Compactness results for flowing to a harmonic map

Author: Edmund A. Paxton Supervisor: Prof. Melanie Rupflin

July 25, 2018

Contents

1 Harmonic maps & applications

1.1 Introduction

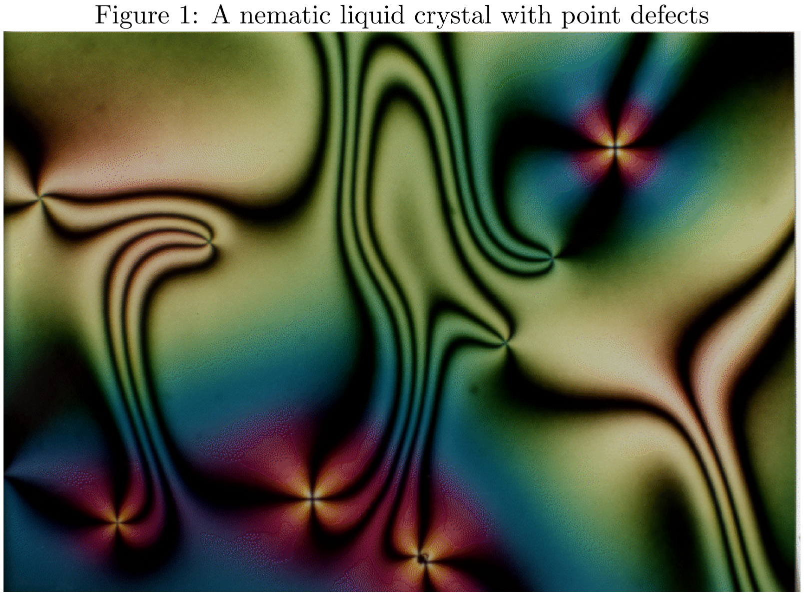
Harmonic maps arise as a generalisation of harmonic functions and are critical points of an "energy" functional. We will motivate the energy functional with three physical examples.

In electrodynamics, electric field lines arise as the gradient of a potential function $V\colon \mathbb{R}^3 \to \mathbb{R}$. The potential energy stored in an electric field, equal to the work required to position the corresponding point charges, is given by the Dirichlet integral $E = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla V|^2 dx$. It is observed that an electric field in a vacuum will form so as to minimise the stored energy amongst all fields satisfying the boundary conditions, and so the potential function satisfies Laplace's equation $\Delta V = 0$. Such functions are classically called *harmonic functions*.

If a rubber band stretched between two points of a smooth, curved surface $\Sigma \subset \mathbb{R}^3$ is described by a path $\gamma\colon [0,1] \to \Sigma$, then up to first order the potential energy stored in the band arising from the tension will be proportional to the integral $E = \frac{1}{2}\int_0^1 |\gamma'(t)|^2 dt$. For an equilibrium position this energy must be a local minimum amongst all paths between the two points whose images are constrained to lie on the surface Σ. The path γ will thus satisfy the geodesic equation for the surface.

A liquid crystal in a nematic phase consists of short, rod-like molecules aligned in the direction of some unit vector $\mathbf{n} \in \mathbb{R}^3$, $|\mathbf{n}| = 1$. It is energetically favourable for the molecules to align with each other and, under a very simple model of a liquid crystal in a domain $\Omega \subset \mathbb{R}^3$, the molecules will orient themselves so as to minimize potential energy by minimizing the integral $\int_\Omega |\nabla \mathbf{n}|^2 dx$ amongst maps $\mathbf{n}$ from Ω to $\mathbb{R}^3$ whose images are constrained to lie on the unit sphere $S^2 = \{x \in \mathbb{R}^3 \colon |x| = 1\}$, subject to the boundary conditions.

Figure 1: A nematic liquid crystal with point defects



These examples highlight how the nature of the minimization problem for the Dirichlet energy integral changes when attention is focused on a class of maps with images constrained to a subset with non-trivial geometry. Laplace's equation always admits unique, smooth solutions, and electrical field lines are uniquely determined by the distribution of charges generating them. The geodesic equation admits solutions which are smooth, but they are only locally unique. For example, between two points of a cylinder or two poles of a sphere are an infinite number of equilibrium positions. In the former there is a unique geodesic in each homotopy class, whereas in the latter we have a continuum of geodesics in a single homotopy class. Figure 1.1 illustrates how it is energetically favourable for nematic liquid crystals to form point defects, and so in this case minima are not smooth and we cannot in general hope for more than the existence of weak solutions to the corresponding Euler-Lagrange equations.

1.2 The Dirichlet integral

The general setting in which we consider the Dirichlet integral is for maps between two Riemannian manifolds (M, g) and $(\mathcal{N}, h)$ of dimensions m and n. For simplicity we will always consider smooth manifolds equipped with smooth metric tensors. There are difficulties defining Sobolev functions between manifolds intrinsically, since a Sobolev function is not necessarily continuous and thus need not map small balls into any one chart in the image. For this reason and for more general ease of notation, it is customary to consider the target $\mathcal{N}$ as isometrically embedded in a flat Euclidean space $\mathbb{R}^N$, which is always possible by the Nash embedding theorem.

Let (M, g) be a Riemannian manifold and let $\mathbb{T}M$ denote its tangent bundle. We write $X \in \Gamma(\mathbb{T}M)$ for a section X of the bundle. Given two tensors of rank (p, q) on M, $X, Y \in \Gamma(\otimes^p \mathbb{T}M \otimes^q \mathbb{T}^* M)$, we can contract with respect to the metric g to give a function $\langle X, Y\rangle_g \colon M \to \mathbb{R}$. We define $|X|_g^2 = \langle X, X\rangle_g$. Letting $d\mu = d\mu_g$ denote the volume form associated to the metric, we can define an inner product on tensors of the same rank by $\langle\langle X, Y\rangle\rangle = \int_M \langle X, Y\rangle_g d\mu$.

Let $\mathcal{C}^\infty(M)$ denote the class of smooth functions on M, $\Gamma(\mathbb{T}M)$ the sections of $\mathbb{T}M$ with $\mathcal{C}^\infty(\mathbb{T}M)$ the smooth sections. The *gradient* is defined as the operator $\nabla_g \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathbb{T}M)$ satisfying $\langle \nabla_g \varphi, X\rangle_g = d\varphi(X)$ for any $X \in \mathcal{C}^\infty(\mathbb{T}M)$. We define the *divergence* as the adjoint operator for $-\nabla_g$ with respect to the inner product $\langle\langle\cdot\rangle\rangle$.
That is, $\operatorname{div}_g \colon \mathcal{C}^\infty(\mathbb{T}M) \to \mathcal{C}^\infty(M)$ so that

$$\int_M \langle \nabla_g \varphi, X\rangle_g d\mu = -\int_M \langle \varphi, \operatorname{div}_g X\rangle_g d\mu$$

We say that a function $f \colon M \to \mathbb{R}$ is of Sobolev class 1 and write $f \in H^1(M, \mathbb{R})$ if there exists $X \in \Gamma(\mathbb{T}M)$ such that

$$\int_M \langle X, Y\rangle_g d\mu = -\int_M f \cdot \operatorname{div}_g(Y) d\mu$$

for all sections $Y \in \mathcal{C}^\infty(\mathbb{T}M)$ whose support on M is a compact set, and such that $\int_M |X|_g^2 d\mu < \infty$. We call such X the weak derivative of f and denote it by $\nabla_g f$. We note that this agrees with the previous definition for the gradient whenever f is a smooth function.

There is a map, which we also denote by ∇_g, given by the Levi-Civita connection associated to the metric g, $\nabla_g \colon \mathcal{C}^\infty(\mathbb{T}M) \to \mathcal{C}^\infty(\mathbb{T}M \otimes \mathbb{T}^* M)$. We may define the adjoint for $-\nabla_g$, $\operatorname{div}_g \colon \mathcal{C}^\infty(\mathbb{T}M \otimes \mathbb{T}^* M) \to \mathcal{C}^\infty(\mathbb{T}M)$

by

$$\int_M \langle \nabla_g X, W\rangle_g d\mu = -\int_M \langle X, \mathrm{div}_g W\rangle_g d\mu$$

We say that a function $f\colon M \to \mathbb{R}$ is of Sobolev class 2 and write $f \in H^2(M,\mathbb{R})$ if it is of Sobolev class 1 with weak derivative $\nabla_g f$ and if there exists $Z \in \Gamma(\mathbb{T}M \otimes \mathbb{T}^*M)$ such that

$$\int_M \langle Z, W\rangle_g d\mu = -\int_M \langle \nabla_g f, \mathrm{div}_g W\rangle_g d\mu$$

for all sections $W \in \mathcal{C}^\infty(\mathbb{T}M \otimes \mathbb{T}^*M)$ whose support on M is a compact set, and such that $\int_M |Z|_g^2 d\mu < \infty$. We denote the second weak derivative Z by $\nabla_g^2 f$

We say a function $u\colon M \to \mathbb{R}^N$ is of Sobolev class s for $s = 1, 2$ and write $f \in H^s(M, \mathbb{R}^N)$ if each component $u^j\colon M \to \mathbb{R}$ is of Sobolev class s for $j = 1, \ldots, N$. Now if $\mathcal{N}$ is a Riemannian manifold isometrically embedded in $\mathbb{R}^N$, we define

$$H^s(M,\mathcal{N}) = \{u \in H^s(M,\mathbb{R}^N)\colon \mathrm{Im}(u) \subset \mathcal{N} \quad \mathcal{H}^m - \text{almost everywhere}\}$$

Now let $u \in H^1(M,\mathcal{N})$. We define the *energy density* by $e(u) = \sum_{j=1}^N \langle \nabla_g u^j, \nabla_g u^j\rangle_g$. If $g_{ij}, h_{\alpha\beta}$ denote the metric tensors on $M,\mathcal{N}$ expressed in local coordinates, then the energy density is given in summation notation by $g^{ij} h_{\alpha\beta} \partial_i u^\alpha \partial_j u^\beta$. We define the Dirichlet energy functional by

$$E\colon H^1(M,\mathcal{N}) \to \mathbb{R}$$
$$E[u] = \frac{1}{2}\int_M e(u) d\mu$$

and if U is an open subset of M, we write $E[u, U] = \frac{1}{2}\int_U e(u) d\mu$.

Critical points of the energy functional are called *harmonic maps*.

On a function $f \in H^2(M,\mathbb{R})$, the Laplace operator is defined by $\Delta_g f = \mathrm{div}_g \circ \nabla_g f$. When f is classically differentiable, in local coordinates it is given by $\Delta_g f = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j) f$, where $\sqrt{g} = \sqrt{\det(g_{ij})}$. For $u \in H^2(M,\mathbb{R}^N)$, we set $(\Delta_g u)^j = \Delta_g(u^j)$.

Lemma 1.1. *Let $u \in H^2(M,\mathcal{N})$ be a harmonic map. Then the tangent space to $\mathcal{N}$ at the point $u(x)$ is orthogonal to $\Delta_g u(x)$ in $\mathbb{R}^N$ for almost all $x \in M$:*

$$\Delta_g u \perp \mathbb{T}_u \mathcal{N}$$

Remark 1.2. This lemma generalises to harmonic maps the classical fact that if a curve is a geodesic on a surface $\Sigma \subset \mathbb{R}^3$ then its curvature vector is perpendicular to Σ.

Proof. Let U be a small neighbourhood of a point on M. We note that since $\mathcal{N}$ is embedded in $\mathbb{R}^N$ we have $\mathbb{T}_p\mathcal{N} \subset \mathbb{T}_p\mathbb{R}^N$ for each $p \in \mathcal{N}$. Choose a smooth vector field $X \in \Gamma(\mathbb{T}\mathbb{R}^N)$ along the image of M, $u(M) \subset \mathbb{R}^N$, so that

1. $X(u(y)) \in \mathbb{T}_{u(y)}\mathcal{N}$ for $y \in U$
2. $X(u(y)) = 0$ for $y \notin U$

Since the Euclidean space $\mathbb{R}^N$ is flat, its tangent space at each point may be canonically associated with $\mathbb{R}^N$ by parallel transport to the origin. Thus we may view this vector field X as a map from M to $\mathbb{R}^N$, which we denote by v. We have then by condition 1 that $u(x) + tv(x) + \mathcal{O}(t^2) \in \mathcal{N}$ for all $x \in M$. Since u in harmonic, this implies

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} E[u + tv + \mathcal{O}(t^2)] \\ &= \sum_{j=1}^{N} \int_M \langle \nabla_g u^j, \nabla_g v^j \rangle_g d\mu \\ &= -\int_M \Delta_g u \cdot v d\mu \\ &= -\int_U \Delta_g u \cdot v d\mu \end{aligned}$$

where $\cdot$ denotes the standard dot product on $\mathbb{R}^N$.

Now, by the fundamental lemma of the calculus of variations, since our vector field X can be chosen to point in any direction tangential to $\mathcal{N}$ in the interior of the image of U, it follows that $\Delta_g u(x) \perp \mathbb{T}_u(x)\mathcal{N}$ for almost all points x as claimed. □

Lemma 1.3. *If $u\colon M \to \mathcal{N}$, then the component of $\Delta_g u$ orthogonal to the tangent space $\mathbb{T}_u\mathcal{N}$ is determined almost everywhere by the second fundamental form, which we denote by A:*

$$(\Delta_g u)^\perp = -A(u)(\nabla u, \nabla u)$$

where $A(u)(\nabla u, \nabla u)$ is given for a classically differential function by $g^{ij} A(u)(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j})$.

Proof. See [12] pg. 154 □

The preceding lemmas give the condition for a harmonic map. $u\colon M \to \mathcal{N}$ is a harmonic map if it satisfies the harmonic map equation:

$$\Delta_g u + A(u)(\nabla u, \nabla u) = 0$$

in the weak sense. The vector field $\tau(u) = \Delta_g u + A(u)(\nabla u, \nabla u)$ is called the *tension field*. The second fundamental form is a symmetric bilinear form and since $\mathcal{N}$ is compact we have $|A(u)(\nabla u, \nabla u)|^2 \leq C|\nabla u|^2$ almost everywhere where C depends only on $\mathcal{N}$ and its embedding in $\mathbb{R}^N$.

Example 1.4 (Nematic liquid crystal)**.** For maps from an open set $\Omega \subset \mathbb{R}^3$ to the unit sphere S^2 equipped with the round metric, the harmonic map equation is

$$\Delta u + u|\nabla u|^2 = 0$$

As discussed, in this example we can only expect solutions to exist in the weak sense.

