This NoteBook was developed in January 2017 on the basis of NoteBooks written in 2004 and 2008

Last updating March 8th 2017

Construction of the adjoint representation of the E_8 Lie Algebra

Pietro G. Frè
University of Torino, Physics Department, Via. P. Giuria 1, 10125, Torino Italy
Embassy of Italy in the Russian Federation, Denezhny Pereulok, 5, Moscow, Russia

Initialization

The name and the address of the working directory

Here the user should specify the address of his/her working directory on his/her computer

```
addro = "C:\\Users\\Lenovo\\Desktop\\CurrentWork\\bookfrefedotov\\frefedotovfin\\
    auxiliaryMathe";
```

Purpose of the NoteBook

The purpose of this Notebook is the construction of the E_8 root system and of the complete E_8 Lie Algebra in a basis adapted to its decomposition with respect to the subalgebra SO(16,ℂ) ⊂ E_8

This notebook contains both routines that are used every time one does calculations and routines that are used only once to construct the 248 x 248 matrices of the adjoint representation and store them on the hard disk. When one evaluates the Notebook these routines are read but not executed. At the beginning the user has to say Y in order to create the library or N in order to skip the creation of the library if that was already done before

Choice of operational regime

```
Ifflagga = InputString["Create library, Y or N"];
```

Description of the content and theory

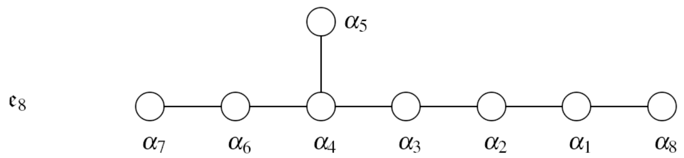
The E_8 root system

Description

In this Notebook we construct an explicit representation of the 8 simple roots of E_8 as euclidean vectors in $\mathbb{R}^8$. The first seven roots are just those of E_7 as introduced by us in previous NoteBooks just extended by the addition of an eigth component in the last position which is zero. The eighth root that completes the correct Dynkin diagram is instead the following vector `{1, -1, 0, 0, 0, 0, 0, 0}` which has instead a non vanishing component in the eight direction.
The enumeration of roots is as follows:

$$\alpha(1)=\{0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0\}$$
$$\alpha(2)=\{0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0\}$$
$$\alpha(3)=\{0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0\}$$
$$\alpha(4)=\{0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0\}$$
$$\alpha(5)=\{0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ 0\}$$
$$\alpha(6)=\{0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0\}$$
$$\alpha(7)=\left\{-\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2}\right\}$$
$$\alpha(8)=\{1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

and we verify that we have the correct Cartan matrix corresponding to the correct Dynkin diagram:



The produced objects

After evaluation of the NoteBook one has the following available objects:

1. The simple roots are encoded in a file named α
2. The set of all positive roots organized by height from height=1 (the simple roots) to height=29 (the highest root). There are two files: one **rootaltorg** contains the integer components of the roots in the simple root basis. The other **euclideroot** containes the components of the roots in the standard euclidian othonormal basis of $\mathbb{R}^8$.
3. The file **Steppi** contains the 120 explicit 248×248 matrices representing the step operators E^α associated with each of the 120 positive roots. In this file the step-operators are organized exactly in the same order as the corresponding roots in the file **rootaltorg**.
4. The file **Cartani** contains the 8 explicit 248×248 matrices representing the Cartan generators $\mathcal{H}^i$ corresponding to the euclidian components of the roots.
5. The generators $E^{-\alpha}$ associated with the negative roots are the transposed of the generators associated with the corresponding positive roots.
6. **Simples** is a file that contains the 8 step operators corresponding to the eight simple roots.
7. The file **racine** contains the enumeration of all the positive roots organized in order of increasing height but not subdived into height groups, just simply listed from 1 to 120.

Summary of the reading instructions for the user

1) the tensor **dimealt** contains the dimension of the sets of roots of heigth h = 1,...,29.

2) the tensor **rootaltorg** contains the roots in Dynkin notation, organized into subspaces corresponding to the various levels of heigth. In concrete terms:

$\texttt{rootaltorg}_{[\![3,8]\!]}$ is the the 8th root of heigth 3 in the Dynkin basis of simple roots.

3) the tensor **euclideroot** contains the roots in Euclidian notation, organized into subspaces corresponding to the various levels of heigth. In concrete terms:

$\texttt{eucliderootot}_{[\![3,8]\!]}$ is the the 8th root of heigth 3 in Euclidean basis.

4) the tensor **eracin** contains the root in euclidean form just enumerated from 1 to 120.

5) the tensor **racine** contains the root in Dynkin form just enumerated from 1 to 120.

