ROYAL HOLLOWAY, UNIVERSITY OF LONDON

Enigma: Prime Numbers and Cryptosystems

by

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"Anyone can design a security system that he cannot break. So when someone announces, Heres my security system, and I cant break it, your first reaction should be, Who are you? If hes someone who has broken dozens of similar systems, his system is worth looking at. If hes never broken anything, the chance is zero that it will be any good."

Bruce Schneier, *The Ethics of Vulnerability Research*

Abstract

Modern cryptography allows us to perform many types of information exchange over insecure channels. One of these tasks is to agree on a secret key over a channel where messages can be overheard. This is achieved by Diffe-Hellman protocol. Other tasks include public key and digital signature schemes; RSA key exchange can be used for them. These protocols are of great importance for bank networks.

Most such algorithms are based upon number theory, namely, the intractability of certain problems involving prime numbers. The project involves implementing basic routines for dealing with prime numbers and then building cryptographic applications using them.

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Abbreviations

AES **A**dvanced **E**ncryption **S**tandard

Chapter 1

Introduction

1.1 What it's about

Secrecy has always been of great importance, not just in modern society, but throughout history. Until very recently, Cryptography was consigned to the sending and receiving of messages, generally using pen and paper. However, increasingly cryptography – the study and implementation of techniques for secure communication – is becoming more vital to the smooth running of even basic systems, whether it's the cliché example of military secrets or simply a micro-payment for an online service. Initially, the usage of provably secure and efficient cryptographic algorithms was limited to governments and related contractors, however the advent of encryption standards, and more relevantly, the creation of public-key cryptography mechanisms, has pushed it in to a wider field of use as researchers gained a better understanding of the area.

1.2 Goals and Intentions

The primary goal of this project can be found in the title and abstract: to research, discuss and create algorithms within or related to the field of prime numbers. To complete this task I will be developing my thoughts and discoveries regarding prime numbers as this report progresses, as well as producing a number of deliverables in the form of software applications, test results and statistics.The topic of number theory is in itself fascinating, and extraordinarily vast, however I will only be scratching the surface.

Nonetheless, I hope to explain throughout this project how prime numbers have become such an important part of modern cryptography, and what they can be used for.

1.3 Project Summary

I will be splitting the project in to four main parts, excluding this report:

1. **Prime numbers in software**
   A discussion of how prime numbers can be efficiently produced programmatically, and software to prove it.
2. **Algorithms**
   The development of algorithms using prime numbers, such as RSA, and complementary cryptographic algorithms such as AES. Alongside this, the development of non-major algorithms in the field of prime number based cryptography that still present an academically interesting concept.
3. **Final Application**
   The production of a software program that utilises the implemented cryptographic algorithms in section 2 to display their efficacy in a real world application.
4. **Cryptanalysis**
   Taking the algorithms created and comparing them with official implementations for statistical analysis, alongside researching and performing "attacks" on them to prove or disprove the implementation's cryptographic security.

The primary algorithms to develop are:

- Asymmetric
  - RSA
  - Diffie-Hellman
- Symmetric
  - AES

- Identification and Authentication
  - RSA Signing
  - Digital Certificates

Other algorithms will be discussed and produced alongside these, however they will not be used in the final application testbed and are purely for research interest. These can be found listed in the contents in the relevant sections.

1.4 Other Information

This document was written and composed with LaTeXusing *Taco Software's* Latexian[1]. The LaTeXsource code for this report should have come bundled with the project documentation and files, however if you are unable to retrieve it please see section 1.4.2.

Eclipse Indigo[2] was used for Java development, the primary language used in this project, along with *MacVim*[3] for general source and text editing.

Git with *GitHub*[4] was used for source control, with *GitHub Issues* used for bug tracking. *Trello*[5] was used for idea and to-do management.

1.4.1 A note on openness

As with any field of study, the quality of research and development is dependent on the open distribution and sharing of ideas. This is *particularly* important with regards to cryptography. As said, cryptography was once reserved to government and research was conducted in secrecy. The open sharing of relevant information in this field is not just for the furthering of knowledge, but also to allow others to inspect and examine algorithms, a process that drastically improves the security of a system. As such, the entirety of this project is licensed under the **GNU Lesser General Public License version 3 or greater** and is available for access publicly online.

[1] http://tacosw.com/latexian/
[2] http://eclipse.org
[3] http://code.google.com/p/macvim/
[4] http://github.com
[5] http://trello.com

1.4.2 Project Repository

If any part of this report is missing, you believe that you do not have a full access to the source code discussed in this document, or if any files have been lost, it is available for full download at `http://cyanoryx.com/files/project.zip`.

Chapter 2

Cryptographic Primitives

2.1 Basics of Information Security

Despite how it is portrayed or colloquially used, Information Security is an entirely different concept and area of study compared to Cryptography. It might seem simple enough to implement a basic cryptographic protocol involving encryption and decryption, however to introduce this into a system and expect the information to be secure, is foolhardy. Cryptography is a *means* to providing information security when following certain rules and guidelines not the be all, end all solution. An understanding of information security, and the related issues, is necessary.

This can be proven using historical evidence: throughout history many complex systems of mechanisms, rules, and protocols have been developed to introduce information security to a system. As with modern day security, this cannot be achieved entirely through mathematical and cryptographic means – it is more than just computational intractability.

As such, stringent criteria for developing secure systems and protocol have been introduced. While institutes such as *The British Computing Society* and *Association for Computing Machinery* ensure their members follow a professional code of ethics, just as a doctor might, these information security criteria are of separate and equal importance. Indeed, there are now several international organisations that exist solely for the overseeing of cryptographic research and development (See: *International Association for Cryptologic Research*).

Often, as we will see, cryptographic systems are simplified for the purposes of presentation particularly for textbooks. This will be discussed further later, with regards to the differences and difficulties involved in developing systems that do not just follow a mathematical "recipe," but also include information security values and other subtleties.

The overall method of dealing with, and ensuring, information security is known as risk management. This encapsulates a large number of countermeasures (including cryptography) that reduce the risk of vulnerabilities in, and threats to, systems. We will only be encountering and discussing the technological areas of mitigation, however some of the solutions include[1]: access control, security policy, physical security, and asset management.

2.2 Objectives

As said, secure systems should follow a guideline, or set of criteria, that ensure the security and integrity of data stored and input. A clear and concise specification should be developed, that will aid the designer in selecting the correct cryptographic primitives, but also help the engineer implement the protocol correctly. There are many of these criteria, however each is derived from four primary objectives:

1. **Confidentiality** is the ability to ensure data is only accessed by those who are allowed to. Maintaining confidentiality of data is an obligation to protect someone else's secret information if you have been entrusted with it.

2. **Authentication** involves identifying both entities and data. Two or more entities wishing to communicate or transmit information to one another must identify each participant to ensure they are who they claim to be - this is known as entity authentication. Data received must be authenticated to ensure the validity of the origin, date sent, contents, etc - this is known as data origin authentication.

3. **Non-repudiation** prevents an entity from denying previous actions they have committed.

[1]For an excellent resource regarding information security, both technical and non-technical, see *Security Engineering*, Ross Anderson

4. **Data Integrity** is how faithfully data compares to it's true state, i.e. proving that a data object has not been altered.

2.3 Key Concepts

As with any discipline, there are a number of fundamental concepts that need to be thoroughly defined. This section will cover the definitions of basic information security and cryptography related concepts. While Computer Scientists and Mathematicians, unlike Biologists, tend to abstract ideas using existing words such as *normal* which ultimately cause confusion to those trying to understand the area of study, the field of Information Security fortunately uses phrases that are succinct and aptly describe notions.

TODO: er, do this bit.

2.4 Primitives

As we discussed in the Objectives, there are certain criteria that must be met for an application to be considered as secure under Information Security guidelines. Excluding physical and psychological measures, there are a number of methods to be implemented cryptographically to guarantee security.

2.4.1 Encryption

Being what is seen as the very 'core' of cryptography, we have defined encryption many times already and what the term means in terms of a process should be apparent. However encryption takes many forms, with each having an appropriate situation for it to be used.

2.4.1.1 Symmetric Key Encryption

Primarily, we will use symmetric key encryption algorithms to encipher data that is to be transmitted between entities.

Mathematically we can formally define symmetric encryption as:

For a message M, algorithm A and key K,

$M' = A(K, M)$

and thus:

$M = A'(K', A(K, M))$

where $A' = A$, $K' = K$ in a symmetric algorithm.

2.4.2 Key Agreement

Key agreement, or key exchange, primarily ensures data integrity and confidentiality. By preventing an attacker from discovering encryption keys used on transmitted data, the attacker should be unable to feasibly read confidential information or modify it (the latter is not entirely true, it may be possible in some cases to modify encrypted data, however we will discuss that later in *Symmetric Cryptography*).

While it might be easier to use asymmetric techniques to encrypt data for transmission, thus allowing us to distribute keys as cleartext, it is slow and inefficient for large quantities of data, such as in an instant messaging application. Because of this, it is prudent to implement an efficient symmetric key encryption algorithm, and share the key (known as a session key) with other entities. However, it would be trivial for an attacker to launch a man-in-the-middle attack and gain access to the encryption keys during the initiation of the conversation, allowing the easy and undetectable decryption of all transmitted messages. As such, session keys will need to be distributed using a key exchange protocol.

2.4.2.1 Key Distribution Centre

The simplest solution is known as a Key Distribution Centre (KDC), which involves the use of a trusted third party (TTP). It is easiest to explain using an example. Alice and Bob are users of a system, attempting to securely communicate. Each share a key securely with third-party Trent (somewhat amusingly, this algorithm does not include

how these keys should be shared. We can assume that it was perhaps conducted through an in-person meeting of entities, or other means), who stores each key.

1. Alice initiates a conversation with Bob.
2. Alice requests a session key from Trent, who makes two copies of an identical key and encrypts one with Alice's stored key, and another with Bob's.
3. Alice receives both encrypted keys, and sends the appropriate one to Bob.
4. Alice and Bob decrypt their session keys, leaving both with a shared key.
5. Alice and Bob can now encrypt and decrypt data using the same key.

2.4.2.2 Asymmetric Cryptography

As can be obviously seen, using a KDC is dependent entirely on the ability of two entities to have previously, and securely, shared an encryption key with a TTP that is known to both entities. A solution to this is to eliminate the third party, and use an asymmetric algorithm to share keys directly. As defined in the *Key Concepts* section, asymmetric cryptography allows Alice to share a public-key, with which any entity can encrypt data that can only be decrypted using Alice's private key.