1.3 Harmonic maps from surfaces

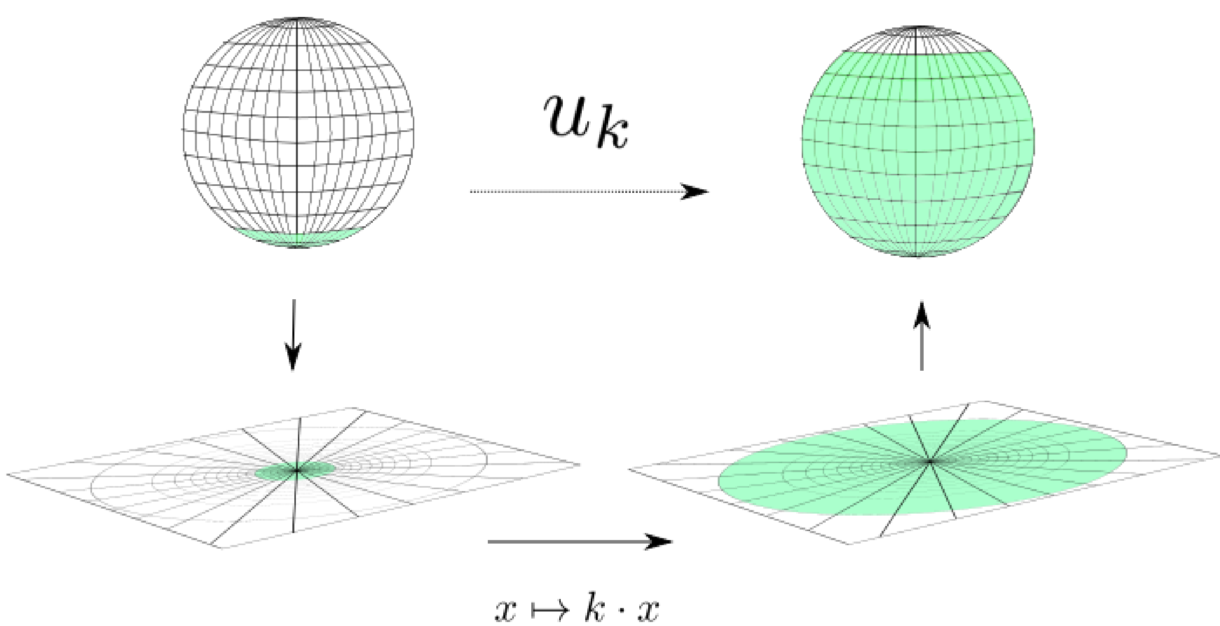
For the large part of this report, we will focus on Harmonic maps from a 2 dimensional Riemannian surface (Σ, g). An important characteristic of such maps is their invariance under a conformal transformation of Σ.

Remark 1.5 (Conformal invariance of the Energy)**.** If $u\colon \Sigma \to \mathcal{N}$ where Σ is a Riemannian surface, then u has the same energy with respect to the metric g as with respect to the metric $f \cdot g$ on Σ for any positive scalar function f.

Proof. Under the conformal transformation $g \mapsto f \cdot g$, we have $\sqrt{g} \mapsto f\sqrt{g}$ (as the dimension is 2) and $g^{ij} \mapsto \frac{1}{f} g^{ij}$. Thus the energy density e_g transforms to $e_{f \cdot g} = \frac{1}{f} e_g$, whilst the volume form $d\mu_g$ transforms to $d\mu_{f \cdot g} = f d\mu_g$, and so the Dirichlet energy is invariant. $\square$

Example 1.6. The preceding remark reveals a range of examples of harmonic maps. The identity map from a Riemannian surface to itself is always harmonic. Immediately we then have by conformal invariance that stereographic projection between the flat plane and the round sphere gives a harmonic map, and moreover we obtain an interesting sequence of harmonic maps $\{u_k\}$ between two round spheres given by a stereographic projection, followed by a rescaling of the plane by a factor k, followed by an inverse stereographic projection.

Figure 2: A sequence of harmonic maps $\{u_k\colon S^2 \to S^2\}$



This sequence is interesting as we notice that, although each map has the same energy, when k is large this energy is concentrated around the south pole. In fact, as k tends to infinity, the energy density will converge to a point measure located at the south pole. Hence, although the sequence is uniformly bounded in H^1, we do not have strong convergence of a subsequence in H^1. This lack of compactness for harmonic maps is what we address in the next section.

Before progressing with this, we remark on two applications of harmonic maps from Riemannian surfaces in geometry.

Remark 1.7 (Minimal surfaces)**.** The classical Plateau problem is to minimise the area among all surfaces with prescribed boundary a simple, closed curve Γ in $\mathbb{R}^3$. This amounts to minimizing the functional

$A[u] = \int_B \sqrt{|u_x|^2|u_y|^2 - u_x \cdot u_y} dxdy$ among all maps from the closed unit ball $B \in \mathbb{R}^2$ to $\mathbb{R}^3$ satisfying $u(\partial B) = \Gamma$. By Cauchy-Schwarz, it is observed that

$$A[u] \leq \frac{1}{2} \int_B |u_x|^2 + |u_y|^2 = E[u]$$

with equality if and only if $u_x \cdot u_y = 0$ and $|u_x|^2 = |u_y|^2$. That is, if and only if u is a conformal map. The difficulty in solving the Plateau problem arises since the functional A is invariant under reparametrization. If $\psi \colon B \to B$ is a diffeomorphism, then $A[u \circ \psi] = A[u]$ and so any minimizers will be far from unique. The Dirichlet functional is less flexible and the only diffeomorphisms under which it is invariant are those which are conformal. As a consequence of this, results in minimal surface theory have been obtained through the study of harmonic maps.

Remark 1.8 (J-holomorphic curves)**.** An almost complex manifold is an even dimensional manifold $\mathcal{N}$ with an almost complex structure J. That is, a smooth family of maps $\{J_p \colon \mathbb{T}_p\mathcal{N} \to \mathbb{T}_p\mathcal{N}\}_{p \in \mathcal{N}}$ such that for all p, $J_p^2 = -\mathrm{Id}$. A Riemannian surface Σ always admits a complex structure compatible with the conformal structure induced by the metric. That is, it can be viewed as a 1-dimensional complex manifold with tangent spaces isomorphic to $\mathbb{C}$, where transition functions between coordinate patches are holomorphic and preserve the angles specified by the metric. A map $u \colon \Sigma \to \mathcal{N}$ is said to be a J-holomorphic curve if the derivative at each point $du_x \colon \mathbb{T}_x\Sigma \to \mathbb{T}_{u(x)}\mathcal{N}$ is complex linear with respect to the given structures i.e. $du_x \circ i = J_{u(x)} \circ du_x$.

A symplectic structure on a manifold $\mathcal{N}$ is given by an exterior 2-form ω satisfying two conditions: firstly, the form is non-degenerate at each point; secondly, the form is closed. An almost complex structure J is said to be compatible with ω if the two-tensor defined by $g(v, w) = \omega(v, Jw)$ is symmetric and positive definite, hence equipping $\mathcal{N}$ with a Riemannian metric. It is known that a symplectic manifold can always be equipped with a compatible almost complex structure and that the space of such structures is contractible, so that the choice of structure is unimportant. When $\mathcal{N}$ is a symplectic manifold with a compatible almost complex structure, it turns out that all J-holomorphic curves $u \colon \Sigma \to \mathcal{N}$ are in fact harmonic maps, and the energy of the map takes the form $E[u] = \int_\Sigma u^*(\omega)$. The situation here is particularly attractive, as is explained nicely by Donaldson in [1], as, on the one hand, the pointwise compatibility between the structures gives the interpretation of $E[u]$ as the "area" of the image of u, whereas the topological condition that ω is closed on the other hand means that E is a homotopy invariant of the map. So the areas of pseudo-holomorphic curves are controlled by straightforward topological data. J-holomorphic curves have been used as a tool to construct new topological invariants, in particular for 4-manifolds, and results here rely on an understanding of the compactness properties of spaces of such curves and an understanding of how they can degenerate, similar to the phenomena we will discuss in the next section.

2 Analysis of the harmonic map equation

2.1 Heat flow

A seminal existence result for harmonic maps was given in the paper of Eells and Sampson [3], using the method of *heat flow*. They showed that in the case of a general, compact domain manifold M and target manifold N having non-positive sectional Riemannian curvature; given any $\mathcal{C}^2$ map $u\colon M \to \mathcal{N}$, there exists a $\mathcal{C}^\infty$ harmonic map $\tilde{u}$ in the same homotopy class as u. Moreover, they showed that this limit map is unique and energy minimizing in the homotopy class. In this section we give a description of heat flow and an overview of the first part of their argument, showing that the flow is defined for all time and flowing to some harmonic map which is not necessarily unique. We do not derive the Bochner identity, which is crucial to the result, and we state a number of results from parabolic PDE without proof. The section is adapted from [3], [11] and [2].

Let (M, g), $(\mathcal{N}, h)$ be compact Riemannian manifolds with $\mathcal{N}$ isometrically embedded in $\mathbb{R}^N$. Let $\mathcal{C}^2(M, \mathcal{N})$ denote the space of maps from M to $\mathcal{N}$ of class $\mathcal{C}^2$. Let $u \in \mathcal{C}^2(M, \mathcal{N})$, then define a *variation* X of u as a map $X\colon M \to \mathbb{T}\mathcal{N}$ such that $\pi \circ X = u$, where $\mathbb{T}\mathcal{N}$ is the tangent bundle of $\mathcal{N}$ with projection map π. The set of variations of u forms a vector space which we denote by $\mathbb{T}_u\mathcal{C}^2(M, \mathcal{N})$.

We can define an inner product on $\mathbb{T}_u\mathcal{C}^2(M, \mathcal{N})$ (and so think of $\mathcal{C}^2(M, \mathcal{N})$ as like a Riemannian manifold) by

$$\langle\langle X, Y\rangle\rangle_u = \int_M \langle X(x), Y(x)\rangle_{h(u(x))} d\mu(x)$$

and for a map $E\colon \mathcal{C}^2(M, \mathcal{N}) \to \mathbb{R}$ we define the directional derivative of E at u in the direction X by

$$DE(u)(X) = \frac{d}{dt}\Big|_{t=0}\big(E(u_t)\big)$$

where $u_t(x) = \exp(tX(x))$. From this viewpoint, the tension $\tau(u)$ is a variation of u and it is the contravariant representative of the differential of the energy E at u with respect to the inner product $\langle\langle\cdot\rangle\rangle_u$; that is

$$DE(u)(X) = -\langle\langle \tau(u), X\rangle\rangle_u$$

Thus, the Euler-Lagrange operator τ may be thought of as a section of $\mathbb{T}\mathcal{C}^2(M, \mathcal{N})$ which takes its zeroes at the critical points of the energy functional, the harmonic maps. We might hope then, that starting at a given point u_0 in $\mathcal{C}^2(M, \mathcal{N})$, we can "flow" along the gradient of E to arrive at a critical point. The point of all this discussion was to motivate the following result

Claim 2.1. *Suppose there exists a $\mathcal{C}^2$, one parameter family of maps $\{u_t \in \mathcal{C}^2(M, \mathcal{N})\}_{t\in[0,T]}$, that is, a $\mathcal{C}^2$ map $u\colon [0, T] \times M \to \mathcal{N}$, satisfying $\frac{\partial}{\partial t}u_t = \tau(u_t)$ for all t. Then $E[u_t]$ will be strictly decreasing in t with exception only at those values of t for which $\tau(t) = 0$.*

Proof.

$$\begin{aligned}\frac{d}{dt}E[u_t] &= DE[u_t][\frac{\partial}{\partial t}u_t]\\ &= -\langle\langle \tau(u_t), \frac{\partial}{\partial t}u_t\rangle\rangle_{u_t}\\ &= -||\tau(u_t)||^2_{u_t}\end{aligned}$$

□

Thus we are led to look for a solution $u\colon [0, T] \times M \to \mathbb{R}^N$ to the system of equations

$$\begin{aligned}(\partial_t - \Delta_g)u &= A(u)(\nabla u, \nabla u) \quad \text{on } (0, T] \times M\\ u(0, \cdot) &= u_0 \qquad\qquad\qquad \text{on } \{0\} \times M\end{aligned} \tag{1}$$

Where ∇ represents the spacial component of D. Note for such a solution that if $u_0 \in \mathcal{C}^2(M,\mathcal{N})$, then $u(t,\cdot) \in \mathcal{C}^2(M,\mathcal{N})$ for all t, since the tension field is always tangent to $\mathcal{N}$. We have gone from an existence problem for an elliptic PDE to one for a parabolic PDE. This is helpful as we can then employ fixed point methods. By the standard parabolic theory, see [11] volume II, we have that the system is equivalent to

$$u(t,\cdot) = \exp\left(t\Delta_g\right) u_0 + \int_0^t \exp\big((t-s)\Delta_g\big) A(u(s,\cdot))ds \tag{2}$$

having abbreviated $A(u) = A(u)(\nabla u, \nabla u)$, where exp is the exponent operator, and $\{\exp(t\Delta_g)\colon \mathcal{C}^2(M) \to \mathcal{C}^2(M)\}$ is a $\mathcal{C}^0$-semigroup of linear operators parametrised by t. This semigroup is particularly well behaved; we have for any $p, q \in \mathbb{N} \cup \{0\}$, $t > 0$

$$\exp(t\Delta_g)\colon \mathcal{C}^p(M) \to \mathcal{C}^{p+q}(M), \quad \text{with } ||\exp(t\Delta_g)||_{\mathcal{B}(\mathcal{C}^p(M),\mathcal{C}^{p+q})} \leq C_q t^{-q/2} \tag{3}$$

and, moreover, $\{\exp(t\Delta_g)\}_{t\geq\varepsilon>0}$ are uniformly bounded operators from $L^1(M)$ to $L^\infty(M)$ for any ε.

Now, we fix some number R and let

$$B = \overline{B_{u_0}(R)} = \{u \in \mathcal{C}([0,T],\mathcal{C}^2(M))\colon u(0,\cdot) = u_0, \sup_{t\in[0,T]} ||u(t,\cdot) - u_0||_{\mathcal{C}^2(M)} \leq R\}$$

be a closed ball in the Banach space $\mathcal{C}([0,T],\mathcal{C}^2(M))$. Since B is a closed subspace of a Banach space, it is a complete metric space. We claim that $\Psi\colon B \to B$ defined by

$$\Psi(u)(t,\cdot) = \exp\left(t\Delta_g\right) u_0 + \int_0^t \exp\big((t-s)\Delta_g\big) A(u(s,\cdot))$$

is well defined and a contraction mapping provided T is sufficiently small. From this it follows that the system (1) admits a unique solution in the ball $\mathcal{C}([0,T],\mathcal{C}^2(M))$ for some small time period T by the Banach fixed point theorem.

Indeed, since the semigroup is $\mathcal{C}^0$, i.e. $\lim_{t\searrow 0} ||\exp(t\Delta_g) - \mathrm{Id}||_{\mathcal{B}(\mathcal{C}^2(M))} = 0$, and since A is bounded considered as a map from $\mathcal{C}^2(M)$ to $\mathcal{C}^1(M)$, $||A(u)||_{\mathcal{C}^1(M)} \leq C$ for $u \in B$, we have

$$\begin{aligned}
||\Psi(u)(t,\cdot) - u_0|| &\leq ||\exp(t\Delta_g)u_0 - u_0||_{\mathcal{C}^2(M)} + \int_0^t ||\exp((t-s)\Delta_g)A(u(s,\cdot))||_{\mathcal{C}^2(M)} \\
&\leq ||\exp(t\Delta_g)u_0 - u_0||_{\mathcal{C}^2(M)} + \int_0^t C_1(t-s)^{-1/2}||A(u(s,\cdot))||_{\mathcal{C}^1(M)} \\
&\leq R
\end{aligned}$$

and so the map is well defined, $\Psi(u) \in B$ for $u \in B$, provided T is chosen smaller than some T_0, depending only on M, $\mathcal{N}$ and R.

