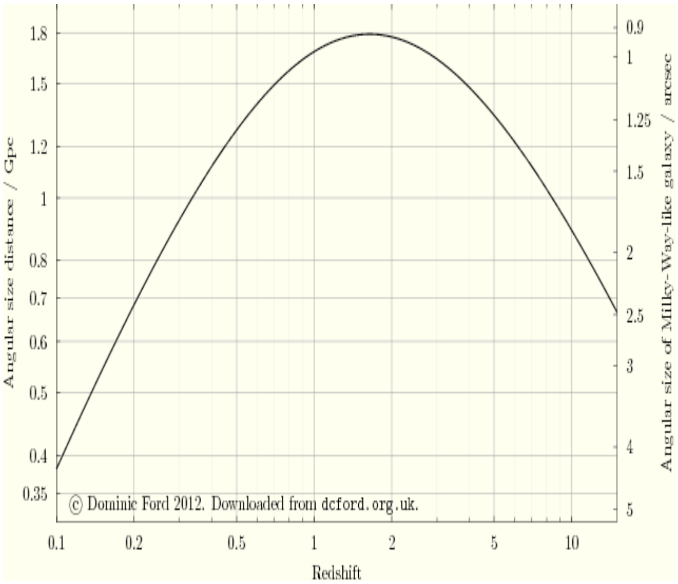

Figure 1.4: The angular distance versus redshift for a flat matter dominated universe – credit Dominic Ford, dcford.org.uk

1.10 Age of the Universe

Determining the age of the Universe is one of the big challenges in cosmology and vital if we are to understand its evolution. Eqn. (1.113) provides the age at a redshift z in any cosmology given by $H(z)$ in Eqn. (1.64). To get the present age we take $z \to 0$ in the integral. In general this can not be solved analytically but a few easy cases can be seen quickly.

At $10 < z < 1000$, where matter dominates, we have $H \simeq \frac{2}{3t}$ hence from Eqn. (1.64) this corresponds to

$$t(z) \simeq \frac{2}{3}H^{-1}(z) \simeq \frac{2}{3}H_0^{-1}\Omega_{m0}^{-\frac{1}{2}}(1+z)^{-\frac{3}{2}} \tag{1.120}$$

For a flat universe, the current age is $H_0 t_0 \simeq (2/3)\Omega_{m0}^{-\frac{1}{2}}$. One of the early pieces of evidence for the need of a cosmological constant type term was when independent tests indicated the product $H_0 t_0 \sim 1$. This required a very low Ω_{m0} to be consistent.

2 Thermal History of the Universe – the Hot Big Bang

As we have seen in section 1, given an equation of state $p = w\rho$, the background evolution of the Universe can be found by solving the Friedmann equation (and *either* the fluid *or* the acceleration equation). Given that the Universe is now expanding the equations tell us that the density ρ and temperature T were much higher in the past: the Early Universe was a hot plasma of interacting particles.

In fact the physics of the Early Universe is surprisingly simple:

- Its background evolution is described by the Friedmann equation.
- Particle abundances are given by Thermodynamics.
- The evolution of structures is well described by *linear* perturbation theory.

You have already covered the first item in other Cosmology courses, and we have briefly reviewed the basics in Section 1. In this section we will be concerned with the second item above: the study of thermodynamics in an expanding Universe, leading to a complete description of the Thermal History of our Universe. We will introduce the basic tools required for obtaining a quantitative understanding of particle abundances in the Early Universe, applicable from the present epoch to at least the TeV scale (less than a ps after the "Big Bang"), and, complemented with a theory of Grand Unification, possibly all the way down to[3] $E_{\rm GUT} \sim 10^{16}\, GeV$.

As we will see, the abundance of any given particle species depends on three things, namely whether it is:

- A boson or a fermion
- Relativistic or non-relativistic ($k_B T \lessgtr mc^2$)
- In thermal equilibrium or decoupled (interaction rate $\Gamma \lessgtr H$)

One of the more useful relations we can derive is that for temperature v time or energy v time. It allows us to quickly estimate when some of the major events occurred in the history of the Universe. If we want to include all forms of relativistic particles, and not just photons, we have to also consider in the sum for today the contribution of neutrinos. Neutrinos are fermions, obeying Fermi-Dirac statistics, and so they have a different number density than photons (there is a famous factor of 7/8). Moreover since they have decoupled from thermal equilibrium, they have a different temperature than the photons (another famous factor of $(\frac{4}{11})^{\frac{1}{3}}$). Let's spend a few minutes discussing the origin of these numbers and the fact that the number of light degrees of freedom is temperature dependent (i.e. there were more than just photons and neutrinos in the early universe). The main tools for this analysis are those of statistical mechanics, both in and out of equilibrium. The key objects we would like to determine are the number densities, energy densities and pressure of the various particle species.

Let's start by considering equilibrium distributions of particles.

2.1 Number densities, energy densities and pressures – relativistic and non-relativistic cases

For a particle species A (with mass m) in statistical equilibrium, the number density n, energy density ρ and pressure p are given as integrals over the distribution function $f_A(\mathbf{p}, t)$ where $\mathbf{p}$ is the 3-momentum of the particle. Different species of particles interact, exchanging energy and momentum. Now the rate of interaction is $\Gamma(t) = n\langle\sigma v\rangle$ where σ is the interaction cross section and v is the velocity of the particles.

