Moscow, Home May. 14th 2016

This is the NoteBook QuotL168.nb.

The simple group $L_{168} \simeq \mathrm{PSL}(2, Z_7)$ and its maximal subgroups.

This Notebook is a background programme. It has to be evaluated. Then its routines can be used in a separate Execution Notebook. It is recommended to read chapters one and two that contain all the explanations about the introduced mathematical items and about the available commands and routines. It is advisable that before EVALUATING this NoteBook you QUIT KERNEL.

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1 Theory and items

WARNING

In this programme the symbols ϵ, ρ,σ,τ, $\mathcal{Y}$,$\mathcal{X}$ are protected and you cannot use them. You can only make substitutions

Description of the items and of their names

The group L168 ≃ PSL(2,Z_7)

Generators

The group L168 has three generators S,T,R with the following relations:

$$R^2 = S^3 = T^7 = \mathrm{RST} = \mathrm{TSR}^4 = 1$$

Abstractly we use the Greek letter for the generators of L168 and we set:

$$\rho^2 = \sigma^3 = \tau^7 = \rho\sigma\tau = (\tau\sigma\rho)^4 = 1$$

In the programme there are two 7-dimensional realizations of these generators:

1. **R,S,T** are 7×7 matrices in the orthonormal basis
2. **RL,SL,TL** are 7×7 matrices in the basis of A7 simple roots where the metric is the A7 Cartan matrix (ROOT LATTICE)
3. **RW,SW,TW** are 7×7 matrices in the basis of A7 simple weights where the metric is the A7 Cartan matrix (WEIGHT LATTICE)

Conjugacy classes

The group L168 has 6 conjugacy classes organized in the following way

Conjugacy Class	C_1	C_2	C_3	C_4	C_5	C_6
representative of the class	e	R	S	TRS	T	SR
order of the elements in the class	1	2	3	4	7	7
number of elements in the class	1	21	56	42	24	24

The abstract form of the group elements as **words** in the generator symbols **ρ,σ,τ** is given in the file **formL168clas** which has the following appearance:

$$\text{formL168clas[[2]]} = \{\mathcal{E},\ 1,\ \{\epsilon\}\} \tag{1.1}$$

$$\text{formL168clas[[2]]} = \{\mathcal{R}, 21, \{\rho.\sigma.\rho.\sigma.\rho.\tau.\tau,\ \tau.\tau.\sigma.\rho.\tau.\tau.\tau,\ \sigma.\rho.\sigma.\sigma,\ \sigma.\rho.\tau.\tau.\sigma.\rho.\tau.\tau,\ \sigma.\rho.\tau.\sigma.\rho.\tau,\ \tau.\sigma,\ \rho.\sigma.\rho.\tau.\sigma.\rho.\tau,\ \tau.\tau.\sigma.\rho.\tau.\sigma.\sigma,\ \rho.\tau.\tau.\sigma.\rho.\sigma.\rho,\ \tau.\sigma.\rho.\tau.\sigma,\ \sigma.\rho.\sigma.\rho.\tau.\sigma.\sigma,\ \rho.\tau.\tau.\sigma.\rho.\tau.\tau,\ \rho.\sigma.\rho.\tau,\ \rho.\tau.\sigma.\rho.\tau.\sigma,\ \tau.\tau.\sigma.\rho.\tau.\tau.\sigma.\rho,\ \tau.\tau.\tau.\sigma.\rho.\tau.\tau,\ \tau.\tau.\sigma.\rho.\sigma,\ \rho.\tau.\sigma.\rho,\ \tau.\sigma.\rho.\tau.\tau.\sigma.\rho.\tau,\ \tau.\sigma.\rho.\sigma.\rho.\tau.\tau.\sigma,\ \rho\}\} \tag{1.2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{1.3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{1.4}$$

