PSTAT 99 Research Paper

Adam Mohib

March 2025

1 Introduction

This paper explores key concepts in time series analysis and applies them to a multivariate time series of the daily returns of five sector-specific Vanguard ETFs. After defining foundational models such as white noise, moving averages, and ARIMA processes, autocorrelation structure was examined using ACF and PACF plots. Stationarity was tested using the Augmented Dickey-Fuller test and identify significant lags to inform potential model selection.

2 Time Series

Consider the probability space $(\Omega, \mathcal{A}, A)$ such that Ω denotes the set of all possible individual outcomes for an event, A is a subset of ω, and $\mathcal{A}$ be a collection of subsets A. Let T denote an index set. A real valued time series (or stochastic proccess) is a real valued function $X(t, \omega)$ defined on $T \times \Omega$ such that for each fixed t, $X(t, \omega)$ is a random variable on $\Omega, \mathcal{A}, P)$.[1, p. 3] The function $X(t, \omega)$ is often written $X_t(\omega)$ or X_t, and a time series can be considered as a collection, $\{X_t : t \in T\}$ of random variables.[1, p. 3]The joint distribution function of a finite set of random variables $\{X_{t_1}, X_{t_2}, ..., X_{t_n}\}$ from the collection $\{X_t : t \in T\}$ is defined by

$$F_{X_{t_1}, X_{t_2} ..., X_{t_n}}(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = P\{\omega : X(t_1, \omega) \leq x_1, \ldots, X(t_n, \omega) \leq x_n\}. \tag{1}$$

[1, p. 3]

2.1 White Noise

A simple kind of generated series might be a collection of uncorrelated random variables, w_t, with mean 0 and finite variance σ_w^2. The time series generated from uncorrelated variables is used as a model for noise in engineering applications, where it is called *white noise*; we shall denote this process as $w_t \sim wn(0, \sigma_w^2)$.[2, p. 9] We will sometimes require the noise to be independent and identically distributed (iid) random variables with mean 0 and variance σ_w^2. We distinguish this by writing $w_t \sim iid(0, \sigma_w^2)$ or by saying *white independent noise* or *iid noise*.[2, p. 9]

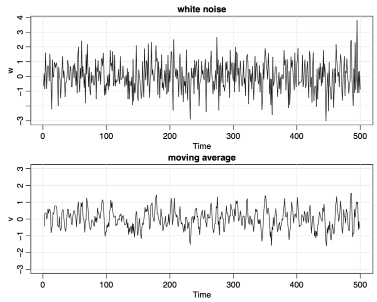

Figure 1: Gaussian white noise series (top) and three-point moving average of the Gaussian white noise series (bottom).[2, p.10]

2.2 Moving Average

We shall call the time series $\{X_t : t \in (0, \pm 1, \pm 2, \dots)\}$, defined by

$$X_t = \sum_{j=-M}^{M} \alpha_j e_{t-j}, \tag{2}$$

where M is a nonnegative integer, α_i are real numbers, $\alpha_{-M} \neq 0$, and the e_t are uncorrelated $(0, \sigma^2)$ random variables, a *finite moving average time series* or a *finite moving average process*.[1, p. 21] Note $e_t \overset{d}{=} w_t$. We might replace the white noise series w_t by a moving average that smooths the series. For example, consider replacing w_t by an average of its current value and its immediate neighbors in the past and future.[2, p. 9] That is, let

$$v_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}) \tag{3}$$

which leads to the series shown in the lower panel of Figure 1.[2, p. 9]

2.3 Random Walk

A model for analyzing trend is the *random walk with drift* model given by

$$x_t = \delta + x_{t-1} + w_t \tag{4}$$

for $t = 1, 2, \dots$, with initial condition $x_0 = 0$, and where w_t is white noise. The constant δ is called the drift, and when $\delta = 0$, (4) is called simply a random walk.[2, p.11] Note we may write (4) as a cumulative sum of white noise variates. That is,

$$x_t = \delta t + \sum_{j=1}^{t} w_j \tag{5}$$

for $t = 1, 2, \dots$.
[2, p. 12]

3 Autocovariance and Autocorrelation

3.1 Autocovariance

The **autocovariance function** is defined as the *second moment product*

$$\gamma_x(s,t) = \text{cov}(x_s, x_t) = \text{E}[(x_s - \mu_s)(x_t - \mu_t)], \tag{6}$$

for all s and t.[2, p. 16] The autocovariance measures the *linear* dependence between two points on the same series observed at different times.[2, p. 16] For example, The white noise series w_t has $E(w_t) = 0$ and

$$\gamma_w(s,t) = \text{cov}(w_s, w_t) = \begin{cases} \sigma^2 & s = t, \\ 0 & s \neq t. \end{cases} \tag{7}$$

[2, p. 16] A realization of white noise with $\sigma_w^2 = 1$ is shown in the top panel of Figure 1.

