Tutorial 7: Solutions

March 7, 2018

Question 8.1.4, Page 325

Pairs of p-values and significance levels, α, are given. For each pair, state whether the observed p-value would lead to rejection of H_0 at the given significance level.

We reject H_0 if $p < \alpha$.

(a) $p = 0.084$, $\alpha = 0.05$ $p \not< \alpha$, so we fail to reject H_0 .

(b) $p = 0.003$, $\alpha = 0.001$ $p \not< \alpha$, so we fail to reject H_0 .

(c) $p = 0.498$, $\alpha = 0.05$ $p \not< \alpha$, so we fail to reject H_0 .

(d) $p = 0.084$, $\alpha = 0.10$ $p < \alpha$, so we reject H_0 .

(e) $p = 0.039$, $\alpha = 0.01$ $p \not< \alpha$, so we fail to reject H_0 .

(f) $p = 0.218$, $\alpha = 0.10$ $p \not< \alpha$, so we fail to reject H_0 .

Question 8.1.12, Page 326

A mixture of pulverized fuel ash and Portland cement to be used for grouting should have a compressive strength of more than 1300 KN/m^2. The mixture will not be used unless experimental evidence indicates conclusively that the strength specification has been met. Suppose compressive strength for specimens of this mixture is normally distributed with $\sigma = 60$. Let μ denote the true average compressive strength.

(a) What are the appropriate null and alternative hypotheses?

$H_0 : \mu \leq 1300$, $H_A : \mu > 1300$

(b) Let $\overline{X}$ denote the sample average compressive strength for $n = 10$ randomly selected specimens. If $\overline{x} = 1340$, should H_0 be rejected using a significance level of .01? [Hint: What is the probability distribution of the test statistic when H_0 is true?]

(i) Use the test statistic method.

We are given that compressive strength is normally distributed. Therefore, $\overline{X} \sim N(\mu_{\overline{X}} = 1300,$ $\sigma_{\overline{X}} = 60^2/10)$ (this is under the assumption that H_0 is true.) Then $Z = \dfrac{\overline{X} - 1300}{60/\sqrt{10}} \sim N(0, 1)$.
Plugging in our values, we obtain $z = \dfrac{1340 - 1300}{60/\sqrt{10}} = 2.108$.

This is an upper tail test (H_A is of the form $\mu > \ldots$) so we need to compute $z_{1-\alpha}$.

$z_{1-\alpha} = z_{0.99} = 2.3263$

We reject H_0 if $z > z_{1-\alpha}$. Since $2.108 \not> 2.3263$, we fail to reject H_0 at the 1% significance level.

There is insufficient evidence to conclude that the true average compressive strength is greater than 1300.

(ii) Consider the test procedure with test statistic $\overline{X}$ itself (not standardized), i.e. find the critical region in terms of $\overline{X}$.

Since this is an upper tailed test, i.e. reject H_0 if $z > z_{1-\alpha}$, under the assumption that H_0 is true:

$$\begin{aligned} z &> z_{1-\alpha} \\ \implies \frac{\overline{x} - 1300}{60/\sqrt{10}} &> 2.3263 \\ \implies \overline{x} &> 1344.138 \end{aligned}$$

Therefore, our critical region is of the form $\{\overline{x} \mid \overline{x} > 1344.138\}$. We are given in the question that $\overline{x} = 1340$. This value is clearly not within our critical region. Once again, we fail to reject H_0 at the 1% significance level. We conclude at the 1% level of significance that there is insufficient evidence that the true mean compressive strength is greater than 1300.

(c) What is the probability distribution of the test statistic when $\mu' = 1350$? For a test with $\alpha = .01$, what is the probability that the mixture will be judged unsatisfactory when in fact $\mu' = 1350$ (a type II error)?

