3 Induction

3.1 The sum of odd numbers

It is time to learn one of the most important tools in discrete mathematics. We start with a question: *We add up the first n odd numbers. What do we get?*

Perhaps the best way to try to find the answer is to experiment. If we try small values of n, this is what we find:

$$\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
1+3+5+7+9+11 &= 36\\
1+3+5+7+9+11+13 &= 49\\
1+3+5+7+9+11+13+15 &= 64\\
1+3+5+7+9+11+13+15+17 &= 81\\
1+3+5+7+9+11+13+15+17+19 &= 100
\end{aligned}$$

It is easy to observe that we get squares; in fact, it seems from this examples that *the sum of the first n odd numbers is n^2*. This we have observed for the first 10 values of n; can we be sure that it is valid for all? Well, I'd say we can be reasonably sure, but not with mathematical certainty. How can we *prove* the assertion?

Consider the sum for a general n. The n-th odd number is $2n-1$ (check!), so we want to prove that

$$1+3+\ldots+(2n-3)+(2n-1)=n^2. \tag{2}$$

If we separate the last term in this sum, we are left with the sum of the first $(n-1)$ odd numbers:

$$1+3+\ldots+(2n-3)+(2n-1)=\Big(1+3+\ldots+(2n-3)\Big)+(2n-1)$$

Now here the sum in the large parenthesis is $(n-1)^2$, so the total is

$$(n-1)^2+(2n-1)=(n^2-2n+1)+(2n-1)=n^2, \tag{3}$$

just as we wanted to prove.

Wait a minute! Aren't we using in the proof the statement that we are proving? Surely this is unfair! One could prove everything if this were allowed.

But in fact we are not quite using the same. What we were using, is the assertion about the sum of the first $n-1$ odd numbers; and we argued (in (3)) that this proves the assertion about the sum of the first n odd numbers. In other words, what we have shown is that if the assertion is true for a certain value of n, it is also true for the next.

This is enough to conclude that the assertion is true for every n. We have seen that it is true for $n=1$; hence by the above, it is also true for $n=2$ (we have seen this anyway by direct computation, but this shows that this was not even necessary: it followed from the case $n = 1$).

In a similar way, the truth of the assertion for $n = 2$ implies that it is also true for $n = 3$, which in turn implies that it is true for $n = 4$, etc. If we repeat this sufficiently many times, we get the truth for any value of n.

This proof technique is called *induction* (or sometimes *mathematical induction*, to distinguish it from a notion in philosophy). It can be summarized as follows.

Suppose that we want to prove a property of positive integers. Also suppose that we can prove two facts:

(a) 1 has the property, and

(b) whenever $n-1$ has the property, then also n has the property ($n \geq 1$).

The *principle of induction* says that if (a) and (b) are true, then every natural number has the property.

Often the best way to try to carry out an induction proof is the following. We try to prove the statement (for a general value of n), and we are allowed to use that the statement is true if n is replaced by $n-1$. (This is called the *induction hypothesis.*) If it helps, one may also use the validity of the statement for $n-2$, $n-3$, etc., in general for every k such that $k < n$.

Sometimes we say that if 0 has the property, and every integer n *inherits* the property from $n-1$, then every integer has the property. (Just like if the founding father of a family has a certain piece of property, and every new generation inherits this property from the previous generation, then the family will always have this property.)

3.1 Prove, using induction but also without it, that $n(n+1)$ is an even number for every non-negative integer n.

3.2 Prove by induction that the sum of the first n positive integers is $n(n+1)/2$.

3.3 Observe that the number $n(n+1)/2$ is the number of handshakes among $n+1$ people. Suppose that everyone counts only handshakes with people older than him/her (pretty snobbish, isn't it?). Who will count the largest number of handshakes? How many people count 6 handshakes?

Give a proof of the result of exercise 3.1, based on your answer to these questions.