1. Alice and Bob share their public keys using a readily available database.
2. Alice downloads Bob's key, and vice versa.
3. Alice generates a session key, encrypts it using Bob's public key and sends it to Bob.
4. Bob can now decrypt the session key using his private key, resulting in both parties being in possession of a secure session key.

There is a security risk: how can you verify that the entity that sent the encrypted key is indeed the one with which you are trying to communicate? It would be easy for an attacker to encrypt their own session key with Alice's public key, and claim that the key is from Bob. The attacker would then be able to decrypt any messages intended for

Bob. The solution to this issue is the use of digital certificates, and signatures, which will be discussed in *Authentication*.

We will discuss specific algorithms further on, however it is worth pointing out that the current public-key algorithms used for these purposes are *RSA* (Rivest, Shamir, Adleman) and the*Diffie-Hellman Key Exchange* (not defined as a public-key algorithm, however it uses the sharing of public cleartext values). There exist a number of interesting algorithms that implement the public key architecture which will also be considered later.

There are other considerations in key management to ensure confidentiality and integrity. Some provisos exist such as changing the key for each session to ensure perfect forward secrecy, however these are trivial to implement and can be considered as part of the overall application security development.

2.4.3 Authentication

There are two types of authentication that are required in a secure application

1. Message authentication
2. Entity authentication

both of which require different protocols and algorithms. These terms have been defined in *Objectives*.

The methods of entity authentication are an interesting topic in themselves, with many, many protocols having been researched and created as the community tries to find a method that is both secure and easy for an entity to use. Some examples are: passwords, two-factor authentication, PINs, smart cards, biometrics, and so on. These are outside of the scope of this project – we will be implementing two broad, yet specific methods of authentication.

2.4.4 Digital Signatures

Digital signatures, as we will see, encompass three of the information security criteria: authentication, data integrity and non-repudiation (a sender cannot claim they did not

send the message). A digital signature is a string that connects a message with its originating entity.

When transmitting a message, the sender signs the message with their private key which can then be verified with their public key, which should be available to the receiver.

1. Alice signs her message A with her private key.
2. Alice sends message A with signature S to Bob $(A|S)$
3. Bob retrieves Alice's public key from an available database, and verifies signature S

The first and most common implementation of digital signatures is RSA.

2.4.5 Public-key Certificates

Digital signatures and a public-key infrastructure, however, are not enough by themselves. A very simple attack can be orchestrated similar to that during key agreement: Mallory, the attacker, could sign a modified message A with her own private-key and then distribute her public-key, claiming it to be Alice's. To counter this, we can introduce a trusted third-party, known as a Certificate Authority (CA), who signs Alice's public-key with their own private-key. Most trusted CA's public keys come bundled with software such as browsers and operating systems.

1. Certificate Authority signs Alice's public key with their private key.
2. The CA distributes its public key with major software.
3. Bob receives Alice's message and signature, as well as her public key and signature.
4. Bob verifies Alice's public key with the CA's public key, and then verifies the message.

2.4.6 Hashing

While not of direct relevance in information security, hashing plays a significant part in cryptographic systems and thus is included as a concept to be implemented. Formally, we can define a hash function as mapping a large domain to a smaller range - in the case of data, mapping a set of bytes to a unique identifier with a set length. A hash function, at the very basics, takes as input a message and produces a fingerprint, or digest, of the input. Within the field of cryptography, they are used for message authentication and data integrity.

This is to say, given a domain D and range R for $f : D \rightarrow R, then |D| > |R|$. This is a many-to-one relationship, the downside of which means that collisions can occur – two input strings resulting with the same output string – however, this varies between algorithms, the more modern of which are less likely to result in collisions.

A hash function can be classed into two categories: keyed and unkeyed, taking both a message and secret key and taking just a message, respectively. Two conditions are necessary for a hash function to be effective:

1. compression – the function f maps input a of arbitrary length to an output of fixed length, n.
2. complexity – it must be easy to compute $f(a)$

Most commonly used are unkeyed, one-way hash functions. Some examples of which are: SHA-1, and MD5. In cryptography, hashes are commonly used for data integrity in combination with digital signatures. A message is hashed, and the fingerprint produced is signed by the entity. There are some algorithms designed specifically for this purpose, known generally as Message Authentication Codes (MACs) and Manipulation Detection Codes (MDCs).

A sample of each of these four primitives will be implemented further in the report.

2.5 Mathematics

This section will cover some of the basic mathematical concepts that will be used throughout the report, and form a foundation of understanding for the more complex abstract methods that will be used.

2.5.1 Notation

1. $\mathbb{Z}$ is the set of all integers.
2. $\mathbb{R}$ is the set of all real numbers.
3. $[a, b]$ is the set of integers n such that $a \leq n \leq b$.
4. $|A|$ is the cardinality of a finite set, i.e. the number of elements.
5. $n \in A$ denotes that an element n exists in set A.
6. $A \subseteq B$ denotes that set A is a subset of set B.
7. $A \subset B$ denotes that A is a subset of B, but $A \neq B$.
8. $\lceil a \rceil$ is the smallest integer greater than or equal to a.
9. $\lfloor a \rfloor$ is the largest integer less than or equal to a.
10. A function (mapping) denoted as $f : A \to B$ signifies that every element a in A is mapped to exactly one element b in B. This can also be written as $f(a) = b$.

2.5.2 Number Theory

We will begin by defining some fundamental rules of number theory.

Definition 2.5.1. Let a, b be integers. a divides b if an integer c exists such that $b = ac$. We define this as a divides b, or $a|b$.

Definition 2.5.2. An integer c is a common divisor if $c|a$ and $c|b$.

Definition 2.5.3. An integer c is the greatest common divisor of a and b, if:

1. c is a common divisor of a, b.

2. when $d|a$ and $d|b$, $d|c$.

3. $c \geq 0$

This is denoted as $c = gcd(a, b)$.

Definition 2.5.4. An integer a is prime if:

1. $a \geq 2$

2. the only positive divisors are 1 and a.

Otherwise, the number is referred to as composite.

Fact 2.5.1. There are an infinite number of prime numbers.

Definition 2.5.5. If a is prime and $a|bc$, then $a|b$ and $a|c$.

Definition 2.5.6. Integers a and b are relatively prime (coprime) to one another, if $gcd(a, b) = 1$.

Definition 2.5.7. The function ϕ is known as the Euler Phi function. Where $n \geq 1$, $\phi(n)$ is the number of integers in the interval $[1, n]$ that are relatively prime to n.

1. If n is prime, then $\phi(n) = n - 1$.

2. $\phi(n)$ is multiplicative: if $gcd(m, n) = 1$, then $\phi(mn) = \phi(m) \cdot \phi(n)$.

2.5.2.1 Modulo Arithmetic

Definition 2.5.8. Where a and b are integers, a is congruent to b modulo n (denoted as $a \equiv b(mod\ n)$. n is known as the modulus. Congruence is reflexive, transitive and and symmetric. That is to say, for every $a, b, c \in \mathbb{Z}$:

1. $a \equiv a(mod\ n)$

2. if $a \equiv b(mod\ n)$, then $b \equiv a(mod\ n)$

3. if $a \equiv b(mod\ n)$ and $c \equiv c(mod\ n)$, then $a \equiv c(mod\ n)$

This is known as an equivalence relation.

Definition 2.5.9. The set of all integers modulo n is denoted as $\mathbb{Z}_n$.

As you can see, there are a great number of theorems regarding prime numbers and modulo arithmetic (we have barely scratched the surface), some of which we will be using.

2.5.3 Abstract Algebra

Further on, particularly in the development of symmetric algorithms like AES, we will be using groups, fields and other abstract algebraic objects. Due to the complex nature of these concepts, they will be explained in tandem with the algorithms themselves.

2.5.4 Complexity

Computational complexity is a vast subject, mostly out of the scope of this report, however we will use the notation occasionally to classify algorithms. Big-oh notation, as it is known, will be used to represent the worst-case running time of an algorithm based on a standard input size. For example:

Definition 2.5.10. A polynomial-time algorithm is an algorithm where the worst-case running time can be represented as $O(n^c)$, where n is the input size, and c is a constant.

An algorithm with a running time that cannot be bounded as such is known as an exponential-time algorithm. It is generally considered that polynomially-time algorithms are efficient, wheres exponential-time algorithms are inefficient.

2.6 Moving On

This is just a basic overview of the cryptographic and mathematical primitives used in Information Security. As the report progresses, each concept will be explained in finer detail alongside their implementations. While it seems hard to apply these abstract definitions, the purpose of each within our algorithms will become quickly clear.

Chapter 3

Number Theory and Public-key Cryptography

3.1 Overview

Number theory, the unit of mathematics that studies integers and their properties, was once a predominantly useless area. However, as cryptographic algorithms became more prevalent, particularly schemes that require the use of large prime numbers. We have already covered the notation used in the majority of number theoretic algorithms, and so this section will build upon this and discuss basic concepts and applications: modular arithmetic, prime number generation, and factorisation.

In this section we will make the assumption that any attackers who may try to break our algorithms or eavesdrop on our messages are extremely powerful and capable, and so we will discuss algorithms with regards to their tractability – in this case, whether or not a significant percentage of all instances of a problem can be solved in polynomial time.

Throughout, where appropriate, programming examples are provided in Java.

3.2 Modular Arithmetic and Congruence

As we know, the core concept of modular arithmetic is to return the remainder of the division of two integers, represented as $a(mod\ n)$. It is said that b is congruent to $a(mod\ n)$ if $b \equiv a(mod\ n)$. Congruences play a great part in cryptography – one of the very first uses was the Caesar Cipher. Messages were made secret – encrypted – by shifting letters forward by an integer n, where $0 < n \leq 25$.

Given a sequence of characters $\{A, B, C, ..., Z\}$ where each letter is represented by a number $\{0, 1, 2, ..., 25\}$. Defined by a function $f(n)$, where n is the integer representation of a character:

$$f(n) = (n + x)\ mod\ 26$$

where x is the number of characters to shift by.

This is a very simple example of how congruences can be used in cryptography. As we will see in the public-key section, congruences can be applied in a similarly simple way as a vital component part of a far more complex algorithm.

Indeed, although the technicalities of this are not relevant, congruences can even be used to generate pseudo-random numbers.

3.3 Prime Numbers

Primes have already been covered, however they are a deeply fascinating subject with many current open problems and conjectures. For example, Goldbach's Conjecture states that every odd integer n, where $n > 2$, is the sum of two primes. This is intriguing because so far this has been *verified* for integers up to 2×10^{17}, but mathematicians have not yet found a formal proof for this, and despite this they continue to believe it to be true [1].

