Constructing protostellar discs for SPH simulations

DIMITRIS STAMATELLOS

Jeremiah Horrocks Institute, University of Central Lancashire, UK

E-mail: dstamatellos@uclan.ac.uk

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1. Introduction

Discs are very common around young stars. Dust continuum observations have shown that at least 50% of PMS stars are surrounded by discs of dust and gas (e.g. Strom et al. 1989, André et al. 1990) and it is believed that most stars start off with discs around them. discs have masses from 10^{-3} to 10^{-1} $M_{\odot}$, radii from 10 to 1000 AU, and lifetimes up to a few Myrs. They are responsible for most of the infrared radiation that is observed in the spectra of PMS stars. disc material slowly circles inwards and finally accretes onto the central star producing UV radiation. The evolution of a protostellar disc around an isolated star is determined by the combined effects of the star's gravity and irradiation, and the angular momentum transport processes in the disc. If the star/disc system is in a cluster then gravitational forces due to passing stars, and radiation from hot neighbouring stars also affect the disc evolution.

2. Disc initial conditions

We set the initial disc density, temperature and rotational velocity using information from theoretical models and observations. In this section we describe the initial conditions in detail.

2.1. Disc surface density

Steady disc theory (e.g. Natta 1993) suggests that the surface density of an accretion disc follows a power law of the form $\Sigma(R) \propto R^{-3/4}\alpha^{-1}$, where R is the distance from the central star on the disc mid-plane, and α is the viscosity parameter in the Shakura-Sunyaev prescription for viscosity (Shakura & Sunyaev 1973). Semi-analytical theoretical studies of cloud collapse and subsequent disc creation (Lin and Pringle 1990) indicate that $\Sigma(R) \sim R^{-p}$, where p is between 1 and 3/2. The same model also predicts a much steeper density profile near the edge of the disc. In this study we assume a surface density profile

$$\Sigma(R) = \Sigma_0 \left(\frac{R_0^2}{R_0^2 + R^2}\right)^{p/2} , \tag{1}$$

where Σ_0 is the surface density at $R = 0$, R_0 is the softening radius, which is used to prevent the surface density from getting nonphysically large near the star, and R denotes the distance from the star on the disc mid-plane. If $x-y$ is the disc mid-plane then $R = \sqrt{x^2 + y^2}$. (We note that in this chapter R is used to denote the distance from the star on the disc mid-plane and $r = \sqrt{x^2 + y^2 + z^2}$ to denote the distance from star in three dimensions). For the models presented in this chapter we use $p = 1$.

We can calculate the disc mass interior to radius R by integrating the surface density:

$$M(R) = \int_0^R \Sigma(R)\, 2\pi R\, dR\,. \tag{2}$$

Substituting for $\Sigma(R)$ and calculating the integral, we obtain

$$M(R) = \frac{2\pi R_0^2}{2-p}\Sigma_0 \left[\left(\frac{R_0^2 + R^2}{R_0^2} \right)^{1-\frac{p}{2}} - 1 \right]. \tag{3}$$

The total mass of the disc is obtained from the above formula by setting $R = R_{\rm disc}$:

$$M_{\rm disc} = \frac{2\pi R_0^2}{2-p}\Sigma_0 \left[\left(\frac{R_0^2 + R_{\rm disc}^2}{R_0^2} \right)^{1-\frac{p}{2}} - 1 \right]. \tag{4}$$

In the case that the disc extends from an inner radius $R_{\rm in}$ to an external radius $R_{\rm out}$ (i.e. there is a gap in the disc around the central star), the mass of the disc within radius $R > R_{\rm in}$ is given by

$$M(R) = \int_{R_{\rm in}}^R \Sigma(R)\, 2\pi R\, dR\,, \tag{5}$$

and the above two equations regarding the disc mass transpire to

$$M(R) = \frac{2\pi R_0^2}{2-p}\Sigma_0 \left[\left(\frac{R_0^2 + R^2}{R_0^2} \right)^{1-\frac{p}{2}} - \left(\frac{R_0^2 + R_{\rm in}^2}{R_0^2} \right)^{1-\frac{p}{2}} \right], \tag{6}$$

and

$$M_{\rm disc} = \frac{2\pi R_0^2}{2-p}\Sigma_0 \left[\left(\frac{R_0^2 + R_{\rm out}^2}{R_0^2} \right)^{1-\frac{p}{2}} - \left(\frac{R_0^2 + R_{\rm in}^2}{R_0^2} \right)^{1-\frac{p}{2}} \right]. \tag{7}$$

