Fast Fourier Transform on $\mathsf{SO}(3)$

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I. Harmonic Analysis on $\mathsf{SO}(3)$

A. Representation on a Lie group

Let $f \in \mathcal{L}^2(\mathbb{R}^n)$. The (left) representation $U(g)$ of G is $\mathsf{GL}(\mathcal{L}^2(\mathbb{R}^n))$, i.e., the set of linear transformation on $\mathcal{L}^2(\mathbb{R}^n)$ defined such that

$$(U(g)f)(x) = f(g^{-1}z). \tag{1}$$

It is straightforward to show the representation is a homomorphism,

$$(U(g_1)U(g_2)f)(z) = f(g_2^{-1}g_1^{-1}x) = (U(g_1g_2)f)(z).$$

By selecting a basis for the invariance subspaces of $\mathcal{L}^2(\mathbb{R}^n)$, $U(g)$ can be represented by a matrix, which is called a matrix representation of G.

Two matrix representations are *equivalent*, if one can obtained by a similarity transform of the other. Any representation can be transformed into a *unitary* representation by a similarity transform so that $U(g)U^*(g) = I$, or $U(g^{-1}) = U^*(g)$. A matrix representation is *reducible*, if it is equivalent to the direct sum of others, or equivalently, it can be block-diagonalized.

B. Irreducible Unitary Representation on $\mathsf{SO}(3)$

Since there is two-to-one homomorphism from $\mathsf{SU}(2)$ to $\mathsf{SO}(3)$, the representation of $\mathsf{SO}(3)$ should be a subset of those for $\mathsf{SU}(2)$, which is a set of linear transformation on $\mathcal{L}^2(\mathbb{C}^2)$. The set of homogeneous polynomials is a basis of $\mathcal{L}^2(\mathbb{C}^2)$, and one can find the matrix representation of $\mathsf{SU}(2)$ with (1), which results in generalizations of the associated Legendre function [1] that is also shown to be unitary, and irreducible. An irreducible unitary representation of $\mathsf{SO}(3)$ is constructed by taking the terms with integer indices.

The wigner-D function is another irreducible unitary representation on $\mathsf{SU}(2)$ (or $\mathsf{SO}(3)$) that is equivalent to the above represented based on the generalized associated Legendre function. It is given by a matrix $D^l_{m,n}(R)$, where the index l is a non-negative integer, and the integer indices m, n vary in $-l \leq m, n \leq l$. As such $D^l(R) \in \mathbb{C}^{2l+1,2l+1}$. More specifically, let $(\alpha,\beta,\gamma) \in [0,2\pi]\times[0,\pi]\times[0,2\pi]$ be the 3-2-3 Euler angles, i.e.,

$$R(\alpha,\beta,\gamma) = \exp(\alpha\hat{e}_3)\exp(\beta\hat{e}_2)\exp(\gamma\hat{e}_3).$$

Then, $D^l_{m,n}(R(\alpha,\beta,\gamma)$ is given by

$$D^l_{m,n}(R(\alpha,\beta,\gamma)) = e^{-im\alpha} d^l_{m,n}(R(\alpha,\beta,\gamma))e^{-in\gamma},$$

where $d^l_{m,n}(R)$ is a real-valued wigner-d function. A recursive formulation to evaluate the wigner-d function is presented in [2].

While this formulation is based on the particular 3-2-3 Euler angles, 3-1-3 Euler angles can be used without any modification, as it will be equivalent anyway [1]. Another nice feature is that $D^l_{m,n}$ is given by a product of three terms, which depend solely on each of three Euler angles. This follows directly from the use of a Gel'fand-Tsetlin basis, i.e., a basis that respects the decomposition of the restriction to the subgroup $\mathsf{SO}(2)$ [3]. Alternatively, the wigner-D matrix is formulated in terms of quaternions and rotation matrices in [4].

C. Fourier Transform on $\mathsf{SO}(3)$

We define an inner product on $\mathcal{L}^2(\mathsf{SO}(3))$ as

$$\langle f(R), g(R)\rangle = \int_{\mathsf{SO}(3)} f(R)g^*(R)dR.$$

For the 3-2-3 Euler angles, the Haar measure is written as $dR = \frac{1}{8\pi^2}\sin\beta d\alpha d\beta d\gamma$ that is normalized such that $\int_{SO} dR = 1$.

Due to the Peter-Weyl theorem, the collection of the irreducible unitary representations on $\mathsf{SO}(3)$ form a complete orthogonal basis for $\mathcal{L}^2(\mathsf{SO}(3))$. Specifically

$$\langle U^{l_1}_{m_1,n_1}(R), U^{l_2}_{m_2,n_2}(R)\rangle = \frac{1}{2l_1+1}\delta_{l_1,l_2}\delta_{m_1,m_2}\delta_{n_1,n_2}.$$

This also follows

$$\int_0^\pi d^{l_1}_{m_1,n_1}(\cos\beta)d^{l_2}_{m_2,n_2}(\cos\beta)\sin\beta d\beta = \frac{2}{2l_1+1}\delta_{l_1,l_2}\delta_{m_1,m_2}\delta_{n_1,n_2}.$$

Consequently, any $f \in \mathcal{L}^2(\mathsf{SO}(3))$ has the following decomposition

$$\begin{aligned} f(R(\alpha,\beta,\gamma)) &= \sum_{l=0}^\infty \sum_{m,n=-l}^{l} (2l+1)\bar{f}^l_{m,n} D^l_{m,n}(\alpha,\beta,\gamma) \\ &= \sum_{l=0}^\infty (2l+1)\mathrm{tr}[(\bar{f}^l_{m,n})^T D^l_{m,n}(\alpha,\beta,\gamma)], \end{aligned} \tag{2}$$

where $\bar{f}^l_{m,n} \in \mathbb{C}^{(2l+1)\times(2l+1)}$ is obtained by

$$\begin{aligned} &\bar{f}^l_{m,n} = \langle f(R), D^l_{m,n}(R)\rangle \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^\pi\int_0^{2\pi} f(R(\alpha,\beta,\gamma))D^{*l}_{m,n}(R(\alpha,\beta,\gamma))d\alpha d\beta d\gamma \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^\pi\int_0^{2\pi} f(R(\alpha,\beta,\gamma))D^{l}_{m,n}(R^T(\alpha,\beta,\gamma))d\alpha d\beta d\gamma \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^\pi\int_0^{2\pi} f(R(\alpha,\beta,\gamma))D^{l}_{m,n}(R(-\gamma,-\beta,-\alpha))d\alpha d\beta d\gamma \end{aligned}$$

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Several variations of the above definition exists. For example, in [5], $D^l_{m,n}$ is normalized such that the factor $2l+1$ does not appear.

References

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