We, however, will be using them for a far more interesting[1] purpose: cryptography based around the intractability of factoring very large composite numbers into their smaller non-trivial prime divisors.

[1] http://xkcd.com/247/

3.3.1 Generation

First and foremost, we must devise an efficient method for generating the prime numbers. Although generating the keys for use in prime number based algorithms is generally not done on a regular basis, it is still a complex task to complete in a reasonable amount of time. As it stands, a number of algorithms already exist (and have done for many hundreds of years) for this purpose, though in a different sense: it is relatively easy to generate small sequences of small primes, however to obtain large, individual primes the fastest method is to generate probable primes using primality tests.

To help understand these algorithms, we must first state a number of theorems that make clearer how these algorithms work and how they will fit in to the public-key implementations later on [2]:

Definition 3.3.1. We define $\pi(x)$ as the number of primes found in the interval $[2, x]$, where $\pi(x) \sim \frac{x}{\ln x}$.

Fact 3.3.1. If $gcd(a, n) = 1$, there are infinitely many primes that are congruent to a mod n.

Fact 3.3.2. We can say that all prime numbers are uniformly distributed across $\phi(n)$ because:

$$\pi(x, n, a) \sim \frac{x}{\phi(n) \ln x}$$

Where $\pi(x, n, a)$ is the number of primes in the interval $[2, x]$ congruent to $a \mod n$.

Fact 3.3.3. We can approximately determine the nth prime number as $\rho_n \sim n \ln n$, where $n \geq 6$.

3.3.1.1 Probable, Provable and Pseudo-primes

There is a subtle yet significant difference between the types of primes that can be generated.

Probable Primes A probably prime is a prime that meets certain conditions that are satisfied by all prime numbers. Testing for probably primes are more appropriately known as *compositeness tests* rather than probable primality tests.

The algorithms we will define as probabilistic primality tests will take arbitrary integers and test them to provide information about their primality.

Pseudo-primes Pseudo-primes are a subset of probably primes. They are composite numbers that pass tests that most composite numbers fail, for example an integer is pseudo-prime if it satisfies Fermat's Little Theorem.

Provable Primes A provable prime is a prime that can be formally proven to be prime using an algorithm, such as the AKS primality test. Generally these are not used in cryptography due to the previous inefficiency of calculating provably-prime numbers, and also as they work most effectively when the input has been passed through a probabilistic primality test first.

3.3.1.2 Sieves

A prime sieve is a fast algorithm to find prime numbers, the most commonly used being the *sieve of Erastothenes* and *sieve of Atkin*. Prime sieves operate by generating a list of integers up to a defined limit n: $\{2, 3, 4, ..., n\}$ and progressively removing the composite integers in line with particular rules. However, this is very slow for the generation of individual large primes, and so is immediately discounted from possible use in a cryptographic algorithm.

3.3.1.3 Fermat's Little Theorem

Note: Fermat's theorem is no longer considered to be a true probabilistic primality test as it cannot determine the difference between probable primes and composite integers known as Carmichael numbers[2]. The theorem can still be used to prove the compositeness of a number.

Fermat's theorem states:

$$\text{If } n \text{ is prime and } a \text{ is an integer, } 1 \leq a \leq n-1, \text{ then } a^{n-1} \equiv 1 \mod n$$

[2]Carmichael numbers are similar to Fermat primes as they satisfy the congruence $b^{n-1} \equiv 1 \mod n$. The conditions for a number to be in the Carmichael set are more complex than this, however the details are out of the scope of this section.

Using this, we can take an integer n and to test for primality find an integer in this interval where the equivalence is not valid, thus proving that n is composite, and is not likely to be prime.

The implementation of this in Java is as so:

```
import java.math.BigInteger;
import java.util.Random;

public class FermatTesting {
  public static boolean checkPrime(BigInteger n, int iterations) {
    Random rng = new Random();

    if (n.equals(BigInteger.ONE)) return false;

    for (int i=0; i<iterations; i++) {
      // Create an integer within the interval [1,n-1]
      BigInteger a = new BigInteger(n.bitLength(),rng);
      while (BigInteger.ONE.compareTo(a) > 0 || a.compareTo(n) >= 0) {
        a = new BigInteger(n.bitLength(),rng);
      }

      // a^(n-1)
      // Repeated until a!=1, thus proving it cannot be prime
      a = a.modPow(n.subtract(BigInteger.ONE),n);

      if (!a.equals(BigInteger.ONE)) return false;
    }

    return true;
  }
}
```

This file can be found at latex_src/Code/primes/FermatTesting.java

This is a very simple algorithm to implement. It selects a random integer in the interval $[1, n-1]$ and computes a^{n-1}. If $a \neq 1$, then a must be composite.

Note: We are using the class `BigInteger` to represent integers, as it allows far larger bit representations of integers than Java's standard 64-bit primitive `integer` type. `BigInteger` offers all the same operators that can be used with `integer` primitives, alongside useful operations such as modular arithmetic, GCD and even built in primality testing. We will use `BigInteger` for all Java-based programming that involves large integers.

3.3.1.4 Miller–Rabin Primality Test

In practice, the Miller–Rabin test is the most used primality test, that also is based on a set of equivalences that are true for primes, and thus if a number to be checked does not pass these equivalences it is not prime.

Definition 3.3.2. Let n be an odd prime: $n-1 = 2^d r$, where d and r are positive integers, with r being odd. For $a \in (\mathbb{Z}/n\mathbb{Z})$:
$a^r \equiv 1 \mod n$ **or**
$a^{2^d r} \equiv -1 \mod n$ assuming $0 \leq d \leq s-1$.

Using this, we can define the Miller–Rabin requirements as:

1. Assume n is an odd composite integer, and $n-1 = 2^s r$, where r is odd.
2. Where a is an integer in the interval $[1, n-1]$, if a does not match the conditions in definition 3.3.2 ($a^r \not\equiv 1 \mod n$, $a^{2^s r} \not\equiv -1 \mod n$), then a is known as a witness for n.
3. Where a is the same as a in point 2, if a matches *either* of the conditions in definition 3.3.2, it is known as a strong liar for n.

When n is determined to be a composite, it is known as a *witness*, meaning it is a definite composite number. When we refer to probable primes, they are known as *strong liars* as they are a likely probably prime to the base integer – it is known as a "strong" liar due to the fact that n could still be composite while the prime equivalences still hold.

The implementation of this in Java is as so:

```
import java.math.BigInteger;
import java.util.Random;

public class MillerRabinTest implements PrimeTest {
  public boolean checkPrime(BigInteger n, int iterations) {
    if (n.equals(BigInteger.valueOf(2L))) return true; // 2 is prime

    if (n.equals(BigInteger.ZERO) || // n==0
        n.equals(BigInteger.ONE)  || // n==1
        n.mod(BigInteger.valueOf(2L)).equals(BigInteger.ZERO)) { // n is even
      return false;
```

```
    }

    // 2 ^ s * r
    int s=0;
    BigInteger r,n_one;
    r = n_one = n.subtract(BigInteger.ONE);

    // Halve r until r is odd
    while (r.mod(BigInteger.valueOf(2L)).equals(BigInteger.ZERO)) {
      r = r.divide(BigInteger.valueOf(2L));
      s++;
    }

    for (int i=0;i<iterations;i++) {
      // Create a random number between 1 and n-1
      BigInteger a;
      do {
        int min = BigInteger.ONE.bitLength();
        int max = n_one.bitLength();
        a = new BigInteger(new Random().nextInt(max - min + 1)+min,new Random());
      } while (BigInteger.ONE.compareTo(a) > 0 || a.compareTo(n) >= 0);

      // y = a^r mod n
      BigInteger y = a.modPow(r,n);

      // While y != 1 and y != n-1
      if (!y.equals(BigInteger.ONE) && !y.equals(n_one)) {
        for (int j=0;j<s;j++) {
          // y = y^2 mod n
          y = y.modPow(BigInteger.valueOf(2L),n);

          // if y == 1, n is prime
          if (y.equals(BigInteger.ONE)) return false;

          // if y == n-1, n is composite
          if (y.equals(n_one)) return true;
        }
      }
    }

    return false;
  }
}
```

This file can be found at latex_src/Code/primes/FermatTesting.java

In some cases the number of iterations is known as the *security parameter*.

3.3.1.5 Solovay–Strassen Primality Test

The Solovay–Strassen test was one of the first primality tests to be made popular by the increasing use of public-key cryptography. It is another probabilistic primality test. It is, in essence, quite a simple algorithm in that the process flow is linearly straightforward.

Definition 3.3.3. Let n be an odd integer, then $a^{\frac{(n-1)}{2}} \equiv (\frac{a}{n}) \mod n$ where $\gcd(a, n) = 1$.

We can use this to define the Solovay–Strassen algorithm requirements:

1. If $\gcd(a, n) > 1$ or $a^{\frac{(n-1)}{2}} \not\equiv (\frac{a}{n}) \mod n$, then a is an Euler witness for n (composite).
2. If $\gcd(a, n) = 1$ and $a^{\frac{(n-1)}{2}} \equiv (\frac{a}{n}) \mod n$ then n is an Euler pseudoprime (prime).

