Optimization and Control, Assignment 1

Introduction

- Answer the questions 1–15 in this assignment concerning unconstrained optimization (1), constrained optimization (2), and linear programming (3).
- You need **80 points** or more to pass the assignment. The maximum is 110 points.
- upload your solutions or email them,
- the deadline is **April 18, 23:59 CEST**, and
- remember that submission after the deadline is not possible.

1 Unconstrained Optimization

1.1 Extreme points (10 points)

Find all the global minimum and maximum points of the following functions (these are the exercises 2.1 and 2.3 on page 34 from [Bec14]):

1. $f(x, y) = x^2 + y^2 + 2x - 3y$ over the circle $g(x, y) = x^2 + y^2 \leq 1$ (substitute $x^2 + y^2 = 1$), and
2. $f(x, y) = 2x - 3y$ over the ellipse $g(x, y) = 2x^2 + 5y^2 \leq 1$.

1.2 Stationary Points Classification (20 points)

Find and classify all the stationary points of the following functions. State whenever they are saddle points, minimum points, or maximum points. For each maximum and minimum point explicit if they are strict or nonstrict and if they are local or global (this is the exercise 2.15 on page 36 from [Bec14]):

3. $f(x, y) = x^4 + y^4$,
4. $f(x, y) = e^{x^2} + e^{y^2} - x^{200} + y^{200}$,
5. $f(x, y) = 2x^2 - 8xy + y^2$,
6. $f(x, y, z) = x^3 + y^3 + z^3$, and
7. $f(x, y) = x^2 - 2xy^2 + y^4$.

Explicit the steps of your analytical solution, and sketch the numerical solution using the scripts introduced in class.

2 Constrained Optimization

2.1 KKT Conditions (20 points)

Find the optimal solution of each of the following minimization problems (in $\mathbb{R}^2$) using KKT conditions (this is the exercise 11.16 on page 234 from [Bec14]):

8. $\min \; 3x_1^2 + x_2^2$
   s. t. $x_1 - x_2 + 8 \leq 0$
   $x_2 \geq 0$,

9. $\min \; 3x_1^2 + x_2^2$
   s. t. $3x_1^2 - x_2^2 + x_1 + x_1 + \frac{1}{10} \leq 0$
   $x_2 + 10 \geq 0$,

10. $\min \; x_1^3 + x_2^3$
    s. t. $x_1^2 - x_2^2 \leq 1$, and

11. $\min \; x_1^4 + x_2^2$
    s. t. $x_1^2 - x_2^2 \leq 1$
    $2x_2 + 1 \leq 0$.

3 Linear Programming

3.1 Manufacture Optimization (30 points)

A factory produces *red* and *yellow* bricks using two different coloring schemes. A *simple coloring* scheme uses artificial colors, while a *permanent coloring* scheme uses a mix of a special coating after the color is applied:

- Each load of red bricks that is produced needs 50 minutes processing time with the simple coloring scheme and 30 minutes processing time with the permanent one.

- Each load of yellow bricks that is produced requires 24 minutes processing time with the simple coloring scheme and 33 minutes processing time with the permanent one.

At the start of the current week, there are 30 loads of red bricks and 90 loads of yellow bricks stocked in the warehouse. Available processing time for the simple scheme is forecast to be 40 hours. For the permanent scheme is forecast to be 35 hours.

The demand for red bricks in the current week is forecast to be 75 units while is forecast to be 95 units for yellow bricks. Company policy is to maximize the combined sum of loads of red and yellow bricks in the warehouse at the end of the week.

12. define the LP problem,

13. formalize the KKT conditions for the problem explicitly (use the values instead of the general form),

14. show which of the points respect all the KKT conditions.

3.2 Dual of LP (30 points)

Consider the standard linear problem (this is the exercise on page 390 number 13.5 in the textbook [NW06]):

$$\min_{\mathbf{x}\in\mathbb{R}^n} J(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \;\text{ s.t. }\; A\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq 0,$$

with $\mathbf{c} \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$:

15. show that its dual is expressed (with $\lambda \in \mathbb{R}^m$):

$$\min_{\mathbf{b}\in\mathbb{R}^m} J(\mathbf{b}) = \mathbf{b}^T\lambda \;\text{ s.t. }\; A^T\lambda \leq \mathbf{c},\, \lambda \geq 0.$$

References

[Bec14] Amir Beck. *Introduction to nonlinear optimization: Theory, algorithms, and applications with MATLAB*, volume 19. Siam, 2014.

[NW06] Jorge Nocedal and Stephen Wright. *Numerical optimization*. Springer Science & Business Media, 2006.