3.4 Give a proof of exercise 3.1, based on figure 3.

3.5 Prove the following identity:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1) \cdot n = \frac{(n-1) \cdot n \cdot (n+1)}{3}.$$

Exercise 3.1 relates to a well-known (though apocryphal) anecdote from the history of mathematics. Carl Friedrich Gauss (1777-1855), one of the greatest mathematicians of all times, was in elementary school when his teacher gave the class the task to add up the integers from 1 to 1000 (he was hoping that he would get an hour or so to relax while his students were working). To his great surprise, Gauss came up with the correct answer almost immediately. His solution was extremely simple: combine the first term with the last, you get $1 + 1000 = 1001$; combine the second term with the last but one,

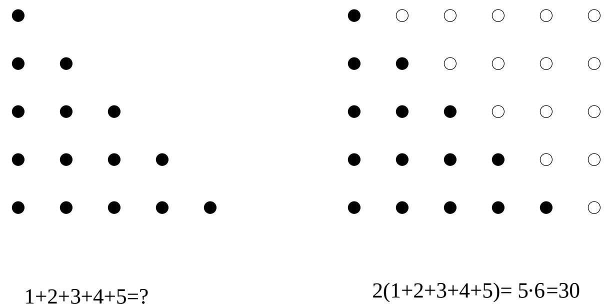

Figure 3: The sum of the first n integers

you get $2+999=1001$; going on in a similar way, combining the first remaining term with the last one (and then discarding them) you get 1001. The last pair added this way is $500+501=1001$. So we obtained 500 times 1001, which makes 500500. We can check this answer against the formula given in exercise 3.1: $1000 \cdot 1001/2 = 500500$.

3.6 Use the method of the little Gauss to give a third proof of the formula in exercise 3.1

3.7 How would the little Gauss prove the formula for the sum of the first n odd numbers (2)?

3.8 Prove that the sum of the first n squares $(1+4+9+\ldots+n^2)$ is $n(n+1)(2n+1)/6$.

3.9 Prove that the sum of the first n powers of 2 (starting with $1=2^0$) is 2^n-1.

3.2 Subset counting revisited

In chapter 2 we often relied on the convenience of saying "etc.": we described some argument that had to be repeated n times to give the result we wanted to get, but after giving the argument once or twice, we said "etc." instead of further repetition. There is nothing wrong with this, if the argument is sufficiently simple so that we can intuitively see where the repetition leads. But it would be nice to have some tool at hand which could be used instead of "etc." in cases when the outcome of the repetition is not so transparent.

The precise way of doing this is using induction, as we are going to illustrate by revisiting some of our results. First, let us give a proof of the formula for the number of subsets of an n-element set, given in Theorem 2.1 (recall that the answer is 2^n).

As the principle of induction tells us, we have to check that the assertion is true for $n=0$. This is trivial, and we already did it. Next, we assume that $n>0$, and that the assertion is true for sets with $n-1$ elements. Consider a set S with n elements, and fix any element $a \in S$. We want to count the subsets of S. Let us divide them into two classes: those containing a and those not containing a. We count them separately.

First, we deal with those subsets which don't contain a. If we delete a from S, we are left with a set S' with $n-1$ elements, and the subsets we are interested in are exactly the subsets of S'. By the induction hypothesis, the number of such subsets is 2^{n-1}.

Second, we consider subsets containing a. The key observation is that every such subset consists of a and a subset of S'. Conversely, if we take any subset of S', we can add a to it to get a subset of S containing a. Hence the number of subsets of S containing a is the same as the number of subsets of S', which, as we already know, is 2^{n-1}. (With the jargon we introduced before, the last piece of the argument establishes as one-to-one correspondence between those subsets of S containing a and those not containing a.)

To conclude: the total number of subsets of S is $2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$. This proves Theorem 2.1 (again).

3.10 Use induction to prove Theorem 2.2 (the number of strings of length n composed of k given elements is k^n) and Theorem 2 (the number of permutations of a set with n elements is $n!$).

3.11 Use induction on n to prove the "handshake theorem" (the number of handshakes between n people in $n(n-1)/2$).

3.12 Read carefully the following induction proof:

ASSERTION: *$n(n+1)$ is an odd number for every n.*

PROOF: *Suppose that this is true for $n-1$ in place of n; we prove it for n, using the induction hypothesis. We have*

$$n(n+1) = (n-1)n + 2n.$$

Now here $(n-1)n$ is odd by the induction hypothesis, and $2n$ is even. Hence $n(n+1)$ is the sum of an odd number and an even number, which is odd.

The assertion that we proved is obviously wrong for $n = 10$: $10 \cdot 11 = 110$ is even. What is wrong with the proof?