3.1.1 Autocovariance of a Moving Average

Consider applying a three-point moving average to the white noise series w_t of the previous example as in (3). In this case,

$$\gamma_v(s,t) = \text{cov}(v_s, v_t) = \begin{cases} \frac{3}{9}\sigma_w^2 & s = t, \\ \frac{2}{9}\sigma_w^2 & |s-t| = 1, \\ \frac{1}{9}\sigma_w^2 & |s-t| = 2, \\ 0 & |s-t| > 2. \end{cases} \tag{8}$$

[2, p. 17]

3.1.2 Autocovariance of a Random Walk

For the random walk model, $x_t = \sum_{j=1}^{t}$, we have

$$\gamma_x(s,t) = \text{cov}(x_s, x_t) = \text{cov}(\sum_{j=1}^{s} w_j, \sum_{k=1}^{t} w_k) = \min\{s,t\}\sigma_w^2, \tag{9}$$

because the w_t are uncorrelated random variables.[2, p. 18]

3.2 Autocorrelation

The **autocorrelation function (ACF)** is defined as

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}. \tag{10}$$

[2, p.18]

The ACF measures the linear predictability of the series at time t, say x_t , using only the value x_s.[2, p. 18] If we can predict x_t perfectly from x_s through a linear relationship, $x_t = \beta_0 + \beta_1 x_s$, then the correlation will be $+1$ when $\beta_1 > 0$, and -1 when $\beta_1 < 0$.[2, p. 18]

4 Stationary Time Series

A **strictly stationary time series** is one for which the probabilistic behavior of every collection of values $\{x_{t_1}, x_{t_2}, \ldots x_{t_k}\}$ is identical to the shifted set $\{x_{t_{1+h}}, x_{t_{2+h}}, \ldots x_{t_{k+h}}\}$.[2, p. 19] That is,

$$\Pr\{x_{t_1}, x_{t_2}, \ldots, x_{t_k}\} = \Pr\{x_{t_{1+h}}, x_{t_{2+h}}, \ldots, x_{t_{k+h}}\} \tag{11}$$

for all $k = 1, 2, \ldots$, all time points $t_1, t_2, \ldots, t_k$, all numbers $c_1, c_2, \ldots, c_k$, and all time shifts $h = 0, \pm 1, \pm 2, \ldots$.[2, p. 19] A **weakly stationary** time series x_t is a finite variance process such that
(i) the mean value function, defined by $\mu_t = \mathrm{E}(x_t) = \int_{-\infty}^{\infty} x f_t(x) dx$ such that $f_t(x)$ denotes marginal density of x at t, is constant and does not depend on time t, and
(ii) the autocovariance function, $\gamma(s, t)$, defined in (6) depends on s and t only through their difference $|s - t|$.[2, p. 20] Because the mean function, $E(x_t) = \mu_t$, of a stationary time series is independent of time t, we will write $\mu_t = \mu$.[2, p. 20] Also, because the autocovariance function, $\gamma(s, t)$, of a stationary time series, x_t, depends on s and t only through their difference

$$|s - t|$$

, we may simplify the notation. Let $s = t + h$, where h represents the time shift or *lag*. Then, $\gamma(t + h, t) = \mathrm{cov}(x_t + h, x_t) = \mathrm{cov}(x_h, x_0) = \gamma(h, 0)$.[2, p. 20] The autocorrelation function of a stationary time series will be written using (10) as $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$.[2, p. 20]

5 ARIMA Models

5.1 Autoregressive Models

An **autoressive model** of order p, abbreviated $\mathbf{AR}(p)$, is one of the form

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t, \tag{12}$$

where x_t is stationary, $w_t \sim \mathrm{wn}(0, \sigma_w^2)$, and $\phi_1, \phi_2, \ldots, \phi_p$ are constants ($\phi_p \neq 0$).[2, p. 78] The mean of x_t in (12) is zero. If the mean, μ, of x_t is not zero, replace x_t by $x_t - \mu$ in (12),

$$x_t - \mu = \phi_1(x_t - 1 - \mu) + \phi_2(x_t - 2 - \mu) + \cdots + \phi_p(x_t - p - \mu) + w_t$$