A particle species, say A, remains in equilibrium with the plasma as long as its rate of interaction $\Gamma(t)$ is larger than the Hubble expansion rate:

$$\Gamma(t) > H(t) \tag{2.1}$$

[3] At $T > E_{\rm GUT}$ cosmic expansion is so fast that no massless gauge boson mediated interactions can maintain thermal equilibrium. This is the so-called thermal state problem.

These interactions are then effective in that they lead to and maintain thermodynamic equilibrium among the interacting particles with some temperature T. In general the interactions have a short range, and we may assume that the role of these interactions is just to provide a mechanism for thermalisation; they do not determine the form of the distribution function. Particles may thus be treated as an ideal (Bose or Fermi) gas, with an equilibrium distribution function:

$$f_A(\mathbf{p},t)d^3\mathbf{p} = \frac{g_A}{(2\pi)^3}\left(\exp[(E_A-\mu_A)/k_BT_A] \pm 1\right)^{-1} d^3\mathbf{p} \tag{2.2}$$

where k_B is the Boltzmann constant, g_A is the spin degeneracy factor (e.g. 2 for photons – two photon polarisations) , μ_A is the chemical potential, T_A is the temperature of this species and $E(\mathbf{p}) = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$. The "+" sign corresponds to fermions (*Fermi-Dirac* distribution) and the "-" sign to bosons (*Bose-Einstein* distribution). You can easily check that at low temperatures both distribution functions in (2.2) reduce to the familiar *Maxwell-Boltzmann* distribution $f(\mathbf{p}) = \exp[-(E_i-\mu_i)/k_BT]$. For a gas in thermal equilibrium the chemical potential is always zero. That is because there are no overall changes in the particle number and if you recall your first law of thermodynamics μ_A is associated with such a change through $dE = TdS - PdV + \mu_A dN_A$.
Given the distribution function, we can obtain the background number density, energy density and pressure of the particles n, ρ and p.

$$\begin{aligned}
n &= \frac{1}{\hbar^3}\int f(\mathbf{p})d^3\mathbf{p} = \frac{g}{2\pi^2\hbar^3}\int \left(\exp[E(p)/k_BT] \pm 1\right)^{-1} p^2 dp \\
&= \frac{g}{2\pi^2c^3\hbar^3}\int_{mc^2}^{\infty}\frac{(E^2-m^2c^4)^{1/2}}{(\exp[E/k_BT]\pm 1)}EdE && (2.3)\\
\rho c^2 &= \frac{1}{\hbar^3}\int E(p)f(\mathbf{p})d^3\mathbf{p} = \frac{g}{2\pi^2c^3\hbar^3}\int_{mc^2}^{\infty}\frac{(E^2-m^2c^4)^{1/2}}{(\exp[E/k_BT]\pm 1)}E^2dE && (2.4)\\
p &= \frac{1}{\hbar^3}\int \frac{|pc|^2}{3E(p)}f(\mathbf{p})d^3\mathbf{p} = \frac{g}{6\pi^2c^3\hbar^3}\int_{mc^2}^{\infty}\frac{(E^2-m^2c^4)^{3/2}}{(\exp[E/k_BT]\pm 1)}dE && (2.5)
\end{aligned}$$

The factors of $\hbar$ are present because we are dealing with identical particles and in quantising them for the energy levels we are going from a discrete to a continuous representation of the momentum ($\sum_{\mathbf{p}} \to \frac{V}{\hbar^3}\int d^3p$). For evaluating the above integrals it is useful to introduce a new dimensionless variable $x \equiv E/k_BT$. In general, one must evaluate the integrals numerically, but we have two analytic limits, which correspond physically to relativistic and non-relativistic species.

1. Relativistic species : $mc^2 \ll k_BT$

$$n = \left(\frac{k_BT}{c}\right)^3 \frac{g}{2\pi^2\hbar^3}\int_0^{\infty}\frac{x^2dx}{(\exp[x]\pm 1)} \propto T^3 \tag{2.6}$$

Now since the Riemann zeta function $\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n} = (1/\Gamma(n))\int_0^{\infty} u^{n-1}du/(\exp(u)-1)$, (note $\zeta(3) \simeq 1.202; \zeta(4) = \frac{\pi^2}{90} \simeq 1.0823$) we have for bosons:

$$n_B = \left(\frac{k_BT}{c\hbar}\right)^3 \frac{g\zeta(3)}{\pi^2} \tag{2.7}$$

whereas for fermions, by using the intriguing identity (which by the way implies that the distribution of fermions looks like a mixture of bosons at two different temperatures, one half the other)

$$\frac{1}{\exp(x)+1} = \frac{1}{\exp(x)-1} - \frac{2}{\exp(2x)-1} \tag{2.8}$$

we then obtain

$$n_F = \left(\frac{k_B T}{c\hbar}\right)^3 \frac{g\zeta(3)}{\pi^2}\left(1-\frac{1}{4}\right) = \frac{3}{4} n_B \propto T^3 \tag{2.9}$$

Putting in some numbers, the cosmic microwave background photons today have a temperature of $T = 2.725$ K, hence from Eqn. (2.7) (with $g = 2$) that gives a number density $n_\gamma = 4.1\times10^8$ m^{-3}.

For the energy density and pressure we obtain

$$\rho_B c^2 = (k_B T)^4 \frac{g}{2\pi^2 c^3\hbar^3}\int_0^\infty \frac{x^3 dx}{(\exp[x]-1)} = \frac{\pi^2}{30c^3\hbar^3} g (k_B T)^4 \propto T^4 \tag{2.10}$$

$$\rho_F c^2 = \left(1-\frac{1}{8}\right)\frac{\pi^2}{30c^3\hbar^3} g (k_B T)^4 = \frac{7}{8}\rho_B c^2 \tag{2.11}$$

with

$$p = \frac{\rho c^2}{3} \tag{2.12}$$

for both cases.

