Computational Methods for FinTech: Midterm Exam

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CONTENTS

This is your take-home midterm exam. This is the major deliverable for the semester.

Instructions

1. In each "chapter" there are several readings. Read them critically and take good notes.
2. Answer the critical thought questions in quarto documents.
3. Answer the numerical/computational questions. Here you can use either quarto documents or jupyter documents.
4. Collect your materials into a github repository.
5. Come see me for a final oral exam at the whiteboard.

- The Political Economy of Financial Technology and Innovation
  - *Foundations*
- Arbitrage, Rationality, and Equilibrium
  - *Arbitrage in the Catallaxy*
- Metallgesellschaft and the Economics of Synthetic Storage
  - *Abitrage, Entrepreneurship, and Creativity*
- Tutorials
  - *Appendix A: Notes on Continuous Compounding*
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  - *Appendix D: A Quick Review of Predictive Distributions*
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Part I

The Political Economy of Financial Technology and Innovation

CHAPTER

ONE

FOUNDATIONS

1.1 The Market Order or Catallaxy

We have examined the claim in this course that FinTech and Data Science should be built upon catallactic foundations. In this section you will be asked to critically examine this claim.

Q1. In a series of articles James M. Buchanan differentiated between political economy and economic science. What is the difference? How does it apply to FinTech and Data Science? Does it make sense to speak of a *Political Economy of Financial Technology*? What would that mean? What are the weaknesses of such an approach? What is your position on this claim? Defend it!

Please see:

- Buchanan [Buc79e]
- Buchanan [Buc79b]
- Buchanan [Buc79a]
- Buchanan [Buc79c]
- Buchanan [Buc79d]
- Buchanan [Buc82]
- Hayek [Hay78]
- Hayek and Buchanan [HB10]

1.2 The Knowledge Problem and Artificial Intelligence

In their paper titled *Economic Reasoning and Artificial Intelligence* Parkes and Wellman [PW15] state the following:

> “The field of artificial intelligence (AI) strives to build rational agents capable of perceiving the world around them and taking actions to advance specified goals. Put another way, AI researchers aim to construct a synthetic *homo economicus*, the mythical perfectly rational agent of neoclassical economics. We review progress toward creating this new species of machine, *machina economicus*, and discuss some challenges in designing AIs that can reason effectively in economic contexts.

In his 1937 paper titled *Economics and Knowledge* F.A. Hayek Hayek [Hay37] outlined what has come to be known as the *knowledge problem* in economics. In his follow up paper *The Use of Knowledge in Society* [Hay45] he developed the concept further.

In a piece that has come to be famous, data scientist Michael I. Jordan Jordan [Jor19a] cast doubt on the human-imatative AI project.[1] Instead he claimed that what we are witnessing is the emergence of a new engineering discipline (that as of yet does not have a good name). Further, he claims that because this new discipline is a uniquely human-centered engineering discipline it should be built upon the foundation of microeconomics.

Q2. Analyze Parkes' and Wellman's *machina economicus* project in terms of the Hayekian knowledge problem. If humans cannot be relied upon to optimally allocate resources in the physical economy, perhaps artificially intelligence machines can?

Q3. Compare and contrast Parkes' and Wellman's vision for AI with Jordan's. What is the right microfoundations for the new engineering discipline that Jordan outlines. What are its necessary elements? What should we call this new engineering discipline?

Q4. Does Hayek's *Knowledge Problem* apply to AI? If so, why? How? If no, why not? Explain!

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[1] See also Jordan [Jor19b]

Part II

Arbitrage, Rationality, and Equilibrium

CHAPTER

TWO

ARBITRAGE IN THE CATALLAXY

2.1 Dutch Books and Arbitrage

Q5. Following Skyrms [Sky12] demonstrate the Dutch book argument for a Tootsie Roll economy with two flavors: chocolate and cherry. What is the relationship between arbitrage in this simple economy and subjective probability? Does this necessarily give us a derivation of Bayes' Rule? Why or why not?

Q6. In an insightful paper Ross [Ros87] discusses the arbitrage approach of neoclassical finance and the supply-and-demand approach of neoclassical economics. Ross distinguishes between the *intuition* of finance and the *theory* of finance? What does he mean by this distinction? Relate his discussion of arbitrage in finance with the Dutch book argument for subjective probability above.

2.2 Operational-Subjective Probability & Statistics

Q7. Nau [Nau01] discusses de Finetti's operational derivation of probability and Nau and McCardle [NM91] outline their theory of *Arbitrage Choice Theory*. Discuss the claim that *Arbitrage Choice Theory* provides an operationally-subjective basis for the catallactic foundation of FinTech and Data Science.

Q8 Poirier [Poi11] discusses de Finetti's representation theorem. The eminent economist David Kreps has called this the *Fundamental Theorem of Statistical Inference*. Discuss.

Part III

Metallgesellschaft and the Economics of Synthetic Storage

CHAPTER

THREE

ABITRAGE, ENTREPRENEURSHIP, AND CREATIVITY

3.1 Futures Markets and Hedging

Q9. In class we stated that the equilibrium price of a forward contract should behave according to the formula $F_0 = S_0 e^{rT}$. Demonstrate that this is true using the arbitrage principle. Assume that you are dealing with an investment asset that pays no dividends unless stated otherwise. Specifically, answer the following:

- Suppose $F_0 > S_0 e^{rT}$. Does this imply an arbitrage opportunity? Explain.
- Now suppose $F_0 < S_0 e^{rT}$. Does this imply an arbitrage opportunity? Explain.
- Now assume that you are dealing with either a dividend-paying asset in the form of discrete dollar dividend payments, or a continuous dividend yield. How does this change the pricing equation? Explain.
- Now assume that you are dealing with a consumption commodity with storage costs, and that holding the physical commodity has certain benefits (i.e. a convenience yield). How does this change the pricing equation? Explain.
- In class we argued that all of this depends upon the possibility of physical delivery of the contract, which guarantees that $F_T = S_T$, that is that the futures price at maturity is equal to the spot price at maturity. First, explain why that is. Second, what would happen if $F_T \neq S_T$? Carefully explain.
- Consider a new kind of forward contract called a *prepaid forward*. It is very similar to a standard forward contract. The only difference is that the long party pays the short party at time $t = 0$, but still takes delivery from the short party at $t = T$. How must the standard pricing formula be modified to properly price the prepaid forward? Carefully explain your logic. Why would anyone ever want to transact in such a contract? Do we actually see prepaid forward contracts in the real world, or only in finance textbooks?

Q10. In the Midterm folder on Canvas there is a subfolder titled Data. In it you will find historical nearby daily settlement futures prices as well as daily New York Harbor spot prices for WTI Crude Oil for the period 01/01/1992 to 12/31/1993. Using these data answer the following questions:

- Test the series for unit roots using the Augmented Dickey-Fuller test. Conduct the tests for prices in levels, and first price differences. Also conduct the test in log-price levels and log-price first differences. What do you conclude? Do any of the series contain a unit root? Which ones? Are the results what you expected, or did they surprise you?
- Make time series plots of the series in price levels, first differences, log-price levels, and log-price first differences.
- Carry out the Engle-Granger two-step procedure to test for cointegration between the series? What are the results? Are the series cointegrated? If so, what is the cointegrating vector? Carefully explain.
- Make a plot of the estimated residuals and comment on the graph as relates to the issue of of cointegration.
- Using linear regression calculate a rolling minimum-variance hedge ratio for each day of 1993 based on 60 prior days observations. Plot a time-series of the hedge-ratios. Comment on the nature of the plotted series. What does this say about the dynamic stability of the hedge ratio?

Q11. Read the paper *Futures markets, Bayesian forecasting and risk modeling* by [QCSC10]. Relate the main idea of the paper to the articles that we have read in this course and the themes that we have developed. There is a technical aspect to this paper. Don't get bogged down in the technical details, instead focus on the main ideas and concepts.