Since $A(u) \leq C|\nabla u|^2$, we have that A is Lipschitz on bounded sets when considered as a map from $\mathcal{C}^2(M)$ to $\mathcal{C}^1(M)$. Let K be the Lipschitz constant so that $||A(u(t,\cdot)) - A(v(t,\cdot))||_{\mathcal{C}^1(M)} \leq K||u(t,\cdot) - v(t,\cdot)||_{\mathcal{C}^2(M)}$

for $u, v \in B$. Then

$$
\begin{aligned}
||\Psi(u)(t,\cdot) - \Psi(v)(t,\cdot)||_{\mathcal{C}^2(M)} &\leq \int_0^t ||\exp((t-s)\Delta_g)\big(A(u(s,\cdot)) - A(v(s,\cdot))\big)||_{\mathcal{C}^2(M)} ds \\
&\leq \int_0^1 C(t-s)^{1/2} ||A(u(s,\cdot)) - A(v(s,\cdot))||_{\mathcal{C}^1(M)} ds \\
&\leq C_1 K \sup_{s\in[0,T]} ||u(s,\cdot) - v(s,\cdot)||_{\mathcal{C}^2(M)} \int_0^1 (t-s)^{-1/2} ds \\
&\leq 2C_1 K T^{1/2} ||u - v||_{\mathcal{C}([0,T],\mathcal{C}^2(M))}
\end{aligned}
$$

And so Ψ is a well defined contraction mapping provided $T < T_1 = \min\{T_0, (2C_1K)^2\}$. This proves short time existence for the flow (1). We can now prove long time existence in the special case of a non-positively curved target.

Theorem 2.2 (Eells-Sampson)**.** *Let M be a compact Riemannian manifold, and $\mathcal{N}$ a compact Riemannian manifold of non-positive sectional curvature. Let $u_0 \colon M \to \mathcal{N}$ be of class $\mathcal{C}^2$, then there exists a smooth harmonic map $\tilde{u}$ in the same homotopy class as u_0.*

Proof. We show that the system (1) admits a solution for all time, and that this solution gives convergence of a subsequence to a harmonic map as $t \to \infty$. The core of the argument is an a priori estimate on the energy density $e = \frac{1}{2}|\nabla u|^2$, which follows immediately from the Bochner identity.

By the Bochner identity:

$$
(\partial_t - \Delta_g)|\nabla u|^2 = -g^{ij}|\nabla^{\#} du(e_i, e_j)|^2 - \frac{1}{2}\langle du \cdot \mathrm{Ric}^M(e_j), du \cdot e_j\rangle + \frac{1}{2}\langle \mathrm{Riem}^N(du \cdot e_j, du \cdot e_k)(du \cdot e_k), du \cdot e_j\rangle
$$

where we sum over repeated indices and $\{e_j\}$ is some local orthonormal frame for $\mathbb{T}M$. $\nabla^{\#}$ here denotes the product connection $\nabla^{\#} = \nabla^M \otimes u^*\nabla^{\mathcal{N}}$ on $\mathbb{T}^*M \otimes u^*\mathbb{T}\mathcal{N} = \mathbb{T}^*M \otimes u^*\mathbb{T}\mathbb{R}^N$. See [11] volume III, pg. 331 for a walkthrough of this calculation.

The second term on the right is bounded by $c|\nabla u|^2$ for $c = c(M)$ since M is compact, and the last term on the right is non-positive by hypothesis. Thus we obtain the estimate:

$$
(\partial_t - \Delta_g)e \leq ce
$$

We may assume without loss of generality that $-c$ is not an eigenvalue of the Laplacian, so that $\exp(\Delta_g + c)$ is a well defined solution operator, then setting $(\partial_t - \Delta_g - c)e = f \leq 0$ we obtain

$$
\begin{aligned}
e(t,\cdot) &= \exp(t(\Delta_g + c))e_0 + \int_0^t \exp(t(\Delta_g + c))f(s,\cdot)ds \\
&\leq \exp(t(\Delta_g + c))e_0
\end{aligned}
$$

where $e_0 = e(u_0)$. Since $\{\exp(t\Delta_g + c) = \exp(ct)\cdot\exp(t\Delta_g)\}_{t\in[T_1,T_2]}$ are uniformly bounded operators from $L^1(M)$ to $L^\infty(M)$ for any fixed interval $[T_1, T_2]$ and $||e_0||_{L^1(M)} = E[u_0]$, this gives us a uniform bound on e on a bounded interval.

Since $|\nabla u|^2$ is uniformly bounded, it follows that the Lipschitz constant K for $A(u)$ may be chosen uniformly on a bounded time interval $[T_1, T_2]$, and so the argument proving short time existence above may be carried out repeatedly to give existence of a unique solution in $u \in \mathcal{C}([0, T_2), \mathcal{C}^2(M))$ to

$$\begin{aligned} (\partial_t - \Delta_g)u &= A(u)(\nabla u, \nabla u) && \text{on } (0, \infty) \times M \\ u(0, \cdot) &= u_0 && \text{on } \{0\} \times M \end{aligned}$$

This holds for any time interval, and so we have

Since we have

$$\begin{aligned} \int_0^\infty \int_M ||\tau(u(t, \cdot))||^2_{L^2(M)} d\mu dt &= -\lim_{T \to \infty} \int_0^T \frac{d}{dt} E[u(t, \cdot)] dt \\ &\leq E[u_0] \end{aligned}$$

it follows that we may pass to a subsequence of times $\{t_k\}$ for which $||\tau(u(t_k, \cdot))||_{L^2(M)} \to 0$ as $t_k \to \infty$. Since $\int_M |\nabla u(t, \cdot)|^2$ is uniformly bounded in t,it follows from the explicit representation of the solution (2), and the properties of the semi-group (3) that $||u(t, \cdot)||_{\mathcal{C}^q}$ is uniformly bounded in t. By the Banach-Alaoglu theorem that we may pass to a subsequence $\{t_k\}$ for which $u(t_k, \cdot) \rightharpoonup \tilde{u}(\cdot)$ weakly in $\mathcal{C}^q(M)$, and by the Rellich-Kondrachov compactness theorem $u(t_k, \cdot) \to \tilde{u}(\cdot)$ strongly in $\mathcal{C}^{q-1}(M)$. This holds for any q and thus we have smooth convergence to a smooth harmonic map $\tilde{u}$. $\square$

The above estimate on e relies on $\mathcal{N}$ having non-positive curvature. In the case of a general target we cannot expect a bound on the energy density and can not expect a solution to the flow which is regular for more than short time. A clue to the type of obstruction that may occur can be seen in the case where the domain is a 2-dimensional surface $M = \Sigma$. Here as we have seen the energy is conformally invariant, and this led to the example of a sequence of harmonic maps between two spheres for which the energy density converged to a Dirac mass as smaller and smaller regions of the domain wrapped around the image. This phenomenon (referred to as bubbling) may occur under the heat flow, and the solution may form singularities in finite time. An existence result for the heat flow from a surface into a general target was first given by Struwe in [9] where it was demonstrated that the heat flow admits a global weak solution which is regular around every point at which as energy does not concentrate in this manner; and that at any point where the energy concentrates and a singularity forms, the singularity must arise due to bubbling with respect to some minimally immersed sphere in the target $\mathcal{N}$.

For the case of such a singularity forming about some point $x \in \Sigma$ at some time T, along the trajectories of the flow at times $t_k \to T$, although energy is necessarily decreasing, we cannot expect the maps $u(t_k, \cdot)$ to converge to a limit strongly in $H^1(\Sigma)$. This is because energy is “lost” in the limit as the singularity forms about the point x. Thus along such trajectories there is no compactness in H^1, and this may be viewed as a characterisation of the singularity. In the next section we will review Struwe’s argument not in full generality, but through a study of this loss of compactness. Although this is a simplified picture of the flow, the main ideas are still addressed.

2.2 Bubbling

In this section we concern ourselves with a sequence of mappings $\{u_k\}$ from a Riemannian surface (Σ, g) into a Riemannian manifold $(\mathcal{N}, h)$ isometrically embedded in $\mathbb{R}^N$, with a uniform bound on the energy $E(u_k) \leq C$ and a uniform bound on the L^2-norm of the tension $||\tau(u_k)||_{L^2} \leq C$ (it will become apparent from the analysis why a uniform L^2 bound is appropriate). We would like to understand the H^1 compactness of such a collection. That is, when we can ensure that there is a subsequence convergent in H^1. It is common to refer to the lack of compactness for a such collection of mappings as *energy loss*. The estimates in this section were first applied in the context of harmonic maps by Struwe and we refer the reader to [9] for the application to heat flow.

The example which concluded the last section demonstrates that we cannot expect compactness in general. Indeed, we see that it is possible for energy to concentrate around a point, in which case it is not possible to extract a limit in H^1. The fundamental estimate of Struwe [9] shows that, as long as the energy does not concentrate, we are guaranteed a uniform bound on the H^2-norm. This implies the desired compactness by the Rellich-Kondrachov compactness theorem.

Lemma 2.3. *There exist universal constants $C_0 = C_0(\Sigma, \mathcal{N})$ and $R_0 = R_0(\Sigma) > 0$ such that for any $u \in H^2(\Sigma, \mathcal{N})$ the following estimate holds for any $R \in (0, R_0]$:*

$$\int_\Sigma |\nabla^2 u|^2 d\mu \leq C \cdot \sup_{x \in \Sigma} \left\{ \int_{B_R(x)} |\nabla u|^2 d\mu \right\} \left(\int_\Sigma |\nabla^2 u|^2 d\mu + \frac{1}{R^2} \int_\Sigma |\nabla u|^2 d\mu \right) + 2 \int_\Sigma |\tau(u)|^2 d\mu$$

The lemma demonstrates that the only obstruction to compactness is a concentration of energy. Setting $\varepsilon_0 = \frac{1}{2C_0}$ gives

Corollary 2.4 (Struwe)**.** *There exists a universal constant $\varepsilon_0 = \varepsilon_0(\Sigma, \mathcal{N}) > 0$ such that if $\{u_k \in H^2(\Sigma, \mathcal{N})\}$ is a sequence of maps with a uniform bound on the energy $E(u_k) \leq C$ and a uniform bound on the L^2-norm of the tension $||\tau(u_k)||_{L^2} \leq C$, and for some R,*

$$\sup_{x \in \Sigma} \left\{ \int_{B_R(x)} |\nabla u_k|^2 d\mu \right\} \leq \varepsilon_0$$

for any k, then we have a uniform bound on the second derivative $||D^2 u_k||_{L^2} \leq C$. Thus the collection is compact in H^1: there exists u_∞ so that $u_k \to u_\infty$ strongly in H^1, passing to a subsequence if necessary.

We will deduce Lemma 2.3 as a consequence of an estimate in a local system of coordinates. Firstly we need a result to control the terms arising in this system as a consequence of the curvature of Σ.

Lemma 2.5. *For a compact Riemannian surface (Σ, g), there exist constants $R_1, Q > 0$ depending only on (Σ, g) such that for any $x_0 \in \Sigma$ and any $r \in (0, R_1]$ there exists a set of coordinates $x = (x^1, x^2)$ on the geodesic ball $B_r(x_0)$ such that $g(x^1, x^2) = \rho(x^1, x^2)\left((dx^1)^2 + (dx^2)^2\right)$ for some scalar function ρ satisfying the bounds*

$$Q^{-1} \leq \rho(x^1, x^2) \leq Q$$
$$|\nabla \rho| \leq Q$$

Lemma 2.6. *For any $x_0 \in \Sigma$ and any $r \in (0, R_1]$ there exists a constant C depending only on Σ and $\mathcal{N}$ such that*

$$\int_{B_{\frac{r}{2}}(x_0)} \chi^4 |\nabla_g^2 u|_g^2 d\mu \leq C\left[\left(\int_{B_r(x_0)} |\nabla_g u|_g^2 d\mu\right)\left(\int_{B_r(x_0)} \chi^4 |\nabla_g^2 u|_g^2 d\mu + \frac{1}{r^2}\int_{B_r(x_0)} |\nabla_g u|_g^2 d\mu\right)\right] + 2\int_{B_r(x_0)} |\tau(u)|^2 d\mu$$

for any $u \in H^2(\Sigma, \mathcal{N})$ where χ is a test function with support on $B_r(x_0)$ so that $\chi = 1$ on $B_{\frac{r}{2}}(x_0)$ and $|\nabla\chi|_g \leq \frac{3}{r}$.

Proof. In the following proof C and $\tilde{C}$ denote constants depending only on Σ and $\mathcal{N}$ which may change from line to line.

The tension field is given by $\Delta_g u + A(u)(\nabla_g u, \nabla_g u) = \tau(u) \in \mathbb{R}^N$ and so a priori we have

$$|\Delta_g u|^2 \leq C|\nabla_g u|_g^4 + 2|\tau(u)|^2$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$.

Letting χ be a test function with support on $B_r(x_0)$ so that $\chi = 1$ on $B_{\frac{r}{2}}(x_0)$ and $|\nabla\chi|_g \leq \frac{3}{r}$ so that $|\nabla\chi| \leq \frac{3Q}{r}$, we have

$$\underbrace{\int_{B_r(x_0)} \chi^4 |\Delta_g u|^2 d\mu}_{\textbf{(I)}} \leq C \underbrace{\int_{B_r(x_0)} \chi^4 |\nabla_g u|_g^4 d\mu}_{\textbf{(II)}} + 2\int_{B_r(x_0)} \chi^4 |\tau(u)|^2 d\mu$$

Choosing isothermal coordinates $x = (x^1, x^2)$ about x_0 with the properties of Lemma 2.5, we have $\Delta_g = \frac{1}{\rho}\Delta = \frac{1}{\rho(x^1,x^2)}((\partial_{x^1})^2 + (\partial_{x^2})^2)$ and $d\mu = \rho dx = \rho(x^1, x^2)dx^1 dx^2$.

