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Abstract

For a general quantum system driven by a slowly time-dependent Hamiltonian, transitions between instantaneous eigenstates are exponentially weak. But a nearby Hamiltonian exists for which the transition amplitudes between any eigenstates of the original Hamiltonian are exactly zero for all values of slowness. The general theory is illustrated by spins driven by changing magnetic fields, and implies that any spin expectation history, including those where the spin never precesses, can be generated by infinitely many driving fields, here displayed explicitly. Asymptotically, the absence of transitions is explained by continuation to complex time, where the complex degeneracies in the transitionless driving fields have a nongeneric structure for which there is no Stokes phenomenon; this is analogous to the explanation of reflectionless potentials.

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1. Introduction

A changing Hamiltonian $\hat{H}_0(t)$ typically induces transitions between quantum states that are driven by it. If the change is slow, it is natural to refer these changes to the adiabatic basis, that is, the basis of eigenstates of the instantaneous ('frozen') $\hat{H}_0(t)$. If the slow change is analytic and described by a small parameter ε, the familiar adiabatic theorem [1] guarantees that if the system starts in one of the instantaneous eigenstates, and if the state remains non-degenerate, then it will follow this state closely. But although the transition amplitude is very small, it is not zero: familiar asymptotic analysis [2–5] indicates that it is of order exp(-constant/ε). The standard example is the exactly solvable Landau–Majorana–Zener (LMZ) Hamiltonian [6–8] for two-state systems.

Nevertheless, in what can be regarded as a 'reverse engineering' perspective, it is easy to find Hamiltonians $\hat{H}(t)$, associated with any chosen $\hat{H}_0(t)$, that drive the instantaneous eigenstates of $\hat{H}_0(t)$ exactly: there are no transitions between them, even if ε is large. We give a general formula for $\hat{H}(t)$ in section 2. An example is spins driven by changing magnetic fields (section 3), for which the result implies the existence of classical spin vectors that do not precess when driven by changing magnetic fields. For the special case of spin 1/2, which

also represents general two-state quantum evolution, the LMZ Hamiltonian can be modified to get a closely related transitionless version (section 4). The explanation of the fact that these transition amplitudes are exactly zero, rather than exponentially small, is given in section 5. It involves asymptotics related to the analytic continuation of the modified $\hat{H}(t)$, which differs from that of $\hat{H}_0(t)$ in a crucial respect. The relation between this reverse engineering view of quantum evolution and the usual direct approach is explained in the concluding section (section 6).

For the special case of a two-state system, the reverse engineering problem has been approached in a different way by Garanin and Schilling [9]: they keep the off-diagonal elements of the 2×2 Hamiltonian matrix constant, and alter the time dependence of the diagonal elements to generate any desired relation between the states at $t = \pm\infty$.

2. General transitionless tracking algorithm

Consider an arbitrary time-dependent Hamiltonian $\hat{H}_0(t)$, with instantaneous eigenstates and energies given by

$$\hat{H}_0(t)|n(t)\rangle = E_n(t)|n(t)\rangle. \tag{2.1}$$

In the adiabatic approximation, the states driven by $\hat{H}_0(t)$ would be

$$|\psi_n(t)\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' E_n(t') - \int_0^t \mathrm{d}t' \langle n(t')|\partial_{t'} n(t')\rangle\right\}|n(t)\rangle. \tag{2.2}$$

For convenience, this incorporates the effective vector potential $\langle n(t)|\partial_t n(t)\rangle$ that generates the geometric phase [10, 11] in cases where the evolution is cyclic, that is $|n(T)\rangle = |n(0)\rangle$ for some T.

In the reverse engineering approach adopted here, we seek a Hamiltonian $\hat{H}(t)$ for which these are the exact evolving states, satisfying

$$\mathrm{i}\hbar\partial_t|\psi_n(t)\rangle = \hat{H}(t)|\psi_n(t)\rangle. \tag{2.3}$$

For this Hamiltonian, the states must follow $|n(t)\rangle$ exactly: there are no transitions between the eigenstates of $\hat{H}_0(t)$, not only after infinite time but for all times.