Q12. In this problem you will simulate values for spot prices and basis for heating oil and gasoline following the models of Bollen and Whaley in their paper *Simulating Supply* (which is in the readings file). Specifically, the follow models will be used:

Spot Prices:

- $\ln\left(\frac{S_{i,t}}{S_{i,t-1}}\right) = \alpha_i(\beta_i - S_{i,t-1}) + \varepsilon_{i,t}$ for $i = 1, 2$
- where $i = 1$ for heating oil and $i = 2$ for gasoline, and $(\varepsilon_{1,t}, \varepsilon_{2,t}) \sim BVN(0, \sigma_1^2, 0, \sigma_2^2, \rho)$. And ρ is the correlation coefficient between heating oil and gas spot returns.

Basis and Futures Prices:

- $b_{i,t} = \alpha_i b_{i,t-1} + \beta_i S_{i,t-1} + \varepsilon_{i,t}$ for $i = 1, 2$
- where again, $i = 1$ for heating oil and $i = 2$ for gasoline and $(\varepsilon_{1,t}, \varepsilon_{2,t}) \sim BVN(0, \sigma_1^2, 0, \sigma_2^2, \rho)$. Again, ρ is the correlation between heating oil and gasoline basis.
- You can then obtain simulated futures prices as $F_{i,t} = S_{i,t}e^{b_{i,t}}$ for $i = 1, 2$

With this as background, do the following:

- Simulate 45 days of prices for spot and daily settlement prices for futures (via the basis equation). Looking at figure 1 in the paper use initial values of $\$0.69$ for heating oil and $\$0.80$ for gasoline. Use an initial value for heating oil basis of -0.02 and -0.01 for gasoline. See Table 2 in the paper for the other parameter values.
- Make time series plots labeling the x-axis as date and the y-axis as dollar prices. Assume a starting date of November 15, 1991. Plot spot prices, basis, and futures prices separately but combining the graphs for heating oil and gasoline together for each. Clearly label the series in each graph.
- For the simulated 45-day period calculate the market-to-market cash flows. Assume and initial margin of 10% and variation margin of 85% of the initial margin. Assume a position of 1 contract for each position.
- For the simulated 45-day period calculate a minimum-variance hedge ratio for both heating oil and unleaded gasoline.
- Do the simulated pairs of futures and spot prices appear to be cointegrated? Why or why not? What does the Bollen-Whaley model used for simulation suggest about cointegration?

3.2 MGRMs Hedging Revisited

Q13. In class we discussed the case of oil hedging by Metalgesellschaft Refining and Marketing (MGRM). In their paper *Metalgesellschaft and the Economics of Synthetic Storage* Culp and Miller [CM95b] defended MGRM's hedging strategy. Pirrong [Pir97] strongly criticized Culp and Miller's findings in his paper *Metalgesellschaft: Prudent Hedger Ruined, or a Wildcatter on NYMEX?*. Please answer the following questions about the debate over MGRM's hedging strategy. In your answers to the following you may also wish to read Culp and Miller [CM95a].

- Culp & Miller called MGRM's main hedging strategy *Synthetic Storage*. Please outline the basics of this strategy, and explain how it differs from risk minimizing hedging. In your explanation, you may want to reference the ideas of the economist Holbrook Working cited by Culp & Miller regarding hedging strategies and the motives for hedging.
- One of the main points of disagreement between Culp & Miller and their critics, especially Pirrong, is over the proper hedge ratio for MGRM's hedging strategy. What do you think the correct hedge ratio was for MGRM? Can you outline an empirical strategy for estimating the proper hedge ratio? Explain.

- At the end of his paper, Pirrong states *"Given the huge losses incurred in late 1993, a Bayesian estimating the probability distribution of MG's information advantage would almost certainly place little weight on the possibility that the firm was well informed, and great weight on the possibility that it did not possess superior information, regardless of the charitability of his priors concerning the prescience of MG's managers."* Why do you think Pirrong invokes Bayes' Rule? Comment on this statement by Pirrong (*Hint:* Don't forget what Dale Poirier has taught you!).
- Relate your answer in part (c) to the concept of the predictive distribution. What is the importance of the predictive distribution in this hedging application?
- Bollen and Whaley [BW98] run simulations to test MGRM's strategy out-of-sample. What is their conclusion? Compare their empirical strategy to Pirrong's. How do the two approaches differ? Do Bollen & Whaley provide a suffiently Bayesian answer to Pirrong? Interpret their results through the lens of Poirier [Poi11] if possible. What is the role of the loss function in proper data analysis? Use the isoelastic utility function to frame your discussion.
- Energy markets such as the oil market are often historically characterized by backwardation, but have also had long periods of contango. Does that matter for MGRM's synthetic storage strategy? Explain.

Part IV

Tutorials

CHAPTER

FOUR

APPENDIX A: NOTES ON CONTINUOUS COMPOUNDING

Based on Appendix B from McDonald

```
import numpy as np
```

4.1 The Language of Interest Rates

We will begin with some defintions:

- **Effective Annual Rate:** If r is quoted as an ***effective annual rate***, this means that if you invest $\$1$, n years later you will have $(1+r)^n$. If you invest x_0 and earn x_n n years later, then the implied effective annual rate is $(x_n/x_0)^{1/n}-1$.
- **Continuously Compounded Rate:** If r is quoted as an annualized ***continously compounded rate***, this means that if you invest $\$1$, n years later you will have e^{rn}. If you invest x_0 and earn x_n n years later, then the implied annual continuously compounded rate is $\ln{(x_n/x_0)}/n$.

```
r = 0.025
n = 2
(1 + r)**n
```

```
1.050625
```

4.2 The Logarithmic and Exponential Functions

Interest are typicall quoted as "$r\%$ per year, compounded n time every year." From your introductory finance classes

$$\left(1+\frac{r}{n}\right)^n$$

In T years you will have

$$\left(1+\frac{r}{n}\right)^{nT}$$

…

- $(\$1+0.01)^3=\$1.331$ with annual compounding.

```
(1. + 0.1)**3
```

```
1.3310000000000004
```

NB: Your attemp goes here in the middle.

4.3 Some Just-in-Time Mathematics

In this section we will look at the $\ln(\cdot)$ and $\exp(\cdot)$ functions.

We will see that these are put to use in calculating log returns and continuously compounded cashflows.

4.3.1 Simple Returns

As a starting point, let's take a look at the very familiar ***simple returns***

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

$$= \frac{P_t}{P_{t-1}} - 1$$

NB: the quantity $\frac{P_t}{P_{t-1}}$ is called the *price relative quotient*.

It is equal to the gross return:

$$1 + R_t = \frac{P_t}{P_{t-1}}$$

4.3.2 Log Returns

- From Hayek we know that it is prices that we mostly care about, because it is prices that carry and convey information/knowledge.
- Prices, however, present statistical challenges that are difficult to ovecome.
- Namely, the issue of *nonstationarity* (which we will define later)

$$r_t = \log(1 + R_t) = \frac{P_t}{P_{t-1}}$$

$$= \log(P_t) - \log(P_{t-1})$$

NB: the last statement is about the differences in log prices. These are continuously compounded returns. This relies upon the idea of the *infinitesimal increment in time*.

```
# Set up some arbitrary prices to demonstrate
# the calculations
p1 = 100.
p2 = 105.
```

```
prq = p2/p1
prq
```

```
1.05
```

```
R = (p2 - p1)/p1
R
```

```
0.05
```

```
np.log(1 + R)
```

```
0.04879016416943205
```

```
r = np.log(p2) - np.log(p1)
r
```

```
0.04879016416943127
```

```
p2 * np.exp(r)
p2
```

```
105.0
```

