Manifold Geometry in Vielbein formalism

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What this Notebook does

In this Notebook we provide a package to calculate the geometry of an arbitrary manifold in arbitrary dimension and with an arbitrary signature using the Vielbein formalism and the intrinsic components of all tensors.

Instructions for the user

The inputs

In order to initialize the calculation the user has to type five lines of inputs providing the following information

1) the dimension n=**dimse**

2) the set of coordinates an n-vector = **coordi**

3) the set of differentials, an n-vector = **diffe**

4) the set of vielbein 1-forms a n-vector = **fform**

5) the signature of the space as n-vector = **signat** of +/- 1**.**

Activating the calculation

After providing the above information the user will start the calculations by typing **mainstart**

The obtained outputs

The MATHEMATICA NoteBook calculates the following objects:

1. The contorsion $c^i{}_{jk}$ defined by the equation $dV^i = c^i{}_{jk}\, V^i \wedge V^k$ and encoded in an array **contens**$_{[\![i,j,k]\!]}$.
2. The spin connection 1-form ω^{ij} defined by the equation $dV^i - \omega^{ij} \wedge V^k \eta_{jk} = 0$ and encoded in an array $\omega_{[\![i,j]\!]}$.
3. The intrinsic components of the spin connection defined by the equation $\omega^{ij} = \omega^{ij}_k$ and encoded in a tensor **ometen**$_{[\![i,j,k]\!]}$.
4. The curvature two-form R^{ij} defined by the equation $R = d\omega - \omega_\wedge \omega$ and encoded in a tensor **RF**$_{[\![i,j]\!]}$.

5. The Riemann tensor with flat indices, defined by $R^{ij} = \mathrm{Rie}^{ij}{}_{pq}\, V^p \wedge V^q$ and encoded in an array **Rie**$_{[[i,j,a,b]]}$.
6. The Ricci tensor with flat indices defined by $\mathrm{Ric}^i_p = \mathrm{Rie}^{iq}{}_{pq}$ and encoded in an array **ricten[[e,b]].**

The MATHEMATICA Code

Input of the Differential Form package and symbol protection

The routines of this package

Main

You start this programme by typing **mainstart**

routine mainstart

Spin connection

This routine is devised to calculate the intrinsic components of the spin connection once the contorsion tensor as already been calculated

$dV^i = c^i{}_{jk}\, V^i \wedge V^k$. The package is named **spinpackgen.**

routine spinpackgen

Routine curvapackgen

This routine is devised to calculate the curvature two form and the Riemann tensor in a general situation for an arbitrary dimensional manifold and with the vielbein depending on all the coordinates. You can start this programme only after having computed the spin connection via the package **spinpackgen**

routine curvapackgen

calculation of the contorsion for general manifolds

This routine is calculates the external differential of the Vielbein $dV^i = c^i{}_{jk}\, V^i \wedge V^k$. It is named **contorgen**

routine contorgen

Example

de Sitter space in the coordinates for spatial sections of positive curvature.

In this example we calculate once again the geometry of de Sitter space using the same coordinates that are used for the same example calculated with metricgrav. Using the Vielbein formalism the homogeneous nature of the manifold is better revealed. Indeed the intrinsic components of the Riemann and Ricci tensor come out constant.

In[42]:=

dimse = 4;

$$\texttt{fform} = \left\{dt, \frac{\texttt{Cosh}[H*t]}{H} * \frac{dr}{\sqrt{1-r^2}}, \frac{\texttt{Cosh}[H*t]}{H} * r * d\theta, \frac{\texttt{Cosh}[H*t]}{H} * r * \texttt{Sin}[\theta] * d\phi\right\};$$

coordi = {t, r, θ, ϕ};

diffe = {dt, dr, dθ, dϕ};

signat = {-1, 1, 1, 1};

In[47]:=

mainstart

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==============================
Welcome this is the Vielbeingrav package that calculates
geometry of a manifold in Vielbein formalism
Your space has dimension n = 4
You gave me the following data
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$$\text{vector of 1-form vielbein} = \left\{dt, \frac{dr\,\texttt{Cosh}[H\,t]}{H\sqrt{1-r^2}}, \frac{d\theta\,r\,\texttt{Cosh}[H\,t]}{H}, \frac{d\phi\,r\,\texttt{Cosh}[H\,t]\,\texttt{Sin}[\theta]}{H}\right\}$$

