University College London

MATHM901: Project in Mathematics

The de Rham Cohomology, Cobordism & Characteristic Classes

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I thank my supervisor, Dr. Jonny Evans, who has been a continuous source of inspiration.

Abstract

This project introduces the theories of de Rham cohomology, cobordism and characteristic classes. The motivation is Milnor's paper of 1956, in which he constructs manifolds that are homeomorphic but not diffeomorphic to S^7.

Contents

0 Introduction

The goal of this project has been to gain an understanding of some of the mathematics that had arisen in the field of differential topology by the end of the 1950s. Differential topology is the study of the properties of a smooth manifold that are preserved under a diffeomorphism. A diffeomorphism is an invertible map, φ, such that both φ and its inverse are continuous (a homeomorphism) and such that both φ and its inverse have well-defined derivatives everywhere. In a paper presented to The Annals of Mathematics in 1956, John W. Milnor showed that there exist smooth manifolds homeomorphic to the 7-sphere which possess, in Milnor's own words, *distinct differentiable structures*. That is, although homeomorphic, they are not diffeomorphic to the 7-sphere. It is this paper which has provided the motivation for the topics under investigation here.

In Section 1 we introduce the theory of fibre bundles and give examples of fibre bundles to illustrate how interesting topological spaces may be constructed. In particular we study S^3 bundles over S^4. We also distinguish the special case of vector bundles, which form the framework for modern differential geometry. The two invariants of algebraic topology developed by Henri Poincaré, homology and homotopy, distinguish spaces as far as homeomorphism. A diffeomorphism between two manifolds may be considered as a homeomorphism which extends, via the functor D, to a homeomorphism between their tangent bundles, and thus in studying differential topology we are led to extend these topological invariants to vector bundles, this is the theory of characteristic classes. Our homology theory of choice is the de Rham cohomology, which we introduce in Section 2 alongside its dual the compactly-supported de Rham cohomology. The de Rham cohomology is a topological invariant, defined for smooth manifolds, in the sense that it is invariant under homeomorphism. We study the duality between these two variants of de Rham cohomology and see how it reveals topological information regarding the intersections of submanifolds. This leads to the signature for a manifold of dimension $4n$. We diverge into the cohomology of a manifold-with-boundary and the theory of cobordism, which turns out to be crucial in distinguishing differentiable structures. Indeed, Milnor exhibited the exotic nature of his spheres in 1956 by first attempting to 'fill them in', and then studying the structure of the filling. In the theory of cobordism, and in developing a relative de Rham theory to prove the cobordism invariance of the signature, much of the research of this project was focused. Once this is developed we are able to present an example of a closed, oriented manifold which cannot be embedded such as to bound an open region of space.

In Section 3 we develop the theory of characteristic classes. These are elements of the de Rham cohomology ring which describe obstructions to the presence of global sections and may be interpreted as a measure of the twisting of a vector bundle. An overview is first given of the main expositions of the theory, before we introduce the Chern classes for a complex vector bundle from two different angles: Chern-Weil theory and the approach of Grothendieck. We then compute the Chern classes of $\mathbb{CP}^n$, before moving on to the Pontrjagin classes for a real vector bundle. We exploit our relative cohomology theory again to demonstrate that the Pontrjagin classes give cobordism invariants, before giving the Hirzebruch signature theorem which expresses the signature of a manifold of dimension $4n$ as a function of its Pontrjagin classes. We conclude with an example of a real 4-dimensional manifold which cannot be smoothly embedded in Euclidean space of dimension less that 8. Our final section concludes the project with an application of all of the theory developed. Following Milnor, the Hirzebruch signature theorem is used to define an invariant for a closed, oriented manifold M satisfying the hypothesis $H^4(M) = H^3(M) = 0$, which is a function of the Pontrjagin classes and signature of a manifold-with-boundary B with boundary M. This invariant is then used to demonstrate that certain of the S^3 bundles introduced in section 1 possess distinct differentiable structures.

It is assumed that the reader has undergraduate level understanding of differential geometry and is well familiar with the definition of a manifold. It is also assumed that the reader has had some prior experience with algebraic topology, specifically with some homology theory. A brief reference is made to category theory, although no prior knowledge is required here beyond the definitions of a category and of a functor.

1 The Language of Bundles

1.1 The Hopf Map

Consider the following construction of a continuous map from the three-sphere S^3 to the two-sphere S^2:

The 3-sphere, S^3, is modelled as the locus of points in a two-dimensional, complex vector space with unit norm, $S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\right\}$. The first complex projective space $\mathbb{CP}^1$ is the space of 1-dimensional complex subspaces of $\mathbb{C}^2$. We define the Hopf map to be the restriction to S^3 of the standard complex projective map:

$$\Pi \colon \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{CP}^1$$
$$\nu \longmapsto \nu\mathbb{C}$$

Note that $\Pi|_{S^3}$ is surjective. There is a homeomorphism, Ψ, from $\mathbb{CP}^1$ to S^2, see for example, Madsen and Tornehave [7] p139/140. Indeed, every complex line except that through one axis may be identified by a chosen complex coordinate, adding in the line through the axis we associate $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$, or the Riemann sphere, and so the Hopf map gives a continuous $\mathrm{H} := \Psi \circ \Pi|_{\mathrm{S}^3} : \mathrm{S}^3 \to \mathrm{S}^2$.

Claim 1.1. *Under the map* H*, the pre-image of points on* S^2 *are circles on* S^3*.*

Proof. The proof is a simple calculation. Take a point on S^2, this corresponds to a unique element of $\mathbb{CP}^1$, say $\nu\mathbb{C}$ where $\nu \in \mathbb{C}^2$ is a vector of unit norm. The pre-image of this element under Π is the set $\{a\nu \colon a \in \mathbb{C}\} \subset \mathbb{C}^2$. The intersection of this set with S^3 is $\{a\nu \colon a \in \mathbb{C}, |a| = 1\}$, a circle. $\square$

In his 1931 paper, Heinz Hopf made the further observation that the circles projecting to any two distinct points on S^2 are linked[1], and that that this map H presents a non-trivial homotopy class of maps $S^3 \to S^2$. For now we will just focus on one implication of the above claim: that S^3 may be realised as a disjoint union of circles, parametrised by the two sphere. We take this as motivation for defining fibre bundles, a tool for building complicated topological spaces out of familiar ones.

[1] A great visualisation of this by stereographic projection can be found at [1]

1.2 Fibre Bundles

Definition 1.2. A fibre bundle consists of three topological spaces B, E, F, called the base space, total space and fibre respectively, together with a continous map $\pi\colon E \to B$ satisfying the following

Condition of local triviality: For each $b \in B$ there is an open neighbourhood $U \subset B$ of b and a homemorphism h such that for each $x \in U$ the correspondence $y \mapsto h(x,y)$ defines a homeomorphism and such that the following diagram commutes:

$$\begin{tikzcd} U \times F \arrow[r, "h"] \arrow[dr, "(x,y)\mapsto x"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

Example 1.3. Any cartesian product of two topological spaces trivially forms a fibre bundle with $E = B \times F$, π the standard projection and h the identity map.
Any fibre bundle in which the total space is homeomorphic to the cartesian product of base space and fibre is said to be *trivial*.

Example 1.4. The Hopf map (or Hopf fibration) as examined earlier, is an example of a circle bundle over S^2. Note that it is non-trivial as S^3 is not homeomorphic to $S^2 \times S^1$. We will discuss a cohomology theory later with which it will be possible to see this.

Example 1.5. The Hopf map has a natural generalisation to any odd dimensional sphere. If S^{2n+1} is modelled as the set of vectors in $\mathbb{C}^{n+1}$ with unit norm, then the generalised Hopf map is $S^{2n+1} \to \mathbb{CP}^n; \nu \mapsto \nu\mathbb{C}$. Again this map squashes circles into points, and we may rephrase claim 1 as the observation, $\mathbb{CP}^n$ *is the orbit space of* S^{2n+1} *under the action of* S^1, where S^1 is the set of unit complex numbers and the action is multiplication.
We may then build fibre bundles over $\mathbb{CP}^n$ by taking some topological space R, endowed with S^1-action, and taking a projection

$$\begin{tikzcd} (S^{2n+1} \times R)_{/S^1} \arrow[d, "\pi"] \\ (S^{2n+1})_{/S^1} \cong \mathbb{CP}^n \end{tikzcd}$$

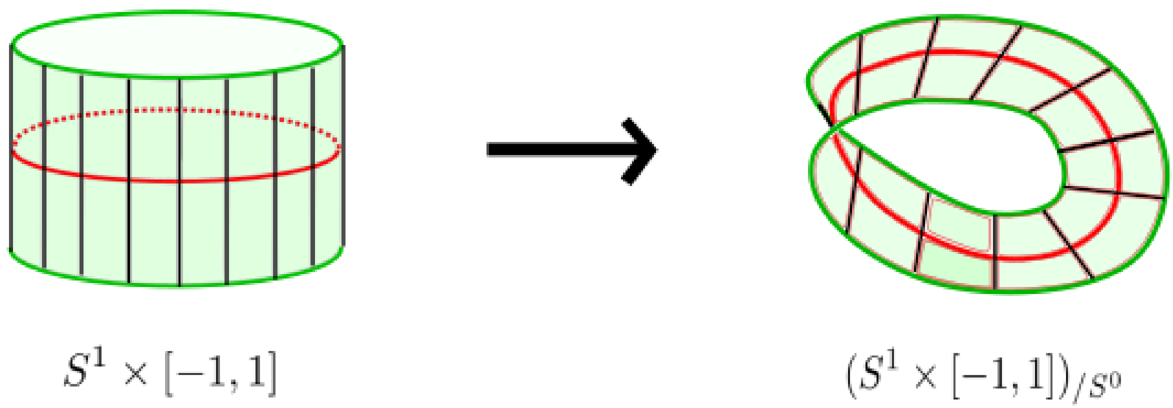
Here the notation $(S^{2n+1} \times R)_{/S^1}$ means the orbit space of the action of S^1 upon $S^{2n+1} \times R$, where the action is multiplication on S^{2n+1} alongside that with which R is endowed.

Example 1.6. We can also build fibre bundles over real projective space $\mathbb{RP}^n$, the space of 1-dimensional real subspaces of $\mathbb{R}^{n+1}$, in this way. Noting that *real* lines through the origin in Euclidean space cut the sphere at two distinct points ($\simeq S^0$), we may observe that $\mathbb{RP}^n$ is the orbit space of S^n under the action of S^0. Here $S^0 \simeq \{\pm 1\}$ has a group structure by multiplication, which makes it isomorphic to $\mathbb{Z}_2$. We thus have fibre bundles of the form

$$\begin{tikzcd} (S^n \times R)_{/S^0} \arrow[d, "\pi"] \\ (S^n)_{/S^0} \cong \mathbb{RP}^n \end{tikzcd}$$

Note $\mathbb{RP}^1$ is homeomorphic to the circle and that $(S^1 \times [-1,1])_{/S^0}$ gives the Möbius band as a fibre bundle over $\mathbb{RP}^1$. It is non-trivial. See figure 1.

Figure 1: S^0 action on the cylinder, giving the Möbius band



The condition of local triviality makes fibre bundles eligible for an algebraic structure describing the transformations between coordinates on the overlap of locally trivialising covers. This is known as the *structure group* of the bundle and is an effective way of classifying bundles.

Definition 1.7. Given a fibre bundle $E \xrightarrow{\pi} B$ and a group G which has a well defined action on the fibre F, we say that the bundle is equipped with structure group G if B is equipped with an open cover $\{U_\alpha\}$ such that there are fibre-preserving diffeomorphisms $\phi_\alpha \colon E\big|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times F$ and continuous *transition functions* $g_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$ which take values in the group G and are defined by $g_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta^{-1}\big|_{\{x\}\times F} \in G$

The Möbius band as a bundle over $\mathbb{RP}^1$ may be equipped with structure group $\mathbb{Z}_2 = \{\pm 1\}$.

Example 1.8. If M is a smooth, n-dimensional real manifold, then the tangent space at a point $p \in M$ is defined by $T_pM = \{\gamma'(0)\}$ where γ denotes any smooth map $\mathbb{R} \to M$ satisfying $\gamma(0) = p$. It is simple to check that this forms a vector space and that the collection of all tangent spaces defines a fibre bundle over M which may be endowed with structure group $GL(n, \mathbb{R})$. This is called the *tangent bundle*, denoted by τ_M with total space denoted by TM. The tangent bundle is an example of a *vector bundle*, a special case of fibre-bundles in which much of the rich structure of linear algebra is inherited. This is what the next section will be devoted to.

1.3 Vector Bundles

A vector bundle is a fibre bundle for which the fibre at each point is a vector space V, endowed with a subgroup of $GL(V)$, the invertible linear transformations of V, as structure group. If ξ is a vector bundle, we will denote the base space by $M = M(\xi)$, the total space by $E = E(\xi)$ and the fibres at each point $x \in M$ by E_x or ξ_x. If U is an open subset of M, denote by $\xi|_U$ the vector bundle obtained by restriction to fibres over U. We adopt the letter M as we will always deal with base space some smooth manifold. We will initially deal with real fibres on real manifolds, and later move on to complex fibres on complex manifolds. In both cases it is a consequence of the condition of local triviality that the total space E is itself a manifold.

Definition 1.9. A vector bundle is said to be *orientable* if it is possible to find a structure group which is a subgroup of $GL^{+}(V)$, the invertible linear transformations of V with positive determinant. A manifold is said to be *orientable* if its tangent bundle is orientable.

Definition 1.10. A map (or morphism) between two vector bundles ξ_1, ξ_2 consists of a pair of continuous maps (Φ, ϕ) such that:

(i) the following diagram commutes:

$$\begin{tikzcd} E(\xi_1) \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E(\xi_2) \arrow[d, "\pi_2"] \\ M(\xi_1) \arrow[r, "\phi"] & M(\xi_2) \end{tikzcd}$$

(ii) for each $x \in M(\xi_1)$ the map $\pi_1^{-1}(x) \to \pi_2^{-1}(\phi(x))$ as induced by Φ is linear.

If we fix some common base space M, then the class of all vector bundles over M along with bundle-morphisms forms a category. The following theorem shows that we may equip this category with constructions from linear algebra, hence attaching meaning to expressions such as

$$\xi \oplus \eta, \xi \otimes \eta, \Lambda^k(\xi), \mathrm{Hom}(\xi, \eta), \xi^*, \text{etc.}$$

Theorem 1.11. *Given a smooth binary functor on the category of vector spaces and linear maps, $F\colon$ **Vect** $\times$ **Vect** $\to$ **Vect**, there is a naturally induced, smooth binary functor on the category of vector bundles over M and bundle maps, $\tilde{F}\colon$ **Vbun** $\times$ **Vbun** $\to$ **Vbun**, which acts like F on individual fibres.*

Proof. This proof is adapted from [7]. Define our induced map $\tilde{F}$ to input two vector bundles, the objects in the category ***Vbun***, and output a new vector bundle with the total space attained by applying the orginal functor F along pairs of fibres.

$$E(\tilde{F}(\xi, \eta)) = \bigsqcup_{x \in M} F(\xi_x, \eta_x)$$

A projection map along each fibre then acts naturally.

Having defined the object images of this induced functor, it is only necessary to introduce a topology on them to make sense of the local triviality condition, and hence show that they are indeed new vector bundles. To do this, we take an open covering for our base space, $\{U_j\}_{j \in J}$, which admits local trivialisations:

$$h_j\colon \xi|_{U_j} \to U_j \times \mathbb{R}^m$$

$$k_j\colon \eta|_{U_j} \to U_j \times \mathbb{R}^n$$

These local trivialisations induce inclusions

$$H_j\colon U_j \times F(\mathbb{R}^m, \mathbb{R}^n) \hookrightarrow E(\tilde{F}(\xi, \eta))$$

And we topologise $E(\tilde{F}(\xi,\eta))$ via

$$A \subset E(\tilde{F}(\xi,\eta)) \text{ open} \Leftrightarrow H_j^{-1}(A) \text{ open for all j}$$

Here $H_j^{-1}(A) = \{x \colon H_j(x) \in A\}$ is the pre-image. Since bundle maps between bundles over the same base space are simply defined to act as linear maps fibrewise and since F is a functor, $\tilde{F}$ may be considered to input two bundle maps, the arrows in the category ***Vbun***, and output a new bundle map by applying F along the maps between pairs of fibres. Hence $\tilde{F}$ is a well defined functor. □

Although the above theorem was stated for only binary functors such as $\otimes$ and $\mathrm{Hom}(-,-)$, we see that the same argument holds for functors such as $\mathrm{Hom}(-,\mathbb{R})$ and Λ^k, and we may make judicious use of the machinery of linear algebra when dealing with vector bundles, so long as we remember that our constructions are manifestly local. One construction which is not immediately apparent is that of the pull-back, it is so important that we state it below.

Definition 1.12. Given a vector bundle $\xi = (E \xrightarrow{\pi} M)$ and a map $g \colon M' \longrightarrow M$, define the pull-back bundle $g^*(\xi) = (g^*(E) \xrightarrow{g^*\pi} M')$ to have total space $g^*(E)$ given by the subset of $M' \times E$ consisting of pairs (x', e) with $g(x') = \pi(e)$ and projection map given by $g^*\pi \colon (x', e) \mapsto x'$.
The spaces and maps just defined fit into a commutative diagram:

$$\begin{tikzcd} g^*(E) \arrow[r, "\tilde{g}"] \arrow[d, "g^*\pi"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "g"] & M \end{tikzcd}$$

Where $\tilde{g}(x', e) = e$. The vector space structure in $g^*\pi^{-1}(x')$ is defined by $(x', e_1) + \lambda(x', e_2) = (x', e_1 + \lambda e_2)$.

We also note that we may define an exact sequence of bundle maps to be a sequence of bundle maps which is exact fibrewise. We have the following immediate consequence of Theorem (1.11)

Corollary 1.13. *A vector bundle ϵ over M can be written as a direct summand $\xi \oplus \eta$ of bundles ξ and η over M if and only if there exists a short exact sequence of bundle maps*

$$0 \longrightarrow \xi \longrightarrow \epsilon \longrightarrow \eta \longrightarrow 0$$

Where 0 denotes the bundle over M with the vector space $\{0\}$ as fibre over each point.

We next introduce an extension of the notion of a basis from linear algebra.

Definition 1.14. A section of a vector bundle $\xi = (E \to M)$ where both M and E are smooth manifolds, is a smooth map

$$s \colon M \longrightarrow E(\xi)$$

which takes each $x \in M$ into the corresponding fibre $F_x(\xi)$

Remark 1.15. The *zero section* is defined as the map which takes every element to the zero element of the fibre. There is an additive operation on sections defined by $(s_1 + s_2)(x) = s_1(x) + s_2(x)$ and we may act by

$\lambda \in \mathbb{R}$ (or $\mathbb{C}$) on sections of real (or complex) vector bundles by $(\lambda s)(x) = \lambda(s(x))$. Hence the set of sections of ξ, denoted by $\Gamma(\xi)$, forms a real (or complex) vector space. If we denote the ring of smooth functions $\{f : M \to \mathbb{R} \text{ (or } \mathbb{C})\}$ by $\Omega^0(M)$, then $\Gamma(\xi)$ forms a module over this ring by $(fs)(x) = f(x)s(x)$.

Remark 1.16. It is a consequence of the condition of local triviality that, given a point $x \in M$, there is a neighbourhood U of x and a set of sections of ξ, $\{e_1, \ldots, e_n\}$, such that for any $y \in U$, $\{e_1(y), \ldots, e_n(y)\}$ is a basis for $F_y(\xi)$. Such a set is called a *frame* over U. It is immediate that a vector bundle is trivial if and only if it admits a global frame.

Remark 1.17. A section of the tangent bundle of a smooth manifold is called a *vector field*.