This is relatively simple to implement:

```
import java.math.BigInteger;
import java.util.Random;

public class SolovayStrassenTest implements PrimeTest {
  public boolean checkPrime(BigInteger n, int iterations) {
    BigInteger n_one = n.subtract(BigInteger.ONE);

    for (int i=0;i<iterations;i++) {
      BigInteger a = new BigInteger(n.subtract(BigInteger.valueOf(2L)).bitLength(),new Random())
      int x=jacobiSymbol(a,n);

      // r = a^(n-1)/2 mod n
      BigInteger r = a.modPow(n_one.divide(BigInteger.valueOf(2L)),n);
      // if (a|n)=0 or r!=x and r!=(n-1)
      if (x==0 || (r.compareTo(BigInteger.valueOf(x))!=0) && (r.compareTo(n_one)!=0)) {
        return false;
      }
    }

    return true;
  }

  // Based on Algorith 2.149 in Alfred-Menezes:1996kx
  public static int jacobiSymbol(BigInteger a, BigInteger n){
      int j = 1;
```

```
      BigInteger ZERO  = BigInteger.ZERO;
      BigInteger ONE   = BigInteger.ONE;
      BigInteger TWO   = BigInteger.valueOf(2L);
      BigInteger THREE = BigInteger.valueOf(3L);
      BigInteger FOUR  = BigInteger.valueOf(4L);
      BigInteger FIVE  = BigInteger.valueOf(5L);
      BigInteger EIGHT = BigInteger.valueOf(8L);

      BigInteger res;

      while (a.compareTo(ZERO) != 0){
        // While a % 2 == 0
        while (a.mod(TWO).compareTo(ZERO) == 0){
          a = a.divide(TWO); // a / 2
          res = n.mod(EIGHT); // n % 8

          if (res.compareTo(THREE) == 0 || res.compareTo(FIVE) == 0) j = -1*j;
        }

        BigInteger temp = a;
        a = n;
        n = temp;

        if (a.mod(FOUR).compareTo(THREE) == 0 && n.mod(FOUR).compareTo(THREE) == 0) j = -1 * j;
        a = a.mod(n);
      }

      if (n.compareTo(ONE) != 0) j = 0;

      return j;
    }
}
```

This file can be found at latex_src/Code/primes/SolovayStrassenTest.java

Definition 3.3.4. $(\frac{a}{n})$ is known as the Jacobi symbol, and plays an important part in the Solovay–Strassen test. Let m, n be odd integers where $m \geq 3$ and $n \geq 3$, and $a, b \in \mathbb{Z}$, then we have the following properties[2]:

1. $(\frac{a}{n}) = 0$ or 1 or -1.
2. $(\frac{a}{n}) = 0$ if $\gcd(a, n) \neq 1$.
3. $(\frac{ab}{n}) = (\frac{a}{n})(\frac{b}{n})$

4. $(\frac{a}{nm}) = (\frac{a}{m})(\frac{a}{n})$
5. If $a \equiv b \mod n$ then $(\frac{a}{n}) = (\frac{b}{n})$. This is one of the most useful properties.
6. $(\frac{1}{n}) = 1$
7. $(\frac{-1}{n}) = (-1)^{\frac{n-1}{2}}$, and so $(\frac{-1}{n}) = 1$ if $n \equiv 1 \mod 4$, or $(\frac{-1}{n}) = -1$ if $n \equiv 3 \mod 4$.
8. $(\frac{2}{n}) = (-1)^{\frac{n^2-1}{8}}$, and so $(\frac{2}{n}) = 1$ if $n \equiv 1$ or $7 \mod 8$, or $(\frac{2}{n}) = -1$ if $n \equiv 3$ or $5 \mod 8$
9. $(\frac{m}{n}) = (\frac{n}{m})$ unless $m, n \cong 3 \mod 4$ which makes $(\frac{m}{n}) = -(\frac{n}{m})$.

The Java implementation of this is given above.

3.3.1.6 AKS Primality Test

The only true-prime test of the four listed so far, AKS was presented as recently as 2002 by three mathematicians: Manindra **A**grawal, Neeraj **K**ayal and Nitin **S**axena (hence AKS) [3]. It is a deterministic primality-proving algorithm, with a complexity relatively similar to that of the faster probabilistic primality tests.

The AKS test is based around Fermat's Little Theorem, however it excels where Fermat's theorem was unable to: it is able to distinguish between pseudoprimes and Carmichael numbers. Recall that Fermat's Little Theorem states that:

Fact 3.3.4. Where integers $a \in \mathbb{Z}$, $n \geq 2$ and $\gcd(a, n) = 1$, n is prime if the following congruence is satisfied:
$(x + a)^n \equiv x^n + a \mod n$

This fails for both pseudoprimes and Carmichael numbers. Instead, the following congruence is used as the base of the AKS algorithm:

Definition 3.3.5. Where r is the smallest possible value that satisfies $O_r(n) > 4log^2 n$,
$(x + a)^n \equiv x^n + a \mod x^r - 1, n$

The algorithm for AKS is fairly simple to follow, however the Java program is somewhat verbose, and thus it is helpful if we first define it in pseudocode (uses the Lenstra and Pomerance Improvements [4]):

1. If $n = a^b$ where $a \in \mathbb{N}$ and $b > 1$ then print "composite" and stop.
2. Find the smallest value of r so that $O_r(n) > 4log^2n$.
3. If $\gcd(a, n) \neq 1$ for all integers a where $a \leq r$ then print "composite" and stop.
4. For $(a = 1)$ to $\lfloor \sqrt{r} \log n \rfloor$:
5. If $((x + a)^n) \equiv x^n + a \mod (x^r - 1, n)$ then print "prime" and stop.

To calculate r, we use the following method:

1. Pick a number q where $q > \lfloor log^2n \rfloor$.
2. For $j = 1$ to $[(log^2n)]$ do
3. Calculate $n^j \mod q$
4. If the residue $= 1 \mod q$, then $q++$.
5. Else $r = q$.

In Java the implementation is as so:

```
import java.math.BigInteger;

public class AKSTest implements PrimeTest {
  public boolean checkPrime(BigInteger n, int iterations) {


    return false;
  }

  /**
   * Generates an appropriate value of r
   *
   */
  public BigInteger generateR(BigInteger n) {
  }

  /**
   * Uses Fermat's Little Theorem to determine is a number is prime.
   *
   */
  public boolean isPrime(BigInteger n) {
    BigInteger TWO = BigInteger.valueOf(2L);
```

```
    while ((TWO.multiply(TWO).compareTo(n) <= 0) {
      if (n.mod(TWO).compareTo(BigInteger.ZERO)==0) return false;
      TWO = TWO.add(BigInteger.ONE);
    }

    return true;
  }

  /**
   * Checks if a number is a power of 2
   *
   */
  public boolean isPower(BigInteger n) {
    return false;
  }
}
```

3.3.1.7 Maurer's Algorithm

3.3.1.8 BigInteger Generation

Using the constructor `BigInteger(int bitLength, int certainty, Random rnd)`, we can generate random, arbitrary probable primes with the given certainty. The `BigInteger` prime generator uses the Miller–Rabin test[5] with a random number generator to produce random probably primes.

3.3.1.9 Primality Test Outcomes

Given these algorithms, which is best suited to our use further in this project? It is best to attempt to do this formally, rather than subjectively.

Primality *proving* algorithms are required to satisfy the following four constraints, however they also apply to probable primality tests:

1. **Generality** – The primality of any general number can be identified.
2. **Polynomial** – The maximum running time of the algorithm is polynomial.

3. **Deterministic** – The algorithm can deterministically identify if the input integer is prime or not.

4. **Unconditional** – The validity of the test result is not dependent on the proof of a current unproven hypothesis.

Algorithm	Constraints matched
Fermat's Little Theorem	Meets only two constraints.
Miller–Rabin	Miller–Rabin in its suggested form meets three constraints, as it is dependent on the proving of the generalised Riemann hypothesis. Basic Miller–Rabin tests are not deterministic, and only meet two constraints.
Solovay–Strassen	Meets all but the third constraint, as it is not deterministic, but probabilistic.
AKS	Meets all four constraints.

Running Times

Algorithm	Running Time
Fermat's Little Theorem	$O(k \times log^2 n \times log\ log\ n \times log\ log\ log\ n)$
Miller–Rabin	$O(k\ log^3 n)$
Solovay–Strassen	$O(k\ log^3 n)$
AKS[3]	$\breve{O}(log^{12} n)$

Based on the above and what we've seen when implementing the algorithms, we can say with confidence that the Miller-Rabin test is the most useful probable primality test due to the balance of complexity and simplicity in implementation, whereas the AKS test is the most useful provable primality test due to its relatively fast execution speed (compared to other provable prime algorithms), and thus the Miller–Rabin test is the best overall algorithm to be used in the efficient generation of primes.

Conclusion Overall, `BigInteger` gives the best results for minimal execution time, minimal effort, support and reduced likelihood of mistakes. Importantly, it implements

the Miller–Rabin test, which as we have determined, is the fastest primality test that can be reasonably used in software. Alongside this, as it is a maintained library that has been in the JDK since 1999, it is mature and any errors or bugs will have been repaired at this point. It is said that a developer should not repeat themselves, or "reinvent the wheel": "Every piece of knowledge must have a single, unambiguous, authoritative representation within a system"[6] and as such, reimplementing Miller–Rabin for no reason other than as proof that you can will result in poor software that is difficult and time-consuming to maintain, as well as containing errors.

3.3.1.10 A Note on Random Number Generators

The random number generators mentioned thus far, and all used later, cannot be considered *truly* random. They are known as *pseudo-random number generators* – sequences generated deterministically based on a seed value. The sequences are produced algorithmically based on a relatively small set of values, Generally these sequence can be considered *practically* random for most uses, assuming the seed is not publicly known and is truly random.

Most truly random numbers are generated by analysing physical methods, such as nuclear decay or cosmic background radiation. However, typically this tends to be costly and is usually reserved only for applications that explicitly need high-security, non-deterministic random numbers. Recently the technology has become more available due to the widespread availability of online services – for example, RANDOM.ORG[4] offers an API that returns truly random numbers on request. This naturally comes with its own security downsides, but can be used for non-cryptographic applications. Even then, RANDOM.ORG charges for clients that go over a quota of requests.

The generation and application of random numbers is an extensive and current area of research. For the purposes of the cryptographic utilities to be developer, and the Enigma application, we will be used the *Java SDK* class `SecureRandom`. `SecureRandom` is a cryptographically strong pseudo-random number generator that complies and follows tests specified by FIPS-140-2[5], and also uses non-deterministic seed material, as defined by *RFC 1750*[6]. Assuming the proper use of this class, it is about as close as we will get

[4] http://random.org
[5] Section 4.9.1
[6] http://www.faqs.org/rfcs/rfc1750.html

to generating secure, non-deterministic random numbers without resorting to physical methods.

In short, and for our purposes, the generation of secure random numbers is based around maintaining the secrecy of a truly random seed on which a random sequence can be generated.

3.3.2 Strong Primes

A prime n is defined as strong if the integers a, b, and c exist such that:

1. $n - 1$ has a large prime factor a.
2. $n + 1$ has a large prime factor b.
3. $r - 1$ has a large prime factor c.

Gordon's algorithm can be used to generate such strong primes based around the output of a probabilistic primality test such as Miller–Rabin, however it is out of the scope of this section.

The utility of random primes is in the generation of moduli for public-key cryptography. It is expected that the integer primes p, q used to generate modulus n be distinct and of sufficient size that factorisation of n is implausible. However, they should be random in the sense that they are selected from a set of prime integers sufficiently large enough to make a brute force attack infeasible.

Recently it has been shown that strong primes do not make much difference in terms of security over their random prime counterparts. However, it is of shared opinion that as they are no *less* secure and have a negligible execution time, strong primes should still be used for cryptographic prime requirements.

For more information, see [7] or [8].

3.4 Integers

Integers are a well-define concept, and you would have had great difficulty getting this far in the report without understanding what one is. Integers are, as with lots of areas of studies, extremely useful in cryptography, or perhaps more specifically: the properties of integers and the operations we can perform on them are particularly useful in cryptography.

