

CHAPTER 2

EXCHANGEABILITY, REPRESENTATION THEOREMS, AND SUBJECTIVITY

DALE J. POIRIER

1 INTRODUCTION

In his popular notes on the theory of choice, (Kreps 1988: 145) opines that Bruno de Finetti's Representation Theorem is the fundamental theorem of statistical inference. De Finetti's theorem characterizes likelihood functions in terms of symmetries and invariance. The conceptual framework begins with an infinite string of $\{Z_n\}_{n=1}^{n=\infty}$ observable random quantities taking on values in a sample space Z . Then it postulates symmetries and invariance for probabilistic assignments to such strings, and finds a likelihood with the prescribed properties for finite strings of length N . This theorem, and its generalizations: (i) provide tight connections between Bayesian and frequentist reasoning, (ii) endogenize the choice of likelihood functions, (iii) prove the existence of priors, (iv) provide an interpretation of parameters which differs from that usually considered, (v) produce Bayes' Theorem as a corollary, (vi) produce the Likelihood Principle (LP) and Stopping Rule Principle (SRP) as corollaries, and (vii) provide a solution to Hume's problem of induction. This is a large number of results. Surprisingly, these theorems are rarely discussed in econometrics.

De Finetti developed subjective probability during the 1920s independently of Ramsey (1926). De Finetti was an ardent subjectivist. He is famous for the aphorism: "Probability does not exist." By this he meant that probability reflects an individual's beliefs about reality, rather than a property of reality itself. This viewpoint is also "objective" in the sense of being operationally measurable, for example, by means of betting

behavior or scoring rules. For example, suppose your true subjective probability of some event A is p and the scoring rule is quadratic $[1(A) - \tilde{p}]^2$, where $1(A)$ is the indicator function and \tilde{p} is your announced probability of A occurring. Then minimizing the expected score implies $\tilde{p} = p$. See Lindley (1982) for more details.

This subjectivist interpretation is close to the everyday usage of the term “probability.” Yet appreciation of the subjectivist interpretation of probability is not widespread in economics. Evidence is the widely used Knightian distinction between risk (“known” probabilities/beliefs) and uncertainty (“unknown” probabilities/beliefs). For the subjectivist, individuals “know” their beliefs; whether these beliefs are well calibrated (i.e. in empirical agreement) with reality or easily articulated are different issues. Subjectivist theory takes such knowledge as a primitive assumption, the same way rational expectations assumes agents know the “true model.” However, unlike frequentists, subjectivists are not assuming knowledge of a property of reality, rather only knowledge of their own perception of reality.¹ This distinction is fundamental.

How knowledge of a subjectivist’s beliefs is obtained is not addressed here, although there is a substantial literature on elicitation (Garthwaite et al. 2005; O’Hagan et al. 2006). More importantly, de Finetti showed that agreement among Bayesian researchers concerning some aspects of prior beliefs for observables can imply agreement over the likelihood function. In terms of Poirier (1988, 1995), intersubjective agreement among a bevy of Bayesians leads them to a common parametric window through which to view the observable world, and a willingness to “agree to disagree” over their priors.

De Finetti’s approach is instrumentalist in nature: past observations are used to make predictions about future observables. Likelihoods, parameters, and random sampling are neither true nor false. They are merely intermediate fictions, that is, mathematical constructs. In contrast, realists seek the “true” data generating process (DGP).

Given the positive integer N , assume an individual’s degrees of belief for N quantities of interest are derived from the specification of a subjective joint cumulative distribution function (cdf) $P(z_1, z_2, \dots, z_N)$ which is representable in terms of a joint probability density function (pdf) $p(z_1, z_2, \dots, z_N)$ (understood as a mass function in the discrete case), where “ z_n ” denotes a realization of the corresponding random quantity “ Z_n .” $P(\cdot)$ and $p(\cdot)$ (and later $F(\cdot)$ and $f(\cdot)$) are used in a generic sense, rather than specifying particular functions. For example, Hume’s problem of induction (why should one expect the future to resemble the past?) requires the predictive pdf of some future observables $\{Z_n\}_{n=N+1}^{n=N+M}$ conditional on having observed $\{Z_n = z_n\}_{n=1}^{n=N}$, i.e.

$$p(Z_{N+1}, Z_{N+2}, \dots, Z_{N+M} | Z_1, Z_2, \dots, Z_N) = \frac{p(Z_1, Z_2, \dots, Z_{N+M})}{p(Z_1, Z_2, \dots, Z_N)}. \quad (2.1)$$

¹ Fuchs and Schack (2004) draw analogies with quantum theory. Is a quantum state an actual property of the system it describes? The Bayesian view of quantum states is that it is not. Rather it is solely a function of the observer who contemplates the predictions or actions he might make with regard to quantum measurements.

The essential building block is $P(\cdot)$ for a variety of arguments. The theorems discussed later put restrictions on $P(\cdot)$. The need for at least some restrictions is obvious since arbitrarily long finite strings will be discussed. But seemingly weak conditions on arbitrarily long finite strings can deliver striking results for finite strings, and computation of (2.1) only involves finite strings of data.

Predictive pdf (2.1) provides a family of solutions to Hume's problem. The only restriction that $P(\cdot)$ must satisfy is *coherence*, that is, use of $P(\cdot)$ avoids being made a sure loser regardless of the outcomes in any betting situation (also known as avoiding *Dutch Book*).² This implies that $P(\cdot)$ obeys the axioms of probability (at least up to finite additivity). Whether (2.1) leads to good out-of-sample predictions is a different question for which there is no guaranteed affirmative answer.

Beyond coherence I consider a variety of restrictions to facilitate construction of $P(\cdot)$. Again, such restrictions should not be thought of as "true" or "false." They are not meant to be properties of reality; rather, they are restrictions on one's beliefs about reality. Other researchers may or may not find a particular restriction compelling. Part of the art of empirical work is to articulate restrictions that other researchers are at least willing to entertain if not outright adopt, that is, to obtain intersubjective agreement among a bevy of Bayesians. The simplest such restriction, exchangeability, is the topic of the next section.

2 EXCHANGEABILITY

De Finetti assigned a fundamental role to the concept of exchangeability. Given a finite sequence $\{Z_n\}_{n=1}^{n=N}$ suppose an individual makes the subjective judgment that the subscripts are uninformative in the sense that the individual specifies the same marginal distributions for the individual random quantities identically, and similarly for all possible pairs, triplets, etc. of the random quantities. Then $P(\cdot)$ satisfies $P(z_1, z_2, \dots, z_N) = P(z_{\pi(1)}, z_{\pi(2)}, \dots, z_{\pi(N)})$ for all positive integers N , where $\pi(n) (n = 1, 2, \dots, N)$ is a permutation of the elements in $1, 2, \dots, N$. Such beliefs are said to be *exchangeable*. In terms of the corresponding density/mass function, exchangeability implies $P(z_1, z_2, \dots, z_N) = P(z_{\pi(1)}, z_{\pi(2)}, \dots, z_{\pi(N)})$. A sequence is *infinitely exchangeable* iff every finite subsequence is exchangeable.