1.4 S^3 Bundles Over S^4

In this section we exploit the algebra of quaternions:

$$\mathbb{H} = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : (q_0, q_1, q_2, q_3) \in \mathbb{R}^4, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}$$

$\mathbb{H}$ has a vector space structure identical to $\mathbb{R}^4$, it has the added structure that any two quaternions may be multiplied by the defining relations to give a new quaternion and that a non-zero quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ has a multiplicative inverse given by

$$\frac{q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}}{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

The part q_0 is called the real part of the quaternion and we denote it $\Re(q)$. The quaternions do not commute and as such do not form a field, however we may define a right action of $\mathbb{H}$ on $\mathbb{H}^n$ by right multiplication $(v_1, \ldots, v_n) \cdot q := (v_1 q, \ldots, v_n q)$ and so given $v = (v_1, \ldots, v_n) \in \mathbb{H}^n$ define a *one-dimensional quaternionic subspace* of $\mathbb{H}^n$ by $v\mathbb{H} = \{(v_1, \ldots, v_n) \cdot q : q \in \mathbb{H}\}$

The first quaternionic projective space $\mathbb{HP}^1$ is the space of one-dimensional quaternionic subspaces of $\mathbb{H}^2$ and it may be viewed as a 4-dimensional real manifold analogously to the way $\mathbb{CP}^1$ may be viewed as a 2-dimensional real manifold. There is a homeomorphism of $\mathbb{HP}^1$ with S^4 which is analogous to that of $\mathbb{CP}^1$ with S^2. Indeed, every one dimensional quaternionic subspace $(v_1, v_2)\mathbb{H}$ with the exception of $(0, 1)\mathbb{H}$ may be identified uniquely by a point of the form $(1, v_2)$ and so we identify $\mathbb{HP}^1$ with $\mathbb{R}^4 \cup \{\infty\} \simeq S^4$. $\mathbb{HP}^1$ has coordinate charts

$$\phi_1 : \{(v_1, v_2)\mathbb{H} : v_1 \neq 0\} \longrightarrow \mathbb{H} \quad ; \quad (v_1, v_2)\mathbb{H} \mapsto v_2 v_1^{-1}$$
$$\phi_2 : \{(v_1, v_2)\mathbb{H} : v_2 \neq 0\} \longrightarrow \mathbb{H} \quad ; \quad (v_1, v_2)\mathbb{H} \mapsto v_1 v_2^{-1}$$

and transition function given by $\mathbb{H} \setminus \{0\} \longrightarrow \mathbb{H} \setminus \{0\}$; $u \mapsto {}^{u}/_{\|u\|^2}$. These coordinate charts correspond to stereographic projections of S^4.

We may model S^7 in quaternionic space via $S^7 = \{(v_1, v_2) \in \mathbb{H}^2 : \|v_1\|^2 + \|v_2\|^2 = 1\}$ and hence define a Hopf map H to be the restriction to S^7 of the quaternionic projective map $\mathbb{H}^2 \longrightarrow \mathbb{HP}^1 \simeq S^4; v \mapsto v\mathbb{H}$. The preimage of a point on S^4 is a copy of S^3 in S^7, and so the Hopf map recognises S^7 as an S^3 bundle over S^4.

$$\begin{tikzcd} S^3 \arrow[r] & S^7 \arrow[d, "\mathrm{H}"] \\ & S^4 \end{tikzcd}$$

We will next introduce a method for building fibre bundles, specifically S^3 bundles over S^4 and show that a number of these bundles have total space homeomorphic to S^7. These examples are those presented by Milnor in the paper "On manifolds homeomorphic to the 7-sphere" [6].

Consider S^4 as the union of two disks, the upper and lower hemispheres D^4_+ and D^4_-, with intersection along the equator, a copy of S^3. It may be shown that any fibre bundle over a contractible base space is trivial, see [7], [15], and so we may presume that any interesting topology may be constructed along the equator. Given two trivial S^3 bundles, one over each hemisphere, $D^4_+ \times S^3$ and $D^4_- \times S^3$ and a map $f\colon S^3 \longrightarrow \mathrm{Diff}(S^3)$ called the *clutching map*, we can glue the two bundles together as follows: by identifying the boundaries ($\simeq S^3$) of D^4_+ and D^4_- by the identity and identifying the pair of fibres at each point $x \in S^3$ in the glued boundary by the diffeomorphism $f(x)$. We then obtain a potentially non-trivial S^3 bundle over S^4

If we model S^3 as the set of unit quaternions then S^3 has a group structure given by quaternion multiplication and we may define a clutching map for each pair of integers $(h, j) \in \mathbb{Z} \times \mathbb{Z}$ by:

$$f_{(h,j)}\colon S^3 \longrightarrow \mathrm{Diff}(S^3)$$
$$f_{(h,j)}(u)(v) = u^h v u^j$$

Define $M_{(h,j)}$ to be the 7-dimensional manifold which is the total space of the bundle obtained from this clutching map. $M_{(h,j)}$ is indeed a manifold, as coordinate neighbourhoods for S^4 take the complement of the north pole and the complement of the south pole, these may be identified with $\mathbb{H} \simeq \mathbb{R}^4$ under stereographic projection and the transition function is given by $u \mapsto u/\|u\|^2$. A cover for $M_{(h,j)}$ is given by $\{U_1 = M_{(h,j)} \setminus (\{\text{north pole}\} \times S^3), U_2 = M_{(h,j)} \setminus (\{\text{south pole}\} \times S^3)\}$ and we have coordinates $\psi_1\colon U_1 \longrightarrow \mathbb{H} \times S^3$, $\psi_2\colon U_2 \longrightarrow \mathbb{H} \times S^3$ with change of coordinates given by:

$$(u, v) \mapsto (u', v') = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|^{h+j}}\right)$$

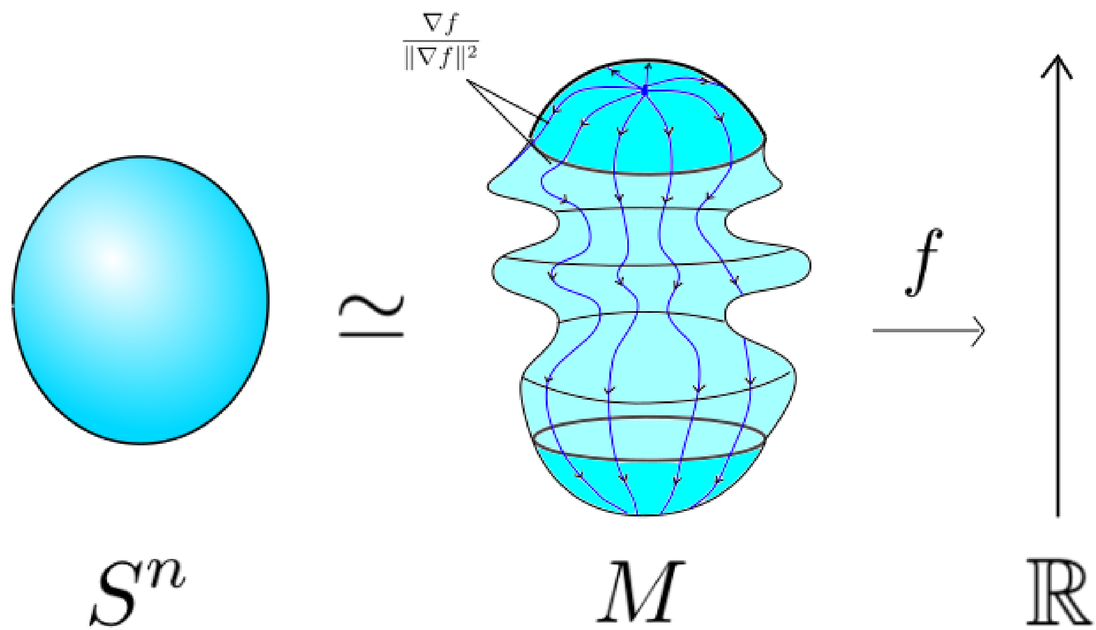
We claim that certain of these manifolds, specifically those for which $h + j = 1$, are homeomorphic to S^7. In order to show this we need to invoke a powerful theorem from Morse theory, which is delicately referred to by Milnor as "a partial characterisation of the n-sphere".

Theorem 1.18 (Reeb)**.** *Let M be an n-dimensional compact manifold. Suppose that there exists a smooth function $f\colon M \longrightarrow \mathbb{R}$ having only two critical points and that moreover these critical points are non-degenerate. Then M is homeomorphic to S^n.*

Proof. Unfortunately we do not have time to set up the necessary machinery and must refer the reader to [2] or [3] for full proof. Here is a brief overview:
Around each critical point one may choose a neighbourhood diffeomorphic to D^n. The two critical points must be a global maximum and a global minimum and outside of the critical points the vector field $\nabla f/\|\nabla f\|^2$ is well defined. The *flow lines* of this vector field connect the two neighbourhoods and may be used to construct a homeomorphism between M and a manifold obtained by gluing together two copies of D^n by a diffeomorphism along their boundaries. A homeomorphism between such a manifold and S^n may be constructed by the Alexander trick. We reserve discussion of this trick until the end of the section. □

We now turn our attention to the manifolds $M_{(h,j)}$ for which $h + j = 1$. Following Milnor we adopt the new

Figure 2: An illustration of Reeb's theorem



coordinate $u'' = u'(v')^{-1}$ and define a function

$$f \colon M_{(h,j)} \longrightarrow \mathbb{R}$$

$$\text{by} \quad f(x) = \frac{\Re(v)}{(1+\|u\|^2)^{1/2}} = \frac{\Re(u'')}{(1+\|u''\|^2)^{1/2}}$$

This function is well defined as the definitions agree on the coordinate overlap. Indeed,

$$\begin{aligned}
\frac{\Re(u'')}{(1+\|u''\|^2)^{1/2}} &= \frac{\Re\left(\frac{u}{\|u\|^2}\left(\frac{u^h v u^j}{\|u\|}\right)^{-1}\right)}{\left(1+\left\|\frac{u}{\|u\|^2}\left(\frac{u^h v u^j}{\|u\|}\right)^{-1}\right\|^2\right)^{1/2}} \\
&= \frac{\Re(u^{1-j}v^{-1}u^{-h})}{\|u\|\left(1+\frac{1}{\|u\|^2}\|u^{1-j}v^{-1}u^{-h}\|^2\right)^{1/2}} \\
&= \frac{\Re(u^{1-j}v^{-1}u^{h})}{(1+\|u\|^2)^{1/2}} && \text{since } v \text{ is a unit quaternion and since } j+h=1 \\
&= \frac{\Re(v)}{(1+\|u\|^2)^{1/2}} && \text{since } \Re(ab)=\Re(ba) \text{ and since } \Re(v^{-1})=\Re(v), \text{ because } v \text{ is a unit quaternion}
\end{aligned}$$

In order to determine the critical points of f, we use the method of Lagrange multipliers. Firstly we look for critical points of f on the coordinates $(u, v) \in \mathbb{H} \times \{v \in \mathbb{H} \colon \|v\| = 1\} \simeq \mathbb{R}^4 \times S^3$. Let $F_1 \colon \mathbb{R}^8 \to \mathbb{R}$ be defined by

$$F_1(x_1,\dots,x_8) = \frac{x_5}{(1+x_1^2+x_2^2+x_3^2+x_4^2)^{1/2}}$$

and let $g\colon \mathbb{R}^8 \to \mathbb{R}$ be defined by

$$g(x_1,\dots,x_8) = x_5^2+x_6^2+x_7^2+x_8^2-1$$

then we have $f = F_1\big|_{g^{-1}(0)}$ on this coordinate patch. We define a *Lagrangian* by $\mathcal{L}_1(x_1,\dots,x_8,\lambda) = F_1(x_1,\dots,x_8) - \lambda g(x_1,\dots,x_8)$. Then our equations to be solved to find a critical point of f on this patch are:

$$\text{for } i=1,2,3,4 \qquad \frac{\partial \mathcal{L}_1}{\partial x_i} = \frac{-x_5 x_i}{(1+x_1^2+x_2^2+x_3^2+x_4^2)^{3/2}} = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}_1}{\partial x_5} = \frac{1}{(1+x_1^2+x_2^2+x_3^2+x_4^2)^{1/2}} - 2\lambda x_5 = 0 \tag{2}$$

$$\text{for } i=6,7,8 \qquad \frac{\partial \mathcal{L}_1}{\partial x_i} = -2\lambda x_i = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}_1}{\partial \lambda} = -x_5^2-x_6^2-x_7^2-x_8^2+1 = 0 \tag{4}$$

Here we see that (2) implies $\lambda \neq 0$, and therefore (3) implies $x_i = 0$ for $i = 6,7,8$. Hence (4) implies $x_5 = \pm 1$ and so (1) implies $x_i = 0$ for $i = 1,2,3,4$. Next we look for critical points of f on the coordinates $(u'', v') \in \mathbb{H} \times \{v \in \mathbb{H} \colon \|v\| = 1\} \simeq \mathbb{R}^4 \times S^3$. Let $F_2\colon \mathbb{R}^8 \to \mathbb{R}$ be defined by

$$F_2(x_1,\dots,x_8) = \frac{x_1}{(1+x_1^2+x_2^2+x_3^2+x_4^2)^{1/2}}$$

so that on this coordinate patch $f = F_2\big|_{g^{-1}(0)}$. We define a Lagrangian by $\mathcal{L}_2(x_1,\dots,x_8,\lambda) = F_2(x_1,\dots,x_8) - \lambda g(x_1,\dots,x_8)$. Our first equation to be solved to find a critical point of f on this patch is

$$\frac{\partial \mathcal{L}_2}{\partial x_1} = \frac{1+x_2^2+x_3^2+x_4^2}{1+x_1^2+x_2^2+x_3^2+x_4^2} = 0$$

but this admits no solutions on $\mathbb{R}^8$. Thus there are no critical points for f on this coordinate patch and we deduce that f has precisely 2 critical points at $(u, v) = (0, \pm 1)$. In order to determine whether these points are non-degenerate, we introduce a coordinate chart over the first coordinates (u, v) by a stereographic projection of the unit quaternions S^3 given by

$$\phi_1^{-1}\colon \mathbb{R}^7 \to \mathbb{R}^4 \times S^3 \hookrightarrow \mathbb{R}^8$$

$$(y_1,\dots,y_7) \mapsto \Big(y_1,y_2,y_3,y_4,\frac{2y_5}{1+y_5^2+y_6^2+y_7^2},\frac{2y_6}{1+y_5^2+y_6^2+y_7^2},\frac{2y_7}{1+y_5^2+y_6^2+y_7^2},\frac{1-y_5^2-y_6^2-y_7^2}{1+y_5^2+y_6^2+y_7^2}\Big)$$

This stereographic projection covers the north pole $(0, 1)$ but misses the south pole $(0, -1)$. Another stereographic projection is given by

$$\phi_2^{-1}\colon \mathbb{R}^7 \to \mathbb{R}^4 \times S^3 \hookrightarrow \mathbb{R}^8$$

$$(y_1,\dots,y_7) \mapsto \Big(y_1,y_2,y_3,y_4,\frac{2y_5}{1+y_5^2+y_6^2+y_7^2},\frac{2y_6}{1+y_5^2+y_6^2+y_7^2},\frac{2y_7}{1+y_5^2+y_6^2+y_7^2},\frac{y_5^2+y_6^2+y_7^2-1}{1+y_5^2+y_6^2+y_7^2}\Big)$$

which covers the south pole but misses the north pole. We can use either of these to determine the Hessian at our critical points as the formula for f is the same

$$f\colon (y_1,\dots,y_7) \mapsto \frac{2y_5}{(1+y_1^2+y_2^2+y_3^2+y_4^2)^{1/2}(1+y_5^2+y_6^2+y_7^2)}$$

It is painful to calculate the Hessian of this by hand, instead we use Mathematica to calculate it and check that the critical points are non-degenerate. We find that they are indeed non-degenerate with a maximum at $(0,1)$ and a minimum at $(0,-1)$ as desired. See the code below.

```
ClearAll[f, potato, spinach, xx, hallelujah];
f[x1_, x2_, x3_, x4_, x5_, x6_, x7_] := ((2 x5) / ((1 + x5^2 + x6^2 + x7^2) ((1 + x1^2 + x2^2 + x3^2 + x4^2) ^ (1 / 2))));
xx = {x1, x2, x3, x4, x5, x6, x7};
potato = Table[D[D[f[x1, x2, x3, x4, x5, x6, x7], xx[[i]]], xx[[j]]], {i, 1, 7}, {j, 1, 7}];
spinach = Det[potato];
hallelujah[y1_, y2_, y3_, y4_, y5_, y6_, y7_] := spinach /. {x1 → y1, x2 → y2, x3 → y3, x4 → y4, x5 → y5, x6 → y6, x7 → y7};
hallelujah[0, 0, 0, 0, 1, 0, 0]
hallelujah[0, 0, 0, 0, -1, 0, 0]

Out[85]= -1

Out[86]= 1
```

Hence we see that the manifolds $M_{(h,k)}$ for which $h+j=1$ are indeed homeomorphic to S^7, quite surprisingly as they are all weirdly twisted S^3 bundles over S^4. As we shall show later, although all are homeomorphic, certain of these spheres can be shown to be not diffeomorphic to S^7. Let us examine briefly how it is that these "spheres" slipped through the net; why does Reeb's theorem only provide "a partial characterisation of the n-sphere"? The reason is that the aforementioned Alexander trick does not carry over to give a diffeomorphism. Suppose two copies of D^n are glued together along their boundary by a diffeomorphism $f\colon S^{n-1} \to S^{n-1}$. The simplest way to extend this map to a homeomorphism of the the entire ball is by extension along some radial coordinate for the disk, setting $F\colon D^n \to D^n$ by $F(rx) = rf(x)$ for $r \in [0,1]$. However, this method of extension does not define a diffeomorphism. As we approach the single point on the disk where $r=0$, the derivative of F is not well defined except for in the special case where f is the identity. Indeed we may approach the point from different directions with grossly different derivatives descending from the boundary[2].

[2]William Thurston summed up this Alexander extension as "combing all the tangles to a single point" [14]

2 The de Rham Cohomology

2.1 Introducing the de Rham Cohomology

We will introduce two topological invariants for a smooth, oriented manifold M and establish a duality between them. Although we start this section by defining differential forms in the language of bundles, a prior working knowledge of the theory of differential forms and of calculus on manifolds is essential, for which Spivak's little book [9] is perfect. Knowledge of at least one prior homology theory is strongly recommended, for which Chapter 2 of Hatcher [4] is a good first-time source. The reader with no prior knowledge of the de Rham cohomology may refer to Bott & Tu [15], as this section is a quick overview of the material there. Another book on de Rham cohomology is Madsen & Tornehave [7], which moves at a gentler pace.

Definition 2.1. A *differential 1-form* ω on a smooth manifold M is defined as a section of its cotangent bundle (the dual to its tangent bundle). That is, $\omega \in \Gamma(\tau_M^*)$. 1-forms are thus dual to vector fields. A *differential k-form* is defined as a section of $\Lambda^k \tau_M^*$. We denote the vector space of differential k-forms on a real manifold M by $\Omega^k(M;\mathbb{R}) = \Gamma(\Lambda^k \tau_M^*)$.

Definition 2.2. Given differential forms $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ we define the *wedge product* $\omega \wedge \eta \in \Omega^{k+l}(M)$ by

$$\omega \wedge \eta(X_1, \dots X_{k+l}) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \operatorname{sign}(\sigma)\, \omega \otimes \eta(X_{\sigma(1)}, \dots, X_{\sigma(k+l)})$$

Example 2.3. Working in local coordinates $x_1, \dots, x_n$ on a coordinate patch U of a manifold M, we have coordinate vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$. We define $dx_1, \dots, dx_n$ to be the 1-forms dual to these vector fields. We find that $\Gamma(\Lambda^k \tau_M^*\big|_U)$ is a free module over the ring of smooth functions with a basis given by $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \colon 1 \le i_1 < \cdots < i_k \le n\}$. Given a smooth function $f \colon M \to \mathbb{R}$ we define a 1-form df on this coordinate patch by $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Definition 2.4. Working in local coordinates, we may write a k-form ω as $\sum\limits_{i_1 < \cdots < i_k} \omega_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $\omega_{i_1 \dots i_k}$ are smooth functions and define its *exterior derivative* to be the $(k+1)$-form $d\omega = \sum_{i_1 < \cdots < i_k} d\omega_{i_1, \dots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. This local definition gives a global operator, the *exterior differential operator*

$$d \colon \Omega^k(M) \to \Omega^{k+1}(M)$$

The main properties of the exterior differential operator are as follows:

1. $d(\omega + \eta) = d\omega + d\eta$
2. If ω is a k-form and η is an l-form, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. $d(d\omega) = 0$
4. If $f \colon M \to N$ is a smooth map between manifolds, then $f^*(d\omega) = d(f^*\omega)$

For proof of these properties and definition of the pull-back f^* see [9].

A k-form ω is said to be *closed* if $d\omega = 0$ and *exact* if there exists a $(k-1)$-form τ such that $d\tau = \omega$. By the identity $d^2 = 0$ we have a *complex*:

$$\cdots \xrightarrow{d} \Omega^{k-1}(M;\mathbb{R}) \xrightarrow{d} \Omega^{k}(M;\mathbb{R}) \xrightarrow{d} \Omega^{k+1}(M;\mathbb{R}) \xrightarrow{d} \cdots$$

Which we denote $\Omega^\star(M)$.

Definition 2.5. The *de Rham cohomology* is the homology of this complex with kth de Rham cohomology group the real, quotient vector space:

$$H^k_{\mathrm{dR}}(M;\mathbb{R}) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

From now on we will suppress the suffix dR. For a closed differential form ω, we will write $[\omega]$ to denote the cohomology class represented by ω. The vector spaces H^k give topological invariants, in the sense that they are invariant under homeomorphism. As an example, consider the zeroth de Rham cohomology group. 0-forms are just functions defined on M and there is only the trivial exact form. So H^0 is the space of functions on M whose gradient vanishes. Calculus tells us that such a function must be constant on any connected component, and so the number of degrees of freedom of H^0 is the number of connected components of M.

