Tutorial 6

March 5, 2020

Question 1

In each case, determine the value of c.

(a) $\Phi(c) = 0.9838$, where $\Phi(\cdot)$ refers to the cdf of the standard normal distribution.

Using Table A.3, the probability value 0.9838 corresponds to row **2.1** and column **0.04**. Therefore $c = 2.14$ and:

$$\Phi(2.14) = \mathbf{P}\left(Z \leq 2.14\right) = 0.9838$$

(b) $P(c \leq Z) = 0.121$

$\mathbf{P}\left(c \leq Z\right) = \mathbf{P}\left(Z \geq c\right) = 0.1210$. Subtracting both sides from 1:

$$\begin{aligned} 1 - \mathbf{P}\left(Z \geq c\right) &= 1 - 0.1210 \\ \mathbf{P}\left(Z \leq c\right) &= 0.8790 \end{aligned}$$

Using Table A.3, the probability value 0.8790 corresponds to row **1.1** and column **0.07**. Therefore $c = 1.17$ and:

$$\mathbf{P}\left(1.17 \leq Z\right) = 0.121$$

Question 2

Suppose that time used by a student at a terminal connected to a time-sharing computer has a Gamma distribution with mean 20 minutes, and variance 80 minutes^2.

(a) Find the parameters α and β of this distribution.

We are given that:

$$\mu = \alpha\beta = 20 \qquad\qquad \sigma^2 = \alpha\beta^2 = 80$$

Then:

$$80 = \alpha\beta^2 = (\alpha\beta)\beta = 20\beta \quad\Longrightarrow\quad \beta = 4$$

From $20 = \alpha\beta = \alpha \cdot 4$, we get that $\alpha = 5$.

If we let X be the time in minutes that a student spends at a terminal, then:

$$X \sim \text{Gamma}(\alpha = 5,\, \beta = 4)$$

(b) What is the probability that the student will use the terminal for at least 24 minutes?

For the gamma distribution we have the following relationship:

$$\text{If} \quad X \sim \text{Gamma}(\alpha = a,\, \beta = b) \quad \text{then} \quad Y := \frac{X}{b} \sim \text{Gamma}(\alpha = a,\, \beta = 1)$$

for $a, b > 0$. $\mathbf{P}\left(Y \leq y\right)$ can then be obtained using the standard (incomplete) gamma function.

From (a), $X \sim \text{Gamma}(\alpha = 5,\, \beta = 4)$. The probability that we seek is $\mathbf{P}\left(X \geq 24\right)$.

$$\begin{aligned}
\mathbf{P}\left(X \geq 24\right) &= 1 - \mathbf{P}\left(X \leq 24\right) \\
&= 1 - \mathbf{P}\left(\frac{X}{4} \leq \frac{24}{4}\right) \\
&= 1 - \mathbf{P}\left(Y \leq 6\right) \\
&= 1 - F(y = 6,\, \alpha = 5)
\end{aligned}$$

[Using Table A.4, we look in row **6**, column **5** and get that $F(y = 6,\, \alpha = 5) = 0.715$]

$$\begin{aligned}
&= 1 - 0.715 \\
&= 0.285
\end{aligned}$$

(c) What is the probability that the student will spend between 20 and 40 minutes using the terminal?

$$\begin{aligned}
\mathbf{P}\left(20 \leq X \leq 40\right) &= \mathbf{P}\left(X \leq 40\right) - \mathbf{P}\left(X \leq 20\right) \\
&= \mathbf{P}\left(\frac{X}{4} \leq \frac{40}{4}\right) - \mathbf{P}\left(\frac{X}{4} \leq \frac{20}{4}\right) \\
&= \mathbf{P}\left(Y \leq 10\right) - \mathbf{P}\left(Y \leq 5\right) \\
&= F(y = 10,\, \alpha = 5) - F(y = 5,\, \alpha = 5) \\
&= 0.971 - 0.560 \\
&= 0.411
\end{aligned}$$

Question 3

A service station has both self-service and full-service islands. Let X be the number of hoses being used on the self-service and Y the number of hoses being used on the full-service at a particular time of the day. The joint pmf of (X, Y) is given as:

$f(x,y)$		**y**		
		0	1	2
	0	0.10	0.04	0.02
x	1	0.08	0.20	0.06
	2	0.06	0.14	0.30

and zero elsewhere.