2.2. Disc temperature

The two major physical processes that heat the disc are irradiation from the central star and energy generated by viscous dissipation within the disc. In the case of a fully reprocessing (or passive) disc, i.e. a disc that just absorbs and reemits radiation from the central star, the flux of stellar radiation at distance R from the star, scales as $1/R^2$ and the incident angle of the radiation on the surface of the disc scales as $1/R$ (assuming a geometrically thin disc). Thus, the total radiation incident on unit area of the disc scales as $1/R^3$ (see Hartmann 1998):

$$F_{\rm incident} \propto \sigma T_\star^4 \left(\frac{R}{R_\star} \right)^{-3}, \tag{8}$$

where σ is the Stefan-Boltzmann constant, $T_\star$ and $R_\star$ the temperature and radius of the star, respectively. Assuming that the disc radiates like a blackbody, then

$$F_{\rm emitted} \propto \sigma T_d^4(R)\,, \tag{9}$$

where $T_d(R)$ is the temperature of the disc at distance R. By equating the absorbed and the emitted radiation, we obtain

$$T_d(R) \propto T_\star \left(\frac{R}{R_\star}\right)^{-3/4} . \tag{10}$$

In the case of an active accretion disc, i.e. a disc that is heated by energy produced by viscous dissipation within the disc, the heating depends on how angular momentum is transferred between parts of the disc that rotate with different angular velocities. If half of the gravitational potential energy released by matter in-spiralling through an annulus between $R + \Delta R$ and R is radiated away as blackbody radiation, then

$$\frac{GM_\star \dot{M}}{2R} \frac{\Delta R}{R} \simeq 2 \times 2\pi R \, \Delta R \, \sigma T_d^4(R) \, , \tag{11}$$

where $\dot{M}$ is the accretion rate, R is the distance from the central star and $T_d(R)$ is the disc temperature at distance R. From the above, we obtain

$$T_d(R) \simeq \left(\frac{GM_\star \dot{M}}{8\pi\sigma R^3}\right)^{1/4} . \tag{12}$$

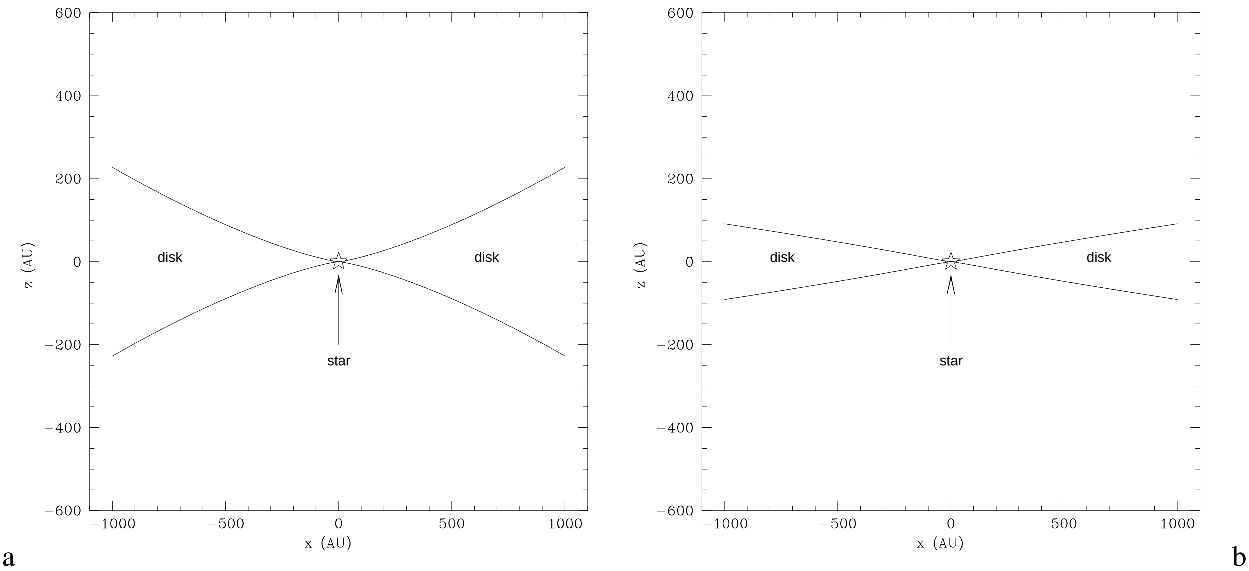
We notice that in both passive and active accretion discs the temperature in the disc is expected to vary as $R^{-3/4}$. The difference is that for active accretion discs the proportionality constant is determined by the mass accretion rate $\dot{M}$ and the mass of the central star, whereas for passive discs it is determined by the luminosity of the central star $L_\star$.

It can be shown (see Hartmann 1998) that for $T_d(R) \propto R^{-3/4}$ the SED varies with the wavelength λ as $\lambda F_\lambda \propto \lambda^{-4/3}$. However, observations show that $\lambda F_\lambda \propto \lambda^{-2/3}$, which suggests that $T_d(R) \propto R^{-3/5}$. Furthermore, SEDs of many systems are almost flat, suggesting even hotter discs. This problem could be solved if discs are flared rather than flat (Kenyon & Hartmann 1987), i.e. the thickness of the disc increases as the distance from the star increases. In this case the disc temperature decreases less rapidly, depending on the degree of flaring. The problem could also be solved by a thin envelope surrounding the star, in addition to the disc. Natta (1993) pointed out that even a small amount of dust distributed above the disc will scatter a considerable amount of radiation back towards the disc mid-plane, heating the disc. Chiang and Goldreich (1997) proposed a more refined disc model in which the outer parts of the disc are optically thin, forming a "disc atmosphere". In this model, dust grains that are in the disc atmosphere absorb unattenuated radiation from the star and become hotter, creating a superheated region in the outer regions of the disc.