3.4.1 Factorisation

The integer factorisation problem is the very foundation of many cryptographic algorithms. Formally, it is defined as:

Definition 3.4.1. Given a positive integer n, find its prime factors: $n = p_1^{e_1} p_2^{e_2} ... p_n^{e_n}$ where p_n are distinct primes, and $e_n \geq 1$.

It is this problem that makes it implausible for RSA and other algorithms to be broken by factoring the modulus and obtaining the private keys. We will see this in the following sections.

There are many current algorithms for factoring large prime multiples, which will be discussed in Chapter 7.

3.5 Public-key Cryptography

3.5.1 RSA

So far we have only covered the concepts of number theory and how they're used in cryptography. In this section we will discuss exactly how they are implemented, and go in to greater detail about the algorithms and techniques required. Chapter 2 provides a brief definition of public-key cryptography.

3.5.1.1 The RSA Problem

The RSA Problem forms the foundation of the RSA public key system:

Definition 3.5.1. Given 3 parameters:

1. a positive integer n, $n = pq$ where p and q are distinct, odd primes.
2. a positive integer e, where $\gcd(e, (p-1)(q-1)) = 1$.
3. an integer c.

Find integer m, where $m^e = c \mod n$.

3.5.1.2 RSA

The RSA cryptosystem – an implementation of the RSA problem – is the most popular asymmetric cryptosystem, mostly due to its ease of implementation and the simple concepts behind it. It's primary purpose is to provide privacy and confidentiality, and we will be using it as part of a key agreement scheme for use in symmetric ciphers.

3.5.1.3 Key Generation

A fundamental property of a public-key cryptosystem is the ability to generate and handle public- and private-keys, both of which come in pairs of corresponding keys. They complete the encryption and decryption transformations, respectively.

When wishing to setup the ability to receive asymmetrically encrypted messages, a user must first generate their keys as so:

1. Generate two large primes, p and q, and calculate $n = pq$.
2. Calculate $\phi = (p-1)(q-1)$, and randomly pick an integer e until $1 < e < \phi$, assuming $\gcd(e, \phi) = 1$.
3. Calculate integer d, $1 < d < \phi$ so that $ed \equiv 1 \pmod{\phi}$

The public key is (n, e) and the private key is d.

The Java implementation of this as a class is:

```
package com.cyanoryx.uni.crypto.rsa;

import java.math.BigInteger;
import java.security.SecureRandom;

/**
 *
 *
 * @author adammulligan
 *
 */
public class KeyGenerator {
        public static int SIZE_DEFAULT = 1024, SIZE_MAX = 16384, SIZE_MIN = 256;

        private int keysize = SIZE_DEFAULT;

        /**
         * Constructor
         *
         * @param size Size of the keys in bits
         */
        public KeyGenerator(int size) throws InternalError {
                if (size<SIZE_MAX) {
                        this.keysize = size;
                } else {
                        throw new InternalError("Key size must adhere to "
                                        +SIZE_MIN+" < k < "+SIZE_MAX);
                }
        }

        public BigInteger[] generatePair() {
                BigInteger[] pq = this.generatePrimes(80);

                BigInteger p = pq[0];
                BigInteger q = pq[1];

                // N = pq
                BigInteger N = p.multiply(q);

                // r = (p-1)*(q-1)
                BigInteger phi = p.subtract(BigInteger.valueOf(1));
                phi = phi.multiply(q.subtract(BigInteger.valueOf(1)));

                BigInteger E;

            do
                {
```

```
                    E = new BigInteger( this.keysize/2, new SecureRandom() ) ;
                }
                while( ( E.compareTo( phi ) != -1 )
                        || ( E.gcd( phi ).compareTo( BigInteger.valueOf( 1 ) ) != 0 ) ) ;

            // D = 1/E mod r
                BigInteger D = E.modInverse(phi);

                return new BigInteger[]{N,E,D};
        }

        private BigInteger[] generatePrimes(int certainty) {
                BigInteger p,q;

                p = new BigInteger(this.keysize/2,certainty,new SecureRandom());
                q = new BigInteger(this.keysize/2,certainty,new SecureRandom());

                if (p == q) this.generatePrimes(certainty);

                BigInteger[] pq = new BigInteger[2];
                pq[0] = p;
                pq[1] = q;

                return pq;
        }
}
```

This file can be found at *src/com/cyanoryx/uni/crypto/rsa/KeyGenerator.java*.

Key Distribution Public-keys, as can be determined from their name, are intended to be made public and widely available. This has its own problems however, as without any capability to verify the origin of a public-key, a user will be susceptible to impersonation attacks. There are various methods to solve this, including certificates, using a trusted server, and other techniques that will be discussed in chapter 5.

3.5.1.4 Encryption

For a user Alice to send an encrypted message m to user Bob, Alice must obtain Bob's public key (n, e). The message is encrypted, $c = m^e \mod n$, and can be sent securely to Bob.

The Java implementation of encryption is:

```
package com.adammulligan.uni;

/**
 *
 * @author adammulligan
 *
 */
public class Encrypt {
  private String key_file;

  private int length;

  private BigInteger E,N;
  private BigInteger[] ciphertext;

  /**
   *
   * @param key_file The public key file for encryption
   */
  public Encrypt(String key_file) {
    this.key_file = key_file;
  }

  /**
   * Takes a String, encrypts using RSA Basic and stores locally
   *
   * @param message Plaintext to be encrypted
   */
  private void encrypt(String message) {
    byte[] temp = new byte[1], bytes;

    bytes = message.getBytes();

    BigInteger[] converted_bytes = new BigInteger[bytes.length];

    // Convert each byte of the message into a bigint
    for(int i=0; i<converted_bytes.length;i++) {
      temp[0] = bytes[i];
      converted_bytes[i] = new BigInteger(temp);
    }

    this.ciphertext = new BigInteger[converted_bytes.length];

    // The actual encryption!
    // Loop through the array of bigint bytes m, and create an array making c = m
    for(int i=0;i<converted_bytes.length;i++) {
      this.ciphertext[i] = converted_bytes[i].modPow(this.E, this.N);
```

```
    }
  }

  private void log(String msg) {
    System.out.println(msg);
  }
}
```

3.5.1.5 Decryption

Decryption, as with encryption, is simple: using private key d, Bob computes $m = c^d \mod n$.

The Java implementation of decryption is:

```
package com.adammulligan.uni;


/**
 * An interactive console application that encrypts a file based on a provided RSA key pair.
 *
 * @author adammulligan
 *
 */
public class Decrypt {
  private String key_file;

  private BigInteger E,N;
  private BigInteger[] plaintext;

  /**
   *
   * @param key_file The public key file for encryption
   */
  public Decrypt(String key_file,String input_file,String output_file) {
    this.key_file = key_file;
  }

  /**
   * Takes a String, encrypts using RSA Basic and stores locally
   *
   * @param message Ciphertext to be decrypted
   */
  private void decrypt(String message) {
    byte[] temp = new byte[1], bytes;
```

```
    bytes = message.getBytes();

    BigInteger[] converted_bytes = new BigInteger[bytes.length];

    // Convert each byte of the message into a bigint
    for(int i=0; i<converted_bytes.length;i++) {
      temp[0] = bytes[i];
      converted_bytes[i] = new BigInteger(temp);
    }

    this.plaintext = new BigInteger[converted_bytes.length];

    // The actual ecryption!
    // Loop through the array of bigint bytes m, and create an array making c = m
    for(int i=0;i<converted_bytes.length;i++) {
      this.plaintext[i] = converted_bytes[i].modPow(this.E, this.N);
    }
  }

  private void log(String msg) {
    System.out.println(msg);
  }
}
```

Both the encryption and decryption Java files have some helper methods removed for brevity and to highlight the actual algorithm.

3.5.1.6 Formal Proof

Given any message m such that $m \in \mathbb{Z}_n$, as the decryption transformation ($D(m)$) is an inverse of the encryption transformation ($E(m)$), we have:

$$E(D(m)) = D(E(m)) = m^{ed} \pmod{n}$$

Exponents e and d are multiplicative inverses of each other, modulo $\phi(n)$, we get:

$$ed = 1 + k(p-1)(q-1)$$

$$\begin{aligned} m^{ed} &\equiv m(m^{p-1})^{k(q-1)} && (\bmod p) \\ &\equiv m((m \bmod p)^{p-1})^{k(q-1)} && (\bmod p) \\ &\equiv m(1)^{k(q-1)} && (\bmod p) \\ &\equiv m && (\bmod p) \end{aligned}$$

And thus $m^{ed} \equiv m \mod p$.

3.5.1.7 Textbooks and OAEP

So far we have looked at what are known as "textbook algorithms" – algorithms that are not suited for us in the real world. Generally they are constructed around the basic core mathematics of the problem, without any additional measures to counteract basic attacks, like frequency analysis.

As a solution to this, a number of standards have been introduced that add extra requirements to the algorithms such as hashing to improve the overall security. In the case of RSA, the most common standard implemented is known as *RSA OAEP* – RSA Optimal Asymmetric Encryption Padding[9]. It is based upon the EME-OAEP encoding method – the purpose of this scheme is to add randomness to the encrypted output, creating a probabilistic result, along with preventing any information leakage from ciphertexts.

More specifically, the use of OAEP protects against chosen-plaintext attacks. There are some concerns about the inability to prove that it is secure from certain types of chosen-plaintext attacks, however this is discussed in more detail in Chapter 7.

Given:

- A hash function.
- A mask generation function (MGF)
- A public key.
- The plaintext message.
- A label to be appended to the message.

There are three steps to the RSA OAEP encryption process:

1. Check the length of input to ensure that it is below the limit for the hash function. In our case, the limit is $2^{61} - 1$ (SHA-1).

2. Encode the message using EME-OAEP:

3. (a) Construct the `byte` array:

```
DB = LHash (the hash of the label)
    || PS (a byte array of zero-bytes [0x00])
    || 0x01
    || M (the message to encrypt)
```

   (b) Generate a random byte array for us as a seed s.