We denote by $\cdot, :, \therefore$ the standard Euclidean inner product and standard Euclidean contractions of 2 and 3-tensors respectively. We denote by ∇ and Δ the operators $(\partial_{x^1}, \partial_{x^2})$ and $\partial_{x^1}^2 + \partial_{x^2}^2$. Then integrating by parts we get

$$\begin{aligned}
\textbf{(I)} &= \int_{B_r(x_0)} (\frac{\chi^4}{\rho})\Delta u \cdot \Delta u dx \\
&= \int_{B_r(x_0)} \nabla((\frac{\chi^4}{\rho})\mathrm{div}(\nabla u)) \colon \nabla u dx \\
&= \int_{B_r(x_0)} \nabla(\frac{\chi^4}{\rho}) \otimes \mathrm{div}(\nabla u) \colon \nabla u dx + \int_{B_r(x_0)} (\frac{\chi^4}{\rho})\mathrm{div}(\nabla^2 u) \colon \nabla u dx \\
&= \int_{B_r(x_0)} (\nabla(\frac{\chi^4}{\rho}) \otimes \mathrm{div}(\nabla u)) \colon \nabla u dx + \int_{B_r(x_0)} (\frac{\chi^4}{\rho})\nabla^2 u \therefore \nabla\left((\frac{\chi^4}{\rho})\nabla u\right) dx \\
&= \int_{B_r(x_0)} (\nabla(\frac{\chi^4}{\rho}) \otimes \mathrm{div}(\nabla u)) \colon \nabla u dx + \int_{B_r(x_0)} (\nabla(\frac{\chi^4}{\rho}) \otimes \nabla U) \therefore \nabla^2 u dx + \int_{B_r(x_0)} (\frac{\chi^4}{\rho})|\nabla^2 u|^2 dx
\end{aligned}$$

We have

$$\begin{aligned}
|\nabla(\frac{\chi^4}{\rho})| &= |\frac{1}{\rho^2}(\rho\nabla\chi^4 + \chi^4\nabla\rho)| \\
&\leq (\frac{12Q^2\chi^3}{r} + \chi^4 Q^3) \\
&\leq \frac{C\chi^2}{r}
\end{aligned}$$

where C is a constant depending only on Σ. Thus

$$\begin{aligned}
\textbf{(I)} &\geq \int_{B_r(x_0)} (\frac{\chi^4}{\rho})|\nabla^2 u|^2 dx - \frac{C}{r}\int_{B_r(x_0)} \chi^2 |\nabla^2 u||\nabla u| dx \\
&\geq Q^{-1}\int_{B_r(x_0)} \chi^4|\nabla^2 u|^2 dx - \frac{Q^{-1}}{2}\int_{B_r(x_0)} \chi^4|\nabla^2 u|^2 dx - \frac{C}{r}\int_{B_r(x_0)} |\nabla u|^2 dx \\
&= \frac{Q^{-1}}{2}\int_{B_{\frac{r}{2}}(x_0)} \chi^4|\nabla^2 u|^2 dx - \frac{C}{r}\int_{B_r(x_0)} |\nabla u|^2 dx
\end{aligned}$$

Now, in local coordinates we have

$$\begin{aligned}
|\nabla_g^2 u|_g^2 &= \delta_{\alpha\beta} g^{ac} g^{bd} \nabla_{ab} u^\alpha \nabla_{cd} u^\beta \\
&= \delta_{\alpha\beta} g^{ac} g^{bd} (\partial_a \partial_b u^\alpha + \Gamma^i_{ab}\partial_i u^\alpha)(\partial_c\partial_d u^\beta + \Gamma^j_{cd}\partial_j u^\beta)
\end{aligned} \tag{4}$$

Since the Christoffel symbols are expressed in terms of derivatives of ρ, we have $\Gamma^c_{ba} \leq \frac{3}{2}Q$. Thus equation (4) implies

$$\begin{aligned}
|\nabla_g^2 u|_g^2 &\leq \frac{1}{\rho^2}(|\nabla^2 u|^2 + 3Q|\nabla u||\nabla^2 u| + (\frac{3Q}{2})^2|\nabla u|^2) \\
\implies \int_{B_r(x_0)} \chi^4 |\nabla^2 u|^2 dx &\geq C\int_{B_r(x_0)} \chi^4 |\nabla_g^2 u|^2 d\mu - \tilde{C}\int_{B_r(x_0)} |\nabla_g u|^2 d\mu
\end{aligned}$$

Putting this together gives

$$\textbf{(I)} \geq C\int_{B_r(x_0)} \chi^4 |\nabla_g^2 u|^2 d\mu - \tilde{C}\int_{B_r(x_0)} |\nabla_g u|^2 d\mu.$$

We estimate the left hand side applying the Sobolev embedding in 2D, $\|\varphi\|_{L^2} \leq \|\nabla\varphi\|_{L^1}$ for functions $\varphi \in \mathcal{C}^1_c$.

$$\begin{aligned}
\textbf{(II)} &= \int_{B_r(x_0)} \chi^4 |\nabla_g u|_g^4 d\mu \\
&= \int_{B_r(x_0)} \chi^4 \rho^{-1} |\nabla u|^2 dx \\
&\leq Q||\chi^2|\nabla u|^2||^2_{L^2(B_r(x_0))} \\
&\leq C||\nabla(\chi^2|\nabla u|^2)||^2_{L^1(B_r(x_0))} \\
&\leq C\left(\int_{B_r(x_0)} \chi|\nabla\chi||\nabla u|^2 dx + \int_{B_r(x_0)} \chi^2|\nabla^2 u||\nabla u| dx\right)^2 \\
&\leq C\left(\frac{3}{r}\int_{B_r(x_0)} |\nabla u|^2 dx + \left(\int_{B_r(x_0)} \chi^4|\nabla^2 u|^2 dx\right)^{\frac{1}{2}}\left(\int_{B_r(x_0)} |\nabla u|^2 dx\right)^{\frac{1}{2}}\right)^2 \\
&\leq C\left(\int_{B_r(x_0)} |\nabla u|^2 dx\right)\left(\frac{1}{r}\int_{B_r(x_0)} \chi^4|\nabla u|^2 dx + \int_{B_r(x_0)} |\nabla^2 u|^2 dx\right)
\end{aligned}$$

By equation (4) we see

$$|\nabla^2 u|^2 \leq \rho^2|\nabla_g^2 u|_g^2 + 3Q|\nabla u||\nabla^2 u| + (\frac{3Q}{2})^2|\nabla u|^2$$

and so

$$\int_{B_r(x_0)} \chi^4|\nabla^2 u|^2 dx \leq C\left(\int_{B_r(x_0)} \chi^4|\nabla_g^2 u|_g^2 d\mu + \int_{B_r(x_0)} |\nabla_g u|_g^2 dx\right)$$

Finally combining our estimates on **(I)** and **(II)** we obtain the desired result

$$\int_{B_{\frac{r}{2}}(x_0)} \chi^4 |\nabla_g^2 u|_g^2 d\mu \leq C \left[\left(\int_{B_r(x_0)} |\nabla_g u|_g^2 d\mu \right) \left(\int_{B_r(x_0)} \chi^4 |\nabla_g^2 u|_g^2 d\mu + \frac{1}{r^2} \int_{B_r(x_0)} |\nabla_g u|_g^2 d\mu \right) \right] + 2 \int_{B_r(x_0)} |\tau(u)|^2 d\mu$$

□

Proof. (of Lemma 2.3). We may deduce the global estimate in Lemma 2.3 from the local estimate of Lemma 2.6 by choosing a cover of Σ by balls of radius $\frac{r}{2}$, where $r \leq R_0$ for some $R_0 \leq R_1$, centred at points $(x_1, \dots, x_n)$ so that each point of Σ lies in no more than K balls of radius r centred at the points $(x_1, \dots, x_n)$. Such a cover may be constructed for K independent of a sufficiently small r, by choosing an embedding of Σ in Euclidean space $\mathbb{R}^n$, and projecting balls which are centred at a sufficiently fine lattice in $\mathbb{R}^n$, of sufficiently small radius to restrict to a tubular neighbourhood of Σ with well defined projection. We obtain the global estimate by summing the local estimate over this cover □

Corollary 2.4 tells us that the only obstruction to H^1-compactness is an energy concentration such as that illustrated in (1.6). We see that for our sequence of mappings $\{u_k\}$ one may pass to a subsequence for which there may exist (at most finitely many, as total energy is bounded) points $\{p_1, \dots, p_n\}$ about which energy concentrates in the sense

$$\lim_{\delta \to 0} \lim_{k \to \infty} E[u_k, B_\delta(p_i)] > \varepsilon_0$$

and such that there is strong convergence to u_∞ in $H^1_{\text{loc}}(\Sigma \setminus \{p_1 \dots p_n\})$.

In other words, the energy densites $e(u_k)$ converge strongly as measures to $e(u_\infty)$ plus a sum of point measures at the p_i with masses $m_i > \varepsilon_0$

$$e(u_k) \to e(u_\infty) + \sum_{i=1}^{n} m_i \delta_{p_i} \tag{5}$$

We now seek to examine precisely the manner in which energy is compressed into a concentration point. The key is to rescale appropriately. Consider a single concentration point p and restrict attention to a ball of radius δ around p, where δ is small. One may always choose local, isothermal coordinates, and we note that a uniform bound on the tension is maintained, whilst energy is invariant, if we consider instead the functions $\{u_k\}$ defined on this coordinate patch in $\mathbb{R}^2$ with the Euclidean metric. Thus it is equivalent to view our sequence on the disk B_δ centred at $0 \in \mathbb{R}^2$.

For each k, let $r_k = \inf\{\tilde{r}_k > 0 \colon \exists \tilde{x}_k \in \overline{B_{\frac{\delta}{2}}}, E[u_k; B_{\tilde{r}_k}(\tilde{x}_k)] \geq \varepsilon_0\}$. Letting $\tilde{r}_k^{(j)} \searrow r_k$, we may take a subsequence $\tilde{x}_k^{(j)}$ which converges to a point $x_k \in \overline{B_{\frac{\delta}{2}}}$ which satisfies $E[u_k; B_{r_k}(x_k)] = \varepsilon_0$ and where $E[u_k; B_{\hat{r}_k}(\hat{x}_k)] \leq \varepsilon_0$ for all $\hat{r}_k \leq r_k, \hat{x}_k \in \overline{B_{\frac{\delta}{2}}}$. Thus we have a sequence of points $x_k \to 0$ about which at least one quantum of energy concentrates.

Next, for each k, a rescaled function $\tilde{u}_k \colon \Omega_k \to \mathcal{N}$ is defined by $\tilde{u}_k(x) = u(r_k x)$, where $\Omega_k \subset \mathbb{R}^2$.

These rescaled functions blow up each small ball of radius r_k to a ball of radius 1, and we see that in the limit as k tends to ∞, as the points x_k tend to the origin and the radii r_k tend to zero, the ball of radius δ is rescaled to the entire plane $\mathbb{R}^2$ and the rescaled domains Ω_k exhaust this $\mathbb{R}^2$. We claim that some energy concentration is measured via this rescaling.

Claim 2.7. *There exists a limit map* $\tilde{u}_\infty \colon \mathbb{R}^2 \to \mathcal{N}$ *, such that* $\tilde{u}_k \to \tilde{u}_\infty$ *strongly in* $H^1_{\text{loc}}(\mathbb{R}^2)$*, with* $E(u_\infty) \geq \varepsilon_0$

Proof. Under the rescaling we have that energy is unchanged by conformal invariance and $\int_{B_1(0)} |\nabla \tilde{u}_k|^2 dx = \int_{B_{r_k}(x_k)} |\nabla u_k|^2 dx = \varepsilon_0$. However, the energy is spread out by by the rescaling and our second derivative relaxes:

$$\begin{aligned}\int_{B_1(0)} |\nabla^2 \tilde{u}_k|^2 dx &= \int_0^{2\pi}\int_0^1 |\nabla^2 \tilde{u}_k(\rho,\theta)|^2 \rho \, d\rho d\theta \\ &= \int_0^{2\pi}\int_0^1 |r_k^2 \nabla^2 u_k(\rho',\theta)|^2 \frac{\rho'}{r_k^2} d\rho' d\theta \qquad \text{where } \rho' = r_k\rho \\ &= r_k^2 \int_{B_{r_k}} |\nabla^2 u|^2 dx\end{aligned}$$

Thus, plugging into our local estimate, Lemma 2.6, we obtain:

$$\frac{1}{r_k^2}\int_{B_1} \chi^2 |\nabla^2 \tilde{u}_k|^2 dx \leq \left(\int_{B_1} \chi |\nabla \tilde{u}_k|^2 dx\right)\left(\frac{1}{r_k^2}\int_{B_1} \chi^2 |\nabla^2 \tilde{u}_k|^2 dx + \frac{1}{r_k^2}\int_{B_1} \chi |\nabla \tilde{u}_k|^2 dx\right) + 2||\tau(u_k)||^2_{L^2(\Sigma)}$$

The r_k's cancel, and since we have chosen a ball with energy ε_0, we can swing over and obtain a uniform bound

$$||\nabla^2(\tilde{u}_k)||_{L^2\left(B_{\frac{1}{2}}(0)\right)} \leq C$$

where C is independent of k, by choosing a cut-off $\chi = 1$ on $B_{\frac{1}{2}}(0)$.

Next, we apply a covering argument to get the local convergence. For a given positive integer $M \in \mathbb{N}$ consider each ball $B_{Mr_k}(x_k)$ as sitting inside a square of side length $2Mr_k$, latticed by $(4M)^2$ points equally spaced so that the Manhatten distance between adjacent lattice points is $\frac{r_k}{2}$.

At each point in the lattice, centre a ball of radius r_k. We note that any point in the square lies in fewer than 25 balls (since shifting any ball laterally by 5 lattice points ensures it is disjoint from its pre-image).

Now because of the way we chose the points x_k, the energy on each of these balls is no greater than ε_0. Thus under the rescaling we obtain the same estimate as above on each ball

$$||D^2(\tilde{u}_k)||_{L^2\left(B_{\frac{1}{2}}(q_l)\right)} \leq C$$

where the $\{q_l\}_{1 \leq l \leq (4M)^2}$ represent the lattice points on the square $[-M, M]^2 \subset \mathbb{R}^2$, with Manhattan distance between two adjacent q's of $\frac{1}{2}$. Thus summing over this cover for the ball of radius M we obtain

$$||D^2(\tilde{u}_k)||_{L^2\left(B_M(0)\right)} \leq C$$

with C independent of k, and so by the Rellich-Kondrachov theorem, there exists $\tilde{u}_\infty \colon \mathbb{R}^2 \to \mathcal{N}$ with $\tilde{u}_k \to \tilde{u}_\infty$ strongly in $H^1_{\text{loc}}(\mathbb{R}^2)$, passing to a subsequence as necessary.

This concludes the proof. $\square$

Remark 2.8. We now make the observation that, alongside the rescaling, the tension $\tau(\tilde{u}_k)$ tends almost everywhere to zero. And so our limit map is in fact harmonic.

Proof. Indeed, the tension rescales as $\tau(\tilde{u}_k) = r_k^2 \tau(u_k)$. Thus for any M

$$\begin{aligned}\int_{B_M} |\tau(\tilde{u}_k)|^2 dx &= \int_{B_M(0)} |\tau(\tilde{u}_k)(\rho,\theta)|^2 \rho \, d\rho d\theta \\ &= r_k^2 \int_{B_{Mr_k}(x_k)} |\tau(u_k)(\rho',\theta)|^2 \rho' \, d\rho' d\theta \\ &\leq r_k^2 C \underset{k\to\infty}{\longrightarrow} 0\end{aligned}$$

And so $\tau(\tilde{u}_k) \to 0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. $\square$

The above remark is enlightening as the work of Sacks and Uhlenbeck [8] shows that any non-constant harmonic map with finite energy from the plane may be extended by a compactification of the plane to give a non-constant harmonic map from the sphere $\varphi\colon S^2 \to \mathcal{N}$. Moreover, φ represents a conformally immersed minimal surface away from a finite number of possible branch points. Thus, we have arrived at a necessary topological condition, namely the presence of non-constant harmonic two-spheres in the target, in order for energy loss to occur. [1] Moreover we have a necessary lower bound on the size of the constant ε_0, call it $\varepsilon_\star$, which may be thought of as representing the infimum size of harmonically immersed spheres in $\mathcal{N}$.