We choose to parametrise the disc temperature using a general profile

$$T_d(R) = \left[T_0^2 \left(\frac{R^2 + R_0^2}{AU^2}\right)^{-q} + T_\infty^2\right]^{1/2} , \tag{13}$$

where R_0 is a softening radius that prevents the temperature from being infinitely large close to the centre of the star, T_0 is the temperature at $R = 1AU$ (provided that $R_0 << 1AU$, which is generally true) and T_∞ is the temperature far away from the star. Beckwith et al. (1990) and Osterloh & Beckwith (1995) observed a large number of PMS stars in the Tau-Aur dark cloud and found temperature power law indices q from 0.35 up to 0.8 . In the models presented here we assume $q = 0.5$, hence we do not examine the effects of different temperature profiles on the disc structure.



a b

Fig. 1. (a) Thickness z against x for a system with $M_{\rm disc}$=0.01 M$_\odot$, $R_{\rm disc}$=1000 AU, $M_\star$=1 M$_\odot$, $p = 1$, $q = 0.5$, T_0=300 K, T_∞=10 K, R_0=0.25 AU. The disc is flared. **(b)** Same as (a) but for a more massive disc ($M_{\rm disc} = 0.5$ M$_\odot$). The more massive disc is less flared.

2.3. Disc thickness

We calculate the thickness z_0 of the disc by balancing the vertical component of the gravitational force of the star and the gravitational force of the underlying disc, against the pressure force due to the disc temperature (disc thermal pressure):

$$\frac{GM_\star}{R^2}\frac{z_0(R)}{R} + \pi G\Sigma(R) \approx \frac{c_s^2(R)}{z_0(R)}, \tag{14}$$

where $c_s^2(R) = kT(R)/(\mu m_p)$ is the local sound speed. μ is the mean molecular weight of the gas in the disc ($\mu \approx 2.3$). The above equation can be written as

$$\frac{GM_\star}{R^3} z_0^2(R) + \pi G\Sigma(R) z_0(R) - c_s^2(R) = 0\,, \tag{15}$$

which is a simple quadratic equation with a positive root

$$z_0(R) = -\frac{\pi\Sigma(R)R^3}{2M_\star} + \left[\left(\frac{\pi\Sigma(R)R^3}{2M_\star}\right)^2 + \frac{R^3}{GM_\star}c_s^2(R)\right]^{1/2} \tag{16}$$

The disc thickness depends on the assumed temperature and density profile, and also on the mass of the central star. In Fig. 1 we plot the disc thickness for a low-mass ($M_{\rm disc}$=0.01 M$_\odot$, Fig. 1a) and a high-mass disc ($M_{\rm disc}$=0.5 M$_\odot$ Fig. 1b). The more massive disc is less flared.

2.4. Disc volume density

Theoretical studies of steady accretion discs (e.g. Frank, King & Raine 1992) suggest that the density of the disc drops with the distance from the disc mid-plane following a gaussian profile. Here, we assume a simple sinusoidal profile

$$\rho(R,z) = \rho(R,0)\cos\left[\frac{\pi z}{2z_0(R)}\right], \quad |z| < z_0(R)\,. \tag{17}$$

We can calculate $\rho(R,0)$ using the fact that the surface density is given by

$$\Sigma(R) = \int_{-z_0(R)}^{z_0(R)} \rho(R,z)dz\,, \tag{18}$$

where $z_0(R)$ is the thickness of the disc at distance R from the star (see Eq. 16). Substituting for $\rho(R,z)$ and integrating, we obtain

$$\Sigma(R) = \frac{4z_0(R)}{\pi}\rho(R,0)\,. \tag{19}$$

Then, we solve for $\rho(R,0)$, using Eq. (1):

$$\rho(R,0) = \frac{\pi\Sigma_0}{4z_0(R)}\left(\frac{R_0^2}{R_0^2+R^2}\right)^{p/2}. \tag{20}$$

Hence, finally the disc volume density profile is

$$\rho(R,z) = \frac{\pi\Sigma_0}{4z_0(R)}\left(\frac{R_0^2}{R_0^2+R^2}\right)^{p/2}\cos\left[\frac{\pi z}{2z_0(R)}\right]. \tag{21}$$

2.5. Disc rotation

We assume that all parts of the disc at distance R from the central axis rotate with the same velocity v, independently of the distance z from the disc mid-plane. (This is not necessarily true as the velocity may be smaller at higher z, leading to vertical momentum transfer and possibly turbulence). We calculate the initial velocity of the disc at distance R from the star by assuming that the centrifugal force is the sum of the gravitational forces of the star and the disc, on the mid-plane of the disc. We assume that there are no vertical motions in the disc i.e. $v_{z,i}=0$, where i runs over all SPH particles. The gravitational acceleration $\mathbf{g}_i$ acting on a given SPH particle i on the $x-y$ plane is calculated using the SPH tree code gravity. We then calculate the modulus v_i of the rotational velocity of the particle i

$$v_i = \sqrt{R_i|\mathbf{g}_i|}\,, \tag{22}$$

where $|\mathbf{g}_i| = \sqrt{{g_{x,i}}^2 + {g_{y,i}}^2}$ is the gravitational acceleration of the particle on the $x-y$ plane (i.e. the disc mid-plane), R_i its the distance from the star, and $v_i = \sqrt{{v_{x,i}}^2 + {v_{y,i}}^2}$.

