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HW 12

1) (a) • $E := \{ f_i \mid$ the nth element of f_i is $H \}$

• $E := \{ f_i \mid \forall n \in \mathbb{N}$ (not including 0), the $2n$th and $(2n-1)$th elements of f_i are equal$\}$

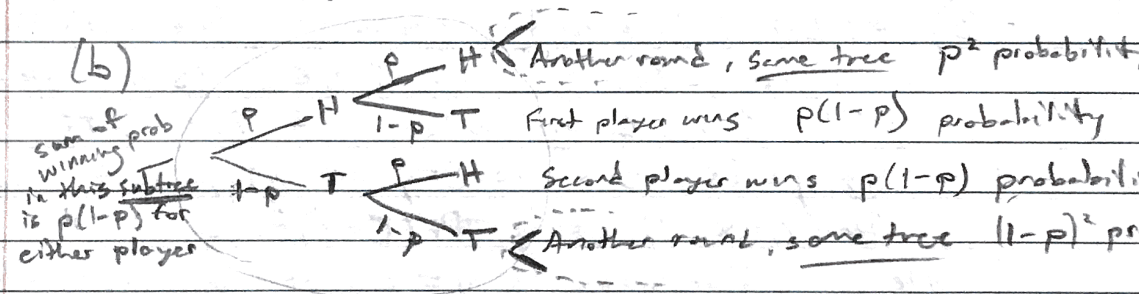
• $E := \{ f_i \mid$ each round is either H,H or T,T until at least round $n \}$

• $E := \{ f_i \mid$ each round before round n is either H,H or T,T, and round n is H,T$\}$

• $E := \{ f_i \mid$ the round H,T occurs, and every round before it is either H,H or T,T$\}$

<u>Note:</u> A round is the $(2n-1)$th and $2n$th elements of a sequence f_i for all $n \in \mathbb{N}, n > 0, n < (\text{length of } f_i)/2$.

(b)



As suggested, let s be the sum of all winning probabilities.

$$s = p^2 s + (1-p)p + (1-p)^2 s$$
$$s(1 - p^2 - (1-p)^2) = p - p^2$$
$$s(1 - p^2 - (1 - 2p + p^2)) = p - p^2$$
$$s(-2p^2 + 2p) = p - p^2$$
$$s = \frac{1}{2}$$

<u>Note:</u> same equation for sum of probs of player 1 winning as for player 2 (for a player)

- Probability that first player wins is $\frac{1}{2}$
- Probability that second player wins is $\frac{1}{2}$
- Sum of probabilities must be 1, and already $\frac{1}{2} + \frac{1}{2} = 1$, so Probability that no one wins must be 0.

by definition

2) Assume A and B are disjoint, so $A \cap B = \emptyset$, and $Pr(A \cap B) = Pr(\emptyset) = 0$.

If A and B are independent, then $Pr(A \cap B) = Pr(A)Pr(B)$ by definition.

Because both sides equal $Pr(A \cap B)$

So $Pr(A)Pr(B) = 0$ (If A and B are independent and disjoint), but the only way for this to be true is if $Pr(A) = 0$ or $Pr(B) = 0$. If $Pr(A) \neq 0$ and $Pr(B) \neq 0$, then $Pr(A \cap B) \neq Pr(A)Pr(B)$, and so A and B cannot be independent.

3) (a) Let C be a constant random variable such that $C : S \to \mathbb{R}$, $C(s) = c$ for all $s \in S$ (S is sample space).

WTS C and C are independent, i.e. WTS $\forall x, y \in \mathbb{R}$, $C = x$ and $C = y$ are independent, i.e. $\forall x, y \in \mathbb{R}$, $Pr(C = x \cap C = y) = Pr(C = x)Pr(C = y)$.

We can either have $x, y = c$ or at least one of x, y is not c. In the first case,

$$Pr(C = x \cap C = y) = Pr(C = c \cap C = c) = Pr(C = c)$$
$$= 1 = (1)(1) = (Pr(C = c)Pr(C = c)) = Pr(C = x)Pr(C = y)$$

In the second case,

$Pr(C = x \cap C = y) = Pr(\emptyset) = 0$ because at least one of $C = x$ or $C = y$ is $\emptyset$.

