LECTURES ON HODGE THEORY AND ALGEBRAIC CYCLES

JAMES D. LEWIS

Abstract. Notes for a mini course at the USTC in Hefei, China, June 23 - July 12, 2014.

Contents

Date: February 6, 2016.

2000 *Mathematics Subject Classification.* Primary: 14C35, 14C25, 19E15; Secondary: 14C30.

Key words and phrases. Chow group, Hodge theory, algebraic cycle, regulator, Deligne cohomology, Beilinson-Hodge conjecture, Abel-Jacobi map, Bloch-Beilinson filtration.

Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

1. **Preface**

These lectures are based on a mini course of 24 lectures presented at the USTC in the summer of 2014. The intent was to present new material beyond some earlier lecture notes the author presented in Mexico in the summer of 2000 on algebraic cycles [Lew4]. The central core of these notes is based on a course on Deligne cohomology that the author offered at the University of Alberta some years ago, but these lecture notes go well beyond that, including unedited subsections of [Lew10] and [Lew11] which may not be easily accessible for the reader to find. Some new material is also included. In summary, there is a strong emphasis on Hodge theory, K-theory, algebraic cycles and regulators.

The author owes deep debt of gratitude to his host Mao Sheng, as well as to Xi Chen, for working out the logistics of arranging this visit. As mathematicians, the author views both of them in the highest regard! A special thanks goes to the referee for a careful read of this paper.

China is now evolving as a major centre for mathematics in the world. Much of this can be attributed to the tireless effort of our friend and colleague S.-T. Yau, as a major influence in that country. This is a monumental accomplishment that the entire mathematical community can take pride in.

2. **Prerequisite reading material**

These lecture notes are fairly advanced. The reader would certainly benefit by looking at the [GH], [Lew1] for the complex analytic side of the subject, and to [Lew4], [Blo3] for that part pertaining to algebraic cycles.

3. **Outline of lecture notes and preliminary material**

3.1. **Motivation.** In the (now) classical literature, there are two cycle class maps used to detect (generalized, viz., motivic) algebraic cycles $\mathrm{CH}^r(X,m)$ of codimension r, (see [Blo1]), on a projective algebraic manifold X. If ξ is such a cycle, the fundamental class determines an element of Betti cohomology $[\xi] \in H^{2r-m}(X,\mathbb{Z}(r))$. Failing this (viz., if $[\xi]=0$), then we say that ξ is null-homologous, and we attempt to detect it via a secondary cycle class map, which takes values in a particular complex torus, $AJ(\xi) \in J^{r,m}(X)$, where AJ is the so-called Abel-Jacobi map. Deligne cohomology in its primitive form incorporates these two processes above in a single object $H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))$. There is a cycle class $\mathrm{cl}_{r,m}(\xi) \in H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))$, and a morphism $H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to H^{2r-m}(X,\mathbb{Z}(r))$, with image $\mathrm{cl}_r(\xi) \mapsto [\xi]$ and kernel $J^{r,m}(X)$, recovering $AJ(\xi)$ in the event $[\xi]=0$. Further, Deligne cohomology has a ring structure, for which $J^{r,m}(X)$ is a ideal whose square is zero. In short, two salient features of Deligne cohomology are:

- Combines the two aforementioned cycle class maps
- Detects higher K-theory classes (This will become apparent later.)
- There is a natural extension of Deligne cohomology to smooth quasi-projective varieties (Deligne-Beilinson cohomology), and quite generally Beilinson's absolute Hodge cohomology $H^{\bullet}_{\mathcal{H}}(-,\bullet)$ (Deligne-Beilinson type cohomology incorporating weights, see [Be1]) for any complex variety.

Central to this is the obvious question:

Question 3.2. *For smooth projective $X/\mathbb{C}$, describe the "complexity" of $CH^r(X,m)$, and more specifically (but not all inclusive to) $CH^r(X,m;\mathbb{Q})$.*

The interesting point is that due to the works of Mumford, Griffiths, Bloch, Beilinson, and others, the map

$$\mathrm{cl}_{r,m}(\xi) : \mathrm{CH}^r(X,m;\mathbb{Q}) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)),$$

for smooth quasi-projective $X/\mathbb{C}$ is neither injective, nor surjective. Further, there is a short exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(X,\mathbb{Q}(r))\big) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r))$$
$$\to \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(X,\mathbb{Q}(r))\big) \to 0,$$