Exchangeability is one of many instances of the use of symmetry arguments in the historical development of probability (Poirier 1995: 17), and more generally, in mathematics (du Sautoy 2008). It provides an operational meaning to the weakest possible notion of a sequence of "similar" random quantities. It is "operational" in the sense that it only requires probability assignments for observable quantities, albeit arbitrarily long sequences. Exchangeability expresses a symmetric type of ignorance: no additional

² Implicitly, de Finetti assumed the utility of money is linear. He and others suggested rationalizations like the "stakes are small." Separation of the concepts of "probability" and "utility" remains a controversial matter (Kadane and Winkler 1988).

information is available to distinguish among the quantities. A sequence of Bernoulli quantities $Z_n (n = 1, 2, \dots, N)$ in $Z = 0, 1$ is exchangeable iff the probability assigned to particular sequences does not depend on the order of zeros and ones. For example, if $N = 3$ and the trials are exchangeable, then the sequences 011, 101, and 110 are assigned the same probability. For applications of exchangeability in economics, see McCall (1991).

Schervish (1995: 7–8) argued that a judgment of exchangeability is a confession by the observer that he cannot distinguish among the quantities, since he believes they are homogeneous. Gelman et al. (1995: 124) remarked: “In practice, ignorance implies exchangeability. Generally, the less we know about the problem, the more confidently we can make claims about exchangeability.” Arguing against an exchangeability assessment is an admission of the existence of non-data-based information on observables for the problem at hand.

Like iid sequences, the quantities in an exchangeable sequence are *identically distributed*. However, unlike in iid sequences, such quantities *need not be independent* for exchangeable beliefs. For example, if the quantities are a sample (without replacement) of size N from a finite population of unknown size $N^* > N$, then they are dependent and exchangeable. Also, the possible dependency in the case of exchangeable beliefs is what enables the researcher to learn from experience using (2.1).

Whereas iid sampling is the foundation of frequentist econometrics, exchangeability is the foundation for Bayesian econometrics. Both serve as the basis for further extensions to incorporate heterogeneity and dependency across observations. For example, in Section 6 exchangeability will be weakened to partial exchangeability and a time series model (a first-order Markov process) arises for the likelihood.

In the Bernoulli case, the sample space Z of $\{Z_n\}_{n=1}^N$ has 2^N elements, and it takes $2^N - 1$ numbers to specify the probabilities of all possible outcomes. In the case of exchangeability, however, the symmetry reduces this number dramatically. All that are needed are the N probabilities $q_1 = P(Z_1), q_2 = P(Z_1 \cap Z_2), \dots, q_N = P(Z_1 \cap Z_2 \cap \dots \cap Z_N)$. By the Inclusion-Exclusion Law (O’Hagan 1994: 113), the probability of any outcome in which r specified Z_i s occur and the other $N - r$ do not occur is $\sum_{k=0}^{N-r} (-1)^k \binom{N-r}{k} q_{r+k}$. Therefore, probabilities of all possible outcomes can be expressed in terms of N of the q_k s. The difference $2^N - 1 - N$ grows rapidly as N increases, suggesting the power of the exchangeability assumption. Exchangeability is also applicable to continuous quantities as the following example shows.

Example 1: Suppose the multivariate normal pdf $p(z) = \phi_N(z | 0_N, \Sigma_N(\rho))$ captures a researcher’s beliefs about $\{Z_n\}_{n=1}^N$, where $\Sigma_N(\rho) = (1 - \rho)I_N + \rho \iota_N \iota_N'$, $\iota_N = [1, 1, \dots, 1]'$, and $\rho > -(N - 1)^{-1}$ is known. It is easy to see that such equicorrelated beliefs are exchangeable. Hereafter assume $\rho > 0$ to accommodate infinite exchangeability. Further suppose these beliefs can be extended across M additional observations $Z^* = [Z_{N+1}, Z_{N+2}, \dots, Z_{N+M}]'$ so that $[Z', Z^{*'}]'$ has pdf $\phi_{N+M}([z, z^*]' | 0_{N+M}, \Sigma_{N+M})$, where

$$\sum_{N+M}(\rho) = (1 - \rho)I_{N+M} + \rho\iota_{N+M}\iota_{N+M}' = \begin{bmatrix} \sum_N(\rho) & \rho\iota_{N\iota_{M'}} \\ \rho\iota_{M\iota_{N'}} & \sum_M(\rho) \end{bmatrix}.$$

Suppose $Z = z$ is observed. Because beliefs between Z and Z^* are dependent, the initial beliefs $Z^* \sim N_M(0_M, \Sigma_M(\rho))$ are updated to $Z^*|Z = z \sim N_M(\mu_{Z^*|z}(\rho), \Sigma_{Z^*|z}(\rho))$, where

$$\mu_{Z^*|z}(\rho) = \left(\frac{N\rho}{(N-1)\rho + 1} \right) \bar{Z}_N \iota_M,$$

$$\Sigma_{Z^*|z}(\rho) = \Sigma_M(\rho) - \rho^2 \iota_{M\iota_{N'}} [\Sigma_N]^{-1} \iota_{N\iota_{M'}} = (1 - \rho) \left[I_M + \left(\frac{\rho}{(N-1)\rho + 1} \right) \iota_{M\iota_{M'}} \right],$$

and $\bar{Z}_N = N^{-1} \iota_{N'} z$. The predictive beliefs $Z^*|Z = z$ are also exchangeable, and the predictive means $\mu_{Z^*|z}(\rho)$ are all shrunk identically in the direction of \bar{Z}_N . Finally, \bar{Z}_N serves as a sufficient statistic summarizing the impact of the past data z on beliefs about the future observables Z^* .

3 THE BERNOULLI CASE

De Finetti's Representation Theorem for Bernoulli sequences can be stated formally as follows (Bernardo and Smith 1994: 172–3).

Theorem 1. (de Finetti's Representation Theorem): Let $\{Z_n\}_{n=1}^{n=N}$ be an infinitely exchangeable sequence of Bernoulli random quantities in $Z = \{0, 1\}$ with probability measure $P(\cdot)$. Define the sum $S_N = Z_1 + Z_2 + \dots + Z_N$, and the average number $\bar{Z}_N = S_N/N$ of occurrences in a string of length N . Let $z = [z_1, z_2, \dots, z_N]'$ denote realized values. Then there exists a cdf $F(\cdot)$ such that the joint mass function $p(z) = P(Z_1 = z_1, Z_2 = z_2, \dots, Z_N = z_N) = p(z_1, z_2, \dots, z_N)$ satisfies

$$p(z) = \int_{\Omega} \mathcal{L}(\theta; Z) dF(\theta), \quad (2.2)$$

where the observed likelihood function corresponding to $S_N = s$ is $\mathcal{L}(\theta; z) \equiv p(z|\theta) = \binom{N}{s} \theta^s (1 - \theta)^{N-s}$, the random variable $\Theta \in \Omega \equiv [0, 1]$ is defined by $\Theta = \lim_{N \rightarrow \infty} \bar{Z}_N$ P —almost surely, and $F(\cdot)$ is the cdf of Θ under $P(\cdot)$, i.e. $F(\theta) \equiv \lim_{N \rightarrow \infty} P(\bar{Z}_N \leq \theta)$.