Alternatively the disc can be set to Keplerian rotation, i.e.

$$v_i = \left(\frac{GM_\star}{R_i}\right)^{1/2}. \tag{23}$$

In the case of a self-gravitating disc (the gravity of the disc is important) the above equations is modified to the following

$$v_i = \left[\frac{G[M_\star + M_{\rm disc}(R<R_i)]}{R_i}\right]^{1/2}, \tag{24}$$

where $M_{\rm disc}(R<R_i))$ is the mass of the disc interior to particle i.

Assuming that the disc rotates counterclockwise we have

$$v_{x,i} = -v_i \frac{y_i}{R_i}\,, \tag{25}$$

$$v_{y,i} = v_i \frac{x_i}{R_i}\,. \tag{26}$$

In Fig. 3, we present the disc velocity profile for a low-mass disc ($M_{\rm disc}$=0.01 M$_\odot$, Fig. 3a) and a high-mass disc ($M_{\rm disc}$=0.5 M$_\odot$ Fig. 3b). The less massive disc rotates with keplerian velocity, because the self-gravity of the disc is not important compared to the gravity of the star. The more massive disc rotates faster than a keplerian disc due to the effect of its self-gravity.

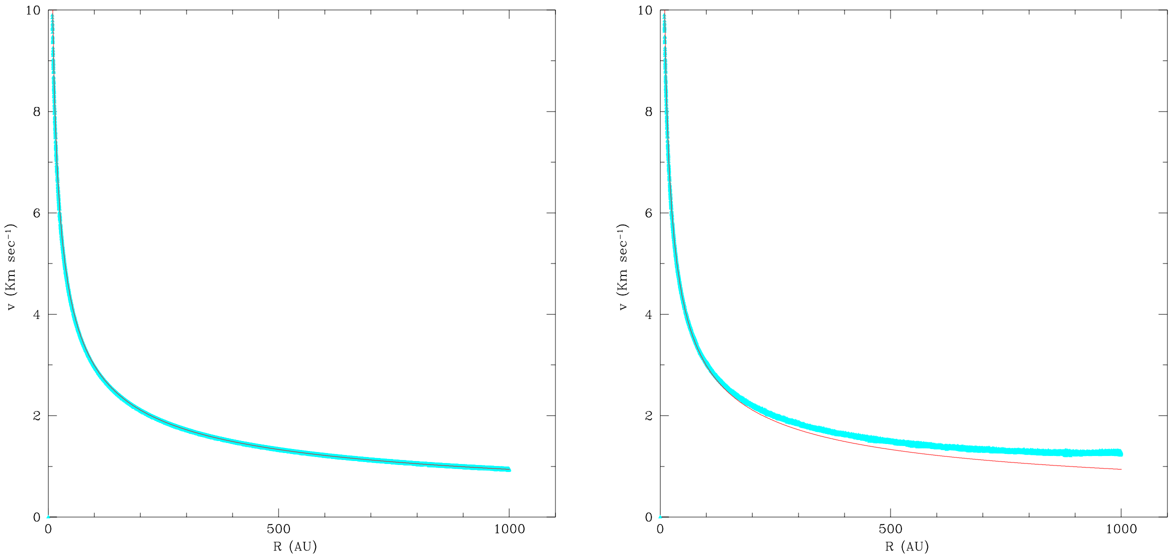


Fig. 2. **(a)** Velocity profile for a representative low-mass disc ($M_{\rm disc}$=0.01 M$_\odot$, $R_{\rm disc}$=1,000 AU, $M_\star$=1 M$_\odot$, $p = 1$, $q = 0.5$, T_0=300 K, T$_\infty$=10 K, R_0=0.25 AU, N=10,000). The solid line is the velocity of the particles if we ignore the disc self gravity (keplerian rotation). The disc rotates with the keplerian velocity. **(b)** Same as (a) but for a more massive disc ($M_{\rm disc} = 0.5$ M$_\odot$). The disc rotates faster than a keplerian disc, due to its self-gravity.

3. SPH disc setup

To construct an SPH disc, we distribute the SPH particles randomly, using a Monte Carlo approach, to reproduce the disc properties as described in the previous section. We also set the mass of each particle and its smoothing length, and impose a perturbation on the disc.