where MHS is the category of $\mathbb{Q}$-mixed Hodge structures (defined later). In terms of failure of surjectivity, this is attributed to both Griffiths ($m=0$) and Beilinson (essentially $m \geq 1$, using "rigidity"). In terms of lack of injectivity, Mumford (and later Bloch) in the case $m=0$ determined this. (There are also refined results in the cases for $m \geq 0$, by Lewis ([Lew8], [Lew9]), Schoen (see [Ja2](Appendix)), Collino-Fakhruddin, [CF], *et al.*) But there are other issues at play here. It turns out, as first observed by Beilinson [Be1], that for any $\mathbb{Q}$-MHS's V,W, $\mathrm{Ext}^{\nu}_{\mathrm{MHS}}(V,W)=0$ for $\nu \geq 2$. This is due in part to J. Carlson's formula [Ca] for $\mathrm{Ext}^1_{\mathrm{MHS}}(V,-)$ as a right exact functor, together with this information: If the category of MHS were to have enough injectives, then this is formal homological algebra. In general, the precise idea works with a Yoneda-Ext argument. This leads to the notion of a Bloch-Beilinson (BB) filtration (originating from Bloch [Blo3], but later fortified by Beilinson), and in it's (paraphrased) generalized form, is that for X/k smooth projective over a field k, there is a descending filtration,

$$\mathrm{CH}^r(X/k, m; \mathbb{Q}) = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^r \supset \{0\},$$

where

$$Gr_F^{\nu}\mathrm{CH}^r(X, m; \mathbb{Q}) \simeq \mathrm{Ext}^{\nu}_{\mathcal{MM}}\big(\mathrm{Spec}(k), h^{2r-m-\nu}(X)(r)\big),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives over k, and $h^{2r-m-\nu}(X)(r)$ is motivic cohomology. The lack of an explicit description of $\mathcal{MM}/k$ has not deterred others from presenting possible candidate BB filtrations. It is the personal prejudice of the author to consider the constructions in [A], and [Lew5], based on the idea of $\overline{\mathbb{Q}}$-spreads. To explain this, consider a smooth projective variety $X/\mathbb{C}$. One can think of $X = X_K \times \mathbb{C}$, where $K/\overline{\mathbb{Q}}$ is finitely generated. Clearly $K = \overline{\mathbb{Q}}(\mathcal{S})$ for some variety $\mathcal{S}/\overline{\mathbb{Q}}$. Let $\eta \in \mathcal{S}/\overline{\mathbb{Q}}$ be the generic point, and observe that $K \subset \mathbb{C}$ corresponds to an embedding $\overline{\mathbb{Q}}(\eta) \hookrightarrow \mathbb{C}$ over $\overline{\mathbb{Q}}$. Correspondingly, there is a spread $\rho : \mathcal{X} \to \mathcal{S}$ which we can assume is given by a smooth proper morphism of smooth quasi-projective varieties, with the property that $X/\mathbb{C} = \mathcal{X}_\eta \times_{\overline{\mathbb{Q}}(\eta)} \mathbb{C}$. Likewise, if $\xi \in \mathrm{CH}^r(X/\mathbb{C}, m; \mathbb{Q})$, then by possibly modifying $\mathcal{S}/\overline{\mathbb{Q}}$, there is a cycle $\tilde{\xi} \in \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}}, m; \mathbb{Q})$ for which $\tilde{\xi}_\eta = \xi$ in $\mathrm{CH}^r(X/\mathbb{C}, m; \mathbb{Q})$. The following illustrates this process.

Example 3.3.

$$Y/\mathbb{C} = \mathrm{Spec}\bigg\{\frac{\mathbb{C}[x,y]}{(\pi y^2 + (\sqrt{\pi}+4)x^3 + \mathrm{e}x)}\bigg\}.$$

$$\mathcal{S}/\mathbb{Q} = \mathrm{Spec}\bigg\{\frac{\mathbb{Q}[u,v,w]}{(u-v^2)}\bigg\},$$

Set:

$$\mathcal{Y}_{\mathcal{S}} = \mathrm{Spec}\bigg\{\frac{\mathbb{Q}[x,y,u,v,w]}{(uy^2 + (v+4)x^3 + wx, u-v^2)}\bigg\}$$

The inclusion

$$\frac{\mathbb{Q}[u,v,w]}{(u-v^2)} \subset \frac{\mathbb{Q}[x,y,u,v,w]}{\big(uy^2 + (v+4)x^3 + wx, u-v^2\big)},$$

defines a morphism $\mathcal{Y}_{\mathcal{S}} \to \mathcal{S}$, as varieties over $\mathbb{Q}$. Let $\eta \in \mathcal{S}$, be the generic point. Then

$$\mathbb{Q}(\eta) = \mathrm{Quot}\bigg(\frac{\mathbb{Q}[u,v,w]}{(u-v^2)}\bigg).$$

Note that the embedding

$$\mathbb{Q}(\eta) \hookrightarrow \mathbb{C}, \quad (u,v,w) \mapsto (\pi, \sqrt{\pi}, \mathrm{e}), \ \Rightarrow \mathcal{Y}_{\mathcal{S},\eta} \times \mathbb{C} = Y/\mathbb{C}.$$

Essentially the [generalized] conjectures of Bloch and Beilinson state that the cycle class map

$$\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}}, m; \mathbb{Q}) \to H^{2r-m}_{\mathcal{H}}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)),$$

should be injective (see [K-L]). Thus a Leray filtration on $H^{2r-m}_{\mathcal{H}}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$ induces a filtration on $\mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}}, m; \mathbb{Q})$, and by passage to the generic point of $\mathcal{S}$, a candidate filtration on $\mathrm{CH}^r(X_K, m; \mathbb{Q})$.[1] This induces a candidate BB filtration on

$$\mathrm{CH}^r(X/\mathbb{C}, m; \mathbb{Q}) = \lim_{\overrightarrow{K\subset\mathbb{C}}} \mathrm{CH}^r(X_K, m; \mathbb{Q}).$$

