ME 370B
Energy Systems II:
Modeling and Advanced Concepts

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More than you ever wanted to know about… Polytropic Efficiency

Polytropic efficiency is nothing more than the differential version of isentropic (*a.k.a.*, adiabatic) efficiency. It is used in situations where the compression (or expansion) ratio is to be varied over a wide range. The reason is that while isentropic efficiency is a suitable metric at any given pressure ratio, it does not hold constant as you change pressure ratio. Polytropic efficiency on the other hand remains almost constant over a wide range of pressure ratios and, as such, is a better metric of the state of the art in turbomachinery design. It is a good metric of how well you can do at any pressure ratio, not just one fixed value.

Polytropic efficiency is defined for turbines and compressors, respectively, as

$$\varepsilon_{turbine} = \frac{\delta w}{\delta w_s} \qquad\qquad \varepsilon_{compressor} = \frac{\delta w_s}{\delta w}$$

where δw is the amount of work required or developed over a differential extent of the process (*i.e.*, a differential pressure change), and δw_s is the corresponding amount for an adiabatic, reversible (isentropic) process of the same extent. For a process using a real fluid, the polytropic efficiency is applied to each differential step of the compression or expansion process. In addition to accounting for the turbomachinery performance in a better way, this has the added advantage of allowing real property effects to be included in a natural way. This is about all you need to know about polytropic efficiency in order to do the assignment, but in case you are interested in how it relates to isentropic efficiency and why it is called polytropic in the first place, we have included a discussion of polytropic efficiency as applied to ideal gases with constant properties below. Read on if you have a few minutes to spare.

For ideal gases with constant properties, the polytropic efficiency—if also assumed to be constant—can be translated analytically into an isentropic efficiency for a specified pressure ratio. Similar to the case of isentropic efficiency, use of a polytropic efficiency as a figure of merit implies that the component is to be analyzed as adiabatic and that the ideal and real machines have the same inlet state and outlet stagnation pressure.

Using the turbine case as an example, the adiabatic requirement gives

$$\varepsilon_t = \frac{\delta w}{\delta w_s} = \frac{dh}{dh_s} = \frac{c_p dT}{c_p dT_s} = \frac{dT}{dT_s}$$

Dividing both numerator and denominator by the differential extent of the expansion dP gives

$$\varepsilon_t = \frac{dT/dP}{dT_s/dP} = \frac{dT/dP}{\left(dT/dP\right)_s}$$

Using Gibbs equation for an ideal gas

$$ds = \frac{dh}{T} - \frac{v}{T}dP = c_p \frac{dT}{T} - \frac{R}{P}dP$$

and applying this to an isentropic process, $ds = 0$, requires that

$$\left(\frac{dT}{T}\right)_s = \frac{R}{c_p}\left(\frac{dP}{P}\right)_s = \left(\frac{k-1}{k}\right)\left(\frac{dP}{P}\right)_s$$

where $k = c_p / c_v$ is the ratio of specific heats. The slope of the *T-P* curve at any point in an isentropic expansion process can then be written as

$$\left(\frac{dT}{dP}\right)_s = \left(\frac{k-1}{k}\right)\frac{T}{P}$$

The expression for the slope of the *T-P* curve of the actual expansion process is then given in terms of the polytropic efficiency as

$$\frac{dT}{dP} = \varepsilon_t \left(\frac{dT}{dP}\right)_s = \varepsilon_t \left(\frac{k-1}{k}\right)\frac{T}{P}$$

or

$$\frac{dT}{T} = \varepsilon_t \left(\frac{k-1}{k}\right)\frac{dP}{P}$$

Integrating this expression across the pressure ratio of the turbine gives the temperature ratio experienced by the fluid under actual conditions

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\varepsilon_t\left(\frac{k-1}{k}\right)} = \pi_t^{-\varepsilon_t\left(\frac{k-1}{k}\right)}$$

where $\pi_t = P_1 / P_2$ is the stagnation pressure ratio of the turbine. These expressions show the origin of the name *polytropic efficiency*. Since the polytropic efficiency appears as a multiplier of the fixed-*k* isentropic exponent, the product effectively defines a polytropic exponent that can be used in place of the isentropic value in process modeling.

Our objective in analyzing a compressor or turbine is to find the work interaction required for the process. Since we are considering ideal gases with constant properties

$$w_{12} = \int_1^2 \delta w = \int_1^2 dh = c_p\left(T_2 - T_1\right) = c_p T_1\left(\frac{T_2}{T_1} - 1\right)$$

Substituting for the temperature ratio in terms of the pressure ratio and polytropic efficiency

$$w_{12} = c_p T_1\left(\pi_t^{-\varepsilon_t\left(\frac{k-1}{k}\right)} - 1\right)$$

While this is the basic result we were after, note that by setting the polytropic efficiency to unity we recover isentropic performance such that

$$w_{12,s} = c_p T_1 \left(\pi_t^{-\left(\frac{k-1}{k}\right)} - 1 \right)$$

The ratio of these two quantities is, by definition, the isentropic efficiency of a turbine such that

$$\eta_{\substack{turbine \\ isentropic}} = \frac{w_{12}}{w_{12,s}} = \frac{\pi_t^{-\varepsilon_t\left(\frac{k-1}{k}\right)} - 1}{\pi_t^{-\left(\frac{k-1}{k}\right)} - 1}$$

can be used to express the relationship between the two efficiencies. Similarly, for ideal gas compression with constant properties and pressure ratio $\pi_c = P_2/P_1$, the expression is

$$\eta_{\substack{compressor \\ isentropic}} = \frac{w_{12,s}}{w_{12}} = \frac{\pi_c^{\left(\frac{k-1}{k}\right)} - 1}{\pi_c^{\frac{1}{\varepsilon_c}\left(\frac{k-1}{k}\right)} - 1}$$

Don't forget that these relationships are for constant-property fluids as well as constant polytropic efficiency. We will not generally be making the constant-property-fluid approximation, so you will have more to do than just plug into these expressions. But they may be of some value in debugging the polytropic part of your code, so we thought you might be interested.