$$\text{formL168clas[[6]]} = \{\mathcal{SR}, 24, \{\rho.\sigma,\ \rho.\tau.\sigma.\rho.\sigma.\rho.\tau.\tau,\ \rho.\sigma.\rho.\tau.\tau.\sigma.\rho,\ \rho.\sigma.\rho.\sigma,\ \sigma.\rho.\tau.\sigma.\rho.\sigma.\rho,\ \rho.\tau.\tau.\sigma.\rho.\sigma.\rho.\tau,\ \tau.\sigma.\rho.\sigma.\sigma,\ \tau.\tau.\sigma.\rho.\tau.\sigma,\ \rho.\sigma.\rho.\tau.\tau.\tau,\ \tau.\sigma.\sigma,\ \tau.\tau.\tau,\ \rho.\sigma.\rho.\sigma.\rho.\tau.\sigma,\ \sigma.\rho.\tau.\tau.\sigma,\ \rho.\tau.\sigma.\rho.\tau,\ \sigma.\rho.\tau.\tau.\sigma.\sigma,\ \rho.\tau.\tau.\tau.\sigma.\rho,\ \rho.\sigma.\rho.\tau.\sigma.\rho.\sigma,\ \tau.\tau.\tau.\sigma,\ \rho.\tau.\tau,\ \tau.\sigma.\rho.\sigma.\rho.\tau.\sigma.\sigma,\ \tau.\sigma.\rho.\tau.\tau.\tau.\sigma,\ \rho.\tau.\tau.\sigma.\sigma,\ \sigma.\rho.\sigma.\rho,\ \sigma.\rho\}\} \tag{1.5}$$

This file is very important in the construction of irreducible representations in order to provide in each case the precise form of the representation of each invidual group element.

In the programme, after evaluation of this notebook the explicit matrices of the group organized by conjugacy classes are stored in two different files:

1. **L168strutclas** contains the 7×7 matrices in the simple root basis, namely those acting on the configuration space of the root lattice.
2. **L168clasW** contains the 7×7 matrices in the simple root basis, namely those acting on the configuration space of the root lattice.

The standard realization of the 7 simple roots as vectors in 8-dimensions orthogonal to the vector {1,1,1,1,1,1,1,1} is encoded in the object **alp** while the simple weights are encoded in the object **lammi**

finally the Cartarn matrix is encoded in the object CC.

The set of all the group elements is provided in two files:

1. **grupL168** contains the 168 matrices 7×7 in the configuration space (root lattice)
2. **L168W** contains the 168 matrices 7×7 in momentum space (weight lattice)

Characters

The 6 irriducible representations of the group L168 are of dimensions **1,6,7,8,3 and 3**, respectively. The character table is the following one:

0	e	R	S	TRS	T	SR
D_1	1	1	1	1	1	1
D_6	6	2	0	0	-1	-1
D_7	7	-1	1	-1	0	0
D_8	8	0	-1	0	1	1
DA_3	3	-1	0	1	$\frac{1}{2}\left(-1+i\sqrt{7}\right)$	$\frac{1}{2}\left(-1-i\sqrt{7}\right)$
DB_3	3	-1	0	1	$\frac{1}{2}\left(-1-i\sqrt{7}\right)$	$\frac{1}{2}\left(-1+i\sqrt{7}\right)$

The character table is encoded in the object **PchiL168**. The populations of conjugacy classes are encoded in the object **PgiL168**. The names of the representations are encoded in the object **namesD168**.

Embedding into G2 and τ-matrices

The embedding of the group L168 into G2 is demonstrated by showing that there is a G2-invariant 3-tensor that is invariant under L168. The tensor which satisfies the G2 - relations and which is L168 invariant is named ϕ in the orthonormal basis and φ in the root basis. The basis of 8×8 gamma matrices in 7-dimensions such that the G2 invariant tensor coincides with $\phi_{ijk} = \eta\, \tau_{ijk}\, \eta$ are constructed and encoded in the file $\tau\tau$

The maximal subgroups G21 and O24A, O24B

The simple group L168 contains maximal subgroups only of index 8 and 7, namely of order 21 and 24. The order 21 subgroup G21 is the unique non-abelian group of that order and abstractly it has the structure of the semidirect product $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$. Up to conjugation there is only one subgroup 21 as we explicitly verified with the computer. On the other hand, up to conjugation there are two different groups of order 24 that are both isomorphic to the octahedral group O_{24}.

They are named O24A, O24B.