Moving in a slightly different direction, is an amended version of the Beilinson-Hodge conjecture:

Conjecture 3.4. *Let $\rho : \mathcal{X} \to \mathcal{S}$ be a smooth proper map of smooth quasi-projective varieties over a subfield $k = \overline{k} \subseteq \mathbb{C}$, with $\eta = \eta_{\mathcal{S}}$ the generic point of $\mathcal{S}/k$. Further, let $r, m \geq 0$ be integers. Then*

$$\mathrm{cl}_{r,m} : CH^r(\mathcal{X}_\eta, m; \mathbb{Q}) = H^{2r-m}_{\mathcal{M}}(\mathcal{X}_\eta, \mathbb{Q}(r)) \to \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_\eta(\mathbb{C}), \mathbb{Q}(r))\big),$$

is surjective.

Here

$$H^{2r-m}(\mathcal{X}_\eta(\mathbb{C}), \mathbb{Q}(r)) := \lim_{\overrightarrow{U\subset\mathcal{S}/k}} H^{2r-m}(\rho^{-1}(U)(\mathbb{C}), \mathbb{Q}(r)),$$

is a limit of mixed Hodge structures (MHS), for which one should not expect finite dimensionality, and for any smooth quasi-projective variety W/k, we identify motivic cohomology $H^{2r-m}_{\mathcal{M}}(W, \mathbb{Q}(r))$ with Bloch's higher Chow group $\mathrm{CH}^r(W, m; \mathbb{Q}) := \mathrm{CH}^r(W, m) \otimes \mathbb{Q}$. Note that if $\mathcal{S} = \mathrm{Spec}(k)$, and $m = 0$, then $\mathcal{X} = X_k$ is smooth, projective over k. Thus in this case Conjecture 3.4 reduces to the (classical) Hodge conjecture. The motivation for this conjecture stems from the following:

Firstly, it is a generalization of similar conjecture in [dJ-L](§1, statement (S3)), where $\mathcal{X} = \mathcal{S}$, based on a generalization of the Hodge conjecture (classical form) to the higher K-groups, and inspired in part by Beilinson's work in this direction.

Secondly, as a formal application of M. Saito's theory of mixed Hodge modules (see [A], [K-L], [SJK-L] and the references cited there), one could conceive of the

[1] For the case $m = 0$, the approach in [Lew5] involved a Leray filtration on $H^{2r}_{\mathcal{H}}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$. The work in [A] is for all $m \geq 0$, based on M. Saito's work on the category of mixed Hodge modules (MHM), and makes use of a natural cycle class map $\mathrm{CH}^r(\mathcal{X}, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\mathcal{X})}(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r))$.

following short exact sequence:

$$
\begin{array}{cc}
 & 0 \\
 & \Big\downarrow \\
 & \mathrm{Ext}^1_{\mathrm{PMHS}}\big(\mathbb{Q}(0), H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big) \\
\begin{smallmatrix}\text{Graded polar-}\\ \text{izable MHS}\end{smallmatrix}\nearrow & \\
 & \Big\downarrow \\
 & \left\{\begin{matrix}\text{Germs of}\\ \text{arithmetic normal functions}\end{matrix}\right\} \\
 & \Big\downarrow \\
 & \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{\nu}(\eta_{\mathcal{S}}, R^{2r-m-\nu}\rho_*\mathbb{Q}(r))\big) \\
 & \Big\downarrow \\
 & 0
\end{array}
\tag{1}
$$

(Warning: As mentioned earlier, passing to the generic point $\eta_{\mathcal{S}}$ of $\mathcal{S}$ is a limit process, which implies that the spaces above need not be finite dimensional over $\mathbb{Q}$. This particularly applies to the case $m \geq 1$, where there are residues.) The key point is, is there lurking a generalized Poincaré existence theorem for higher normal functions? Namely, modulo the "fixed part" $\mathrm{Ext}^1_{\mathrm{PMHS}}\big(\mathbb{Q}(0), H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)$, are these normal functions cycle-induced? A formal conjecture along these lines is presented in [dJ-L-P]. In another direction, this diagram is related to a geometric description of the notion of a Bloch-Beilinson (BB) filtration. As a service to the reader, and to make sense of this all, we elaborate on all of this.

