Eigen values, Taylor Series, and Stability Review

July 13, 2019

1 Linear stability analysis

This review sheet has been adapted from a set of notes written by Gordon Brown in 1990, titled "Stability Analysis of Continuous Single-Species Population Models".

A model of population growth such as:

$$\frac{dN}{dt} = F(N) \tag{1}$$

will have equilibrium point N^* when:

$$\frac{dN}{dt} = 0 \tag{2}$$

Assuming that N^* is found, one might wish to determine its stability - i.e. if N is perturbed away from N^*, will it return to N^* or move to some other value? Linear stability analysis can help answer this question, provided that the perturbation is "small" (unfortunately, what constitutes a "small" perturbation depends on the equation being analyzed). This handout will attempt to illustrate some of the logic behind linear stability analysis, and is designed to complement Dr. Tilman's discussion. Let:

$$N = N^* + a(t) \tag{3}$$

therefore:

$$a(t) = N - N^* \tag{4}$$

Linear stability analysis determines the fate of the function $a(t)$, which indicates the magnitude of the deviation from equilibrium, as t approaches infinity. If $a(t) \to 0$, then N returns to N^*. To examine the dynamics of $a(t)$, we find its derivative:

$$\begin{aligned} \frac{d}{dt}(a(t)) &= \frac{d}{dt}(N - N^*) \\ &= \frac{dN}{dt}(\text{sinceN}^*\text{isaconstant}) \end{aligned} \tag{5}$$

An expression for $\frac{dN}{dt}$ in terms of $a(t)$ and t is now needed. Recall that $\frac{dN}{dt} = F(N)$, and that we are examining values of N which are close to N^*. In order to derive our function of $a(t)$ at t, we will expand $F(N)$ about the value N^* using a Taylor Series approximation. The formula for the Taylor Series is:

$$f(x) = f(a) + \frac{df}{dx}|_{x=a}(x-a) + \frac{\frac{d^2f}{dx^2}}{2!}|_{x=a}(x-a)^2 + \frac{\frac{d^3f}{dx^3}}{3!}|_{x=a}(x-a)^3 + ... \tag{6}$$

where a is a value for x. When applied to our case, we get:

$$F(N) = F(N)|_{N^*} + \frac{dF}{dN}|_{N^*}(N-N^*) + \frac{\frac{d^2F}{dN^2}}{2!}|_{N^*}(N-N^*)^2 + \frac{\frac{d^3F}{dN^3}}{3!}|_{N^*}(N-N^*)^3 + ... \tag{7}$$

We only use the first two terms of this series to yield a linear equation which approximates $F(N)$ (this is why this procedure is called linear stability analysis).

$$F(N) \approx F(N)|_{N^*} + \frac{dF}{dN}|_{N^*}(N - N^*) \tag{8}$$

However, since $F(N^*) = 0$, this simplifies further to $F(N) \approx \frac{dF}{dN}|_{N^*}(N - N^*)$.

Recall that $\frac{d}{dt}(a(t)) = \frac{dN}{dt} = F(N)$, and $a(t) = N - N^*$. Substituting these relationships into the linear approximation for $F(N)$ yields:

$$\frac{d}{dt}(a(t)) = \frac{d}{dN}(F(N))|_{N^*}a(t) \tag{9}$$

If we let $\frac{d}{dN}(F(N))|_{N^*} = \lambda$, then we simply get:

$$\frac{d}{dt}(a(t)) = \lambda a(t) \tag{10}$$

We now have our differential equation in terms of $a(t)$ and t, and can use it to determine the dynamics of $a(t)$, the deviation of population size from its equilibrium value. First, we must solve for $a(t)$:

$$\begin{aligned}
(\frac{1}{a(t)})d(a(t)) &= \lambda dt \\
\int(\frac{1}{a(t)})d(a(t)) &= \int \lambda dt \\
\log(a(t)) + K_1 &= \lambda t + K_2
\end{aligned} \tag{11}$$

let $K = K_2 - K_1$

$$\begin{aligned}
\log(a(t)) &= \lambda t + K \\
e^{\log(a(t))} &= e^{\lambda t + K}
\end{aligned} \tag{12}$$

let $e^K = K'$

$$a(t) = K'e^{\lambda t} \tag{13}$$

let $t = 0$ to find the initial perturbation

$$\begin{aligned}
a(0) &= K'e^{\lambda 0} \\
K' &= a(0) \\
a(t) &= a(0)e^{\lambda t}
\end{aligned} \tag{14}$$

Here is our equation to determine the dynamics of the perturbation with time. Obviously, if $c < 0$, then the perturbation decays away exponentially with time, and the population returns to its equilibrium value; the equilibrium is therefore stable. If $c > 0$, then the perturbation grows exponentially with time; the equation is unstable. If $c = 0$, then the equilibrium is neutrally stable; the perturbation does not grow or shrink with time (at least, according to our linearized system). Remember that:

$$c = \frac{d}{dN}(F(N))|_{N^*} \tag{15}$$

Therefore the sign of $\frac{d}{dN}(F(N))$ at the equilibrium point determines the stability of that equilibrium.

2 A Worked Example

Consider the standard logistic growth model $F(N) = \frac{dN}{dt} = rN(1 - \frac{N}{K}) = rN - \frac{rN^2}{K}$. This model has two equilibria, at $N = 0$ and $N = K$. Differentiating by N, we get:

$$\frac{d}{dN}(F(N)) = r - \frac{2rN}{K} \tag{16}$$

For equilibrium $N = 0$, we solve $\frac{d}{dN}(F(N)) = r - 0 = r = \lambda$. Thus, provided that r is positive, we expect deviations from this equilibrium to grow roughly following e^r, and the point is thus unstable (leftmost blue line in figure).

For equilibrium $N = K$, we solve $\frac{d}{dN}(F(N)) = r - \frac{2rK}{K} = r - 2r = -r = \lambda$. Thus, provided that r is positive, we expect deviations from equilibrium to shrink back to K roughly following e^{-r} (right red and blue lines in the figure).

Since all known organisms have a positive growth rate in an unlimited environment, this analysis indicates that the equilibrium at zero is unstable ($\lambda > 0$), and that the equilibrium at K is locally stable ($\lambda < 0$). The equilibrium at K is also globally stable, but that's another story.

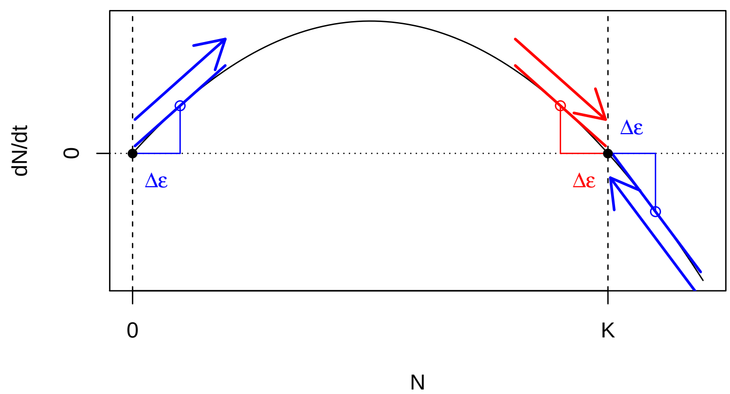

Figure 1: Linearized stability analysis. Dynamics following small perturbations around equilibrium $\Delta\epsilon$ can be estimated using a first-order Taylor expansion. Thick lines show these derivatives, and arrows show the resulting directions of flow on either side of the equilibrium.