Define the support of a differential form to be $\mathrm{supp}(\omega) = \overline{\{p \in M : \omega(p) \neq 0\}}$, where the bar denotes topological closure, and let $\Omega^k_c(M,\mathbb{R})$ denote the vector space of differential k-forms whose support is a compact set. It can be seen that the exterior derivative preserves compact support and we have another complex:

$$\cdots \xrightarrow{d} \Omega^{k-1}_c(M;\mathbb{R}) \xrightarrow{d} \Omega^{k}_c(M;\mathbb{R}) \xrightarrow{d} \Omega^{k+1}_c(M;\mathbb{R}) \xrightarrow{d} \cdots$$

Which we denote $\Omega^\star_c(M)$.

Definition 2.6. The *compactly-supported de Rham cohomology* is the homology of this complex, with kth homology group:

$$H^k_c(M;\mathbb{R}) = \frac{\{\text{compactly-supported closed } k\text{-forms}\}}{\{\text{compactly-supported exact } k\text{-forms}\}}$$

The de Rham cohomology satisfies the *Eilenberg-Steenrod axioms*, see [12]. In particular, we have the following property:

Theorem 2.7 (Homotopy axiom for de Rham cohomology)**.** *Homotopic maps induce the same map on de Rham cohomology.*

Proof. See Bott & Tu [15] □

To demonstrate the difference between the two invariants, consider the manifold $\mathbb{R}$. Again, the closed 0-forms on $\mathbb{R}$ are the constant functions, but the only one of these with compact support is zero. So $H^0(\mathbb{R}) \simeq \mathbb{R}$ and $H^0_c(\mathbb{R}) \simeq 0$. It can be seen easily that any 1-form on $\mathbb{R}$ is exact, so $H^0(M) = \{0\}$. Now consider the surjective integration map:

$$\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \longrightarrow \mathbb{R}$$

By Stokes' theorem, this map vanishes on those forms exact with compact support, and if $g(x)dx \in \ker \int_{\mathbb{R}}$, then $f(x) := \int_{-\infty}^{x} g(u)du$ will have compact support and $df = g(x)dx$. So $\ker \int_{\mathbb{R}}$ is precisely the space of compactly supported exact forms and hence $H^1_c(\mathbb{R}) = \frac{\Omega^1_c(\mathbb{R})}{\ker \int_{\mathbb{R}}} \simeq \mathbb{R}$

These results are a special case of the Poincaré Lemmas for de Rham cohomology:

Lemma 2.8 (Poincaré)**.**

$$H^k(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R} & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases}$$

$$H^k_c(\mathbb{R}^n) \simeq \begin{cases} 0 & \text{if} \quad k \neq n \\ \mathbb{R} & \text{if} \quad k = n \end{cases}$$

Proof. See Bott & Tu [15]. The argument is by induction on n and uses the homotopy axiom for de Rham cohomology applied to a retraction map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ in the de Rham case, and uses a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which integrates along one dimension, called *integration along the fibre* in the compactly supported de Rham case. The argument in the second case also uses a *homotopy operator*. □

This cohomology theory has its limitations; unless one is working with the family of smooth manifolds then the complexes are not even defined. However, as long as we keep within this family, the de Rham theory is exceptionally rich. This is because it inherits much structure from the theory of differential forms.

Given two closed forms $\omega \in \Omega^r$, $\tau \in \Omega^s$, it can be seen that $[\omega_1] \wedge [\omega_2] := [\omega_1 \wedge \omega_2]$ is well defined, giving $H^\star$ the structure of a graded ring. Moreover, if S is a closed submanifold (a submanifold $S \hookrightarrow M$ which is compact, and hence satisfies $\partial S = 0$) then integration of forms over S descends to cohomology. Indeed, if $[\omega_1] = [\omega_2]$, then $\int_S (\omega_1 - \omega_2) = \int_S d\tau = \int_{\partial S} \tau = 0$ by Stokes' theorem.

One point that the reader will notice: for the class of compact manifolds, our two complexes are identical. We distinguish the two by endowing them with opposing functorial natures. A smooth map $f \colon M \to N$ on manifolds induces a pull-back map on forms $f^* \colon \Omega^k(N) \to \Omega^k(M)$. However, the pull-back of a form with compact support may not itself have compact support. If $j \colon U \to W$ is the inclusion of an open set, then we may define $j_* \colon \Omega^k_c(U) \to \Omega^k_c(W)$ as the map which extends forms by zero.

Now we may follow the standard Mayer-Vietoris argument, see Hatcher [4] for the general argument, Bott & Tu [15] for explicit use in our situation. For two open sets, U and V in our manifold, the opposing functorial natures of the complexes $\Omega^\star$ and $\Omega^\star_c$ give rise to reverse short exact sequences of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\star(U \cup V) & \longrightarrow & \Omega^\star(U \oplus V) & \longrightarrow & \Omega^\star(U \cap V) & \longrightarrow & 0 \\ 0 & \longleftarrow & \Omega^\star_c(U \cup V) & \longleftarrow & \Omega^\star_c(U \oplus V) & \longleftarrow & \Omega^\star_c(U \cap V) & \longleftarrow & 0 \end{array}$$

Which induce long exact sequences in cohomology:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^k(U \cup V) & \longrightarrow & H^k(U \oplus V) & \longrightarrow & H^k(U \cap V) & \longrightarrow & H^{k+1}(U \cup V) & \longrightarrow & \cdots \\ \cdots & \longleftarrow & H^k_c(U \cup V) & \longleftarrow & H^k_c(U \oplus V) & \longleftarrow & H^k_c(U \cap V) & \longleftarrow & H^{k-1}_c(U \cup V) & \longleftarrow & \cdots \end{array}$$

Suppose that M is an n-dimensional manifold with an orientation, so that integration makes sense. Then we have a well defined bilinear pairing on cohomology groups, descended from the theory of differential forms, which exposes the duality between our two constructions.

$$\int_M : H^k(M) \times H_c^{n-k}(M) \longrightarrow \mathbb{R}$$

$$([\omega], [\tau]) \mapsto \int_M \omega \wedge \tau$$

Theorem 2.9 (Poincaré Duality)**.** *If M is an orientable manifold of finite type*[3] *then the above pairing is non-degenerate.*

Proof. We will sketch the proof as it is in [15], see there to check the definitions of the connecting morphisms in the long Mayer-Vietoris sequences d^* and d_*. We let U, V be contractible open sets in M and claim that our two long Mayer-Vietoris sequences can be paired to form a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cap V) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[r, "d^*"] & H^{q+1}(U \cup V) \arrow[r] & \cdots \\
& \times & \times & \times & \times & \\
\cdots & H_c^{n-q}(U \cup V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q}(U) \oplus H_c^{n-q}(V) \arrow[l] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U \cap V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q-1}(U \cup V) \arrow[l, "d_*"'] \arrow[d, "\int_{U \cup V}"] & \cdots \arrow[l] \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

where the first two squares are commutative and the last square is sign-commutative in the sense that

$$\int_{U \cap V} \omega \wedge d_* \tau = \pm \int_{U \cup V} d\omega \wedge \tau \tag{5}$$

Now, we have that a bilinear pairing on finite dimensional real vector spaces $\langle\ ,\ \rangle \colon V_1 \times V_2 \to \mathbb{R}$ induces a map $V_1 \to V_2^*$ by $v \mapsto \langle v, - \rangle$ and that the pairing is non degenerate if and only if this map defines an isomorphism $V_1 \simeq V_2^*$. Hence the above claim is equivalent to saying that the pairing induces a map from the upper exact sequence to the dual of the lower exact sequence such that the resulting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(U \cap V) \arrow[r] \arrow[d] & H^q(U) \oplus H^q(V) \arrow[r] \arrow[d] & H^q(U \cap V) \arrow[r] \arrow[d] & H^{q+1}(U \cup V) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_c^{n-q}(U \cup V)^* \arrow[r] & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* \arrow[r] & H_c^{n-q}(U \cap V)^* \arrow[r] & H_c^{n-q-1}(U \cup V)^* \arrow[r] & \cdots
\end{tikzcd}$$

is commutative on the first two squares and sign-commutative on the last square. Showing that the first two squares are commutative is straightforward, so we shall just demonstrate (5). Choosing a partition of unity ρ_U, ρ_V subordinate to U, V, we have by definition of the connecting morphism that $d^*\omega\big|_U = -d\rho_V \omega$, $d^*\omega\big|_V = d\rho_U \omega$ and that $d_*\tau$ is such that

$$\big(-\text{(extension by 0 of } d_*\tau \text{ to } U), \text{(extension by 0 of } d_*\tau \text{ to } V) \big) = \big(d(\rho_U \tau), d(\rho_V \tau) \big)$$

[3] A manifold of dimension n is said to be of finite type if there exists a finite cover of open sets of the manifold such that all non-empty intersections of the cover are diffeomorphic to $\mathbb{R}^n$. It follows that any compact manifold is of finite type.

Since τ, ω are closed, we have $d(\rho_V \tau) = (d\rho_V)\tau$ and $d(\rho_V \omega) = (d\rho_V)\omega$. Thus

$$\begin{aligned}\int_{U\cap V} \omega \wedge d_* \tau &= \int_{U\cap V} \omega \wedge (d\rho_V)\tau \\ &= (-1)^{\deg(\omega)} \int_{U\cap V} (d\rho_V)\omega \wedge \tau\end{aligned}$$

and, since $d^*\omega$ takes support in $U \cap V$,

$$\int_{U\cup V} d^*\omega \wedge \tau = -\int_{U\cap V} (d\rho_V)\omega \wedge \tau$$

which demonstrates (5). Now, we know that our pairing is non-degenerate for contractible open sets by the Poincaré lemmas, and so the argument follows from this by induction on the cardinality of a finite good cover for M by applying the five lemma. See [4] for statement and proof of the five lemma. □

For a manifold of finite type. It can be shown by induction on the cardinality of a good cover, alongside the Poincaré Lemmas and Mayer-Vietoris argument, that the cohomology groups are finite dimensional. Thus Poincaré duality gives an isomorphism $H^k(M)^* \simeq H_c^{n-k}(M)$. Now, suppose we have some closed, k-dimensional submanifold $S \hookrightarrow M$. Integration over S gives a linear functional on k-forms, and so under Poincaré duality this corresponds to a unique compactly supported cohomology class. We refer to both this class and any representative element of it as the Poincaré dual to S.

Corollary 2.10. *To any compact, k-dimensional submanifold $S \overset{i}{\hookrightarrow} M$ there exists a unique compactly supported cohomology class, represented by an $(n-k)$-form η_S satisfying:*

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

Example 2.11. Consider the unit circle S^1 embedded in $\mathbb{R}^2 \setminus \{0\}$. Working in polar coordinates, we would like to find $\eta_{S^1} = g_\theta d\theta + g_r dr$ satisfying

$$\int_{S^1} i^*\omega = \int_{\mathbb{R}^2\setminus\{0\}} \omega \wedge \eta_{S^1} \quad \text{for an arbitrary 1-form} \quad \omega = f_\theta d\theta + f_r dr$$

$$\text{i.e.} \int_0^{2\pi} f_\theta d\theta = \int_0^\infty \int_0^{2\pi} (f_\theta g_r - f_r g_\theta) d\theta dr$$

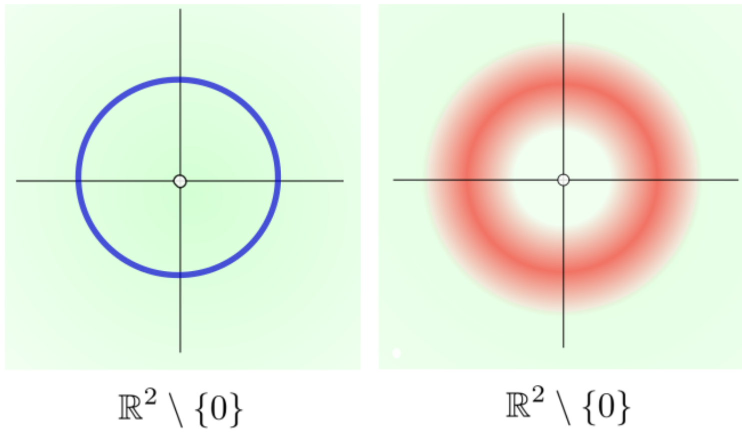
We see from the second line that setting $g_\theta = 0, g_r = \rho(r)$ where $\int_0^\infty \rho(r)dr = 1$ suffices. So Poincaré dual to the circle is a radial "bump form" of total integral one. See figure 3.

The argument of induction on the cardinality of a finite good cover is extremely powerful, as it allows up to write down global results about the topology of a manifold of finite type from purely local results about Euclidean space. As a further application of this argument, Bott & Tu present a proof of the well-known Künneth formula, as well as its partner the Leray-Hirsch theorem, see page 47, which we state below. These will both be important to our later arguments.

Theorem 2.12 (Künneth formula for products of manifolds)**.**

$$H^k(M \times N) \simeq \bigoplus_{p+q=k} (H^p(M) \otimes H^q(M))$$

Figure 3: In blue on the left we have the submanifold S^1, red shading on the right illustrates its dual bump form.



Theorem 2.13 (Leray-Hirsch theorem for twisted products)**.** *Let $E \xrightarrow{\pi} M$ be a fibre bundle with fibre F. If there exist global cohomology classes $e_1, \ldots, e_r$ on E which when restricted to each fibre freely generate the cohomology of the fibre, then*

$$H^k(E) \simeq \bigoplus_{p+q=k} (H^p(M) \otimes H^q(F))$$

2.2 The Relative de Rham Theory

It is a standard construction to introduce a relative homology theory, in which one may isolate the topology of a chosen submanifold. This is useful when dealing with a *manifold-with-boundary* [see [9] for definition and discussion of orientation and of differential forms on a manifold-with-boundary] and in particular in the theory of cobordism. The relative homology may be obtained in the singular and cellular theories by collapsing the chosen submanifold to a point, however such an operation will often not yield a manifold and differential forms are not even defined unless we are dealing with smooth manifolds, so care is needed in the de Rham case. Although a manifold-with-boundary M with boundary ∂M is not strictly a manifold, as it is not an open set, it may be treated as a manifold by taking its *interior set* ie. by shaving off the boundary to form the open set $M \setminus \partial M$. The complex of differential forms on $M \setminus \partial M$ is effectively the same as the complex of forms which may be defined on M as in [9], and we will *define* the de Rham cohomology of M to be just the de Rham cohomology of $M \setminus \partial M$. However those forms with compact support on $M \setminus \partial M$ must vanish in some neighbourhood of the boundary. Hence the compactly supported cohomology suffices to pick out the boundary and we have the following immediate consequence of Poincaré duality

Theorem 2.14 (Alexander-Lefschetz duality)**.** *Let M be an oriented manifold-with-boundary, then there is a non-degenerate pairing*

$$\int_M : H^k(M) \times H_c^{n-k}(M \setminus \partial M) \longrightarrow \mathbb{R}$$

This duality will be crucial to our later arguments on cobordism and in order to utilise it we need to place our relative homology in a long exact sequence. We first derive the long exact sequence of the relative theory as presented in [Bott & Tu [15]] and then we give a proof that this homology is in fact isomorphic to that which we require in order to exploit Alexander-Lefschetz duality.

Let $f\colon S \to M$ be a smooth map between manifolds. Define $\Omega^k(f) := \Omega^k(M) \oplus \Omega^k(S)$ with the map $\delta\colon \Omega^k(f) \to \Omega^{k+1}(f)$ by $\delta(\omega, \theta) = (d\omega, f^*\omega - d\theta)$. It is simple to check that $\delta^2 = 0$ and so this forms a complex $\Omega^\star(f)$. There is an obvious short exact sequence:

$$0 \longrightarrow \Omega^{k-1}(S) \xrightarrow[\theta \mapsto (0,\theta)]{\alpha} \Omega^k(f) \xrightarrow[(\omega,\theta)\mapsto\omega]{\beta} \Omega^k(M) \longrightarrow 0$$

It can be shown that β is a chain map, whereas α is not quite, it anticommutes with the chain $\alpha\delta = -\delta\alpha$. However this is still enough to induce a long exact sequence on homology. It turns out that the connecting morphism in our sequence is nothing but the pull back of f. Indeed, consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(S) \arrow[r, "\alpha"] & \Omega^{k+1}(f) \arrow[r, "\beta"] & \Omega^{k+1}(M) \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{k-1}(S) \arrow[r, "\alpha"] \arrow[u, "d"] & \Omega^k(f) \arrow[r, "\beta"] \arrow[u, "\delta"] & \Omega^k(M) \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

Let $\omega \in \Omega^k(M)$ and choose $(\omega, \theta) \in \Omega^k(f)$ which maps to ω. Then $\delta(\omega, \theta) = (0, f^*\omega - d\theta) = \alpha(f^*\omega - d\theta)$ and so the connecting morphism sends $[\omega]$ to $[f^*\omega - d\theta] = [f^*\omega]$.

Now, if $i : \partial M \hookrightarrow M$ is the inclusion of the boundary into M, we define the relative de Rham cohomology to be $H^k(M, \partial M) = H^k(i)$. We then have our long exact sequence:

$$\cdots \longrightarrow H^{k-1}(\partial M) \xrightarrow{\alpha} H^k(M, \partial M) \xrightarrow{\beta} H^k(M) \xrightarrow{i^*} H^k(\partial M) \longrightarrow \cdots$$

Claim 2.15. $H^k_c(M \setminus \partial M) \simeq H^k(M, \partial M)$

Proof. Our isomorphism is to be given by

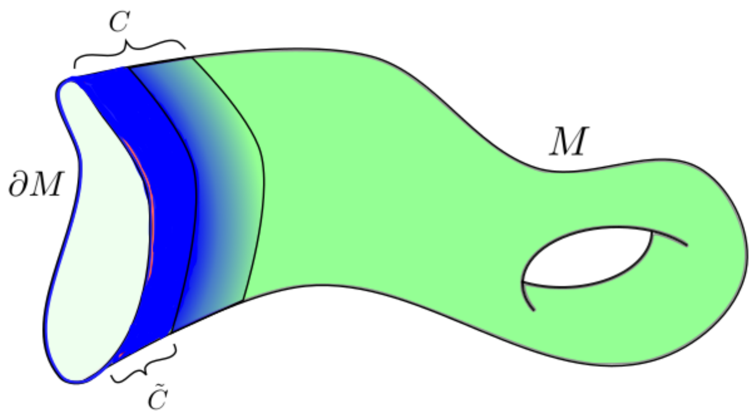
$$j\colon H^k_c(M \setminus \partial M) \to H^k(M, \partial M)$$
$$[\omega] \mapsto [(\omega, 0)]$$

j is clearly linear, and if $(\omega, 0)$ is δ-exact then ω is d-exact a fortiori, and so $\ker(j) = \{0\}$ and so j is injective. It remains to show that j is surjective. Let (ω, θ) be a δ-closed form in $\Omega^k(M, \partial M)$. We wish to find a d-closed form $\tilde{\omega}$ in $\Omega^k(M)$ which vanishes in some neighbourhood of the boundary and so that (ω, θ) and $(\tilde{\omega}, 0)$ differ by a δ-exact form.

Choose a collar neighbourhood for the boundary ∂M, that is a neighbourhood C of ∂M in M which is diffeomorphic to $C \times [0, 1]$. Such a neighbourhood may always be chosen [10]. Then choose a smaller collar neighbourhood $\tilde{C}$ for ∂M in C so that we may define a smooth bump function $\rho\colon M \to [0, 1]$ which takes the value 1 on $\tilde{C}$ and takes the value 0 outside of C.

Now, we have an inclusion $\partial M \overset{i_C}{\hookrightarrow} C$ of the boundary into the collar and a retraction $C \overset{r_C}{\to} \partial M$ of the collar onto the boundary so that $r_C \circ i_C$ is the identity on ∂M and $i_C \circ r_C$ is homotopic to the identity on C. Note that $1 - \rho$ gives a function which takes the value 0 on $\tilde{C}$ and takes the value 1 outside of C, and

Figure 4: The cut off function ρ is illustrated in blue



we may define a smooth map $r\colon M \to M$ as follows: for points in the collar $(q,x) \in \partial M \times [0,1] \simeq C$ by $r\colon (q,x) \mapsto (p, 1-\rho(q,x))$ and for points outside of the collar $p \in M \setminus C$ by the identity map $r\colon p \mapsto p$. This map retracts $\tilde{C}$ to the boundary and leaves points outside of C unchanged.

The map r is defined so that **(i)** r is homotopic to the identity on M, and **(ii)** r is homotopic to the identity through maps r_t which fix the boundary pointwise. **(i)** implies that r^* is *chain homotopic* to the identity, see Hatcher [4], i.e. there is a sequence of linear maps (a chain homotopy) $\{P_k\colon \Omega^k(M) \to \Omega^{k-1}(M)\}$ satisfying

$$dP_k + P_k d = r^* - \mathrm{Id}$$

Moreover, **(ii)** implies that a chain homotopy descends to the relative de Rham complex: there is a sequence of linear maps $\{Q_k\colon \Omega^k(M,\partial M) \to \Omega^{k-1}(M,\partial M)\}$ satisfying

$$\delta Q_k + Q_k \delta = r^* - \mathrm{Id}$$

Then since (ω,θ) is δ-closed we have that $\delta(Q_k(\omega,\theta)) = (r^*\omega,\theta) - (\omega,\theta)$ and so may deduce that (ω,θ) is δ-cohomologous to $(r^*\omega,\theta)$.