To find $\hat{H}(t)$, we first note that any time-dependent unitary operator $\hat{U}(t)$ is the solution of

$$\mathrm{i}\hbar\partial_t\hat{U}(t) = \hat{H}(t)\hat{U}(t), \tag{2.4}$$

where

$$\hat{H}(t) = \mathrm{i}\hbar(\partial_t\hat{U}(t))\hat{U}^\dagger(t). \tag{2.5}$$

Then choosing

$$\hat{U}(t) = \sum_n \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' E_n(t') - \int_0^t \mathrm{d}t' \langle n(t')|\partial_{t'} n(t')\rangle\right\}|n(t)\rangle\langle n(0)|, \tag{2.6}$$

we find from (2.5) that the Hamiltonian driving the eigenstates $|n(t)\rangle$ according to (2.3) is

$$\hat{H}(t) = \sum_n |n\rangle E_n\langle n| + \mathrm{i}\hbar\sum_n(|\partial_t n\rangle\langle n| - \langle n|\partial_t n\rangle|n\rangle\langle n|) \equiv \hat{H}_0(t) + H_1(t) \tag{2.7}$$

in which the explicit t-dependence of quantities has been omitted and all kets represent $|n(t)\rangle$ not $|n(0)\rangle$. This Hamiltonian has been obtained before [12, 13], as a tool in proofs of the adiabatic theorem.

To get a more convenient form, we eliminate the derivative $|\partial_t n\rangle$ by differentiating (2.1):

$$\langle m|\partial_t n\rangle = \frac{\langle m|\partial_t \hat{H}_0|n\rangle}{E_n - E_m}. \tag{2.8}$$

Now $\hat{H}_1(t)$ can be written as

$$\hat{H}_1(t) = \mathrm{i}\hbar \sum_{m\neq n}\sum \frac{|m\rangle\langle m|\partial_t \hat{H}_0|n\rangle\langle n|}{E_n - E_m}. \tag{2.9}$$

This algorithm gives the Hamiltonian $\hat{H}(t)$ that drives the eigenstates $|n(t)\rangle$ of $\hat{H}_0(t)$ exactly, that is without generating transitions between them. Note that this single $\hat{H}(t)$works for all the states $|n(t)\rangle$. In a sense, we can regard the set of states $|n(t)\rangle$ as 'moving eigenstates' of $\hat{H}(t)$.

An unanticipated feature of the result (2.9) emerges in the adiabatic regime of small ε, in which the original Hamiltonian varies slowly and so can be written as $\hat{H}_0(\varepsilon t)$. It might seem that the correction $\hat{H}_1(t)$, which eliminates the transitions of order $\exp(-1/\varepsilon)$ that would be generated by $\hat{H}_0(t)$ alone, would itself be of order $\exp(-1/\varepsilon)$. But it is not: from (2.9), the correction is of order ε.

Instead of specifying $\hat{H}_0(t)$, we could specify the complete orthonormal set of states $|n(t)\rangle$. Then the freedom to choose $E_n(t)$ shows that there are infinitely many Hamiltonians that can generate the evolution $|n(t)\rangle$; they are distinguished by the choice of phases in (2.2). The simplest choice is $E_n(t) = 0$, for which the bare states $|n(t)\rangle$, with no phase factors, are driven by

$$\hat{H}(t) = \mathrm{i}\hbar \sum_n |\partial_t n(t)\rangle\langle n(t)| \tag{2.10}$$

reflecting the identity

$$|\partial_t n(t)\rangle = \sum_m |\partial_t m(t)\rangle\langle m(t)|n(t)\rangle. \tag{2.11}$$

As an immediate application, transitionless driving provides a mechanism for producing the cyclic evolution postulated in the Aharonov–Anandan (AA) version of the geometric phase [14]: simply choose a Hamiltonian satisfying $\hat{H}_0(T) = \hat{H}_0(0)$, and single-valued eigenstates $|n(T)\rangle = |n(0)\rangle$; then (2.7) will generate this cyclic evolution exactly, with the AA phase given by the vector potential integral in (2.2).