Note 2.9. It is this observation that has led to the term *bubbling* in describing this energy loss; as well as in describing singularities for other critical, parabolic PDE which may be analysed by a rescaling argument. As we follow the energy concentration for our sequence, the rescaled maps $\tilde{u}_k$ are visualised on domains under inverse stereographic projection, which exhaust the 2-sphere, with $\tilde{u}_\infty$ corresponding to a "fully blown bubble" $S^2 \to \mathcal{N}$.

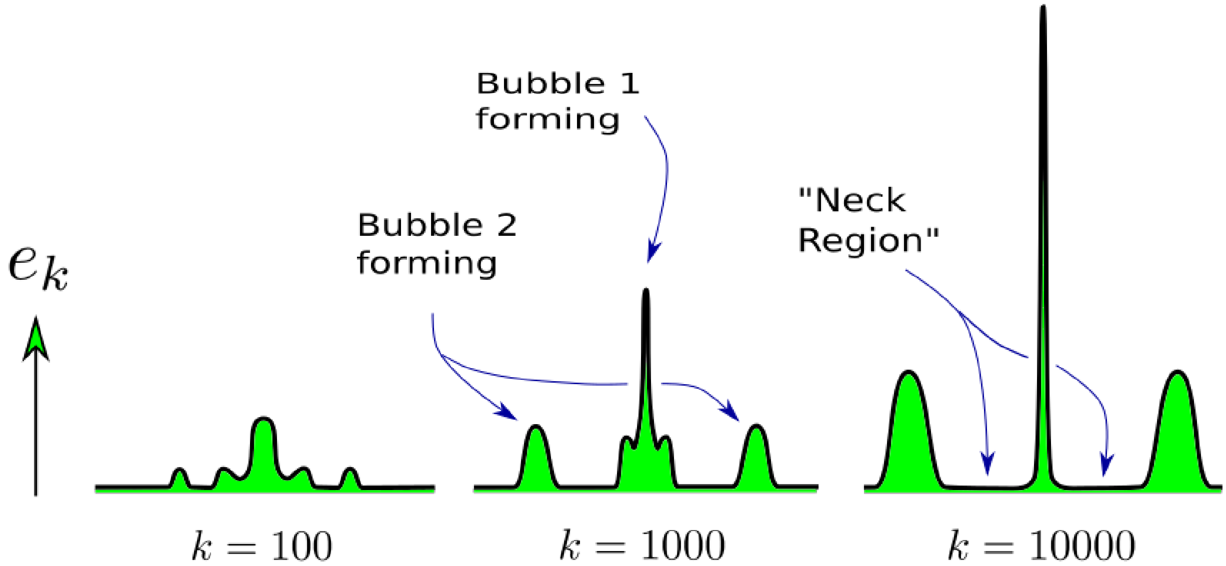
Recalling that we only prove local convergence under this rescaling, it must be remembered that we are not guaranteed to capture all of the energy lost with a single bubble as energy could well be lost "at infinity" (consider the case of the indicator functions $\{\mathbb{1}_{[k,k+1]}\colon \mathbb{R} \to \mathbb{R}\}_{k\in\mathbb{N}}$, which converge locally to zero). It may well be that multiple bubbles form at a point which concentrate at different rates and from different directions. Crucially though, we observe that the number of bubbles possible forming is necessarily finite, as each must contain at least one quantum of energy $\varepsilon_\star$ and the total energy of our sequence is uniformly bounded. A question which now arises is this: does bubbling fully account for the energy loss for our sequence? That is, in the expression (5), can we ensure that each point mass may be written as

$$m_i = \sum_{j=1}^{N_i} E[\varphi_j^i]$$

where the $\varphi_j^i\colon S^2 \to \mathcal{N}$ are harmonic maps representing minimal 2-spheres in $\mathcal{N}$? The hypothesis is referred to as the *energy identity* and the question was first answered, in the affirmative, in full generality in [13]. The next section is dedicated to a review of this argument.

[1] In particular, by the Gauss-Bonnet theorem, it follows from the work of Struwe that energy loss will not occur in the case of a target $\mathcal{N}$ with non-positive sectional curvature, which implies the result of Eells-Sampson in the case of a 2 dimensional Σ.

Figure 3: Two bubbles forming around a concentration point



2.3 The energy identity

Again restricting ourselves to a small neighbourhood $B_\delta = B_\delta(0)$, $0 < \delta << 1$ in local isothermal coordinates about a concentration point $p \in \Sigma$, we have that proving the energy identity is equivalent to proving

$$\lim_{k\to\infty} E[u_k] = E[u_\infty] + \sum_{j=1}^{m} E[\varphi_j]$$

where $u_k \to u_\infty$ strongly in $H^1_{\mathrm{loc}}(B_\delta \setminus \{0\})$ and $\varphi_j : S^2 \to \mathcal{N}$ are harmonic maps (bubbles).

Let φ_1 correspond to a bubble blown around the points $x_k \to 0$ with scaling factor $r_k \to 0$. Following Ding and Tian [13], we define the annuli $\mathcal{A}_{\delta,M,k} = \{x \in \mathbb{R}^2 : r_k M \le |x - x_k| \le \delta\}$ and note that the energy identity is equivalent to

$$\lim_{M\to\infty} \lim_{\delta\to 0} \lim_{k\to\infty} E[u_k, \mathcal{A}_{\delta,M,k}] = \sum_{j=2}^{m} E[\varphi_j]$$

where attention is on the bubble φ_1 without loss of generality. Next we define a sequence of transformations from our annuli to a sequence of cylinders tending to infinite length in one direction

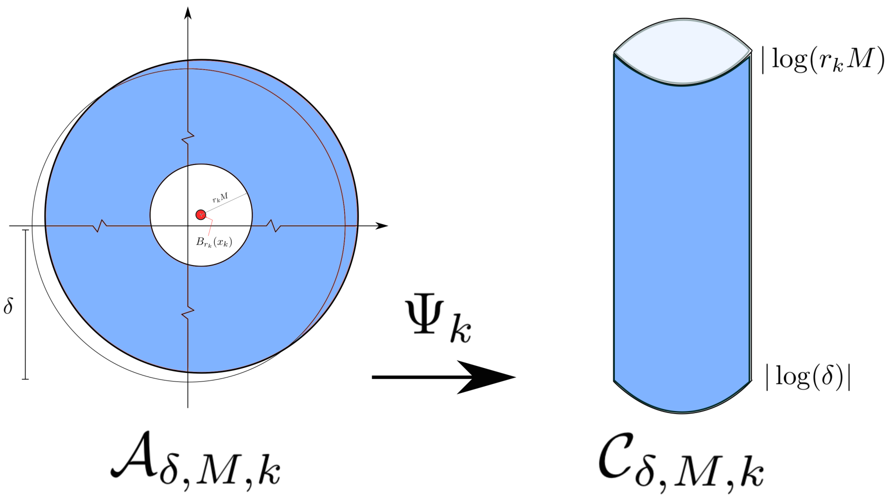
$$\begin{aligned} \Psi_k : \mathcal{A}_{\delta,M,k} &\longrightarrow \mathcal{C}_{\delta,M,k} = [|\log(\delta)|, |\log(r_k M)|] \times S^1 \\ (\rho_k, \theta_k) &\mapsto (t_k, \theta_k) \end{aligned}$$

where $t_k = |\log(\rho_k)|$, and the $\mathcal{C}_{\delta,M,k}$ are equipped with product metrics $g_k(t_k, \theta_k) = dt_k^2 + d\theta^2$.

We note that these transformations are conformal, indeed we calculate $(\Psi_k^{-1})^* g_k = \frac{1}{r_k^2} g_{\mathrm{euclid}}$, and so the energy is invariant.

Note 2.10. It is common to refer to these annular regions or their cylindrical counterparts as "neck regions". In the case of a single bubble, in the limit as M and k tend to ∞ and δ tends to 0, the limiting region $\mathcal{A}_{\delta,M,k}$

Figure 4: Viewing the necks as long cylinders



is visualised as a neck connecting the extracted bubble S^2 to the domain Σ about the point of concentration p. Proving the energy identity in this case is then equivalent to proving that no energy is "lost in the neck".

The problem is thus recast onto the study of "almost harmonic" maps from long cylinders. We use the expression "almost harmonic" as, under this conformal transformation, we see again that whilst energy is invariant, the tension is tempered

$$\begin{aligned} \int_{\mathcal{C}_{\delta,M,k}} |\tau(v_k)|^2 d\mu(\mathcal{C}_{\delta,M,k}) &= \int_{\mathcal{A}_{\delta,M,k}} r_k^2 |\tau(u_k)|^2 d\mu(\mathcal{A}_{\delta,M,k}) \\ &\leq 2\delta C \to 0 \\ &\text{as } \delta \to 0 \end{aligned}$$

This observation is crucial to the analysis as is made clear by exposing some theory from complex geometry.

Definition 2.11. For any map $u\colon \mathbb{R}^2 \to \mathcal{N}$, define the Hopf differential $\phi(u)$ in Cartesian coordinates x, y by

$$\phi(u) = |u_x|^2 - |u_y|^2 + 2iu_x \cdot u_y$$

The Hopf differential is not independent of coordinates, but selecting a complex coordinate $z = x+iy$ we have that the corresponding *quadratic differential* $\phi(u)(z)dz$ is independent of coordinates. The Hopf differential satisfies the following two properties which can be checked by direct calculation:

(i) The L^1-norm of $\phi(u)$ is invariant under a conformal transformation.

(ii) Defining the operator $\bar{\partial}$ in cartesian coordinates by $\bar{\partial} = \partial_x + i\partial_y$, we have

$$\bar{\partial}(\phi(u)) = (u_x - iu_y)\tau(u)$$

We will show that the L^1-norm of $\phi(u)$ is small on D_δ for sufficiently small δ. It then follows from property (i) that the L^1-norm of $\phi(v) := |v_\theta|^2 - |v_t|^2 + 2iv_\theta \cdot v_t$ is small on the corresponding cylinder (neck) for sufficiently small delta. Thus it suffices to show that one component of the energy, $\int |v_\theta|^2$ or $\int |v_t|^2$, is small on the neck to deduce that the total energy $\int(|v_\theta|^2 + |v_t|^2)$ is small on the neck for sufficiently small delta. It turns out to be possible to estimate the angular component of energy, $\int |v_\theta|^2$.

Claim 2.12. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that $||\phi(u_k)||_{L^1(D_\delta)} < \varepsilon$.*

Proof. This proof is adapted from [13].

Let $\varepsilon > 0$ be given. Abbreviate $\phi(u_k)$ by ϕ_k. Labelling points by a complex coordinate $z = x + iy$, by the generalised Cauchy integral formula we may write

$$2\pi i\phi_k(z) = \int_{\partial D_1(z)} \frac{\phi_k(\theta)}{\theta - z} d\theta - \int_{D_1(z)} \frac{\bar{\partial}\phi_k(\xi)}{\xi - z} d\xi$$

where $D_1(z)$ is the disc of radius 1 centred at z.

For the first integral, since $u_k \to u_\infty$ strongly in $H^1_{\text{loc}}(D_\delta \setminus \{0\})$, we have that the energy of u_k is locally small on a neighbourhood U of $\partial D_1(z)$; at least for z close to the origin. By Lemma 2.3, it follows that u_k have uniformly bounded H^2-norms on U. Thus, by the trace embedding theorem, it follows that u_k are uniformly bounded in $H^1(\partial D_1(z))$ and so ϕ_k are uniformly bounded in $L^1(\partial D_1(z))$. Hence for δ_1 small enough we have

$$\int_{D_{\delta_1}(0)} \int_{\partial D_1(z)} \left|\frac{\phi_k(\theta)}{\theta - z}\right| d\theta dz < \frac{\varepsilon}{2}$$

For the second integral, by property (ii), $|\bar{\partial}\phi_k| \leq |Du_k||\tau(u_k)|$ and so

$$\begin{aligned} ||\bar{\partial}\phi_k||_{L^1(D_{\frac{3}{2}}(0))} &\leq E[u_k]^{\frac{1}{2}} ||\tau(u_k)||_{L^2(D_{\frac{3}{2}}(0))} \\ &\leq C \end{aligned}$$

Hence for δ_2 small enough

$$\begin{aligned} \int_{D_{\delta_2}(0)} \int_{D_1(z)} \left|\frac{\bar{\partial}\phi_k(\xi)}{\xi - z}\right| d\xi dz &\leq 2 \int_{D_{\delta_2}(0)} ||\bar{\partial}\phi_k||_{L^1(D_{\frac{3}{2}}(0))} dz \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Setting $\delta = \min\{\delta_1, \delta_2\}$ then completes the proof. $\square$

Now, still following [13], we first deal with the case of a single bubble $m = 1$, in order to treat the case of multiple bubbles inductively later. We wish to establish then that no energy resides in the neck region in the limit. That is, we wish to show that for given $\varepsilon > 0$, there exists a large M, a small δ and a large $I = I(M, \delta)$ so that

$$\begin{aligned} E[v_k, \mathcal{C}_{\delta,M,k}] &= E[u_k, \mathcal{A}_{\delta,M,k}] \\ &< \varepsilon \end{aligned}$$

for any k greater than I.

Firstly, in analysing the behaviour of our sequence of maps from an elongating cylinder, it is observed that we need not worry about energy near the ends of the cylinder:

On the one hand, any energy which is to be lost must be "moving along the cylinder" as $k \to \infty$. That is, since there is strong H^1 convergence to u_∞ away from the concentration point, we may take δ small enough so that $E[u_\infty, D_\delta] < \frac{\varepsilon}{2}$. It follows that for any large $\Lambda > 0$ there is an $I = I(\Lambda)$ so that

$$\begin{aligned} E[v_k, [T_0, T_0 + \Lambda] \times S^1] &= E[u_k, D_\delta \setminus D_{\delta e^{-\Lambda}}] \\ &< \frac{\varepsilon}{2} \end{aligned}$$

where $T_0 := |\log(\delta)|$ and $[T_0, T_0 + \Lambda] \times S^1 \subset \mathcal{C}_{\delta,M,k}$, for all $k > I$.

On the other hand, any energy which "lingers close to the end of the cylinder" $\mathcal{C}_{\delta,M,k}$ for some δ, M, is sure to be extracted in the bubble. That is, since the bubble has finite energy, an *ad absurdum* argument shows that for any Λ, M may be chosen large enough so that

$$\begin{aligned} E[v_k, [T_k - \Lambda, T_k] \times S^1] &= E[u_k, D_{e^\Lambda r_k M} \setminus D_{r_k M}] \\ &< \frac{\varepsilon}{2} \end{aligned}$$

where $T_k := |\log(r_k M)| \to \infty$ and $[T_k - \Lambda, T_k] \times S^1 \subset \mathcal{C}_{\delta,M,k}$; for any k greater than some $I = I(\Lambda)$.