The group G21

The group G21 has two generators $\mathcal{X}$ and $\mathcal{Y}$ that satisfy the following relations:

$$\mathcal{X}^3 = \mathcal{Y}^7 = 1\ ;\ \mathcal{X}\mathcal{Y} = \mathcal{Y}^2\mathcal{X}$$

Conjugacy classes of G21

Conjugacy Class	C_1	C_2	C_3	C_4	C_5
representative of the class	e	$\mathcal{Y}$	$\mathcal{X}^2\,\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
order of the elements in the class	1	7	7	3	3
number of elements in the class	1	3	3	7	7

Characters

The 5 irriducible representations of the group G21 are of dimensions **1,1,1,3, and 3**, respectively. The character table is the following one:

0	e	$\mathcal{Y}$	$\mathcal{X}^2\,\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
D_1	1	1	1	1	1
DX_1	1	1	1	$-(-1)^{1/3}$	$(-1)^{2/3}$
DY_1	1	1	1	$(-1)^{2/3}$	$-(-1)^{1/3}$
DA_3	3	$\frac{1}{2}\,i\left(i+\sqrt{7}\right)$	$-\frac{1}{2}\,i\left(-i+\sqrt{7}\right)$	0	0
DB_3	3	$-\frac{1}{2}\,i\left(-i+\sqrt{7}\right)$	$\frac{1}{2}\,i\left(i+\sqrt{7}\right)$	0	0

The character table is encoded in the object **Pchi21**. The populations of conjugacy classes are encoded in the object **Pgi21**. The names of the representations are encoded in the object **names21**.

Our choice of the representative for the entire conjugacy class of G21 maximal subgroups is given, abstractly, by setting the following generators:

$$\mathcal{Y} = \rho.\tau.\tau.\tau.\sigma.\rho \quad ; \quad \mathcal{X} = \sigma.\rho.\sigma.\rho.\tau.\tau$$

In the programme the entire group G21 is encoded in the following files.

1. **G21W** contains all the 21 elements as 7×7 matrices in the weight basis
2. **G21clasW** contains all the conjugacy classes in the weight basis (momentum space)
3. **G21clasR** contains all the conjugacy classes in the root basis (configuration space)
4. **FormalG21clas** contains the group generators written symbolically as words in ρ,σ,τ organized in conjugacy classes with the mention of the choice of $\mathcal{Y},\mathcal{X}$ generators:

```
FormalG21clas[[1]] = {𝒴, 𝒳};                                                    (1.6)
```

```
FormalG21clas[[2]] =
{{1, {ϵ}}, {3, {ρ.σ.ρ.σ.ρ.τ.σ, ρ.τ.τ.τ.σ.ρ, τ.σ.ρ.σ.ρ.τ.σ.σ}},
 {3, {σ.ρ.σ.ρ.τ.τ.σ, ρ.σ.ρ.τ.τ.σ.σ, τ.σ.ρ.τ.τ}},                                   (1.7)
 {7, {σ, ρ.σ.ρ.τ.τ, σ.ρ.σ.ρ.τ.τ.σ.σ, ρ.σ.ρ.σ.ρ.τ.σ.σ, ρ.τ.τ.τ.σ.ρ.σ,
   τ.σ.ρ.σ.ρ.τ, τ.σ.ρ.τ.τ.σ}}, {7, {ρ.σ.ρ.τ.τ.σ, ρ.σ.ρ.σ.ρ.τ,
   σ.ρ.σ.ρ.τ.τ, τ.σ.ρ.τ.τ.σ.σ, ρ.τ.τ.τ.σ.ρ.σ.σ, σ.σ, τ.σ.ρ.σ.ρ.τ.σ}}}
```

```
FormG21YX = {{1, {ϵ}}, {3, {𝒴.𝒴, 𝒴, 𝒳.𝒳.𝒴.𝒳}}, {3, {𝒴.𝒴.𝒴, 𝒳.𝒴.𝒳.𝒴.𝒳, 𝒴.𝒳.𝒳.𝒴.𝒳}},
 {7, {𝒴.𝒳.𝒳.𝒴, 𝒳.𝒳.𝒴, 𝒴.𝒳.𝒳, 𝒳.𝒳, 𝒳.𝒴.𝒳.𝒴, 𝒳.𝒴.𝒳, 𝒴.𝒳.𝒴.𝒳}},              (1.8)
 {7, {𝒴.𝒳.𝒴, 𝒳.𝒴.𝒴.𝒴, 𝒳, 𝒳.𝒴, 𝒴.𝒳.𝒴.𝒴, 𝒳.𝒴.𝒴, 𝒴.𝒳}}}
```

is a file that contains all the group elements of the G21 group organized into cnjugacy classes in the standard order and expressed as formal words in the generators