3. Spins driven by magnetic fields

For a spin driven by a magnetic field $\boldsymbol{B}_0(t)$, we choose

$$\hat{H}_0(t) = \gamma \mathbf{B}_0(t)\cdot\hat{\mathbf{S}}, \tag{3.1}$$

where γ is the gyromagnetic ratio and $\hat{\mathbf{S}}$ is the vector spin operator for a particle with arbitrary spin quantum number s. For $s = 1/2$,

$$\hat{\mathbf{S}} = \frac{1}{2}\hbar\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}, \tag{3.2}$$

involving the Pauli matrices.

For general s, (2.9) is

$$\hat{H}_1(t) = \mathrm{i}\hbar\gamma\,\partial_t\mathbf{B}_0\cdot\sum_{m\neq n}\sum \frac{|m\rangle\langle m|\hat{\mathbf{S}}|n\rangle\langle n|}{E_n - E_m}, \tag{3.3}$$

in which the states $|n\rangle$ are eigenstates of (3.1), with energies

$$E_n = \gamma\hbar n B_0(t), \tag{3.4}$$

where $B_0(t)$ is the length $|\boldsymbol{B}_0(t)|$. To get an explicit form for $\hat{H}_1(t)$, we use manipulations similar to those in the analogous calculation of geometric phases [10], and temporarily rotate axes so that $\boldsymbol{B}_0$ is in the z-direction. Then the only nonzero matrix elements are

$$\begin{aligned}
&\langle n|\hat{S}_z|n\rangle = \hbar n,\\
&\langle n \pm 1|\hat{S}_x|n\rangle = \tfrac{1}{2}\hbar\sqrt{(s(s+1) - n(n \pm 1))},\\
&\langle n \pm 1|\hat{S}_y|n\rangle = \mp\tfrac{1}{2}i\hbar\sqrt{(s(s+1) - n(n \pm 1))}.
\end{aligned} \tag{3.5}$$

Substituting in (3.3) we get, after some manipulation,

$$\hat{H}_1 = \frac{1}{B_0}(\partial_t B_{0x}\hat{S}_y - \partial_t B_{0y}\hat{S}_x), \tag{3.6}$$

and so, reverting to general axes

$$\hat{H}_1 = \frac{1}{B_0^2}\partial_t\mathbf{B}_0 \times \hat{\mathbf{S}} \cdot \mathbf{B}_0 = \frac{1}{B_0^2}\mathbf{B}_0 \times \partial_t\mathbf{B}_0 \cdot \hat{\mathbf{S}} \tag{3.7}$$

Thus the Hamiltonian that drives the spin states of (3.1) transitionlessly is

$$\begin{aligned}
\hat{H}(t) &= \left[\gamma\mathbf{B}_0(t) + \frac{1}{B_0(t)^2}\mathbf{B}_0(t) \times \partial_t\mathbf{B}_0(t)\right] \cdot \hat{\mathbf{S}}\\
&= [\gamma\mathbf{B}_0(t) + \mathbf{b}_0(t) \times \partial_t\mathbf{b}_0(t)] \cdot \hat{\mathbf{S}} \equiv \gamma\mathbf{B}(t) \cdot \hat{\mathbf{S}}
\end{aligned} \tag{3.8}$$

involving a modified magnetic field

$$\mathbf{B}(t) = \mathbf{B}_0(t) + \frac{1}{\gamma}\mathbf{b}_0(t) \times \partial_t\mathbf{b}_0(t). \tag{3.9}$$

Note that the modification is independent of γ and involves only the field direction

$$\mathbf{b}_0(t) = \frac{\mathbf{B}_0(t)}{B_0(t)}. \tag{3.10}$$

The result (3.9) holds for any spin s. For the special case $s = 1/2$, (3.8) is a 2×2 matrix Hamiltonian that generates transitionless evolution in any quantum two-state system.

The modified field (3.9) has a classical interpretation, in terms of the vector spin expectation value

$$\mathbf{S}(t) \equiv \langle\psi_n(t)|\hat{\mathbf{S}}|\psi_n(t)\rangle, \tag{3.11}$$

in which $|\psi_n(t)\rangle$ is one of the states (2.2), satisfying (2.3) with the Hamiltonian (3.8). As is well known, this satisfies the classical equation

$$\partial_t\mathbf{S}(t) = \gamma\mathbf{B}(t) \times \mathbf{S}(t). \tag{3.12}$$

Since this preserves the length $|\boldsymbol{S}(t)|$, it is convenient in the following to choose $\boldsymbol{S}(t)$ as a unit vector.