So the analysis is firstly reduced to the "middle" of the cylinder. Secondly, it is observed that, within the middle, energy cannot linger in any fixed interval; as this would mean energy concentration allowing for the extraction of a second bubble:

Claim 2.13. *Fix some δ, M. For any $\varepsilon > 0$, there exists $I = I(\varepsilon)$ such that*

$$\int_{[t,t+1]\times S^1} |\nabla v_k|^2 < \varepsilon$$

for any $k \geq I$, $t \in [T_0, T_k - 1]$.

Proof. This proof is from [13].

Suppose the claim to be false. Then as $k \to \infty$ there exist t_k such that

$$\int_{[t_k,t_k+1]\times S^1} |\nabla v_k|^2 \geq \varepsilon$$

By translating coordinates $t \mapsto t - t_k$ may consider v_k on a new cylinder $[-Y_k, Z_k] \times S^1$ where $Y_k, Z_k \to \infty$ by previous discussion, and

$$\int_{[0,1]\times S^1} |\nabla v_k|^2 \geq \varepsilon$$

Since energy is bounded above, we may assume $v_k \rightharpoonup v_\infty$ weakly in $H^1([0,1] \times S^1)$.

If the convergence is H^1-strong, then v_∞ is a non-constant harmonic map on $[0,1] \times S^1$ (since $\tau(v_k) \to 0$ in $L^2_{\text{loc}}([0,1] \times S^1)$). Thus on the entire cylinder $[-Y_k, Z_k] \times S^1$ we may pass to a subsequence to obtain weak

H^1-convergence to some non-constant harmonic map also denoted v_∞ on $(-\infty, \infty) \times S^1$. As $(-\infty, \infty) \times S^1$ is conformally equivalent to a sphere minus its two poles, this gives rise to a new non-constant, harmonic sphere (a new bubble) φ_2 by the Sacks-Uhlenbeck removable singularity theorem [8]. A contradiction.

If the convergence is not H^1-strong, then by Lemma 2.3 there is necessarily a point in $[0, 1] \times S^1$ about which energy concentrates, and a second bubble φ_2 may be obtained by rescaling about this point. A contradiction. $\square$

We are not yet close to seeing that the energy on the neck is arbitrarily small in the limit. We have shown that it is arbitrarily small in arbitrarily long neighbourhoods of the ends, and on any interval of a fixed length as we pass to the limit, but so far this is useless as we are considering a domain tending to infinite length. This is the difficulty in proving the energy identity and an explicit estimate on the energy is required to go further.

Recalling that we have reduced a control of the total energy on our neck to a control of just one component on the neck, it suffices to develop an estimate for the angular component of energy $\int |(v_k)_{\theta_k}|^2$. For this we prefer the approach of Huxol, Rupflin & Topping [10] where the angular energy on a neck region is estimated explicitly in terms of the tension.

Lemma 2.14. *Let $v \colon \mathcal{C} \to \mathcal{N}$ be a map from a long cylinder. Defining the angular energy $\vartheta(t) = \int_{\{t\} \times S^1} |v_\theta|^2 d\theta$, then there exists an $\varepsilon > 0$ such that if*

$$\int_{[s,s+1] \times S^1} |\nabla v|^2 dt d\theta < \varepsilon \qquad \text{and} \qquad \int_{[s,s+1] \times S^1} |\tau(v)|^2 dt d\theta < \varepsilon$$

for any t in range, then ϑ satisfies the differential inequality

$$\vartheta''(t) \geq \vartheta(t) - 2 \int_{\{t\} \times S^1} |\tau|^2 d\theta$$

Proof. This proof is from [10]. We will omit writing volume forms in the following proof, which are always those induced by the flat cylinder.

The tension on the cylinder is $\tau(v) = v_{\theta\theta} + v_{tt} + A(v)(v_\theta, v_\theta) + A(v)(v_t, v_t)$ since the metric is $ds^2 = d\theta^2 + dt^2$. Thus a priori we have

$$\begin{aligned}
\vartheta'(t) &= 2 \int_{\{t\} \times S^1} v_\theta \cdot v_{\theta t} \\
\vartheta''(t) &= 2\Big(\int_{\{t\} \times S^1} |v_{\theta t}|^2 - \int_{\{t\} \times S^1} v_{\theta\theta} \cdot v_{tt} \Big) \\
&= 2\Big(\int_{\{t\} \times S^1} |v_{\theta t}|^2 + \int_{\{t\} \times S^1} |v_{\theta\theta}|^2 \Big) \\
&\qquad \underbrace{-2 \int_{\{t\} \times S^1} v_{\theta\theta} \tau(v)}_{\textbf{(I)}} + \underbrace{2 \int_{\{t\} \times S^1} v_{\theta\theta} (A(v)(v_t, v_t) + A(v)(v_\theta, v_\theta))}_{\textbf{(II)}} \\
&\geq 2\Big(\int_{\{t\} \times S^1} |v_{\theta t}|^2 + |v_{\theta\theta}|^2 \Big) - \big|\textbf{(I)}\big| - \big|\textbf{(II)}\big|
\end{aligned}$$

We estimate **(I)** by Young's inequality

$$\big|\mathbf{(I)}\big| \le \frac{1}{2}\int_{\{t\}\times S^1} |v_{\theta\theta}|^2 + 2\int_{\{t\}\times S^1} |\tau(v)|^2$$

And for **(II)**, since $\mathcal{N}$ is compact, the derivatives of A are bounded and we have $[A(u)(X,Y)]_\theta \le C(|X_\theta||Y| + |X||Y_\theta|)$ and so integrating by parts gives

$$\big|\mathbf{(II)}\big| \le C\big(\int_{\{t\}\times S^1} |v_\theta||v_t||v_{t\theta}| + \int_{\{t\}\times S^1} |v_{\theta\theta}||v_\theta|^2\big)$$

thus by Young's inequality

$$\big|\mathbf{(II)}\big| \le 2\int_{\{t\}\times S^1} |v_{t\theta}|^2 + \frac{1}{4}\int_{\{t\}\times S^1} |v_{\theta\theta}|^2 + C\int_{\{t\}\times S^1} |v_\theta|^2(|v_t|^2 + |v_\theta|^2)$$

Now, we note that so long as ε is chosen smaller than $\varepsilon_0 = \varepsilon_0(\mathcal{C}, \mathbb{N})$, then by Lemma 2.3 we have

$$\begin{aligned}||v - \bar{v}||_{H^2([t,t+1]\times S^1)} &\le C(||Dv||_{L^2([t,t+1]\times S^1} + ||\tau(v)||_{L^2([t,t+1]\times S^1}\\ &\le C\cdot 2\varepsilon\end{aligned}$$

where $\bar{v}$ denotes the mean value of v. Hence by the trace embedding theorem, ε may be chosen sufficiently small so that

$$C\int_{\{t\}\times S^1} (|v_t|^2 + |v_\theta|^2) \le C\cdot 2\varepsilon \le \frac{1}{4}$$

and so

$$\begin{aligned}\int_{\{t\}\times S^1} |v_\theta|^2(|v_t|^2 + |v_\theta|^2) &\le \frac{1}{4}\sup_{\{t\}\times S^1} |v_\theta|^2\\ &\le \frac{1}{4}\int_{\{t\}\times S^1} |v_{\theta\theta}|^2\end{aligned}$$

Putting this together gives

$$\big|\mathbf{(I)}\big| + \big|\mathbf{(II)}\big| \le 2\int_{\{t\}\times S^1} |v_{t\theta}|^2 + \int_{\{t\}\times S^1} |v_{\theta\theta}|^2 + 2\int_{\{t\}\times S^1} |\tau(v)|^2$$

and so

$$\begin{aligned}\vartheta''(t) &\ge \int_{\{t\}\times S^1} |v_{\theta\theta}|^2 - 2\int_{\{t\}\times S^1} |\tau(v)|^2\\ &\ge \int_{\{t\}\times S^1} |v_\theta|^2 - 2\int_{\{t\}\times S^1} |\tau(v)|^2\end{aligned}$$

by Wirtinger/Poincaré's inequality since $\int_{\{t\}\times S^1} v_\theta = 0$.

This is the desired result. □

Writing $\mathcal{T}(t) := \int_{\{t\}\times S^1} |\tau(v)|^2$, we now examine the obtained differential inequality

$$-\vartheta''(t) + \vartheta(t) \le 2\mathcal{T}(t)$$

Setting $\xi = \vartheta - f$ where f satisfies $-f'' + f = 2\mathcal{T}$ gives $-\xi'' + \xi \le 0$. We may express f explicitely as a sum of a homogenous solution and a particular solution

$$f(t) = Ae^{S_0 - t} + Be^{t - S_1} + \int_{S_0}^{S_1} e^{-|t-q|}\mathcal{T}(q)dq$$

for some arbitrary constants A, B. We choose our A, B large enough so that $f \geq \vartheta$ at the ends of the cylinder on the boundary $\partial\mathcal{C}$. Then the maximum principle implies $\xi \leq 0$ i.e.

$$\vartheta(t) \leq Ae^{S_0-t} + Be^{t-S_1} + \int_{S_0}^{S_1} e^{-|t-q|}\mathcal{T}(q)dq$$

Returning now to our sequence of almost harmonic maps from an elongating cylinder $\{v_k\}$, we set $\vartheta_k(t) = \vartheta_k(t_k) = \int_{\{t_k\}\times S^1} |(v_k)_{\theta_k}|^2$; $\mathcal{T}_k(t) = \mathcal{T}_k(t_k) = \int_{\{t_k\}\times S^1} |\tau(v_k)|^2$.

For δ small enough we have $\int_{T_0}^{T_k} \mathcal{T}_k(t)dt$ small, and so by Claim 2.13 v_k satisfies the conditions for Lemma 2.14 for sufficiently large k. Thus

$$\vartheta_k(t) \leq A_k e^{T_0+M-t} + B_k e^{t-T_k+M} + \int_{T_0+M}^{T_k-M} e^{-|t-q|}\mathcal{T}(q)dq$$

and so

$$\begin{aligned}
\int_{\mathcal{C}_{\delta,M,k}} |(v_k)_{\theta_k}|^2 &= (\int_{T_0}^{T_0+M} + \int_{T_0+M}^{T_k-M} + \int_{T_k-M}^{T_k})\vartheta_k \\
&= \int_{T_0+M}^{T_k-M} \vartheta_k(t)dt + 2\varepsilon \\
&\leq A_k \int_{T_0+M}^{T_k-M} e^{T_0+M-t}dt + B_k \int_{T_0+M}^{T_k-M} e^{t-T_k+M}dt + \int_{T_0+M}^{T_k-M}\int_{T_0+M}^{T_k-M} e^{-|t-q|}dt\mathcal{T}(q)dq + 2\varepsilon \\
&\leq A_k + B_k + \int_{T_0+M}^{T_k-M}\int_{T_0+M}^{T_k-M} e^{-|t-q|}dt\mathcal{T}(q)dq + 2\varepsilon \\
&\leq A_k + B_k + 2||\mathcal{T}(v_k)||_{L^2} + 2\varepsilon
\end{aligned}$$

where Fubini's theorem is applied to change order of integration. It is now observed that the constants A_k and B_k here may be chosen to represent the values of the angular energy near the ends of the cylinder, at $T_0 + M$ and $T_k - M$.

Indeed, selecting $A_k = 2\vartheta(T_0 + M)$ and $B_k = 2\vartheta(T_k - M)$, since we have a uniform bound on energy at the ends $A_k, B_k \leq C$ we see that

$$\begin{aligned}
f_k(T_0 + M) &= A_k + B_k e^{-|T_k-T_0-2M|} + \int_{T_0+M}^{T_k-M} e^{-|T_0+M-q|}\mathcal{T}(q)dq \\
&\geq 2\vartheta_k(T_0 + M) - 2e^{-|T_k-T_0-2M|}C - ||\tau(v_k)||^2_L(\mathcal{C}_k) \\
&\geq \vartheta_k(T_0 + M)
\end{aligned}$$

and similarly $f_k(T_k - M) \geq \vartheta_k(T_k - M)$ for sufficiently small δ and large k.

We have already shown that the energy is arbitrarily small in these end regions for suitably small δ and large M, for all large k. Hence applying the Lemma 2.3 alongside the trace embedding theorem, we have that A_k and B_k may be taken less that ε, and so

$$\int_{\mathcal{C}_{\delta,M,k}} |(v_k)_{\theta_k}|^2 \leq 5\varepsilon$$

Hence we have shown that, there exists an ε' such that for any $0 < \varepsilon < \varepsilon'$, there exists a large M, a small δ, and a number I so that $\int_{\mathcal{C}_{\delta,M,k}} |(v_k)_{\theta_k}|^2 < \epsilon$. Since we have shown that the total energy is controlled by the angular energy we deduce

$$\lim_{M\to\infty}\lim_{\delta\to 0}\lim_{k\to\infty} E[u_k;\mathcal{A}_{\delta,M,k}] = 0$$

This establishes the energy identity in the case of a single bubble. We can now progress to the identity in full generality by simply identifying more bubble regions and neck regions, and applying the above analysis on the neck regions in order to reduce the case of m bubbles to the case of $m-1$ bubbles. For m bubbles, we can no longer argue that no energy will concentrate, rather we argue that if it does concentrate in a region $[s, s+1] \times S^1$ in the middle of a neck, then the convergence must either be H^1 strong, in which a bubble may be extracted by the Sacks-Uhlenbeck removable singularity theorem, or energy must concentrate and thus we can extract a bubble by rescaling. Either way we identify a bubble and no energy on the necks connecting it to the other $m-1$. Proceeding by induction we obtain the general result.

Theorem 2.15. *Let $\{u_k \colon \Sigma \to \mathcal{N}\}$ be a sequence of maps with uniformly bounded energy $E[u_k]$ and tension $\tau(u_k)$. Suppose that $p \in \Sigma$ is a concentration point at which m bubbles may be extracted by rescaling $\{\varphi_j \colon S^2 \to \mathcal{N}\}$, and that these bubbles are counted only once. Then in some neighbourhood U of p we may extract a subsequence such that $u_k \rightharpoonup u_\infty$ weakly in $H^1(U)$ and the energy identity*

$$\lim_{k\to\infty} E[u_k, U] = E[u_\infty, U] + \sum_{j=1}^{m} E[\varphi_j, S^2]$$

holds.

3 Maps from a degenerating surface

3.1 A new compactness result

The harmonic map flow is not guaranteed to converge towards an energy minimizer even locally. Consider for example the initial condition of a map between two round spheres, $u_0 \colon S^2 \to S^3$, given in angular coordinates by $(\theta, \varphi) \mapsto (\theta, \varphi, 0)$. In this case the initial map is already critically immersed, yet the flow will run if the metric on the domain is perturbed slightly.

Thus, in order to probe the geometry of a given Riemannian manifold $\mathcal{N}$, one may view the energy as a functional not confined to the space of maps from a surface with a fixed metric, but as a functional depending also on the metric imposed on the domain. This is a viewpoint which has been adopted in recent years by Rupflin, Topping et. al. [7], [6] [5], in developing a new flow in the search for existence results for minimal surfaces.