In other words, Theorem 1 implies it is *as if*, given $\Theta = \theta$, $\{Z_n\}_{n=1}^{n=N}$ are iid Bernoulli trials with likelihood function $\mathcal{L}(\theta; z)$, and where the probability Θ of a success is assigned a prior cdf $F(\theta)$ that can be interpreted as the researcher's beliefs about the long-run relative frequency of $Z_N \leq \theta$ as $N \rightarrow \infty$. From de Finetti's standpoint, both the parameter Θ and the notion of independence are “mathematical fictions”

implicit in the researcher's subjective assessment of arbitrarily long sequences of observable successes and failures. The “P—almost surely,” or equivalently, “with probability one” qualification on the existence of Θ in Theorem 1 refers to the researcher's predictive beliefs (i.e. the left-hand side of (2.2), which may not be reflected in reality. De Finetti's Theorem commits the researcher to believe almost surely in the existence of Θ in his/her personal world, not necessarily in the physical universe. In standard cases where $F(\theta)$ is absolutely continuous with pdf $f(\theta)$, (2.2) can be replaced with the more familiar form

$$p(z) = \int_{\Omega} \mathcal{L}(\theta; Z) f(\theta) d\theta. \quad (2.3)$$

The pragmatic value of de Finetti's Theorem depends on whether it is easier to assess the left-hand side $p(z)$ of (2.3), which only involves observable quantities, or instead, the integrand on the right-hand side of (3.3) which involves the likelihood, the prior, and the mathematical fiction θ . Most researchers think in terms of the right-hand side. Non-Bayesians implicitly do so with a degenerate distribution that treats Θ as equal to a constant θ_o with probability one, that is, a degenerate “prior” distribution for Θ at the “true value” θ_o . I am promoting an attitude that emerges from the left-hand side of (3.3), but which can be used to help researchers work on the right-hand side of (2.3).

De Finetti's theorem suggests an isomorphism between two worlds, one involving only observables z and the other involving the parameter Θ . De Finetti put parameters in their proper perspective: they are mathematical constructs that provide a convenient index for a probability distribution, they induce conditional independence for a sequence of observables, and they are “lubricants” for fruitful thinking and communication. Their “real-world existence” is a question only of metaphysical importance.

Example 2: Suppose N Bernoulli trials $\{Z_n\}_{n=1}^{n=N}$ yield r ones and $N - r$ zeros. Assume $F(\theta)$ is absolutely continuous with pdf $f(\theta)$. Applying Theorem 1 to the numerator and denominator of (2.1) (see Poirier 1995; 216) with $M = 1$ yields the predictive probability

$$P(Z_{N+1} = z_{N+1} | N\bar{Z}_N = r) = \begin{cases} E(\Theta|z), & \text{if } z_{N+1} = 1 \\ 1 - E(\Theta|z), & \text{if } z_{N+1} = 0 \end{cases},$$

where $E(\Theta|z) = \int_{\Omega} \theta f(\theta|z) d\theta$, and

$$f(\theta|z) = \frac{f(\theta)\mathcal{L}(\theta; z)}{p(z)}, 0 \leq \theta \leq 1, \quad (2.4)$$

is the posterior pdf of Θ . From (2.4) it is clear that experiments with proportional likelihoods yield the same posterior, implying the Likelihood Principle. The fiction Θ and its posterior mean $E(\Theta | z)$ have a conceptually useful role in updating beliefs about $Z_{N+1} = z_{N+1}$ after observing $N\bar{Z}_N = r$ ones in N trials. \square

The existence of the prior $F(\cdot)$ is a *conclusion* of Theorem 1, not an *assumption*. The updating of prior beliefs captured in (2.4) corresponds to Bayes' Theorem. Although

$\{Z_n\}_{n=1}^{n=N}$ are conditionally independent given $\Theta = \theta$, unconditional on Θ they are dependent. Putting further restrictions on the observable Bernoulli quantities $\{Z_n\}_{n=1}^{n=\infty}$ beyond infinite exchangeability can help pin down the prior $F(\cdot)$. For example, the assumption that $\{Z_n\}_{n=1}^{n=\infty}$ correspond to draws from a Polya urn process implies the prior $F(\cdot)$ belongs to the conjugate beta family (Freedman 1965).³ Hill et al. (1987) proved that an exchangeable urn process can only be Polya, Bernoulli iid, or deterministic.

Example 3: Consider the Bernoulli case in Example 2 for a Polya urn. Then for some hyperparameters $\underline{\alpha} > 0$ and $\underline{\delta} > 0$ the implied prior for Θ is the conjugate beta density $f_b(\theta|\underline{\alpha}, \underline{\delta}) = \frac{\Gamma(\underline{\alpha}+\underline{\delta})}{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})} \theta^{\underline{\alpha}-1} (1-\theta)^{\underline{\delta}-1}$. Posterior pdf (2.4) is $f_b(\theta|\bar{\alpha}, \bar{\delta})$ with hyperparameters $\bar{\alpha} = \underline{\alpha} + N\bar{z}_N$ and $\bar{\delta} = \underline{\delta} + N(1 - \bar{z}_N)$. Note that the posterior mean of Θ is $E(\Theta|z) = \frac{\bar{\alpha}}{\bar{\alpha}+\bar{\delta}} = \frac{\underline{\alpha}+N\bar{z}}{\underline{\alpha}+\underline{\delta}+N}$, demonstrating posterior linearity in \bar{z} . Diaconis and Ylvisaker (1979: 279–80) prove that the beta family is the unique family of distributions allowing linear posterior expectation of success in exchangeable binomial sampling. Infinite exchangeability for observables is enough to pin down the likelihood, and the addition of the Polya urn interpretation for the observable process identifies a beta prior up to the two free hyperparameters $\underline{\alpha}$ and $\underline{\delta}$. \square