A few points worth mentioning: Note the factor of 7/8 which appears in the fermion energy density. It arises simply from the fact Fermions satisfy Fermi-Dirac statistics as opposed to the Bose-Einstein statistics satisfied by Bosons. Eqn. (2.12) allows us to obtain the equation of state for radiation but derived from statistical mechanics, and as expected we find that $w = p/(\rho c^2) = 1/3$. Eqn. (2.10) is the famous Stefan-Boltzmann law $\rho_\gamma = \sigma_{SB} T^4$. Since we know from earlier lectures that the energy density in radiation scales as $\rho_\gamma \propto a^{-4}$, then combing the two results leads to the result presented in class that the temperature of the radiation (and of any relativistic species) scales like

$$T_\gamma \propto \frac{1}{a} \tag{2.13}$$

This of course means the universe was much hotter when it was smaller. We are thinking of the temperature of the radiation as the 'temperature of the universe', because it is well defined as the radiation has a thermal spectrum, so a well defined temperature and because at early times the other particle species interact with the radiation and so share its temperature. In fact, equation (2.13) breaks down in the Early Universe as the number of relativistic species changes near "mass thresholds", when certain particle species become non-relativistic. It is, however, valid for all times later than about 1 sec after the Big Bang (when electrons and positron became non-relativistic). One of the main goals of this section is to explain in detail the physics of such changes in the number of relativistic degrees of freedom and to understand how this formula gets modified in the Early Universe.

2. Non-relativistic (massive) species : $mc^2 \gg k_B T$

In this limit we have:

$$\begin{aligned} n &= \frac{g(k_BT)^3}{2\pi^2 c^3 \hbar^3}\int_{mc^2/k_BT}^{\infty} \exp(-x)(x^2-(mc^2/k_BT)^2)^{1/2} x dx \\ &\simeq \frac{g}{\hbar^3}\left(\frac{mk_BT}{2\pi}\right)^{3/2} \exp(-mc^2/k_BT) \qquad (2.14) \\ \rho &= mn \qquad (2.15) \\ p &\simeq n(k_BT) \ll \rho \quad (p \simeq 0) \qquad (2.16) \end{aligned}$$

for both bosons and fermions.

For the maths geeks amongst us, we briefly discuss how we to get these results. Consider the number density Eqn. (2.14). Introduce $x = \mu y$ where $\mu = mc^2/k_BT$ we see that we have

$$n = \frac{g(k_BT)^3}{2\pi^2 c^3 \hbar^3}\left(\frac{mc^2}{k_BT}\right)^3 \int_1^{\infty} \exp(-\mu y)(y^2-1)^{1/2} y dy \qquad (2.17)$$

Now using

$$I_\nu(\mu) \equiv \int_1^{\infty} \exp(-\mu y)(y^2-1)^{(\nu-1)} y dy = \frac{2^{\nu-1/2}}{\sqrt{\pi}}\mu^{1/2-\nu}\Gamma(\nu)K_{(\nu+1/2)}(\mu) \qquad (2.18)$$

where $\Gamma(\mu)$ is a Gamma function and $K_\nu(\mu)$ is a Bessel function of the second kind, we see

$$n = \frac{g(mc)^3}{2\pi^2 \mu \hbar^3} K_2(\mu). \qquad (2.19)$$

We are in the regime $\mu \gg 1$, hence we look for asymptotic expansions. In this regime

$$\mathrm{Lim}_{\mu\to\infty}\ K_\nu(\mu) \simeq \sqrt{\frac{\pi}{2\mu}}\exp(-\mu) \qquad (2.20)$$

then we see

$$n \simeq \frac{g(mc)^3}{2\pi^2 \mu \hbar^3}\sqrt{\frac{\pi}{2\mu}}\exp(-\mu) = g\left(\frac{mk_BT}{2\pi}\right)^{3/2} \exp(-mc^2/k_BT) \qquad (2.21)$$

as in Eqn. (2.14). Lets look to derive Eqn. (2.15). Starting with Eqn. (2.4), under the substitution $x = E/k_BT$, it becomes

$$\rho c^2 = \frac{g}{2\pi^2 c^3 \hbar^3}(k_BT)^4 \int_{mc^2/k_BT}^{\infty} \exp(-x)(x^2-(mc^2/k_BT)^2)^{1/2} x^2 dx$$

which as before under $x = \mu y$ becomes

$$\rho c^2 = \frac{g}{2\pi^2 c^3 \hbar^3}(mc^2)^4 \int_1^{\infty} \exp(-\mu y)(y^2-1)^{1/2} y^2 dy \qquad (2.22)$$

Hence

$$\rho = \frac{gm}{2\pi^2\hbar^3}(mc)^3\int_1^\infty \exp(-\mu y)(y^2-1)^{1/2}y^2 dy = -\frac{gm}{2\pi^2}(mc)^3\frac{d}{d\mu}\left(\frac{K_2(\mu)}{\mu}\right) \tag{2.23}$$

from Eqn. (2.18). Now from the integral definition of the Bessel function

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt \tag{2.24}$$

it follows by direct differentiation that $\frac{dK_2(z)}{dz} = -\frac{1}{2}(K_3(z)+K_1(z))$. Therefore in Eqn. (2.23) we have

$$\rho = \frac{gm}{2\pi^2\hbar^3}(mc)^3\left(-\frac{1}{2\mu}(K_3(\mu)+K_1(\mu)) - \frac{1}{\mu^2}K_2(\mu)\right) \tag{2.25}$$