3.13 Read carefully the following induction proof:

ASSERTION: *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*

PROOF: *The assertion is true for one line (and also for 2, since we have assumed that no two lines are parallel). Suppose that it is true for any set of $n-1$ lines. We are going to prove that it is also true for n lines, using this induction hypothesis.*

So consider a set of $S = \{a, b, c, d, \ldots\}$ of n lines in the plane, no two of which are parallel. Delete the line c, then we are left with a set S' of $n-1$ lines, and obviously no two of these are parallel. So we can apply the induction hypothesis and conclude that there is a point P such that all the lines in S' go through P. In particular, a and b go through P, and so P must be the point of intersection of a and b.

Now put c back and delete d, to get a set S'' of $n-1$ lines. Just as above, we can use the induction hypothesis to conclude that these lines go through the same point P'; but just like above, P' must be the point of intersection of a and b. Thus $P' = P$. But then we see that c goes through P. The other lines also go through P (by the choice of P), and so all the n lines go through P.

But the assertion we proved is clearly wrong; where is the error?

3.3 Counting regions

Let us draw n lines in the plane. These lines divide the plane into some number of regions. How many regions do we get?

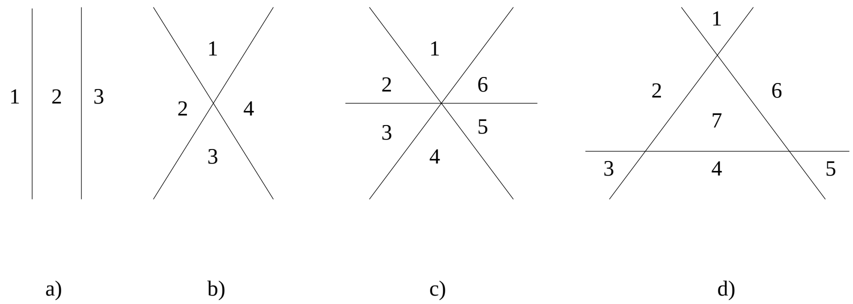

Figure 4:

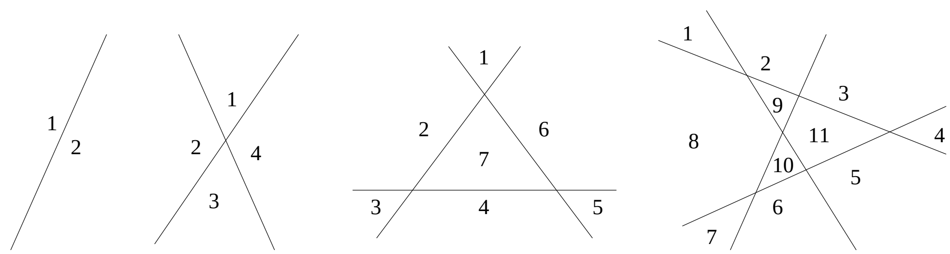

Figure 5:

A first thing to notice is that this question does not have a single answer. For example, if we draw two lines, we get 3 regions if the two are parallel, and 4 regions if they are not.

OK, let us assume that no two of the lines are parallel; then 2 lines always give us 4 regions. But if we go on to three lines, we get 6 regions if the lines go through one point, and 7 regions, if they do not (Figure 4).

OK, let us also exclude this, and assume that no 3 lines go through the same point. One might expect that the next unpleasant example comes with 4 lines, but if you experiment with drawing 4 lines in the plane, with no two parallel and no three going threw the same point, then you invariably get 11 regions (Figure 5). In fact, we'll have a similar experience for any number of lines.

A set of lines in the plane such that no two are parallel and no three go through the same point is said to be *in general position*. If we choose the lines "randomly" then accidents like two being parallel or three going through the same point will be very unlikely, so our assumption that the lines are in general position is quite natural.

Even if we accept that the number of regions is always the same for a given number of lines, the question still remains: what is this number? Let us collect our data in a little table (including also the observation that 0 lines divide the plane into 1 region, and 1 line divides the plane into 2):

0	1	2	3	4
1	2	4	7	11

Staring at this table for a while, we observe that each number in the second row is the sum of the number above it and the number before it. This suggests a rule: the n-th entry is n plus the previous entry. In other words: *If we have a set of* $n-1$ *lines in the plane* *in general position, and add a new line (preserving general position), then the number of regions increases by n.*

Let us prove this assertion. How does the new line increase the number of regions? By cutting some of them into two. The number of additional regions is just the same as the number of regions intersected.