   (c) Create a mask m = MGF(s, rsa modulus length − hash length −1)

   (d) Mask DB: $mdb = DB \oplus m$

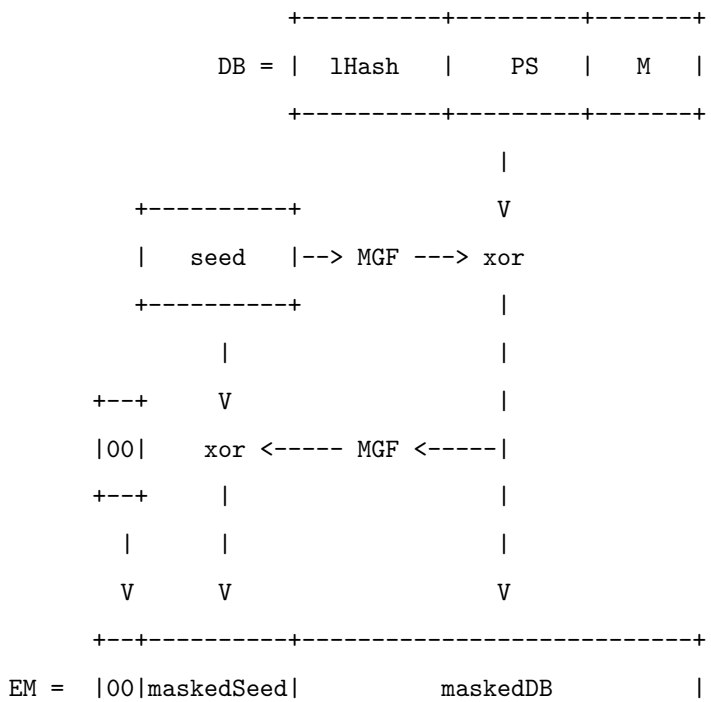
   (e) Create a seed mask sm = MGF(masked DB, hash length)

   (f) Mask the seed: $ms = s \oplus sm$

   (g) Finally, concatenate the generated masked byte arrays, prefixed with `0x00`: `EM = 0x00 || ms || mdb`.

4. Encrypt the encoded `byte` array EM using the RSA method as defined previously.

After all the components have been created, they are assembled as so[10]:



```
+--+----------+---------------------------+
```

However, there is one function with this that is not yet explained.

The Mask Generation Function (MGF) is a function that takes (in Java terminology) a `byte` array of arbitrary length and a desire output length, and gives a `byte` array of that length. As it is based on a hash function, the output is deterministic. It is defined the RSA PKCSv2 specification [10] that the MGF should be pseudorandom, in that given one piece of the output, you are unable to predict another part of the output. The entire security of RSA OAEP is dependent on this property.

The MGF in Java is:

```
package com.cyanoryx.uni.crypto.rsa;

import java.security.MessageDigest;

import com.cyanoryx.uni.common.Bytes;

/**
 *  A mask generation function takes an octet string of variable length
 *  and a desired output length as input, and outputs an octet string of
 *  the desired length. There may be restrictions on the length of the
 *  input and output octet strings, but such bounds are generally very
 *  large.  Mask generation functions are deterministic; the octet string
 *  output is completely determined by the input octet string.
 *
 * @author adammulligan
 *
 */
public class MGF1 {
  private final MessageDigest digest;

  public MGF1(MessageDigest digest) {
    this.digest = digest;
  }


  /**
   *  MGF1 is a Mask Generation Function based on a hash function.
   *
   *  MGF1 (mgfSeed, maskLen)
   *
   *  Options:
   *  Hash     hash function (hLen denotes the length in octets of the hash
```

```
   *           function output)
   *
   *  Input:
   *  mgfSeed  seed from which mask is generated, an octet string
   *  maskLen  intended length in octets of the mask, at most 2^32 hLen
   *
   *  Output:
   *  mask     mask, an octet string of length maskLen
   *
   * @param mgfSeed
   * @param maskLen
   * @return
   */
  public byte[] generateMask(byte[] mgfSeed, int maskLen) {
        // (maskLen / hLen) - 1
    int hashCount = (maskLen + this.digest.getDigestLength() - 1)
                        / this.digest.getDigestLength();

    byte[] mask = new byte[0];

    // For counter from 0 to \ceil (maskLen / hLen) - 1, do the following:
    for (int i=0;i<hashCount;i++) {
      /*
       * a. Convert counter to an octet string C of length 4 octets (see
       *    Section 4.1):
       *
       *     C = I2OSP (counter, 4) .
       */
      this.digest.update(mgfSeed);
      this.digest.update(new byte[3]);
      this.digest.update((byte)i);
      byte[] hash = this.digest.digest();

      /*
       * b. Concatenate the hash of the seed mgfSeed and C to the octet
       *    string T:
       *
       *     T = T || Hash(mgfSeed || C)
       */
      mask = Bytes.concat(mask, hash);
    }

    //  4. Output the leading maskLen octets of T as the octet string mask.
    byte[] output = new byte[maskLen];
    System.arraycopy(mask, 0, output, 0, output.length);
    return output;
  }
}
```

This file can be found in *com.cyanoryx.uni.crypto.rsa*.

Decryption, as with standard RSA, is effectively the reverse of encryption. The ciphertext is decrypted, the concatenated string of properties is split in to the components created in encryption, and the message output.

Because OAEP is a purely technical standard, rather than mathematical, we will list the Java implementation of it with inline comments as explanation.

```
package com.cyanoryx.uni.crypto.rsa;

import java.math.BigInteger;
import java.security.MessageDigest;
import java.security.NoSuchAlgorithmException;
import java.security.SecureRandom;
import java.util.zip.DataFormatException;

import com.cyanoryx.uni.common.Bytes;

public class RSA_OAEP {
  private BigInteger E,N;

  private Key key;

  private MessageDigest md;

  public RSA_OAEP(Key key) {
    this.key = key;

    this.E = this.key.getExponent();
    this.N = this.key.getN();

    try {
      this.md = MessageDigest.getInstance("sha1");
    } catch (NoSuchAlgorithmException e1) {
      e1.printStackTrace();
    }
  }

  /**
   * Encrypts a byte[] using RSA-OAEP and returns a byte[] of encrypted values
   *
   * @param byte[] M byte array to be encrypted
   * @return byte[] C byte array of encrypted values
   * @throws DataFormatException
   */
```

```
public byte[] encrypt(byte[] M) throws DataFormatException, InternalError {
  if (this.key instanceof PrivateKey) {
    throw new InternalError("A public key must be passed for encryption");
  }

  // The comments documenting this function are from RFC 2437 PKCS#1 v2.0

  /*
     1. Length checking:

        a. If the length of L is greater than the input limitation for the
           hash function (2^61 - 1 octets for SHA-1), output "label too
           long" and stop.

        b. If mLen > k - 2hLen - 2, output "message too long" and stop.
   */
  int mLen = M.length;

  int k = (this.N.bitLength()+7)/8;

  if (mLen > (k - 2*this.md.getDigestLength() - 2)) {
    throw new DataFormatException("Block size too large");
  }

  /*
     b. Generate an octet string PS consisting of k - mLen - 2hLen - 2
         zero octets.  The length of PS may be zero.
   */
  byte[] PS = new byte[k - mLen - 2*this.md.getDigestLength() - 2];

  /*
      c. Concatenate lHash, PS, a single octet with hexadecimal value
         0x01, and the message M to form a data block DB of length k -
         hLen - 1 octets as
               DB = lHash || PS || 0x01 || M.
   */
  byte[] DB = Bytes.concat(this.md.digest(),PS, new byte[]{0x01},M);

  // d. Generate a random octet string seed of length hLen
  // (hLen = hash function output length in octets)
  SecureRandom rng = new SecureRandom();
  byte[] seed = new byte[this.md.getDigestLength()];
  rng.nextBytes(seed);

  // e. Let dbMask = MGF(seed, k - hLen - 1).
  MGF1 mgf1 = new MGF1(this.md);
  byte[] dbMask = mgf1.generateMask(seed, k - this.md.getDigestLength() - 1);
```

```
// f. Let maskedDB = DB \xor dbMask.
byte[] maskedDB = Bytes.xor(DB, dbMask);

// g. Let seedMask = MGF(maskedDB, hLen).
byte[] seedMask = mgf1.generateMask(maskedDB, this.md.getDigestLength());

// h. Let maskedSeed = seed \xor seedMask.
byte[] maskedSeed = Bytes.xor(seed, seedMask);

/*
   i. Concatenate a single octet with hexadecimal value 0x00,
       maskedSeed, and maskedDB to form an encoded message EM of
       length k octets as

           EM = 0x00 || maskedSeed || maskedDB.
 */
byte[] EM = Bytes.concat(new byte[]{ 0x00 }, maskedSeed, maskedDB);

/*
   a. Convert the encoded message EM to an integer message
         representative m (see Section 4.2):

           m = OS2IP (EM).
 */
BigInteger m = new BigInteger(1,EM);

/*
   b. Apply the RSAEP encryption primitive (Section 5.1.1) to the RSA
         public key (n, e) and the message representative m to produce
          an integer ciphertext representative c:

           c = RSAEP ((n, e), m).
 */
BigInteger c = m.modPow(this.E,this.N);

/*
   c. Convert the ciphertext representative c to a ciphertext C of
         length k octets (see Section 4.1):

           C = I2OSP (c, k).
 */
byte[] C = Bytes.toFixedLenByteArray(c, k);

if(C.length != k) {
  throw new DataFormatException();
}

return C;
```

```
}

/**
 * Takes a byte[] of encrypted values and uses RSAES-OAEP-DECRYPT to decrypt them
 *
 * @param byte[] C Array of encrypted values
 * @return byte[] M Array of decrypted values
 * @throws DataFormatException
 */
public byte[] decrypt(byte[] C) throws DataFormatException, InternalError {
  if (this.key instanceof PublicKey) {
    throw new InternalError("A private key must be passed for decryption");
  }

  /*
     1. Length checking:

        a. If the length of L is greater than the input limitation for the
           hash function (2^61 - 1 octets for SHA-1), output "decryption
           error" and stop. (NOTE: labels are not used in this implementation)

        b. If the length of the ciphertext C is not k octets, output
           "decryption error" and stop.
   */
  int k = (this.N.bitLength()+7)/8;

  if(C.length != k) {
    throw new DataFormatException();
  }

  //c. If k < 2hLen + 2, output "decryption error" and stop

  if (k < (2*this.md.getDigestLength()+2)) {
    throw new DataFormatException("Decryption error");
  }

  /*
     2.    RSA decryption:

          a. Convert the ciphertext C to an integer ciphertext
             representative c (see Section 4.2):

                c = OS2IP (C).
   */
  BigInteger c = new BigInteger(1, C);

  /*
     b. Apply the RSADP decryption primitive (Section 5.1.2) to the
```

```
        RSA private key K and the ciphertext representative c to
        produce an integer message representative m:

              m = RSADP (K, c).
 */
BigInteger m = c.modPow(this.E,this.N);

/*
   c. Convert the message representative m to an encoded message EM
       of length k octets (see Section 4.1):

              EM = I2OSP (m, k).
 */
byte[] EM = Bytes.toFixedLenByteArray(m, k);
if(EM.length != k) {
    throw new DataFormatException();
}

/*
   3. EME-OAEP decoding:

      a. If the label L is not provided, let L be the empty string. Let
         lHash = Hash(L), an octet string of length hLen (see the note
         in Section 7.1.1).

      b. Separate the encoded message EM into a single octet Y, an octet
         string maskedSeed of length hLen, and an octet string maskedDB
         of length k - hLen - 1 as

              EM = Y || maskedSeed || maskedDB.
 */
if(EM[0] != 0x00) throw new DataFormatException();

byte[] maskedSeed = new byte[this.md.getDigestLength()];
System.arraycopy(EM, 1, maskedSeed, 0, maskedSeed.length);

byte[] maskedDB = new byte[k - this.md.getDigestLength() -1];
System.arraycopy(EM, 1 + this.md.getDigestLength(), maskedDB, 0, maskedDB.length);

//  c. Let seedMask = MGF (maskedDB, hLen).
MGF1 mgf1 = new MGF1(this.md);
byte[] seedMask = mgf1.generateMask(maskedDB, this.md.getDigestLength());

//  d. Let seed = maskedSeed ^ seedMask.
byte[] seed = Bytes.xor(maskedSeed, seedMask);

//  e. Let dbMask = MGF (seed, k - hLen - 1).
byte[] dbMask = mgf1.generateMask(seed, k - this.md.getDigestLength() -1);
```

```
    //  f. Let DB = maskedDB ^ dbMask.
    byte[] DB = Bytes.xor(maskedDB, dbMask);

    /*
       g. Separate DB into an octet string lHash' of length hLen, a
           (possibly empty) padding string PS consisting of octets with
           hexadecimal value 0x00, and a message M as

                DB = lHash' || PS || 0x01 || M.
     */
    byte[] lHash1 = new byte[this.md.getDigestLength()];
    System.arraycopy(DB, 0, lHash1, 0, lHash1.length);
    if(!Bytes.equals(this.md.digest(), lHash1))
        throw new DataFormatException("Decryption error");

    /*
       If there is no octet with hexadecimal value 0x01 to separate PS
        from M, if lHash does not equal lHash', or if Y is nonzero,
        output "decryption error" and stop.  (See the note below.)
     */
    int i;
    for(i = this.md.getDigestLength(); i < DB.length; i++) {
        if(DB[i] != 0x00) break;
    }

    if(DB[i++] != 0x01) throw new DataFormatException();

    // 4. Output the message M.
    int mLen = DB.length - i;
    byte[] M = new byte[mLen];
    System.arraycopy(DB, i, M, 0, mLen);
    return M;
  }
}
```