For a general state and a general driving field $\boldsymbol{B}(t)$, that is, a field not generated by the reverse engineering formula (3.9), the dynamical equation (3.12) would describe a spin precessing round the instantaneous direction $\boldsymbol{b}(t)$ with angular velocity $\gamma B(t)$. The adiabatic approximation corresponds to large $\gamma B(t)$, in which the precession is fast compared with the variation of the direction $\boldsymbol{b}(t)$. If at some initial time we choose an extreme eigenstate, corresponding to a quantum number $n = \pm s$, the corresponding instantaneous spin vector is directed along $\boldsymbol{b}(t)$ and does not precess. Nevertheless, changes of the direction $\boldsymbol{b}(t)$ will

eventually cause $\boldsymbol{S}(t)$ to deviate from $\boldsymbol{b}(t)$ and then it will precess. Such weak precession is the classical analogue of the weak quantum transitions induced by slowly changing fields.

But we are considering fields $\boldsymbol{B}(t)$ that are not arbitrary. The arbitrary field is $\boldsymbol{B}_0(t)$, which is related to $\boldsymbol{B}(t)$ by (3.9). Therefore for an extreme eigenstate the unit spin vector is given by

$$\mathbf{S}(t) = \pm\mathbf{b}_0(t), \tag{3.13}$$

not only initially but forever: the modification (3.9) ensures that when driven by $\boldsymbol{B}(t)$ the spin is perpetually precessionless. This can be stated in a different way: any choice of spin evolution $\boldsymbol{S}(t)$, that is any time-dependent path on the unit spin sphere, can be generated with (3.12) exactly by infinitely many driving fields, namely, from (3.9),

$$\mathbf{B}(t) = B_0(t)\mathbf{S}(t) + \frac{1}{\gamma}\mathbf{S}(t) \times \partial_t\mathbf{S}(t), \tag{3.14}$$

in which $B_0(t)$ can be any function of time. If $B_0(t) = 0$, the evolution equation (3.12) reduces to the identity

$$\partial_t\mathbf{S}(t) = (\mathbf{S}(t) \times \partial_t\mathbf{S}(t)) \times \mathbf{S}(t). \tag{3.15}$$

An elementary application of (3.14) is to reproduce fields that drive a spin uniformly rotating on a cone with opening angle 2θ: the evolution

$$\mathbf{S}(t) = \sin\theta(\mathbf{e}_x\cos\Omega t + \mathbf{e}_y\sin\Omega t) + \cos\theta\mathbf{e}_z \tag{3.16}$$

can be generated by a driving field rotating on a different cone:

$$\mathbf{B}(t) = \left(B_0(t) - \frac{1}{\gamma}\Omega\cos\theta\right)\sin\theta(\mathbf{e}_x\cos\Omega t + \mathbf{e}_y\sin\Omega t) + \left(B_0(t)\cos\theta + \frac{1}{\gamma}\Omega\sin^2\theta\right)\mathbf{e}_z. \tag{3.17}$$

The spin rotation should not be regarded as precession, because there need be no relation between Ω and the instantaneous precession angular velocity $\gamma B(t)$.

If $B_0(t)$ is constant, (3.17) represents a uniformly rotating field as employed in NMR [15], but the same evolution is generated for any $B_0(t)$. Note that the freedom to choose $B_0(t)$ means that there is no unique relation between the cone angles of $\boldsymbol{B}(t)$ and $\boldsymbol{S}(t)$; for a specified driving cone, the spin cone angle depends on the initial condition.

4. Transitionless Landau–Majorana–Zener model

In the standard two-state LMZ model [6–8], the driving field varies along a straight line in $\boldsymbol{B}$ space, with uniform speed V and closest approach $B_{\min}$ to the origin $\boldsymbol{B} = 0$. For transitionless evolution (nonprecessing spin) we require $\boldsymbol{B}_0(t)$ to have this behaviour, so we choose

$$\mathbf{B}_0(t) = B_{\min}\mathbf{e}_x + Vt\mathbf{e}_z. \tag{4.1}$$

The unit spin vector evolves according to

$$\mathbf{S}(t) = \frac{B_{\min}\mathbf{e}_x + Vt\mathbf{e}_z}{\sqrt{B_{\min}^2 + V^2t^2}}, \tag{4.2}$$

so its path on the spin (Bloch) sphere is a great circle, starting at the south pole $\{0, 0, -1\}$ at $t = -\infty$ and ending at the north pole $\{0, 0, +1\}$at $t = +\infty$.