Next we define a form on M by

$$\eta = \begin{cases} \rho \cdot r_C^*\theta & \text{on} \quad C \\ 0 & \text{outside of} \quad C \end{cases}$$

and note that $i^*\eta = \theta$ and therefore $\delta(\eta,0) = (d\eta,\theta)$.

Restricting attention to $\tilde{C}$ we have

$$\begin{aligned} d\eta &= dr_C^*\theta \\ &= r_C^* d\theta && \text{since pullback commutes with the exterior derivative} \\ &= r_C^* \circ i^*\omega && \text{since } (\omega,\theta) \text{ is } \delta\text{-closed} \\ &= r^*\omega && \text{since } r \text{ retracts } \tilde{C} \text{ to the boundary} \end{aligned}$$

And so setting $\tilde{\omega} = r^*\omega - d\eta$ we have a closed form which vanishes in a neighbourhood of the boundary, namely $\tilde{C}$, and since $\delta(\eta,0) = (r^*\omega - \omega,\theta)$ we have that $(\tilde{\omega},0)$ is δ-cohomologous to $(r^*\omega,\theta)$, which in turn is δ-cohomologous to (ω,θ). This completes the proof. $\square$

Now we may rewrite Alexander-Lefschetz duality in its usual form:

Theorem 2.16 (Alexander-Lefschetz duality)**.** *Let M be an n-dimensional, oriented manifold-with-boundary, then there is a non-degenerate pairing*

$$\int_M : H^k(M) \times H^{n-k}(M, \partial M) \longrightarrow \mathbb{R}$$

2.3 The Thom Class

In order to understand further the relationship between submanifolds and forms, we must devote time to a construction known as the Thom class, with which the de Rham cohomology may be seen to encode topological information concerning the intersections of submanifolds. This section is a brief summary of Bott & Tu [15], where the detailed exposition is to be found. From now on, to avoid having to specify which homology we are using and for general convenience, all manifolds will be assumed to be oriented and compact.

Definition 2.17. Let $E \xrightarrow{\pi} M$ be a real k-vector bundle over a smooth manifold. We say that a differential form ω on E has compact support in the vertical direction if, for any compact set $K \subset M$, $\pi^{-1}(K) \cap \text{supp}(\omega)$ is compact. The forms with compact support in the vertical direction form a complex which we denote by $\Omega^\star_{cv}(E)$ and the homology of this complex is the *vertically-compactly-supported de Rham cohomology*, $H^\star_{cv}(E)$.

Let $t_1, \ldots, t_k$ be coordinates on the fibre $\mathbb{R}^k$, over some local, trivialising coordinate patch. A form on this patch may be expressed as a linear combination of two types of form: **(I)** those which do not contain as a factor the k-form $dt_1 \cdots dt_k$, and **(II)** those which do. We may define a linear map between complexes, "integration along the fibre", $\pi_* \colon H^\star_{cv}(E) \to H^{\star-k}(M)$ as follows: by sending forms of type **(I)** to zero, and integrating along the fibre those of type **(II)**. Although this definition is local, it may be made global by choosing some locally trivialising cover and using a smooth partition of unity.
The main properties of π_* are presented below:

Theorem 2.18. *(1) Integration along the fibre commutes with the exterior derivative: $\pi_* \circ d = d \circ \pi_*$*
(2) (Projection formula)
if τ is a form on M, and ω is a form on E with compact support in the vertical direction, then

$$\pi_*((\pi^*\tau) \cdot \omega) = \tau \cdot \pi_*(\omega)$$

Supposing, in addition, that M is oriented of dimension n, $\omega \in \Omega^q_{cv}(E), \tau \in \Omega^{n+k-q}_c(M)$. Then with the local product orientation on E, we have

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_*\omega$$

Proof. See [15] □

It follows over from the proof of the Poincaré Lemma for compactly supported cohomology that integration along the fibre defines an isomorphism $\pi_* \colon H^\star_{cv} \to H^{\star-n}(M)$, and yet again by induction on the cardinality of a good cover we can piece together this local result to obtain a global one: the Thom isomorphism theorem.

Theorem 2.19 (Thom)**.** *If $\pi\colon E \to M$ is an orientable vector bundle over a smooth, n-dimensional manifold of finite type, then integration along the fibre gives an isomorphism:*

$$H^\star_{cv}(E) \simeq H^{\star-n}(M)$$

Define the *Thom class*, Φ to be the unique vertically-compactly-supported cohomology class which maps to the unit element $[1] \in H^0(M)$. By the projection formula we have $\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_*\Phi = \omega$ and so the Thom isomorphism, which is inverse to integration along the fibre, is given by

$$\mathcal{T}\colon H^{\star-n}(M) \to H^\star_{cv}(E)\ ;\ \mathcal{T}(-) = \pi^*(-) \wedge \Phi$$

Lemma 2.20. *If E, F are oriented real vector bundles over M and if p_1, p_2 are the projections:*

$$\begin{tikzcd} & E \oplus F \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E & & F \end{tikzcd}$$

then $\Phi(E \oplus F) = p_1^*\Phi(E) \wedge p_2^*\Phi(F)$

Proof. This proof is from [15]. The Künneth isomorphism $H_c^{m+n}(\mathbb{R}^m \times \mathbb{R}^n) \simeq H_c^m(\mathbb{R}^m) \otimes H_c^n(\mathbb{R}^n)$ is given by the wedge product of generators, hence $p_1^*\Phi(E) \wedge p_2^*\Phi(F)$ is a cohomology class whose restriction to each fibre generates the compact cohomology of the fibre. □

If S is an oriented, closed submanifold of dimension k, embedded in an oriented manifold M of dimension n, the tubular neighbourhood theorem states that S has a tubular neighbourhood in M which is diffeomorphic to the normal bundle $\mathcal{N}_S$ [see Spivak [10]]. We therefore have an inclusion $\mathcal{N}_S \overset{j}{\hookrightarrow} M$. Applying the Thom isomorphism theorem, we obtain a sequence of maps:

$$H^\star(S) \xrightarrow[\wedge\Phi(S)]{\simeq} H^{\star+n-k}_{cv}(\mathcal{N}_S) \overset{j_*}{\hookrightarrow} H^{\star+n-k}(M)$$

Claim 2.21. *Let M be a smooth, oriented manifold. The Poincaré dual to an embedded, closed submanifold $S \overset{j}{\hookrightarrow} M$ is the Thom class of its normal bundle.*

$$\textit{i.e.} \quad \eta_S = j_*\Phi(S)$$

Proof. This proof is from [15]. Let ω be a closed form on M, and $S \overset{i}{\hookrightarrow} T \simeq \mathcal{N}_S$ the inclusion, regarded as the zero section of the bundle $T \overset{\pi}{\to} S$. Since π is a deformation retraction of T onto S, π^* and i^* are inverse isomorphisms in cohomology. Therefore $\omega = \pi^* i^* \omega + d\tau$, and so:

$$\begin{aligned}
\int_M \omega \wedge j_* \Phi &= \int_T \omega \wedge \Phi, && \text{since } j_*\Phi \text{ has support in T} \\
&= \int_T (\pi^* i^* \omega + d\tau) \wedge \Phi \\
&= \int_T (\pi^* i^* \omega) \wedge \Phi && \text{as } \int_T d\tau \wedge \Phi = \int_T d(\tau \wedge \Phi) = 0 \text{ by Stokes' theorem,} \\
& && \text{since } \Phi \text{ has compact support in the vertical direction} \\
&= \int_S i^* \omega \wedge \pi_* \Phi && \text{by the projection formula} \\
&= \int_S i^* \omega && \square
\end{aligned}$$

Moreover, suppose E is an oriented vector bundle over an oriented manifold M. M may be considered as diffeomorphically embedded as the zero section and we have an exact bundle sequence:

$$0 \longrightarrow \tau_M \longrightarrow (\tau_E)\big|_M \longrightarrow E \longrightarrow 0$$

or in other words, the normal bundle of M in E is E itself. Thus we see that the Thom class of an oriented vector bundle over an oriented manifold and the Poincaré dual to the zero section of E may be represented by the same form.

We also note that it follows from the tubular neighbourhood theorem that a representative of the Poincaré dual of a submanifold may be chosen to take support in an arbitrarily small neighbourhood. This is called the localisation principle.

Now, let $R \hookrightarrow M, S \hookrightarrow M$ be embedded submanifolds. We say R and S intersect transversally if $TR \oplus TS \simeq TM$ along the intersection. For such an intersection, the codimension in M is additive and so $\mathcal{N}_{R\cap S} \simeq \mathcal{N}_R \oplus \mathcal{N}_S$. Therefore by the above Lemma, $\eta_{R\cap S} \simeq \eta_R \wedge \eta_S$. We can now see how this construction has provided a powerful interpretation for the de Rham cohomology. We state the result as a theorem:

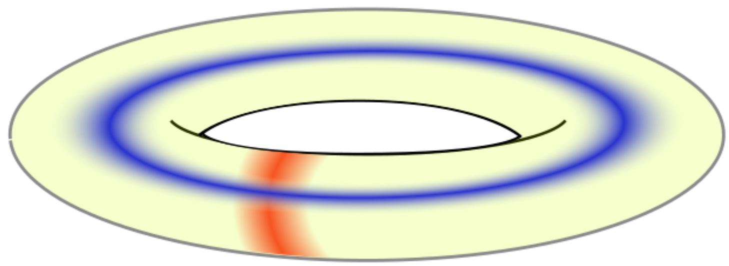
Theorem 2.22. *Under Poincaré duality, the transversal intersections of closed, oriented submanifolds corresponds to the wedge product of forms.*

Example 2.23 (The Torus)**.** The torus $\mathbb{T}^2 \simeq S^1 \times S^1$ has two generators for the 1st de Rham cohomology group. These generators are Poincaré dual to two non-contractible loops in the torus and may be represented in coordinates (θ, φ) by $\rho(\theta)d\theta$ and $\tau(\varphi)d\varphi$ where ρ and τ are bump functions of total integral one which are supported in the fibre of the normal bundle (a tubular neighbourhood) to their respective loops. These loops intersect transversally and the wedge product of the forms generates the 2nd de Rham cohomology group. See Figure 5.

When working with manifolds of dimension $4n$, the intersection of submanifolds is of critical interest, and we have a special case of study. By Poincaré duality, $H^{2n} \simeq H^{2n}_c$ and the wedge product is commutative on even dimensional forms, so we have a symmetric bilinear form called *the intersection form*:

$$\int_M : H^{2n}(M) \times H^{2n}(M) \to \mathbb{R}$$

Figure 5: The red and blue densities indicate the presence of differential forms which generate the 1st de Rham cohomology of $\mathbb{T}^2$



Non-degenerate, symmetric bilinear forms are often useful objects, and the intersection form for a $4n$-dimensional manifold reveals interesting results in topology, as we will see in the next section.

2.4 Signature and the Oriented Cobordism Ring

Definition 2.24. We define the signature $\mathfrak{sig}(M)$ of a $4n$-dimensional manifold M to be the signature of its intersection form. That is, the number of positive eigenvalues minus the number of negative eigenvalues when the form is diagonalised over the real numbers. We will also adopt the convention of taking the signature of a manifold whose dimension is not a multiple of 4 to be zero.

Definition 2.25. Two closed, oriented manifolds M and N are said to be *oriented cobordant* if there exists a manifold-with-boundary W oriented in such a way that $\partial W = M \sqcup -N$.
W is called a cobordism[4] between M and N.
A manifold which forms the boundary of a manifold-with-boundary is called *null-bordant.*

Claim 2.26. *Oriented cobordism is an equivalence relation on the family of closed, oriented manifolds and the set of equivalence classes forms a graded ring Ω^+, called the oriented cobordism ring, with grading by dimension, addition by disjoint union and multiplication by cartesian product.*

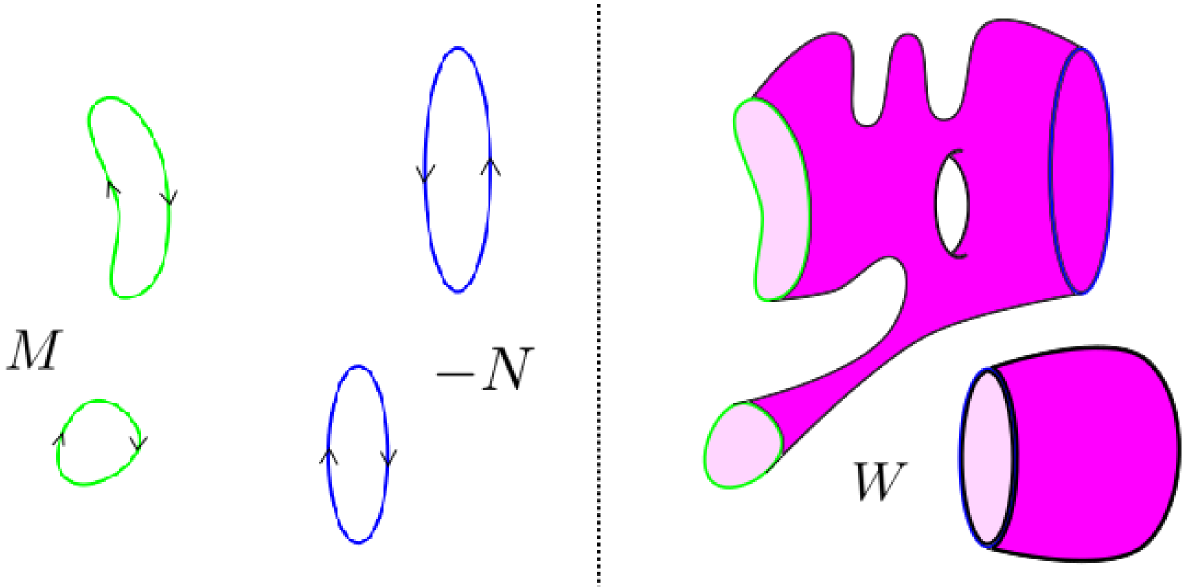
Proof. It is simple to verify that cobordism defines an equivalence relation. If M is cobordant to M' via W and M is cobordant to M' via V, then $M \sqcup -M'$ is cobordant to $N \sqcup -N'$ via $W \sqcup -V$ and so addition by disjoint union is well defined. We also have that $M \times N$ is cobordant to $M' \times N$ via $W \times N$ and that $M' \times N$ is cobordant to $M' \times N'$ via $M' \times V$ and so $M \times N$ is cobordant to $M' \times N'$ and thus multiplication by cartesian product is well defined. $\square$

Cobordism is such a crude equivalence equivalence relation that it may initially appear to be devoid of any useful information. Indeed, one can quickly convince oneself that all closed, oriented manifolds of dimension 1 are cobordant, as are those of dimension 2, and it is not at all obvious that there exist closed, oriented manifolds that cannot bound an open region of space. In this section we will prove that two cobordant

[4]The term cobordism comes from the French *bord*, meaning boundary

manifolds must have the same signature, and hence reveal such a class, alongside other useful theory.

Figure 6: A cobordism between two oriented 1-manifolds



Definition 2.27. Let Φ be a non-degenerate, symmetric bilinear form on a real vector space V. A subspace $L \leq V$ is said to be *Lagrangian* if it satisfies the two properties:

1. *Isotropy:* if $l_1, l_2 \in L$, then $\Phi(l_1, l_2) = 0$
2. *Maximality:* if $v \in V$ and $l \in L$ implies $\Phi(v, l) = 0$, then $v \in L$

Lemma 2.28. *If a non-degenerate, symmetric bilinear form Φ on a finite-dimensional real vector space V admits a non-trivial Lagrangian subspace, then its signature must be zero.*

Proof. The trick is to notice that the presence of a Lagrangian subspace splits the ambient space in half. Indeed, with respect to the form Φ, the isotropy condition ensures that a complement of L is included in L and the maximality condition ensures that this complement is in fact isomorphic to L.
Formally, we have a sequence which we claim to be exact:

$$0 \longrightarrow L \xrightarrow{\text{inclusion}} V \xrightarrow[v \mapsto \Phi(v,-)]{\varphi} L^* \longrightarrow 0$$

Where $*$ denotes the dual space of real linear functionals. This sequence is exact at its centre precisely by the Lagrangian condition, and so to show it is exact everywhere we must show that the map φ is surjective.

Since Φ is non-degenerate, there is an isomorphism $\psi \colon V \to V^*$, given by $v \mapsto \Phi(v, -)$. Choose a projection map $p \colon V \to L$, and define an embedding $e \colon L^* \hookrightarrow V^*$ by $f \mapsto f \circ p$. Then the following diagram commutes

$$\begin{tikzcd} V^* & \\ V \arrow[u, "\psi", "\simeq"'] \arrow[r, "\varphi"'] & L^* \arrow[ul, "e"'] \end{tikzcd}$$

and we have that $\varphi \circ (\psi^{-1} \circ e) = \mathrm{Id}_{L^*}$, which proves that φ is surjective.
Since short exact sequences of finite dimensional vector spaces always split, we conclude that $V \simeq L \oplus L^*$. And hence that

$$\dim(V) = 2\dim(L) = 2N.$$

Note that we have shown the dimension of V to be necessarily even.

Now, since Φ is non-degenerate, we may choose a basis of Φ so that the matrix with respect to that basis takes the form:

$$\begin{pmatrix} 1 & 0 & \ddots & & & \\ 0 & \ddots & 0 & & & \\ \ddots & 0 & 1 & & & \\ & & & -1 & 0 & \ddots \\ & & & 0 & \ddots & 0 \\ & & & \ddots & 0 & -1 \end{pmatrix}$$

Suppose this basis is $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ with $\Phi(\alpha_i, \alpha_j) = \delta_{ij}$, $\Phi(\beta_i, \beta_j) = -\delta_{ij}$, $\Phi(\alpha_i, \beta_j) = 0$ and assume without loss of generality that $m \leq n$. Let $A = \mathrm{span}\{\alpha_1, \dots, \alpha_n\}$.

If $v \in A \cap L$, then we have $v = \sum a_i \alpha_i$ and $\Phi(v, v) = \sum |a_i|^2 = 0$, and so $v = 0$
Therefore we deduce $A \cap L = \{0\}$.

Then by dimension counting: $\mathrm{codim}(A \cap L) \leq \mathrm{codim}(A) + \mathrm{codim}(L) \implies 2N \leq m + N \leq N + N$

And so $N = m = n$ and $\mathfrak{sig}(\Phi)$=0 $\square$

Theorem 2.29. *If ∂M is a $4n$-dimensional manifold bounding a $(4n+1)$-dimensional manifold-with-boundary M, then the intersection form of ∂M must admit a Lagrangian subspace, hence the signature of ∂M must be zero.*

Proof. Let $i \colon \partial M \hookrightarrow M$ be the inclusion of the boundary. We define our subspace L to be the image of $i^* \colon H^{2k}(M) \to H^{2k}(\partial M)$ and claim that L is Lagrangian with respect to the intersection form.