Notes on the Natural Exp and Log Functions

$$\log_e(\cdot) = \ln(\cdot)$$

and

$$\exp(\cdot) = e^{(\cdot)}$$

NB: From Leonhard Euler (1700's)

$$e \equiv lim_{s \to \infty} \left(1 + \frac{1}{s}\right)^s \approx 2.71828$$

```
# Define a function to represent (1 + 1/s)^s
def f(s):
    return (1 + 1/s)**s
```

```
f(10.)
```

```
2.5937424601000023
```

```
# Argument values increase by powers of 10
inputs = [1000., 10000., 100000., 1000000., 10000000.,100000000.]

# The function values converge on 2.71828 as we
# increase the argument value by powers of 10
for s in inputs:
    print(f(s))
```

```
2.7169239322355936
2.7181459268249255
2.7182682371922975
2.7182804690957534
2.7182816941320818
2.7182817983473577
```

Let m be the compounding period and r_t the return.

$$P_t = P_{t-1}\left(1 + \frac{r_t}{m}\right)^m \qquad \text{discrete}$$

$$P_t = P_{t-1} lim_{m\to\infty}\left(1 + \frac{r_t}{m}\right)^m$$

Now let $s = \frac{m}{r_t}$, and we obtain the following:

$$P_t = P_{t-1}\lim_{s\to\infty}\left[\left(1 + \frac{1}{s}\right)^{sr_t}\right]$$

$$= P_{t-}\left[\lim_{s\to\infty}\left(1 + \frac{1}{s}\right)^{s}\right]^{r_t}$$

$$= P_{t-1}e^{r_t}$$

This gives us continuous compounding between period $t-1$ and t at the rate r_t. This is where the $\exp(\cdot)$ function comes into continuous compounding. You will see it all throughout the course!

The $\ln(\cdot)$ and $\exp(\cdot)$ functions are mutual inverses, so that:

$$e^{r_t} = \frac{P_t}{P_{t-1}}$$

In other words:

$$\ln\left(e^{r_t}\right) = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

$$r_t = \ln\left(P_t\right) - \ln\left(P_{t-1}\right)$$

So we've come full circle from the statement: $P_t = P_{t-1}e^{r_t}$.

CHAPTER

FIVE

APPENDIX B: COMPUTING MINIMUM-VARIANCE HEDGE RATIOS

5.1 Calculating Sample Statistics for Hedge Ratios

The formula for the optimal static hedge ratio is given in Chapter 3 as:

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

In practice this will require statistical inference from historical data to **estimate** the values of the paramters: ρ, σ_S, and σ_F.

We will look at how this is to be done.

Recall the formula for the sample standard deviation as the estimator for the standard deviation:

$$\hat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \bar{x})^2}$$

where n is the sample size, the x_i are the particular historical values of each of the data points for $i = 1, \ldots, n$.

Recall also that the estimator for the sample mean is given by:

$$\bar{x} = \frac{1}{n}\sum_{i=t}^{n} x_t$$

Below I have reproduced the data in Table 3.2 from the book. We will use these data to calculate the optimal minimum variance hedge ratio.

Lets first compute the sample means for each of the change in futures prices and the change in jet fuel prices.

The mean for the change in heating oil futures prices is:

$$\bar{\Delta F} = \frac{1}{15}(0.021 + 0.035 - 0.046 + \ldots + 0.029) = -0.000867$$

By the way, this can also be easily done in `Python` using the `Numpy` module as:

```
import numpy as np
from scipy import stats
```

```
delta_f = np.array([0.021, 0.035, -0.046, 0.001, 0.044, -0.029, -0.026, -0.029,
                    0.048, -0.006, -0.036, -0.011, 0.019,-0.027, 0.029])
```

```
delta_f
```

```
array([ 0.021,  0.035, -0.046,  0.001,  0.044, -0.029, -0.026, -0.029,
        0.048, -0.006, -0.036, -0.011,  0.019, -0.027,  0.029])
```

```
delta_f.shape
```

```
(15,)
```

```
fbar = delta_f.mean()
```

```
print("The mean of heating oil futures {:f}".format(fbar))
```

```
The mean of heating oil futures -0.000867
```

```
np.mean(delta_f)
```

```
-0.0008666666666666659
```

Also, the sample standard deviation is:

$$\hat{\sigma}_{\Delta F} = \sqrt{\frac{1}{14}[(0.021 + 0.000867)^2 + (0.035 + 0.000867)^2 + ... + (0.029 + 0.000867)^2]} = 0.0313$$

Here is how to do it in `Python`:

```
sigma_f = delta_f.std(ddof=1) # note: use ddof = 1 to get the unbiased estimator!
```

```
print("The standard deviation of heating oil futures is {:f}".format(sigma_f))
```

```
The standard deviation of heating oil futures is 0.031343
```

The sample mean for jet fuel spot prices is:

$$\bar{\Delta S} = \frac{1}{15}(0.029 + 0.020 - 0.044 + ... + 0.023) = 0.0002$$

Here it is in `Python`:

```
delta_s = np.array([0.029, 0.020, -0.044, 0.008, 0.026, -0.019, -0.010, -0.007,
                    0.043, 0.011, -0.036, -0.018, 0.009, -0.032, 0.023])
```

```
sbar = delta_s.mean()
print("The mean of jet fuel spot prices is: {:f}".format(sbar))
```

```
The mean of jet fuel spot prices is: 0.000200
```

Here is the sample standard deviation for jet fuel spot prices:

$$\hat{\sigma}_{\Delta S} = \sqrt{\frac{1}{14}[(0.029 - 0.0002)^2 + (0.020 - 0.0002)^2 + ... + (0.023 - 0.0002)^2]} = 0.0263$$

Here is how to do it in `Python`:

```
sigma_s = delta_s.std(ddof=1)
print("The sample standard deviation of jet fuel prices is: {:f}".format(sigma_s))
```

```
The sample standard deviation of jet fuel prices is: 0.026255
```

5.2 The Sample Correlation Coefficient

We also need to be able to estimate ρ in order to calculate the hedge ratio. Let's recall that the formula for the sample (estimator) correlation coefficient is equal to:

$$\hat{\rho}_{\Delta S,\Delta F} = \frac{\hat{Cov}(\Delta S, \Delta F)}{\hat{\sigma}_{\Delta S} \times \hat{\sigma}_{\Delta F}}$$

So in order to calculate this we need to know how to calculate the sample covariance $\hat{Cov}(\Delta S, \Delta F)$. That formula is:

$$\hat{Cov}(\Delta S, \Delta F) = \hat{\Sigma}_{\Delta S,\Delta F} = \frac{1}{n-1}\sum_{i=1}^{n}(\Delta S_i - \bar{\Delta S})(\Delta F_i - \bar{\Delta F})$$

```
rho = stats.pearsonr(delta_s, delta_f)[0]
```

```
h_star = rho * (sigma_s / sigma_f)
print(f"The estimated optimal hedge ratio is: {h_star : 0.4f}")
```

```
The estimated optimal hedge ratio is:  0.7777
```

We can also estimate this value using the estimated slope coefficient in a linear regression.