Irreducible Representations of G21

The irreducible representations of G21 are provided in the form of substitution rules named **repG21D1, repG21D2, repG21D3, repG21D4, repG21D5** for the generators $\{\mathcal{Y},\mathcal{X}\}$. On the other hand the object

formRepra21=$\{\epsilon,\mathcal{Y},\mathcal{X}.\mathcal{X}.\mathcal{Y}.\mathcal{X}.\mathcal{Y}.\mathcal{Y},\mathcal{Y}.\mathcal{X}.\mathcal{X},\mathcal{X}\}$

contains a representative for each conjugacy class.

The groups O24A and O24B

The octahedral group O24 has two generators S and T that satisfy the following relations:

$$S^2 = T^3 = (ST)^4 = 1$$

Conjugacy classes of O24

Conjugacy Class	C_1	C_2	C_3	C_4	C_5
representative of the class	e	T	STST	S	ST
order of the elements in the class	1	3	2	2	4
number of elements in the class	1	8	3	6	6

Characters

The 5 irriducible representations of the group O24 are of dimensions **1,1,2,3, and 3**, respectively. The character table is the following one:

0	e	S	T	ST	?
D_1	1	1	1	1	1
DX_1	1	1	1	−1	−1
D_2	2	−1	2	0	0
DA_3	3	0	−1	−1	1
DB_3	3	0	−1	1	−1

The character table is encoded in the object **Pchi24**. The populations of conjugacy classes are encoded in the object **Pgi24**. The names of the representations are encoded in the object **names24**.

In the program the entire group O24 are encoded in the following objects **O24A** and **O24B** that contain all the 7×7 matrices representing these maximal subgroups in the weight basis. Furthermore

In the program the entire group O24 are encoded in the following objects:

1. **O24AclasW** contains all the 24 elements as 7×7 matrices in the weight basis, organized by conjugacy classes for the group **O24A.** The file **O24AclasR** containes instead the same matrices in the root lattice basis.
2. **O24BclasW** contains all the 24 elements as 7×7 matrices in the weight basis, organized by conjugacy classes for the group **O24B.** The file **O24BclasR** containes instead the same matrices in the root lattice basis.
3. The formal encoding of the of the groups **O24A** and **O24B** is given by the objects **formO24Aclas** and **formO24Bclas** where every group element in each conjugacy class is expressed in terms of the generators ρ,σ,τ

Irreducible representations

The irreducible representations of **O24** are named **DD1classa, DD2classa, DD3classa, DD4classa, DD5classa**.
They are organized by conjugacy classes in the standard order.

Generators of O24A and O24B

In the case of the group **O24A** the formal generators are:

$$\mathbf{fT} = \rho.\sigma.\rho.\tau.\tau.\sigma.\rho.\tau \quad ; \qquad \mathbf{fS} = \tau.\tau.\sigma.\rho.\tau.\sigma.\sigma \tag{1.9}$$

In the case of the group **O24B** the formal generators are:

$$\mathbf{fT} = \rho.\tau.\sigma.\rho.\tau.\tau.\sigma.\rho.\tau \quad ; \qquad \mathbf{fS} = \sigma.\rho.\tau.\sigma.\rho.\tau \tag{1.10}$$

The substitution rules for the embedding are **QO24A** and **QO24B**.

2 List of routines and commands

The *Mathematica* package and its available commands

The basic commands of the present package are the following ones:

Construction of the representation of the group L168 and of its maximal subgroups and decomposition into irreps

For these tasks we have the following commands:

1. **brutcaratterL168.** This routine calculates the character of any linear representation D of the group L168 provided in the following way:
$$\text{Repra} = \{D[1],\ D[\rho],\ D[\sigma],\ D[\tau\rho\sigma],\ D[\tau],\ D[\sigma\rho]\}$$
where D[1], etc are the the matrix realization of the standard representatives of the six conjugacy classes in terms of the generators of the group, satisfying the defining relations. The routine derives the multiplicity vector encoded in the object **decompo** and it constructs the decomposition of the representation D into irreps of L168. The projectors onto the irreducible representations are named **PP168.**
2. **brutcaratter21.** This routine calculates the character of any linear representation D of the group G21 provided in the following way:
$$\text{Repra21} = \left\{ \boxed{D[e]} \boxed{D[\mathcal{Y}]} \boxed{D[\mathcal{X}^2 \mathcal{Y}\mathcal{X}\mathcal{Y}^2]} \boxed{D[\mathcal{Y}\mathcal{X}^2]} \boxed{D[\mathcal{X}]} \right\}$$
where D[e], etc are the the matrix realization of the standard representatives of the five conjugacy classes in terms of the generators of the group $\mathcal{Y}$, and $\mathcal{X}$ satisfying the defining relations. The routine derives the multiplicity vector encoded in the object **decompo21** and it constructs the decomposition of the representation D into irreps of G21. The projectors onto the irreducible representations are named **PP21.**
3. **brutcaratter24.** This routine calculates the character of any linear representation D of the group O24 provided in the following way:
$$\text{Repra24} = \left\{ \boxed{D[e]} \boxed{D[T]} \boxed{D[\text{STST}]} \boxed{D[S]} \boxed{D[\text{ST}]} \right\}$$
where D[e], etc are the the matrix realization of the standard representatives of the five conjugacy classes in terms of the generators of the group T, and S satisfying the defining relations. The routine derives the multiplicity vector encoded in the object **decompo24** and it constructs the decomposition of the representation D into irreps of O24. The projectors onto the irreducible representations are named **PP24.**
4. **belcaratter21.** This routine requires a substitution rule **passarulla** =**{$\mathcal{Y}\to$ D[$\mathcal{Y}$], $\mathcal{X}\to$ D[$\mathcal{X}$]}** and calculates the character of the corresponding representation and its explicit decomposition into irreps. The projection operators onto irreps are explicitly calculated**.**

Auxiliary group theoretical routines used by the package but available also to the user

Besides the basic commands described in the previous section this package contains also some general group-theoretical routines that are internally utilized but available to the user. These are

1. **generone**. Given a set of matrices named **Allgroup** the routine generone generates the set of all their products. Repeated use of generone arrives at a set that closes under multiplication if the original matrices were elements of a finite group.
2. **generoneName**. Given a set of matrices named **AllgroupN** , associated, each of them with a name, the routine generone generates the set of all their products keeping track of the non--commutative product of names. Repeated use of generoneName arrives at a set that closes under multiplication if the original matrices were elements of a finite group
3. **coniugatoL** (or **coniugatoM**, they are equivalent). If you give a set of matrices forming a finite group and you name it **gruppone**, **coniugatoL** produces the set of conjugacy classes into which the finite group is organized. The output of this calculation is named **orgclas**.

4. **verifiosub**. Given a set of matrices that form a finite group, named **gruppone** and a subset named **settino**, verifiosub verifies whether settino is a subgroup and moreover it verifies whether it is a normal subgroup.
5. **quozientus**. Given a set of matrices forming a finite group, named gruppone and a normal subgroup named gruppino, quozientus constructs the equivalence classes G/H namely the quotient group. The output of this calculation is named **equaclass.**

Additional specialized routines for the analysis of subgroups and orbits in the 7-dimensional momentum lattice

1. **cercatoreorbo**. Given a subgroup G of L168 named **sottogruppo** (it should be given in the weight basis) the routine cercatoreorbo generates the general form of a 7-vector invariant under G. The output is named **invarvec** and it is displayed.
2. **stabilio.** Given a 7-vector named **veicolo** the routine stabilio constructs the subgroup of L168 (in the weight basis) that leaves it invariant. The output of this calculation is named **stab**.
3. **stabilione.** You have to call **gruppone** the set of **n×n** matrices representing a given group G. You have to name **veicolo** an **n**-vector. The routine stabilione constructs the subgroup G that leaves it invariant. The output of this calculation is named **stab.**
4. **rovnastab**. Given a 7-vector **v**, named **vectus** the routine rovnastab constructs first the L168 orbit O_v of vectus, named **orballo.** Next for each vector $v_i \in O_v$ the routine constructs its stability subgroup Γ_i named **stab**. The program verifies whether this subgroup is contained in one of the three standard representatives of the maximal subgroups, namely O24A, O24B or G21. By definition of orbit it is obligatory that one of the Γ_i should be inside one of the three groups O24A, O24B or G21. The routine finds the standard vector named repvec whose stability subgroup is included in one of the three standard maximal subgroup and determines which one. The final output is a subgroup of one of the three maximal ones that represents the orbit and is named **standgroup**.
5. **dihedraleA**. (**dihedraleB**) These two routines respectively construct the dihedral subgroup $\mathrm{Dih}_3 \subset O_{24}$ in the case of the maximal subgroup O24A and O24B. The outputs of the two routines are respectively named **DH3A** and **DH3B.** The construction is done fixing first the generator A of order 3 and then looking for a generator B of order 2 which satisfies the defining relation of the dihedral group with A. Fixing A amounts to choosing a fixed representative of the conjugacy class of Dih_3 groups inside the octahedral one.

