NoteBook created July 1st 2017 in Moscow on the basis of a package tested by the author for many years

MATHEMATICA PACKAGE METRICGRAV

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Einstein Equations in metric formalism

In this Notebook we provide a package to calculate Einstein equations for any given metric in arbitrary dimensions using the metric formalism.

Description of the programme

What the NoteBook does

Given a n-dimensional manifold $\mathcal{M}$ whose coordinates we denote x_i and a metric defined over it and provided in the form

$$ds^2 = g_{ij}(x)\, dx^i \otimes dx^j \tag{0.1}$$

The programme extracts the metric tensor $g_{ij}\,(x)$ calculates its inverse $g^{ij}\,(x)$ calculates the Christoffel symbols $\Gamma_{ij}{}^{k}(x)$, then the Riemann tensor, the Ricci tensor and the Einstein tensor.

Initialization and inputs to be supplied

After reading the NoteBook, calculations are intialized in the following way

1. First the user types **nn** = positive integer number (which is going to be the dimension **n** of the considered manifold)
2. First the user types **mainmetric.** The computer will ask the user to supply three inputs in the following form:
   a) the set of coordinates as n-vector. That vector must be named **coordi =** $\{x_1,....,x_n\}$
   b) the set of coordinates differentials as n-vector. That vector must be named **diffe =** $\{dx_1,....,dx_n\}$
   c) the metric given as a quadratic differential that must be named ds2. The user will type **ds2**$=g_{[[i,j]]}dx^i\, dx^j$
3. After providing these inputs the user will type the command **metricresume.**

Produced outputs

1. The Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ are encoded in an array **Gam[[λ,μ,ν]].**
2. The Riemann tensor $R^{\lambda}_{\mu\nu\rho}$ is encoded in an array **Rie[[λ,μ,ν,ρ]].**

3. The curvature 2-form **R** = $d\Gamma + \Gamma \wedge \Gamma$ is encoded in an array named **RR[[λ,μ]].**

4. The Ricci tensor $R_{\mu\rho} \equiv R^{\lambda}_{\mu\lambda\rho}$ is encoded in an array named **ricten[[μ,ρ]].**

5. The Einstein tensor $G_{\mu\rho} \equiv R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} R$ is encoded in an array **einst[[μ,ρ]].**

The *Mathematica* code

Examples

The Schwarzschild metric

In this section we exemplify the use of the package metricgrav with the case of the Schwarzschild metric that we write in the following form

$$ds^2 = -\left(1 - 2\frac{\mu}{r}\right) dt^2 + \left(1 - 2\frac{\mu}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \mathrm{Sin}[\theta]^2 d\phi^2\right) \qquad (0.2)$$

computer calculation of the Riemann, Ricci and Einstein tensors

```
In[5]:= nn = 4;
        mainmetric

        OK I calculate your space, Give me the data
        Give me the dimension of your space
        Your space has dimension n = 4
        Now I stop and you give me two vectors of dimension 4
        vector coordi = vector of coordinates
        vector diffe = vector of differentials
        Next you give me the metric as ds2 =
        Then to resume calculation you print metricresume

Out[6]= {Null}

In[7]:= coordi = {t, r, θ, ϕ};
        diffe = {dt, dr, dθ, dϕ};
        ds2 = -(1 - 2 μ/r) dt^2 + (1 - 2 μ/r)^-1 dr^2 + r^2 (dθ^2 + Sin[θ]^2 dϕ^2);

In[10]:= metricresume

        I resume the calculation
        First I extract the metric coefficients from your data
        Then I calculate the inverse metric
```

Done !

and I calculate also the metric determinant

Done

I perform the calculation of the Christoffel symbols

I finished

the Levi Civita connection is given by:

$\Gamma[11] = \frac{dr\,\mu}{r^2 - 2\,r\,\mu}$

$\Gamma[12] = \frac{dt\,\mu}{r^2 - 2\,r\,\mu}$

$\Gamma[13] = 0$

$\Gamma[14] = 0$

$\Gamma[21] = \frac{dt\,(r - 2\,\mu)\,\mu}{r^3}$

$\Gamma[22] = -\frac{dr\,\mu}{r^2 - 2\,r\,\mu}$

$\Gamma[23] = d\theta\,(-r + 2\,\mu)$

$\Gamma[24] = -d\phi\,(r - 2\,\mu)\,\mathrm{Sin}[\theta]^2$

$\Gamma[31] = 0$

$\Gamma[32] = \frac{d\theta}{r}$

$\Gamma[33] = \frac{dr}{r}$

$\Gamma[34] = -d\phi\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma[41] = 0$

$\Gamma[42] = \frac{d\phi}{r}$

$\Gamma[43] = d\phi\,\mathrm{Cot}[\theta]$

$\Gamma[44] = \frac{dr}{r} + d\theta\,\mathrm{Cot}[\theta]$

Task finished

The result is encoded in an array Gam[[λ,μ,ν]]