Recall that the OLS slope coefficient estimator is equal to the following:

$$\begin{aligned}\hat{\beta} &= \frac{Cov(y, x)}{Var(x)} \\ &= \rho\frac{\sigma_y \sigma_x}{\sigma_x \sigma_x} \\ &= \rho\frac{\sigma_y}{\sigma_x}\end{aligned}$$

```
slope, intercept, r_value, p_value, std_err = stats.linregress(delta_f, delta_s)
```

```
print(f"The estimated minimum-variance hedge ratio (via OLS) is: {slope : 0.4f}")
```

```
The estimated minimum-variance hedge ratio (via OLS) is:  0.7777
```

CHAPTER

SIX

APPENDIX C: UNIT ROOTS, COINTEGRATION, AND ERROR-CORRECTION

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

6.1 Unit Roots and Stationarity

A simple starting model for efficient log-prices of assets is the ***Random Walk with Drift*** model:

$$y_t = \mu + y_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2)$$

with something like $y_t = \ln(p_t)$ with p_t a transaction price observed in some market. The expected value of this process is:

$$E(y_t) = \mu + y_{t-1}$$

To get the variance it is helpful to solve recursively as follows, assuming $y_0 = 0$ for simplicity:

$$y_t = t\mu + \sum_{i=0}^{t} \epsilon_{t-i}$$

We can now state the variance of the process as:

$$Var(y_t) = \sum_{i=0}^{t} Var(\epsilon_{t-i}) = Var(\epsilon_t) + Var(\epsilon_{t-1}) + \cdots + Var(\epsilon_0) = t\sigma_\epsilon^2$$

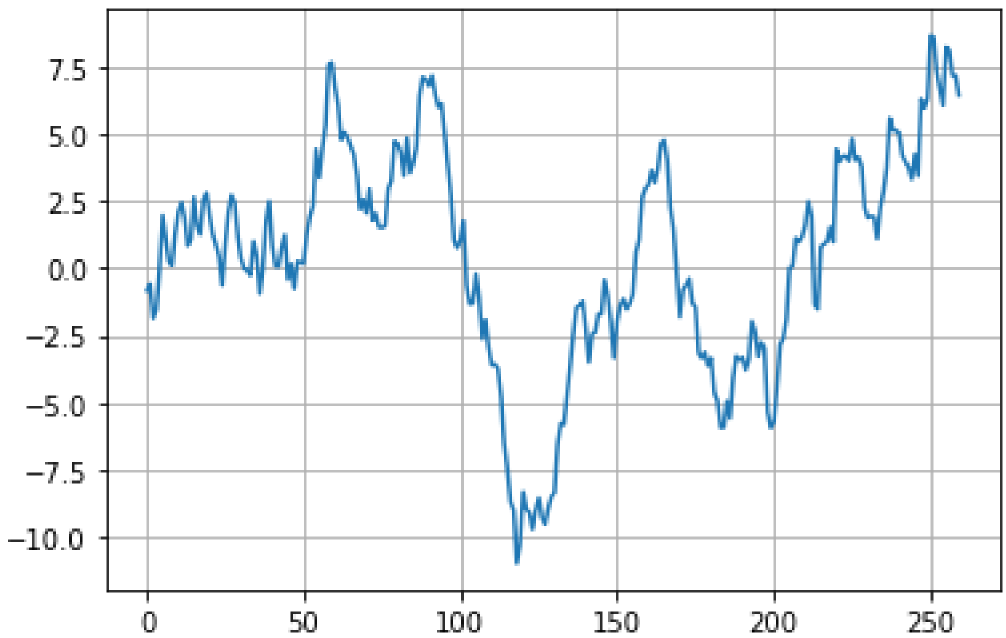
From this it is easy to see that this is an explosive process as the variance is proportional to time. This translates to the value of the process at any time t being unpredictable based on the information known up to that time (I_{t-1}). This is a good starting place for a model of informationally efficient prices as the definition of such is one that incorporates all available information up to that point in time.

NB: Samuelson's paper: Proof That Properly Anticipated Prices Fluctuate Randomly

We can simulate this process as follows (setting $\mu = 0$ for convenience):

```
y = pd.Series(np.cumsum(np.random.normal(size=52*5)))
y.plot(grid=True)
```

```
<AxesSubplot:>
```



6.1.1 Weak Stationarity

In the time series literature a process is known as weakly stationary if the mean and autocovariance are not time varying.

A few notes:

- Clearly the random walk process is **NOT** weakly stationary.
- A weakly stationary time series process will exhibit mean reversion
- Mean reversion in a time series can be such that there is some predictibility in the process

It makes sense that informationally efficient prices should behave as a random walk model (with possible other extensions).

The random walk model is a special case of the AR(1) model (y_t is now thought of as an arbitrary random variable):

$$y_t = \phi y_{t-1} + \epsilon_t$$

It won't be shown here, but a technical requirement for the AR(1) model to be weakly stationary is $|\phi| < 1$. For the random walk model $\phi = 1$, thus the alternative name ***unit root***.

6.1.2 Some Notation

A random walk model is also known as a unit-root non-stationary process. In the literature this is often denoted as $y_t \sim I(1)$. We state this as: "*the process y_t is **integrated of order one***."

We can tranform and $I(1)$ process to a stationary process by ***first differencing*** the process like so:

$$\begin{aligned} y_t - y_{t-1} &= y_{t-1} - y_{t-1} + \epsilon_t \\ y_t - y_{t-1} &= \epsilon_t \\ \Delta y_t &= \epsilon_t \\ r_t &= \epsilon_t \end{aligned}$$

In this case we can denote that Δy_t is now weakly stationary with the notation $\Delta y_t \sim I(0)$, i.e. "*Δy_t is **integrated of order zero.***"

6.2 Spurious Regression

It is important to understand the properties of unit root processes, because they can be problematic to work with in applying econometrics to finance.

For example, there is a well-known problem of ***spurious regression*** when one unit root process is regressed on an independent unit root process:

$$y_t = \alpha + \beta x_t + u_t, \quad u_t \sim N(0, \sigma_u^2)$$

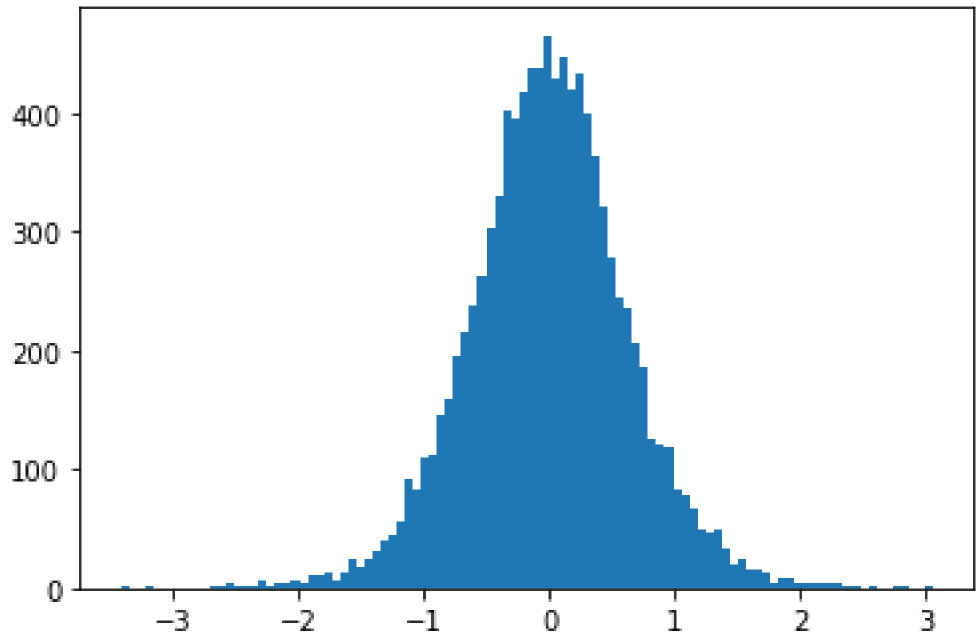
This regression is not valid because the homoscedasticity assumption of the error term is violated (recall that $Var(y_t) = t\sigma_\epsilon^2$)

This can be easily demonstrated by a simple Monte Carlo study as follows:

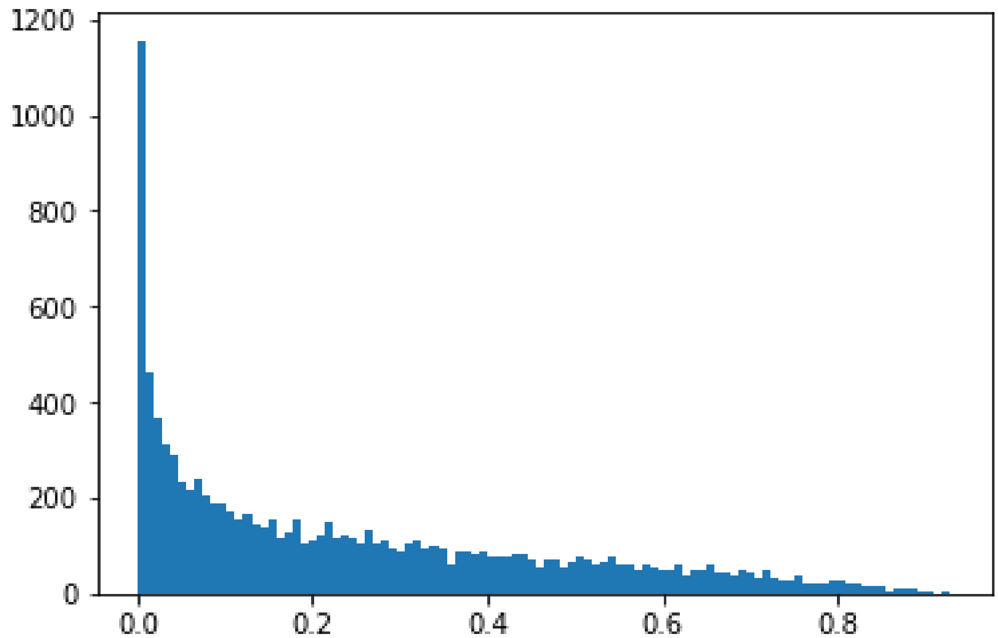
```
from scipy import stats
```

```
M = 10000
N = 52 * 5
betaHat = np.empty(M)
Rsqrd = np.empty(M)

for i in range(M):
    y = np.cumsum(np.random.normal(size=N))
    x = np.cumsum(np.random.normal(size=N))
    reg = stats.linregress(x,y)
    betaHat[i] = reg.slope
    Rsqrd[i] = reg.rvalue ** 2
```

```
plt.hist(betaHat, bins=100);
```




```
plt.hist(Rsqrd, bins=100);
```



While the central tendency of sampling distribution appears to be zero, the distribution is extremely wide, so while we will fail to reject the null hypothesis on average we will fail to do so far too often.

We can also see from histogram of the R^2 that there are some extremely high values even though we know that the processes are independent!

6.3 The Dickey-Fuller Test for Unit Roots

We can test for unit roots in a time series process with the so-called ***Dickey-Fuller Test***, named for the statisticians who invented it.

It would seem natural to test the following hypothesis:

$$H_0 : \phi = 1$$
$$H_a : \phi \neq 1$$

for the regression: $y_t = \phi y_{t-1} + \epsilon_t$, but because the model under the null hypothesis leads to spurious regression we cannot conduct this direct test.

D-F had the bright idea to transform the model to render it amenable to such testing. We start by subtracting y_{t-1} from both sides:

$$\begin{aligned} y_t - y_{t-1} &= \phi y_{t-1} - y_{t-1} + \epsilon_t \\ \Delta y_t &= (\phi - 1) y_{t-1} + \epsilon_t \\ \Delta y_t &= \theta y_{t-1} + \epsilon_t \end{aligned}$$

where $\theta = \phi - 1$. With this transformation we can now conduct the test:

$$H_0 : \theta = 0$$
$$H_a : \theta \neq 0$$

Because $\Delta y_t \sim I(0)$ and when $\phi = 1$ it means that $\theta = 0$ this model is now valid under the null hypothesis. We can form the standard t-ratio as our test statistic, but D-F showed that the asymptotic sampling distribution of t is no longer the Standard Normal distribution. Instead they provide critical values via a Monte Carlo method.

6.3.1 The Augmented Dickey-Fuller Test

D-F added one extension to the test to account for possible serial correlation in Δy_t. The model under the null hypothesis now becomes:

$$\Delta y_t = \theta y_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta y_{t-i} + \epsilon_t$$

This tends to make the test more robust to short-term serial correlations in the process.

We can use the Statsmodels model to conduct the ADF test as follows. See Statsmodels documentation on adfuller for more details.

```
import statsmodels.api as sm

N = 52 * 5
y = np.cumsum(np.random.normal(size=N))
results = sm.tsa.stattools.adfuller(y)
print(f"The value of the ADF statistic is {results[0]:0.4f}, with a p-value of:
↪{results[1]: 0.4f}")
```

```
The value of the ADF statistic is -1.9319, with a p-value of:  0.3172
```

```
results
```

```
(-1.9318612942578632,
 0.31722414597504,
 0,
 259,
 {'1%': -3.4558530692911504,
  '5%': -2.872764881778665,
  '10%': -2.572751643088207},
 678.1857534946926)
```

Here we can see that we fail to reject the null hypothesis of the presence of a unit-root in y_t (which is not surprising since we simulated it as a random walk).

6.3.2 Cointegration

Fortunately, there is an upside to unit-root non-stationarity for financial modeling. It turns out that there is a relationship that is even stronger when pairs of asset prices are $I(1)$, but move together in a way. This concept is ***cointegration***.

If there is a linear combination of two processes that are separately $I(1)$ that is itself $I(0)$, we say that the two processes are ***cointegrated***. Cointegration is a stronger concept than mere correlation. It has a causal explanation. I often like to say that (at least in financial applications) ***cointegration is the statistical footprint of an arbitrage relationship.***

We can test for cointegration using the ADF test developed above. To begin, set up and run the following regression:

$$y_t = \alpha + \beta x_t + u_t$$

If the variables are cointegrated this is not a spurious regression. In fact, it has the property of superconsistency (a kind of uber statistical efficiency).

Once the model is estimated, we can form the fitted residuals:

$$\hat{u}_t = y_t - \hat{\alpha} - \hat{\beta} x_t$$
$$\hat{u}_t = y_t - \hat{y}_t$$

We can now submit these fitted residuals to the ADF test as above.

NB: the null hypothesis of the ADF test is that there ***is*** a unit-root. Cointegration exists between y_t and x_t if there ***is not*** a unit-root in $\hat{u}_t$. So we conclude that there is cointegration if reject the null hypothesis of the ADF test.

We can simulate this as follows.

```
N = 52 * 5
x = np.cumsum(np.random.normal(size=N))
u = np.random.normal(size=N, loc=0.0, scale=2.0)
y = 0.22 + 2.45 * x + u

df = pd.DataFrame(dict(y=pd.Series(y), x=pd.Series(x)))
```

```
df.head()
```