According to (3.14), the field that would generate this evolution is

$$\mathbf{B}(t) = B_{\min}\mathbf{e}_x + Vt\mathbf{e}_z - \frac{\varepsilon B_{\min}V}{B_{\min}^2 + V^2t^2}\mathbf{e}_y, \tag{4.3}$$

so the driving trajectory in $\boldsymbol{B}$ space is a plane curve that coincides with $\boldsymbol{B}_0$ at $t = \pm\infty$ but includes an excursion out of the xz-plane near $t = 0$.

The standard LMZ evolution, in which $\boldsymbol{B}(t)$, rather than $\boldsymbol{B}_0(t)$, is given by (4.1), is much more complicated (see section 6). The solution to (2.3) for the quantum state (with $s = 1/2$), or (3.12) for the classical spin vector, involves parabolic cylinder functions [14], leading to a transition amplitude

$$\exp\left\{-\frac{\pi\gamma B_{\min}^2}{4V}\right\}. \tag{4.4}$$

How can it be that the more complicated driving field (4.3) generates the much simpler spin evolution (4.2), in which there are no transitions? This is the subject of the following section.

5. Adiabatic explanation of transitionlessness

A general theory for the development of exponentially weak transitions (i.e. spin precession), in the adiabatic regime of large γB_0, has been developed in detail for the quantum $s = 1/2$ case and where the components of the driving field $\boldsymbol{B}(t)$ are analytic functions. This involves continuation to pairs of complex-conjugate times t_c, t_c^* for which $B(t_c) = 0$ [2, 5]. These are branchpoints of

$$B(t) = \sqrt{B_x^2(t) + B_y^2(t) + B_z^2(t)}. \tag{5.1}$$

At the complex instant t_c, the complexified Hamiltonian in (3.8) is degenerate and the adiabatic approximation breaks down. A Stokes line connects t_c and t_c^* and the time when this crosses the real axis marks the appearance of the exponentially weak transition [5, 16].

The explanation of the failure of the standard argument when the driving field is given by (3.9) lies in the radically different nature of the zero at t_c, compared with that of the zeros t_{c0} of $B_0(t)$. Since the two terms in (3.9) are orthogonal, the zeros are given by

$$\gamma^2 B_0^6(t_c) + |\mathbf{B}_0(t_c) \times \dot{\mathbf{B}}_0(t_c)|^2 = 0. \tag{5.2}$$

The components of $\boldsymbol{B}_0(t)$ are assumed to be smooth functions of t, so the second term on the left-hand side will be finite. Thus, close to t_{c0},

$$\begin{aligned} B_0(t) &= \sqrt{A(t - t_{c0}) + B(t - t_{c0})^2 + \cdots}, \\ |\mathbf{B}_0(t) \times \dot{\mathbf{B}}_0(t)|^2 &= C + D(t - t_{c0}) + \cdots. \end{aligned} \tag{5.3}$$

For large γ there are three solutions to (4.2) close to t_{c0}, namely

$$t_{cn} = t_{c0} + \frac{\gamma^{-2/3}}{A}(-C)^{1/3}\exp\left(\frac{2}{3}\mathrm{i}\pi n\right) \quad (n = 1, 2, 3). \tag{5.4}$$

Thus the term modifying $\boldsymbol{B}_0(t)$ in (3.8), ensuring the absence of transitions, splits each branchpoint into three, separated by a distance of order $\gamma^{-2/3}$. Precisely this situation occurs in the analogous problem of reflectionless potentials [17] so it is not necessary to repeat the argument in detail. The result is that local expansion of the associated second-order differential equation near the cluster of zeros does not give an Airy function (involving Bessel function of order $\pm 1/3$), whose Stokes phenomenon would generate the small exponential responsible for adiabatic quantum transitions and spin precession. Rather, the local expansion gives a 'fake Airy function': a Bessel function of order $1/2$, whose asymptotic expansion terminates at the first term so there is no Stokes phenomenon and therefore no small exponential.