3.1. Position, mass and initial smoothing length of SPH particles

To calculate the distance R of the particle on the disc mid-plane we assume

$$\frac{M(R)}{M_{\rm disc}} = \mathcal{R}_1\,, \tag{27}$$

where $\mathcal{R}_1$ is a random number between 0 and 1. We set $\omega = R^2/R_0^2$, $\omega_0 = R_{\rm disc}^2/R_0^2$, and then substituting in Eq. (27), using Eq. (3) and Eq. (4), we obtain

$$\left[(1+\omega)^{1-(p/2)} - 1\right] = \left[(1+\omega_0)^{1-(p/2)} - 1\right]\mathcal{R}_1\,. \tag{28}$$

Solving for ω, we get

$$\omega = \left\{1 + \mathcal{R}_1\left[(1+\omega_0)^{1-(p/2)} - 1\right]\right\}^{2/(2-p)} - 1\,. \tag{29}$$

In the case of a disc with gap around the central star, we assume $\omega_{\rm in} = R_{\rm in}^2/R_0^2$, $\omega_{\rm out} = R_{\rm out}^2/R_0^2$, and using Eqs. (6), (7), we obtain

$$\left[(1+\omega)^{1-(p/2)} - (1+\omega_{\rm in})^{1-(p/2)}\right] = \left[(1+\omega_0)^{1-(p/2)} - (1+\omega_{\rm in})^{1-(p/2)}\right]\mathcal{R}_1\,. \tag{30}$$

Thus, solving for ω we get for this (more general) case

$$\omega = \left\{(1+\omega_{\rm in})^{1-(p/2)} + \mathcal{R}_1\left[(1+\omega_{\rm out})^{1-(p/2)} - (1+\omega_{\rm in})^{1-(p/2)}\right]\right\}^{2/(2-p)} - 1\,. \tag{31}$$

Finally, R is calculated from the equation

$$R = R_0\omega^{1/2}\,. \tag{32}$$

Then, we generate a second random number $\mathcal{R}_2$ to calculate the azimuthal angle ϕ:

$$\phi = 2\pi\mathcal{R}_2\,. \tag{33}$$

Thus, the x and y coordinates of the SPH particle are

$$x = R\cos(\phi)\,, \tag{34}$$

$$y = R\sin(\phi). \tag{35}$$

To calculate the vertical distance z of the SPH particle from the disc mid-plane we assume

$$\frac{\Sigma(R,z)}{\Sigma(R,z_0)} = \mathcal{R}_3, \tag{36}$$

where

$$\Sigma(R,z) = \int_{-z_0}^{z}\rho(R,z)dz \tag{37}$$

Substituting for $\rho(R,z)$ (Eq. 21), and solving the integral, we obtain

$$\Sigma(R,z) = \rho(R,0)z_0\frac{2}{\pi}\left[\sin\left(\frac{\pi z}{2z_0}\right) + 1\right]\,. \tag{38}$$

$\Sigma(R,z_0)$ is calculated by setting $z = z_0$ in the previous equation:

$$\Sigma(R,z_0) = \rho(R,0)z_0\frac{4}{\pi}\,. \tag{39}$$

After substituting the two previous equations in Eq. (36), and solving for z, we obtain for the vertical distance of the particle:

$$z(R) = z_0(R)\frac{2}{\pi}\sin^{-1}(2\mathcal{R}_3 - 1)\,, \tag{40}$$

where $z_0(R)$ is calculated from Eq. (16).

The particle mass is calculated from the total disc mass, $M_{\rm disc}$, by assuming that all particles have the same masses, hence

$$m_i = \frac{M_{\rm disc}}{N}, \tag{41}$$

where N is the total number of SPH particles used in the simulation.

The initial smoothing length of each particle is calculated from the mean number of neighbours $N_{\rm neigh}$ that each particle must have ($N_{\rm neigh} \simeq 50 \pm 5$). If h_i is the smoothing length of particle i, then the density inside a sphere of radius $2h_i$ centred on the particle is

$$\rho(R_i, z_i) = \frac{N_{\rm neigh} m_i}{\frac{4}{3}\pi(2h_i)^3}\,. \tag{42}$$

Solving the above for h_i, we obtain

$$h_i = \left[\frac{3N_{\rm neigh} m_i}{32\pi\rho(R_i, z_i)}\right]^{1/3}\,. \tag{43}$$

3.2. Azimuthal density perturbations

Azimuthal density perturbations are imposed on the disc by moving the particles along ϕ, keeping their distance from the centre constant. Consider a ring of radius r with N particles on it. Suppose that $\mu(\phi)$ is the line particle density (particles per unit length). Then for an unperturbed disc we have that

$$\mu(\phi) = \frac{N}{2\pi r}\,. \tag{44}$$

After imposing a perturbation, we want the new line density $\mu^\star$ to very with the new azimuthal angle $\phi_\star$, as

$$\mu^\star(\phi^\star) = \frac{N}{2\pi r}[1 + A\cos(m\phi^\star)]\,, \tag{45}$$

where m is the mode and A is the amplitude of the perturbation. According to the above equation, the line density is larger at specific azimuthal angles and lower at other azimuthal angles, depending on the mode m of the simulation. We can find the relationship between the old and the new azimuthal angle, using the fact that the total number of particles on the ring from 0 to ϕ before the perturbation is the same as the number of particles from 0 to $\phi^\star$ after the perturbation. Before the perturbation the number of particles from 0 to ϕ is

$$N(\phi) = \int_0^\phi \mu(\phi) r\, d\phi = \int_0^\phi \frac{N}{2\pi r} r d\phi = \frac{N\phi}{2\pi}\,. \tag{46}$$