In this case, $\overline{X} \sim N(\mu' = 1350, \sigma_{\overline{X}} = 60^2/10)$

$$\begin{aligned} \mathbf{P}\left(\text{Type II Error}\right) &= \mathbf{P}\left(\frac{\overline{X} - \mu_0}{\sigma_{\overline{X}}} < z_{1-\alpha} \middle| \mu' = 1350\right) \\ &= \mathbf{P}\left(\overline{X} < z_{1-\alpha} \cdot \sigma_{\overline{X}} + \mu_0 \middle| \mu' = 1350\right) \\ &= \mathbf{P}\left(\frac{\overline{X} - \mu'}{\sigma_{\overline{X}}} < \frac{(z_{1-\alpha} \cdot \sigma_{\overline{X}} + \mu_0) - \mu'}{\sigma_{\overline{X}}}\right) \\ &= \mathbf{P}\left(Z < z_{1-\alpha} + \frac{\mu_0 - \mu'}{\sigma_X}\right) \\ &= \mathbf{P}\left(Z < 2.3263 + \frac{1300 - 1350}{60/\sqrt{10}}\right) \\ &= \mathbf{P}\left(Z < -0.3089\right) \\ &= 0.37869 \end{aligned}$$

Question 8.2.25, Page 334

Body armor provides critical protection for law enforcement personnel, but it does affect balance and mobility. An article reported that for a sample of 52 male enforcement officers who underwent an acceleration task that simulated exiting a vehicle while wearing armor, the sample mean was 1.95 sec, and the sample standard deviation was 0.20 sec. Does it appear that true average task time is less than 2 sec? Carry out a test of appropriate hypotheses using a significance level of 0.01.

$H_0 : \mu \geq 2$, $H_A : \mu < 2$

We are given: $n = 52$, $\overline{x} = 1.95$, $s = 0.20$, $\alpha = 0.01$

We typically require $n > 30$ for anything related to large samples, but since we are using s instead of σ, we require $n > 40$, which we do have ($n = 52$). Therefore we can proceed to obtain a large sample test statistic with distribution approximately standard normal.

$$z = \frac{\overline{x} - \mu}{s/\sqrt{n}} = \frac{1.95 - 2}{0.20/\sqrt{52}} = -1.8028$$

$$-z_{1-\alpha} = -z_{0.99} = -2.3263$$

We reject H_0 if $z < -z_{1-\alpha}$. Since $-1.8028 \not< -2.3263$, we fail to reject H_0 at the 1% significance level and conclude that there is insufficient evidence that the true average task time is less than 2 seconds.

Question 8.3.36, Page 345

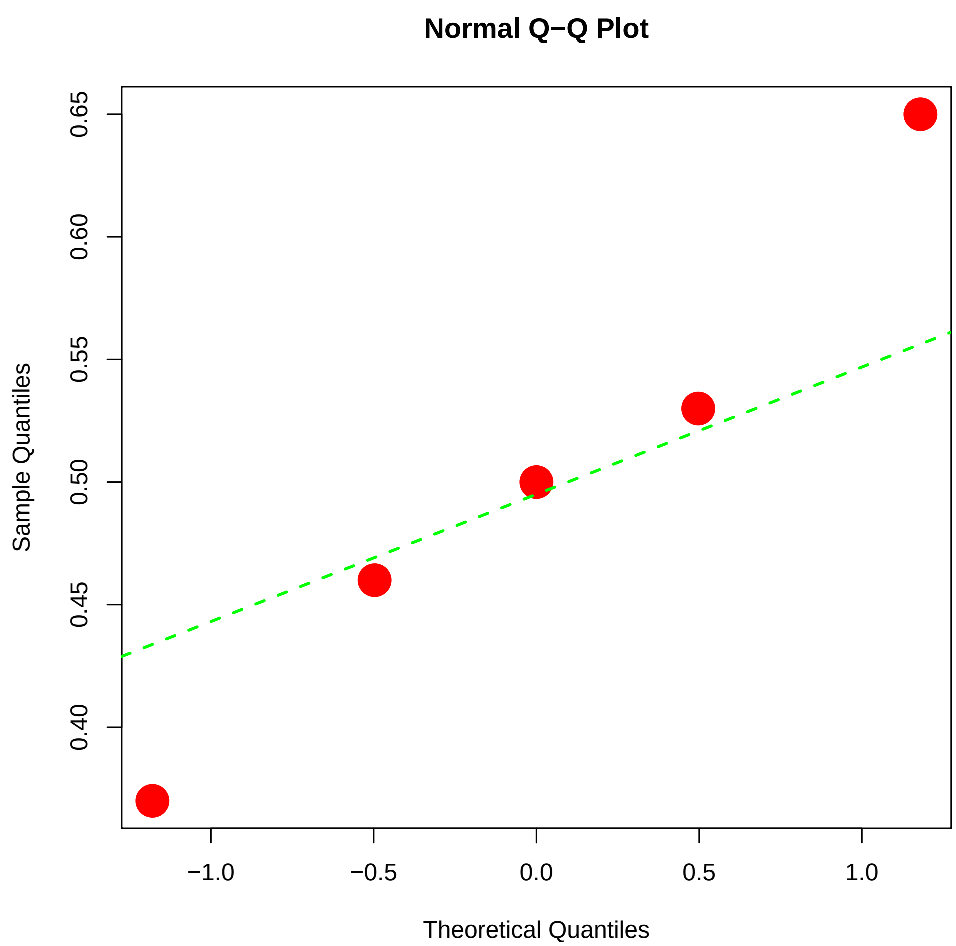
Have you ever been frustrated because you could not get a container of some sort to release the last bit of its contents? An article reported on an investigation of this issue for various consumer products. Suppose five 6.0 oz tubes of toothpaste of a particular brand are randomly selected and squeezed until no more toothpaste will come out. Then each tube is cut open and the amount remaining is weighed, resulting in the following data (consistent with what the cited article reported): .53, .65, .46, .50, .37. Does it appear that the true average amount left is less than 10% of the advertised net contents?