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vector of coordinates = {t, r, θ, ϕ}
vector of differentials = {dt, dr, dθ, dϕ}
 I resume the calculation and I evaluate the contorsion
I calculate the exterior differentials of the vielbeins
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I finished!
Next I calculate the inveverse vielbein
Done!
I resume the calculation of the contorsion
 I calculate the contorsion c[i,j,k] for
i = 1
i = 2
i = 3
i = 4
I have finished!
The result, encoded in a vector dV[[i]] is the following:
dV[1] = 0
dV[2] = H Tanh[H t] V[1] ∧ V[2]
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$$dV[3] = 2\left(\frac{1}{2} H\,\texttt{Tanh}[H\,t]\,V[1]\wedge V[3] + \frac{H\sqrt{1-r^2}\,\texttt{Sech}[H\,t]\,V[2]\wedge V[3]}{2\,r}\right)$$

dV[4] =

$$2\left(\frac{1}{2}\,\mathrm{H}\,\mathrm{Tanh}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[1]\wedge\mathrm{V}[4]+\frac{\mathrm{H}\sqrt{1-\mathrm{r}^2}\;\mathrm{Sech}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[2]\wedge\mathrm{V}[4]}{2\,\mathrm{r}}+\frac{\mathrm{H}\,\mathrm{Cot}[\theta]\,\mathrm{Sech}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[3]\wedge\mathrm{V}[4]}{2\,\mathrm{r}}\right)$$

The contorsion is encoded in a tensor named contens

Now I can begin the calculation of the spin connection

I start

I resume the calculation of the spin connection

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the result is

$\omega[12] = \mathrm{H}\,\mathrm{Tanh}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[2]$

$\omega[13] = \mathrm{H}\,\mathrm{Tanh}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[3]$

$\omega[14] = \mathrm{H}\,\mathrm{Tanh}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[4]$

$\omega[23] = -\frac{\mathrm{H}\sqrt{1-\mathrm{r}^2}\;\mathrm{Sech}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[3]}{\mathrm{r}}$

$\omega[24] = -\frac{\mathrm{H}\sqrt{1-\mathrm{r}^2}\;\mathrm{Sech}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[4]}{\mathrm{r}}$

$\omega[34] = -\frac{\mathrm{H}\,\mathrm{Cot}[\theta]\,\mathrm{Sech}[\mathrm{H}\,\mathrm{t}]\,\mathrm{V}[4]}{\mathrm{r}}$

Task finished

The result is encoded in a tensor ω[[i,j]]

Its components are encoded in a tensor ometen[i,j,m]

I calculate the Riemann tensor

I tell you my steps :

a = 1

b = 1

b = 2

b = 3

b = 4

a = 2

b = 1

b = 2

b = 3

b = 4

a = 3

b = 1

b = 2

b = 3

b = 4

a = 4

b = 1

b = 2

b = 3

b = 4

Finished

Now I evaluate the curvature 2-form of your space

I find the following answer

R[12] = H^2 V[1] ∧ V[2]

R[13] = H^2 V[1] ∧ V[3]

R[14] = H^2 V[1] ∧ V[4]

R[23] = H^2 V[2] ∧ V[3]

R[24] = H^2 V[2] ∧ V[4]

R[34] = H^2 V[3] ∧ V[4]

The result is encoded in a tensor RF[i,j]

Its components are encoded in a tensor Rie[i,j,a,b]

Now I calculate the Ricci tensor

1 1   non zero

2 2   non zero

3 3   non zero

4 4   non zero

I have finished the calculation

The tensor ricten[[a,b]] giving the Ricci tensor

is ready for storing on hard disk

Ricci[a,b] = $\begin{pmatrix} \frac{3 H^2}{2} & 0 & 0 & 0 \\ 0 & \frac{3 H^2}{2} & 0 & 0 \\ 0 & 0 & \frac{3 H^2}{2} & 0 \\ 0 & 0 & 0 & \frac{3 H^2}{2} \end{pmatrix}$

Out[47]= {Null}