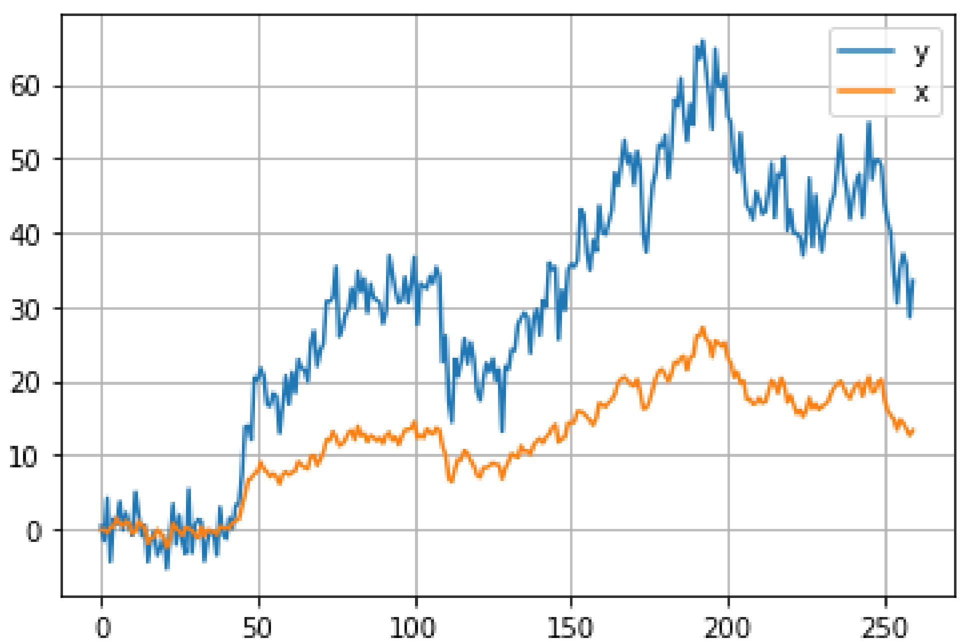
```
          y         x
0  0.424795 -0.103200
1 -1.586530 -0.009573
2  4.220291 -0.492609
3 -4.377467  0.211619
4  1.395756  0.433135
```

```
df.tail()
```

```
             y          x
255  35.267996  14.798878
256  37.143072  14.269424
257  35.574052  13.527909
258  28.618259  12.656532
259  33.511301  13.339086
```

```
df.plot(grid=True)
```

```
<AxesSubplot:>
```



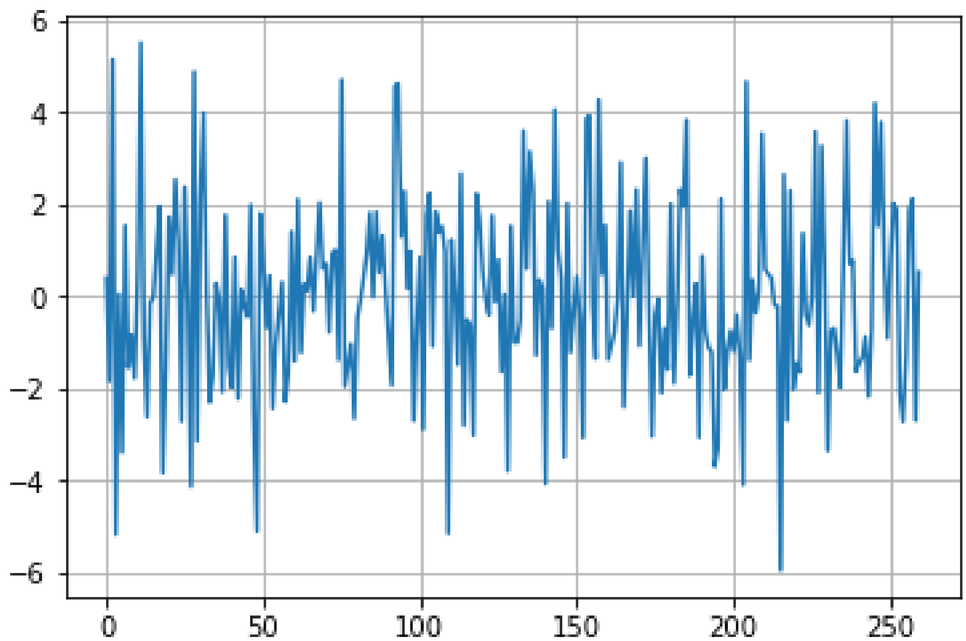
Now let's run the regression and form the residuals:

```
reg = stats.linregress(x,y)
uhat = y - reg.intercept - reg.slope * x
```

```
resids = pd.Series(uhat)
```

```
resids.plot(grid=True)
```

```
<AxesSubplot:>
```



The residuals clearly look to be mean-reverting and stationary. Let's see the results of the ADF test:

```
results = sm.tsa.stattools.adfuller(uhat)
```

```
results
```

```
(-18.177750990319065,
 2.4375466077798407e-30,
 0,
 259,
 {'1%': -3.4558530692911504,
  '5%': -2.872764881778665,
  '10%': -2.572751643088207},
 1036.7134263348507)
```

```
print(f"The ADF test statistic is: {results[0]: 0.4f}, with a p-value of: {results[1]: 0.4f}")
```

```
The ADF test statistic is: -18.1778, with a p-value of:  0.0000
```

That's an awefully small p-value, so we reject the null hypothesis of the ADF test of a unit-root and conclude that y_t and x_t are cointegrated. Again, we're not surprised since we engineered it.

6.4 Error-Correction Models

Whenever two (or more) asset prices are cointegrated, we can also write down an error-correction model. That is, cointegration implies an error-correction form.

We state the simplest form of the error-correction model as follows:

$$\Delta y_t = \lambda(y_{t-1} - \alpha - \beta x_{t-1}) + \nu_t$$
$$\Delta y_t = \lambda(z_{t-1}) + \nu_t$$

with $z_t = \hat{u}_t$.

This model form relates the changes in y_t, that is Δy_t to the ***spread*** between y_{t-1} and x_{t-1} in ***levels***. This is a valid time series regression, given that $\Delta y_t \sim I(0)$ via first differencing, and $z_{t-1} \sim I(0)$ via cointegration.

Let's see if we can develop some intuition for this model. Let's start by interpreting the coefficient λ, which we call the ***error-correction coefficient***. Its value will be such that when there is a large past deviation between y_{t-1} and x_{t-1} (i.e. a large error) it will cause an ***error correction*** in the change in y_t, or Δy_t. In other words, Δy_t will adjust based on a lagged error in the spread. There is now a stationary relationship (i.e. mean-reverting) that exists, and can even be predicted. It's easy to see now why we call this an error-correction model, and also its relationship with cointegration.

- **Q:** What causes the error-correction in Δy_t?
- **A:** in financial markets between related asset prices that are cointegrated, the answer is ***arbitrage***!

We can now think about the error-correction model in terms of some kind of equilibrium concept. When dynamic market forces are such that related asset prices are temporarily driven apart, an arbitrage relationship between the asset prices acts to restore the spread between the two to a long-run equilibrium level.

- **Q:** what kind of equilibrium concept fits this description?
- **Q:** is it a static neo-classical equilibrium?
- **Q:** is it more like the neo-Austrian type of equilibrium that has been mentioned in this class?

6.4.1 A More General Error-Correction Model

We can also account for possible short-run variation in Δy_t by adding lagged terms on the right-hand side of the model as follows (as well as a drift term):

$$\Delta y_t = \mu + \sum_{j=1}^{p} \delta_j \Delta y_{t-j} + \lambda z_{t-1} + \nu_t$$

6.4.2 A Vector Error-Correction Model

Now we can think in terms of systems of equations, and think about a multivariate relationship between y_t and x_t called a ***vector error-correction model*** (vecm).

Here is a VECM(1) model in y_t and x_t:

$$\Delta y_t = \mu + \delta_1 \Delta y_{t-1} + \gamma_1 \Delta x_{t-1} + \lambda_1 z_{t-1} + \nu_{1,t}$$
$$\Delta x_t = \mu + \delta_2 \Delta y_{t-1} + \gamma_2 \Delta x_{t-1} + \lambda_2 z_{t-1} + \nu_{2,t}$$

CHAPTER

SEVEN

APPENDIX D: A QUICK REVIEW OF PREDICTIVE DISTRIBUTIONS

Typically, the concept of a **predictive distribution** is associated with Bayesian statistical methods. In a Bayesian model the answer to any problem of statistical inference comes in the form of an entire distribution of outcomes rather than simply a *point estimate* (a single numerical value). This is very nature for Bayesians because the only "estimator" in the Bayesian world is Bayes' Rule:

$$p(\theta|x) \propto p(x|\theta)p(\theta)$$

$p(\theta|x)$ is called the *posterior distribution* and is obtained by taking the product of the *likelihood function* (i.e. $p(x|\theta)$) and the *prior distribution* (i.e. $p(\theta)$).