(a) Compute $\mathbf{P}(X = 1,\, Y = 1)$.

$\mathbf{P}(X = 1,\, Y = 1) = 0.20$, obtained from the above table in row $\mathbf{x} = 1$ and column $\mathbf{y} = 1$.

(b) Compute $\mathbf{P}(X \leq 1,\, Y \leq 1)$.

$$\begin{aligned}\mathbf{P}(X \leq 1,\, Y \leq 1) &= \mathbf{P}(X = 0,\, Y = 0) + \mathbf{P}(X = 0,\, Y = 1) \\ &\quad + \mathbf{P}(X = 1,\, Y = 0) + \mathbf{P}(X = 1,\, Y = 1) \\ &= 0.1 + 0.04 + 0.08 + 0.20 \\ &= 0.42\end{aligned}$$

(c) What is the probability that both islands are in use?

Both islands are in use when there is at least one customer in the self-service and at least one customer in the full service simultaneously. The probability we seek is $\mathbf{P}(X \geq 1,\, Y \geq 1)$.

$$\begin{aligned}\mathbf{P}(X \geq 1,\, Y \geq 1) &= \mathbf{P}(X = 1,\, Y = 1) + \mathbf{P}(X = 1,\, Y = 2) \\ &\quad + \mathbf{P}(X = 2,\, Y = 1) + \mathbf{P}(X = 2,\, Y = 2) \\ &= 0.20 + 0.06 + 0.14 + 0.30 \\ &= 0.70\end{aligned}$$

(d) Find the marginal pmfs of X and Y.

To obtain the marginal pmf of X, we want to get rid of Y. We can do this by summing the probabilities for a fixed x while varying y. The marginal pmf of X is a function solely of x.

$$f_X(x) = \mathbf{P}(X = x) = \begin{cases} 0.10 + 0.04 + 0.02 & x = 0 \\ 0.08 + 0.20 + 0.06 & x = 1 \\ 0.06 + 0.14 + 0.30 & x = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.16 & x = 0 \\ 0.34 & x = 1 \\ 0.50 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that this is the same as computing the row sums of the above table.

The marginal pmf of Y is obtained similarly by summing the probabilities for a fixed y while varying x. The marginal pmf of Y is a function solely of y.

$$f_Y(y) = \mathbf{P}(Y = y) = \begin{cases} 0.10 + 0.08 + 0.06 & y = 0 \\ 0.04 + 0.20 + 0.14 & y = 1 \\ 0.02 + 0.06 + 0.30 & y = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.24 & y = 0 \\ 0.38 & y = 1 \\ 0.38 & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that this is the same as computing the column sums of the above table.

(e) Do the two service islands operate independently of each other?

X and Y are independent if

$$f(x, y) = f_X(x) \cdot f_Y(y),$$

or equivalently for the discrete case,

$$\mathbf{P}(X = x, Y = y) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$$

for **all** pairs of x and y values.

Therefore if we can find even one pair of x and y where this relationship does not hold, then we can conclude that X and Y are dependent.

Take $x = 0$ and $y = 0$.

$$\mathbf{P}(X = 0, Y = 0) = 0.10$$

$$\mathbf{P}(X = 0) \cdot \mathbf{P}(Y = 0) = 0.16 \cdot 0.24 = 0.0384$$

Since $\mathbf{P}(X = x, Y = y) \neq \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$, X and Y are not independent. In other words, the two service islands do not operate independently of each other.