Now I calculate the Riemann tensor

I tell you my steps :

a = 1

b = 1

b = 2

b = 3

b = 4

a = 2

```
 b = 1
 b = 2
 b = 3
 b = 4
 a = 3
 b = 1
 b = 2
 b = 3
 b = 4
 a = 4
 b = 1
 b = 2
 b = 3
 b = 4
Finished
-------------------------
Now I evaluate the curvature 2-form of your space
I find the following answer
```

R[11] = 0

R[12] = $\frac{2\,\mu\,\mathrm{dt} ** \mathrm{dr}}{r^2\,(r-2\,\mu)}$

R[13] = $-\frac{\mu\,\mathrm{dt} ** \mathrm{d}\theta}{r}$

R[14] = $-\frac{\mu\,\mathrm{dt} ** \mathrm{d}\phi\,\mathrm{Sin}[\theta]^2}{r}$

R[21] = $\frac{2\,(r-2\,\mu)\,\mu\,\mathrm{dt} ** \mathrm{dr}}{r^4}$

R[22] = 0

R[23] = $-\frac{\mu\,\mathrm{dr} ** \mathrm{d}\theta}{r}$

R[24] = $-\frac{\mu\,\mathrm{dr} ** \mathrm{d}\phi\,\mathrm{Sin}[\theta]^2}{r}$

R[31] = $-\frac{(r-2\,\mu)\,\mu\,\mathrm{dt} ** \mathrm{d}\theta}{r^4}$

R[32] = $\frac{2\,\mu\,\mathrm{dr} ** \mathrm{d}\theta}{2\,r^3-4\,r^2\,\mu}$

R[33] = 0

R[34] = $\frac{2\,\mu\,\mathrm{d}\theta ** \mathrm{d}\phi\,\mathrm{Sin}[\theta]^2}{r}$

R[41] = $-\frac{(r-2\,\mu)\,\mu\,\mathrm{dt} ** \mathrm{d}\phi}{r^4}$

R[42] = $\frac{2\,\mu\,\mathrm{dr} ** \mathrm{d}\phi}{2\,r^3-4\,r^2\,\mu}$

$$R[43] = -\frac{2\,\mu\, d\theta ** d\phi}{r}$$

R[44] = 0

The result is encoded in a tensor RR[[λ,μ]]

Its components are encoded in a tensor Rie[[λ,μ,ν,ρ]]

Now I calculate the Ricci tensor

I have finished the calculation

The Ricci tensor is zero

The Einstein tensor is zero

Out[10]= {Null}

The de Sitter metric for a manifold with positive spatial curvature

In this section we exemplify the use of the package metricgrav with the case of the de Sitter metric for a manifold of positive spatial curvature that we write as follows:

$$ds^2 = -dt^2 + \frac{\mathrm{Cosh}[H * t]^2}{H^2}\left(\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \mathrm{Sin}[\theta]^2\, d\phi^2\right)\right) \tag{0.3}$$

computer calculation of the Riemann, Ricci and Einstein tensors

In[11]:=
```
nn = 4;
mainmetric
```

OK I calculate your space, Give me the data

Give me the dimension of your space

Your space has dimension n = 4

Now I stop and you give me two vectors of dimension 4

vector coordi = vector of coordinates

vector diffe = vector of differentials

Next you give me the metric as ds2 =

Then to resume calculation you print metricresume

Out[12]= {Null}

In[13]:=
```
coordi = {t, r, θ, ϕ};
diffe = {dt, dr, dθ, dϕ};
```

$$ds2 = -dt^2 + \frac{\mathrm{Cosh}[H * t]^2}{H^2}\left(\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \mathrm{Sin}[\theta]^2\, d\phi^2\right)\right);$$

In[16]:=
```
metricresume
```

I resume the calculation

First I extract the metric coefficients from your data

Then I calculate the inverse metric

Done !

and I calculate also the metric determinant

Done

I perform the calculation of the Christoffel symbols

I finished

the Levi Civita connection is given by:

$\Gamma[11] = 0$

$\Gamma[12] = \dfrac{\text{dr Cosh}[H\,t]\ \text{Sinh}[H\,t]}{H - H\,r^2}$

$\Gamma[13] = \dfrac{\text{d}\theta\ r^2\ \text{Cosh}[H\,t]\ \text{Sinh}[H\,t]}{H}$

$\Gamma[14] = \dfrac{\text{d}\phi\ r^2\ \text{Cosh}[H\,t]\ \text{Sin}[\theta]^2\ \text{Sinh}[H\,t]}{H}$