Once again taking the limit $\mu \gg 1$ we obtain

$$\rho \simeq -\frac{gm}{2\pi^2\hbar^3}(mc)^3\frac{1}{\mu}\sqrt{\frac{\pi}{2\mu}}\exp(-\mu) = mn \tag{2.26}$$

from Eqn. (2.21) as advertised in Eqn. (2.15). To determine the pressure in Eqn. (2.16) from Eqn. (2.5) we need the additional piece of information

$$\int_1^\infty \exp(-\mu y)(y^2-1)^{(\nu-1)}dy = \frac{2^{\nu-1/2}}{\sqrt{\pi}}\mu^{1/2-\nu}\Gamma(\nu)K_{(\nu-1/2)}(\mu). \tag{2.27}$$

It follows from Eqn. (2.5) that $\nu = 5/2$ hence

$$p = \frac{g}{6\pi^2c^3\hbar^3}(mc^2)^4\frac{K_2(\mu)}{\mu^2}\frac{4}{\sqrt{\pi}}\Gamma(5/2) \tag{2.28}$$

This simplifies recalling Eqn. (2.19) to give

$$p = n(k_BT) \tag{2.29}$$

as given in Eqn. (2.16).

A few comments are in order:

- Equation (2.16) is in fact the ideal gas law $PV = Nk_BT$, with $N = nV$ the total number of particles in volume V. Note that in the non-relativistic limit we have $P \ll \rho c^2$ so the fluid is practically pressureless $P \simeq 0$ (cf. the 'dust' equation of state $p = w\rho$ with $w = 0$ in Section 1). To get an idea of how good this approximation is consider for example how small is the pressure compared to the energy density in the solar system.
- The computation of (2.15) in the non-relativistic limit gives $\rho = mn + (3/2)nk_BT$, but since $mc^2 \gg k_BT$ the kinetic energy $(3/2)nk_BT$ in negligible and to a good approximation we have equation (2.15).

We can summarise. In general for relativistic species the number densities go as T^3 and the energy density behaves as T^4, with relative factors of (3/4) and (7/8) respectively for fermions. For massive species (both bosonic and fermionic) the number and energy densities are exponentially suppressed by the Boltzmann factor $\exp(-mc^2/k_BT)$. In fact it is Eqn. (2.14) that plays an important role in **nucleosynthesis**. The exponential suppression of the number density means that non-relativistic particles soon drop below the limit where they interact sufficiently often to stay in equilibrium.

The main results of the above discussion are shown in Table 1. Note that, due to the Boltzmann suppression in the energy densities of non-relativistic particles, the total energy density ρ in the Early Universe is dominated by relativistic species.

Physical Quantity	Relativistic Bosons	Relativistic Fermions	Non-relativistic (either)
n_i	$\frac{\zeta(3)}{\pi^2} g_i T^3$	$\left(\frac{3}{4}\right) \frac{\zeta(3)}{\pi^2} g_i T^3$	$g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$
ρ_i	$\frac{\pi^2}{30} g_i T^4$	$\left(\frac{7}{8}\right) \frac{\pi^2}{30} g_i T^4$	$m_i n_i$
p_i	$\frac{1}{3}\rho_i$	$\frac{1}{3}\rho_i$	$n_i T$

Table 1: Number density, energy density and pressure of different particle species (labeled by i) in thermal equilibrium. We use natural units $\hbar = c = k_B = 1$.

2.2 Dealing with several relativistic species : number of degrees of freedom

We just saw that the Early Universe can essentially be described as a mixture of relativistic species. Summing up the contributions of all relativistic particles gives the *total energy density*:

$$\rho c^2 = \frac{k_B^4 \pi^2}{30 c^3 \hbar^3} \left(\sum_{\text{bosons}} g_i T_i^4 + \frac{7}{8} \sum_{\text{fermions}} g_i T_i^4 \right) \tag{2.30}$$

where we have allowed different species to have equilibrium distributions with different temperatures. This is to account for decoupled relativistic species and it will soon become clear how it works in detail. For now let's just note that we can express the total density (2.30) in terms of the photon temperature $T_\gamma \equiv T$ as:

$$\rho c^2 = \frac{\pi^2}{30 c^3 \hbar^3} g_\star (k_B T)^4 \tag{2.31}$$

with

$$g_\star(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \tag{2.32}$$

the *effective number of relativistic degrees of freedom* at temperature T. As the (photon) temperature decreases, the effective number of degrees of freedom in radiation will decrease, as massive particles become non-relativistic when their mass becomes larger than T_i.

Note that (since the energy density of non-relativistic species is negligible) this is the total energy density that enters the Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3} \rho(t) = \frac{8\pi G}{3} \frac{\pi^2}{30 c^3 \hbar^3} g_\star (k_B T)^4 \tag{2.33}$$

It is conventional to work in natural units, where $\hbar = c = k_B = 1$ and $8\pi G = M_{\rm Pl}^{-2}$, and the Friedmann equation is just

$$H^2 = \frac{\pi^2}{90} g_\star \frac{T^4}{M_{\rm Pl}^2}\,. \tag{2.34}$$

From the Friedmann equation it becomes clear how the effective number of relativistic degrees of freedom affects the expansion history. (This is why Observational Cosmology can constrain the number of neutrino families.)

To solve equation (2.33/2.34) we need one more equation to relate $a(t)$ to T. This will come from entropy conservation.