There are connections between reflectionlessness, and therefore the special case of transitionlessness in the two-state case, and Ermakov–Lewis invariants [18, 19], but I do not pursue these here.

6. Comparison of reverse and direct engineering

The reverse engineering approach answers the question: what Hamiltonian $\hat{H}(t)$ will generate a given evolution, that is, evolving states that are instantaneous eigenstates of a given Hamiltonian $\hat{H}_0(t)$? In theoretical physics it is customary to pose the question in the opposite way: we seek the evolution driven by a specified Hamiltonian $\hat{H}(t)$, which is equivalent to finding $\hat{H}_0(t)$ by solving the time-dependent Schrödinger equation.

Determining $\hat{H}_0(t)$ is equivalent to inverting (2.7). One way to do this is by iteration (cf (2.9)) of

$$\hat{H}_0(t) = \hat{H}(t) - \mathrm{i}\hbar \sum_{m\neq n} \sum \frac{|m\rangle\langle m|\partial_t \hat{H}_0|n\rangle\langle n|}{E_n - E_m}. \tag{6.1}$$

This generates a version of the familiar adiabatic series, with each iteration introducing a higher derivative of $\hat{H}(t)$—and similar to other schemes, for example iteration to give the time-ordered evolution operator [20].

But the adiabatic iteration diverges, and it is instructive to illustrate the 'reverse reverse engineering' with the exact solution of the LMZ example, in the classical formulation involving the spin expectation $\boldsymbol{S}(t)$. To save writing, we choose $B_{\min} = V = 2$ in (4.1), so the evolution equation is of the form (3.12) with

$$\mathbf{B}(t) = 2\{1, 0, t\}. \tag{6.2}$$

We can solve for the spin using the solution of the quantum equation

$$\mathrm{i}\partial_t |\psi(t)\rangle = \gamma \begin{pmatrix} t & 1 \\ 1 & -t \end{pmatrix} |\psi(t)\rangle. \tag{6.3}$$

An exact solution can be expressed in terms of parabolic cylinder functions D:

$$|\psi(t)\rangle = \begin{Bmatrix} u(t) \\ v(t) \end{Bmatrix} = \exp\left(-\frac{\pi\gamma}{8}\right) \begin{Bmatrix} -\mathrm{i}\left(\frac{1}{2}\mathrm{i}\gamma\right)^{1/2} D_{\mathrm{i}\gamma/2-1}(t\exp(3\pi\mathrm{i}/4)\sqrt{2\gamma}) \\ D_{\mathrm{i}\gamma/2}(t\exp(3\pi\mathrm{i}/4)\sqrt{2\gamma}) \end{Bmatrix}, \tag{6.4}$$

and satisfies

$$|u(t)|^2 + |v(t)|^2 = 1. \tag{6.5}$$

The corresponding evolving unit spin is given exactly by (3.11):

$$\mathbf{S}(t) = \{2\,\mathrm{Re}[u*(t)v(t)],\, 2\,\mathrm{Im}[u*(t)v(t)],\, |u(t)|^2 - |v(t)|^2\}. \tag{6.6}$$

From this exact solution we seek $B_0(t)$ in (3.14), from which

$$\mathbf{B}_0(t) = (\mathbf{B}(t)\cdot\mathbf{S}(t))\mathbf{S}(t) = 2(2\,\mathrm{Re}[u*(t)v(t)] + t(|u(t)|^2 - |v(t)|^2))\mathbf{S}(t). \tag{6.7}$$

