Tutorial 4

February 13, 2020

Question 1

An interviewer is conducting a phone survey. Each call has probability 0.4 to be answered.

(a) What is the average number of calls that need to be placed before the first answer?

From last time, $X \sim \text{Geometric}(p = 0.4)$. The mean of the geometric distribution is:

$$\mathbf{E}\left(X\right) = \frac{1-p}{p}$$

With $p = 0.4$, the average number of (no-answer) calls that need to be placed before the first answer is:

$$\mathbf{E}\left(X\right) = \frac{1-0.4}{0.4} = \frac{0.6}{0.4} = 1.5$$

(b) What is the probability that the interviewer will have to make 10 calls before getting 3 answered?

Let X represent the number of failures before the third answered call (success). Then

$$X \sim \text{NBinom}(r = 3,\, p = 0.4)$$

To make 10 calls before the third success means that a total of 11 calls are placed. Of these 11 calls, the first 10 calls consists of 2 successes and 8 failures (in any order), followed by a success.

Since X counts the number of failures, we seek $\mathbf{P}\left(X = 8\right)$.

$$\begin{aligned}
\mathbf{P}\left(X = 8\right) &= \binom{x+r-1}{r-1}(1-p)^x p^r \\
&= \binom{8+3-1}{3-1}(1-0.4)^8(0.4)^3 \\
&= \binom{10}{2}(0.6)^8(0.4)^3 \\
&= 0.0484
\end{aligned}$$

Note that we deviate slightly from the textbook in how we interpret the phrase "10 calls before 3 answered". (Compare the above solution with example 3.37 on page 129)

(c) What is the average number of non-answered calls before he or she gets 3 calls answered?

The mean of the negative binomial distribution is:

$$\mathbf{E}\left(X\right) = \frac{r(1-p)}{p}$$

With $r = 3$ and $p = 0.4$, the average number of non-answered calls before getting 3 answered is:

$$\mathbf{E}\left(X\right) = \frac{3(1-0.4)}{0.4} = \frac{3 \cdot 0.6}{0.4} = 4.5$$

Question 2

Aircraft arrive at a small airport according to a Poisson process at a rate of $\alpha = 5$ per hour.

(a) What is the probability that during the next hour there will be 6 arrivals?

Let X represent the number of arrivals in the next hour ($t = 1$). Then $\mu = \alpha t = 5 \cdot 1 = 5$ and $X \sim \text{Poisson}(\mu = 5)$. The probability we seek is $\mathbf{P}\left(X = 6\right)$.

$$\begin{aligned}\mathbf{P}\left(X = 6\right) &= \frac{e^{-\mu} \cdot \mu^x}{x!} \\ &= \frac{e^{-5} \cdot 5^6}{6!} \\ &= 0.1462\end{aligned}$$

(b) What is the probability that during the next two hours there will be 7 arrivals?

Let Y represent the number of arrivals in the next two hours ($t = 2$). Then $\mu = \alpha t = 5 \cdot 2 = 10$ and $Y \sim \text{Poisson}(\mu = 10)$. The probability we seek is $\mathbf{P}\left(X = 7\right)$.

$$\begin{aligned}\mathbf{P}\left(X = 7\right) &= \frac{e^{-10} \cdot 10^7}{7!} \\ &= 0.0901\end{aligned}$$

(c) What is the average number of arrivals during the next 3 hours?

Let W represent the number of arrivals in the next three hours ($t = 3$). Then $\mu = \alpha t = 5 \cdot 3 = 15$ and $W \sim \text{Poisson}(\mu = 15)$. The mean of a Poisson distribution with parameter μ is μ. Therefore the average number of arrivals during the next three hours is:

$$\mathbf{E}\left(W\right) = \mu = 15$$

Question 3

Of the people passing through an airport metal detector, 0.5% activate it. Let X be the number among a randomly selected group of 500 who activate the detector.

(a) What is the exact distribution of X? What is the approximate distribution of X?

The exact distribution of X is $\text{Binom}(n = 500,\, p = 0.005)$. As a rule of thumb, for $n > 50$ and $np < 5$, we can approximate the binomial distribution with a Poisson distribution. Here, $n = 500 > 50$ and $np = 500 \cdot 0.005 = 2.5 < 5$. The Poisson parameter $\mu = np = 2.5$. Then the approximate distribution of X is:

$$X \mathrel{\dot\sim} \text{Poisson}(\mu = 2.5)$$

(b) Compute $\mathbf{P}(X = 5)$ (using the approximate distribution).

$$\begin{aligned} \mathbf{P}(X = 5) &\approx \frac{e^{-2.5} \cdot 2.5^5}{5!} \\ &= 0.0668 \end{aligned}$$

(c) Compute $\mathbf{P}(X \leq 5)$ (using the approximate distribution).

$$\begin{aligned} \mathbf{P}(X \leq 5) &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1) + \ldots + \mathbf{P}(X = 5) \\ &\approx \sum_{x=0}^{5} \frac{e^{-2.5} \cdot 2.5^x}{x!} \\ &= e^{-2.5} \sum_{x=0}^{5} \frac{2.5^x}{x!} \\ &= 0.9580 \end{aligned}$$