Also, $Pr(C = x)Pr(C = y) =$ will either be $(0)Pr(C = y) = 0$ or $Pr(C = x)(0) = 0$ because either $C = x$ or $C = y$ is $\emptyset$ and $Pr(\emptyset) = 0$.

So $Pr(C = x \cap C = y) = Pr(C = x)Pr(C = y)$ in this case.

$Pr(C = x \cap C = y) = Pr(C = x)Pr(C = y)$ in both cases, so C must be independent from itself.

(b) Assume a random variable X is independent from itself, so $\forall x, y \in \mathbb{R}$, $Pr(X=x \cap X=y) = Pr(X=x)\,Pr(X=y)$.
WTS if $Pr(X=x) > 0$ and $Pr(X=y) > 0$, then $x = y$.
Assume $Pr(X=x) > 0$ and $Pr(X=y) > 0$. Then
$Pr(X=x \cap X=y) = Pr(X=x)\,Pr(X=y) > 0$.
If $x \neq y$, then $X=x \cap X=y$ is $\emptyset$, so for $Pr(X=x \cap X=y)$ to be greater than 0, as required, we can't have $x \neq y$, so it <u>must</u> be true that $x = y$, i.e. if X is independent from itself, if $Pr(X=x) > 0$ and $Pr(X=y) > 0$, then $x = y$.

4) (a) The function given is shown to satisfy the 3 axioms below

1. For all events E, $Pr(E) \geq 0$.
   - We are given for all $s \in S$, $P(s) \geq 0$, and $Pr(E) = \sum_{s \in E} P(s)$. A sum of numbers greater than or equal to 0 must also be ≥ 0, so for all events E, $Pr(E) \geq 0$ ✓
2. $Pr(S) = 1$
   - Let $E = S$, then $Pr(E) = \sum_{s \in S} P(s)$, and we are given $\sum_{s \in S} P(s) = 1$, so $Pr(S) = 1$ ✓
3. For all disjoint events E_1 and E_2; $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$
   - $Pr(E_1 \cup E_2) = \sum_{s \in E_1 \cup E_2} P(s) = \sum_{s \in E_1} P(s) + \sum_{s \in E_2} P(s)$ because E_1 and E_2 are disjoint, sharing no common elements, so summing over the elements in $E_1 \cup E_2$ is the same as first summing over the elements in E_1 and adding to the result of summing the elements in E_2.
     $\sum_{s \in E_1} P(s) + \sum_{s \in E_2} P(s) = Pr(E_1) + Pr(E_2)$ by definition ✓

(b) Given $Pr'(\{s\}) = P(s)$, then $Pr'(\{s\}) = \sum_{s \in \{s\}} P(s)$ because $P(s)$ is the result of summing $P(s)$ over just $\{s\}$ ($\sum_{s \in \{s\}} P(s) = P(s)$).
By definition, $Pr(\{s\}) = \sum_{s \in \{s\}} P(s)$, so $Pr(\{s\})$ must equal $Pr'(\{s\})$ ✓
This means Pr is unique.

(c) We show the measure satisfies the 3 axioms below

1. For all E, $Pr(E) \geq 0$
   - $|E| \geq 0$ because $E \subseteq S$ and any subset has either some positive # of elements ($|E| > 0$) or is $\emptyset$ ($|E| = 0$). We know $S \neq \emptyset$, and there are a positive # of elements in S, so $|S| > 0$. $Pr(E) = \frac{|E|}{|S|}$, so $Pr(E)$ must also be ≥ 0 as required ✓
2. $Pr(S) = 1$
   - Let $E = S$, then $Pr(E) = Pr(S) = \frac{|S|}{|S|}$ by definition, and $\frac{|S|}{|S|} = 1$, so $Pr(S) = 1$ ✓
3. For all disjoint events E_1, E_2, $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$
   - $Pr(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|}$

     $|E_1 \cup E_2| = |E_1| + |E_2|$ because E_1 and E_2 share no elements in common, so

     $$\frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|}$$

     $= Pr(E_1) + Pr(E_2)$ by def'n

     So $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$ ✓