1. For the moment, let us replace $\eta_{\mathcal{S}}$ by $\mathcal{S}$, (ν, m) by $(1, 0)$ in diagram (1), and where $\mathcal{S}$ is chosen to be a curve. Then this diagram represents the schema of the original Griffiths program aimed at generalizing Lefschetz's famous $(1,1)$ theorem, via normal functions.[2] This program was aimed at solving the Hodge conjecture inductively. Unfortunately, the lack of a Jacobi inversion theorem for the jacobian of a general smooth projective variety involving a Hodge structure of weight > 1 led to limited applications towards the Hodge conjecture. However the qualitative aspects of his program led to the non-triviality of the now regarded Griffiths group. In that regard, the aforementioned diagram represents a generalization of this idea to the higher K-groups of $\mathcal{X}$ and the general fibers of $\rho : \mathcal{X} \to \mathcal{S}$.

2. In the case $k = \overline{\mathbb{Q}}$, one relates this to graded pieces of a candidate BB filtration.

[2]Technically speaking, Griffiths worked with normal functions that extended to the boundary $\overline{\mathcal{S}} \backslash \mathcal{S}$, but apart from Remark 7.24, let's not go there.

Finally, the famous Bloch-Kato theorem (involving the norm residue map in Milnor K-theory - see subsection 9.27), can be thought of as a "topological" version of a special case of the Beilinson-Hodge conjecture, which we discuss in §9.

3.5. **Notation.** Throughout these notes, and unless otherwise specified, $X = X/\mathbb{C}$ is a projective algebraic manifold, of dimension d. A projective algebraic manifold is the same thing as a smooth complex projective variety. If $V \subseteq X$ is an irreducible subvariety of X, then $\mathbb{C}(V)$ is the rational function field of V, with multiplicative group $\mathbb{C}(V)^\times$. Depending on the context (which will be made abundantly clear in the text), $\mathcal{O}_X$ will either be the sheaf of germs of holomorphic functions on X in the analytic topology, or the sheaf of germs of regular functions in the Zariski topology.

3.6. **Some Hodge theory.** Some useful reference material for this section is [GH] and [Lew1].

Let $E^k_X = \mathbb{C}$-valued C^∞ k-forms on X. (One could also use the common notation of $A^k(X)$ for C^∞ forms, but let's not.) We have the decomposition:

$$E^k_X = \bigoplus_{p+q=k} E^{p,q}_X, \quad \overline{E^{p,q}_X} = E^{q,p}_X, \quad \overline{\{\cdots\}} = \text{complex conjugation},$$

where $E^{p,q}_X$ are the C^∞ (p,q)-forms which in local holomorphic coordinates $z = (z_1, \ldots, z_n) \in X$, are of the form:

$$\sum_{|I|=p, |J|=q} f_{IJ} dz_I \wedge d\overline{z}_J, \quad f_{IJ} \ \mathit{are}\ \mathbb{C} - \mathit{valued}\ C^\infty \mathit{functions},$$

$$I = 1 \leq i_1 < \cdots < i_p \leq d, \quad J = 1 \leq j_1 < \cdots < j_q \leq d,$$

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

One has the differential $d : E^k_X \to E^{k+1}_X$, and we define

$$H^k_{\mathrm{DR}}(X, \mathbb{C}) = \frac{\ker d : E^k_X \to E^{k+1}_X}{dE^{k-1}_X}.$$

The operator d decomposes into $d = \partial + \overline{\partial}$, where $\partial : E^{p,q}_X \to E^{p+1,q}_X$ and $\overline{\partial} : E^{p,q}_X \to E^{p,q+1}_X$. Further $d^2 = 0 \Rightarrow \partial^2 = \overline{\partial}^2 = 0 = \partial\overline{\partial} + \overline{\partial}\partial$, by (p,q) type.

The above decomposition descends to the cohomological level, viz.,

Theorem 3.7 (Hodge decomposition)**.**

$$H^k_{\mathrm{sing}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) = d$*-closed* (p,q)*-forms (modulo coboundaries), and*

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Furthermore:

$$H^{p,q}(X) \simeq \frac{E^{p,q}_{X, d-\mathrm{closed}}}{\partial\overline{\partial} E^{p-1,q-1}_X}.$$

Some more terminology: *Hodge filtration.* Put

$$F^r H^i(X,\mathbb{C}) = \bigoplus_{p\geq r} H^{p,i-p}(X).$$

Now recall $\dim X = d$.

Theorem 3.8 (Poincaré and Serre duality)**.** *The following pairings induced by*

$$(w_1, w_2) \mapsto \int_X w_1 \wedge w_2,$$

are non-degenerate:

$$H^r_{\mathrm{DR}}(X,\mathbb{C}) \times H^{2d-r}_{\mathrm{DR}}(X,\mathbb{C}) \to \mathbb{C},$$
$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}.$$

Therefore $H^r(X) \simeq H^{2d-r}(X)^\vee,\ H^{p,q}(X) \simeq H^{d-p,d-q}(X)^\vee$

Corollary 3.9.

$$\frac{H^i(X,\mathbb{C})}{F^r H^i(X,\mathbb{C})} \simeq F^{d-r+1} H^{2d-i}(X,\mathbb{C})^\vee.$$

3.10. **Formalism of mixed Hodge structures.**

Definition 3.11. Let $\mathbb{A} \subset \mathbb{R}$ be a subring. An $\mathbb{A}$-Hodge structure (HS) of weight $N \in \mathbb{Z}$ is given by the following datum:

• A finitely generated $\mathbb{A}$-module V, and either of the two equivalent statements below:

$\bullet_1$ A decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=N} V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p},$$

where $^-$ is complex conjugation induced from conjugation on the second factor $\mathbb{C}$ of $V_{\mathbb{C}} := V \otimes \mathbb{C}$.