While the Polya urn formulation is a predictive argument for a beta prior, there remains the choice of $\underline{\alpha}$ and $\underline{\delta}$. Recall that $S_N \equiv N\bar{z}_N$ is the number of ones. Bayes advocated $P(S_N = s) = (N+1)^{-1}(s = 0, 1, \dots, N)$, implying $\underline{\alpha} = \underline{\delta} = 1$. Chaloner and Duncan (1983) recommended eliciting $\underline{\alpha}$ and $\underline{\delta}$ predictively by putting additional restrictions on the implied beta-binomial mass function for S_N :

$$\begin{aligned} p(S_N = s) &= \int_0^1 \binom{N}{s} \theta^s (1-\theta)^{N-s} \left[\frac{\Gamma(\underline{\alpha} + \underline{\delta})}{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})} \right] \theta^{\underline{\alpha}-1} (1-\theta)^{\underline{\delta}-1} d\theta \\ &= \binom{N}{s} \left[\frac{\Gamma(\underline{\alpha} + \underline{\delta})}{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})} \right] \int_0^1 \theta^{s+\underline{\alpha}-1} (1-\theta)^{N-s+\underline{\delta}-1} d\theta \\ &= \binom{N}{s} \left[\frac{\Gamma(\underline{\alpha} + \underline{\delta})}{\Gamma(\underline{\alpha})\Gamma(\underline{\delta})} \right] \left[\frac{\Gamma(s + \underline{\alpha})\Gamma(N-s + \underline{\delta})}{\Gamma(\underline{\alpha} + \underline{\delta} + N)} \right], \quad s = 0, 1, \dots, N, \end{aligned} \quad (2.5)$$

with mean $\frac{N\underline{\alpha}}{\underline{\alpha}+\underline{\delta}}$ and variance $\frac{N\underline{\alpha}\underline{\delta}(N+\underline{\alpha}+\underline{\delta})}{(\underline{\alpha})^2(\underline{\alpha}+\underline{\delta}+1)}$. Specifically, Chaloner and Duncan (1983) argued (assuming $\underline{\alpha} > 1$ and $\underline{\delta} > 1$) for elicitation in terms of the mode $m = \frac{\underline{\alpha}-1}{(\underline{\alpha}+\underline{\delta}-2)}$ of (2.5) and the ratios of probability at m relative to $m-1$ and $m+1$. In contrast, Geisser (1984) discussed “noninformative” priors for θ including two members of the

³ Suppose an urn initially contains r red and b black balls and that, at each stage, a ball is selected at random, then replaced by two of the same color. Let Z_n be 1 or 0 accordingly as the n th ball selected is red or black. Then the Z_n ($n = 1, 2, 3, \dots$) are infinitely exchangeable and comprise a Polya urn process. However, not all urn processes are exchangeable. Neither can all exchangeable processes be represented as urn processes. See Hill et al. (1987).

beta family other than the uniform: the limiting improper prior of Haldane ($\underline{\alpha} = \underline{\delta} = 0$) and the proper prior of Jeffreys ($\underline{\alpha} = \underline{\delta} = 1/2$).

For extension to the multinomial case, see Bernardo and Smith (1994: 176–7). Johnson (1924) gave a predictive argument for the multinomial case, similar to Bayes’ argument in the binomial case.

4 NONPARAMETRIC REPRESENTATION THEOREM

De Finetti’s Representation Theorem has been extended to cover exchangeable beliefs involving random variables more complicated than Bernoulli random variables. The initial nonparametric case for Euclidean spaces was studied by de Finetti (1938). Hewitt and Savage (1955) extended the result to arbitrary compact Hausdorff spaces, and Aldous (1985) extended it to random elements with values in a standard Borel space. Dubins and Freedman (1979) showed that without any topological assumptions, the result need not hold. The following theorem covers the general case for real-valued exchangeable random quantities (Bernardo and Smith 1994: 178–9 outlined its proof).

Theorem 2. (General Representation Theorem): Consider an infinitely exchangeable sequence $\{Z_n\}_{n=1}^{n=\infty}$ of real-valued random quantities with probability measure $P(\cdot)$. Then there exists a probability measure F over \mathcal{F} , the space of all distribution functions on \mathfrak{R} , such that the joint cdf of $\{Z_n\}_{n=1}^{n=N}$ has the form

$$P(Z_1, Z_2, \dots, Z_N) = \int_{\mathcal{F}} \prod_{n=1}^N Q(Z_n) dF(Q),$$

where

$$F(Q) = \lim_{N \rightarrow \infty} P(Q_N), \tag{2.6}$$

and Q_N is the empirical distribution function corresponding to $\{Z_n\}_{n=1}^{n=N}$. □

In other words, it is *as if* the observations $\{Z_n\}_{n=1}^{n=N}$ are independent conditional on Q , an unknown cdf (in effect an infinite-dimensional parameter), with a belief distribution $F(\cdot)$ for Q , having the operational interpretation in (2.6) of what we believe the empirical distribution function would look like for a large sample.

Theorem 2 is a general existence theorem. Unfortunately, it is of questionable pragmatic value because it is hard to think of specifying a prior on all probabilities on \mathfrak{R} . Diaconis and Freedman (1986) discussed the difficulties of specifying priors over infinite dimensional spaces. Also see Sims (1971), Schervish (1995: 52–72) and Ferguson (1974). Therefore, in the next section attention turns to additional restrictions required to specify intermediate familiar finite-dimensional parametric sampling models. Unlike

in the simple Bernoulli case of Section 3, however, predictive arguments for choosing particular priors are harder to obtain.

5 GENERALIZATIONS

Diaconis and Freedman (1981: 205) noted that an equivalent formulation of exchangeability of Bernoulli random quantities is the following. For every N , given the sum $S_N = s$, the joint distribution of $\{Z_n\}_{n=1}^{n=N}$ is uniformly distributed over the $\binom{N}{s}$ sequences having s ones and $(N - s)$ zeros. In other words, $\{Z_n\}_{n=1}^{n=N}$ are exchangeable iff the partial sums are sufficient with an “equiprobable” conditional distribution for Z_1, Z_2, \dots, Z_N given $S_N = s$. This section explores invariance and sufficiency restrictions that deliver familiar sampling distributions in cases more complicated than Bernoulli variables. In the process, these restrictions on observables will yield parametric families and operationally useful results falling between Theorems 1 and 2.

An example of such a restriction is spherical symmetry. Beliefs regarding z are *spherically symmetric* iff $p(z) = p(Az)$ for any $N \times N$ orthogonal matrix A (i.e. $A^{-1} = A'$) satisfying $A\iota_N = \iota_N$ (i.e. which preserves the unit N -vector ι_N). This restriction amounts to rotational invariance of the coordinate system which fixes distances from the origin. Exchangeability is one form of spherical symmetry since permutation is one form of orthogonal transformation.

Example 4: The exchangeable beliefs captured by $p(z) = \phi_N(z \mid 0_N, \Sigma_N(\rho))$ in Example 1 are characterized by spherical symmetry because $A\Sigma_N(\rho)A' = \Sigma_N(\rho)$ for any $N \times N$ orthogonal matrix A satisfying $A\iota_N = \iota_N$. Even without assuming infinite exchangeability, $p(z)$ has the representation

$$p(z) = \int_{-\infty}^{\infty} \prod_{n=1}^N \phi(z_n \mid \theta, 1 - \rho) \phi(\theta \mid 0, \rho) d\theta, \quad z \in \Re^N. \quad \square$$

Dropping the multivariate normality assumption in Example 1, maintaining $p(z) = p(Az)$ for any $N \times N$ orthogonal matrix A , not requiring $A'\iota_N = \iota_N$, and strengthening the exchangeability assumption to infinite exchangeability, leads to the following theorem (see Schoenberg 1938; Freedman 1962b; Kingman 1972; and Bernardo and Smith 1994: 182).