After the perturbation the number of SPH particles from 0 to $\phi^\star$ is

$$N^\star(\phi^\star) = \int_0^{\phi^\star} \mu^\star(\phi^\star) r\, d\phi^\star = \int_0^{\phi^\star} \frac{N}{2\pi r}[1 + A\cos(m\phi^\star)] r\, d\phi^\star = \frac{N}{2\pi}[\phi^\star + \frac{A\sin(m\phi^\star)}{m}]\,. \tag{47}$$

Hence, since $N^\star(\phi) = N(\phi^\star)$,

$$\phi = \phi^\star + \frac{A}{m}\sin(m\phi^\star)\,. \tag{48}$$

4. Practical issues

Free/input parameters for disc construction

- Number of SPH particles
- Disc inner radius
- Disc outer radius
- Disc mass
- Disc centre
- Density radial profile exponent p
- Temperature radial profile exponent q
- $T(1AU)$
- R_0 (see density and temperature equations)
- $T_{\rm inf}$
- Amplitude of perturbation
- Mode of perturbation
- Stellar mass
- Stellar temperature
- Stellar sink size

5. Flowchart

You need to create a file (its format depends on the code) to be read by the SPH code that has the following: particle position, particle velocity, particle temperature, particle smoothing length, particle density, particle mass, and particle type.

- Generate 3 random numbers for each SPH particle to define its position on xy plane (Eqs. 32-35) and z (Eq. 40), where z_0 is calculated from Eq. 16.
- Calculate velocities using Eqs. 24-26.
- Calculate temperatures using Eq. 13.
- Calculate masses using Eq. 41.
- Calculate densities using Eq. 21.
- Calculate smoothing lengths using Eq. 43.
- Set particle types (1 for gas, -1 for sinks, i.e. stars or planets).

6. Visualizing data

You can use the visualization software SPLASH that is installed on the starlink (linux) machines. The userguide can be found at `http://users.monash.edu.au/~dprice/splash/`. Initialization files (splash.*) are needed to be present in your file directory. These are included in the test run directory. You can load a simulation file to splash by typing: `dsplash filename`. To plot, e.g column density (surface density) you choose ”2”, ”1”, ”9” and then hit [return].

7. File formats for code SEREN

runid.df.# file (e.g. `PL.df.000001`). The file corresponds to one snapshot of the simulation. It contains the following information:

```
150001          --> number of particles
I
I               ---> 19 integers (nothing important)
I
I
........
R
R
R              <---- 50 real numbers  (still nothing important)
R
........

 x1 y1 z1       <-- positions (pc)
 x2 y2 z2
 x3 y3 z3
 ........
 xN xN xN
 v1 v1 v1       <-- velocities (km/s)
 v2 v2 v2
 v3 v3 v3
 ........
 vN vN vN
 temp1          <-- temperatures (K)
 temp2
 .....
 tempN
 h1             <-- smoothing lenghts (pc)
 h2
 ...
 hN
 rho1           <-- densities (g cm-3)
 rho2
 ...
 rhoN
 mass1          <-- masses (Msun)
 mass2
 ....
 massN
 itype1         <-- particle type 1: gas, -1: sink (stars/planets)
 itype2
 ....
 itypeN
```

8. New disc initial conditions

Start with the momentum equation (in Eulerian form):

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\nabla)\bar{v} = -\frac{\nabla P}{\rho} + \bar{g} \tag{49}$$

The flow is steady, i.e. $\frac{\partial \bar{v}}{\partial t} = 0$, there is no radial motion $v_R = 0$ and no vertical motion $v_z = 0$. In cylindircal coordinates the momentum equation becomes (no ϕ components due to axisymmetry):

$$\frac{1}{\rho}\frac{\partial P}{\partial R} - \frac{v_\phi^2}{R} = g_R \tag{50}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = g_z \tag{51}$$

8.1. Disc without self gravity

If we neglect disc self-gravity then g is only due to the central star, therefore

$$g_R = -\frac{GM_\star}{R^2+z^2}\cos\theta\,, \tag{52}$$

and

$$g_z = -\frac{GM_\star}{R^2+z^2}\sin\theta\,, \tag{53}$$

where $\theta = \arctan(z/R)$.

For a geometrically thin disc $z << R$, therefore $R^2+z^2 \approx R^2$, $\cos\theta \approx 1$ and $\tan\theta \approx \sin\theta = z/R$. The above two equations then become

$$g_R = -\frac{GM_\star}{R^2}\,, \tag{54}$$

and

$$g_z = -\frac{GM_\star}{R^2}\frac{z}{R} = -\frac{GM_\star}{R^3}z\,. \tag{55}$$

8.1.1. Radial direction

We substitute g_R in Eq. 50, taking into account that for an isothermal disc $P(R) = c_s(R)^2\rho(R,z)$, and we get

$$\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{v_\phi^2}{R} = -\frac{GM_\star}{R^2} \Rightarrow v_\phi^2 = \frac{GM_\star}{R} + \frac{c_s^2 R}{\rho}\frac{\partial \rho}{\partial R} \tag{56}$$

