NoteBook developed in Moscow in June 2017 on the basis of old NoteBooks written in 2005 and upgraded in 2012

The $F_{(4,4)}$ Lie algebra, its maximal compact subalgebra usp(6) ⊗ su(2) and the non-compact subalgebra sp(2,ℝ)×sp(6,ℝ)

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Preface

This NoteBook is essentially a reading file. It reads from the repository included in the NoteBook all structures that have been separately calculated and stored relative to the quaternionic manifold

$$\frac{F_{(4,4)}}{\mathrm{Usp}(6)\times \mathrm{SU}(2)} \supset \frac{\mathrm{Sp}(6,\,\mathbb{R})}{\mathrm{SU}(3)\times U(1)} \tag{1}$$

The names of all the loaded structures are summarized below. Evaluating this Notebook the user obtains all the items pertaining to this algebra and also more information concerinig the quaternionic coset mentioned above.

Preliminary Set up

Repository of files to be loaded

input from previous calculations and definitions of needed new items

Executing the loading

```
laudando
```

```
=======================================
Now I load all the needed items
I finished loading
{Null}
```

Theory and Instructions for the User

Description of commands and available objects

- **The F4 Lie algebra**

1) The file containing the roots of F4 is named **rutteF4**
2) The file containing the Cartan operators of F4 is named **Carti**
3) The file containing the step operators of positive roots of F4 (in the same order as the root is rutteF4) is named **stepsi**
4) The file containing the step operators of negative roots of F4 (in the same order as the root is rutteF4) is named **antistepsi**

All the operators are in the fundamental 26 ×26 dimensional representation of $F_{(4,4)}$ the roots of the Lie algebra are of the form:

$$
\text{roots} = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_1+\alpha_2 \\
\alpha_2+\alpha_3 \\
\alpha_3+\alpha_4 \\
\alpha_1+\alpha_2+\alpha_3 \\
\alpha_2+2\,\alpha_3 \\
\alpha_2+\alpha_3+\alpha_4 \\
\alpha_1+\alpha_2+2\,\alpha_3 \\
\alpha_1+\alpha_2+\alpha_3+\alpha_4 \\
\alpha_2+2\,\alpha_3+\alpha_4 \\
\alpha_1+2\,\alpha_2+2\,\alpha_3 \\
\alpha_1+\alpha_2+2\,\alpha_3+\alpha_4 \\
\alpha_2+2\,\alpha_3+2\,\alpha_4 \\
\alpha_1+2\,\alpha_2+2\,\alpha_3+\alpha_4 \\
\alpha_1+\alpha_2+2\,\alpha_3+2\,\alpha_4 \\
\alpha_1+2\,\alpha_2+3\,\alpha_3+\alpha_4 \\
\alpha_1+2\,\alpha_2+2\,\alpha_3+2\,\alpha_4 \\
\alpha_1+2\,\alpha_2+3\,\alpha_3+2\,\alpha_4 \\
\alpha_1+2\,\alpha_2+4\,\alpha_3+2\,\alpha_4 \\
\alpha_1+3\,\alpha_2+4\,\alpha_3+2\,\alpha_4 \\
2\,\alpha_1+3\,\alpha_2+4\,\alpha_3+2\,\alpha_4
\end{pmatrix} = \begin{pmatrix}
-1 & -1 & -1 & 1 \\
0 & 0 & 2 & 0 \\
0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 \\
-1 & -1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 1 & -1 & 1 \\
0 & -1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
-1 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 \\
2 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & -1 & -1 & 1 \\
0 & 1 & 0 & 1 \\
1 & -1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}
$$

■ Maximal compact subalgebra su(2) × sp(6,R) ⊂ $F_{(4,4)}$

The generators of the maximal compact subalgebra are 24 and are of the form:

H_i = E[α_i] - E[-α_i]

■ The algebra su(2)