$\bullet_2$ A finite descending filtration

$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\},$$

satisfying

$$V_{\mathbb{C}} = F^r \bigoplus \overline{F^{N-r+1}},\ \forall\ r \in \mathbb{Z}.$$

Remark 3.12. The equivalence of $\bullet_1$ and $\bullet_2$ can be seen as follows. Given the decomposition in $\bullet_1$, put

$$F^r V_{\mathbb{C}} = \bigoplus_{p+q=N, p\geq r} V^{p,q}.$$

Conversely, given $\{F^r\}$ in $\bullet_2$, put $V^{p,q} = F^p \bigcap \overline{F^q}$.

Example 3.13. $X/\mathbb{C}$ smooth projective. Then $H^i(X,\mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of weight i.

Example 3.14. $\mathbb{A}(r) := (2\pi \mathrm{i})^r \mathbb{A}$ is an $\mathbb{A}$-Hodge structure of weight $-2r$ and of pure Hodge type $(-r,-r)$, called the Tate twist.

Example 3.15. $X/\mathbb{C}$ smooth projective. Then $H^i(X,\mathbb{Q}(r)) := H^i(X,\mathbb{Q}) \otimes \mathbb{Q}(r)$ is a $\mathbb{Q}$-Hodge structure of weight $i - 2r$.

To extend these ideas to singular varieties, one requires the following terminology.

Definition 3.16. An $\mathbb{A}$-mixed Hodge structure ($\mathbb{A}$-MHS) is given by the following datum:

- A finitely generated $\mathbb{A}$-module $V_{\mathbb{A}}$,
- A finite descending "Hodge" filtration on $V_{\mathbb{C}} := V_{\mathbb{A}} \otimes \mathbb{C}$,
$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\},$$
- An increasing "weight" filtration on $V_{\mathbb{A}} \otimes \mathbb{Q} := V_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$,
$$\{0\} \subset \cdots \subset W_{\ell-1} \subset W_{\ell} \subset \cdots \subset V_{\mathbb{A}} \otimes \mathbb{Q},$$

such that $\{F^r\}$ induces a (pure) HS of weight ℓ on $Gr_{\ell}^{W} := W_{\ell}/W_{\ell-1}$.

Theorem 3.17 (Deligne [De])**.** *Let Y be a complex variety. Then $H^i(Y,\mathbb{Z})$ has a canonical and functorial $\mathbb{Z}$-MHS.*

Remark 3.18. (i) A morphism $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ of $\mathbb{A}$-MHS is an $\mathbb{A}$-linear map satisfying:

- $h\big(W_{\ell}V_{1,\mathbb{A}\otimes\mathbb{Q}}\big) \subseteq W_{\ell}V_{2,\mathbb{A}\otimes\mathbb{Q}}, \quad \forall\ \ell,$
- $h\big(F^rV_{1,\mathbb{C}}\big) \subseteq F^rV_{2,\mathbb{C}}, \quad \forall\ r.$

Deligne ([De] (Theorem 2.3.5)) shows that the category of $\mathbb{A}$-MHS is abelian; in particular if $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ is a morphism of $\mathbb{A}$-MHS, then $\ker(h)$, $\mathrm{coker}(h)$ are endowed with the induced filtrations. Let us further assume that $\mathbb{A}\otimes\mathbb{Q}$ is a field. Then Deligne (*op. cit.*) shows that h is strictly compatible[3] with the filtrations $W_{\bullet}$ and $F^{\bullet}$, and that the functors $V \mapsto Gr_{\ell}^{W}V$, $V_{\mathbb{C}} \mapsto Gr_{F}^{r}V_{\mathbb{C}}$ are exact.

(ii) Roughly speaking, the functoriality of the MHS in Deligne's theorem translates to the following yoga: the "standard" exact sequences in singular (co)homology, together with push-forwards and pullbacks by morphisms (wherever permissible) respect MHS. In particular for a subvariety $Y \subset X$, the localization cohomology sequence associated to the pair (X,Y) is a long exact sequence of MHS. Here is where the Tate twist comes into play: Suppose that $Y \subset X$ is an inclusion of projective algebraic manifolds with $\mathrm{codim}_X Y = r \geq 1$. One has a Gysin map $H^{i-2r}(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ which involves Hodge structures of different weights. To remedy this, one considers the induced map $H^{i-2r}(Y,\mathbb{Q}(-r)) \to H^i(X,\mathbb{Q}(0)) = H^i(X,\mathbb{Q})$ via (twisted) Poincaré duality (see §6), which is a morphism of pure Hodge structures (hence of MHS). A simple proof of this fact can be found in §7 of [Lew1]. Note that the morphism $H^i_Y(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is a morphism of MHS, and that accordingly $H^i_Y(X,\mathbb{Q}) \simeq H^{i-2r}(Y,\mathbb{Q}(-r))$ is an isomorphism of MHS (with Y still smooth).