In order to understand the types of singularities that might form under a flow on a space, as discussed earlier it is a nice first step to study the compactness properties of that space. This section will be devoted to such a compactness result for maps from Riemannian surfaces.

The first point to observe from this viewpoint is that the energy functional is invariant under a conformal transformation of the domain. Thus it is equivalent to consider the functional restricted to the space of metrics on a surface, modulo conformal equivalence. This is the Riemann moduli space[2]. By the uniformization theorem, each conformal equivalence class (point in the moduli space) admits a representative of constant curvature.

Theorem 3.1 (Uniformization theorem). *In a given conformal class of metrics on a Riemann surface Σ, there is a representative with constant Gauss curvature:*

- *1 if Σ is of genus 0*
- *0 if Σ is of genus 1*
- *-1 if Σ is of genus ≥ 2*

The interesting case is genus ≥ 2, and the moduli spaces of genus 0 and 1 are 0 and 1 dimensional, and it is surfaces in this class which we study. Such a surface with a metric of curvature -1 is called *hyperbolic*, and the compactifications of the hyperbolic moduli space are well studied. The volume of a hyperbolic surface with given genus is fixed (by Gauss-Bonnet), however surfaces may "degenerate" in the sense that the length of a closed geodesic may shrink to a point, whilst its annular neighbourhood elongates into an infinitely long region called a *collar*.

A hyperbolic Riemannian surface may be viewed as split into a "δ-thin" part consisting of points about which the injectivity radius is less than δ, we write $\mathcal{C} = \{p \colon \mathrm{inj}(p) < \delta\}$, and a complementary "$\delta$-thick" part $\Sigma \setminus \mathcal{C}$. For sufficiently small δ, the compactness properties of the moduli space may be fully understood via this splitting as the δ-thick parts admit convergent subsequences whilst the δ-thin parts admit a canonical model for degeneration, the collar. We will state the following two results informally, and refer the reader to the appendix of [5], specifically sections A1 and A2, for a more comprehensive statement.

Lemma 3.2 (Keen-Randol collar lemma). *For $0 < \delta < \arcsin(1)$, the δ-thin part of a Riemannian surface may be decomposed as a disjoint union of collar regions, whose number is bounded above by a function of the genus, each based around a unique closed geodesic σ. Each collar region is conformally equivalent to a cylinder $[-X, X] \times S^1$, where the number X depends only on the length of σ and tends to ∞ as this length tends to 0.*

Theorem 3.3 (Deligne-Mumford compactness theorem). *Given a sequence of hyperbolic Riemannian surfaces (Σ, g_k) of the same genus, there exist a finite collection of closed curves $\{\sigma^{(j)}\}$ and a subsequence upon which the δ-thick parts converge to a limit $(\Sigma \setminus \bigcup_j \sigma^{(j)}, g_\infty)$ where convergence is understood in the sense of the Gromov-Hausdorff topology.*

In light of these results, there is only one type of degeneration within the moduli space that need be considered. We will work in the much simplified setting of a sequence of harmonic maps, and into a manifold

[2] In the work of Rupflin, Topping et. al., they focus instead on the *Teichmüller space* of conformal equivalences which are conformally equivalent under the action of a diffeomorphism homotopic to the identity. This is the universal cover of the Riemann moduli space, and it is a prefered object for defining a flow as it is a (contractible) manifold

of negative curvature. We deal with this setting in order to understand the first mechanism by which a flow might degenerate. The rest of this report is dedicated to a proof of the following theorem.

Theorem 3.4. *Let Σ be a Riemannian surface and let $\{g_k\}$ be a sequence of hyperbolic metrics on that surface. Suppose that (Σ, g_k) is degenerating in the sense that there is single closed curve σ on Σ such that the unique geodesics σ_k homotopic to σ have lengths $L(\sigma_k)$ tending to zero. Consider a sequence of harmonic maps $\{u_k\colon (\Sigma, g_k) \to (\mathcal{N}, h)\}$ in a distinguished homotopy class $[u_0]$, with a uniform bound on the energy. Suppose further that $(\mathcal{N}, h)$ is of negative sectional curvature. Then the images of the degenerating collars tend to curves whose lengths are uniformly bounded in the limit as $k \to \infty$.*

We will prove the theorem in the end by contradiction, assuming a subsequence of maps for which the image of the collars do not tend to curves with uniformly bounded length. Thus in what follows we will regularly pass to a subsequence.

Firstly, we note that since N is negatively curved, no bubbles form and, after extracting a convergent subsequence of metrics on the δ-thick part by the Deligne-Mumford compactness theorem, by the results of Section 2.1 we may then extract a subsequence of maps strongly convergent in $H^1\left((\Sigma \setminus \sigma, g_\infty), (\mathcal{N}, h)\right)$ on $(\Sigma \setminus \sigma)$. The degenerating δ-thin part, the collar, we model in cylindrical coordinates $(t, \theta) \in [-X_k, X_k] \times S^1 = \mathcal{C}_k$.

Lemma 3.5. *There exists $\Lambda_1 > 0$ such that for all sufficiently large k, for any $t \in [-X_k + \Lambda_1, X_k - \Lambda_1]$, we have $E[u_k, [s, s+1] \times S^1] \leq \varepsilon_0$.*

Proof. For each $k \in \mathbb{N}$, let $s_k^{(i)} = -X_k + 2i$ for $i = 1, \dots, \lceil \frac{1}{X_k} \rceil$.

Let $\{t_r^{(j)}\}$ be the subset formed of precisely those values of $s_k^{(i)}$ for which

$$E[u_k, [s_k^{(i)}, s_k^{(i)} + 1] \times S^1] > \varepsilon_0$$

Suppose there exists a subseqence, also denoted $\{t_r^{(j)}\}$, for which $|X_r - t_r^{(j)}| \to \infty$ as $r \to \infty$. In this case, we may argue as in Claim 2.13 to deduce the existence of a non-constant harmonic 2-sphere either by energy concentration about a point on the collar or by the Sacks-Uhlenbeck removable singularity theorem.

As $\mathcal{N}$ is negatively curved, it follows that this cannot be the case. Thus there exists Λ such that $|X_r - t_r^{(j)}| \leq \Lambda$ for all r, j. Setting $\Lambda_1 = \Lambda + 2$ then gives the desired result. $\square$

Now, the result of Lemma 3.5 means we can apply Lemma 2.14 to our sequence of elongating collars $\mathcal{C}_k$. Thus for k sufficiently large we have an estimate on the angular energy

$$\vartheta_k(t) = \int_{\{t\} \times S^1} |(u_k)_\theta|^2 \leq A_k e^{-X_k + \Lambda_1 - t} + B_k e^{t - X_k + \Lambda_1}$$

As $k \to \infty$, the constants A_k, B_k tend to the values of angular energy of u_k at the ends of the $\mathcal{C}_k$. By the Deligne-Mumford compactness theorem and the results of Section 2.1 we have the values of u_k in $H^2(\Sigma \setminus ([-X_k + 1, X_k - 1] \times S^1), \mathcal{N})$ are bounded and thus by trace embedding we deduce that the angular energy is uniformly bounded at the ends of the collars, $A_k, B_k < C$.

Since $\mathcal{N}$ is compact, there is a lower bound on the size of the injectivity radius of the exponential map about a point in $\mathcal{N}$, $\delta_0 = \mathrm{inj}(N)$. It follows from the above that there exists $\Lambda_0 \geq \Lambda_1$ such that $\vartheta_k(t) < \frac{\delta_0^2}{2\pi}$ for all $t \in [-X_k + \Lambda_0, X_k - \Lambda_0]$. Moreover, by the collar lemma we see that Λ_0 may be chosen large enough so that the length of a circle $\{t\} \times S^1 \subset [-X_k + \Lambda_0, X_k - \Lambda_0] \times S^1$ is less than 1.

Lemma 3.6. *Redefining $\mathcal{C}_k = [-X_k + \Lambda_0, X_k - \Lambda_0] \times S^1$, we have that the image of $\mathcal{C}_k$ under u_k may be deformed continuously by a distance of $(2\pi\vartheta_k(t))^{\frac{1}{2}} < \delta_0$ at each point $u_k(\theta, t)$, giving a curve ω_k.*

Proof. To show this, first note that if $\gamma_0, \gamma_1 \colon [-X_k + \Lambda_0, X_k - \Lambda_0] \to \mathcal{N}$ are two curves such that

$$\begin{aligned}\mathrm{dist}(\gamma_0, \gamma_1) &:= \max_{s \in [-X_k+\Lambda_0, X_k-\Lambda_0]} \min_{t \in [-X_k+\Lambda_0, X_k-\Lambda_0]} \{d(\gamma_0(t), \gamma_1(s))\} \\ &< \delta_0\end{aligned}$$

where d is the distance function induced by the metric h, then γ_0 may be perturbed to γ_1 by moving points no further than a distance δ_0.

Indeed, for each $s \in [-X_k+\Lambda_0, X_k-\Lambda_0]$, let t be chosen so that $d(\gamma_0(t), \gamma_1(s)) = \min_{\tau \in [-X_k+\Lambda_0, X_k-\Lambda_0]} \{d(\gamma_0(\tau), \gamma_1(s))\}$.

Then by definition of δ_0 we have a unique geodesic $\xi_s \colon [0,1] \to \mathcal{N}$ defined so that $\xi_s(0) = \gamma_0(s), \xi_s(1) = \gamma_1(s)$. Since these geodesics vary continuously, we thus obtain a continuous perturbation $\Gamma \colon [-X_k + \Lambda_0, X_k - \Lambda_0] \times [0,1] \to \mathcal{N}$ by $\Gamma(q, r) = \xi_q(r)$.

Secondly, we observe by Cauchy-Schwarz that the image of $\{t\} \times S^1$ under u_k has length

$$\begin{aligned}L &= \int_{\{t\} \times S^1} |(u_k)_\theta| d\mu_k(\{t\} \times S^1) \\ &\leq (2\pi)^{\frac{1}{2}} (\vartheta(t))^{\frac{1}{2}}\end{aligned} \tag{6}$$

From this we deduce the lemma ☐

We would like to compare energy of the maps with a competitor whose image on the middle of the collar is a curve.

Lemma 3.7. *For a choice of $\mu_k \to \infty$, there exist maps $\tilde{u}_k$ in the same homotopy class as u_k such that $\tilde{u}_k = u_k$ on $\Sigma \setminus \mathcal{C}_k$, and on $\mathcal{C}_k$ we have*

$$\begin{aligned}\tilde{u}_k(\theta, t) &= u_k(\theta, t) \qquad \textit{for } t \in [-X_k + \Lambda_0, X_k + \Lambda_0 + \mu_k] \cup [X_k - \Lambda_0 - \mu_k, X_k - \Lambda_0] \\ \tilde{u}_k(\theta, t) &= u_k(\theta_0, t) \qquad \textit{for } t \in [-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1]\end{aligned}$$

and such that $E[\tilde{u}_k] \leq E[u_k] + C(e^{-\mu_k} + \frac{1}{X_k - \mu_k})$ for a constant C, when k is sufficiently large.

Proof. (Sketch). Choose a value θ_0 such that for $\tilde{u}_k \colon [-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1] \times S^1 \to \mathcal{N}$ defined by $\tilde{u}_k(\theta, t) = u_k(\theta_0, t)$ we have

$$E[\tilde{u}_k; [-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1] \times S^1] \leq E[u_k; [-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1] \times S^1]$$

which is possible by Fubini's theorem.

We then define $\tilde{u}_k = u_k$ outside of $[-X_k + \Lambda_0 + \mu_k, x_k - \Lambda_0 - \mu_k] \times S^1 \subset \mathcal{C}_k$.

Let Γ be the curve traced by $u_k(\{-X_k + \Lambda_0 + \mu_k\} \times S^1)$. Let S be the minimal surface in N whose boundary is Γ. Let $\tilde{u}_k \colon [-X_k + \Lambda_0 + \mu_k, -X_k + \Lambda_0 + \mu_k + 1] \times S^1$ be the map which collapses S at a uniform speed to the point $u_k(X_k + \Lambda_0 + \mu_k, \theta_0)$. Since the area of S is controlled by some function of the length of Γ squared by the isoperimetric inequality, we have that the energy of this map may be controlled by a function of the length squared. Thus

$$E[\tilde{u}_k; [-X_k + \Lambda_0 + \mu_k, -X_k + \Lambda_0 + \mu_k + 1] \times S^1] \leq C\vartheta(-X_k + \Lambda_0 + \mu_k) \leq Ce^{-\mu_k}$$

We define the map $\tilde{u}_k$ the same way on the right hand region $[X_k - \Lambda_0 - \mu_k - 1, X_k - \Lambda_0 - \mu_k] \times S^1$.

Next, for $t \in [-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1]$ we set $\tilde{u}_k(t, \theta) = u_k(\xi_k t, \theta_0)$ where $\xi_k = \frac{X_k - \Lambda_0 - \mu_k}{X_k - \Lambda_0 - \mu_k - 1}$.

Putting this together gives $E[\tilde{u}_k] \leq \xi_k E[u_k] + Ce^{-\mu_k} \leq E[u_k] + C(e^{-\mu_k} + \frac{1}{X_k - \mu_k})$. □

The previous lemma shows that, choosing to "pinch" the image of u_k to form a curve at a distance μ_k into the middle of the collar will cost an energy of order $e^{-\mu_k}$. For the rest of the calculation we will set $\mu_k = (X_k)^{\frac{1}{3}}$. In order to compare u_k to a homotopic map which pinches to a curve at some point, we need a control of the images of the ends of the collar. For this we develop an estimate on the speed of a harmonic map from a thin collar, which is analagous to showing that geodesics travel at a constant speed.

Lemma 3.8. *For a sequence of harmonic maps $\{u_k \colon \mathcal{C}_k \to \mathcal{N}\}$, we have*

$$\int_{\{t\} \times S^1} |(u_k)_t|^2 d\theta = \vartheta_k(t) + R_k$$

where R_k is a sequence of constants satisfying $R_k \leq \frac{E}{2X_k}$

Proof. For a harmonic map $u \colon \Sigma \to \mathcal{N}$, we have $u_{\theta\theta} + u_{tt} + A(u)(u_\theta, u_\theta) + A(u)(u_t, u_t) = 0$. Since the second fundamental form is orthogonal to $\mathcal{N}$ in $\mathbb{R}^N$, taking the dot product with $u_t \in \mathbb{T}\mathcal{N}$ gives $u_{\theta\theta} \cdot u_t + u_{tt} \cdot u_t = 0$.