So, how many regions does the new line intersect? At a first glance, this is not easy to answer, since the new line can intersect very different sets of regions, depending on where we place it. But imagine to walk along the new line, starting from very far. We get to a new region every time we cross a line. So the number of regions the new line intersects is one larger than the number of crossing points on the new line with other lines.

Now the new line crosses every other line (since no two lines are parallel), and it crosses them in different points (since no three lines go through the same point). Hence during our walk, we see $n-1$ crossing points. So we see n different regions. This proves that our observation about the table is true for every n.

We are not done yet; what does this give for the number of regions? We start with 1 region for 0 lines, and then add to it $1,2,3,\ldots,n$. This way we get

$$1+(1+2+3+\ldots+n) = 1+\frac{n(n+1)}{2}.$$

Thus we have proved:

Theorem 3.1 *A set of n lines in general position in the plane divides the plane into $1+n(n+1)/2$ regions.*

3.14 Describe a proof of Theorem 3.1 using induction on the number of lines.

Let us give another proof of Theorem 3.1; this time, we will not use induction, but rather try to relate the number of regions to other combinatorial problems. One gets a hint from writing the number in the form $1+n+\binom{n}{2}$.

Assume that the lines are drawn on a vertical blackboard (Figure 6), which is large enough so that all the intersection points appear on it. We also assume that no line is horizontal (else, we tilt the picture a little), and that in fact every line intersects the bottom edge of the blackboard (the blackboard is very long).

Now consider the lowest point in each region. Each region has only one lowest point, since the bordering lines are not horizontal. This lowest point is then an intersection point of two of our lines, or the intersection point of line with the lower edge of the blackboard, or the lower left corner of the blackboard. Furthermore, each of these points is the lowest point of one and only one region. For example, if we consider any intersection point of two lines, then we see that four regions meet at this point, and the point is the lowest point of exactly one of them.

Thus the number of lowest points is the same as the number of intersection points of the lines, plus the number of intersection points between lines and the lower edge of the blackboard, plus one. Since any two lines intersect, and these intersection points are all different (this is where we use that the lines are in general position), the number of such lowest points is $\binom{n}{2}+n+1$.

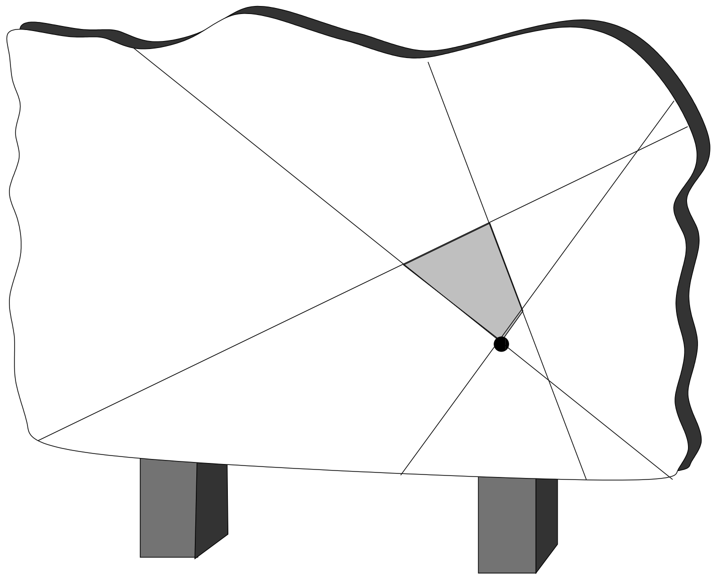

Figure 6:

4 Counting subsets

4.1 The number of ordered subsets

At a competition of 100 athletes, only the order of the first 10 is recorded. How many different outcomes does the competition have?

This question can be answered along the lines of the arguments we have seen. The first place can be won by any of the athletes; no matter who wins, there are 99 possible second place winners, so the first two prizes can go $100 \cdot 99$ ways. Given the first two, there are 98 athletes who can be third, etc. So the answer is $100 \cdot 99 \cdot \ldots \cdot 91$.

4.1 Illustrate this argument by a tree.

4.2 Suppose that we record the order of all 100 athletes.

(a) How many different outcomes can we have then?

(b) How many of these give the same for the first 10 places?

(c) Show that the result above for the number of possible outcomes for the first 10 places can be also obtained using (a) and (b).

There is nothing special about the numbers 100 and 10 in the problem above; we could carry out the same for n athletes with the first k places recorded.