Theorem 3. (Normal Sampling with zero mean): Consider an infinitely exchangeable sequence, $\{Z_n\}_{n=1}^{n=\infty}$, $Z_n \in \Re$, with cdf $P(\cdot)$. If for any N , $z = [z_1, z_2, \dots, z_N]'$ is characterized by spherical symmetry, then there exists a distribution $F(\theta)$, $\theta \in \Re_+$, such that

$$p(z) = \int_0^{\infty} \prod_{n=1}^N \Phi(z_n / \sqrt{\theta}) dF(\theta),$$

where $\Phi(\cdot)$ is the standard normal cdf, $\tilde{s}_N^2 = (z_1^2 + z_2^2 + \dots + z_N^2)/N$, $\theta \equiv \lim_{N \rightarrow \infty} \tilde{s}_n^{-2}$, and

$$F(\theta) = \lim_{N \rightarrow \infty} P(\tilde{s}_N^{-2} \leq \theta). \quad \square \quad (2.7)$$

Theorem 3 implies that if predictive beliefs are characterized by infinite exchangeability and spherical symmetry, then it is *as if*, given $\Theta = \theta$, $\{Z_n\}_{n=1}^{n=N}$ are iid $N(0, \theta^{-1})$ with a prior distribution $F(\cdot)$ in (2.7) for the precision θ . $F(\cdot)$ can be interpreted as beliefs about the reciprocal of the limiting means sum of squares of the observations.

Diaconis and Freedman (1981: 209–10) provided the equivalent condition: for every N , given the sufficient statistic $T = \left(\sum_{n=1}^N z_n^2 \right)^{1/2} = t$, the joint distribution of $\{Z_n\}_{n=1}^{n=N}$ is uniform on the $(N-1)$ -sphere of radius t . Arellano-Valle et al. (1994) showed that if the condition $E(Z_2^2|Z_1) = aZ_1^2 + b$, where $0 < a < 1$ and $b > 0$, is added to the spherical symmetry judgment, then $F(\cdot)$ is the conjugate inverted-gamma distribution. Consequently, the distribution of z is a spherical multivariate Student- t model. Loschi et al. (2003) extended this result to the matrix-variate case. Dawid (1978) considered the multivariate extension of Theorem 3.

If beliefs about $z_1 - \bar{z}_N, z_2 - \bar{z}_N, \dots, z_N - \bar{z}_N$ possess spherical symmetry, then beliefs about z are said to be characterized by *centered spherical symmetry*. Centered spherical symmetry fixes distances from the mean of the observations, that is, identical probabilities are asserted for all outcomes z_1, z_2, \dots, z_N leading to the same value of $(z_1 - \bar{z}_N)^2 + (z_2 - \bar{z}_N)^2 + \dots + (z_N - \bar{z}_N)^2$. When infinite exchangeability is augmented with centered spherical symmetry, then the familiar normal random sampling model with unknown mean and unknown precision emerges in the following theorem of Smith (1981). For a proof, see Bernardo and Smith (1994: 183–5). Also see Eaton et al. (1993: 4) for an important qualification.

Theorem 4. (Centered Normal Sampling): Consider an infinitely exchangeable sequence $\{Z_n\}_{n=1}^{n=\infty}$ of real-valued random quantities with probability measure $P(\cdot)$. If for any N , $z = [z_1, z_2, \dots, z_N]'$ is characterized by centered spherical symmetry, then there exists a distribution function $F(\theta)$, with $\theta = [\mu, \sigma^{-2}]' \in \Re \times \Re_+$, such that the joint distribution of z has the form

$$P(z) = \int_{\Re \times \Re_+} \prod_{n=1}^N \Phi[(z_n - \mu)/\sigma] dF(\mu, \sigma^{-2}),$$

where $\Phi(\cdot)$ is the standard normal cdf,

$$F(\mu, \sigma^{-2}) = \lim_{N \rightarrow \infty} P[(\bar{z}_N \leq \mu) \cap (s_N^{-2} \leq \sigma^{-2})], \quad (2.8)$$

$$\mu \equiv \lim_{N \rightarrow \infty} \bar{z}_N, \quad (2.9)$$

where $s_N^2 = [(z_1 - \bar{z}_N)^2 + (z_2 - \bar{z}_N)^2 + \dots + (z_N - \bar{z}_N)^2]/N$, and

$$\sigma^2 \equiv \lim_{N \rightarrow \infty} s_N^2. \quad \square \quad (2.10)$$

Theorem 4 implies that if predictive beliefs are characterized by infinite exchangeability and centered spherical symmetry, then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid $N(\mu, \sigma^2)$ given μ and σ^{-2} defined in (2.9) and (2.10), and with prior distribution $F(\cdot)$ in (2.8). As in the Bernoulli case, adding the restriction of linearity of the poster mean in terms of \bar{z}_N implies a conjugate normal-gamma prior.

Example 5: Under the conditions of Theorem 4 and a conjugate normal-gamma prior density

$$f(\mu, \sigma^{-2}) = \phi(\mu | \underline{\mu}, \underline{q}\sigma^2) f_g(\sigma^{-2} | \underline{v}/2, 2/\underline{v}\underline{s}^2), \quad \underline{\mu} \in \Re, \underline{q}, \underline{s}, \underline{v} > 0,$$

(2.3) is the centered spherically symmetric multivariate- t pdf $f_t^N(y | \underline{\mu}, \underline{s}^2(I_N + \underline{q}\underline{t}_N\underline{t}_N')^{-1}, \underline{v})$. \square

The multivariate analog of Theorem 4 follows (see Bernardo and Smith 1994: 186 and Diaconis et al. 1992).

Theorem 5. (Multivariate Normal Sampling): Consider an infinitely exchangeable sequence $\{Z_n\}_{n=1}^{n=\infty}$ of real-valued random vectors in \Re^K with cdf $P(\cdot)$, such that for any N and $c \in \Re^K$, the random quantities $c'Z_1, c'Z_2, \dots, c'Z_N$ are characterized by centered spherical symmetry. Then the predictive beliefs $P(z)$ are *as if* $\{Z_n\}_{n=1}^{n=N}$ were iid multivariate normal vectors, conditional on a random mean μ and covariance matrix Σ , with a distribution over μ and Σ induced by $P(\cdot)$, where

$$\begin{aligned} \mu &\equiv \lim_{N \rightarrow \infty} \bar{z}_N, \\ \Sigma &\equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (z_n - \bar{z}_N)(z_n - \bar{z}_N)'. \end{aligned} \quad \square$$

Additional results based on infinite exchangeability and particular invariance properties yield other familiar sampling models. Two examples of characterizations of discrete distributions over nonnegative integers are:

- (a) Freedman (1962b) showed that if for every N , the joint distribution of $\{Z_n\}_{n=1}^{n=N}$, given the sum S_N is multinomial on N -tuples with uniform probabilities equal to N^{-1} (the Maxwell-Boltzman distribution), then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid Poisson random variables.
- (b) Diaconis and Freedman (1981: 214) noted that if for every N , the joint distribution of $\{Z_n\}_{n=1}^{n=N}$, given S_N , is uniform on N -tuples with uniform probabilities equal to J^{-1} , where J is the total number of N -tuples with sums S_N , then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid geometric random variables.