The `Key` class is a method for objectively storing key information that is imported from file. It can be viewed in the same package, *com.cyanoryx.uni.crypto.rsa*.

3.5.2 Diffie-Hellman

3.5.2.1 Implementation

3.6 Other Variations of Public-key Cryptography

Chapter 4

Symmetric Cryptography

Chapter 5

Identification and Authentication

5.1 Overview and Intentions

5.2 Basic and Common Schemes

5.3 Digital Signatures

5.4 Certificates

5.5 Other Methods

Chapter 6

Enigma: A Testbed

6.1 Overview and Intentions

The intention of this project is to produce an application that allows two users to securely identify and authenticate one another, and communicate with text messages via a presumed-insecure network. However, the primary *goal* is to create a testbed that allows any cryptographic algorithms to be integrated in place, to provide a platform for further analysis, such as comparison of run-times between open and closed source implementations.

6.2 Engineering Methodologies and Planning

6.2.1 Methodology

The main focus of the development process was to break the application down in to its component pieces, and iteratively develop them so that minimalist functionality sets could be produced and then combined in the shortest possible times. The overarching category for this style of production is known as *agile development*: a practice based around iterative and incremental development. More specifically, a subset of agile development will be used – Scrum. The goals of Scrum fit neatly with the desired development cycle in that programming is done in "sprints," with a deadline organised at the start of these sprints and a set of tasks that must be completed by the end. A sprint can last anywhere between one week and one month. This, combined with test driven development (TDD, writing tests for functionality before producing the functionality), produces a set of components matching a well-defined requirements document that can be combined in to the final product.

However, without results, methodologies are meaningless

6.2.2 Documentation

Documentation is an extremely important part of software engineering. It is a broad concept, and encompasses: requirements and specification, design overview, technical details, user manuals and even marketing information. However, in terms of the software development process, it is only the first three that are of direct importance.

It is said a program listing should be documentation unto itself: the programming style should allow an overall structure and purpose to be easily determined from examining the code [11]. However, this is an idealistic view and design and technical documentation are of great importance, not only for the developer currently producing the software, but for those possible developers in the future who have to maintain the code.

As a major component of this project is to develop a library of cryptographic algorithms, having a clear and accurate listing of all available methods in the API is vital. Conveniently, Java and the Eclipse IDE are tightly integrated with *JavaDoc*[1], a documentation generator that automatically produces standardised API documentation in HTML format based on comments inserted in the code. The comment blocks – distinguished using the format `/** ... */` – contain a method description, and a number of control sequences that give detailed information about what the method returns, the parameters it takes and other details such as exceptions thrown.

JavaDoc outputs are bundled with the appropriate packages in the file tree, and manual technical documentation is found in the appendices. A requirements specification has been compiled in *Appendix A*.

6.3 Application Development

6.3.1 User Interface

*Packages covered in this section: com.cyanoryx.uni.enigma.gui.**

[1] http://java.sun.com/j2se/javadoc

6.3.1.1 GUI Frameworks

6.4 Protocol Implementation

*Packages covered in this section: com.cyanoryx.uni.enigma.net.**

6.5 Algorithm Implementation

*Packages covered in this section: com.cyanoryx.uni.crypto.**

6.6 Usage

A user's instruction manual is included as *Appendix D*. This manual also briefly covers compilation, required dependencies, and other build related information.

6.7 Program Listing

Due to the size of the project a detailed, printed code listing is impossible, and thus it is recommended that the code be viewed using a text editor or other text environment on a device. If you do not have a digital copy of this project, please see *Section 1.4.2*.

Chapter 7

Cryptanalysis

7.1 Public-key Cryptography

7.1.1 RSA

A distinction must be made between the security of the RSA *algorithm* and the RSA *system*. This is known as semantic security [12], the use of other non-algorithmic techniques to resist attacks and make it, for example, difficult to recover information from the public key. As we discussed in Chapter 3, standards exist such as RSA-OAEP that introduce elements of randomness in to the RSA algorithm that limit certain basic attacks – this is the RSA algorithm implemented in to a cryptosystem. The RSA algorithm itself is distinct from this at is the core mathematics, rather than an implementation of a secure system.

In this section, we will discuss both attacks on the RSA *algorithm* and attacks against RSA *systems*, such as the one we have implemented.

7.1.1.1 Brute Force

The first attack we should mention is known as the *brute-force* attack: factorising the integer modulus N to determine the prime numbers used to generate the keys. This has been discussed throughout the paper, and due to the simplicity (to an extent) of the attack, we will not extensively cover it. It is interesting to note that there are no (publicly available) efficient algorithms with an acceptable running time for factoring large prime multiples. The current fastest algorithm is the *General Number Field Sieve*, which runs on $\exp\left(\left(\sqrt[3]{\frac{64}{9}} + o(1)\right)(\log n)^{\frac{1}{3}}(\log\log n)^{\frac{2}{3}}\right) = L_n\left[\frac{1}{3}, \sqrt[3]{\frac{64}{9}}\right]$ time, with an n-bit integer. See [13] for an interesting introduction to GNFS.

If an algorithm for factoring large integers in reasonable time periods is ever discovered, RSA will be rendered entirely insecure. However, until then we will only consider attacks that can decrypt RSA ciphertext *without* factoring the modulus N.

7.1.1.2 Elementary Attacks

Small Exponent

Private Exponent

Public Exponent It is common belief that using a small encryption exponent e does not adversely affect the capabilities of the RSA algorithm. For example, it is suggested $e = 3$.

This is best shown as an example. If Alice wishes to send a message m to three associates with the public moduli $n_{\{1,2,3\}}$ and exponent $e = 3$, she would send $c_i = m^3 \bmod n_i$. Through a eavesdropping, an attacker Mallory could use Gauss's Algorithm all three values of c to find:

$$\begin{cases} x \equiv c_1 \ mod \ n_1 \\ x \equiv c_2 \ mod \ n_2 \\ x \equiv c_3 \ mod \ n_3 \end{cases}$$

Using the Chinese Remainder Theorem, we can determine that $x = m^3$, and thus $m = \sqrt[3]{x}$. Also, if a message $m < n^{\frac{1}{e}}$ we can calculate the e^{th} root of $(c = m^e)$ to get the plaintext message.

It should be noted that this attack cannot be considered a "break" of the algorithm. Both these attacks can be mitigated through the use of salting – adding randomly generated bits to messages before encryption. Another simple, but not recommended, solution is to make $e \geq 2^{16} + 1$.

Due to the use of hashing and salting, this attack does not affect RSA cryptosystems such as RSA OAEP, the system implemented for Enigma.

Key Exposure

Coppersmith's Short Pad Attack

Hastad's Broadcast Attack

Related Message Attack

Forward Search Attack

Common Modulus Attack A previously common solution to managing the keys of multiple entities within one network was creating a central authority that would select a modulus N, and then share exponent pairs e_i and d_i with the entities. While this attack relies on the ability to factor prime multiples, given (e_i, d_i), an attacker could determine the decryption exponents for all other entities within the network.

Cycling Attack

Message Concealing

7.1.1.3 System and Implementation Attacks

Timing

Random Faults

Bleichenbacher's Attack

7.1.2 Certificates and Authentication

7.1.2.1 Implementation Attacks

SSL BEAST

Authority Security and Impersonation The underlying security of the certification system is based upon a trusted third party (TTP) that verifies the chain of trust – the certificate authority (CA) – and so "a chain is only as strong as its weakest link" appears to be quite apt. If, through social or technical means, an attacker can gain access to the private keys or construction information for the private keys of a CA, they will be able to issue certificates as they wish until the breach is noticed and the CA can be removed from the chain.

This is not a theoretical attack by any means. In July 2011, Mozilla – the developers of many open source products such as Firefox and Thunderbird – was informed that a fraudulent SSL certificate belonging to Google, Inc. had been issued by CA *DigiNotar*. As it happens, DigiNotar's network

had been breached allegedly without their knowledge, giving the attacker access to their private keys and thus allowing certificates to be issued arbitrarily. In this case, it was shown to be used by unknown entities in Iran to conduct a man-in-the-middle against Google services [14].