If μ, ν are closed $2k$-forms, then

$$\begin{aligned} \int_{\partial M} i^*\mu \wedge i^*\nu &= \int_{\partial M} i^*(\mu \wedge \nu) \\ &= \int_M d(\mu \wedge \nu) \qquad \text{by Stokes' theorem} \\ &= 0 \end{aligned}$$

and so isotropy is indeed satisfied. In order to show maximality, we need to remember our discussion on the relative de Rham theory. We had a long exact sequence

$$\cdots \longrightarrow H^{k-1}(M) \xrightarrow{i^*} H^{k-1}(\partial M) \xrightarrow[{[\theta] \mapsto [(0,\theta)]}]{\alpha} H^k(M, \partial M) \longrightarrow \cdots$$

And an isomorphism $j\colon H^k_c(M \setminus \partial M) \to H^k(M, \partial M)$. The inverse of this isomorphism was given by $j^{-1}\colon [(\omega, \theta)] \mapsto [r^*\omega - d\eta]$ where η was a form on obtained by a kind of extension of θ to a form on M with support in a collar neighbourhood C. By Stokes' theorem, we have that

$$\int_M d\eta = \int_{\partial M} i^*\eta = \int_{\partial M} \theta$$

And from this we obtain the following formula, for any $[\theta] \in H^k(\partial M)$:

$$\int_{\partial M} \theta = -\int_M j^{-1} \circ \alpha(\theta)$$

So, suppose that γ is a closed form on ∂M satisfying $\int_{\partial M} \gamma \wedge i^*\mu = 0$ for an arbitrary closed form μ on M. Applying the above we obtain

$$\begin{aligned}
\int_{\partial M} \gamma \wedge i^*\mu &= -\int_M j^{-1} \circ \alpha(\gamma \wedge i^*\mu) \\
&= -\int_C d(\rho \cdot r_C^*(\gamma \wedge i^*\mu)) && \text{by inputting the definition of } j^{-1} \circ \alpha \\
&= -\int_C d(\rho \cdot r_C^*\gamma \wedge r_C^* \circ i^*\mu) \\
&= -\int_C d(\rho \cdot r_C^*\gamma) \wedge r_C^* \circ i^*\mu && \text{since } d(r_C^* \circ i^*\mu) = r_C^* \circ i^* d\mu = 0 \\
&= -\int_C j^{-1} \circ \alpha(\gamma) \wedge r_C^* \circ i^*\mu \\
&= 0
\end{aligned}$$

But $j^{-1} \circ \alpha(\gamma)$ has support in C and r_C^* and $i^*\big|_C = i_C^*$ are maps inducing inverse isomorphisms on the cohomology of C, and so this implies, for an arbitrary closed form ω on M:

$$\begin{aligned}
\int_M j^{-1} \circ \alpha(\gamma) \wedge \omega &= \int_C \alpha(\gamma) \wedge (\omega\big|_C) \\
&= \int_C j^{-1} \circ \alpha(\gamma) \wedge (r_C^* \circ i_C^*) \circ ((r_C^* \circ i_C^*)^{-1} \omega\big|_C) \\
&= 0
\end{aligned}$$

So by Alexander-Lefschetz duality we may deduce that $[\alpha(\gamma)] = 0$, and so by the exactness of our sequence we may deduce that $\gamma \in \mathrm{Im}(i^*)$. This demonstrates that maximality is indeed satisfied, and hence concludes the proof. □

Corollary 2.30. *Signature gives a cobordism invariant. Moreover, it defines a ring homomorphism*

$$\mathfrak{sig}\colon \Omega^+ \to \mathbb{Z}$$

Proof. The above theorem demonstrates that signature gives a cobordism invariant, and so we have a well defined map $\mathfrak{sig}\colon \Omega^+ \to \mathbb{Z}$. Clearly $\mathfrak{sig}(M \sqcup N) = \mathfrak{sig}(M) + \mathfrak{sig}(N)$. The identity $\mathfrak{sig}(M \times N) = \mathfrak{sig}(M)\mathfrak{sig}(N)$ can be seen to follow immediately the Hirzebruch signature theorem, which we shall reach at the end of Section 3. □

Example 2.31 (A manifold which cannot bound an open region of space)**.** The second complex projection space $\mathbb{CP}^2$ has the underlying structure of a real manifold of dimension 4, it is closed and since is derived from a complex manifold it admits an orientation. The de Rham cohomology groups are given by:

$$H^k(\mathbb{CP}^2;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k=0 \\ 0 & \text{if } k=1 \\ \mathbb{R} & \text{if } k=2 \\ 0 & \text{if } k=3 \\ \mathbb{R} & \text{if } k=4 \end{cases}$$

We do not develop the tools for the computation of these homology groups in this project and refer to [7] or [15]. Intuitively, $\mathbb{CP}^2$ is obtained by taking a copy of $\mathbb{C}^2$ and attaching a copy of $\mathbb{CP}^1 \simeq S^2$ "at infinity", so these real cohomology groups are as expected. Since the middle cohomology is one dimensional and the intersection form is non-degenerate, it follows that the signature of $\mathbb{CP}^2$ equals $+1$ or -1. Either way, we deduce that $\mathbb{CP}^2$ represents a non-trivial element of the oriented cobordism ring, and so we have found a closed oriented manifold which cannot be embedded such as to bound an open region of space. In fact the same argument may be applied for $\mathbb{CP}^n$ when n is even.

We will now present a quick method for exposing more manifolds which cannot be embedded such as to bound an open region of space. We note that this method also relies on the relative long exact sequence and Alexander-Lefschetz duality which took us such pains to arrive at.

Definition 2.32. Given a manifold M [or a relative pair $(M, \partial M)$] of finite dimension n, define the *Euler characteristic* of M [or $(M, \partial M)$] by

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim(H^k(M))$$
$$[\text{or } \chi(M, \partial M) = \sum_{k=0}^{n} (-1)^k \dim(H^k(M, \partial M))]$$

Our relative long exact sequence is given by

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(M,\partial M) & \longrightarrow & H^0(M) & \longrightarrow & H^0(\partial M) & \\ \hookrightarrow & H^1(M,\partial M) & \longrightarrow & H^1(M) & \longrightarrow & H^1(\partial M) & \\ \hookrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \\ \hookrightarrow & H^n(M,\partial M) & \longrightarrow & H^n(M) & \longrightarrow & 0 & \end{array}$$

A property of long exact sequences is that the alternating sum of the dimensions of the terms equals zero, sometimes referred to as Whitehead's Lemma; it is not hard to prove this by induction on the length of the sequence and is a good exercise in exact sequences. Applying the property we have

$$\begin{array}{llllll} + & \dim(H^0(M,\partial M)) & - & \dim(H^0(M)) & + & \dim(H^0(\partial M)) \\ - & \dim(H^1(M,\partial M)) & + & \dim(H^1(M)) & - & \dim(H^1(\partial M)) \\ + & \cdots & - & \cdots & + & \cdots \\ + & (-1)^n \dim(H^n(M,\partial M)) & - & (-1)^n \dim(H^n(M)) & & & = 0 \end{array}$$

Which is equivalent to

$$\chi(M,\partial M) - \chi(M) + \chi(\partial M) = 0$$

But, by Alexander-Lefschetz duality $\dim(H^k(M,\partial M)) = \dim(H^{n-k}(M))$ and so $\chi(M,\partial M) = (-1)^n \chi(M)$ and we deduce that

$$\chi(\partial M) = 0 \qquad \text{(modulo 2)}$$

It follows that any manifold with odd Euler characteristic cannot be embedded such as to bound an open region of space. And so we immediately have our result for $\mathbb{CP}^2$, in fact we have it for $\mathbb{CP}^n$ when n is even. Furthermore this method reveals instantly that the second real projective space $\mathbb{RP}^2$, which has Euler characteristic 1, see [4], cannot be embedded such as to bound an open region of space. In fact neither can any $\mathbb{RP}^n$ when n is even. The surface $\mathbb{RP}^2$ can be constructed by taking a Möbius band and gluing a disk along its boundary. This result is less surprising as $\mathbb{RP}^2$ is non-orientable, there is no way to choose an 'inside' or 'outside', but still to prove it gives vindication for our relative de Rham theory.

3 Characteristic Classes

3.1 Overview

Today there are three main expositions of characteristic classes: Chern-Weil theory; the approach of Grothendieck; and the study of universal bundles. We will focus initially on the Chern-Weil approach as presented in Madsen & Tornehave [7] in which characteristic classes are derived by taking invariant polynomials in the curvature form, a construction well studied in differential geometry. This approach is preferred as it leads to simple proofs of the important naturality and Whitney-sum properties. We will then discuss briefly the approach of Grothendieck as presented in Bott & Tu [15] which deserves attention simply for its elegance. Here a class is constructed for a rank 2, oriented vector bundle, living in the cohomology of the base manifold, which is seen to pull back its information from the fibre above, counting the number of obstructions to the presence of a global section. Then, using the Leray-Hirsch theorem and some clever algebra as well as a result from cohomology, characteristic classes are derived from this for general complex vector bundles. This approach allows us to present a clear computation.

Neither of these first two expositions really address the important question of *why* such characteristic classes should have been presumed to exist, nor do they address uniqueness, which is ultimately what unites the approaches. This is best understood from the third exposition which is presented in Milnor & Stasheff [11]. The underlying idea is that one can define a *universal bundle*—so manifestly large and twisted a beast that all vector bundles maybe be embedded in it. This universal bundle lies over a base space, called the *classifying space*. The universal bundle is so contorted that any two embeddings of a given bundle are homotopic, and the generators of its cohomology ring may be pulled back from the classifying space along the embeddings to give a unique set of cohomology classes which may distinguish between bundles which are not isomorphic. We have avoided this approach as it does not involve the explicit geometrical concepts which are helpful in an introduction to the theory and as it requires extensive techniques to calculate the cohomology of the beast.

3.2 Chern-Weil Theory

A connection is a construction for connecting together neighbouring fibres. Analogously to the way the differential operator D is built to push tangent vectors along curves in Euclidean space, we define a connection as an operation on sections, which gives a notion of how vectors in the fibre may be pushed along paths in the base space. The theory in this section is mostly from [7].

Definition 3.1. Let ξ be a smooth vector bundle, with base a smooth manifold M. A *connection* on ξ is a linear map:

$$\nabla \colon \Gamma(\xi) \to \Omega^1(M) \otimes \Gamma(\xi)$$

which satisfies the Leibniz rule $\nabla(f \cdot s) = \mathrm{d}f \otimes s + f \cdot \nabla s$ for a smooth function f on M. The tensor product is taken over the ring of smooth functions on M, $\Omega^0(M)$.

Remark 3.2. Given a vector field $X \in \Gamma(\tau)$ there is an evaluation map

$$\mathrm{Ev}_X \colon \Omega^1(M) \otimes \Gamma(\xi) \to \Gamma(\xi);$$

$$\omega \otimes s \mapsto \omega(X) \cdot s$$

and hence we have a connection along X

$$\nabla_X := \mathrm{Ev}_X \circ \nabla \colon \Gamma(\xi) \to \Gamma(\xi)$$

Remark 3.3. The Leibnitz rule ensures that if a section s vanishes on some open set then so does $\nabla(s)$. We say that ∇ is a *local operator*. Let $e_1, \dots, e_k$ be a frame over an open subset $U \subset M$, then restricted to U we have

$$\nabla(e_i) = \sum_{j=1}^{k} \varphi_{ij} \otimes e_j$$

where $\varphi_{ij} \in \Omega^1(M)$. The matrix (φ_{ij}) is called the *connection form* of ∇ with respect to the frame.
This local exposition is enough to consider all connections. Given a connection, we may push vectors around globally with connection forms by first of all taking an open cover which admits local trivialisations, then choosing a connection form with respect to frames over each patch and defining appropriate transition functions on the coordinate overlaps to switch between connections forms.

Example 3.4. Suppose ϵ is a trivial bundle. Define the connection ∇^0 on ϵ by choosing a global frame $e_1, \dots, e_k$ and setting the connection form to be $\varphi_{ij} = 0$. Therefore,

$$\nabla^0(s) = \nabla^0(\sum s_i e_i) = \sum ds_i \otimes e_i$$

and, applying the evaluation map, we have a connection along a vector field given by the standard directional derivative $\nabla^0_X = d_X$

Example 3.5. Next, let ξ be a vector bundle over a base manifold M and suppose there exists a vector bundle η over M such that $\xi \oplus \eta$ is trivial. There are natural induced inclusion and projection maps $i \colon \Gamma(\xi) \to \Gamma(\xi \oplus \eta)$ and $p \colon \Gamma(\xi \oplus \eta) \to \Gamma(\xi)$ respectively, and we have a connection on ξ via the composition with ∇^0:

$$\nabla \colon \Gamma(\xi) \xrightarrow{\ i\ } \Gamma(\xi \oplus \eta) \xrightarrow{\nabla^0} \Omega^1(M) \otimes \Gamma(\xi \oplus \eta) \xrightarrow{\mathrm{Id} \otimes p} \Omega^1(M) \otimes \Gamma(\xi)$$

Any vector bundle over a compact base space has a complementary bundle as above, see [7] p153.

Lemma 3.6. *Given a smooth map between two base manifolds $g \colon M' \to M$, where M is equipped with a connection ∇, there exists a unique connection $g^*\nabla$ on the induced bundle $g^*\xi$ so that the following diagram commutes:*

$$\begin{tikzcd}
\Gamma(\xi) \arrow[r, "\nabla"] \arrow[d, "g^*"] & \Omega^1(M) \otimes \Gamma(\xi) \arrow[d, "g^*"] \\
\Gamma(g^*\xi) \arrow[r, "g^*\nabla"] & \Omega^1(M') \otimes \Gamma(g^*\xi)
\end{tikzcd}$$

Proof. For a detailed proof, see [7]. The construction is briefly as follows: taking some frame $\{e_1, \dots, e_n\}$ over a patch of M, along with the connection form for ∇ relative to this frame, we obtain the pulled back connection form by taking the standard pullback operation on forms and working with respect to the induced frames g^*e_i given at $p \in M'$ by $g^*e_i(p) = e_i(g(p))$. By the locality of connections, this treatment uniquely defines a global connection. $\square$

A choice of connection determines a notion of *parallel transport*, a method for pushing vectors in the fibre along paths in the base space. Conversely, given some notion of a parallel transport, one may obtain a corresponding connection. It is through parallel transport that connections may be seen to connect neighbouring fibres. See [7] for a discussion of parallel transport.

We now follow a construction congruent to the notions of curvature with which we are familiar. For a vector bundle ξ over a base manifold M, we start by introducing the suggestive notation:

$$\Omega^0(\xi) := \Gamma(\xi)$$
$$\Omega^k(\xi) := \Omega^k(M) \otimes \Gamma(\xi), \quad k \in \mathbb{N}$$

Where the tensor product is taken over the ring of smooth functions on M, $\Omega^0(M)$. We define an $\Omega^0(M)$-bilinear product $\wedge\colon \Omega^i(\nu)\otimes\Omega^j(\xi) \to \Omega^{i+j}(\nu\otimes\xi)$ induced from the exterior product by setting $(\omega\otimes s)\wedge(\tau\otimes t) = \omega \wedge \tau \otimes (s \otimes t)$. We may then write a connection on ξ as a linear map $\nabla\colon \Omega^0(\xi) \to \Omega^1(\xi)$ and analytically extend this map to make a *differential graded algebra* $\Omega^*(\xi)$ by defining

$$\tilde{\nabla}\colon \Omega^k(\xi) \to \Omega^{k+1}(\xi)$$
$$\tilde{\nabla}(\omega \otimes s) = \mathrm{d}\omega \wedge s + (-1)^k \omega \wedge \nabla s$$

Note that the sequence $\Omega^0(\xi) \to \Omega^1(\xi) \to \cdots \to \Omega^k(\xi) \to \cdots$ is by not necessarily a complex. The failure of this sequence to be a complex is measured by *the curvature*.

Definition 3.7. We set the *curvature map* to be $\mathcal{K} := \tilde{\nabla} \circ \nabla\colon \Omega^0(\xi) \to \Omega^2(\xi)$.

Lemma 3.8. *The curvature map $\mathcal{K}$ satisfies the following two properties:*

(i). $\mathcal{K}$ is $\Omega^0(M)$-linear.

(ii). $\mathcal{K}$ may be uniquely represented, locally, having chosen some frame, by an $n \times n$ matrix of 2-forms Φ defined on M. This matrix, called the curvature form, may be expressed in terms of the connection form via $\Phi_{ij} = d\varphi_{ij} - \sum_k \varphi_{ik} \wedge \varphi_{kj}$. i.e. $\Phi = d\varphi - \varphi \wedge \varphi$.

Proof. The proof is a simple calculation. For part *(i)*, let $s \in \Gamma(\xi)$, $f \in \Omega^0(M)$

$$\begin{aligned}\tilde{\nabla} \circ \nabla(f \cdot s) &= \tilde{\nabla}(df \otimes s + f \cdot \nabla s) \\ &= d(df) \otimes s - df \otimes \nabla s + df \otimes \nabla s + f \cdot \tilde{\nabla} \circ \nabla s \\ &= f(\tilde{\nabla} \circ \nabla s)\end{aligned}$$

For part *(ii)*, let $\{e_1, \ldots, e_n\}$ be a local frame over some patch of M

$$\begin{aligned}\tilde{\nabla} \circ \nabla(e_i) &= \tilde{\nabla}(\sum_j \varphi_{ij} \otimes e_j) \\ &= \sum_j d\varphi_{ij} \otimes e_j - \sum_j \varphi_{ij} \wedge (\sum_k \varphi_{jk} \otimes e_k) \\ &= \sum_k d\varphi_{ik} \otimes e_k - \sum_k (\sum_j \varphi_{ij} \wedge \varphi_{jk}) \otimes e_k \qquad \text{(here we had to fiddle the dummy indices)} \\ &= \sum_k (d\varphi_{ik} - \sum_j \varphi_{ij} \wedge \varphi_{jk}) \otimes e_k\end{aligned}$$

This conclude the properties of $\mathcal{K}$ $\square$

It is from this map, or rather from its corresponding local matrices, that we may extract a topological invariant. The theory of characteristic classes is usually first presented for complex vector bundles, and for the rest of this section we will work over $\mathbb{C}$. Note that differential forms may be considered over $\mathbb{C}$ by tensoring the complex $\Omega^\star$ with $\mathbb{C}$, and we obtain a corresponding de Rham cohomology with complex vector spaces as invariants. Moreover, if a manifold admits a complex structure, that is admits complex coordinates $z_k = x_k + iy_k$, then we have complex differential forms generated by $dz_k = dx_k + idy_k$.

Definition 3.9. Let $M_n(\mathbb{C})$ denote the algebra of complex, $n \times n$-matrices. An *invariant polynomial* is a function $P\colon M_n(\mathbb{C}) \to \mathbb{C}$ which may be expressed as a complex polynomial in the entries of the input matrix and which satisfies $P(XY) = P(YX)$ for arbitrary matrices X, Y.

Over a coordinate patch on our base manifold, with respect to some chosen frame, the curvature form Φ takes as entries differential 2-forms which we know to commute. Therefore, given some invariant polynomial, it makes sense to evaluate it on the curvature form and we obtain a single differential form defined locally on our coordinate patch, $P(\Phi)$. Given some other choice of frame, we know from linear algebra that our new curvature form may be written as $T\Phi T^{-1}$ for some non-singular matrix T, and so $P(T\Phi T^{-1}) = P(\Phi)$ as P is invariant. Therefore we have defined a form independent of the choice of frame, and so these forms defined locally over coordinate patches piece together to give a single, global form which we denote $P(\nabla)$. The following theorem may justifiably be titled the Fundamental Theorem for the Chern-Weil theory of characteristic classes.

Theorem 3.10. *Let ξ be a vector bundle over a smooth base manifold M, equipped with a connection ∇. Given an invariant polynomial P, the corresponding differential form defined on M, $P(\nabla)$, is a closed form.*

Proof. This proof is taken from [11]. Given an invariant polynomial P in indeterminates A_{ij}, we may form the matrix of formal derivatives $(\frac{\partial P}{\partial A_{ij}})$ and denote it by $P'(A)$.
Let $\Phi = (\Phi_{ij})$ be the curvature form with respect to some local frame. By the chain rule we have:

$$dP(\Phi) = \sum_{ij} \frac{\partial P}{\partial \Phi_{ij}} d\Phi_{ij}$$

Noting the formula for the trace, $\mathrm{trace}(AB) = \sum_{kl} A_{kl}B_{lk}$, we have that the above may be rewritten as

$$dP(\Phi) = \mathrm{trace}(P'(\Phi)^\intercal d\Phi)$$

where $\intercal$ denotes matrix transpose. We now invoke two results, one from differential geometry and the other from the theory of invariant polynomials.
Result 3.11 (Bianchi Identity)**.**

$$d\Phi = \varphi \wedge \Phi - \Phi \wedge \varphi$$

Proof. Computing the differential, the result follows immediately:

$$\begin{aligned} d\Phi &= d(d\varphi - \varphi \wedge \varphi) \\ &= -d\varphi \wedge \varphi + \varphi \wedge d\varphi \\ &= -(d\varphi - \varphi \wedge \varphi) \wedge \varphi + \varphi \wedge (d\varphi - \varphi \wedge \varphi) \\ &= -\Phi \wedge \varphi + \varphi \wedge \Phi \end{aligned}$$

□

Result 3.12. *For an arbitrary matrix A and an invariant polynomial P, we have that $P'(A)^\intercal$ commutes with A*

Proof. Given an invariant polynomial P, we have, for any real number t

$$P((\mathrm{Id} + tE_{ji})A) = P(A(\mathrm{Id} + tE_{ji}))$$

where E_{ji} denotes the matrix with a 1 in the (j,i)th place and zeros elsewhere. Differentiating with respect to t and setting $t = 0$ gives

$$\begin{aligned}
&& \frac{d}{dt}\Big|_{t=0} P((\mathrm{Id} + tE_{ji})A) &= \frac{d}{dt}\Big|_{t=0} P(A(\mathrm{Id} + tE_{ji})) \\
\iff && \sum_k A_{ik}\frac{\partial P}{\partial A_{jk}} &= \sum_k \frac{\partial P}{\partial A_{kj}} A_{kj} \\
\iff && A(P'(A))^\intercal &= (P'(A)^\intercal)A
\end{aligned}$$

□

Now, from the second result we have that $\Phi \wedge P'(\Phi)^\intercal = P'(\Phi)^\intercal \wedge \Phi$ and so applying the Bianchi identity we see

$$\begin{aligned}
dP(\Phi) &= \mathrm{trace}(P'(\Phi)^\intercal \wedge (\varphi \wedge \Phi - \Phi \wedge \varphi)) \\
&= \mathrm{trace}((P'(\Phi)^\intercal \wedge \varphi) \wedge \Phi - \Phi \wedge (P'(\Phi)^\intercal \wedge \varphi)) \\
&= 0
\end{aligned}$$

having used the properties of the trace, $\mathrm{trace}(A+B) = \mathrm{trace}(A)+\mathrm{trace}(B)$ and $\mathrm{trace}(AB) = \mathrm{trace}(A)\mathrm{trace}(B)$. This completes the proof of the Fundamental Theorem. □

Corollary 3.13. *The de Rham cohomology class of $P(\nabla)$ is independent of the choice of connection ∇, hence justifying the choice of notation $[P(\xi)]$ for this cohomology class.*

Proof. This proof is taken from [11]. Let ∇_0, ∇_1 be two connections on ξ. We first must note that, although the set of connections on a given vector bundle possesses no natural vector space structure, for a smooth function on the base manifold g we have that $g\nabla_0 + (1-g)\nabla_1$ gives a connection (to see this, just check the Leibnitz rule).