$p(\theta|x)$ is a probability density function (or many thousands of draws from one as a proxy). It represents infinitely many possible outcomes and their associated probability density. But the random variable(s) being modeled here is θ (possibly a vector).

In finance, we are not primarily interested in any particular θ's but rather with future, but *uncertain* y values:

- What will the value of the investment portfolio be at retirement?
- What will the value of the spot price be at delivery?
- What will the value of the underlying be at maturity of the option contract? I.e. what will be the payoff of the option at expiry?
- What will the settlement price of the futures contract?

And on and on. In finance we care about forward-looking outcomes of random variables.

Bayesian methods aid this objective by providing a *predictive distribution* in the following way:

$$p(x^*|D) \propto \int\limits_{\Theta} p(x|\theta)p(\theta|D)d\theta$$

where x^* represents future values of x as of yet unobserved, but anticipated and $D = \{x_1, x_2, \dots, x_N\}$ (i.e. the observed data).

This is typically done by Monte Carlo simulation in any model beyond moderate complexity.

This approach can be approximated for a *diffuse* prior distribution $p(\theta)$ with classical statistical tools and Monte Carlo simulation. This is what Bollen and Whaley do with their model of synthetic energy supply estimated via maximum likelihood and implemented with Monte Carlo simulation.

Consider the *simulating supply* log-spot price model from Bollen & Whaley:

$$\ln\left(\frac{S_{i,t}}{S_{i,t-1}}\right) = \alpha_i(\beta_i - S_{i,t-1}) + \varepsilon_{i,t}$$

Where $\varepsilon_1, \varepsilon_2 \sim BVN(0, \sigma_1^2, 0, \sigma_2^2, \rho)$. For convenience we can rewrite this as:

$$\ln(S_{i,t}) = \ln(S_{i,t-1}) + \alpha_i(\beta_i - S_{i,t-1}) + \varepsilon_{i,t}$$

Let's implement this in a function that can return many thousand simulated paths as follows:

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = [20, 10]

from numpy import size, log, exp, pi, sum, diff, array, zeros, diag, mat, asarray,␣
↪sqrt, copy
from numpy.linalg import inv
```

```
def simulated_supply(params, numObs=252, numReps=10000):
    a1 = params[0]
    b1 = params[1]
    s1 = params[2]
    S1 = params[3]

    paths = np.empty((numReps, numObs))

    for i in range(numReps):
        paths[i,0] = np.log(S1)
        z = np.random.normal(size=numObs)

        for j in range(1, numObs):
            paths[i,j] = paths[i,j-1] + a1 * (b1 - exp(paths[i,j-1])) + z[j] * s1

    return paths
```

```
a1 = 0.342
b1 = 0.539
s1 = 0.11
S1 = 0.69
numObs = 45
numReps = 10
params = array([a1, b1, s1, S1])

paths = simulated_supply(params, numObs, numReps)
```

```
## let's look at the last column (i.e. simulated terminal prices)
paths[:,-1]
```

```
array([-0.64392195, -0.36709185, -0.66192076, -0.55071768, -0.74635585,
       -0.919873  , -0.55147313, -0.66937168, -0.71615994, -0.18680618])
```

```
## let's look at dollar prices
np.exp(paths[:,-1])
```

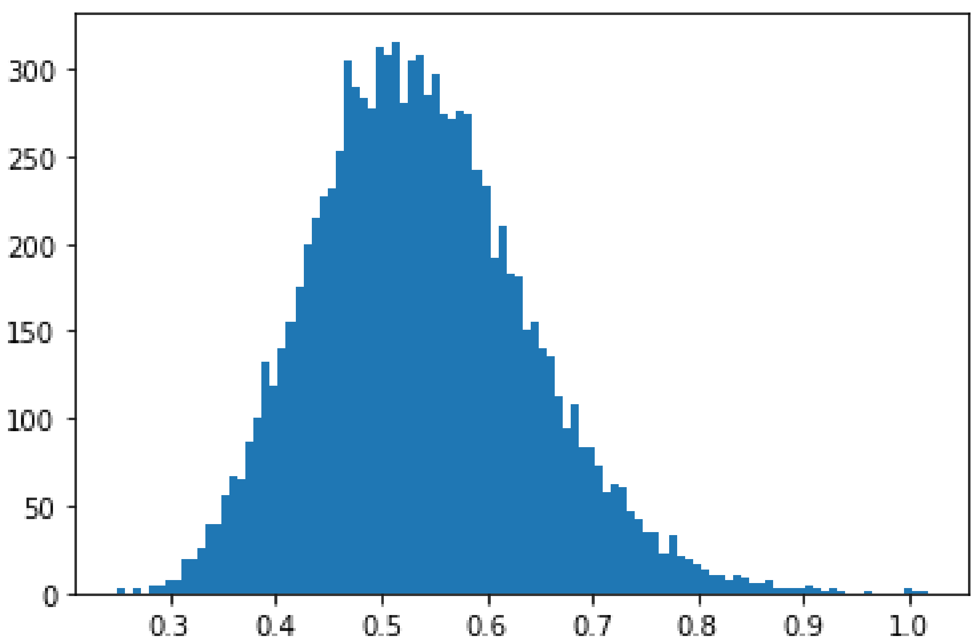
```
array([0.52522846, 0.69274602, 0.51585954, 0.57653589, 0.47409107,
       0.39856966, 0.57610051, 0.5120302 , 0.48862501, 0.82960451])
```

```
## now let's do many thousand reps
paths = simulated_supply(params)
```

```
## let's look at the first 10 rows of the last column
np.exp(paths[:,:10])
```

```
array([[0.69      , 0.75635433, 0.71487835, ..., 0.55554817, 0.51543837,
        0.49688827],
       [0.69      , 0.62698267, 0.61239374, ..., 0.71371029, 0.7799327 ,
        0.75206533],
       [0.69      , 0.55459105, 0.61091835, ..., 0.44547325, 0.4356088 ,
        0.498973  ],
       ...,
       [0.69      , 0.57105493, 0.45167625, ..., 0.49632953, 0.47276177,
        0.47770396],
       [0.69      , 0.70097637, 0.5955739 , ..., 0.66935661, 0.65919282,
        0.59748901],
       [0.69      , 0.66576024, 0.79349134, ..., 0.60144765, 0.47689664,
        0.46998021]])
```