, or write

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t. \tag{13}$$

[2, p. 78]

5.2 Moving Average Models

The **moving average model** of order q, or **MA**(q) model, is defined to be

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}, \tag{14}$$

where $w_t \sim \text{wn}(0, \sigma_w^2)$, and $\theta_1, \theta_2, \ldots, \theta_q (\theta_q \neq 0)$ are parameters.[2, p. 83]

5.3 Autoregressive Moving Average Models

A time series $x_t; t = 0, \pm 1, \pm 2, \ldots$ is ARMA(p, q) if it is stationary and

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 wt - 1 + \cdots + \theta_q w_{t-q}, \tag{15}$$

with $\phi_p \neq 0$,$\theta_q \neq 0$, and $\sigma_w^2 > 0$.[2, p. 85] If x_t has a nonzero mean μ, we set $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$ and write the model as

$$x_t = \alpha + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}, \tag{16}$$

where $w_t \sim \text{wn}(0, \sigma_w^2)$.[2, p. 86] An ARMA$(p, q)$ model is said to be **causal**,if the time series $x_t; t = 0, \pm 1, \pm 2, \ldots$ can be written as a one-sided linear process:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t, \tag{17}$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.[2, p.87] An ARMA(p, q) model is causal if and only if $\phi(z) \neq 0 for |z| \leq 1$. The coefficients of the linear process given in (17) can be determined by solving

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)} \quad |z| \leq 1$$

.[2, p.87]

6 Sample Autocorrelation and Partial Autocorrelation Function

6.1 Sample Autocorrelation Function

Consider a weakly stationary return series r_t. The correlation coefficent between r_t and r_{t-l} is called the *lag-l autocorrelation* of r_t and is commonly denoted by ρ_l, which under the weak stationarity assumption is a function of l only.[3, p. 31] This is an extension of (10). Specifically, we define

$$\rho_l = \frac{\gamma(r_t, r_{t-l})}{\sqrt{\gamma(r_t, r_t)\gamma(r_{t-l}, r_{t-l})}} = \frac{\gamma(r_t, r_{t-l})}{\gamma(r_t, r_t)} = \frac{\gamma_l}{\gamma_0}, \tag{18}$$

where the property $\mathrm{Var}(r_t) = \mathrm{Var}(r_{t-l})$ for a weakly stationary time series is used.[3, p. 31] From the definition, we have $\rho_0 = 1$, $\rho_l = \rho_{-l}$, and $-1 \leq \rho_l \leq 1$.[3, p. 31] Consider a given sample of returns $\{r_t\}_{t=1}^{T}$, such that $\bar{r}$ is the sample mean. That is, $\bar{r} = \sum_{t=1}^{T} r_t/T$. In general, the lag-l sample autocorrelation of r_t is defined as

$$\hat{\rho_l} = \frac{\Sigma_{t=l+1}^{T}(r_t - \bar{r})(r_{t-l} - \bar{r})}{\Sigma_{t=1}^{T}(r_t - \bar{r})^2}, \quad 0 \leq l < T - 1. \tag{19}$$

[3, p. 31]
Box and Pierce (1970) propose the Portmanteau statistic $Q^*(m) = T\Sigma_{l=1}^{m}\hat{\rho_l}^2$ as a test statistic for the null hypothesis $H_0 : \rho_1 = \cdots = \rho_m = 0$ against the alternative hypothesis $H_a : \rho_i \neq 0$ for some $i \in \{1, \ldots, m\}$.[3, p. 32] Ljung and Box modify the $Q^*(m)$ statistic as below to increase the power of the test in finite samples,

$$Q(m) = T(T+2)\sum_{l=1}^{m}\frac{\hat{\rho}_l^2}{T-l}. \tag{20}$$

[3, p. 32]
The decision rule is to reject H_0 if $Q(m) > \chi_\alpha^2$, where χ_α^2 denotes the $100(1-\alpha)$th percentile of a chi-square distribution with m degrees of freedom.[3, p. 32] The statistics $\hat{\rho_1}, \rho_2, \ldots$ defined in (18) is called the *sample autocorrelation function* (ACF) of r_t.[3, p. 33] Under general conditions, if x_t is white noise, then for n large, the sample ACF, distributed with zero mean and standard deviation given by

$$\sigma_{\hat{\rho}_x(h)} = \frac{1}{\sqrt{n}} \tag{21}$$

[2, p. 28]