To give a more mathematical form to the result, we can replace the athletes by any set of size n. The list of the first k places is given by a sequence of k elements of n, which all have to be different. We may also view this as selecting a subset of the athletes with k elements, and then ordering them. Thus we have the following theorem.

Theorem 4.1 *The number of ordered k-subsets of an n-set is $n(n-1)\ldots(n-k+1)$.*

(Note that if we start with n and count down k numbers, the last one will be $n-k+1$.)

4.3 If you generalize the solution of exercise 4.1, you get the answer in the form

$$\frac{n!}{(n-k)!}$$

Check that this is the same number as given in theorem 4.1.

4.4 Explain the similarity and the difference between the counting questions answered by theorem 4.1 and theorem 2.2.

4.2 The number of subsets of a given size

From here, we can easily derive one of the most important counting results.

Theorem 4.2 *The number of k-subsets of an n-set is*

$$\frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Recall that if we count *ordered* subsets, we get $n(n-1)\dots(n-k+1) = n!/(n-k)!$, by Theorem 4.1. Of course, if we want to know the number of *unordered* subsets, then we have overcounted; every subset was counted exactly $k!$ times (with every possible ordering of its elements). So we have to divide this number by $k!$ to get the number of subsets with k elements (without ordering).

The number of k-subsets of an n-set is such an important quantity that one has a separate notation for it: $\binom{n}{k}$ (read: 'n choose k'). Thus

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Thus the number of different lottery tickets in $\binom{90}{5}$, the number of handshakes is $\binom{7}{2}$ etc.

4.5 Which problems discussed during the party were special cases of theorem 4.2?

4.6 Tabulate the values of $\binom{n}{k}$ for $n, k \le 5$.

In the following exercises, try to prove the identities by using the formula in theorem 4.2, and also without computation, by explaining both sides of the equation as the result of a counting problem.

4.7 Prove that $\binom{n}{2} + \binom{n+1}{2} = n^2$.

4.8 (a) Prove that $\binom{90}{5} = \binom{89}{5} + \binom{89}{4}$.

(b) Formulate and prove a general identity based on this.

4.9 Prove that $\binom{n}{k} = \binom{n}{n-k}$.

4.10 Prove that
$$1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=2^n.$$

4.11 Prove that for $0<c\le b\le a$,
$$\binom{a}{b}\binom{b}{c}=\binom{a}{a-c}\binom{a-c}{b-c}$$

4.3 The Binomial Theorem

The numbers $\binom{n}{k}$ also have a name, *binomial coefficients*, which comes from a very important formula in algebra involving them. We are now going to discuss this theorem.

The issue is to compute powers of the simple algebraic expression $(x+y)$. We start with small examples:
$$(x+y)^2=x^2+2xy+y^2,$$
$$(x+y)^3=(x+y)\cdot(x+y)^2=(x+y)\cdot(x^2+2xy+y^2)=x^3+3x^2y+3xy^2+y^3,$$
and, going on like this,
$$(x+y)^4=(x+y)\cdot(x+y)^3=x^4+4x^3y+6x^2y^2+4xy^3+y^4.$$
You may have noticed that the coefficients you get are the numbers that we have seen, e.g. in exercise 4.2, as numbers $\binom{n}{k}$. Let us make this observation precise. We illustrate the argument for the next value of n, namely $n=5$, but it works in general.

Think of expanding
$$(x+y)^5=(x+y)(x+y)(x+y)(x+y)(x+y)$$
so that we get rid of all parentheses. We get each term in the expansion by selecting one of the two terms in each factor, and multiplying them. If we choose x, say, 2 times then we choose y 3 times, and we get x^2y^3. How many times do we get this same term? Clearly as many times as the number of ways to select the two factors that supply x (the remaining factors supply y). Thus we have to choose two factors out of 5, which can be done in $\binom{5}{2}$ ways.