Five examples of characterizations of continuous distributions are:

- (c) Suppose for every N , the joint distribution of $\{Z_n\}_{n=1}^{n=N}$, given $M_N \equiv \max\{Z_1, Z_2, \dots, Z_N\}$, are independent and uniform over the interval $[0, M_N]$. Diaconis and Freedman (1981: 210) noted that this condition is necessary and

- sufficient for the representation that $\{Z_n\}_{n=1}^{n=N}$ are iid uniform over the interval $[0, \Theta]$, with a prior distribution for Θ .
- (d) Diaconis and Freedman (1987) showed that if for every N , given the sum $S_N = s$, the joint distribution of $\{Z_n\}_{n=1}^{n=N}$ is uniformly distributed over the simplex $\{Z_n \geq 0, s\}$, then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid exponential random variables.⁴
 - (e) Singpurwalla (2006: 54) noted that if for every N , given the sum $S_N = s$, the joint distribution of $\{Z_n^{\Theta_1}\}_{n=1}^{n=N}$ is uniformly distributed over the simplex $\{Z_n \geq 0, s\}$, then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid gamma random variables with pdf $p(z_n) = \Theta_2^{\Theta_1} z_n^{\Theta_1-1} \exp(-\Theta_2 z_n) / \Gamma(\Theta_1)$.
 - (f) Singpurwalla (2006: 55) noted that if the uniformity in (e) is over the simplex $\{Z_n \geq 0, \sum_{n=1}^N Z_n^{\Theta_1}\}$, then it is *as if* $\{Z_n\}_{n=1}^{n=N}$ are iid Weibull random variables.

Further restrictions on $\{Z_n\}_{n=1}^{n=\infty}$ to analytically derive the prior distributions are difficult to find once we leave the Bernoulli sample space of Section 3. A more common approach is to elicit moments and quantiles for the left-hand side of (2.3), and assuming a parametric family for $f(\cdot)$ (usually conjugate), to then back-out a prior on the right-hand side of (2.3). Usually the process is iterative.⁵ Rationalization for restricting attention to conjugate priors is provided by Diaconis and Ylvisaker (1979), who characterized conjugate priors through the property that the posterior expectation of the mean parameter of Z_n , $E[E(Z_n | \Theta = \theta) | Z_n = z_n]$, is linear in z_n . While not analytical, the approach is predictively motivated.

There are many other representation theorems available. Bernardo and Smith (1994: 215–16) outline how two-way ANOVA (analysis of variation) specifications and hierarchical specifications can be rationalized. But these are only partly predictively motivated. Bernardo and Smith (1994: 219–22) cover binary choice models, growth curves, and regression. In these extensions, the parameter Θ becomes a function of the regressors, but the specification of $\Theta(\cdot)$ is done in an ad hoc manner to provide common specifications instead of providing transparent restrictions on observables for the left-hand side of a representation theorem. Diaconis et al. (1992) characterize normal models for regression and ANOVA in terms of symmetry or sufficiency restrictions. Arnold (1979) considered multivariate regression models with exchangeable errors.

Finally, although de Finetti's theorem does not hold exactly for finite sequences, it does hold approximately for sufficiently large finite sequence. Diaconis and Freedman (1980) showed that for a binary exchangeable sequence of length K which can be extended to an exchangeable sequence of length N , then de Finetti's theorem

⁴ Alternative formulations are possible. For example, Diaconis and Ylvisaker (1985) showed that in the case of an infinitely exchangeable sequence $\{Z_n\}_{n=1}^{n=\infty}$ of positive real quantities with probability measure $P(\cdot)$ that exhibit a certain “lack of memory” property with respect to the origin, then $p(\cdot)$ has a representation as a mixture of iid exponential random quantities. A similar result holds for an infinitely exchangeable sequence of positive integers leading to a representation as a mixture of iid geometric random quantities. In both of the latter cases, the predictive “lack of memory” property is reminiscent of similar properties for the parametric exponential and geometric distributions. See Bernardo and Smith (1994: 187–90) for more details.

⁵ Indeed the method of Chaloner and Duncan (1983), discussed in Example 3, is an example.

“almost” holds in the sense that the total variation distance between the distribution of Z_1, Z_2, \dots, Z_K and the approximating mixture is $\leq 2K/N$. Diaconis et al. (1992: 292) provided a finite version of Dawid (1978). Diaconis and Freedman (1987) discussed numerous extensions with K/N continuing to play a key role. Finite versions of de Finetti’s theorem for Markov chains are given by Diaconis and Freedman (1980) and Zaman (1986).

6 PARTIAL EXCHANGEABILITY

Exchangeability involves complete symmetry in beliefs. Often such beliefs are not warranted across all observables, but are reasonable for subsets. This leads to *partial exchangeability*. Partial exchangeability takes on a variety of guises (see de Finetti 1938; Aldous 1981; and Diaconis and Freedman 1980, 1981), but an essential ingredient is that the sequence $\{Z_n\}_{n=1}^{n=\infty}$ is broken down into exchangeable subsequences. For example, suppose $\{Z_n\}_{n=1}^{n=\infty}$ are the employment status after undergoing job training. If both males and females are included, one might be reluctant to make a judgment of exchangeability for the entire sequence of results. However, within subsequences defined by gender, an assumption of exchangeability might be reasonable.

Alternatively, consider the case of Markov chains for Bernoulli quantities as initially studied by de Finetti (1938) and then Freedman (1962a). Consider three subsequences: the first observation, observations following a zero, and observations following a one. Two binary sequences are said to be *equivalent* if they begin with the same symbol and have the same number of transitions from 0 to 0, 0 to 1, 1 to 0, and 1 to 1. A probability on binary sequences is partially exchangeable iff it assigns equal probability to equivalent strings. Freedman (1962a) showed that a stationary partially exchangeable process is a mixture of Markov chains. Diaconis and Freedman (1980) eliminated the stationary assumption. To get a mixture of Markov chains in this case, infinitely many returns to the starting state are needed. Extensions to countable situations are straightforward, but extensions to more general spaces are more complex (see Diaconis 1988).

Yet another form of partial exchangeability is described by Bernardo and Smith (1994: 211). The M infinite sequences of 0–1 random quantities Z_{m1}, Z_{m2}, \dots ($m = 1, 2, \dots, M$) are *unrestrictedly infinitely exchangeable* iff each sequence is infinitely exchangeable and, in addition, for all $n_m \leq N_m$ and $z_m(n_m) = [z_{m1}, z_{m2}, \dots, z_{mn_m}]'$ ($m = 1, 2, \dots, M$),

$$p[z_1(n_1), \dots, z_M(n_M) | w_1(N_1), \dots, w_M(N_M)] = \prod_{m=1}^M p[z_m(n_m) | w_m(N_m)],$$

where $w_m(N_m) = z_{m1} + z_{m2} + \dots + z_{mN_m}$ is the number of successes in the first N_m observation from the m th sequence ($m = 1, 2, \dots, M$). In other words, unrestrictedly infinite exchangeability adds to infinite exchangeability the requirement that, given

$w_m(N_m)(m = 1, 2, \dots, M)$, only the total for the m th sequence is relevant for beliefs about the outcomes of any subset of n_m of the N_m observations from that sequence. Therefore, unrestrictedly infinite exchangeability involves a conditional irrelevance judgment. With this definition in hand, the following theorem of Bernardo and Smith (1994: 212–13) can be proved.