The irriducible representation of L_{168}

The irreducible representations of the group L_{168} are defined by giving the form of the three generators ρ,σ,τ

The 3-dimensional complex representation

The 3-dimensional complex representation has been constructed in two different basis by Pierre Ramond et al and by Markusevich

1. In Pierre Ramond basis the three standard generators are given by the matrices **R3,S3,T3** the entire representation organized in conjugacy classes can be obtained from the file **formL168clas** performing the substitution **ramond3.** Applying the same substitution rule to the formal-presentation of a subgroup we obtain its realization inside the 3-dimensional realization of the bigger group.
2. In Murkusevich basis the three standard generators are given by the matrices **RP,SP,TP** the entire representation organized in conjugacy classes can be obtained from the file **formL168clas** performing the substitution **markus3.** Applying the same substitution rule to the formal-presentation of a subgroup we obtain its realization inside the 3-dimensional realization of the bigger group

The 6-dimensional representation

In the 6-dimensional representation the three standard generators are given by the matrices **R6,S6,T6** the entire representation organized in conjugacy classes can be obtained from the file **formL168clas** performing the substitution **ramond6.** Applying the same substitution rule to the formal-presentation of a subgroup we obtain its realization inside the 6-dimensional realization of the bigger group

The 8-dimensional representation

In the 8-dimensional representation the three standard generators are given by the matrices **R8,S8,T8** the entire representation organized in conjugacy classes can be obtained from the file **formL168clas** performing the substitution **ramond8.** Applying the same substitution rule to the formal-presentation of a subgroup we obtain its realization inside the 8-dimensional realization of the bigger group.

Prepared characters

The file **Reprano** contains the formal definition of the representatives of the 6-conjugacy classes for the group L168. Substituting the formal generators with one of the irrep-substitutions one obtains the 6-matrices whose trace provide the character of the representation. These file can be utilized with the routine brutcaratter168.

The file **Reprano21** contains the formal definition of the representatives of the 6-conjugacy classes for the group G21. Substituting the formal generators with one of the irrep-substitutions one obtains the 6-matrices whose trace provide the character of the representation. These file can be utilized with the routine brutcaratter21.

Generation of orbits in the C^3:

Given a complex three-vector that should be named **vectus** one can generate its orbit under L168T or one of its three maximal subgroubs using the following commands:

1. **orbitando168** generates the orbit of **vectus** under the full group L168. The output is named **orbita**.
2. **orbitando21** generates the orbit of **vectus** under the subgroup G21. The output is named **orbita**.
3. **orbitando24A** generates the orbit of **vectus** under the subgroup O24A. The output is named **orbita**.
4. **orbitando24B** generates the orbit of **vectus** under the subgroup O24B. The output is named **orbita**

Constructing the Group L_{168} acting on $\mathbb{C}^3$

We construct explicitly the 3×3 complex matrices that represent the simple finite group L_{168} inside SL(3,ℂ). In order to be able to compare with Markusevich's paper, we utilize its basis for the generators that was constructed in the background Note Book. It is important to note that the form given by Markusevich of the generators which he calls $\mathcal{Y}$, χ and ω, respectively of order 7, 3 and 2, are not the standard generators in the presentation of the group L_{168} utilized by other authors and by Pietro in his recent paper and in the background MATHEMATICA Notebook. Yet there is no problem since we have the vocabulary. Setting:

$$R = \omega.\chi \quad ; \quad S = \chi.\mathcal{Y} \quad ; \quad T = \chi^2.\omega$$

These new generators satisfy the standard relations of the presentation used by Pietro and other authors:

$$R^2 = S^3 = T^7 = \mathbf{RST} = (\mathrm{TSR})^4 = \mathbf{1}$$

Construction of the complete group L_{168} in the 3-dimensional complex representation.
We also construct explicitly all the group elements of the group L_{168} in the 3-dimensional complex representation. They are encoded in two different files
1) **gruppo3C** contains the group elements organized in conjugacy classes exactly in the same order as in the file **formL168** which writes each element as a word in the generators. This is very important in order to identify each group element if needed in another representation.
2) **L168C3** contains the 168 elements just in one stock. This file is useful to calculate orbits of given vectors or lines and to find out stability subgroups.