Hence the expansion of $(x+y)^5$ looks like this:
$$(x+y)^5=\binom{5}{0}y^5+\binom{5}{1}xy^4+\binom{5}{2}x^2y^3+\binom{5}{3}x^3y^2+\binom{5}{4}x^4y+\binom{5}{5}x^5.$$

We can apply this argument in general to obtain

Theorem 4.3 (The Binomial Theorem) *The coefficient of x^ky^{n-k} in the expansion of $(x+y)^n$ is $\binom{n}{k}$. In other words, we have the identity:*
$$(x+y)^n=y^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\ldots+\binom{n}{n-1}x^{n-1}y+\binom{n}{n}x^n.$$

This important theorem is called the Binomial Theorem; the name comes from the Greek word *binome* for an expression consisting of two terms, in this case, $x+y$. The appearance of the numbers $\binom{n}{k}$ in this theorem is the source of their name: *binomial coefficients.*

The Binomial Theorem can be applied in many ways to get identities concerning binomial coefficients. For example, let us substitute $x = y = 1$, then we get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}. \tag{4}$$

Later on we are going to see trickier applications of this idea. For the time being, another twist on it is contained in the next exercise.

4.12 Give a proof of the Binomial Theorem by induction, based on exercise 4.2.

4.13 (a) Prove the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \ldots = 0$$

(The sum ends with $\binom{n}{n} = 1$, with the last depending on the parity of n.)

(b) This identity is obvious if n is odd. Why?

4.14 Prove identity 4, using a combinatorial interpretation of the two sides (recall exercise 4.2).

4.4 Distributing presents

Suppose we have n different presents, which we want to distribute to k children. For some reason, we are told how many presents should each child get; so Adam should get n_{Adam} presents, Barbara, n_{Barbara} presents etc. In a mathematically convenient (though not very friendly) way, we call the children $1, 2, \ldots, k$; thus we are given the numbers (non-negative integers) $n_1, n_2, \ldots, n_k$. We assume that $n_1 + n_2 + \ldots + n_k = n$, else there is no way to distribute the presents.

The question is, of course, how many ways can these presents be distributed?

We can organize the distribution of presents as follows. We lay out the presents in a single row of length n. The first child comes and picks up the first n_1 presents, starting from the left. Then the second comes, and picks up the next n_2; then the third picks up the next n_3 presents etc. Child No. k gets the last n_k presents.

It is clear that we can determine who gets what by choosing the order in which the presents are laid out. There are $n!$ ways to order the presents. But, of course, the number $n!$ overcounts the number of ways to distribute the presents, since many of these orderings lead to the same results (that is, every child gets the same set of presents). The question is, how many?

So let us start with a given distribution of presents, and let's ask the children to lay out the presents for us, nicely in a row, starting with the first child, then continuing with the second, third, etc. This way we get back *one* possible ordering that leads to the current distribution. The first child can lay out his presents in $n_1!$ possible orders; no matter which order he chooses, the second child can lay out her presents in $n_2!$ possible ways, etc. So the

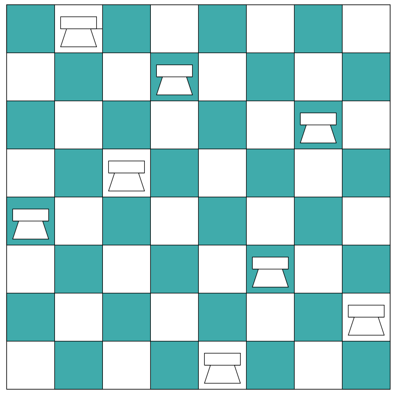

Figure 7: Placing 8 non-attacking rooks on a chessboard

number of ways the presents can be laid out (given the distribution of the presents to the children) is a product of factorials:

$$n_1! \cdot n_2! \cdot \ldots \cdot n_k!.$$

Thus the number of ways of distributing the presents is

$$\frac{n!}{n_1!n_2!\ldots n_k!}.$$

4.15 We can describe the procedure of distributing the presents as follows. First, we select n_1 presents and give them to the first child. This can be done in $\binom{n}{n_1}$ ways. Then we select n_2 presents from the remaining $n - n_1$ and give them to the second child, etc.

Complete this argument and show that it leads to the same result as the previous one.

4.16 The following special cases should be familiar from previous problems and theorems. Explain why.

(a) $n = k, n_1 = n_2 = \ldots = n_k$;

(b) $n_1 = n_2 = \ldots = n_{k-1} = 1, n_k = n - k + 1$;

(c) $k = 2$;

(d) $k = 3, n = 6, n_1 = n_2 = n_3 = 2$.

4.17 (a) How many ways can you place n rooks on a chessboard so that no two attack each other (Figure 7)? We assume that the rooks are identical, so e.g. interchanging two rooks does not count as a separate placement.

(b) How many ways can you do this if you have 4 black and 4 white rooks?

(c) How many ways can you do this if all the 8 rooks are different?