6.2 Sample Partial Autocorrelation Function

The PACF of a stationary time series is a function of its ACF and is a useful tool for determining the order p of an AR model. Consider the following AR models in consecutive orders:

$$\begin{aligned}
r_t &= \phi_{0,1} + \phi_{1,1}r_{t-1} + e_{1t},\\
r_t &= \phi_{0,2} + \phi_{1,2}r_{t-1} + \phi_{2,2}r_{t-2} + e_{2t},\\
r_t &= \phi_{0,3} + \phi_{1,3}r_{t-1} + \phi_{2,3}r_{t-2} + \phi_{3,3}r_{t-3} + e_{3t},\\
r_t &= \phi_{0,4} + \phi_{1,4}r_{t-1} + \phi_{2,4}r_{t-2} + \phi_{3,4}r_{t-3} + \phi_{4,4}r_{t-4} + e_{4t},\\
&\vdots
\end{aligned}$$

where $\phi_{0,j}, \phi_{i,j}$, and $\{e_{jt}\}$ are, respectively, the constant term, the coefficient of r_{t-i}, and the error term of an AR(j) model.[3, p. 47] The estimate $\hat{\phi}(1,1)$ of the first equation is called the lag-1 sample PACF of r_t. The estimate $\hat{\phi}(2,2)$ of the second equation is called the lag-2 sample PACF of r_t. The estimate $\hat{\phi}(3,3)$ of the second equation is called the lag-3 sample PACF of r_t, and so on.[3, p. 47]

7 Unit Root Testing

7.1 Standard Brownian Motion

A continuous time process $W(t); t \geq 0$ is called **standard Brownian motion** if it satisfies the following conditions:
(i)$W_0 = 0$;
(ii) $\{W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})\}$ are independent for any collection of points $0 \leq t_1 < t_2 \cdots < t_n$, and integer $n > 2$;
(iii) $W(t + \Delta t) - W(t) \nabla N(0, \Delta t)$ for $\Delta t > 0$.[2, p. 251]

7.2 Dickey Fuller Test

Consider a casual AR(1) process with Gaussian noise,

$$x_t = \phi x_{t-1} + w_t \tag{22}$$

Applying $(1 - B)$ to both sides shows that differencing, $\nabla x_t = \nabla x_{t-1} + \nabla w_t$, or

$$y_t = \phi y_{t-1} + w_t - w_{t-1},$$

A unit root test provides a way to test whether (22) is a random walk (the null case) as opposed to a causal process (the alternative). That is, it provides a procedure for testing

$$H_0 : \phi = 1 \text{ versus } H_1 |\phi| < 1$$

.[2, p. 250] To examine the behavior of $(\hat{\phi} - 1)$ under the null hypothesis that $\phi = 1$, or more precisely that the model is a random walk, $x_t = \sum_{j=1}^{t} w_j$, or $x_t = x_t - 1 + w_t$ with $x_0 = 0$, consider the least squares estimator of ϕ. Noting that $\mu_x = 0$, we have that

$$\hat{\phi} - 1 = \frac{\frac{1}{n\sigma_w^2} \sum_{t=1}^{n} w_t x_{t-1}}{\frac{1}{n\sigma_w^2} \sum_{t=1}^{n} x_{t-1}^2}. \tag{23}$$

[2, p. 250]
As $n \to \infty$, we have that

$$n(\hat{\phi} - 1) = \frac{\frac{1}{n\sigma_w^2} \sum_{t=1}^{n} w_t x_{t-1}}{\frac{1}{n^2\sigma_w^2} \sum_{t=1}^{n} x_{t-1}^2} \xrightarrow{d} \frac{\frac{1}{2}(\chi_1^2 - 1)}{\int_0^1 W^2(t)dt}. \tag{24}$$

where $W(t)$ is standard Brownian motion on $[0, 1]$.[2, p. 252] The test statistic $n(\hat{\phi} - 1)$ is known as the unit root or Dickey-Fuller (DF) statistic, although the actual DF test statistic is normalized a little differently.[2, p. 252] Because the distribution of the test statistic does not have a closed form, quantiles of the distribution must be computed by numerical approximation or by simulation.[2, p. 252] Toward a more general model, we note that the DF test was established by noting that if $x_t = \gamma x_{t-1} + w_t$, then$\nabla x_t = (\phi - 1)x_t - 1 + w_t = \gamma x_t - 1 + w_t$, and one could test $H_0 : \gamma = 0$ by regressing ∇x_t on $x_t - 1$.[2, p. 252]