We first map $M \times \mathbb{R} \to M$ by the projection $(x,t) \mapsto x$ and consider bundle ξ' which is induced under pullback of this map, with pulled back connections ∇_0', ∇_1'. We then define a new connection for $t \in [0,1]$ by $\nabla_t = t\nabla_1' + (1-t)\nabla_0'$.

We next map back again by $i_\mu \colon M \to M \times \mathbb{R}$ sending $x \mapsto (x,\mu)$ where $\mu = 0$ or 1. The induced connection $i_\mu^*\nabla$ on $i_\mu^*\xi'$ can be identified with the connection ∇_μ on ξ, and so $i_\mu^*(P(\nabla)) = P(\nabla_\mu)$.

Since the map i_0 is homotopic to i_1, the cohomology classes of $P(\nabla_0)$ and $P(\nabla_1)$ are equal by a chain homotopy on cohomology. □

We are now in position to define characteristic classes, and to prove the important properties of Naturality and Whitney-sum. Later we will see an example of a non trivial characteristic class, demonstrating the worth of the theory.

Definition 3.14. For a square matrix A, let $\sigma_k(A)$ denote the kth *elementary symmetric function* of the eigenvalues of A, that is so that $\det(\mathrm{Id}+tA) = 1 + t\sigma_1(A) + \cdots + t\sigma_n(A)$. It is well known that the σ_k form invariant polynomials. See [Madsen & Tornehave] for example. [note. need a general reference for theory of invariant polynomials. Lang?]

Define the kth *Chern class* of a complex vector bundle ξ over a base manifold M by

$$c_k(\xi) = \frac{(-1)^k}{k!(2\pi i)^k}[\sigma_k(\xi)] \in H^{2k}(M)$$

By a Chern class of a smooth, complex manifold, we mean that of its tangent bundle $c_k(M) := c_k(\tau_M)$. Define the *total Chern class* of a complex n-vector bundle to be the sum of its Chern classes

$$c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots + c_n(\xi)$$

Theorem 3.15. *(Naturality) Let $g\colon M' \to M$ be a smooth map between smooth manifolds and ξ a vector bundle over M. For every invariant polynomial P we have $f^*[P(\xi)] = [P(f^*\xi)]$*

Proof. The result follows from the naturality of connections. Let ∇ be a connection on M. Endow $f^*\xi$ with the induced connection $f^*\nabla$. There is a comutative diagram analogous to that from Lemma 1.6 with ∇ replaced by $\tilde{\nabla}$ and combining these we have a commutative diagram:

$$\begin{tikzcd}
\Gamma(\xi) \arrow[r, "\nabla"] \arrow[d, "g^*"] & \Omega^1(M)\otimes\Gamma(\xi) \arrow[r, "\tilde{\nabla}"] \arrow[d, "g^*"] & \Omega^2(M)\otimes\Gamma(\xi) \arrow[d, "g^*"] \\
\Gamma(g^*\xi) \arrow[r, "g^*\nabla"] & \Omega^1(M')\otimes\Gamma(g^*\xi) \arrow[r, "g^*\tilde{\nabla}"] & \Omega^2(M')\otimes\Gamma(g^*\xi)
\end{tikzcd}$$

From commutativity of this diagram, $g^*(\tilde{\nabla}\circ\nabla) = (g^*\tilde{\nabla})\circ(g^*\nabla)$, we see that the characteristic classes are natural. ☐

Theorem 3.16. *(Whitney sum formula) For the direct product of two vector bundles we have $c_k(\xi\oplus\nu) = \sum_{r=0}^{k} c_r(\xi)c_{k-r}(\nu)$.*

Proof. Given connections on vector bundles ξ,ν, ∇_ξ,∇_ν, we take the obvious direct sum connection $\nabla_\xi\oplus\nabla_\nu$ on $\xi\oplus\nu$. The Whitney sum formula then follows immediately from the determinant formula:

$$\det(\mathrm{Id}+t(A_1\oplus A_2)) = \det(\mathrm{Id}+tA_1)\cdot\det(\mathrm{Id}+tA_2)$$ ☐

3.3 The Euler Class and the Approach of Grothendieck

Let ξ be a rank 2, real, oriented vector bundle over a smooth, compact manifold M and let $\{U_1,\dots,U_n\}$ be a cover of M so that $\xi\big|_{U_i}$ is trivial for each i. Choose a frame over each patch and let $\{\rho_1,\dots,\rho_n\}$ be a smooth partition of unity subordinate to the cover. Since ξ is orientable, its structure group ay be reduced to $SO(2)$, and by the 'accidental isomorphism' of $SO(2)$ with $U(1)$, whose elements look like $e^{i\phi}$, we may assign to each element of the structure group a real number, unique up to a multiple of 2π, which corresponds to the degree of rotation when transforming coordinates from one frame to another. Hence, on the overlap between two covering patches $U_\alpha \cap U_\beta$ we may define the real valued function $\phi_{\alpha\beta}$ which assigns to each point the degree of rotation taking the frame over U_α to that over U_β. Note that $\phi_{\alpha\beta} = -\phi_{\beta\alpha}$ and that on $U_\alpha \cap U_\beta \cap U_\gamma$ a rotation from U_α to U_β followed by a rotation from U_β to U_γ should equal a rotation from U_α to U_γ so we have

$$\phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\alpha\gamma} \in 2\pi\mathbb{Z} \tag{6}$$

The collection of these functions may be seen to contain hard information regarding the obstruction to the presence of a global section. The following two propositions are from [15].

Proposition 3.17. *There exist 1-forms $\{\xi_\alpha\}$ defined on M which satisfy $\frac{1}{2\pi} d\phi_{\alpha\beta} = \xi_\beta - \xi_\alpha$*

Proof. Defining $\xi_\alpha = \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\gamma\alpha}$, we have:

$$\begin{aligned}
\xi_\beta - \xi_\alpha &= \frac{1}{2\pi}\sum_\gamma \rho_\gamma (d\phi_{\gamma\beta} - d\phi_{\gamma\alpha}) \\
&= \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\alpha\beta} \qquad \text{by (6)} \\
&= \frac{1}{2\pi} d\phi_{\alpha\beta}
\end{aligned}$$

□

Proposition 3.18. *The 2-forms $\{d\xi_\alpha\}$ piece together to give a globally defined, closed form, whose cohomology class is independent of the choice of $\{\xi_\alpha\}$.*

Proof. Clearly $d\xi_\alpha, d\xi_\beta$ agree on the overlap and so the forms piece together. If $\frac{1}{2\pi} d\phi_{\alpha\beta} = \xi_\beta - \xi_\alpha = \tilde{\xi}_\beta - \tilde{\xi}_\alpha$ then $d\xi_\alpha d\tilde{\xi}_\alpha = d\xi_\beta d\tilde{\xi}_\beta = d(\xi_\alpha - \tilde{\xi}_\alpha)$ and so $d\xi_\alpha$ and $d\tilde{\xi}_\alpha$ differ by an exact global form. □

This construction for real, rank two vector bundles yields a cohomology class which we call the *Euler class*. If we are handed from the start bundle transition functions $g_{\alpha\beta}$ then it is given by

$$e(\xi) = \Big[\frac{1}{2\pi i} d(\sum_\gamma \rho_\gamma \log g_{\alpha\beta})\Big]$$

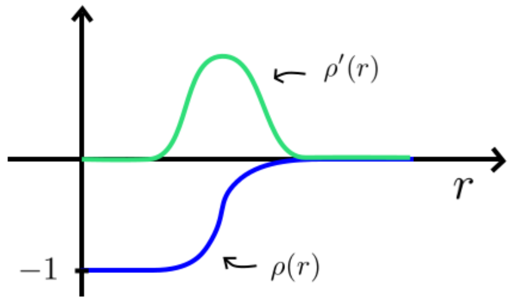
Firstly we can see that if our vector bundle is trivial, its Euler class automatically vanishes, we shall see in the later example that this is not always the case. Secondly note that the transition functions for a pulled back bundle are given by the pull backs of the transition functions of the original bundle, and so the Euler class automatically satisfies naturality. Thirdly note that if ξ_1, ξ_2 are bundles over the same base, then $e(\xi_1 \otimes \xi_2) = e(\xi_1) + e(\xi_2)$.

The fact that the Euler class vanishes for a trivial bundle is a reassurance that we are on the right track. In order to fully appreciate how it measures the obstruction to the presence of a global section, we need to study its relationship to the Thom class. If θ_α and θ_β are respective angular coordinates on U_α and U_β, then they satisfy by definition the relation $\theta_\beta = \theta_\alpha + \pi^*\phi_{\alpha\beta}$. Since we have $\frac{1}{2\pi} d\phi_{\alpha\beta} = \xi_\beta - \xi_\alpha$, putting these together we arrive at

$$\frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha = \frac{d\theta_\beta}{2\pi} - \pi^*\xi_\beta$$

And so these forms piece together to give a form which is defined globally on $E \setminus M$ (it is not defined on the zero section as the angular coordinates are not defined there) which we denote $\psi = \frac{d\theta_\alpha}{2\pi} - \pi^* \xi_\alpha$. We note that $d\psi = -\pi^* d\xi_\alpha = -\pi^* e$ and we claim that the Thom class is represented by $\Phi = d(\rho \cdot \psi) = d\rho \cdot \psi - \rho \cdot \pi^* e$ where $\rho = \rho(r)$ is a radial function on the fibre as illustrated in figure 7.

Figure 7: ρ in blue with its derivative in green



Note that although ψ is defined only outside the zero function, we choose ρ so that $d\rho$ vanishes near the zero section and so Φ is defined everywhere on E. We see that Φ satisfies the following properties:

- Φ has compact support in the vertical direction.
- Φ is closed.
- The restriction of Φ to any fibre has total integral $\int_0^\infty \int_0^{2\pi} d\rho \cdot \frac{d\theta}{2\pi} = 1$

and so Φ indeed represents the Thom class. Furthermore, if $s\colon M \to E$ is the zero section of E then $s^*\Phi = d(\rho)(0) \cdot s^*\psi - \rho(0) \cdot s^*\pi^* e = e$ and thus we have made the following observation: *The pullback of the Thom class to M by the zero section is the Euler class.* However, we note that any two sections of a vector bundle are always homotopic, and since homotopic maps induce identical maps on cohomology, the Euler class may be represented by the pullback of the Thom form to M under *any* section. On top of this we recall two facts from our discussion of the Thom class:

(i) the Thom class of E is the Poincaré dual to the zero section in E.

(ii) (localisation principle) a representative of the Poincaré dual may be chosen to take support in an arbitrarily small tubular neighbourhood. Putting all of this information together and we have enough to interpret the Euler class as decisively measuring the obstruction to the presence of a global section.

Definition 3.19. We call a section s of a complex vector bundle $E \to M$ *generic* if its intersection with the zero section is transversal. A section s is generic if and only if the set of zeros for s, Z, is a submanifold of complex codimension 1 in M.

Theorem 3.20. *Let E be an oriented vector bundle over an oriented manifold M and let s be a generic section of E with zero set Z. Then the Euler class of E is Poincaré dual to Z in M.*

Proof. Let s be the given generic section with zero set Z. Let ω be an arbitrary $(n-2)$-form defined on M, and let i be the inclusion of Z into M. Let Φ be the Thom class of E. By the above observation we may write the Euler class as $s^*\Phi$. We want to show

$$\int_Z i^*\omega = \int_M s^*\Phi \wedge \omega$$

Our proof will exploit the localisation principle. The idea is to shrink the support of Φ to an arbitrarily small neighbourhood and demonstrate that the above relation holds in the limit. Firstly, we may shrink the support of Φ enough so that the support of $s^*\Phi$ in M is contained in a tubular neighbourhood of Z, say $U \simeq Z \times \mathbb{R}^2$, then

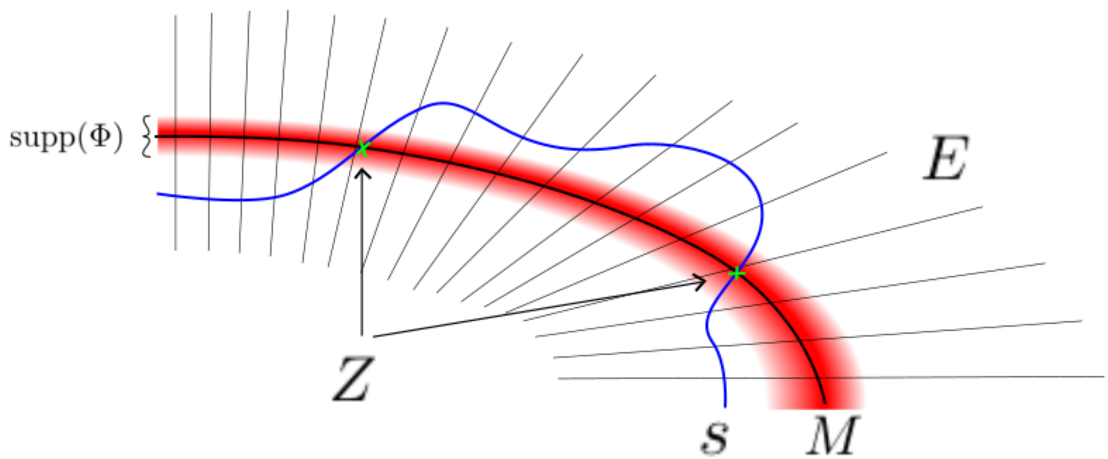
$$\int_M s^*\Phi \wedge \omega = \int_{Z\times\mathbb{R}^2} s^*\Phi \wedge \omega$$

We notice that shrinking the support of Φ corresponds to shrinking the support of $s^*\Phi$ and hence the size of the required tubular neighbourhood U. Moreover, along Z $s^*\Phi$ is constant and since ω is a smooth form we may choose the support of Φ to be small enough such that ω is approximately constant along $\{p\} \times \mathbb{R}^2$ for any $p \in Z$, this approximation becomes exact in the limit as the support of Φ is shrunk infinitesimally small. Thus we have

$$\int_{Z\times\mathbb{R}^2} s^*\Phi \wedge \omega \sim \int_Z i^*\omega \int_{\{p\}\times\mathbb{R}^2} s^*\Phi$$

With equality in the limit. Now as the intersection of s with the zero section is transversal, we may choose the support of Φ to be small enough such that s gives a diffeomorphism from $\{p\} \times \mathbb{R}^2$ to the fibre over p and so we have $\int_{\{p\}\times\mathbb{R}^2} s^*\Phi = \int_{\pi^{-1}(p)} \Phi = 1$. This is the desired result. □

Figure 8: A generic section s of a vector bundle $E \to M$ is drawn in blue. Red shading indicated the support of the Thom class which may be shrunk arbitrarily small



Having thoroughly discussed the Euler Class, we come to the method of Alexander Grothendieck in which the higher Chern classes are constructed from the Euler class by some very clever algebra. We call a complex vector bundle of rank 1 a *complex line bundle*. Any complex vector bundle ν of rank n has an underlying real vector bundle $\nu_{\mathbb{R}}$ of rank $2n$ obtained by discarding the complex structure. The isomorphism of $SO(2)$ with $U(1)$ reveals a 1-1 correspondence between complex line bundles and rank 2, oriented, real vector bundles. For the Grothendieck approach , the first Chern class of a complex line bundle is defined to be the Euler class of its underlying real bundle $c_1(\nu) := e(\nu_{\mathbb{R}})$.

If a complex line bundle admits a global section s, we see that $\{s, is\}$ gives a global frame for its underlying real bundle. This demonstrates that an orientable real vector bundle of rank 2 will admit a global section if and only if it admits a global frame. Now, if ν is a complex line bundle and ν^* is its dual, then we see $\nu \otimes \nu^* \simeq \mathrm{Hom}(\nu, \nu)$ admits a global section given by the identity map, and so is trivial. Thus $c_1(\nu) = -c_1(\nu^*)$.

Let V denote a complex vector space and let $\mathbb{P}(V)$ denote its *projectivisation*, that is the space of one dimensional subspaces of V. Over $\mathbb{P}(V)$ let $\epsilon_{\mathbb{P}(V)}$ denote the trivial bundle with total space $\mathbb{P}(V) \times V$. There is a well studied sub-line bundle of ϵ called the *universal sub-bundle* which consists of pairs $\{(l, v) \colon v \in l\} = \{(\textit{subspace of } V, \textit{vector in that subspace})\}$. We denote the universal sub-bundle by $\nu_{\mathbb{P}(V)}$. The universal sub-bundle has a complement, which we will denote by $\mu_{\mathbb{P}(V)}$, defined by the short exact bundle sequence:

$$0 \longrightarrow \nu_{\mathbb{P}(V)} \longrightarrow \epsilon_{\mathbb{P}(V)} \longrightarrow \mu_{\mathbb{P}(V)} \longrightarrow 0$$

We must unfortunately present the following theorem as a black box, as its derivation requires background material for which we do not have the time. The relevant theory may be found from sections 8 to 12 of [15].

Theorem 3.21. *The cohomology of a complex projective space $\mathbb{P}(V)$ is a graded ring with one generator given by $x = c_1(\nu^*_{\mathbb{P}(V)})$.*

Let $E \to M$ be a complex vector bundle of dimension n. The projectivisation of E is defined to be the vector bundle over M whose fibre at a point $p \in M$ is the projectivised $\mathbb{P}(E_p)$. This projectivisation forms a total space $\mathbb{P}(E)$ over which we can perform a construction as above. We have a trivial bundle $\epsilon_{\mathbb{P}(E)}$ with total space $\{(\textit{subspace of } Ep, Ep)\}$ and a universal sub-bundle $\nu_{\mathbb{P}(E)}$ with total space $\{(\textit{subspace of } Ep, \textit{vector in that subspace})\}$ alongside its complement $\mu_{\mathbb{P}(E)}$. See the following illustration:

$$\begin{tikzcd}
0 \arrow[r] & \nu_{\mathbb{P}(E)} \arrow[r] \arrow[dr, dotted] & \epsilon_{\mathbb{P}(E)} \arrow[r] \arrow[d, dotted] & \mu_{\mathbb{P}(E)} \arrow[r] \arrow[dl, dotted] & 0 \\
E \arrow[rr, "projectivisation"'] \arrow[dr] & & \mathbb{P}(E) \arrow[dl] & & \\
& M & & &
\end{tikzcd}$$

Now we define $x = c_1(\nu^*_{\mathbb{P}(E)})$, then x is a cohomology class in $H^2(\mathbb{P}(E))$. Since the restriction of the universal sub-bundle on $\mathbb{P}(E)$ to a fibre of the vector bundle $\mathbb{P}(E) \to M$ is precisely the universal sub-bundle of the projective space $\mathbb{P}(E_p)$, it follows by naturality that $c_1(\nu_{\mathbb{P}(E_p)})$ is the restriction of x to $\mathbb{P}(E_p)$. Hence, by the Leray-Hirsch theorem and the above black box we have that $H^\star(\mathbb{P}(E))$ is a free module over $H^\star(M)$ with basis $\{1, x, \dots, x^{n-1}\}$. So x^n may be written as a linear combination if the basis elements with coefficients in $H^\star(M)$. These coefficients are then defined as the Chern classes of E.

$$x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0$$

We will not discuss the Whitney-sum and Naturality properties of these coefficients, for a full exposition see [15].

3.4 Computations of Chern Classes

3.4.1 $\mathbb{CP}^1$

It is time for an explicit computation. A first test for a theory in topology is the two-sphere S^2, which we shall model as the first complex projective space $\mathbb{CP}^1$ recalling section (1.1) where a homeomorphism between these two was discussed. It is a very appropriate example as we have introduced the theory of characteristic classes in the hope of detecting an obstruction to the presence of global sections, and the tangent space of S^2 is well known to be non-trivial[5].