Using the disc scale height $h(R) = c_s(R)/\Omega_K(R)$, where $\Omega_K(R) = (GM_\star/R^3)^{1/2}$ we get that

$$v_\phi^2 = \frac{GM_\star}{R} + \frac{GM_\star}{R^3}\frac{h^2 R}{\rho}\frac{\partial \rho}{\partial R}\,, \tag{57}$$

and finally

$$v_\phi^2 = \frac{GM_\star}{R}\left[1 + \left(\frac{h}{R}\right)^2 \frac{R}{\rho}\frac{\partial \rho}{\partial R}\right] \Rightarrow v_\phi = v_K\left[1 + \left(\frac{h}{R}\right)^2 \frac{R}{\rho}\frac{\partial \rho}{\partial R}\right]^{1/2}\,, \tag{58}$$

where $v_K = (GM_\star/R^3)^{1/2}$ is the Keplerian velocity. Since the density drops with radius it follows that $\frac{\partial\rho(R,z)}{\partial R} < 0$, and therefore the velocity of the gas is sub-keplerian. We see that for a thin disc ($h << R$), $v_\phi \approx v_K$. We also see that in the general case, we will need to take into account the variation of the disc density profile with radius to accurately calculate the azimuthal velocity (see Section 8.5).

8.1.2. Azimuthal direction

We substitute g_R in Eq. 51

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GM_\star}{R^3}z = -\Omega_K^2 z\,. \tag{59}$$

We use $P(R) = c_s(R)^2\rho(R,z)$ for a locally isothermal disc and integrate to height z to get a Gaussian vertical density profile

$$\rho(R,z) = \rho(R,0)\,\mathrm{e}^{-\frac{z^2}{2h^2}}\,, \tag{60}$$

where h is the disc scale height

$$h(R) = c_s(R)/\Omega_K(R)\,. \tag{61}$$

The density at the midplane, $\rho(R,0)$, is calculated from the surface density profile (see Eq. 18) :

$$\Sigma(R) = \int_{-\infty}^{\infty}\rho(R,z)dz = \sqrt{2\pi}h(R)\rho(R,0) \Rightarrow \rho(R,0) = \frac{\Sigma(R)}{\sqrt{2\pi}}\frac{1}{h(R)}\,. \tag{62}$$

8.2. Disc with self-gravity

Assuming the disc is geometrically thin as before, we get

$$g_R = -\frac{G\left[M_\star + M(<R)\right]}{R^2}\,, \tag{63}$$

and

$$g_z = -\frac{G\left[M_\star + M(<R)\right]}{R^3}z + g_z^D\,, \tag{64}$$

where $M(<R)$ is the mass within radius R, and g_z the self gravity of the disc in the z-direction. We therefore assume that the radial component of the disc self gravity can be approximated by considering that the disc mass within radius R is located in the position of the star. This is true for a spherical mass distribution but I am not sure it is valid for a mass distribution in a disc. With the above assumption the azimuthal velocity is given by

$$v_\phi = \left(\frac{G\left[M_\star + M(<R)\right]}{R^3}\right)^{1/2} \tag{65}$$

The more general case of self gravitating discs in different contexts is examined in Bertin & Lodato (1999), Bardou et al. (1998) and Binney & Tremaine (2008).

In short, in the case of homogeneous self-gravitating infinite slab but without a central star (Spitzer, 1942) the vertical density profile is given by

$$\rho(R,z) = \rho(R,0)\,\mathrm{sech}^2\left(\frac{z}{h}\right)\,, \tag{66}$$

where $h \equiv h(R) = c_s(R)^2/\pi G\Sigma(R)$, and $\rho(R,0) = \Sigma(R)/2h$.

In the case of a self-gravitating disc with a central star the vertical density profile can be approximated (see Bertin & Lodato, 1999) by a Gaussian function (see Eq. 60)

$$\rho(R, z) = \rho(R, 0)\, \mathrm{e}^{-\frac{z^2}{2h^2}}\,, \tag{67}$$

where the scale height h is given by

$$h(R) = \sqrt{\frac{2}{\pi}} \frac{c_s(R)^2}{G\Sigma(R)} \frac{1}{4Q^2(R)} \left(\sqrt{1 + 8Q^2(R)/\pi} - 1 \right) \Rightarrow \tag{68}$$

$$h(R) = \sqrt{\frac{\pi}{8}} \frac{c_s}{\Omega} \left(\sqrt{\frac{1}{Q^2} + \frac{8}{\pi}} - \frac{1}{Q} \right)\,, \tag{69}$$

where

$$Q(R) = \frac{c_s(R)\Omega(R)}{\pi G\Sigma(R)}\,, \tag{70}$$

is the Toomre parameter (note sure about whether Ω should include the mass within radius R). We that for a non self-gravitating disc ($Q \to \infty$) then $h(R) \to c_s/\Omega$, as expected.

Note that these are approximations only (not sure how accurate they are). Ideally the gravitational acceleration should be computed from the hydrodynamics code and then used to assign velocities. However, the gravity actually determines the vertical positions, so an iterative method to settle the disc is needed for optimal accuracy.