```
## let's plot a histogram
plt.hist(np.exp(paths[:,-1]), bins=100);
```



```
spot_t = np.exp(paths[:,-1])
```

```
spot_t.mean()
```

```
0.5381264222726476
```

```
spot_t.std()
```

```
0.10167706265346829
```

CHAPTER

EIGHT

APPENDIX E: SIMULATING SUPPLY

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

8.1 Simulating the Spot Prices for Heating Oil and Gasoline

To simulate a few values from the spot price equation, which is stated as follows:

$$\ln\left(\frac{S_{i,t}}{S_{i,t-1}}\right) = \alpha_i(\beta_i - S_{i,t-1}) + \varepsilon_{i,t}$$

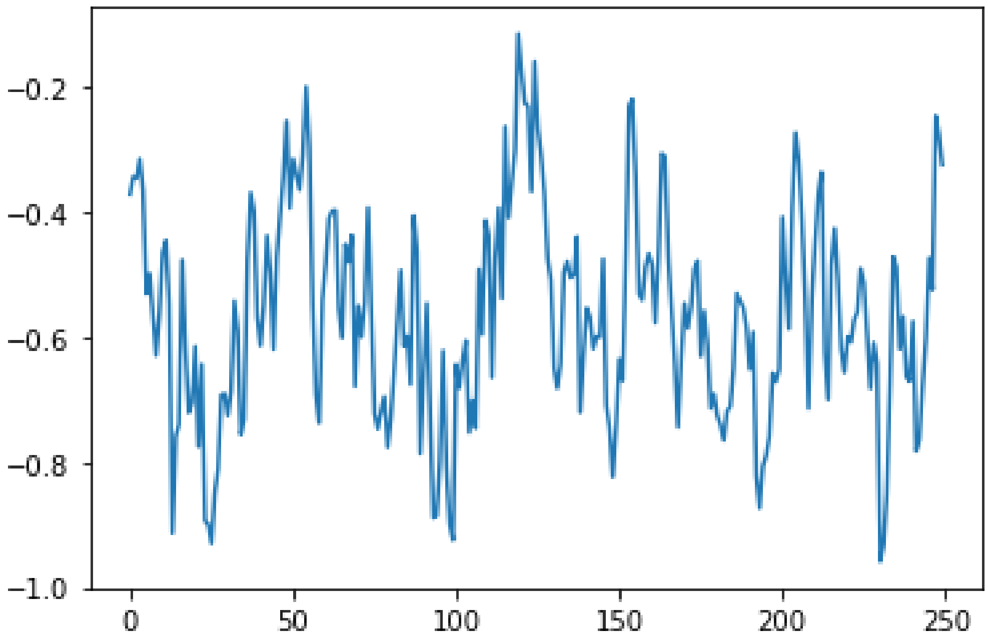
Where $\varepsilon_1, \varepsilon_2 \sim BVN(0, \sigma_1^2, 0, \sigma_2^2, \rho)$. For convenience we can rewrite this as:

$$\ln\left(S_{i,t}\right) = \ln\left(S_{i,t-1}\right) + \alpha_i(\beta_i - S_{i,t-1}) + \varepsilon_{i,t}$$

```
a1 = 0.342
b1 = 0.539
s1 = 0.11
S1 = 0.69
num_reps = 250

ln_spot1 = np.zeros(num_reps)
ln_spot1[0] = np.log(S1)

z1 = np.random.normal(size=num_reps)

for t in range(1, num_reps):
    ln_spot1[t] = ln_spot1[t-1] + a1 * (b1 - np.exp(ln_spot1[t-1])) + z1[t] * s1
```

```
plt.plot(ln_spot1)
```

```
[<matplotlib.lines.Line2D at 0x7fe228bd9820>]
```



8.2 Simulating Correlated Normals

Since the disturbance terms are distributed jointly Normal, we need a way to draw from the BVN distribution. It turns out this is pretty easy.

Here are the necessary steps to draw two correlated normal random variables.

1. Draw z_1 independently from $N(0, \sigma_{z_1})$
2. Next draw z_{tmp} from a standard normal $N(0, 1)$
3. Create z_2 that is correlated with z_1 according to the correlation coeficient ρ

We can use the following equation to accomplish the third step:

$$z_2 = z_1 * \rho + \sqrt{(1 - \rho^2)} * z_{tmp} * \sigma_2$$

We can do this in Python as follows:

```
s1 = 0.11      ## std of heating oil
s2 = 0.116     ## std of gasoline
rho = 0.705    ## the correlation coef between HO and Gas
num_reps = 10000000

z1 = np.random.normal(size=num_reps) * s1
ztmp = np.random.normal(size=num_reps)
z2 = z1 * rho + np.sqrt((1 - rho**2)) * ztmp * s2
```

```
np.corrcoef(z1,z2)
```

```
array([[1.        , 0.68581256],
       [0.68581256, 1.        ]])
```

I prefer to create a function to handle it:

```
def draw_correlated_normals(mn1 = 0.0, sd1 = 1.0, mn2 = 0.0, sd2 = 1.0, rho = 0.5,␣
↪num_reps = 100):
    z1 = np.random.normal(size=num_reps, loc=mn1, scale=sd1)
    z2 = np.random.normal(size=num_reps, loc=mn2, scale=sd2)
    z2 = rho * z1 + np.sqrt((1.0 - rho**2.0)) * z2

    return (z1,z2)
```

```
x1, x2 = draw_correlated_normals(sd1=0.11, sd2=0.116, rho=0.705, num_reps=45000)
np.corrcoef(x1, x2)
```

```
array([[1.        , 0.68300394],
       [0.68300394, 1.        ]])
```

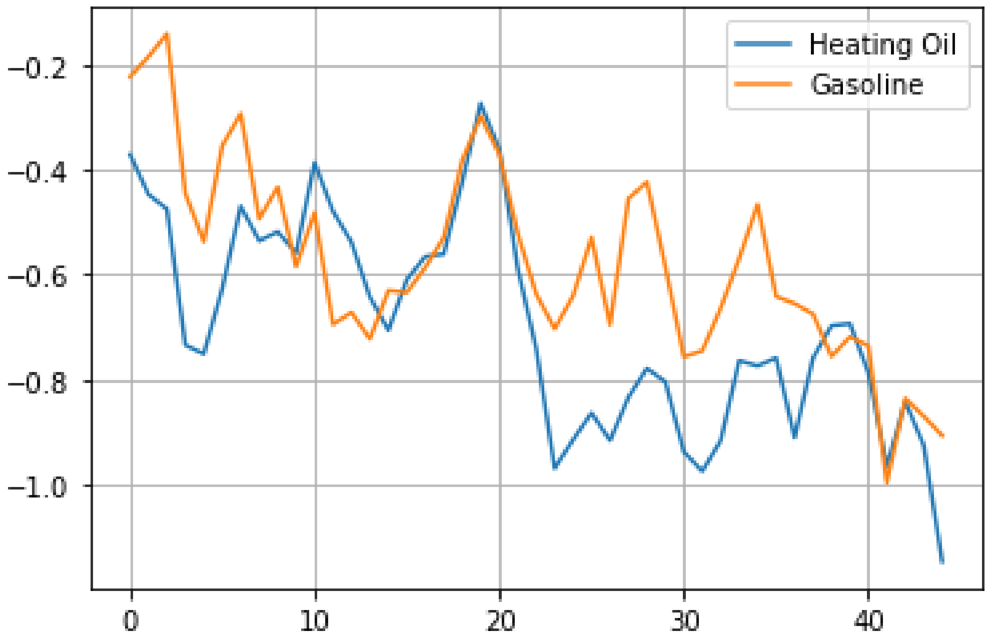
8.3 Simulating Spot Prices for Oil and Gasoline Jointly

With this in place, we can now simulate spot prices jointly as follows:

```
## See B&W Table Panel A on page 145
a1 = 0.342
b1 = 0.539
s1 = 0.11
S1 = 0.69
a2 = 0.391
b2 = 0.560
s2 = 0.116
S2 = 0.80
rho = 0.705
num_reps = 45

ln_spot1 = np.zeros(num_reps) ## HO
ln_spot2 = np.zeros(num_reps) ## Gas

ln_spot1[0] = np.log(S1)
ln_spot2[0] = np.log(S2)


z1, z2 = draw_correlated_normals(sd1=s1, sd2=s2, rho=rho, num_reps=num_reps)

for t in range(1, num_reps):
    ln_spot1[t] = ln_spot1[t-1] + a1 * (b1 - np.exp(ln_spot1[t-1])) + z1[t]
    ln_spot2[t] = ln_spot2[t-1] + a2 * (b2 - np.exp(ln_spot2[t-1])) + z2[t]

ts = pd.DataFrame({'Heating Oil' : ln_spot1, 'Gasoline' : ln_spot2})
```

```
ts.plot(grid=True)
```

```
<AxesSubplot:>
```




```
ts.head(5)
```

```
   Heating Oil  Gasoline
0    -0.371064 -0.223144
1    -0.447224 -0.185105
2    -0.475107 -0.141527
3    -0.733948 -0.445842
4    -0.750985 -0.537235
```

```
ts.tail(5)
```

```
    Heating Oil  Gasoline
40    -0.786624 -0.735310
41    -0.964276 -0.995696
42    -0.841541 -0.835507
43    -0.924789 -0.869291
44    -1.146027 -0.906146
```

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