6) the tensor **Nalfa** contains the cocycle $N_{\alpha\beta}$

Calculation of the Results

Setting the Working Directory

This section contains the initial commands needed to start the programme, in particular the computer prompts to read the name of the working directory.

Routine setdir

Constructing the E8 simple roots

Description

In this section we construct an explicit representation of the 8 simple roots of E(8) as euclidean vectors in $\mathbb{R}^8$. The first seven roots are just those of E(7) as introduced in previous notebooks us in old papers just extended by the addition of an eighth component in the last position which is zero. The eighth root that completes the correct Dynkin diagram is instead the following vector `{1, -1, 0, 0, 0, 0, 0, 0}` which has instead a non vanishing component in the eight direction and we verify that we have the correct Cartan matrix corresponding to such a Dynkin diagram

routine iniziali

Algorithm to construct all the roots

This section contains an induction routine that constructs all the positive roots of E(8) as integer valued linear combination of the simple roots and obtaines them by induction on their height.

The result is organized in a tensor

racine *of* ***120*** *integer valued* ***8****-vectors*

routine induczia and commands to run it

Organizing roots by height and converting them to the euclidean basis

This section is finalized to rewrite the roots in the euclidean basis adapted to SO(16,C) basis and to organize them by height. The result of the computation are three objects:

1) the tensor **dimealt** that contains the dimension of the sets of roots of heigth h = 1,...,29

2) the tensor **rootaltorg** that contains the roots in Dynkin notation, organized into subspaces corresponding to the various levels of heigth.

In concrete terms:

rootaltorg$_{[[3,8]]}$ is the the 8th root of heigth 3 in a Dynkin basis of simple roots.

3) the tensor **eucliderоot** that contains the roots in Dynkin notation, organized into subspaces corresponding to the various levels of heigth. In concrete terms:

eucliderоot$_{[[3,8]]}$ is the the 8th root of heigth 3 in Euclidean basis.

4) the tensor **eracin** that contains the root in euclidean form just enumerated from 1 to 120.

routine organizia

Constructing the matrices of the adjoint = fundamental representation of E8

We order the axis of the representation in the following way:

$$248 = \underbrace{1,2,3,4,5,6,7,8}_{\text{Cartan generators}},\ \underbrace{120}_{\text{Positive roots}},\ \underbrace{120}_{\text{Negative roots}}$$

Furthermore as Cartan generators we choose those corresponding to the Euclidean axis and we order the roots according to the order generated by the computer in the vector racine, namely we order them by height.

The procedure is divided in several steps. In a first step we create the step operators associated with simple roots:

$\alpha_i \Longrightarrow E^{\alpha_i}$ where α_i is a simple root i=1,...,8

the routine that does this job is named **simplestep**.

They are requested to satisfy the Lie algebra:

$[\mathcal{H}, E^{\alpha_i}] = \alpha_i E^{\alpha_i}$ (algebra 1)

$[E^{\alpha_i}, E^{-\alpha_j}] = \delta^{ij}\ \alpha_i \cdot \mathcal{H}$ (algebra 2)

where $\mathcal{H}$ are the Cartan generators that are stored in the file **Cartani.ma**

The simple root step operators are instead saved in the file **Simples.ma**

The two routines **chcksimple1** and **chcksimple2** perform the checking of the algebras 1 and 2 written above.

routine to create simple root step operators

routine simplestep

routine to check the above construction

Routine stepcreator

Routine newceccus and newceccus2 to check the above construction

Execution of the construction

Running the routine setup

running iniziali

```
iniziali
```

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

```
{Null}
```

Finding and displaying the roots

```
runinduz
```

```
==============================

{Null}
```

Organizing the roots

executing the construction of the simple root step operators

Running stepcreator

Derivation of the simple weights

Construction of $N_{\alpha\beta}$ cocycle

In this Notebook we first construct the $N_{\alpha\beta}$ cocycle on positive roots calculating commutators of Step operators. Then we calculate the extension to the negative roots and then we check the cocycle condition on the whole tensor $N_{\alpha\beta}$

Setting Directory

Loading inputs

routine lodo

running the routine lodo

Programme to calculate $N_{\alpha\beta}$ for positive roots

In this section we construct the $N_{\alpha\beta}$ cocycle for positive roots

routine nalfaprogra

running the routine nalfa

Programme to extend $N_{\alpha\beta}$ also to negative roots

In this section we extend the $N_{\alpha\beta}$ cocycle also to negative roots

routine lodus2

Check of the cocycle

routine nalfacec

The following routine nalfacec performs the check that Nalfa is a cocycle (on positive roots)

routine nalfacec

running the routine nalfacec

Displaying the number of errors

```
sbagli
```

```
0
```

Checks of cocycle condition also on the negative roots