2.3 Entropy in an Expanding Universe

Recall the 1st Law of Thermodynamics:

$$dE = T dS - p\, dV \tag{2.35}$$

where we have neglected the chemical potential. In thermodynamics we think of entropy and energy as extensive quantities (i.e. they are additive for subsystems, being proportional to the amount of material in the system). This means $\partial S/\partial V = S/V$ and $\partial E/\partial V = E/V$ where we have $E(T, V)$ and $S(T, V)$. For an expanding universe let us write the energy E and entropy S in terms of densities $\rho(T)$ and $s(T)$, which are both functions of temperature:

$$E = c^2 \rho(T)\, V \qquad (\rho \text{ is the energy density}) \tag{2.36}$$

$$S = s(T)\, V \qquad (s \text{ is the entropy density}) \tag{2.37}$$

Then:

$$dE = (V d\rho + \rho\, dV) c^2 \tag{2.38}$$

$$dS = V ds + s\, dV \tag{2.39}$$

and the 1st Law becomes:

$$\underbrace{c^2 d\rho - T ds}_{\propto dT} = (Ts - c^2\rho - p)\frac{dV}{V}\,. \tag{2.40}$$

First, observe that, since ρ and s are functions of temperature, the left hand side is proportional to dT. Then, note that the coefficients of dT (LHS) and of dV (RHS) must vanish separately, as this equation is valid for arbitrary variations of T and V. (Indeed, one quantity is intensive and the other is extensive. Consider for example a volume change at $T =$ const. Then $dV \neq 0$ with $dT = 0$.) In particular, the RHS gives for the *entropy density* s:

$$s = \frac{c^2\rho + p}{T}\,. \tag{2.41}$$

Alternatively (to make a closer connection to standard thermodynamics) we can start from

$$dE = TdS - PdV \tag{2.42}$$

and then substitute for dE and dT giving

$$\frac{\partial E}{\partial T}dT + \frac{\partial E}{\partial V}dV = T\frac{\partial S}{\partial T}dT + T\frac{\partial S}{\partial V}dV - PdV \tag{2.43}$$

This is true for arbitrary changes dT and dV so collecting terms we end up with

$$\frac{\partial E}{\partial T} = T\frac{\partial S}{\partial T} \qquad \text{from dT term} \tag{2.44}$$

$$S = \frac{E+PV}{T} \qquad \text{from dV term} \tag{2.45}$$

In the ultrarelativistic limit we have been considering ($k_BT \gg mc^2$), we have, using Eqn. (2.10) and (2.12) in Eqn. (2.41), that the *entropy density of a bosonic species* is

$$s = \frac{4}{3}\frac{\rho_B c^2}{T} = \frac{2\pi^2 k_B^4}{45c^3\hbar^3} gT^3 \tag{2.46}$$

with a *factor of* (7/8) *of this for fermionic species.* Recalling the number density of relativistic particles as given in Eqn. (2.6) also scales as T^3 then we see that the entropy density also counts the number of particles. This is why we can say that the ratio of the number density of photons in the universe to the number density of baryons is called the entropy per baryon. It has a value of order 10^9 today.

As we have seen, in thermal equilibrium massive (i.e. non-relativistic) species are exponentially suppressed ($\rho \propto \exp(-m/T)$ in natural units) and have negligible pressure $p \ll \rho c^2$. Thus, *the entropy density s is dominated by relativistic species.*

Degrees of Freedom in Entropy

For a mix of relativistic species it is convenient to express s in terms of the photon temperature $T = T_\gamma$, as we did above with energy density (2.30):

$$s = \sum_i \frac{c^2\rho_i + p_i}{T_i} = \frac{2\pi^2 k_B^4}{45c^3\hbar^3} g_{\star S} T^3 \tag{2.47}$$

where

$$g_{\star S}(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 \tag{2.48}$$

is the *effective number of relativistic degrees of freedom in entropy.* Note the different scaling with temperature (3rd power compared to 4th power for $g_\star$ in equation (2.32)). We have $g_{\star S} = g_\star$ only if all species have the same temperature. As we will see, this is only true until about $t \lesssim 1\,s$ in our Universe.

Entropy Conservation

The magic of the quantity s in (2.47) is that it scales simply as $a(t)^{-3}$ because entropy S is conserved[4]: $S = \text{const} \Rightarrow sa^3 = \text{const}$. Thus from (2.47):

$$\frac{2\pi^2 k_B^4}{45 c^3 \hbar^3} g_{\star S} T^3 a^3 = \text{const} \quad \Rightarrow$$
$$T \propto (g_{\star S})^{-1/3} a^{-1} \tag{2.49}$$

This relates temperature to the scalefactor and is just the what we needed to close the system with the Friedmann equation. It is actually the generalisation of the heuristic formula (2.13). It now becomes clear why that heuristic formula can fail, as it does not take into account changes in the effective number of relativistic degrees of freedom in the early universe.

Plugging (2.49) into the Friedmann equation yields:

$$a(t) \propto (g_{\star S})^{-1/12} \left(\frac{g_\star}{g_{\star S}}\right)^{1/4} t^{1/2} \tag{2.50}$$

during radiation domination in the early universe. Thus, we see that the standard relation $a(t) \propto t^{1/2}$ during radiation domination receives *corrections as jumps every time the effective number of degrees of freedom changes.*

Using equation (2.49) again we find

$$T \propto g_\star^{-1/4} t^{-1/2} \tag{2.51}$$

In fact, tracking all the numerical factors yields

$$\frac{T}{1MeV} \simeq 1.5 g_\star^{-1/4} \left(\frac{1s}{t}\right)^{1/2} . \tag{2.52}$$

This is a good rule of thumb to get a sense of how temperatures/energy scales are related to times in the Early Universe: around $1s$ after the Big Bang the Universe had a temperature of roughly $1MeV$ or $10^{10}K$.