$\mathbb{CP}^1$ is the space of 1-dimensional subspaces of $\mathbb{C}^2$, that is $\{\mathbb{CP}^1 = \{[z_1, z_2] \colon (z_1, z_2) \in \mathbb{C}^2\}$ where $[z_1, z_2] = [\tilde{z}_1, \tilde{z}_2]$ if $(z_1, z_2) = (\lambda\tilde{z}_1, \lambda\tilde{z}_2)$ for some $\lambda \in \mathbb{C}$. We take the standard open cover of $\mathbb{CP}^1$:

$$U_1 = \{[z_1, z_2] \in \mathbb{CP}^1 \colon z_1 \neq 0\} \quad ; \quad U_2 = \{[z_1, z_2] \in \mathbb{CP}^1 \colon z_2 \neq 0\}$$

with the standard charts:

$$\begin{aligned} \sigma_1 \colon U_1 &\longrightarrow \mathbb{C} \\ [z_1, z_2] &\mapsto z_2/z_1 \end{aligned} \qquad ; \qquad \begin{aligned} \sigma_2 \colon U_2 &\longrightarrow \mathbb{C} \\ [z_1, z_2] &\mapsto z_1/z_2 \end{aligned}$$

These charts correspond to stereographic projections of S^2. The coordinate-transition function on the overlap $U_1 \cap U_2$ is given by

$$\begin{aligned} \sigma_2 \circ \sigma_1^{-1} \colon \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\mapsto {}^1\!/_z \end{aligned}$$

We are interested in the bundle-transition function for the tangent bundle on the overlap $U_1 \cap U_2$ and we determine this by passing into the differential category via the functor D. We bear in mind the following commutative diagram:

$$\begin{tikzcd}
& U_1 \cap U_2 \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] \arrow[drrr, dashed, "D"] & & & & \\
\mathbb{C} \setminus \{0\} \arrow[rr, "\text{coordinate transition}"] \arrow[drrr, dashed, "D"] & & \mathbb{C} \setminus \{0\} \arrow[drrr, dashed, "D"] & & T(U_1 \cap U_2) \arrow[dl, "(\sigma_1)_*"'] \arrow[dr, "(\sigma_2)_*"] & \\
& & & T(\mathbb{C} \setminus \{0\}) \arrow[rr, "\text{bundle transition}"] & & T(\mathbb{C} \setminus \{0\})
\end{tikzcd}$$

[5]This is the famous 'hairy ball theorem' which states that there cannot exist a nowhere zero vector field on S^2, or 'every cow has at least one cowlick', it is usually first proved as in [7] via the Poincaré-Hopf theorem, but our calculation will also provide a proof.

As we are working over $\mathbb{C} \setminus \{0\}$, ${}^{\partial}/_{\partial z}$ and ${}^{\partial}/_{\partial w}$ give basis vectors for the tangent bundle with respect to coordinates z and w on U_1 and U_2 respectively. The bundle-transition function may now be seen to be given clearly by the chain rule. Indeed, since the coordinate-transition is $z \mapsto w = {}^{1}/_{z}$, we have ${}^{\partial}/_{\partial z} = ({}^{\partial w}/_{\partial z})({}^{\partial}/_{\partial w}) = ({}^{-1}/_{z^2})({}^{\partial}/_{\partial w})$ and thus at a point on $\mathbb{CP}^1$ corresponding to $z = re^{i\theta} \in \mathbb{C} \setminus \{0\} = \sigma_1(U_1 \cap U_2)$ our transition function is multiplication by ${}^{-1}/_{z^2} = -r^{-2}e^{-2i\theta}$, which involves a rotation by angle -2θ. Therefore we have:

$$\begin{aligned} \phi_{12} \colon U_1 \cap U_2 &\longrightarrow \mathbb{R} \\ \sigma_1^{-1}(re^{i\theta}) &\mapsto -2\theta \end{aligned}$$

Next, we need a partition of unity subordinate to our cover. It suffices to find a pair of functions on $\mathbb{C} \setminus \{0\}$ which sum to 1 and such that the first vanishes at infinity and the second vanishes at the origin. We see that

$$\rho_1(z) = \frac{1}{1+|z|^2} \quad ; \quad \rho_2(z) = \frac{|z|^2}{1+|z|^2}$$

suffices.

Putting all of this together, working over U_2 we see that the first Chern class may be represented by:

$$\begin{aligned} c_1(\mathbb{CP}^1) &= \frac{1}{2\pi} d\Big(\sum_{\gamma} \rho_\gamma d\phi_{\gamma 2} \Big) \\ &= \frac{1}{2\pi} d\Big(\frac{1}{1+r^2} d(-2\theta) \Big) \\ &= \frac{2}{\pi} \frac{r}{(1+r^2)^2} dr \wedge d\theta \end{aligned}$$

Since integration descends to cohomology, in order to demonstrate that the first Chern class is non-trivial it suffices to integrate it over $\mathbb{CP}^1$, giving us the first *Chern number*, and check that this is non-zero. We do not even need to consider the form of our class over both coordinate patches as they have been chosen so that each patch covers everywhere except for a single point, so the integral of the Chern class over one patch is equal to the integral over the entire manifold. We have:

$$\begin{aligned} \int_{\mathbb{CP}^1} c_1(\mathbb{CP}^1) &= \int_{\mathbb{C}\setminus\{0\}} \frac{2}{\pi} \frac{r}{(1+r^2)^2} dr \wedge d\theta \\ &= \frac{2}{\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{r}{(1+r^2)^2} dr \\ &= \frac{2}{\pi} \cdot 2\pi \cdot \frac{1}{2} = 2 \end{aligned}$$

Where the last integral was calculated using Mathematica. This demonstrates that there is an obstruction to a global section of the tangent space of $\mathbb{CP}^1$, 'you cannot comb the hair on a coconut', and the theory of characteristic classes is vindicated.

We have opted for following the calculation through the method of examining transition functions as this gives a clear picture of what is going on. We should note however that the same result is obtained if the calculation is followed through the Chern-Weil theory—after all this is the theory with which we have proved the main properties of Chern classes. Taking a complex coordinate z on $\mathbb{CP}^1$ and choosing the connection from Example (3.5), the curvature 2-form may be calculated to be $\Phi = \frac{2dz\wedge d\bar{z}}{(1+|z|^2)^2}$ and we obtain the same Chern number from the Chern-Weil definition. For a more in depth discussion of the curvature form, and this specific example for S^2, see [16].

3.4.2 $\mathbb{CP}^n$

We now turn to the case of $\mathbb{CP}^n$. In general we would always rather avoid direct computations of the higher Chern classes as they can be painful, and we shall use this example of $\mathbb{CP}^n$ as an illustration of the power of the theory developed so far. In particular it is an illustration of the power of the Whitney sum formula. From here on we will omit the $\wedge$ when multiplying characteristic classes wherever there is no possibility of confusion. The following trick is from [15]. We consider the tautological exact sequence over $\mathbb{CP}^n$ as discussed on page 41.

$$0 \longrightarrow \nu_{\mathbb{CP}^n} \longrightarrow \epsilon_{\mathbb{CP}^n} \longrightarrow \mu_{\mathbb{CP}^n} \longrightarrow 0$$

A tangent vector to a point $[v] \in \mathbb{CP}^n$ may be regarded as an infinitesimal motion of the line $[v]$ into the complement of $[v]$ in $\mathbb{C}^{n+1}$ with some magnitude and direction, and so we may associate to each tangent vector a unique linear map from the line $[v]$ to its complement in $\mathbb{C}^{n+1}$. Thus we may associate $\tau_{\mathbb{CP}^n} \simeq \mathrm{Hom}(\nu_{\mathbb{CP}^n}, \mu_{\mathbb{CP}^n}) = \nu^*_{\mathbb{CP}^n} \otimes \mu_{\mathbb{CP}^n}$. Tensoring our exact sequence with $\nu^*_{\mathbb{CP}^n}$ we obtain:

$$0 \longrightarrow \nu^*_{\mathbb{CP}^n} \otimes \nu_{\mathbb{CP}^n} \longrightarrow \nu^*_{\mathbb{CP}^n} \otimes \epsilon_{\mathbb{CP}^n} \longrightarrow \nu^*_{\mathbb{CP}^n} \otimes \mu_{\mathbb{CP}^n} \longrightarrow 0$$

Or equivalently $(\nu^*_{\mathbb{CP}^n} \otimes \nu_{\mathbb{CP}^n}) \oplus (\nu^*_{\mathbb{CP}^n} \otimes \mu_{\mathbb{CP}^n}) \simeq \nu^*_{\mathbb{CP}^n} \otimes \epsilon_{\mathbb{CP}^n}$. But $\nu^*_{\mathbb{CP}^n} \otimes \nu_{\mathbb{CP}^n}$ is trivial as it has a global section given by the identity map, and $\nu^*_{\mathbb{CP}^n} \otimes \epsilon_{\mathbb{CP}^n} \simeq \oplus_{i=1}^{n+1} \nu^*_{\mathbb{CP}^n}$ since $\epsilon_{\mathbb{CP}^n}$ is trivial. Thus applying the Whitney sum formula gives

$$\begin{aligned} & c(\nu^*_{\mathbb{CP}^n} \otimes \mu_{\mathbb{CP}^n}) = c(\oplus_{i=1}^{n+1} \nu^*_{\mathbb{CP}^n}) \\ \implies \qquad & c(\tau_{\mathbb{CP}^n}) = c(\nu^*_{\mathbb{CP}^n})^{n+1} \\ & \qquad\qquad = (1+\eta)^{n+1} \end{aligned}$$

Where $\eta = c_1(\nu_{\mathbb{CP}^n})$. The Chern classes of $\mathbb{CP}^n$ are thus given by $c_k(\mathbb{CP}^n) = \binom{n+1}{k}\eta^k$

So what is this generator for the Chern classes of $\mathbb{CP}^n$? Well, if we take some $\mathbb{C}$-linear functional $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$, we have that for each line $[v] \in \mathbb{CP}^n$, $f\big|_{[v]}\colon [v] \to \mathbb{C}$ defines a linear functional on the fibre of $\nu_{\mathbb{CP}^n}$ over $[v]$. Thus f defines a section of $\nu^*_{\mathbb{CP}^n}$. Now, since f is linear its kernel must have complex dimension n, and thus the space of lines on which f is zero is a copy of $\mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$. Thus f gives a generic section and applying Theorem 3.20 we deduce that η must be the Poincaré dual to a copy of $\mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$. $\eta = \eta_{\mathbb{CP}^{n-1}}$.

As in the case of $\mathbb{CP}^1$, it is often the enough to cash in our characteristic classes via the integration homomorphism $\int_M\colon H^n(M) \to \mathbb{R}$.

Definition 3.22. Let M be a compact complex manifold of complex dimension n. Let $(i_1, \ldots, i_k)$ be a collection of integers whose sum is n, that is, a *partition* of n. We define the $(i_1, \ldots, i_k)$th *Chern number* of M to be

$$c_{(i_1,\ldots,i_l)}(M) = \int_M c_{i_1}(M) \cdots c_{i_l}(M)$$

We have already calculated our first Chern number, that is $c_{(1)}(\mathbb{CP}^n) = 2$. By our above formula we note that $c_1(\mathbb{CP}^1) = 2\eta$ and so $\int_{\mathbb{CP}^1} \eta = 1$. But, η is Poincaré dual to $\mathbb{CP}^{n-1}$ inside $\mathbb{CP}^n$ so for any complex n-form ω we have $\int_{\mathbb{CP}^{n-1}} i^*\omega = \int_{\mathbb{CP}^n} \omega \wedge \eta$. Setting $\omega = \eta^{n-1}$ gives $1 = \int_{\mathbb{CP}^{n-1}} \eta^{n-1} = \int_{\mathbb{CP}^n} \eta^n$ by induction and thus we may deduce all of the Chern numbers of $\mathbb{CP}^n$

$$c_{(i_1,\ldots,i_k)}(\mathbb{CP}^n) = \binom{n+1}{i_1} \cdots \binom{n+1}{i_k}$$

3.5 Pontrjagin Numbers and the Hirzebruch Signature Theorem

Chern classes are invariants defined for complex vector bundles. We may arrive at an invariant for a real vector bundle by a simple recipe: first 'complexify' the bundle, then take its Chern class. The result of this is called the Pontrjagin class. The best reference on Pontrjagin classes is Milnor & Stasheff [11].

Let V be a real vector space, the tensor product $V \otimes \mathbb{C}$ is defined to be its *complexification*. Given a real vector bundle ξ, this operation may be applied fibrewise to give a new vector bundle $\xi \otimes \mathbb{C}$, the complexification.

Lemma 3.23. *If ν is a complex vector bundle and $\bar{\nu}$ is its conjugate, then $c_k(\nu) = (-1)^k c_k(\bar{\nu})$.*

Proof. See Milnor & Stasheff [11] □

We have that $F \otimes \mathbb{C} = F \oplus iF$ for each fibre, and so the underlying real vector bundle for $\xi \otimes \mathbb{C}$ is canonically isomorphic to $\xi \oplus \xi$. The map $\xi \otimes \mathbb{C} \to \overline{\xi \otimes \mathbb{C}}$ which acts like $x + iy \mapsto x - iy$ on each fibre maps the total space homomorphically onto itself and is $\mathbb{R}$-linear on each fibre and so we have

$$c(\xi \otimes \mathbb{C}) = 1 + c_1(\xi \otimes \mathbb{C}) + c_2(\xi \otimes \mathbb{C}) + \cdots + c_n(\xi \otimes \mathbb{C})$$

$$c(\overline{\xi \otimes \mathbb{C}}) = 1 - c_1(\overline{\xi \otimes \mathbb{C}}) + c_2(\overline{\xi \otimes \mathbb{C}}) - \cdots + (-1)^n c_n(\overline{\xi \otimes \mathbb{C}})$$

and thus, since we are working over a field with no torsion, we may conclude that the odd Chern classes of $\xi \otimes \mathbb{C}$ vanish.

Definition 3.24. Given a real vector bundle ξ over a smooth manifold M, we define the kth *Pontrjagin class* to be

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}) \in H^{4k}(M).$$

By the Pontrjagin class of a smooth manifold, we mean that of its tangent bundle $p_k(M) := p_k(\tau_M)$.
The *total Pontrjagin class* is $p(\xi) = 1 + p_1(\xi) + \cdots + p_n(\xi)$

Remark 3.25. It follows by definition that Pontrjagin classes satisfy naturality and the Whitney-sum formula.

Remark 3.26. Since the underlying real manifold for a complex manifold has a canonical preferred orientation, it follows that the Pontrjagin numbers are independent of orientation.

Lemma 3.27. *For any complex vector bundle ν, the complexification $\nu_{\mathbb{R}} \otimes \mathbb{C}$ of the underlying real bundle is canonically isomorphic to the direct sum $\nu \oplus \bar{\nu}$*

Proof. This proof is taken from [11]. A complex structure on a real $2n$-dimensional vector space W is a continuous map $J\colon W \to W$ satisfying $J^2 = -\mathrm{Id}$. For any real vector space V, $V \otimes \mathbb{C}$ can be identified with $V \oplus V$ equipped with the complex structure $J(x, y) = (y, x)$.

Now, suppose $V = F_{\mathbb{R}}$ where F is the fibre of a complex vector bundle. Then we have that the correspondence $g\colon F \to V \oplus V$; $x \mapsto (x, -ix)$ is complex linear, that is $g(ix) = J(g(x))$. On top of this, the correspondence $h\colon F \to V \oplus V$; $x \mapsto (x, ix)$ is conjugate linear. Thus, since every point $(x, y) \in V \oplus V \simeq F_{\mathbb{R}} \otimes \mathbb{C}$ can be written uniquely as the sum $g(\frac{1}{2}(x + iy)) + h(\frac{1}{2}(x - iy))$ of an element in $g(F)$ and an element in $h(F)$, it follows that $F_{\mathbb{R}} \otimes \mathbb{C}$ is canonically isomorphic, as a complex vector space, to $F \oplus \bar{F}$.

Since this is true for each fibre of ν, combining all of the isomorphisms it follows that $\nu_{\mathbb{R}} \otimes \mathbb{C} \simeq \nu \oplus \bar{\nu}$ □

Corollary 3.28. *For any complex vector bundle ν of complex rank n, the Pontrjagin classes $p_i = p_i(\nu)$ are completely determined by the Chern classes $c_i = c_i(\nu)$ by the formula*

$$1 - p_1 + p_2 - \cdots \pm p_n = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n)$$

Proof. Follows immediately from Lemma (3.25) by the Whitney-sum formula and Lemma (3.23). □

Example 3.29. Since the total Chern class of $\mathbb{CP}^n$ is $(1 + \eta)^{n+1}$, we have that the Pontrjagin classes are given by

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n) \\ &= (1 - \eta)^{n+1}(1 + \eta)^{n+1} \\ &= (1 - \eta^2)^{n+1} \end{aligned}$$

Thus the total Pontrjagin class is equal to $(1 + \eta^2)^{n+1}$ and so $p_k(\mathbb{CP}^n) = \binom{n+1}{k}\eta^{2k}$.

Definition 3.30. Let M be a smooth, compact, oriented manifold of dimension $4n$. Let $(i_1, \ldots, i_k)$ be a partition of n. We define the $(i_1, \ldots, i_k)$th Pontrjagin number of M to be:

$$p_{(i_1,\ldots,i_k)}(M) = \int_M p_{i_1}(M) \cdots p_{i_k}(M)$$

Example 3.31. $\mathbb{CP}^{2n}$ has underlying structure a real manifold of dimension $4n$, and we see that it has Pontrjagin numbers given by

$$p_{(i_1,\ldots,i_k)}(\mathbb{CP}^{2n}) = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_k}$$

Theorem 3.32. *If ∂M is a $4n$-dimensional manifold bounding a $(4n + 1)$-dimensional manifold-with-boundary M, then the Pontrjagin numbers of ∂M are all zero. Hence Pontrjagin numbers give cobordism invariants.*

Proof. Consider the tangent bundle of M restricted to the boundary ∂M, taking a small collar neighbourhood and imposing a Riemannian metric we have that

$$\tau_M\big|_{\partial M} \simeq \tau_{\partial M} \oplus \nu_{\partial M}$$

where $\nu_{\partial M}$ is the normal bundle diffeomorphic to the collar. Complexifying, we arrive at

$$\begin{aligned}
\tau_M\big|_{\partial M} \otimes \mathbb{C} &\simeq (\tau_{\partial M} \oplus \nu_{\partial M}) \otimes \mathbb{C} \\
\implies (\tau_M \otimes \mathbb{C})\big|_{\partial M} &\simeq (\tau_{\partial M} \otimes \mathbb{C}) \oplus (\nu_{\partial M} \otimes \mathbb{C})
\end{aligned}$$

The normal bundle has a nowhere vanishing section and hence so does its complexification, so it must be trivial with total Chern class 1. Thus applying naturality and Whitney-sum we have that the Pontrjagin classes of M restricted to the boundary give the Pontrjagin classes of ∂M.

Now we need to recall again our discussion on the relative de Rham theory. We had a long exact sequence

$$\cdots \longrightarrow H^{k-1}(M) \xrightarrow{i^*} H^{k-1}(\partial M) \xrightarrow[{[\theta]\mapsto[(0,\theta)]}]{\alpha} H^k(M, \partial M) \longrightarrow \cdots$$

And an isomorphism $j\colon H^k_c(M \setminus \partial M) \to H^k(M, \partial M)$. In the proof of the cobordism invariance of signature we noted the identity for arbitrary $[\theta] \in H^k(\partial M)$

$$\int_{\partial M} \theta = \int_M j^{-1} \circ \alpha(\theta)$$

Let $i_1 \dots i_l$ be a partition of $4n$ and $p_{(i_1,\dots,i_l)}(\partial M)$ the corresponding Pontrjagin number. We have explained above that the Pontrjagin classes on ∂M are restrictions of Pontrjagin classes on M and so we may write $p_{(i_1,\dots,i_l)}(\partial M) = i^* A$ for some $[A] \in H^k(M)$. Putting everything together we see

$$\int_{\partial M} p_{(i_1,\dots,i_l)}(\partial M) = \int_M j^{-1} \circ (\alpha \circ i^* A) = 0$$

Since $\alpha \circ i^* = 0$ by the exactness of our sequence. $\square$

We now have two seemingly independent invariants for cobordism: the signature and the Pontrjagin numbers. A great triumph of 20th century mathematics was the unearthing of connections between number theory and topology, which began to be exposed alongside the discovery of a relationship between these two invariants[6]. René Thom first showed that the signature could be expressed as some linear combination of Pontrjagin numbers in 1954, and Friedrich Hirzebruch in 1956 gave the explicit relationship. We state the theorem below:

Theorem 3.33 (Hirzebruch Signature Theorem)**.** *For a $4n$-dimensional manifold M, the signature is a universal linear combination of Pontrjagin numbers. The first few formulas are as follows:*

for 4-manifolds $$\mathfrak{sig}(M) = \frac{1}{3}\int_M p_1$$

for 8-manifolds $$\mathfrak{sig}(M) = \frac{1}{3^2 \cdot 5}\int_M (7p_2 - p_1^2)$$

for 12-manifolds $$\mathfrak{sig}(M) = \frac{1}{3^3 \cdot 5 \cdot 7}\int_M (62p_3 - 13p_2p_1 + 2p_1^3)$$

for 16-manifolds $$\mathfrak{sig}(M) = \frac{1}{3^4 \cdot 5^2 \cdot 7}\int_M (381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4)$$

Where $p_i = p_i(M)$.