$\Gamma[21] = \text{dr}\ H\ \text{Tanh}[H\,t]$

$\Gamma[22] = \dfrac{\text{dr}\ r}{1 - r^2} + \text{dt}\ H\ \text{Tanh}[H\,t]$

$\Gamma[23] = \text{d}\theta\ r\ (-1 + r^2)$

$\Gamma[24] = \text{d}\phi\ r\ (-1 + r^2)\ \text{Sin}[\theta]^2$

$\Gamma[31] = \text{d}\theta\ H\ \text{Tanh}[H\,t]$

$\Gamma[32] = \dfrac{\text{d}\theta}{r}$

$\Gamma[33] = \dfrac{\text{dr}}{r} + \text{dt}\ H\ \text{Tanh}[H\,t]$

$\Gamma[34] = -\text{d}\phi\ \text{Cos}[\theta]\ \text{Sin}[\theta]$

$\Gamma[41] = \text{d}\phi\ H\ \text{Tanh}[H\,t]$

$\Gamma[42] = \dfrac{\text{d}\phi}{r}$

$\Gamma[43] = \text{d}\phi\ \text{Cot}[\theta]$

$\Gamma[44] = \dfrac{\text{dr}}{r} + \text{d}\theta\ \text{Cot}[\theta] + \text{dt}\ H\ \text{Tanh}[H\,t]$

Task finished

The result is encoded in an array Gam[[λ,μ,ν]]

Now I calculate the Riemann tensor

I tell you my steps :

a = 1

b = 1

b = 2

b = 3

b = 4

a = 2

b = 1

b = 2

b = 3

b = 4

a = 3

b = 1

b = 2

b = 3

b = 4

a = 4

b = 1

b = 2

b = 3

b = 4

Finished

Now I evaluate the curvature 2-form of your space

I find the following answer

R[11] = 0

R[12] = $\frac{2\,\mathrm{Cosh}[H\,t]^2\,dt ** dr}{2 - 2\,r^2}$

R[13] = $r^2\,\mathrm{Cosh}[H\,t]^2\,dt ** d\theta$

R[14] = $r^2\,\mathrm{Cosh}[H\,t]^2\,dt ** d\phi\,\mathrm{Sin}[\theta]^2$

R[21] = $H^2\,dt ** dr$

R[22] = 0

R[23] = $r^2\,\mathrm{Cosh}[H\,t]^2\,dr ** d\theta$

R[24] = $r^2\,\mathrm{Cosh}[H\,t]^2\,dr ** d\phi\,\mathrm{Sin}[\theta]^2$

R[31] = $H^2\,dt ** d\theta$

R[32] = $\frac{2\,\mathrm{Cosh}[H\,t]^2\,dr ** d\theta}{-2 + 2\,r^2}$

R[33] = 0

R[34] = $r^2\,\mathrm{Cosh}[H\,t]^2\,d\theta ** d\phi\,\mathrm{Sin}[\theta]^2$

R[41] = $H^2\,dt ** d\phi$

R[42] = $\frac{2\,\mathrm{Cosh}[H\,t]^2\,dr ** d\phi}{-2 + 2\,r^2}$

R[43] = $-r^2\,\mathrm{Cosh}[H\,t]^2\,d\theta ** d\phi$

R[44] = 0

The result is encoded in a tensor RR[[λ,μ]]

Its components are encoded in a tensor Rie[[λ,μ,ν,ρ]]

Now I calculate the Ricci tensor

1 1 non zero

$$\text{Ricci[11]} = -\frac{3\,H^2}{2}$$

2 2 non zero

$$\text{Ricci[22]} = \frac{3\,\text{Cosh}[H\,t]^2}{2-2\,r^2}$$

3 3 non zero

$$\text{Ricci[33]} = \frac{3}{2}\,r^2\,\text{Cosh}[H\,t]^2$$

4 4 non zero

$$\text{Ricci[44]} = \frac{3}{2}\,r^2\,\text{Cosh}[H\,t]^2\,\text{Sin}[\theta]^2$$

I have finished the calculation

The tensor ricten[[μ,ρ]] giving the Ricci tensor

is ready for inspection or for storing on hard disk

The Einstein tensor is encoded in an array einst and it is ready for inspection

Out[16]= {Null}

In[17]:= **MatrixForm[einst]**

Out[17]//MatrixForm=

$$\begin{pmatrix} \frac{3\,H^2}{2} & 0 & 0 & 0 \\ 0 & \frac{3\,\text{Cosh}[H\,t]^2}{2\,(-1+r^2)} & 0 & 0 \\ 0 & 0 & -\frac{3}{2}\,r^2\,\text{Cosh}[H\,t]^2 & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\,r^2\,\text{Cosh}[H\,t]^2\,\text{Sin}[\theta]^2 \end{pmatrix}$$

In[18]:= **scalaron**

Out[18]= $6\,H^2$