The su(2) part of the **H subalgebra is generated by**

$$
\{J_x,\ J_y,\ J_x\} = \left\{\frac{H_1 - H_{14} + H_{20} - H_{22}}{4\sqrt{2}},\ \frac{H_5 + H_{11} - H_{18} + H_{23}}{4\sqrt{2}},\ -\frac{H_2 - H_9 + H_{16} + H_{24}}{4\sqrt{2}}\right\} \tag{3}
$$

and it is contained in the following files

1) Formal expression in **formSO3gen**

2) Explicit matrix representation **SO3gen**

All generators given in the 26 × 26 fundamental representation

■ The algebra USp(6)

The other factor of the compact subalgebra is generated by 21 one generators encoded in the following files

1) Formal expression in **formUsp6gen** =

$$\begin{pmatrix} -\frac{H_2}{2}-\frac{H_9}{2}+\frac{H_{16}}{2}-\frac{H_{24}}{2} \\ -\frac{H_2}{2}+\frac{H_9}{2}+\frac{H_{16}}{2}+\frac{H_{24}}{2} \\ \frac{H_2}{2}+\frac{H_9}{2}+\frac{H_{16}}{2}-\frac{H_{24}}{2} \\ H_{10} \\ H_7 \\ H_4 \\ -H_{13} \\ H_6 \\ -H_3 \\ -H_1+H_{14}+H_{20}-H_{22} \\ -H_5-H_{11}-H_{18}+H_{23} \\ H_{21} \\ -H_8 \\ H_1+H_{14}+H_{20}+H_{22} \\ H_5-H_{11}-H_{18}-H_{23} \\ -H_1-H_{14}+H_{20}+H_{22} \\ H_5-H_{11}+H_{18}+H_{23} \\ H_{17} \\ H_{15} \\ H_{12} \\ H_{19} \end{pmatrix}$$

2) Explicit matrix representation **Usp6gen** (26 ×26)

- **The Sl(2,R) × Sp(6,R) subalgebra and the W-representation**

- **The algebra sl(2,R)**

The sl(2,R) **subalgebra is generated by**

$$\{\; L_+ \;,\; L_- \;,\; L_0 \;\} \;=\; \left\{ \frac{\mathbf{E}^{\alpha_{24}}}{\sqrt{2}},\; \frac{\mathbf{E}^{-\alpha_{24}}}{\sqrt{2}},\; \frac{\mathcal{H}_4}{2} \right\}$$

and it is contained in the following files

1) Formal expression in **LLp**

2) Explicit matrix representation **LLm**

3) Explicit matrix representation **LL0**

All generators given in the 26 × 26 fundamental representation

- **The Lie algebra sp(6,R)**

1) The generators of the Sp(6,R) Lie algebra are in the file **Sp6RgenF4**

2) Their formal expression in terms of F4 generators is contained in the file **formSp6RgenF4** =

$$\begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ E\alpha[4] \\ E\alpha[3] \\ E\alpha[2] \\ E\alpha[7] \\ -E\alpha[6] \\ E\alpha[10] \\ -E\alpha[9] \\ E\alpha[13] \\ E\alpha[16] \\ E\alpha[-4] \\ E\alpha[-3] \\ E\alpha[-2] \\ E\alpha[-7] \\ -E\alpha[-6] \\ E\alpha[-10] \\ -E\alpha[-9] \\ E\alpha[-13] \\ E\alpha[-16] \end{pmatrix}$$

4) The W-generators transforming in the 14 of Sp(6,R) are encoded in the file **Wgen**

5) The formal expression of the W-generators in terms of the F4 generators are encoded in the file **formWgen=** $\begin{pmatrix} E\alpha[5] \\ E\alpha[20] \\ E\alpha[14] \\ -E\alpha[23] \\ E\alpha[21] \\ E\alpha[19] \\ -E\alpha[17] \\ -E\alpha[22] \\ -E\alpha[11] \\ -E\alpha[18] \\ -E\alpha[1] \\ -E\alpha[8] \\ -E\alpha[12] \\ -E\alpha[15] \end{pmatrix}$

6) The matrices of the 14 dimensional representation of Sp(6,R) are encoded in the file **D14 . The generators are ordered in the following way: first the 3 Cartan, then the 9 stepup operator finally the 9 stepdown operators associated with the negative roots of sp(6)**