8.3. Setting the vertical particle positions for a Gaussian vertical density profile

We assume that the disc vertical density profile is given by a Gaussian

$$\rho(R, z) = \rho(R, 0)\, \mathrm{e}^{-\frac{z^2}{2h^2}}\,, \tag{71}$$

where $h \equiv h(R)$ is given from Eq. 61 (when ignoring disc self-gravity) or from Eq. 69 (for a self-gravitating disc). As in Eq. 36 we have

$$\frac{\Sigma(R, z)}{\Sigma(R, \infty)} = \frac{\int_{-\infty}^{z} \rho(R, z) dz}{\int_{-\infty}^{\infty} \rho(R, z) dz} = \mathcal{R}_3\,, \tag{72}$$

where $\mathcal{R}_3$ is random number between 0 and 1. We calculate the two integrals in the above fraction:

$$\Sigma(R, z) = \int_{-\infty}^{z} \rho(R, z) dz = \rho(R, 0) \int_{-\infty}^{z} \mathrm{e}^{-\frac{z^2}{2h^2}} dz = \rho(R, 0)\, h \frac{\sqrt{2\pi}}{2} \left[\mathrm{erf} \left(\frac{z}{\sqrt{2}h} \right) + 1 \right] \tag{73}$$

and

$$\Sigma(R, \infty) = \rho(R, 0)\, h\sqrt{2\pi}\,. \tag{74}$$

Therefore from Eq. 72 we obtain

$$z(R) = \sqrt{2}h(R)\, \mathrm{erf}^{-1}(2\mathcal{R}_3 - 1)\,. \tag{75}$$

Using the above equation we can assign the particles's vertical position using a sequence of random numbers. Note that erf^{-1} is the inverse erf function. To use it in python you need to import the special functions from the scipy package: `from scipy import special`. Then the command would be: `special.erfinv(2*R3-1)`.

8.4. Comparison of different vertical density profiles

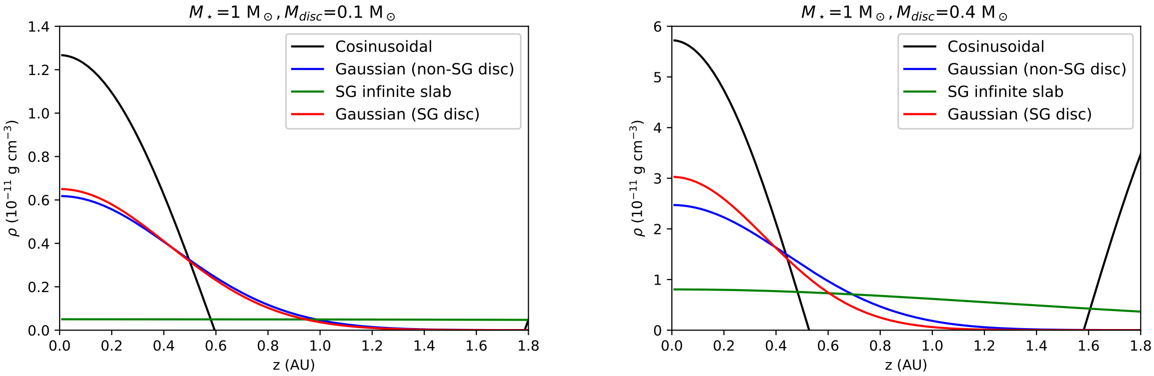


Fig. 3. Comparison of the vertical density profiles (plotted at R=10 AU) with the different approaches described above.

8.5. Azimuthal velocity correction

As we found in Section 8.1.1, in the general case the azimuthal velocity is

$$v_\phi^2 = \frac{GM_\star}{R} + \frac{c_s^2 R}{\rho}\frac{\partial\rho(R,z)}{\partial R}\,, \tag{76}$$

where

$$\rho(R,z) = \rho(R,0)\ \mathrm{e}^{-\frac{z^2}{2h(R)^2}}\,, \tag{77}$$

and

$$\rho(R,0) = \frac{\Sigma(R)}{\sqrt{2\pi}}\frac{1}{h(R)}\,. \tag{78}$$

Since there is not much difference between the self-gravitating and the non self-gravitating case, we will assume that $h(R) \to c_s/\Omega$, and then we can find that

$$\frac{\partial\rho(R,z)}{\partial R} = -\frac{\rho}{R}\left[1 + p - \left(\frac{z}{h(R)}\right)^2\right]\,, \tag{79}$$

and finally

$$v_\phi = v_K \left\{1 - \left[\frac{h(R)}{R}\right]^2 \left[1 + p - \left(\frac{z}{h(R)}\right)^2\right]\right\}^{1/2}\,, \tag{80}$$

where p is the surface density index ($\Sigma \propto R^{-p}$).

8.6. Comments

To set the initial disc conditions, we have made a number of simplifying assumptions:

- Disc is geometrically thin ($R >> z$)
- Disc is locally isothermal, i.e. the temperature is only a function of the distance from the star (there is no vertical variation of the temperature)

- Approximations about the disc self gravity (cannot be calculated analytically). This approximation affects the computed vertical density profile, but also the computed azimuthal velocity (more specifically through the assumption that we include the disc mass within radius R in our calculations).

References

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