Temperature v Redshift

The temperature was hotter in the past, a result that follows form the evidence that the universe has been expanding adiabatically. Given that the temperature of the present day cosmic microwave background (CMB) has been measured to be

$$T_0 = 2.725 \pm .001\text{K} \tag{2.53}$$

then we usually assume the temperature evolves as

$$T(z) = 2.725(1+z). \tag{2.54}$$

[4]This is true to a good approximation in the Early Universe because particle interactions are so effective that (most) particles are in thermal equilibrium, maximising entropy. In this sense the expansion is slow, allowing thermal equilibrium to be approached.

We have argued for this based on the fact the energy density of photons scales as $\rho_{\rm rad} \propto (1+z)^4$ and also from the statistical discussion earlier where we saw from Eqn. (2.10) that $\rho_{\rm rad} \propto T^4$. However, this is not a safe way to argue for this relationship, as the same reasoning in terms of the scalefactor gave us earlier the relationship $T \propto a^{-1}$, which as we just discussed gets corrected by changes in $g_{\star S}$, see equation (2.49). Actually, a more accurate argument for the temperature-redshift relationship (2.54) is based on the adiabatic expansion assumption, in other words that entropy is conserved. We have seen that the entropy density scales as $s \propto a^{-3}$ and $s \propto T^3$, hence this requires

$$T \propto (1+z)\,. \tag{2.55}$$

As above, one must generally be careful with the range of applicability of such formulae, due to changes in the effective number of degrees of freedom $g_{\star S}$, so if we were referring to the total entropy of relativistic species we should really use $s \propto g_{\star S}T^3$ instead of just $s \propto T^3$. However, when we talk about the CMB temperature, we mean the temperature of CMB photons alone, which were released at a redshift of $z \sim 1000$ (surface of last scattering) and have free-streamed to us at $z = 0$ today (see later detailed discussion on CMB). Thus, $s \propto T^3$ is perfectly valid for CMB photons in this redshift range and equation (2.55) does not really involve $g_{\star S}$. This simple relationship between the CMB temperature and redshift is a robust prediction of the Standard Cosmological Model.

2.4 Changes in relativistic degrees of freedom

We just saw that changes in the relativistic degrees of freedom have a significant effect on the expansion history. But when does $g_\star$ (and $g_{\star S}$) change? As we have already hinted, the answer is near *mass thresholds*, i.e. when $k_BT = m_ic^2$ for some species i. We already have all the information we need to understand what happens then: that species becomes non-relativistic, so its energy density gets Boltzmann suppressed $\rho_i \propto \exp - m_i/T$ and does not contribute significantly to the total energy (and entropy) density anymore $\longrightarrow g_\star$ decreases (and so does $g_{\star S}$ too).

However, we know that entropy is conserved and therefore that species must somehow transfer its entropy to the thermal bath. So what happens to this particle species? The answer is that the particle annihilates (with its antiparticle) producing radiation. Consider for example electron-positron annihilation:

$$e^- + e^+ \longleftrightarrow \gamma + \gamma \tag{2.56}$$

At $T > m_e \simeq 0.5\,MeV$ photons have enough energy to produce e^-e^+ pairs so the reaction (2.56) goes both ways. However, for $T < m_e$ there isn't enough energy in the photon bath to produce e^-e^+ pairs so the reaction only goes one way ($\longrightarrow$) and the electrons and positrons annihilate producing photons. We say that 'they heat the photon bath', i.e. the photon temperature decreases more slowly during this process. (Remember the entropy $a^3s = g_{\star S}T^3a^3$ is constant, so if $g_{\star S}$ decreases then Ta must increase, which just means that T falls more slowly than a^{-1}.)

Electron-positron annihilation is just one example that happens at $T \simeq 0.5 MeV \simeq$ few seconds. As the Universe cools down more species become non-relativistic, and to find $g_\star$ at any given time we must account for all particles with $m < T$. For $T > 1MeV$ ($t \lesssim 1sec$) this is easy because all species are in equilibrium. For example:

- **Between 1 MeV $\lesssim$ T $\lesssim$ 100 MeV**, corresponding to times $1s \gtrsim t \gtrsim 50\mu s$ or redshift $6\times10^9 \lesssim z \lesssim 10^{12}$, we have

$$g_\star = 2 + \frac{7}{8}(3\times2 + 2\times2) = 10.75 \tag{2.57}$$

The first term counts the photon degrees of freedom (2 photon polarisations). The factor 7/8 is just the fermion factor of equation (2.11). The next term is for neutrinos: 3 neutrino families times two degrees of freedom per family (neutrino and antineutrino, each having only one spin degree of freedom – left handed helicity). The last term is for electrons: 2 spins times 2 electron degrees of freedom (electron and positron).

We see that photons, neutrinos and electrons/positrons are in equilibrium at the same temperature so $T_i = T$ for all relativistic species in equations (2.32) and (2.48) and we have $g_\star = g_{\star S}$. Note that at these energies all other species are non-relativistic, e.g. for muons we have $m_\mu \simeq 105MeV$.