7.3 Augmented Dickey-Fuller Test

The test was extended to accommodate $AR(p)$ models, $x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + w_t$, as follows. Subtract x_{t-1} from both sides to obtain

$$\nabla x_t = \gamma x_{t-1} + \sum_{j=1}^{p-1} \psi_j \nabla x_{t-j} + w_t,, \tag{25}$$

where $\gamma = \sum_{j=1}^{p} \phi_j - 1$ and $\psi_j = -\sum_{i=j}^{p} \phi_i$ for $j = 2, \ldots, p$.[2, p. 252] We can test $H_0 : \gamma = 0$ by estimating γ in the regression of ∇x_t on $x_{t-1}, \nabla x_{t-1}, \ldots, \nabla x_{t-p+1}$, and forming a Wald test based on $t_\gamma = \frac{\hat{\gamma}}{se(\hat{\gamma})}$.[2, p. 252] This test leads to the so-called augmented Dickey-Fuller test (ADF).[2, p. 252] One can extend the model to include a constant, or even non-stochastic trend.[2, p. 252] For example, consider the model

$$x_t = \beta_0 + \beta_1 t + \phi x_{t-1} + w_t. \tag{26}$$

[2, p. 252]
If we assume $\beta_1 = 0$, then under the null hypothesis, $\phi = 1$, the process is a random walk with drift β_0.[2, p. 252] Under the alternate hypothesis, the process is a causal $AR(1)$ with mean $\mu_x = \beta_0(1 - \phi)$.[2, p. 252] If we cannot assume $\beta_1 = 0$, then the interest here is testing the null that $(\beta_1, \phi) = (0, 1)$, simultaneously, versus the alternative that $\beta_1 \neq 0$ and $|\phi| < 1$.[2, p. 252] In this case, the null hypothesis is that the process is a random walk with drift, versus the alternative hypothesis that the process is trend stationary.[2, p. 252] In the ADF test, the default number of AR components included in the model, say k, is $\lfloor (n-1)^{\frac{1}{3}} \rfloor$, which corresponds to the suggested upper bound on the rate at which the number of lags, k, should be made to grow with the sample size for the general ARMA(p, q) setup.[2, p. 253]

8 GARCH Models

8.1 ARCH Models

Models such as the *autoregressive conditionally heteroscedastic* or ARCH model, first introduced by Engle (1982), were developed to model changes in volatility.[2, p. 253] These models were later extended to generalized ARCH, or GARCH models by Bollerslev (1986).[2, p. 253] Either value, $\Delta log(x_t)$ or $\frac{(x_t - x_{t-1})}{x_{t-1}}$, will be called the *return*, and will be denoted by r_t.[2, p. 253] The simplest ARCH model, the ARCH(1), models the return as

$$r_t = \sigma_t \epsilon_t \tag{27}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2, \tag{28}$$

where ϵ_t is standard Gaussian white noise, $\epsilon_t \sim$ iid $\mathrm{N}(0,1)$.[2, p. 254] The ARCH(1) model can be extended to the general ARCH(p) model in an obvious way.That is, (27), $r_t = \sigma_t \epsilon_t$, is retained, but (28) is extended to

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \cdots + \alpha_p r_{t-p}^2. \tag{29}$$

[2, p. 257]

8.2 GARCH Model

Another extension of ARCH is the generalized ARCH or GARCH model developed by Bollerslev (1986).[2, p. 258] For example, a GARCH(1,1) model retains (27), $r_t = \sigma_t \epsilon_t$,but extends (28) as follows:

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \tag{30}$$

[2, p. 258]
The GARCH(p, q) model retains (27) and extends (30) to

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q}]\beta_j \sigma_{t-j}^2. \tag{31}$$

[2, p. 258]

9 Methods and Results

The following methodology and analysis were performed in R studio, with R version 4.4.2 (2024-10-31).