---

[3] Strict compatibility means that $h(F^rV_{1,\mathbb{C}}) = h(V_{1,\mathbb{C}}) \cap F^rV_{2,\mathbb{C}}$ and $h(W_{\ell}V_{1,\mathbb{A}\otimes\mathbb{Q}}) = h(V_{1,\mathbb{A}\otimes\mathbb{Q}}) \cap W_{\ell}V_{2,\mathbb{A}\otimes\mathbb{Q}}$ for all r and ℓ. An nice explanation of Deligne's proof of this fact can be found in [St], where a quick summary goes as follows: For any $\mathbb{A}$-MHS V, $V_{\mathbb{C}}$ has a $\mathbb{C}$-splitting into a bigraded direct sum of complex vector spaces $I^{p,q} := F^p \cap W_{p+q} \cap \big[\overline{F^q} \cap W_{p+q} + \sum_{i\geq 2} \overline{F^{q-i+1}} \cap W_{p+q-i}\big]$, where one shows that $F^rV_{\mathbb{C}} = \oplus_{p\geq r} \oplus_q I^{p,q}$ and $W_{\ell}V_{\mathbb{C}} = \oplus_{p+q\leq \ell} I^{p,q}$. Then by construction of $I^{p,q}$, one has $h(I^{p,q}(V_{1,\mathbb{C}}) \subseteq I^{p,q}(V_{2,\mathbb{C}})$. Hence h preserves both the Hodge and complexified weight filtrations. Now use the fact that $\mathbb{A}\otimes\mathbb{Q}$ is a field to deduce that h preserves the weight filtration over $\mathbb{A}\otimes\mathbb{Q}$.

Example 3.19. Let $\overline{U}$ be a compact Riemann surface, $\Sigma \subset \overline{U}$ a finite set of points, and put $U := \overline{U}\backslash\Sigma$. According to Deligne, $H^1(U, \mathbb{Z}(1))$ carries a $\mathbb{Z}$-MHS. The Hodge filtration on $H^1(U, \mathbb{C})$ is defined in terms of a filtered complex of holomorphic differentials on U with logarithmic poles along Σ ([De], but also see subsection 5.10 below). One can "observe" the MHS via weights as follows. Poincaré duality gives us $H^1_\Sigma(\overline{U}, \mathbb{Z}) \simeq H_1(\Sigma, \mathbb{Z}) = 0$, and the localization sequence in cohomology below is a sequence of MHS:

$$0 \to H^1(\overline{U}, \mathbb{Z}(1)) \to H^1(U, \mathbb{Z}(1)) \to H^0(\Sigma, \mathbb{Z}(0))^\circ \to 0,$$

where

$$H^0(\Sigma, \mathbb{Z}(0))^\circ := \ker\big(H^2_\Sigma(\overline{U}, \mathbb{Z}(1)) \to H^2(\overline{U}, \mathbb{Z}(1))\big) \simeq \mathbb{Z}(0)^{|\Sigma|-1}.$$

Put $W_0 = H^1(U, \mathbb{Z}(1))$, $W_{-1} = \mathrm{Im}\big(H^1(\overline{U}, \mathbb{Z}(1)) \to H^1(U, \mathbb{Z}(1))\big)$, $W_{-2} = 0$. Then $Gr^W_{-1} H^1(U, \mathbb{Z}(1)) \simeq H^1(\overline{U}, \mathbb{Z}(1))$ has pure weight -1 and $Gr^W_0 H^1(U, \mathbb{Z}(1)) \simeq \mathbb{Z}(0)^{|\Sigma|-1}$ has pure weight 0.

The following notation will be introduced:

Definition 3.20. Let V be an $\mathbb{A}$-MHS. We put

$$\Gamma_{\mathbb{A}} V := \hom_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), V),$$

and

$$J_{\mathbb{A}}(V) = \mathrm{Ext}^1_{\mathbb{A}-\mathrm{MHS}}(\mathbb{A}(0), V).$$

In the case where $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$, we simply put $\Gamma = \Gamma_{\mathbb{A}}$ and $J = J_{\mathbb{A}}$.

Example 3.21. Suppose that $V = V_{\mathbb{Z}}$ is a $\mathbb{Z}$ (pure) HS of weight $2r$. Then $V(r) := V \otimes \mathbb{Z}(r)$ is of weight 0, and (up to the twist) one can identify ΓV with $V_{\mathbb{Z}} \cap F^r V_{\mathbb{C}} = V_{\mathbb{Z}} \cap V^{r,r} := \epsilon^{-1}\big(V^{r,r}\big)$, where $\epsilon : V \to V_{\mathbb{C}}$.