Theorem 6. (Representation Theorem for Several Sequences of 0–1 Random Quantities): Suppose $\{Z_{mn}\}_{n=1}^{n=\infty}(m = 1, \dots, M)$ are unrestrictedly infinitely exchangeable sequences of $\{0, 1\}$ random quantities with joint probability measure $P(\cdot)$. Then there exists a cdf $F(\cdot)$ such that

$$p[z_1(n_1), \dots, z_M(n_M)] = \int_{[0,1]^M} \prod_{m=1}^M \prod_{j=1}^{n_m} \theta_m^{z_{mj}} (1 - \theta_m)^{1-z_{mj}} dF(\theta),$$

where $w_m(n_m) = z_{m1} + z_{m2} + \dots + z_{mn_m}(m = 1, 2, \dots, M)$, and

$$F(\theta) = \lim_{\substack{n_m \rightarrow \infty \\ (m = 1, \dots, M)}} P \left[\left(\frac{w_1(n_1)}{n_1} \leq \theta_1 \right) \cap \dots \cap \left(\frac{w_M(n_M)}{n_M} \leq \theta_M \right) \right]. \quad \square$$

To appreciate Theorem 6, consider the case of $M = 2$ subsequences as do Bernardo and Smith (1994: 214, 223). Then Theorem 6 implies we can proceed *as if* (i) the $\{Z_{mn}\}_{n=1}^{n=N}(m = 1, 2)$ are judged to be independent Bernoulli random quantities conditional on $\Theta_m(m = 1, 2)$, (ii) Θ_1 and Θ_2 have bivariate cdf $F(\theta_1, \theta_2)$, and according to the SLLN, $\Theta_m = \lim_{n_m \rightarrow \infty} \bar{W}_{n_m}/n_m(m = 1, 2)$ P -almost surely. Specification of $F(\theta_1, \theta_2)$ depends on the application at hand. Four possibilities are: (i) belief that knowledge of the limiting relative frequency for one of the sequences will not change beliefs about the other sequence (prior beliefs about Θ_1 and Θ_2 are independent), (ii) belief that the limiting relative frequency of the second sequence is necessarily greater than for the first sequence implies $F(\theta_1, \theta_2)$ is zero outside of $0 \leq \theta_1 < \theta_2 \leq 1$, (iii) belief that there is a positive non-unitary probability that the limits of the two sequences are the same, and (iv) belief that the long-run frequencies $\Theta_m(m = 1, 2)$ are themselves exchangeable, leading to a hierarchical model.

The mathematics behind de Finetti's Theorem and its generalizations has many cousins. Diaconis and Freedman (1981) discussed the mathematical similarity to statistical-mechanical studies of "Gibbs states". Lauritzen (1988) developed extreme point models in the language of projective systems. Ladha (1993) used de Finetti's Theorem to relax Condorcet's assumption of independent voting while preserving the result that a majority of voters is more likely than any single voter to choose the better of two alternatives. The mathematics of representation theorems is both deep and broad.

7 CONCLUSIONS

... one could say that for him (de Finetti) Bayesianism represents the crossroads where pragmatism and empiricism meet subjectivism. He thinks one needs to be Bayesian in order to be subjectivist, but on the other hand subjectivism is a choice to be made if one embraces a pragmatist and empiricist philosophy.

(Galavotti 2001: 165)

I believe the case has been made for Bayesianism in the sense of de Finetti. I have promoted a subjective attitude emerging from the left-hand side of (2.3), but which can be used to help researchers work on the more customary right-hand side. This change of emphasis from parameters to observables puts the former in a subsidiary role. Less fascination with parameters can be healthy. The prior $f(\cdot)$, which is implied (not assumed) by a representation theorem, is always proper. This rules out the usual priors used in *objective Bayesian analysis* (see Berger 2004). But interestingly, exchangeability, which is an admission of no additional information regarding otherwise similar observable quantities, implies the use of a *proper* prior for the mathematical fiction Θ . To pin down this prior further requires additional assumptions about observable sequences (e.g. the Polya urn assumption in Example 3).

If the researcher finds it more convenient to think in terms of the right-hand side of (2.3) (say, because the researcher has a theoretical framework in which to interpret Θ), then by all means elicit a personal prior for Θ or consider the sensitivity of the posterior with respect to professional viewpoints of interest defined parametrically. But given that most people find prior choice difficult, a representation theorem provides an alternative way to use subjective information about observables to facilitate prior choice for Θ .

Even when a representation theorem is not available for a given parametric likelihood (say, because it is unclear what restrictions on the left-hand side of (2.3) would be sufficient to imply this choice of likelihood), the spirit of this discussion has hopefully not been lost on the reader. Representation theorems reflect a healthy attitude toward parametric likelihoods: they are not intended to be “true” properties of reality, but rather useful windows for viewing the observable world, communicating with other researchers, and making inferences regarding future observables.

REFERENCES

- Aldous, D. (1981). “Representations for Partially Exchangeable Arrays of Random Variables”. *Journal of Multivariate Analysis*, 11: 581–98.
- (1985). “Exchangeability and Related Topics”, in P. L. Hennequin (ed.), *École d’Été de Probabilités de Saint-Flour XIII—1983*. Berlin: Springer, 1–198.