7) The roots of Sp(6,R) are encoded in the file **ruttasp6 =** $\begin{pmatrix} \alpha_1 & \{1, -1, 0\} \\ \alpha_2 & \{0, 1, -1\} \\ \alpha_3 & \{0, 0, 2\} \\ \alpha_1+\alpha_2 & \{1, 0, -1\} \\ \alpha_2+\alpha_3 & \{0, 1, 1\} \\ \alpha_1+\alpha_2+\alpha_3 & \{1, 0, 1\} \\ 2\,\alpha_2+\alpha_3 & \{0, 2, 0\} \\ \alpha_1+2\,\alpha_2+\alpha_3 & \{1, 1, 0\} \\ 2\,\alpha_1+2\,\alpha_2+\alpha_3 & \{2, 0, 0\} \end{pmatrix}$

7) The invariant symplectic matrix of Sp(6,R) corresponding to the 14 representation is encoded in the file **C14**

- **Special geometry**

1) The holomorphic section of special geometry is named **Hol**
2) The complex conjugate of holomorphic section of special geometry is named **Holb**
3) The Kahler potential of special geometry is named **KK**
4) The coset representative of $\frac{\mathrm{Sp}(6,R)}{\mathrm{SU}(3)\times U(1)}$ in the 6-dimensional representation is named **Lcoset6**
5) The coset representative of $\frac{\mathrm{Sp}(6,R)}{\mathrm{SU}(3)\times U(1)}$ in the 14-dimensional representation is named **Lcoset14**
6) The real and imaginary parts of the matrix Z providing the projective parametrization of the coset are given by **ReZ and ImZ (they are expressed in terms of the solvable parameters h and p)**
7) The transformation from complex coordinates to real ones is performed by the rule **toreal**
8) The localization on the origin of the manifold is performed by the command **toorigin**

- **Comment on the Inputs**

In this way the file Usp6gen contains the generators of Usp(6) arranged in the following way. The first 3 are the Cartan, the second 18 are the step operators associated wityh the 9 roots. They are written one after the other in pairs such that T_{2a-1} and T_{2a} are rotated one into the other by the Cartans.

Construction of the F4 coset Manifold

Here we recall in text form (without calculating it) the expression for the coset representative of F4 in solvable coordinates.

This is done in order to know how the coset representative LLF4 is organized

L = Exp[a L_+] ▪ Exp[$\sum_{i=1}^{7}$ $Z_{2*i-1} * W_{2*i-1}$] ▪ Exp [$\sum_{i=1}^{7}$ $Z_{2*i} * W_{2*i}$] ▪ $\prod_{i=1}^{9}$ Exp[$p_i\, E^{\beta_i}$]▪ $\prod_{i=1}^{3}$ Exp[$h_i \mathcal{H}^i$]▪Exp[U L_0]

The coset representative is encoded in the file **LLF4**. Its inverse is encoded in the file **ILLF4**

- **routine cosetf4construo**
- **The coordinates and the differentials are encoded in the file coordi:**

coordi={U, h_i, p_j, a, Z^α}

diffe={dU, dh$_i$, dp$_j$, da, dZ$^\alpha$}

Calculation of Momentum Maps and Potentials

```
maurocartano := {Print["Now I construct the Maurer Cartan form"];
   MCF = 0;
   Do[{Print[" y = ", y]; MCF = MCF + diffe[[y]] * (ILLF4.(∂coordi[[y]] LLF4));}, {y, 1, 28}];
   Print["I have finished calculating"];};
coordi
```

```
{U, h1, h2, h3, p1, p2, p3, p4, p5, p6, p7, p8, p9,
 a, Z1, Z2, Z3, Z4, Z5, Z6, Z7, Z8, Z9, Z10, Z11, Z12, Z13, Z14}
```

Exercises with the momentum map

- **The compact O(2) momentum map**

```
GG = LLp - LLm;
```

Expansions of the coset representatives

- **Preparations for moment map calculations**
- **Routine for the calculation of the momentum map for some parabolic translation + GG**
- **routine inspecto**