Example 3.22. Let V be a $\mathbb{Z}$-MHS. There is the identification due to J. Carlson (see [Ca], [Ja2]),

$$J(V) \simeq \frac{W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V},$$

where in the denominator term, $V := V_{\mathbb{Z}}$ is identified with its image $V_{\mathbb{Z}} \to V_{\mathbb{C}}$ (viz., quotienting out torsion). For example, if $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), V)$ corresponds to the short exact sequence of MHS:

$$0 \to V \to E \xrightarrow{\alpha} \mathbb{Z}(0) \to 0,$$

then one can find $x \in W_0 E$ and $y \in F^0 W_0 E_{\mathbb{C}}$ such that $\alpha(x) = \alpha(y) = 1$. Then $x - y \in V_{\mathbb{C}}$ descends to a class in $W_0 V_{\mathbb{C}}/\{F^0 W_0 V_{\mathbb{C}} + W_0 V\}$, which defines the map from $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), V)$ to $W_0 V_{\mathbb{C}}/\{F^0 W_0 V_{\mathbb{C}} + W_0 V\}$.

3.23. **Algebraic cycles (classical).** Recall $X/\mathbb{C}$ smooth projective, $\dim X = d$. For $0 \le r \le d$, put $z^r(X)$ $(= z_{d-r}(X))$ = free abelian group generated by subvarieties of codim r $(= \dim\ d - r)$ in X.

Example 3.24. (i) $z^d(X) = z_0(X) = \{\sum_{j=1}^M n_j p_j \mid n_j \in \mathbb{Z},\ p_j \in X\}$.

(ii) $z^0(X) = z_d(X) = \mathbb{Z}\{X\} \simeq \mathbb{Z}$.

(iii) Let $X_1 := V(z_2^2 z_0 - z_1^3 - z_0 z_1^2) \subset \mathbb{P}^2$, and $X_2 := V(z_2^2 z_0 - z_1^3 - z_1 z_0^2) \subset \mathbb{P}^2$. Then $3X_1 - 5X_2 \in z^1(\mathbb{P}^2) = z_1(\mathbb{P}^2)$.

(iv) $\mathrm{codim}_X V = r-1$, $f \in \mathbb{C}(V)^\times$. $\mathrm{div}(f) := (f) := (f)_0 - (f)_\infty \in z^r(X)$ (principal divisor). (Note: $\mathrm{div}(f)$ is easy to define, by first passing to a normalization $\widetilde{V}$ of V, then using the fact that the local ring $\mathcal{O}_{\widetilde{V},\wp}$ of regular functions at $\wp$ is a discrete valuation ring for a codimension one "point "$\wp$ on $\widetilde{V}$, together with the proper push-forward associated to $\widetilde{V} \to V$.)

Divisors in (iv) generate a subgroup,

$$z^r_{\mathrm{rat}}(X) \subset z^r(X),$$

which defines the rational equivalence relation on $z^r(X)$.

Definition 3.25.

$$\mathrm{CH}^r(X) := z^r(X)/z^r_{\mathrm{rat}}(X),$$

is called the r-th Chow group of X.

Remark 3.26. On can show that $\xi \in z^r_{\mathrm{rat}}(X) \Leftrightarrow \exists\ w \in z^r(\mathbb{P}^1 \times X)$, each component of the support $|w|$ flat over $\mathbb{P}^1$, such that $\xi = w[0] - w[\infty]$. (Here $w[t] := \mathrm{pr}_{2,*}\big(\langle \mathrm{pr}_1^*(t) \bullet w\rangle_{\mathbb{P}^1\times X}\big)$.) If one replaces $\mathbb{P}^1$ by any choice of smooth connected curve B (not fixed!) and $0,\ \infty$ by any 2 points $P,\ Q \in B$, then one obtains the subgroup $z^r_{\mathrm{alg}}(X) \subset z^r(X)$ of cycles that are algebraically equivalent to zero[4]. There is the fundamental class map (see for example [Lew1], [Lew4]) $z^r(X) \to H^{2r}(X,\mathbb{Z})$ whose kernel is denoted by $z^r_{\mathrm{hom}}(X)$. One has inclusions:

$$z^r_{\mathrm{rat}}(X) \subseteq z^r_{\mathrm{alg}}(X) \subseteq z^r_{\mathrm{hom}}(X) \subset z^r(X).$$

Definition 3.27. Put

(i) $\mathrm{CH}^r_{\mathrm{alg}}(X) := z^r_{\mathrm{alg}}(X)/z^r_{\mathrm{rat}}(X)$,

(ii) $\mathrm{CH}^r_{\mathrm{hom}}(X) := z^r_{\mathrm{hom}}(X)/z^r_{\mathrm{rat}}(X)$,

(iii) $\mathrm{Griff}^r(X) := z^r_{\mathrm{hom}}(X)_{/}z^r_{\mathrm{alg}}(X) = \mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X)$, called the Griffiths group.

The Griffiths group is known to be trivial in the cases $r = 0,\ 1,\ d$.

[4]The fact that a smooth connected B will suffice in the definition follows from the transitive property of algebraic equivalence (see [Lew1] (p. 180)).