- Arellano-Valle, R. B., Bolfarine, H., and Iglesias, P. (1994). "A Predictivistic Interpretation of the Multivariate t Distribution". *Test*, 3: 221–36.
- Arnold, S. F. (1979). "Linear Models with Exchangeably Distributed Errors". *Journal of the American Statistical Association*, 74: 194–9.
- Berger, J. O. (2004). "The Case for Objective Bayesian Analysis". *Bayesian Analysis*, 1: 1–17.
- Bernardo, J. M., and Smith, A. F. M. (1994). *Bayesian Theory*. New York: Wiley.
- Chaloner, K. M., and Duncan, G. T. (1983). "Assessment of a Beta Prior Distribution: PM Elicitation". *Statistician*, 32: 174–80.
- Dawid, A. P. (1978). "Extendibility of Spherical Matrix Distributions," *Journal of Multivariate Analysis*, 8: 559–66.
- de Finetti, B. (1938). "Sur la Condition d'équivalence Partielle". *Actualités Scientifiques et Industrielles*, 739. Paris: Herman and Cii.
- Diaconis, P. (1988). "Recent Progress on de Finetti's Notions of Exchangeability", in J. M. Bernardo, M. H. DeGroot, D. V. Lindley, and A. F. M. Smith (eds.), *Bayesian Statistics*, iii. Oxford: Oxford University Press, 111–25.
- Eaton, M. L. and Lauritzen, S. L. (1992). "Finite de Finetti Theorems in Linear Models and Multivariate Analysis". *Scandinavian Journal of Statistics*, 19: 289–315.
- and Freedman, D. (1980). "de Finetti's Theorem for Markov Chains". *Annals of Probability*, 8: 115–30.
- (1981). "Partial Exchangeability and Sufficiency", in J. K. Ghosh and J. Roy (eds.), *Proceedings of the Indian Statistical Institute Golden Jubilee International Conference on Statistics: Applications and New Directions*. Calcutta: Indian Statistical Institute, 205–36.
- (1986). "On the Consistency of Bayes Estimates (with discussion)". *Annals of Statistics*, 14: 1–67.
- (1987). "A Dozen de Finetti-style Results in Search of a Theory". *Annals of the Institute Henri Poincaré*, 23: 394–423.
- and Ylvisaker, D. (1979). "Conjugate Priors for Exponential Families". *Annals of Statistics*, 7: 269–81.
- (1985). "Quantifying Prior Opinion" (with discussion), in J. M. Bernardo, M. H. DeGroot, D. V. Lindley, and A. F. M. Smith (eds.), *Bayesian Statistics*, ii. Amsterdam: North-Holland, 133–56.
- du Sautoy, M. (2008). *Finding Moonshine: A Mathematician's Journey through Symmetry*. London: Fourth Estate.
- Dubins, L. E., and Freedman, D. A. (1979). "Exchangeable Processes Need Not be Mixtures of Independent Identically Distributed Random Variables". *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 48: 115–32.
- Eaton, M. L., Fortini, S., and Regazzini, E. (1993). "Spherical Symmetry: An Elementary Justification". *Journal of the Italian Statistical Association*, 1: 1–16.
- Ferguson, T. S. (1974). "Prior Distributions on Spaces of Probability Measures". *Annals of Statistics*, 2: 615–29.
- Freedman, D. (1962a). "Mixtures of Markov Processes". *Annals of Mathematical Statistics*, 33: 114–18.
- (1962b). "Invariants under Mixing which Generalize de Finetti's Theorem". *Annals of Mathematical Statistics*, 33: 916–23.
- (1965). "Bernard Friedman's Urn". *Annals of Mathematical Statistics*, 36: 956–70.

- Fuchs, C. A., and Schack, R. (2004). "Unknown Quantum States and Operations, a Bayesian View", in M. G. A. Paris and J. Řeháček (eds.), *Quantum Estimation Theory*. Berlin: Springer.
- Galavotti, M. C. (2001). "Subjectivism, Objectivism and Objectivity in Bruno de Finetti's Bayesianism". in D. Cornfield and J. Williamson (eds.), *Foundations of Bayesianism*. Dordrecht: Kluwer, 161–74.
- Garthwaite, P. H., Kadane, J. B., and O'Hagan, A. (2005). "Statistical Methods for Eliciting Probability Distributions". *Journal of the American Statistical Association*, 100: 680–700.
- Geisser, S. (1984). "On Prior Distributions for Binary Trials (with discussion)". *American Statistician*, 38: 244–51.
- Gelman, A., Carlin, J. B., Stern, H. S., and Rubin, D. B. (1995). *Bayesian Data Analysis*. London: Chapman & Hall.
- Hewitt, E., and Savage, L. J. (1955). "Symmetric Measures on Cartesian Products". *Transactions of the American Mathematical Society*, 80: 470–501.
- Hill, B., Lane, D., and Suddreth, W. (1987). "Exchangeable Urn Processes". *Annals of Probability*, 15: 1586–92.
- Johnson, W. E. (1924). *Logic, Part III: The Logical Foundations of Science*, Cambridge: Cambridge University Press.
- Kadane, J. B., and Winkler, R. L. (1988). "Separating Probability Elicitation from Utilities". *Journal of the American Statistical Association*, 83: 357–63.
- Kingman, J. F. C. (1972). "On Random Sequences with Spherical Symmetry". *Biometrika*, 59: 492–4.
- Kreps, D. M. (1988). *Notes on the Theory of Choice*. Boulder, Colo.: Westview Press.
- Ladha, K. K. (1993). "Condorcet's Jury Theorem in Light of de Finetti's Theorem: Majority-Rule Voting with Correlated Votes". *Social Choice and Welfare*, 10: 69–85.
- Lauritzen, S. L. (1988). "Extremal Families and Systems of Sufficient Statistics", *Lecture Notes in Statistics*, 49. New York: Springer.
- Lindley, D. V. (1982). "Scoring Rules and the Inevitability of Probability". *International Statistical Review*, 50: 1–26.
- Loschi, R. H., Iglesias, P. L., and Arellano-Valle, R. B. (2003). "Predictivistic Characterizations of Multivariate Student-t Models". *Journal of Multivariate Analysis*, 85: 10–23.
- McCall, J. J. (1991). "Exchangeability and its Economic Applications". *Journal of Economic Dynamics and Control*, 15: 549–68.
- O'Hagan, A. (1994). *Kendall's Advanced Theory of Statistics, Vol. 2B, Bayesian Inference*. London: Halsted Press.
- Buck, C. E., Daneshkhah, A., Eiser, J. R., Garthwaite, P. H., Jenkinson, D. J., Oakley, J. E., and Rakow, T. (2006). *Uncertain Judgements: Eliciting Experts' Probabilities*. New York: Wiley.
- Poirier, D. J. (1988). "Frequentist and Subjectivist Perspectives on the Problems of Model Building in Economics (with discussion)". *Journal of Economic Perspectives*, 2: 121–70.
- (1995). *Intermediate Statistics and Econometrics: A Comparative Approach*. Cambridge, Mass.: MIT Press.
- Ramsey, F. P. (1926). "Truth and Probability", in Ramsey (1931), *The Foundations of Mathematics and other Logical Essays*, Ch. VII, ed. R. B. Braithwaite, London: Kegan, Paul, Trench, Trubner & Co. (New York: Harcourt, Brace and Company), 156–98.
- Schervish, M. J. (1995). *Theory of Statistics*. New York: Springer-Verlag.
- Schoenberg, I. J. (1938). "Metric Spaces and Positive Definite Functions". *Transactions of the American Mathematical Society*, 44: 522–36.

- Sims, C. A. (1971). "Distributed Lag Estimation When the Parameter Space is Explicitly Infinite-Dimensional". *Annals of Mathematical Statistics*, 42: 1622–36.
- Singpurwalla, N. (2006). *Reliability and Risk: A Bayesian Perspective*. Chichester: Wiley.
- Smith, A. F. M. (1981). "On Random Sequences with Centred Spherical Symmetry". *Journal of the Royal Statistical Society*, 43, Series B: 208–9.
- Zaman, A. (1986). "A Finite Form of de Finetti's Theorem for Stationary Markov Exchangeability". *Annals of Probability*, 14: 1418–27.