HMS 112 FORMULAS

STATIONARY POINTS ON A SURFACE: if a surface $z = f(x, y)$ has $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ zero at the point P $(a, b, f(a, b))$, then P is a stationary point, and

(i) the surface has a local maximum at P if $\frac{\partial^2 z}{\partial x^2} < 0$ and $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - (\frac{\partial^2 z}{\partial x \partial y})^2 > 0$;

(ii) the surface has a local minimum at P if $\frac{\partial^2 z}{\partial x^2} > 0$ and $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - (\frac{\partial^2 z}{\partial x \partial y})^2 > 0$;

(iii) the surface has a saddle point at P if $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - (\frac{\partial^2 z}{\partial x \partial y})^2 < 0$)

CHAIN RULES:

(i) if $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(ii) if $z = f(x, t)$ and $x = x(t)$ then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial t}$

(iii) if $z = f(x, y)$ and $y = y(x)$ then $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$

(iv) if $z = f(x, y)$ and $x = x(u, v)$ and $y = y(u, v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Complex Impedance in RLC Circuits

$$\omega = 2\pi f$$
$$V = V_0 \sin(\omega t + \phi)$$
$$I = I_0 \sin(\omega t)$$
$$Z = R + j(\omega L - \frac{1}{\omega C})$$
$$Z = |Z| cis\phi$$
$$I_0 = \frac{V_0}{|Z|}$$

LINEAR HOMOGENEOUS 2ND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

i.e. $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$ (1)

IF THE ROOTS OF THE CHARACTERISTIC (AUXILIARY) EQUATION $am^2+bm+c=0$ ARE	THEN THE SOLUTION OF (1) IS
2 real and different roots m_1 and m_2	$y=Ae^{m_1x}+Be^{m_2x}$
2 real and equal roots (i.e. $m=m_1=m_2$) (also called a double root)	$y=e^{m_1x}(A+Bx)$
2 conjugate complex roots m_1 and m_2, where $m_1=\alpha+j\beta$ and $m_2=\alpha-j\beta$, (where $j=\sqrt{-1}$)	$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

LINEAR NONHOMOGENEOUS 2ND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

i.e. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where $f(x) \neq 0$ (2)

The solution of (2) is $y = y_h + y_p$, where $y_h\ (\equiv y_c)$ (the COMPLEMENTARY FUNCTION or C.F.) is the GENERAL SOLUTION (i.e. it contains 2 arbitrary constants) of the corresponding homogeneous equation, i.e.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

and y_p, the PARTICULAR INTEGRAL, can be found by the table below:

IF $f(x)$ HAS A TERM A CONSTANT MULTIPLE OF	AND IF	THEN y_p IS
e^{px}	p N.R.A.E (i.e. p not a root of auxiliary equation)	Ce^{px}
	p S.R.A.E (i.e. p a single root of the auxiliary equation)	Cxe^{px}
	p D.R.A.E (i.e. p a double root of the auxiliary equation)	Cx^2e^{px}
$\sin qx$ or $\cos qx$	jq N.R.A.E (where $j = \sqrt{-1}$)	$C\cos qx + D\sin qx$
	jq S.R.A.E	$x(C\cos qx + D\sin qx)$
$a_0 + a_1x + ... + a_nx^n$	0 N.R.A.E.	$b_0 + b_1x + ... + b_nx^n$
	0 S.R.A.E.	$x(b_0 + b_1x + ... + b_nx^n)$
	0 D.R.A.E.	$x^2(b_0 + b_1x + ... + b_nx^n)$

Note: the constants in y_p can always be found.

Derivatives

FUNCTION	DERIVATIVE
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Integrals

FUNCTION	INTEGRAL
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}+c$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$
$\sinh kx$	$\dfrac{\cosh kx}{k}+c$
$\cosh kx$	$\dfrac{\sinh kx}{k}+c$
$\operatorname{sech}^2 kx$	$\dfrac{\tanh kx}{k}+c$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\dfrac{x}{a}+c$ or $\ln(x+\sqrt{x^2+a^2})+c$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\dfrac{x}{a}+c$ or $\ln(x+\sqrt{x^2-a^2})+c$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}+c$ or $\dfrac{1}{2a}\ln\dfrac{a+x}{a-x}+c$
$\dfrac{1}{x^2-a^2}$	$-\dfrac{1}{a}\coth^{-1}\dfrac{x}{a}+c$ or $\dfrac{1}{2a}\ln\dfrac{x-a}{x+a}+c$
$e^{\alpha x}\sin\beta x$	$\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x-\beta\cos\beta x)$ $+c$
$e^{\alpha x}\cos\beta x$	$\dfrac{e^{\alpha x}}{\alpha^2+\beta^2}(\alpha\cos\beta x+\beta\sin\beta x)$ +c

INTEGRATION BY PARTS: $\int u\dfrac{dv}{dx}dx = uv-\int v\dfrac{du}{dx}dx$

SUBSTITUTION RULE: $\int f(u)\dfrac{du}{dx}\,dx=\int f(u)du$ or $\int f(x)dx=\int f(x)\dfrac{dx}{d\theta}d\theta$

RADIUS OF CURVATURE: $\dfrac{\left(1+(y')^2\right)^{3/2}}{y''}$

CURVATURE:

$$\kappa = \frac{|\underset{\sim}{v} \times \underset{\sim}{a}|}{|\underset{\sim}{v}|^3}$$

CENTRE OF CURVATURE:

$$h = x_1 - R\sin\theta$$
$$k = y_1 + R\cos\theta$$

EULER SPIRAL TRANSITION CURVE:

$\theta = \dfrac{s^2}{2rL}$ (θ is in radians).

$$x = s - \frac{s^5}{40(rL)^2}$$

$$y = \frac{s^3}{6rL} - \frac{s^7}{336(rL)^3}$$

PLANES:

For a plane $ax + by + cz = d$

Perpendicular distance from the origin to the plane : $p = \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

L, the distance between parallel planes

$$ax + by + cz = d_1$$

and

$$ax + by + cz = d_2$$

is given by

$$L = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$