Figures 1(*a*) and (*b*) show the components of $\boldsymbol{B}_0(t)$ for several values of γ. For small γ, the curves are very different from those for $\boldsymbol{B}(t)$ (equation 6.2), but they approach $\boldsymbol{B}(t)$ for large γ. When substituted into (3.14), $\boldsymbol{B}_0(t)$ reproduces $\boldsymbol{B}(t)$ exactly, whatever the value of γ. In fact the redundancy is much greater, because different solutions of (6.3) will give different spin evolutions. The general time-dependent solution of (6.3) can be written in terms of (6.4) as follows:

$$|\psi(t)\rangle = \cos\left(\tfrac{1}{2}\theta\right)\exp\left(\tfrac{1}{2}\mathrm{i}\phi\right)\begin{Bmatrix} u(t) \\ v(t) \end{Bmatrix} + \sin\left(\tfrac{1}{2}\theta\right)\exp\left(-\tfrac{1}{2}\mathrm{i}\phi\right)\begin{Bmatrix} -v*(t) \\ u*(t) \end{Bmatrix} \tag{6.8}$$

in which θ and ϕ are arbitrary. Each choice of θ and ϕ (and γ) gives a different function $\boldsymbol{B}_0(t)$ (figures 1(*b*) and (*c*)), but all reproduce the same $\boldsymbol{B}(t)$ when substituted into (3.14). Note that none of the fields $\boldsymbol{B}_0(t)$ are asymptotic to $\boldsymbol{B}(t)$ for large $|t|$, reflecting the inevitabilty of transitions between the adiabatic states.

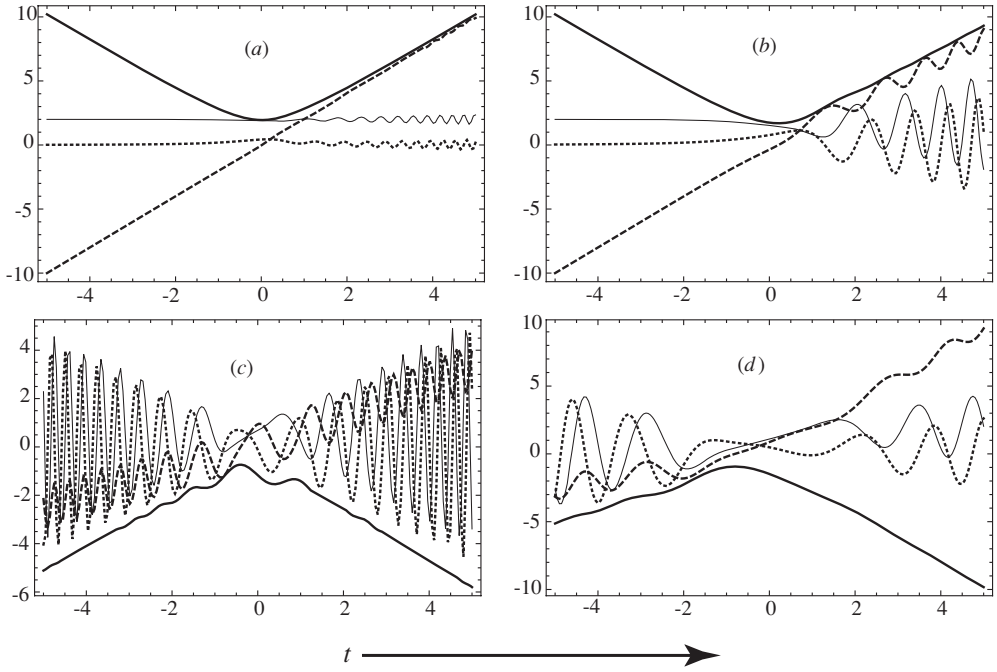

Figure 1. Magnetic field $\boldsymbol{B}_0(t)$ (equation (6.7)) for Landau–Majorana–Zener Hamiltonian whose eigenstates are exact time-dependent states for driving field $\boldsymbol{B}(t) = 2\{1, 0, t\}$. Thick curve: length $B_0(t) = \boldsymbol{B}(t) \cdot \boldsymbol{S}(t)$; thin curve: $B_{0x}(t)$; dotted curve: $B_{0y}(t)$; dashed curve: $B_{0z}(t)$. The fields are generated from the states (6.8) with (*a*) $\gamma = 2.5, \theta = 0$; (*b*) $\gamma = 1, \theta = 0$; (*c*) $\gamma = 2, \theta = 30^\circ, \phi = 22.5^\circ$; (*d*) $\gamma = 0.59, \theta = 30^\circ, \phi = 22.5^\circ$. Note that the oscillations in the components are much stronger than the oscillations in the length $B_0(t)$.

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