- **Above Electroweak Unification T $\gtrsim$ 200 GeV** (corresponding to $t \lesssim 20ps$ or in terms of redshift $z > 10^{15}$) all particles of the Standard Model are massless and we have

$$g_\star = 106.75 \tag{2.58}$$

At much higher energies we could in principle have more degrees of freedom, i.e. in addition to those of the Standard Model. For example, in Grand Unified Theories $g_\star \approx 10^3$ for $T \gtrsim 10^{16}GeV$.

But what about **today**? At the present epoch ($T \simeq 2.7K \simeq 0.24\,meV$) the massless particles are photons and neutrinos. However, neutrinos have long ago decoupled (at around $T \simeq 1MeV$) and they now have a different temperature than photons. Thus, for $T < 1MeV$ ($t > 1s$) we have to take into account that neutrinos have decoupled from equilibrium. As we will see (in fact derive) below, they are slightly colder than the photons, with a temperature $T_\nu = (4/11)^{1/3}T_\gamma$. We then have

$$g_\star = 2 + \frac{7}{8} \times 3 \times 2 \times \left(\frac{4}{11}\right)^{4/3} \simeq 3.36$$

We have thus reached the third important ingredient for understanding the abundance of a given species in the early Universe: the physics of decoupling. (Remember we said earlier that the abundance of any given particle species depends on three things: whether it is a boson or a fermion, relativistic or non-relativistic, in thermal equilibrium or decoupled.)

2.5 Decoupling

If all species had remained in thermal equilibrium there would be no matter today. (Recall equations (2.14)-(2.15), non-relativistic species are exponentially suppressed.) Fortunately, nature has been more creative. As the Universe expands (scalefactor $a(t)$ increases), the temperature $T \propto (g_{\star S})^{-1/3}a(t)^{-1}$ drops. The expansion rate $H(t)$ then falls as $H(t) \propto T^2$. (This follows directly from the Friedmann equation $H^2 = (8\pi G/3)\rho$ with $\rho \propto T^4$.)

The key point is that particle interaction rates Γ fall at a different power of T than the expansion. For example, the weak interaction $\Gamma_w \sim G_F^2 T^5$, where G_F is Fermi's constant, falls much faster. Eventually, $\Gamma(t) < H(t)$ so the particle stops interacting with the plasma (decouples) and evolves independently. As a more concrete example, consider the interaction $e^- + e^+ \longrightarrow 2\gamma$. It freezes out soon after electrons become non-relativistic, because $\Gamma \simeq \langle\sigma v\rangle n$ decreases dramatically as the number density n gets exponentially suppressed. We will see this in more detail later (when we will study the Boltzmann equation) but it is actually intuitively clear: if the species stops interacting the number of particles is conserved and so the number density must fall like a^{-3} with cosmic expansion.

For a decoupled species i (either relativistic or non-relativistic) the number of particles is conserved and so

$$n_i \propto a^{-3} \tag{2.59}$$

Note, therefore, that the number density behaves in the same way for both relativistic and non-relativistic species. This is not the case for the energy densities. For *non-relativistic species* we have the following:

A species that decouples while non-relativistic has

$$\rho_i = m_i n_i \propto a^{-3} \tag{2.60}$$

Note that decoupling of non-relativistic species kills the exponential suppression factor of the equilibrium distribution so the decoupled massive species can later dominate the energy density. (Recall that the radiation energy density falls faster, as a^{-4}.)

On the other hand, for *relativistic species* we have:

A species that decouples (at temperature T_D, time t_D) while relativistic maintains an equilibrium distribution with

$$T_i = T_D \frac{a(t_D)}{a(t)} \tag{2.61}$$

and so has energy density

$$\rho_i \propto T_i^4 \propto a^{-4} \tag{2.62}$$

So after decoupling, a relativistic species maintains an equilibrium distribution but at a temperature that can in general be different to the photon temperature. This is why we included species with different temperatures than T_γ in our definition of ρ and s in equations (2.30) and (2.47), leading in general to different $g_\star$ and $g_{\star S}$ (see equations (2.32) and (2.48)) at a given (photon) temperature.

The reason that the temperature of decoupled species can be different is clear with the physics we have already seen. After a relativistic species decouples, it stops interacting with the hot plasma, and so, when another species becomes non-relativistic and heats up the plasma, the temperature of the decoupled species is unaffected. Thus, *a decoupled relativistic species can have a colder temperature because it does not heat up when massive species transfer their entropy to the hot plasma in equilibrium.*

We just saw how the energy density of decoupled species behaves in both the relativistic and non-relativistic cases. What about the entropy of decoupled species? Well, we can immediately see the answer since the entropy density s is related to the energy density ρ and pressure p from equation (2.41). At decoupling:

Non-relativistic species have already transferred their entropy to the heat bath so they decouple having negligible entropy.

while

Relativistic species take their entropy share with them.

In other words, relativistic species that have decoupled do not interact with the plasma anymore so *their entropy is independently conserved.* They still contribute their share to the total entropy density of the Universe, but, since they have decoupled, their temperature evolves independently according to equation (2.61) and is in general different than the temperature of the heat bath (which repeatedly 'heats up' near mass thresholds as we have seen).

2.6 Neutrino Temperature

We now examine neutrino decoupling (and the effects of electron-positron annihilation that followed soon after it) as an important example of the general story described above. In doing so, we will explain the origin of the mysterious factor of $(11/4)^{1/3}$ associated with the temperature of neutrinos today.