9.1 Data Preprocessing

Daily Open, High, Low, Close (OHLC) and Volume data for assets Vanguard Materials ETF (VAW), Vanguard Energy ETF (VDE), Vanguard Financials ETF (VFH), Vanguard Industrials ETF (VIS), Vanguard Utilities ETF (VPU) from (March 13, 2023) to (March 11, 2025), was obtained using Yahoo Finance API, through R package quantmod. Asset data was only recorded on trading days. The simple returns for each asset were then calculated using the following formula $r_{i,t} = \frac{p_{i,t} - p_{i,t-1}}{p_{i,t-1}}$, where $r_{i,t}$ denotes the return at time t for the marginal asset i, and $p_{i,t}$ denotes the closing price at time t for marginal asset i. The simple returns of each asset were stored in a common eXtensible time series (xts) object. The number of entries in each marginal asset's time series is $n = 500$.

The returns of each marginal asset were then plotted using ggplot2 as shown in Fig. (2).

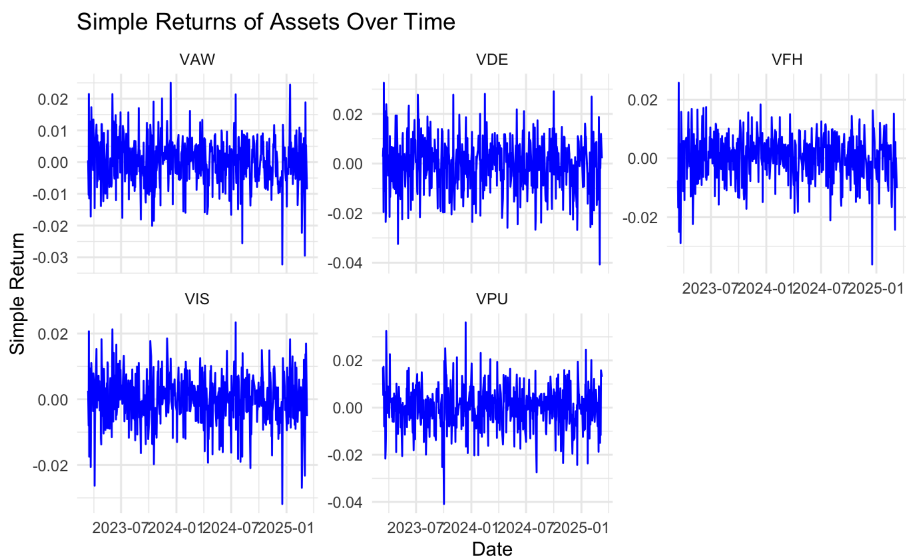

Figure 2: Simple returns of assets VAW, VDE, VFH, VIS, VPU from (time 1) to (time 2).

9.2 Augmented Dickey-Fuller Test

The R package tseries was used to run the Augmented Dickey-Fuller test on each marginal asset's returns time series. The null and alternative hypotheses for the test were each marginal is non-stationary versus each marginal asset is weakly stationary, respectively. That is, $H_0 : \gamma_i = 0$ versus $H_1 : |\gamma_i| < 0$, where γ_i represents γ in (25) for marginal asset i. The test was run with 7 lags for all assets. The general regression equation for the test incorporated a constant and a linear trend. That is, $\beta_0 \neq 0$ and $\beta_1 \neq 0$ for β_0, β_1 in (26). The dickey-fuller t-statistic and p-value for the test on each marginal asset is shown in Table 1.

VAW	VDE	VFH	VIS	VPU
$t_\gamma = -7.758$	$t_\gamma = -8.4815$	$t_\gamma = -7.5424$	$t_\gamma = -8.1925$	$t_\gamma = -7.646$
$p < .01$	$p < .01$	$p < .01$	$p < .01$	$p < .01$

Table 1: Dickey-Fuller t-statistic and P-value returned by ADF test for each marginal asset.

The p-values for each marginal asset is less than .01, therefore we reject H_0 at the $\alpha = .01$ signficance level for all assets. We have strong statistical evidence suggesting that each asset is weakly stationary around a constant term and trend term.