Integrating by parts over the circle $\{t\} \times S^1$, we have

$$\begin{aligned} 0 &= -\int_{\{t\} \times S^1} u_{\theta t} \cdot u_\theta d\theta + \int_{\{t\} \times S^1} u_{tt} \cdot u_t d\theta \\ &= \frac{d}{dt}\left(-\int_{\{t\} \times S^1} |u_\theta|^2 d\theta + \int_{\{t\} \times S^1} |u_t|^2 d\theta \right) \end{aligned}$$

which gives the desired result for our sequence

$$\int_{\{t\} \times S^1} |(u_k)_t|^2 d\theta = \vartheta_k(t) + R_k$$

Moreover, our uniform energy bound gives $E \geq \int_{-X_k}^{X_k} |(u_k)_t|^2 d\theta dt \geq 2X_k R_k$ □

From the above lemma we deduce

Lemma 3.9. *In the limit as $k \to \infty$, for $\mu_k = (X_k)^{\frac{1}{3}}$ as above, the images*

$$\mathrm{Im}\big(u_k\big|_{[-X_k+\Lambda_0,-X_k+\Lambda_0+\mu_k+1]\times S^1}\big),\ \ \mathrm{Im}\big(u_k\big|_{[X_k-\Lambda_0-\mu_k-1,X_k-\Lambda_0]\times S^1}\big)$$

are constrained to balls of diameter $< \delta_0$ in $\mathcal{N}$.

Proof. We will just deal with the left hand image, $\mathrm{Im}\big(u_k\big|_{[-X_k+\Lambda_0,-X_k+\Lambda_0+\mu_k+1]\times S^1}\big)$.

Choosing angles $\theta_1^{(k)}$ such that the curves $\xi_k\colon [-X_k+\Lambda_0, -X_k+\Lambda_0+\mu_k+1] \to \mathcal{N}$ defined by $\xi_k(t) = u_k(\theta_1^{(k)}, t)$ satisfy

$$\begin{aligned}\int_{-X_k+\Lambda_0}^{-X_k+\Lambda_0+\mu_k+1} |\dot{\xi}_k|^2 dt &\leq \frac{1}{2\pi}\int_0^{2\pi}\int_{-X_k+\Lambda_0}^{-X_k+\Lambda_0+\mu_k+1} |(u_k)_t|^2 dt d\theta \\ &\leq \frac{1}{2\pi} R_k(\mu_k+1)\end{aligned}$$

Then by Cauchy-Schwarz we have that the length of ξ_k tends to zero.

$$\begin{aligned}L(\xi_k) &\leq (\mu_k+1)^{\frac{1}{2}}(\frac{1}{2\pi}R_k(\mu_k+1))^{\frac{1}{2}} \\ &\leq (\frac{E}{4\pi X_k})^{\frac{1}{2}}((X_k)^{\frac{1}{3}}+1) \to 0\end{aligned}$$

Thus, since circles on the collar $\{t\} \times S^1$ map under u_k into balls of diameter δ_0 by Lemma 3.6, the result follows. □

We have now gathered enough results to prove the main theorem.

Proof. (of Theorem 3.4). We assume the presence of a subsequence of maps for which collars tend to curves with length tending to ∞. We will arrive at contradiction by constructing a new map which is fixed on the collar and which has less energy, this map will be constructed to be homotopic to the original u_k by careful gluing.

By the Deligne-Mumford compactness theorem, passing to a subsequence, we have that for a sufficiently large fixed k_1, for all $k > k_1$ the Gromov-Hausdorff distance between the respective δ-thick parts of (Σ, g_k) and (Σ, g_{k_1}) is sufficiently small. That is, for an arbitrarily small $\varepsilon > 0$, k_1 may be chosen so that we may define a sequence of diffeomorphisms on the δ-thick parts

$$\Phi_k\colon \Sigma \setminus \mathcal{C}_k \to \Sigma \setminus \mathcal{C}_{k_1}$$

satisfying $|d_{(\Sigma,g_k)}(x,y) - d_{(\Sigma,g_{k_1})}(\Phi_k(x),\Phi_k(y))| < \varepsilon$ for all $x, y \in \Sigma \setminus \mathcal{C}_k$, where $d_{(\Sigma,g_k)}, d_{(\Sigma,g_{k_1})}$ are the respective distance functions.

The results of section 2.1 tell us that, since $\mathcal{N}$ is negatively curved and no bubbles can form, it follows that, for any $k > k_1$, the maps $u_k \circ \Phi_k^{-1}$ are uniformly bounded in $H^2(\Sigma \setminus \mathcal{C}_{k_1}, \mathcal{N})$ which implies convergence of a subsequence $\{u_k \circ \Phi_k^{-1}\}$ in $\mathcal{C}^0(\Sigma \setminus \mathcal{C}_{k_1}, \mathcal{N})$).

Thus, combining these results, we obtain that there exist sufficiently large values k_1, k_2, for all $k > k_2 \geq k_1$, we have $\mathrm{dist}_{(\mathcal{N},h)}(u_k \circ \Phi_k^{-1}, u_{k_2}\big|_{\Sigma\setminus\mathcal{C}_{k_2}}) < \delta_0$, so that the image of $u_{k_2}\big|_{\Sigma\setminus\mathcal{C}_{k_2}} = (u_{k_2} \circ \Phi_k^{-1}) \circ \Phi_k\colon \Sigma \setminus \mathcal{C}_{k_2} \to \mathcal{N}$

can be deformed by moving points a no further than a distance δ_0 in $\mathcal{N}$ to give the image of u_k. Choosing a large enough $k_0 = k_1 = k_2$, we have derived the following result.

Lemma 3.10. *There exists k_0 such that for all $k > k_0$, restricted to $\Sigma \setminus \mathcal{C}_k$, the image of $u_{k_0} \circ \Phi_k$ can be deformed by moving points no further than a distance δ_0 in $\mathcal{N}$ to give the image of u_k.*

We first extend the definition of the Φ_k to the collars to give diffeomorphisms on the entire of Σ by

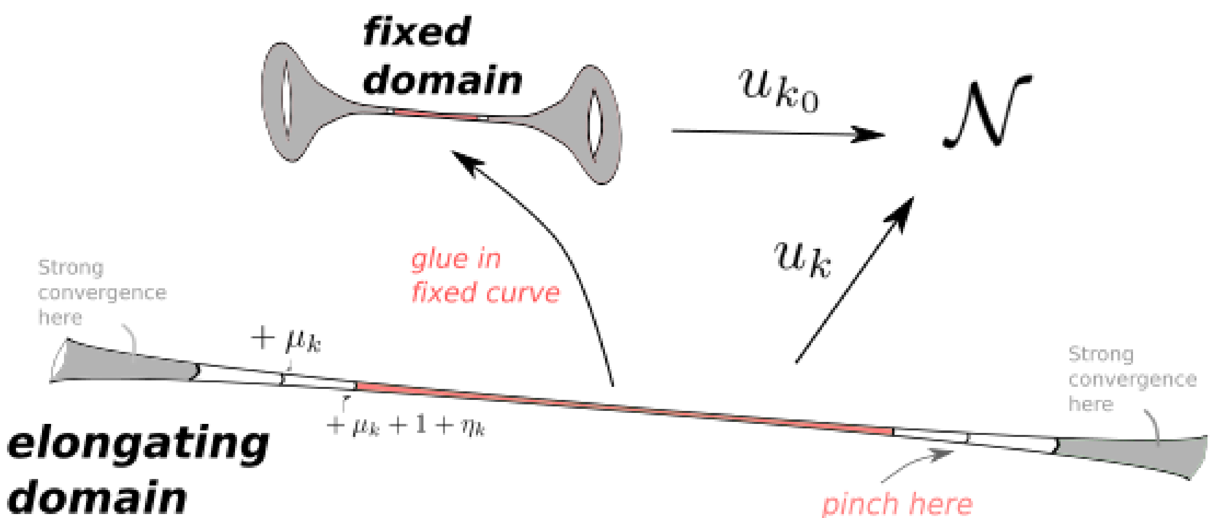
$$\Phi_k \colon (\mathcal{C}_k, g_k) \to (\mathcal{C}_{k_0}, g_{k_0}) \quad \Phi_k(\theta, t) = (\theta, \lambda_k t)$$

where $\lambda_k = \frac{X_{k_0} - \Lambda_0}{X_k - \Lambda_0}$.

Next we define maps

$$\hat{u}_k = \begin{cases} \tilde{u}_k & \text{on } \Sigma \setminus ([-X_k + \Lambda_0 + \mu_k + 1, X_k - \Lambda_0 - \mu_k - 1] \times S^1) \\ \alpha_k \circ \Phi_k & \text{on } [-X_k + \Lambda_0 + \mu_k + 1, -X_k + \Lambda_0 + \mu_k + 1 + \eta_k] \times S^1 \\ \nu_{k_0} \circ \Phi_k & \text{on } [-X_k + \Lambda_0 + \mu_k + 1 + \eta_k, X_k - \Lambda_0 - \mu_k - 1 - \eta_k] \times S^1 \\ \beta_k \circ \Phi_k & \text{on } [X_k - \Lambda_0 - \mu_k - 1 - \eta_k, X_k - \Lambda_0 - \mu_k - 1] \times S^1 \end{cases}$$

Figure 5: Constructing a homotopic map



where $\tilde{u}_k$ are defined as in Lemma 3.7; $\alpha_k, \beta_k : \mathcal{C}_{k_0} \to \mathcal{N}$ are defined to trace some curves of bounded length $\leq 2X_{k_0}E + 2\delta_0$ which follow $\mathrm{Im}(u_{k_0})$ connecting $\tilde{u}_k(\theta_0, -X_k + \Lambda_0 + \mu_k + 1)$ with $u_{k_0} \circ \Phi_k(\theta_0, -X_k + \Lambda_0 + \mu_k + 1 + \eta_k))$ and $\tilde{u}_k(\theta_0, X_k - \Lambda_0 - \mu_k - 1)$ with $u_{k_0} \circ \Phi_k(\theta_0, X_k - \Lambda_0 - \mu_k - 1 - \eta_k)$ respectively in times η_k. Such curves will exist for sufficiently large k by an analogous argument to that in Lemma 3.9 so long as we have $\frac{\eta_k}{X_k} \to 0$ as $k \to \infty$.

$\nu_{k_0} \colon \mathcal{C}_{k_0} \to \mathcal{N}$ is defined to trace a curve by $\nu_{k_0}(\theta, \tau) = u_{k_0}(\theta_0, \tau)$.

By Lemma 3.6, Lemma 3.7, Lemma 3.9 and Lemma 3.10, we have that the defined maps $\hat{u}_k$ are homotopic to $u_{k_0} \circ \Phi_k$, which is in turn homotopic to u_{k_0} (as Φ_k are diffeomorphisms), which is in turn homotopic u_k by hypothesis.

Firstly, we see that $E[\alpha_k \circ \Phi_k], E[\beta_k \circ \Phi_k] \leq \frac{E}{\eta_k}$.

Secondly, we observe $E[\nu_{k_0} \circ \Phi_k] \leq \lambda_k E$.

Thirdly, by Cauchy Schwarz, the lengths of the curves traced by the collar under $\tilde{u}_k$ satisfy

$$L[\tilde{u}_k\big|_{[-X_k+\Lambda_0+\mu_k+1,X_k-\Lambda_0-\mu_k-1]\times S^1}]^2 \leq 2(-X_k+\Lambda_0+\mu_k+1)E[\tilde{u}_k\big|_{[-X_k+\Lambda_0+\mu_k+1,X_k-\Lambda_0-\mu_k-1]\times S^1}]$$

So by the assumption $L[\tilde{u}_k\big|_{[-X_k+\Lambda_0+\mu_k+1,X_k-\Lambda_0-\mu_k-1]\times S^1}]^2 \to \infty$ we have for sufficiently large k

$$\begin{aligned} E[\tilde{u}_k\big|_{[-X_k+\Lambda_0+\mu_k+1,X_k-\Lambda_0-\mu_k-1]\times S^1}] &> \frac{E(X_{k_0}-\Lambda_0)}{X_k-\Lambda_0-\mu_k-1} \\ &= \frac{E(X_{k_0}-\Lambda_0)}{X_k-\Lambda_0}\left(\frac{1}{1-\frac{\mu_k+1}{X_k-\Lambda_0}}\right) \\ &= \frac{E(X_{k_0}-\Lambda_0)}{X_k-\Lambda_0}\left(\sum_{j=0}^{\infty}\left(\frac{\mu_k+1}{X_k-\Lambda_0}\right)^j\right) \\ &\geq E[\nu_{k_0}\circ\Phi_k] + \frac{\mu_k+1}{X_k-\Lambda_0}(X_{k_0}-\Lambda_0) \end{aligned}$$

Thus it follows with Lemma 3.7

$$\begin{aligned} E[\hat{u}_k] &= E[\tilde{u}_k\big|_{\Sigma\setminus([-X_k+\Lambda_0+\mu_k+1,X_k-\Lambda_0-\mu_k-1]\times S^1)}] + E[\alpha_{k_0}\circ\Phi_k] + E[\beta_{k_0}\circ\Phi_k] + E[\nu_{k_0}\circ\Phi_k] \\ &\leq \frac{2E}{\eta_k} + E[\tilde{u}_k] - \frac{\mu_k+1}{X_k-\Lambda_0}(X_{k_0}-\Lambda_0) \\ &\leq E[u_k] + \frac{2E}{\eta_k} + C(e^{-\mu_k} + \frac{1}{X_k-\mu_k}) - \frac{\mu_k+1}{X_k-\Lambda_0}(X_{k_0}-\Lambda_0) \end{aligned}$$

Choosing $\eta_k = (X_k)^{3/4}$, with $\mu_k = X_k^{\frac{1}{3}}$ then gives $E[\hat{u}_k] < E[u_k]$ for a suitably large k.

Since the paper of Eells-Sampson [3] tells us that harmonic maps into a negatively curved target are always energy minimizing in their homotopy class, this gives the contradiction. $\square$

References

[1] Simon Donaldson. What is... a pseudoholomorphic curve? *Notices of the AMS, Vol. 52.*

[2] B. Andrews & C. Hopper. The ricci flow in riemannian geometry. 2010.

[3] Jr. J. Eells and J. H. Sampson. Harmonic mappings of riemannian manifolds. *American Journal of Mathematics*, 1964.

[4] E. Sanabria-Codesal M. G. Monera, A. Montesinos-Amilibia. The taylor expansion of the exponential map and geometric applications. *arXiv:1210.5971 [math.DG]*, 2012.

[5] P. Topping & M. Zhu M. Rupflin. Asymptotics of the teichmüller harmonic map flow. *arXiv:1209.3783 [math.DG]*, 2012.

[6] M. Rupflin & P.Topping. Flowing maps to minimal surfaces. *arXiv:1205.6298 [math.DG]*, 2012.

[7] M. Rupflin. Flowing maps to minimal surfaces: Existence and uniqueness of solutions. *arXiv:1205.6982 [math.DG]r*, 2012.

[8] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Annals of Mathematics*, 1981.

[9] M. Struwe. On the evolution of harmonic mappings of riemannian surfaces. *Comment. Math. Helvetici*, 1985.

[10] M. Rupflin T. Huxol and P. M. Topping. Refined asymptotics of the teichmüller harmonic map flow into general targets. *arXiv:1502.05791 [math.DG]*, 2015.

[11] ME Taylor. Partial differential equations, volumes ii & iii. 1997.

[12] H. Urakawa. *Calculus of variations and harmonic maps*. 1993.

[13] G Tian W Ding. Energy identity for a class of approximate harmonic maps from surfaces. *Communications in Analysis & Geometry*, 1995.