7.2 Symmetric Cryptography

This sections covers attacks affecting the AES symmetric cipher.

7.2.1 Brute-force

7.2.2 XSL Attack

7.2.3 Biryukov and Khovratovich

7.2.4 Related Key Attack

7.2.5 Known-key Distinguishing Attack

7.2.6 Bogdanov, Khovratovich, and Rechberger

7.2.7 Side-channel attacks

7.3 Hash Functions

7.3.1 Birthday Attacks

7.3.2 Collision and Compression

7.3.3 Chaining Attack

7.4 Emerging Threats

7.4.1 Quantum Cryptography and Cryptanalysis

Quantum computing is an emerging technology.

As well as affecting algorithms based around the intractability of integer factorisation, quantum cryptanalysis also affects algorithms that utilise the discrete logarithm problem such as ElGamal, Diffie-Hellman and DSA.

Quantum cryptography, in its current form, does not affect modern symmetric cryptographic algorithms – neither ciphers or hash functions. Grover's algorithm, an algorithm that improves the efficiency of searching an unsorted database, improves the speed at which a symmetric cipher key can be cracked, however this is preventable using standard counter-actions such as increasing the key size.

An interesting current field of study is post-quantum cryptography: algorithms designed such that a cryptanalyst with a powerful quantum computer (should they become prevalent, or even plausibly workable in the future) cannot easily break. See [15].

7.4.2 Ron was wrong, Whit is right

In early February 2012, a paper entitled "Ron was wrong, Whit was right" – a slight jab at RSA co-creator Ronald Rivest and nod towards cryptographer Whitfield Diffie – swept through the headlines of the cryptographic communities. Written by researchers at the *School of Computer and Communication Studies, École Polytechnique Fédérale De Lausanne.* The paper detailed a "sanity check" of a subsection of RSA public keys that can be found online, and analysed them to test the randomness of the inputs used to calculate the keys.

As we are now well aware, RSA is based entirely on the inability to efficiently factor prime numbers. However, there are other requirements for the good security of the algorithm. Namely, it is crucial that when the keys are generated, previous random numbers are not reused. [16] states that:

Given a study of 11.7 million public keys,

Conversely, the researchers were unable to find any of the common exponents, used in RSA public keys, being used for the other two major public-key algorithms: DSA and ElGamal. ECDSA was also investigated, however only one certificate was found to be using ECDSA.

7.4.2.1 Sony PlayStation 3

An unrelated, yet high-profile, realisation of the risks of poor entropy in random number generators is the cracking of Sony's PlayStation 3 console. Sony used public-key cryptography to sign its bootloaders and games, which prevents unauthorised software being executed on the device. The overall system used to protect the device consists of many different techniques and

algorithms, however it was the implementation of *Elliptic Curve Digital Signature Algorithm (ECDSA)* that was flawed. As we will show below, a poorly executed random number generator design led to the extraction of the private key stored securely within the device, previously implausible to retrieve.

ECDSA uses 11 parameters in its cryptographic algorithms: 9 public, and 2 private:

Public

$p, a, b, G, N =$ curve parameters

$Q =$ public key

$e =$ data hash

$R, S =$ signature

Private

$m =$ random number

$k =$ private key

The signature, R, S is computed:

$$R = (mG)_x$$

$$S = \frac{e+kR}{m}$$

Because of this, it is absolutely vital that a high-entropy random number generator be used to produce m, otherwise given two signatures with the same m, we are able to determine m *and* private key k:

$$\begin{array}{ll} R = (mG)_x & R = (mG)_x \\ S_1 = \frac{e_1+kR}{m} & S_2 = \frac{e_2+kR}{m} \end{array}$$

Rearranging:

$$S_1 - S_2 = \frac{e_1-e_2}{m}$$

$$m = \frac{e_1-e_2}{S_1-S_2}$$

$$\therefore k = \frac{e_1S_2-e_2S_1}{R(S_1-S_2)} = \frac{mS_n-e_n}{R}$$

Given k, we have now have the capability to sign arbitrary data, meaning: the chain of trust is broken, signed executables are no longer useful, encrypted storage can be accessed, and many other features of the security system rendered ineffective [17].

While the human-impact was virtually zero in terms of loss of life, injury, etc. this is an excellent example of how something seemingly easy to implement like an RNG can affect the overall security of an entire system and the impact it can have.

7.5 A comment on theory

It is interesting to note, based upon history, that in all likelihood these theoretical attacks are currently being implemented. Though

However, the topic of real-world security flaws and those that take advantage of them is extensive and could easily be covered by its own paper, if not several. It is left up to the reader's imagination to consider the many possible vulnerabilities currently being utilised unbeknownst to the vast majority of the community, and also to forget this idea lest they never use a networked device again.

Chapter 8

Outcomes and Further Research

8.1 Fulfilment of Specification

8.2 Testing

Various testing methodologies

8.2.1 Test Driven Development

8.2.2 Unit Testing

8.2.3 Code Coverage

8.2.4 Functional Testing

Functional testing is a subset of black box testing. However, it was decided in the case of Enigma that due to the low complexity of the software and use-flow,

8.3 Problems and Evaluation

8.4 Limitations

8.4.1 Standards Compliance

8.4.2 Hardware Implementation

8.5 Project Management

8.6 Summary

Appendix A

Enigma Application Specification

A.1 Introduction

A.1.1 Purpose

A.1.2 Scope

A.1.3 Definitions

A.1.4 Overview

A.2 Overall Description

A.2.1 Product Perspective

A.2.2 Product Functions

A.2.3 User Characteristics

A.2.4 Constraints

A.2.5 Assumptions and dependencies

Appendix B

Enigma Protocol Specification

B.1 Introduction

The Enigma Protocol is a communications protocol intended for use in Instant Messaging applications. It is very loosely based around the concept of *Jabber/XMPP*[1], primarily as it is an application of the Extensible Markup Language (XML) to allow for near-real-time communications between networked devices. The XMPP protocol is considerably more complex than Enigma, and the two should not be considered comparable.

This document covers the basic XML elements that must be implemented by an Enigma Protocol-based application. As its primary purpose is to provide a simple way of sending text with metadata, it should only be used for research purposes. The Enigma Application can be considered a reference implementation, however it is not a complete implementation.

B.2 Requirements

An IM application for use in the testing of sending encrypted messages should have the capability to:

1. Be able to exchange brief text messages in near-real-time.
2. Transmit information that can be used to securely generate and exchange encryption keys.
3. Transmit information that can be used to identify and authenticate.

[1] http://xmpp.org/about-xmpp/history/

And thus, any application implementing the Enigma Protocol must be able to adhere to the above requirements.

B.3 History

This document covers version 1.0 of the Enigma protocol. There are no prior implementations.

B.4 Terminology

No specialised terminology is used without explanation in this document.

B.5 Format

A conversation between two users should be considered as the building of a valid XML document using the `enigma:client` namespace. The document should consist of a valid root element occurring only once, valid child elements matching those listed in the document, and finish with a closing tag matching the root element. Each packet, that is not a root element, must consist of a start-tag and end-tag, or an empty-element tag otherwise it **should not** be parsed and **should** be dropped.

The order of the messages is important if the `time` attribute is not included, in which case they should be handled and displayed in a first-in-first-out order.

B.6 Commands

B.6.1 Connection

A connection is represented by a streaming XML document, with the root element being `<stream>`.

- **Opening a connection:**

```
<stream  to="SERVER_NAME"
         from="USER_NAME"
         id="SESSION_ID"
         return-port="LOCALHOST_INBOUND_PORT"
         xmlns="enigma:client">
```

- **Closing a connection:**

```
</stream>
```

B.6.2 Authentication

- **Toggling encryption:**

```
<auth stage="streaming"
      id="SESSION_ID"
      type="toggle">
      [off|on]
</auth>
```

Note: this requires the agreement of both users. The connection should be closed if one user disagrees to change the current encryption status.

- **Asserting the key agreement method:**

```
<auth stage="agreement"
      id="SESSION_ID"
      type="method">
      [method type identifier]
</auth>
```

- **Publishing a certificate:**

```
<auth stage="agreement"
      id="SESSION_ID"
      type="cert">
      [Base64 encoded Enigma Certificate]
</auth>
```

- **Publish an encrypted symmetric cipher key:**

```
<auth stage="agreement"
      id="SESSION_ID"
      type="key"
      [method="CIPHER_ALGORITHM"]>
      [Base64 encoded encrypted key]
</auth>
```

B.6.3 Messaging

- **Sending a message:**

```
<message  [to="REMOTE_USER_NAME"]
          from="USER_NAME"
          id="SESSION_ID"
          [type=""]>
    <body>MESSAGE_CONTENT</body>
</message>
```

B.6.4 Errors

Errors do not have a specific element themselves, but are included as a subelement of any other tag, setting `att:type` to `error`.

- **Sending an error:**

```
<element  [other attributes]
          type="error">
    <error type="ERROR_NUMBER">
        ERROR_MESSAGE
    </error>
    [element contents]
</element>
```

Appendix C

Licenced Software Usage

The Enigma application and associated utilities make use of several libraries and projects, all of which are used legally and in accordance with the matching software licence. The individual licence declarations are included within the files, or where appropriate. Below each library/project is listed, with their locations and other details.

1. **Base64 Encoder/Decoder**
   Author: Robert Harder
   Website: http://www.iharder.net/current/java/base64/
   Licence: None. Public domain release, full usage allowed.
   Location: *com.cyanoryx.uni.common.Base64*

2. **Cryptix Byte Utilities**
   Author: The Cryptix Foundation Ltd.
   Website: http://www.cryptix.org/
   Licence: Cryptix General Licence - http://www.cryptix.org/LICENSE.TXT
   Location: *com.cyanoryx.uni.common.Bytes*
   Notes: Most methods only used partially.

3. **Apache Xerces (XML parsing)**
   Author: Apache Software Foundation
   Website: http://xerces.apache.org/
   Licence: Apache Licence, v2.0 - See licence at *dep/APACHE-LICENSE-2.0.txt*
   Location: *dep/xerces.jar*

4. **Instant Messaging in Java**
   Author: Iain Shigeoka (Manning)
   Website: http://www.manning.com/shigeoka/

Licence: The Shigeoka Software Licence - See licence at *dep/SHIGEOKA-LICENCE.txt*
Location: *com.cyanoryx.uni.enigma.net.io.XercesReader.java*, plus occasional usage of code fragments and methods, which are noted within the code.

Appendix D

Enigma: User Manual

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