Contents

Pre	Preface			
I	The N	Nature Trail	1	
1	△ C	oin Tosses and Polynomials	3	
	1.1	Probability and experiments	4	
	1.2	A Law of Large Numbers	8	
	1.3	The Weierstrass Approximation Theorem	13	
	1.4	Worked problems	16	
	Exerc	ises	17	
2	Sets and Functions			
	2.1	Sets	20	
	2.2	Functions	24	
	2.3	Cardinality	28	
	2.4	Sequences of sets	31	
	2.5	The extended real number system	33	
	2.6	Worked Problems	35	
	Exerc	ises	36	
3	▲ A Probability Model for Sequences of Coin Tosses			
	3.1	A probability model for sequences of coin tosses	38	
	3.2	A Weak Law of Large Numbers	45	
	3.3	Sets of measure zero	50	
	3.4	A Strong Law of Large Numbers	56	
	3.5	Worked problems	58	
	Exerc	ises	60	
П	The	Foothills	61	
			63	
4	Construction of a General Measure Structure			
	4.1	Sigma algebras	64	
	4.2	Measure	71	
	4.3	Sets of measure zero and completion of measure	78	
	4.4	Outer measures	80	
	4.5	A Hausdorff measure	91	
	4.6	Premeasures	96	

vi

	4.7	Approximation of measures	. 101
	4.8	A zoology of measure creatures	. 103
	4.9	Worked problems	. 103
	Exerci	ises	. 104
5	Measi	ure Structure in Euclidean Space	107
	5.1	Approximation of open sets	. 109
	5.2	Generating the σ - algebra	
	5.3	Borel and Lebesgue-Stieljes measures in \mathbb{R}	
	5.4	Borel and Lebesgue-Stieljes measures in \mathbb{R}^n	. 126
	5.5	The regularity of the Lebesgue-Stieljes measure	. 135
	5.6	△ Properties of Lebesgue measure	. 137
	5.7	Approximation of Lebesgue-Stieljes measure	. 139
	5.8	Worked problems	. 141
	Exerci	ises	. 142
6	Proba	ability and Measure	145
	6.1	Probability spaces	. 146
	6.2	Examples of probability models	
	6.3	"Infinitely often" and the First Borel Cantelli Lemma	
	6.4	Conditional probability and independence	. 176
	6.5	The Second Borel-Cantelli Lemma	. 181
	6.6	\triangle Tail σ -algebras	. 182
	6.7	Worked problems	. 184
	Exerci	ises	. 187
7	Measi	urable Functions and Random Variables	189
	7.1	σ - algebras induced by functions	. 190
	7.2	Measurable functions	. 193
	7.3	Approximation of measurable functions by simple functions	. 200
	7.4	A Relation of measurability to continuity	. 206
	7.5	Measures induced by measurable functions	. 208
	7.6	Probability measures of random variables	. 210
	7.7	Random vectors	. 218
	7.8	\triangle Properties of the Hausdorff measure in \mathbb{R}^n	. 223
	7.9	Worked problems	. 224
	Exerci	ises	. 226
8	Integr	ration and Expectation	229
	8.1	Simple functions	. 231
	8.2	Nonnegative measurable functions	. 235
	8.3	General measurable functions	
	8.4	Measures induced by integration of measurable functions	
	8.5	Sequences of measurable functions	
	8.6	Some important inequalities	
	8.7	Moments, variance, and covariance	
	8.8	Change of variables	
	8.9	Integration of functions that depend on parameters	. 273
	8.10	Riemann and Lebesgue integration	
	8.11	Worked problems	. 281

Contents

	Exerci	ses		282
9	L^p Spa	aces		285
	9.1	L^p spaces for $1 \leq p < \infty$		286
	9.2	L^{∞}		290
	9.3	Completeness		292
	9.4	Approximation by simple functions and separability		296
	9.5	Comparison of L^p spaces		
	9.6	Convergence in measure		
	9.7	The Hilbert Space structure of L^2		
	9.8	Worked problems		
		ses		
III	The	Peaks		307
10	Measi	ares on Product Spaces		309
10	10.1	A simple example		
	10.1	Product σ -algebras		
	10.2	The Monotone Class Theorem		
	10.3	Product measure for two factors		
	10.4	Integration for a product space with two factors		
	10.5			
	10.0	Extension to product spaces with n factors		
	10.8	The multivariate normal distribution		
	10.9	Convolution and sums of independent random variables		
	10.10	Countable collections of independent random variables		
	10.11 Exerci	An application to a stochastic process model		
11	C			
11	_	ences of Random Variables		355
	11.1	Sequences of random variables		
	11.2	Laws of Large Numbers		
	11.3	Empirical distribution functions, medians, and quantiles		
	11.4	Convergence of random series		
	11.5	\triangle A Kolmogorov $0-1$ Law for random variables		
	Exerci	ises	•	382
12	Sequences of Measures 38.			
	12.1	Basic notions of weak convergence		
	12.2	Convergence in distribution		391
	12.3	Tightness and relative compactness		398
	12.4	Convergence of densities and Scheffe's Theorem		402
	12.5	△ Weak convergence in metric spaces		404
	Exerci	ises		404
13	Deriva	atives of Measures		407
•	13.1	The Hahn and Jordan Decompositions		
	13.2	Absolute continuity and mutually singular measures		
	13.3	The Lebesgue-Radon-Nikodym Theorem		
	13.4	Properties of the Radon-Nikodym derivative		

viii Contents

Bibl	liograph	505
В	To Do	
	A.8	Solutions to problems in Chapter 8
	A.7	Solutions to problems in Chapter 7
	A.6	Solutions to problems in Chapter 6
	A.5	Solutions to problems in Chapter 5
	A.4	Solutions to problems in Chapter 4
	A.3	Solutions to problems in Chapter 3
	A.2	Solutions to problems in Chapter 2
_	A.1	Solutions to problems in Chapter 1
A	Soluti	ons to Worked Problems 485
	Exerci	ises
18	Excha	angeability 483
		ises
17	Disint	tegration of Measures 481
		ises
16	The L	indeberg-Feller Central Limit Theorem 479
	Exerci	ises
	15.5	Moments and regularity
	15.4	Sums of independent r.v
	15.3	Inversion
	15.2	Characteristic functions and simple properties
15	15.1	Acteristic Functions 465 Review of complex numbers
15	Chan	acteristic Functions 465
	Exerci	ises
	14.7	Conditional probability and orthogonal projections
	14.6	Regular conditional probability measure
	14.5	Properties of conditional expectation
	14.4	Conditional expectation
	14.3	Properties of conditional probability
	14.1	Conditional probability
14	14.1	itional Probability and Conditional Expectation A special case
	Exerci	ises
	13.6	△ Duality and the Riesz Representation Theorem
	13.5	△ Differentiation of a measure as a limit