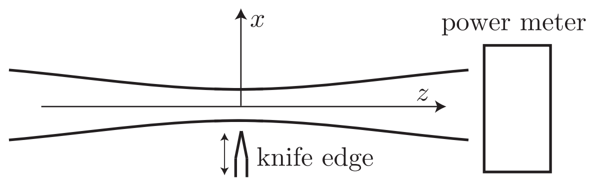
A common technique in the laboratory for measuring the beam waist parameter of a Gaussian beam is illustrated in the diagram. A Gaussian beam is incident on an optical power meter, which registers the total power of the incident beam. A knife edge can be translated in the transverse direction to block part of the beam (i.e., if the position of the knife edge is x_{knife}, then the parts of the beam with $x < x_{\text{knife}}$ is blocked from reaching the power meter). The "10-90" rule is to measure the knife edge position $x_{10\%}$ where the power meter reads 10% of the total beam power, and then the position $x_{90\%}$ where the power meter reads 90% of the total beam power. Then the beam radius $W(z)$ at the knife-edge location z along the beam is given by

$$w(z) = \alpha |x_{10\%} - x_{90\%}|,$$

where α is some constant factor. Calculate the numerical value of α.



Solution. Nothing happens in the y-direction, so we only need to consider the x-dependence of the Gaussian beam profile. Also, we will consider the intensity only at the z-position of the knife edge, so we will simply use w to denote $w(z)$. Then the intensity profile of the beam is

$$I(x) = A \exp\left(-\frac{2x^2}{w^2}\right),$$

where A is an arbitrary constant. The knife edge blocks the beam in the region $x < x_{\text{knife}}$, so the fraction of the beam that makes it past the knife edge is:

$$\frac{\displaystyle\int_{x_{\text{knife}}}^{\infty} I(x)dx}{\displaystyle\int_{\infty}^{\infty} I(x)dx} = \frac{\frac{1}{2}\sqrt{\frac{\pi w^2}{2}}\operatorname{erfc}\left(\sqrt{\frac{2x_{\text{knife}}^2}{w^2}}\right)}{\sqrt{\frac{\pi w^2}{2}}} = \frac{1}{2}\operatorname{erfc}\left(\sqrt{2}\frac{x_{\text{knife}}}{w}\right).$$

I used Mathematica to do this, but the integral in the denominator is in any standard integral table and the numerator is a change of variables to write the integral in terms of the complementary error function,

$$\operatorname{erfc}(z) := 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\, dt,$$

for which tabulated values are available (Mathematica spits them out quite happily).

The two special values are then given by:

$$10\% = \frac{1}{2}\operatorname{erfc}\left(\sqrt{2}\frac{x_{10\%}}{w}\right), \qquad 90\% = \frac{1}{2}\operatorname{erfc}\left(\sqrt{2}\frac{x_{90\%}}{w}\right).$$

Solving for $x_{10\%}$ and $x_{90\%}$,

$$x_{10\%} = \frac{\operatorname{erfc}^{-1}(0.2)}{\sqrt{2}}2 \approx 0.6408\, w, \qquad x_{90\%} = \frac{\operatorname{erfc}^{-1}(1.8)}{\sqrt{2}}2 \approx -0.6408\, w$$

Thus,

$$x_{10\%} - x_{90\%} = 1.2816\, w,$$

or $w = \alpha(x_{10\%} - x_{90\%})$ where $\alpha = 0.7803$.

Sometimes the tails of the Gaussian beam are distorted, and a 20-80 measurement might be more robust (or a good consistency check). A similar analysis shows that $w = 1.188\,(x_{20\%} - x_{80\%})$.