9.3 ACF and PACF

9.3.1 ACF and PACF of Returns

The ACF and PACF of each asset's returns was then plotted using R package stats with

```
lag.max = 50
```

. Under the null hypothesis that each marginal return series follows white noise, that is each marginal returns' series $r_{i,t} \sim \text{wn}(0, \sigma_w^2)$, with $H_0 : \rho(h) = 0$, we have from (21), that the sample acf is normally distributed with zero mean and $SE_{\hat{\rho}_x} = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{500}}$. Using the same standard error for the sample PACF of each marginal asset, a 95% confidence interval of acf and pacf lags for each time series was calculated, and plotted, an example of which is shown in Fig. (3). Statistically significant lags for each asset are shown in Table (2)

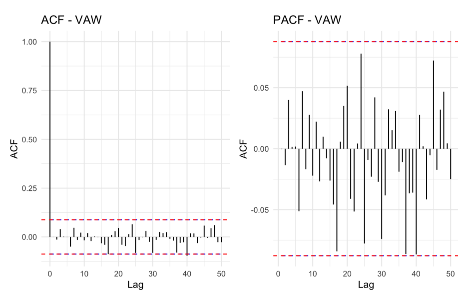

Figure 3: Sample ACF and PACF of marginal asset VAW returns, with 95% confidence interval bounds ($\pm 1.96\frac{1}{\sqrt{500}}$) for significant lags.

VAW	VDE	VFH	VIS	VPU
$\hat{\rho}_{17}, \hat{\rho}_{40}$	$\hat{\rho}_{21}$	$\hat{\rho}_{17}$	$\hat{\rho}_{40}, \hat{\rho}_{43}$	$\hat{\rho}_{10}$
None	$\hat{\phi}_{21,21}$	$\hat{\phi}_{25,25}$	$\hat{\phi}_{43,43}$	$\hat{\phi}_{10,10}$

Table 2: Statistically signficant ACF and PACF lags for each marginal asset's returns using a 95% confidence interval.

9.3.2 ACF and PACF of Squared Returns

Under the same assumptions previously, the sample ACF and PACF of each asset's squared returns were calculated and plotted, an example of which is shown in Fig. (4). Statistically significant lags for each asset are shown in Table (3).

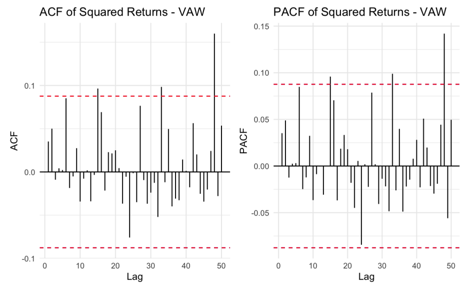

Figure 4: Sample ACF and PACF of marginal asset VAW squared returns, with 95% confidence interval bounds ($\pm 1.96\frac{1}{\sqrt{500}}$) for significant lags.

VAW	VDE	VFH	VIS	VPU
$\hat{\rho}_{50}$	$\hat{\rho}_{13}, \hat{\rho}_{20}, \hat{\rho}_{29}, \hat{\rho}_{33}$	$\hat{\rho}_{1}, \hat{\rho}_{29}, \hat{\rho}_{43}$	$\hat{\rho}_{29}$	$\hat{\rho}_{2}, \hat{\rho}_{31}, \hat{\rho}_{33}, \hat{\rho}_{35}$
$\hat{\phi}_{50,50}$	$\hat{\phi}_{13,13}, \hat{\phi}_{20,20}, \hat{\phi}_{29,29}$	$\hat{\phi}_{1,1}, \hat{\phi}_{29,29}, \hat{\phi}_{43,43}$	$\hat{\phi}_{29,29}$	$\hat{\phi}_{2,2}, \hat{\phi}_{31,31}$

Table 3: Statistically signficant ACF and PACF lags for each marginal asset's squared returns using a 95% confidence interval.

10 Conclusion

This paper confirms that each asset is weakly stationary through the ADF test, and exhibits distinct autocorrelation patterns. Statistically significant ACF and PACF lags suggest underlying AR or MA dynamics, which can guide future modeling with ARIMA or GARCH frameworks. Overall, the analysis demonstrates how classical time series tools can reveal structure in financial return series.

10.1 Future Implications

In the future, modeling with ARIMA or GARCH frameworks should be implemented to asset data in the univariate setting. Then, a Bernstein copula will be used to the standardized residuals to model nonlinear dependence between marginal assets.

References

[1] Wayne A. Fuller. *Introduction to statistical time series.* John Wiley & Sons, 2 edition, dec 29 1995. [Online; accessed 2025-03-13].

[2] Robert H. Shumway and David S. Stoffer. *Time series analysis and its applications: With R examples.* Springer, 4 edition, apr 25 2017. [Online; accessed 2025-03-16].

[3] Ruey S. Tsay. *Analysis of Financial Time Series.* John Wiley & Sons, 3 edition, oct 26 2010. [Online; accessed 2025-03-20].