[6] In Hirzebruch's own words, writing in 'Prospects in Mathematics—The Signature Theorem: Reminiscences and Recreation' in 1971: "[the] signature theorem has many more number theoretical connections... As a theme (familiar to most topologists) under the general title of *Prospects in Mathematics*, we propose *more and more number theory in topology.*" [8]

The coefficients in these formulas come from something called the *multiplicative sequence of polynomials* associated to the formal power series expansion of the expression

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{3^2 \cdot 5}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}B_k}{(2k)!}t^k + \cdots$$

where $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \ldots$ are the famous Bernoulli numbers.

The theory here is exceptionally weird, and unfortunately it is just beyond the reach of this project. A full treatment of the theory alongside a proof of the signature theorem can be found in Milnor & Stasheff [11].

Although we do not derive the signature theorem from first principles, we have in fact already done enough to check to any degree of accuracy. Indeed, a theorem of Thom [13] states that the oriented cobordism ring, if working over a field with no torsion, is generated by the elements $\mathbb{CP}^{2n}$. If we accept this result as a black box, then we see that in order to verify the signature formulas it is only a matter of checking the numbers for $\mathbb{CP}^{2n}$. Indeed, we check the first few relations

$$\mathfrak{sig}(\mathbb{CP}^2) = \frac{1}{3}\binom{3}{1} = 1$$

$$\mathfrak{sig}(\mathbb{CP}^4) = \frac{1}{3^2 \cdot 5}(7\binom{5}{2} - \binom{5}{1}\binom{5}{1}) = \frac{1}{45}(7 \cdot 10 - 5 \cdot 5) = 1$$

$$\mathfrak{sig}(\mathbb{CP}^6) = \frac{1}{3^3 \cdot 5 \cdot 7}(62\binom{7}{3} - 13\binom{7}{2}\binom{7}{1} + 2\binom{7}{1}\binom{7}{1}\binom{7}{1}) = \frac{1}{945}(31 \cdot 5 \cdot 2 \cdot 7 - 13 \cdot 3 \cdot 7^2 + 2 \cdot 7^3) = 1$$

Example 3.34 (The Pontrjagin classes of the sphere)**.** The normal bundle ν of S^n with respect to its standard embedding in $\mathbb{R}^{n+1}$ is trivial, from the short exact bundle sequence

$$0 \longrightarrow \tau_{S^n} \longrightarrow \tau_{\mathbb{R}^{n+1}}\big|_{S^n} \longrightarrow \nu_{S^n} \longrightarrow 0$$

since $\tau_{\mathbb{R}^{n+1}}$ is also trivial, the total Pontrjagin classes of both ν and $\tau_{\mathbb{R}^{n+1}}$ are 1 and so we have by Whitney sum that the total Pontrjagin class of the sphere is also 1.

Example 3.35 (An application to embeddings of $\mathbb{CP}^2$)**.** As an application of Pontrjagin classes, we will show that $\mathbb{CP}^2$, which we know has the underlying structure of a real 4-dimensional manifold, cannot be smoothly embedded in Euclidean space of dimension less that 8. Indeed, suppose that we have a smooth embedding of $\mathbb{CP}^2$ in $\mathbb{R}^7$, then we have a short exact bundle sequence:

$$0 \longrightarrow \tau_{\mathbb{CP}^2} \longrightarrow \tau_{\mathbb{R}^7}\big|_{\mathbb{CP}^2} \longrightarrow \nu \longrightarrow 0$$

Where ν here is the normal to $\mathbb{CP}^2$ in $\mathbb{R}^7$. Since ν is a rank 3 real bundle, its total Pontrjagin class is just 1. Since $\tau_{\mathbb{R}^7}$ is trivial, its total Pontrjagin class is 1 and so we have by naturality that the total Pontrjagin class of $\tau_{\mathbb{R}^7}\big|_{\mathbb{CP}^2}$ is also just 1. Thus applying Whitney-sum we have $1 = 1 + 3\eta^2 + 3\eta^4$, a contradiction. We deduce that $\mathbb{CP}^2$ cannot be smoothly embedded in Euclidean space of dimension less than 8. This argument carries over to show that $\mathbb{CP}^{2n}$ cannot be smoothly embedded in Euclidean space of dimension less than $4(n+1)$.

4 Milnor's Exotic Spheres

4.1 Milnor's Invariant λ

We are now in a position to discuss Milnor's 'exotic spheres' as presented in [6]. Milnor constructed his invariant λ for a 7-dimensional, closed, oriented, compact manifold M satisfying the hypothesis

$$(*) \qquad H^3(M) = H^4(M) = 0$$

Given such a manifold, Thom's theorem [13] states that there exists an 8-dimensional manifold-with-boundary B such that $\partial B = M$. Hence our relative long exact sequence

$$\cdots \longrightarrow H^3(M) \longrightarrow H^4(B, M) \xrightarrow{\beta} H^4(B) \longrightarrow H^4(M) \longrightarrow \cdots$$

implies by the hypothesis $(*)$ that β gives an isomorphism $H^4(B, M) \xrightarrow{\simeq} H^4(B)$. Thus applying Alexander-Lefschetz duality we have a non-degenerate, symmetric bilinear form

$$\int_B : H^4(B) \times H^4(B) \longrightarrow \mathbb{R}$$

and we define the signature of B, $\mathfrak{sig}(B)$, to be the signature of this form. Moreover, by 'shaving off the boundary' as discussed in (2.2) we may talk of the Pontrjagin classes and Pontrjagin numbers of B. Milnor's invariant is then given by the following:

$$\lambda(M) = 2 \int_B p_1(B)^2 - \mathfrak{sig}(B) \qquad \text{(modulo 7)}$$

In order to show that λ is well defined, we must check that it does not depend on the choice of manifold-with-boundary B. Let B_1, B_2 be two manifolds-with-boundary, each with boundary M. Then $C = B_1 \cup_{Id_M} B_2$ is a closed, 8-dimensional manifold with a smooth structure compatible with that of B_1 and B_2 and we may chose an orientation on C which coincides with that of B_1 and hence that of $-B_2$. Now by the Hirzebruch signature theorem, we have

$$\mathfrak{sig}(C) = \frac{1}{45} \int_C (7p_2(C) - p_1(C)^2)$$

which rearranged gives

$$45\,\mathfrak{sig}(C) + \int_C p_1(C)^2 = 7 \int_C p_2(C)$$

$$\Longrightarrow \qquad 2 \int_C p_1(C)^2 - \mathfrak{sig}(C) = 0 \qquad \text{(modulo 7)}$$

and so to show λ is well defined it suffices to prove the following lemma:

Lemma 4.1. *The following two relations hold*

$$\mathfrak{sig}(C) = \mathfrak{sig}(B_1) - \mathfrak{sig}(B_2)$$

$$\int_C p_1(C)^2 = \int_{B_1} p_1(B_1)^2 - \int_{B_2} p_1(B_2)^2$$

Proof. Following Milnor, we note a commutative square of isomorphisms

$$\begin{tikzcd} H^4(B_1,M)\oplus H^4(B_2,M) \arrow[d, "\beta_1\oplus\beta_2"'] & H^4(C,M) \arrow[l, "h"'] \arrow[d, "\beta"] \\ H^4(B_1)\oplus H^4(B_2) & H^4(C) \arrow[l, "k"] \end{tikzcd}$$

Here h is obtained by first noting that our proof that $H^k(M,\partial M)\simeq H^k_c(M)$ extends in fact to any pair and gives an isomorphism between $H^k(C,M)$ and the homology of the complex of forms on C which vanish in some neighbourhood of M. The forms on C which vanish in some neighbourhood of M is in a linear bijection with pairs of forms on (B_1,B_2) which vanish in some neighbourhood of their respective boundaries and this correspondence clearly descends to give an isomorphism on cohomology. k is obtained from the Mayer-Vietoris sequence:

$$H^3(M) \longrightarrow H^4(C) \xrightarrow{k} H^4(B_1)\oplus H^4(B_2) \longrightarrow H^4(M)$$

where the terms on the end vanish by the hypothesis (*). k is a restriction of forms on C to B_1 and B_2 respectively, see [15] page 22. If we define $a=\beta h^{-1}(a_1,a_2)\in H^4(C)$, then

$$\begin{aligned}\int_C a^2 &= \int_C \beta\circ h^{-1}(a_1^2\oplus a_2^2)\\ &= \int_{B_1} a_1^2 - \int_{B_2} a_2^2\end{aligned}$$

and the second relation follows immediately. Next, by defining $a_1=\beta_1^{-1}p_1(B_1)$, $a_2=\beta_2^{-1}p_1(B_2)$, we have by naturality that $k(p_1(C))=p_1(B_1)\oplus p_1(B_2)$ and the above computation then implies the first relation. $\square$

As an immediate consequence of this, we have

Corollary 4.2. *If $\lambda(M)\neq 0$, then M is not the boundary of any manifold-with-boundary having fourth cohomology group $H^4\simeq\{0\}$*

4.2 Milnor's Exotic Spheres

We now continue to follow Milnor in demonstrating that certain of the earlier manifolds from Section (1.4), $M_{(h,j)}$, for which $h+j=1$, are homeomorphic but not diffeomorphic to S^7.

When building our S^3 bundles over S^4 in Section (1.4), we were happy to label our structure group as $\mathrm{Diff}(S^3)$. We note now however that multiplication by a unit quaternion gives an orientation preserving isometry of S^3, and so our structure group may be reduced to $SO(4)$. In fact Milnor points out that the equivalence classes of such bundles lie in a 1-1 correspondence with $\pi_3(SO(4))\simeq\mathbb{Z}\times\mathbb{Z}$ and that the correspondence is given by our clutching map $f_{(h,j)}\in\pi_3(SO(4))$ for $(h,j)\in\mathbb{Z}\times\mathbb{Z}$. To each of our S^3 bundles over S^4, therefore, we may associate a corresponding $\mathbb{R}^4$ bundle over S^4, with structure group $SO(4)$, where the original fibres S^3 sit in the new fibres $\mathbb{R}^4$ as the set of vectors with unit norm. We denote each of these corresponding vector bundles by $\xi_{(h,j)}$.

If we denote the set of these bundles by $\Xi = \{\xi_{(h,j)} : (h,j) \in \mathbb{Z} \times \mathbb{Z}\}$, then we may define an operation

$$\# : \Xi \times \Xi \longrightarrow \Xi$$

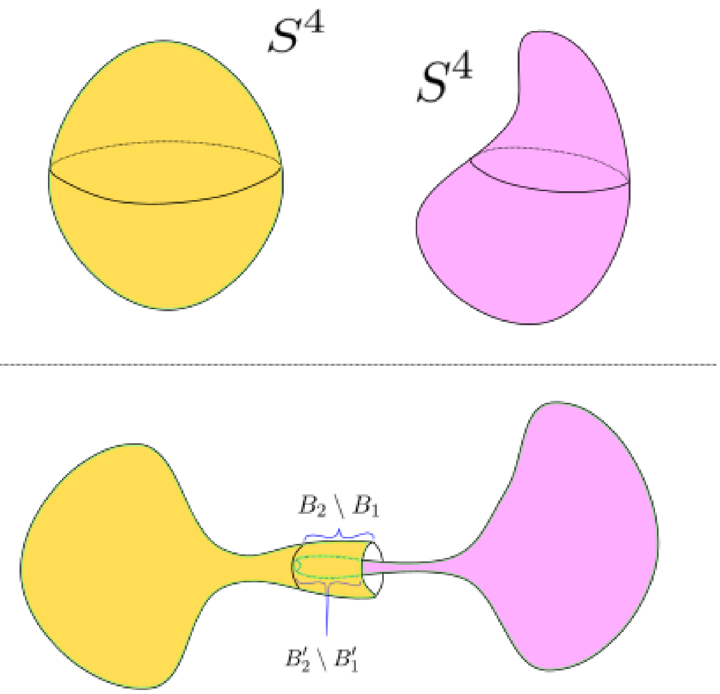
by "connected sum on the base space" as follows. Given $\xi_{(h,j)}$,$\xi_{(h',j')}$, choose small open 4-balls in the base spaces $B_1 \subset M(\xi_{(h,j)})$, $B_1' \subset M(\xi_{(h',j')})$ and note that the fibres may be perturbed such that $\xi_{(h,j)}\big|_{B_1}$, $\xi_{(h',j')}\big|_{B_1'}$ are trivial. i.e.

$$\xi_{(h,j)}\big|_{B_1} \simeq B_1 \times \mathbb{R}^4$$

$$\xi_{(h',j')}\big|_{B_1'} \simeq B_1' \times \mathbb{R}^4$$

Then choose smaller open balls $B_2 \subset B_1$, $B_2' \subset B_1'$ and glue together the bundles $\xi_{(h,j)}\big|_{S^4 \setminus B_2}$, $\xi_{(h',j')}\big|_{S^4 \setminus B_2'}$ in the following way: by associating the boundary of B_2 with the boundary of B_1', and associating the boundary of B_1 with the boundary of B_2' to give a smooth connected sum of the two base spaces, and by identifying the fibres over each point in the glue by the identity map. This is possible as the bundles are trivial over $B_2 \setminus B_1$ and $B_2' \setminus B_1'$. This defines our operation $\#$.

Figure 9: A connected sum on the base space



Now consider the case of connected sum on the base space of an arbitrary bundle $\xi_{(h,j)}$ with $\xi_{(1,0)}$, that is $\#(-, \xi_{(1,0)}) : \Xi \to \Xi$. Since we may perform this connected sum over a small section of the equator, around the base point of S^3 in $\pi_3(SO(4))$, and since composition in π_3 is given by traversing the two copies of S^3 in turn, we have that $\#(\xi_{(h,j)}, \xi_{(1,0)})$ has clutching function $f_{(h+1,j)}$ and thus $\#(\xi_{(h,j)}, \xi_{(1,0)}) = \xi_{(h+1,j)}$, and by a symmetric argument $\#(\xi_{(h,j)}, \xi_{(0,1)}) = \xi_{(h,j+1)}$

Moreover, we note that representatives of the Pontrjagin classes of $\xi_{(h,j)}$, $\xi_{(h',j')}$ can be chosen to take support away from the balls B_1, B_1', and so we have that $p_1(\#(\xi_{(h,j)}, \xi_{(h',j')})) = p_1(\xi_{(h,j)}) + p_1(\xi_{(h',j')})$ in $H^4(S^4)$. Putting this together, we have demonstrated the following

Lemma 4.3. *The Pontrjagin class $p_1(\xi_{(h,j)})$ is a linear function of h and j.*

Furthermore, we know that the Pontrjagin classes are independent of the orientation, but if the orientation of the fibre S^3 is reversed, then $\xi_{(h,j)}$ is replaced with $\xi_{(-j,-h)}$. Thus we have that the Pontrjagin class is given by

$$p_1(\xi_{(h,j)}) = r(h-j)\iota$$

where r is some integer and ι is a generator of $H^4(S^4)$.

Now, for each odd integer k, let $(h,j) \in \mathbb{Z} \times \mathbb{Z}$ be determined by the formulas

$$h + j = 1,\ h - j = k$$

and denote $\xi_{(h,j)} = \xi_k$, $M_{(h,j)} = M_k$.

Associated to each S^3 bundle $M_k \longrightarrow S^4$ there is a 4-ball bundle $B_k \xrightarrow{\rho_k} S^4$. The total space of this bundle is a smooth manifold-with-boundary with boundary M_k and we have that the total space of ξ_k retracts onto the interior of B_k. The cohomology group $H^4(B_k)$ is generated by $\alpha = \rho_k^* \iota$. We choose orientations for M_k and B_k so that $\mathfrak{sig}(B_k) = +1$. We have a short exact bundle sequence

$$0 \longrightarrow \tau_{S^4} \longrightarrow \tau_{B_k} \longrightarrow \xi_k \longrightarrow 0$$

And since $p(S^4) = 1$ by Example (3.34), we have by Whitney-sum that $p_1(B_k) = rk\alpha$.

Now, we return to the quaternions for a study of the quaternionic projective space $\mathbb{HP}^2$, which is obtained by taking a copy of $\mathbb{H}^2 \simeq \mathbb{R}^8$ and attaching a copy of $\mathbb{HP}^1 \simeq S^4$ "at infinity". Indeed, writing

$$\mathbb{HP}^2 = \{(q_0, q_1, q_2)\mathbb{H} \colon (q_0, q_1, q_2) \in \mathbb{H}^3 \setminus (0,0,0)\} = \{[q_0, q_1, q_2]\}$$

we have that $\mathbb{HP}^2$ is a union of $\{[1, q_1, q_2]\}$ and

$$\{(0, q_1, q_2)\mathbb{H} \colon (q_1, q_2) \in \mathbb{H}^2 \setminus (0,0)\} = \{[0, q_1, q_2]\} \simeq \mathbb{HP}^1$$

.

We associate $\{[1, q_1, q_2]\}$ with $\mathbb{H}^2 \simeq \mathbb{R}^8$ by $[1, q_1, q_2] \mapsto (q_1, q_2)$. And we may associate to every point $[0, q_1, q_2] \in \mathbb{HP}^1$ a copy of $\mathbb{H}$ in $\mathbb{H}^2$ by

$$[0, 1, v] \longmapsto \{(w, v) \colon w \in \mathbb{H}\}$$

$$[0, u, 1] \longmapsto \{(u, w) \colon w \in \mathbb{H}\}$$

and this association realises $\mathbb{HP}^2$ with the point $[0,0,1]$ removed as an $\mathbb{H} \simeq \mathbb{R}^4$ bundle over $\mathbb{HP}^1 \simeq S^4$.

We have coordinate charts on $\mathbb{HP}^1$ by $[0, 1, v] \mapsto v$ and $[0, u, 1] \mapsto u$. And we see that, switching between coordinates, the fibre transition over the point $[0, 1, u] = [0, u^{-1}, 1]$ is given by multiplication by u. Hence we see that $\mathbb{HP}^2$ with a point removed is in fact the space B_k for $k = 1$. The Pontrjagin class $p_1(\mathbb{HP}^2 \setminus \{\text{point}\})$ is known to be twice a generator of $H^4(\mathbb{HP}^2)$, see Hirzebruch [5] pp. 301-312. Therefore we conclude that the integer r must be ± 2. We now have all of the numbers available to compute $\lambda(M_k)$.

$$
\begin{aligned}
2\int_{B_k} p_1(B_k)^2 - \mathfrak{sig}(B_k) &= 8k^2 - 1 \\
&= k^2 - 1 \qquad \text{(modulo 7)}
\end{aligned}
$$

And we arrive at the final statement of Milnor's theorem.

Theorem 4.4 (Milnor)**.** *For $k^2 \neq 1$ (modulo 7) the manifold M_k is homeomorphic to S^7 but not diffeomorphic to S^7.*

In particular, the manifolds M_3, M_5, M_7 all exhibit *distinct differentiable structures*. Milnor remarks on the following property of the invariant λ: if the orientation of M is reversed, then $\lambda(M)$ is multiplied by -1, it follows that if $\lambda(M) \neq 0$, then M possesses no orientation reversing diffeomorphism onto itself. Milnor's exotic spheres cannot be turned inside out.

References

[1] https://www.youtube.com/watch?v=akotmpgfjyk.

[2] Michèle Audin & Mihai Damian. *Morse Theory and Floer Homology.* Springer, 2014.

[3] Joseph Driscoll. Morse homology. *UCL MATHM901: Project in Mathematics*, 2015.

[4] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[5] F. Hirzebruch. Ueber die quaternionalen projektiven räume. *S.-Ber. math.- naturw. Kl. Bayer. Akad. Wiss. München*, 1953.

[6] John Willard Milnor. On manifolds homeomorphic to the 7-sphere. *The Annals of Mathematics, 2nd Ser., Vol. 64*, 1956.

[7] Ib Madsen & Jørgen Tornehave. *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes.* Tsinghua University Press, 1997.

[8] F. Hirzebruch; Lars H ormander; John Milnor; Jean-Pierre Serre; I. M. Singer. *Prospects in Mathematics.* Princeton University Press, 1971.

[9] Michael Spivak. *Calculus On Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus.* Westview Press, 1965.

[10] Michael Spivak. *A Comprehensive Introduction to Differential Geometry: Volume 1.* Publish or Perish, inc., 1979.

[11] John Willard Milnor & James D. Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[12] Samuel Eilenberg & Norman Earl Steenrod. *Foundations of Algebraic Topology.* Princeton University Press, 1952.

[13] René Thom. Quelques propriétés globale des variétés différentiables. *Actual. sci. industr. 1183, Paris*, 1952.

[14] William P. Thurston. *Three-dimensional Geometry and Topology, Volume 1.* Princeton University Press, 1997.

[15] Raoul Bott & Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer New York, 1995.

[16] Liam Williams. 4-dimensional Einstein metrics. *UCL MATHM901: Project in Mathematics*, 2015.