The Fisher equation and Fisher waves

Let us consider the following, reaction diffusion, equation in 2 dimensions, $\varphi(\mathbf{r})$ due to a (scalar) field of electric charges $\rho(\mathbf{r})$ is the solution of Poissons equation

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi + \alpha\varphi\left(1-\varphi\right), \tag{1}$$

where φ indicates the concentration of a chemical and α the rate of reaction, while D is the diffusion coefficient. Eq. 1 is known as the Fisher equation.

1. Write down a finite difference algorithm to integrate Eq. 1 subject to a suitable initial condition, and periodic boundary conditions in space.

2. Based on this algorithm, write a Java code to solve the Fisher equation on an $N \times N$ grid, subject to the initial condition that $\varphi = 1$ for $|\mathbf{r}| < R$, where $\mathbf{r}$ is the position vector linking any point of the grid to the grid centre, and R is the "radius" of the initial droplet. You should see that with a suitable value of R (sufficiently large), the initial nucleus increases and spreads in a wavelike fashion, and invades the whole system. You can set $D = \alpha = 1$.

3. Now consider an effectively 1D system, in which initially φ is 1 for $x < x_0$, and 0 otherwise. The dynamics does not introduce any dependence on y, so you can simulate an $N \times 1$ system to make your simulation more efficient. Why is this?

4. Calculate the integral over space of φ, and plot this value over time. Use this to compute the "wave velocity", and show that this is equal to 2 asymptotically.

5. Repeat the calculation using an initial condition which, instead of being a Heaviside-function, starts from 1 and decays to zero exponentially. For simplicity, instead of using periodic boundary condition in space you can use no flux boundary conditions, setting $\frac{\partial \varphi}{\partial x} = 0$ at the boundary of your 1D grid. Plot the wave velocity as a function of the decay rate of the exponential.