Lab 3 Notes

Week of October 4, 2021

Review of expectation, variance, and covariance operators

Expectation

Let X, Y be two random variables whose expectations exist and let $a \in \mathbb{R}$.

1. $\mathbf{E}(aX) = a\mathbf{E}(X)$
2. $\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$
3. $\mathbf{E}(a) = a$

Through properties 1 and 2, the expectation is known as a *linear operator*. If you have taken first year calculus, you have already seen two examples of linear operators: differentiation and integration!

For example, let $\mathbf{D}(\cdot)$ be the differentiation operator. Then

$$\mathbf{D}(3x^2) = 3\mathbf{D}(x^2)$$

$$\mathbf{D}(x^2 + x^3) = \mathbf{D}(x^2) + \mathbf{D}(x^3)$$

You can remember the linear operator properties of expectation by recalling the linear operator properties of differentiation.

Variance

Let X be a random variable with $\mathbf{E}(X) = \mu$.

$$\begin{aligned}
\mathbf{Var}(X) &= \mathbf{E}\left((X - \mu)^2\right) \\
&= \mathbf{E}\left(X^2 - 2\mu X + \mu^2\right) \\
&= \mathbf{E}\left(X^2\right) - 2\mu\mathbf{E}(X) + \mathbf{E}\left(\mu^2\right) \\
&= \mathbf{E}\left(X^2\right) - 2\mu^2 + \mu^2 \\
&= \mathbf{E}\left(X^2\right) - \mu^2 \\
&= \mathbf{E}\left(X^2\right) - (\mathbf{E}(X))^2
\end{aligned}$$

Let X and Y be two random variables whose expectations exist and let $a \in \mathbb{R}$.

1. $\mathbf{Var}(aX) = a^2\mathbf{Var}(X)$
2. $\mathbf{Var}(X + Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$ for X, Y **independent**
3. $\mathbf{Var}(a) = 0$.

Covariance

Let X and Y be two random variables with $\mathbf{E}(X) = \mu_X$ and $\mathbf{E}(Y) = \mu_Y$.

$$\begin{aligned}
\mathbf{Cov}(X, Y) &= \mathbf{E}((X - \mu_X)(Y - \mu_Y)) \\
&= \mathbf{E}(XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y) \\
&= \mathbf{E}(XY) \,-\, \mu_Y \mathbf{E}(X) \,-\, \mu_X \mathbf{E}(Y) \,+\, \mathbf{E}(\mu_X \mu_Y) \\
&= \mathbf{E}(XY) - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X \mu_Y \\
&= \mathbf{E}(XY) - \mu_X \mu_Y \\
&= \mathbf{E}(XY) - \mathbf{E}(X)\,\mathbf{E}(Y)
\end{aligned}$$

Note that $\mathbf{Cov}(X, X) = \mathbf{Var}(X)$.