At around $T \simeq 10^{12} K \simeq \mathcal{O}(100)$ MeV, the energy density of the universe is almost all in relativistic particles $e^{\pm}$, ν, $\bar{\nu}$ and photons. They are in equilibrium with the same temperature, hence the effective number of degrees of freedom is $g_\star = 10.75$ as we have just seen. The corresponding rate of expansion in this radiation dominated regime is

$$H^2(T) = \frac{8\pi G}{3}\rho_R \tag{2.63}$$

where ρ_R is given by Eqn. (2.31).

Neutrinos feel only the weak interaction are kept in equilibrium via weak interaction proceses like $\nu\bar{\nu} \leftrightarrow e^+e^-$, with a cross section given by

$$\sigma_F \simeq G_F^2 E^2 \simeq G_F^2 T^2 \tag{2.64}$$

where $G_F = 1.1664 \times 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant. The interaction rate per (massless) neutrino is:

$$\Gamma_F = n\langle\sigma_F v\rangle \simeq 1.3\, G_F^2 T^5. \tag{2.65}$$

The factor of T^5 comes from the number density (T^3) and the cross-section (T^2).

Since they only feel the weak interaction they decouple when the weak interaction rate becomes equal to the expansion rate

$$\Gamma_F = H\,. \tag{2.66}$$

From the expressions for Γ_F and $H(T)$ we obtain (after substituting in the correct numbers)

$$\frac{\Gamma_F}{H(T)} \simeq \frac{1.2\, T^3 G_F^2}{\sqrt{8\pi G}} \sim \left(\frac{T}{1\ \text{MeV}}\right)^3 . \tag{2.67}$$

Therefore neutrinos decouple from the rest of the matter when $T_D \simeq 1$ MeV. They are, of course, relativistic at decoupling. (Their mass, even though not precisely known, is constrained by data to be over 1000 times smaller than T_D.) Now the key point is that, soon after neutrinos decouple, electrons become non-relativistic (at $T \simeq m_e \simeq 0.5\, MeV$) and the photons get 'heated' by electron-positron annihilation. Therefore, *neutrinos today have a different (lower) temperature T_ν than photons $T_\gamma \equiv T$.*

Let's compute this temperature. We use entropy conservation and the fact that the effective number of relativistic degrees of freedom $g_\star$ changes at $T \simeq 0.5\, MeV$ when electrons (and positrons) become non-relativistic. In fact, since the entropy of the neutrinos is separately conserved (they decouple relativistic) it is convenient to only consider the effective number of relativistic degrees of freedom *in equilibrium* $g_{\star\text{equil}}$. After neutrino decoupling and while electrons are relativistic (i.e. before photons get heated by e^+e^- annihilation) we have:

$$g_{\star\text{eq}}(T_D > T > m_e) = 2 + \frac{7}{8} \times 2 \times 2 = \frac{11}{2} \equiv g_{\star\text{before}}\,, \tag{2.68}$$

where the first term is for photons and the second term for electrons (and positrons). After electrons become non-relativistic and the photons are heated we have:

$$g_{\star\text{equil}}(T < m_e) = 2 \equiv g_{\star\text{after}}\,. \tag{2.69}$$

Now, since the entropy of neutrinos S_ν and that of the species in equilibrium S_{equil} are separately conserved, we have:

$$S_{\text{equil}} \propto g_{\star\text{equil}}(aT)^3 = \text{const} \tag{2.70}$$

and so:

$$\begin{aligned} g_{\star\text{after}}(aT_\gamma)^3_{\text{after}} &= g_{\star\text{before}}(aT_\gamma)^3_{\text{before}} \\ \Longrightarrow \frac{(aT_\gamma)^3_{\text{after}}}{(aT_\gamma)^3_{\text{before}}} &= \frac{g_{\star\text{before}}}{g_{\star\text{after}}} = \frac{11}{4} \end{aligned} \tag{2.71}$$

Finally, we use the fact that before e^+e^- annihilation photons and neutrinos had the same temperature:

$$(aT_\gamma)_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} (aT_\gamma)_{\text{before}} = \left(\frac{11}{4}\right)^{1/3} (aT_\nu)_{\text{before}} = \left(\frac{11}{4}\right)^{1/3} (aT_\nu)_{\text{after}} \tag{2.72}$$

Therefore for all $T < 0.5\, MeV$ (times later than about $6\, sec$ after the Big Bang) we have:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{2.73}$$

Thus we have found that the neutrino temperature is scaled with resect to the CMB temperature by the famous factor $(4/11)^{1/3}$. Today $T_\gamma \simeq 2.725K$ and so $T_\nu = 1.945K$.

Note that in the above calculation we have only considered the entropy of relativistic species in equilibrium and so used $g_{\star\text{equil}}$, which is the same as $g_{\star S\text{equil}}$ because all species in equilibrium have the same temperature (refer to equations (2.32) and (2.48)). We could have instead considered the total entropy and we would of course have found the same result after a slightly longer computation[5].

[5]For completeness we sketch the computation here. We have $(g_{\star S})_{\text{before}} = 2 + \frac{7}{8}(2 \times 2 + 3 \times 2) = 10.75$ and $(g_{\star S})_{\text{after}} = 2 + \frac{7}{8}[3 \times 2 \times (T_\nu/T_\gamma)^3]$. Setting them equal we have $10.75(aT_\gamma)^3_{\text{before}} = 2(aT_\gamma)^3_{\text{after}} + 5.25(aT_\nu)^3_{\text{after}}$. Finally, using $(aT_\gamma)_{\text{before}} = (aT_\nu)_{\text{before}} = (aT_\nu)_{\text{after}}$ gives $10.75 = 2(T_\gamma/T_\nu)^3_{\text{after}} + 5.25$ which is the same as (2.73).