(a) Check the validity of any assumptions necessary for testing the appropriate hypotheses.

In order to use the t-distribution, the underlying distribution of the data must be normal. We use the following code in `R` to check the normality of the data set:

```
> data <- c(0.53, 0.65, 0.46, 0.50, 0.37)
> qqnorm(data, pch = 19, col = "red", cex = 3)
> qqline(data, col = "green", lty = 2, lwd = 2)
```

This produces the following plot:



There is somewhat of a linear trend so we can assume normality in the data set.

(b) Carry out a test of the appropriate hypotheses using a significance level of 0.05. Would your conclusion change if a significance level of 0.01 had been used?

10% of 6.0 oz is 0.6 oz. Our hypotheses are: $H_0 : \mu \geq 0.6$, $H_A : \mu < 0.6$.

We calculate $\overline{x} = 0.502$, and $s = 0.010232\ldots$ (use the exact number in your calculations!)

$$t = \frac{\overline{x} - \mu}{s/\sqrt{n}} = \frac{0.502 - 0.6}{0.10232/\sqrt{5}} = -2.1416$$

Case 1: $\boldsymbol{\alpha} = 0.05$
This is a lower-tailed test so we need $-t_{n-1,0.05} = -t_{4,0.05} = -2.132$.

We reject H_0 if $t < -t_{4,0.05}$. Since $-2.1416 < -2.132$, we reject H_0 at the 5% significance level and conclude that data supports the claim that the true average amount left is less than 10% of the advertised net contents.

Case 2: $\boldsymbol{\alpha} = 0.01$
This is a lower-tailed test so we need $-t_{n-1,0.01} = -t_{4,0.01} = -3.747$.

We reject H_0 if $t < -t_{4,0.01}$. Since $-2.1416 \not< -3.747$, we fail to reject H_0 at the 1% significance level and conclude that there is insufficient evidence that the true average amount left is less than 10% of the advertised net contents.

(c) Describe in context type I and II errors, and say which error might have been made in reaching a conclusion.

A Type I Error would occur if we concluded there is less than 10% remaining when there is actually more than 10% remaining.
A Type II Error would occur if we concluded there is greater than 10% remaining when there is actually less than 10% remaining.
Since we have concluded in case 1 that there is less than 10% remaining in the tubes, it is possible that we have committed a Type I Error.