We utilize the internal Mathematical command *FullSimplify* in order to write the numerical matrices representing the group elements in the simplest possible way. The matrix entries are all elementary transcendental numbers or algebraic numbers that have an algebraic representation as roots of algebraic equations with rational coefficients. For this reason *Mathematica* some-times writes them as follows Root[1 + 7 #1^2 - 14 #1^3 + 49 #1^6 &, 1]. One should not be afraid. Originally these numbers where produced by multiplication of rational numbers extended with elementary trascendentals that are the seventh roots of unity $\sqrt[7]{1}$ or their real or imaginary parts.

In Markusevich basis the analytic form of the generators contained in the substitution rule markus3 (computed in this background Notebook) is the following one:

$$\epsilon \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho \to \begin{pmatrix} -\frac{2\,\text{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} & -\frac{2\,\text{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} & \frac{2\,\text{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} \\ -\frac{2\,\text{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} & \frac{2\,\text{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} & -\frac{2\,\text{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} \\ \frac{2\,\text{Sin}\left[\frac{\pi}{7}\right]}{\sqrt{7}} & -\frac{2\,\text{Cos}\left[\frac{\pi}{14}\right]}{\sqrt{7}} & -\frac{2\,\text{Cos}\left[\frac{3\pi}{14}\right]}{\sqrt{7}} \end{pmatrix}$$

$$\sigma \to \begin{pmatrix} 0 & 0 & -(-1)^{1/7} \\ (-1)^{2/7} & 0 & 0 \\ 0 & (-1)^{4/7} & 0 \end{pmatrix}$$

$$\tau \to \begin{pmatrix} \frac{\text{i}+(-1)^{13/14}}{\sqrt{7}} & -\frac{(-1)^{1/14}\,(-1+(-1)^{2/7})}{\sqrt{7}} & \frac{(-1)^{9/14}\,(1+(-1)^{1/7})}{\sqrt{7}} \\ \frac{(-1)^{11/14}\,(-1+(-1)^{2/7})}{\sqrt{7}} & \frac{\text{i}+(-1)^{5/14}}{\sqrt{7}} & \frac{(-1)^{3/14}\,(1+(-1)^{3/7})}{\sqrt{7}} \\ -\frac{(-1)^{11/14}\,(1+(-1)^{1/7})}{\sqrt{7}} & -\frac{(-1)^{9/14}\,(1+(-1)^{3/7})}{\sqrt{7}} & -\frac{-\text{i}+(-1)^{3/14}}{\sqrt{7}} \end{pmatrix}$$

We remind the reader that ρ,σ,τ are the abstract names for the generators of L_{168} whose 168 elements are written as words in these letters (modulo relations). Substituting explicit matrices satisfying the relations for these letters one obtains an explicit representation of the group. In the present case the irreducible 3-dimensional representation is DA_3.

For possible use in future calculation, in the present section we rewrite the generators of L_{168} in the Markusevich basis in terms of 6×6 real matrices where the imaginary unit i has been replaced by the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the real part is proportional to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Essentially this is the canonical embedding $SU(3) \mapsto SO(6)$. Since the group $L_{168} \subset SU(3)$. We can make such a construction. This representation might be useful in the sequel.

The generators are named:

RP6 , SP6, TP6

The *Mathematica* code

Input of the Differential Form package and symbol protection

Some General Group Theoretical routines

Additional routines for orbit and subgroup analysis

The L168 Group and its G2 embedding

The formal structure of the group L168 in terms of words

Subgroups G21 and O24

Maximal subgroups O24 construction

Maximal subgroups G21 construction

Representations of L168 and of its subgroupsnd decompostion into irreps

Irrepses of L168

Constructing the Group $\mathbb{L}_{168}$ acting on $\mathbb{C}^3$