4. Cohomological machinery

Much of this section is based on the corresponding treatment in [GH].

4.1. A primer on spectral sequences.

Spectral sequences were invented by Jean Leray. Let us consider a bounded complex $(K^\bullet, d)$ of abelian groups, where for simplicity $K^{\bullet<0} = 0$. [Note that $K^{\bullet>>1} = 0$.]

$$K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots, \quad d^2 = 0.$$

Thus we have

$$H^p(K^\bullet) := \frac{\ker d : K^p \to K^{p+1}}{dK^{p-1}}.$$

Next, we will assume that this complex has a descending filtration[5] of subcomplexes:

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \cdots \supset F^{N+1}K^\bullet = \{0\},$$

where again $F^{\bullet\geq 0}$ is out of convenience, and being a subcomplex means that $dF^\nu K^p \subset F^\nu K^{p+1}$. This induces a corresponding associated ν-th graded complex $(Gr_F^\nu K^\bullet, d)$. Now put

$$F^\nu H^p(K^\bullet) := \frac{F^\nu K^p_{d-\text{closed}}}{F^\nu \cap (dK^{p-1})}.$$

This gives

$$H^p(K^\bullet) = F^0H^p(K^\bullet) \supset F^1H^p(K^\bullet) \supset \cdots \supset F^\nu H^p(K^\bullet) \supset F^{\nu+1}H^p(K^\bullet) \supset \cdots$$

<u>Warning.</u> $Gr_F^\nu H^p(K^\bullet) \neq H^p(Gr_F^\nu K^\bullet)$. It is the LHS that we want to compute!

Definition 4.2. A spectral sequence is a sequence $\{E_r, d_r\}$, $(r \geq 0)$, of bigraded groups

$$E_r = \bigoplus_{p,q} E_r^{p,q},$$

with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad d_r^2 = 0,$$

such that $H^*(E_r) = E_{r+1}$.

Proposition 4.3. *Given a filtered complex $(K^\bullet, d, F^\bullet)$, then there exists a spectral sequence $\{E_r\}$ with:*

$$E_0^{p,q} = \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}} =: Gr_F^pK^{p+q}$$

$$E_1^{p,q} = H^{p+q}(Gr_F^pK^\bullet)$$

$$E_\infty^{p,q} = Gr_F^p\big(H^{p+q}(K^\bullet)\big)$$

We say that the spectral sequence abuts to $H^\bullet(K^\bullet)$ and write

$$E_r \Rightarrow H^{p+q}(K^\bullet).$$

[5] A one-step filtration leads to a subcomplex, and associated quotient complex, with gives a LES in cohomology. The idea being is that $H^p(K^\bullet)$ may be better understood in terms of the cohomology of the two other complexes. Spectral sequences are a generalization of this.

Proof. The $E_0^{p,q}$ term is already defined. Let d_0 be induced by d:

$$\begin{array}{ccccc} E_0^{p,q-1} & \xrightarrow{d_0} & E_0^{p,q} & \xrightarrow{d_0} & E_0^{p,q+1} \\ \Big\| & & \Big\| & & \Big\| \\ \frac{F^pK^{p+q-1}}{F^{p+1}K^{p+q-1}} & \xrightarrow{d} & \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}} & \xrightarrow{d} & \frac{F^pK^{p+q+1}}{F^{p+1}K^{p+q+1}} \end{array}$$

Then $E_1^{p,q}$ is by definition the cohomology in the middle part, which is precisely $H^{p+q}(Gr_F^pK^\bullet)$. Next, let us define

$$E_r^{p,q} := \frac{\big\{\xi \in F^pK^{p+q} \;\big|\; d\xi \in F^{p+r}K^{p+q+1}\big\}}{\big\{d\big(F^{p-r+1}K^{p+q-1}\big) + F^{p+1}K^{p+q}\big\} \bigcap \text{Numerator}},$$

which is consistent with $E_0^{p,q}$ and $E_1^{p,q}$. Obviously, for $r >> 1$,[6]

$$E_r^{p,q} = E_\infty^{p,q} = \frac{\big\{\xi \in F^pK^{p+q} \;\big|\; d\xi = 0\big\}}{\big\{d\big(K^{p+q-1}\big) + F^{p+1}K^{p+q}\big\} \bigcap \text{Numerator}}$$
$$=: Gr_F^pH^{p+q}(K^\bullet).$$

Therefore it suffices to show that for all $r \geq 0$:

$$E_{r+1}^{p,q} = \frac{\ker d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}}{d_r\big(E_r^{p-r,q+r-1}\big)}.$$

But this follows from the definitions (drop "$\bigcap$ Numerator" for notational convenience):

$$\begin{array}{ccc} E_r^{p-r,q+r-1} & == & \frac{\{\xi\in F^{p-r}K^{p+q-1} \,|\, d\xi\in F^pK^{p+q}\}}{d\left(F^{p-2r+1}K^{p+q-2}\right)+F^{p-r+1}K^{p+q-1}} \\ d_r\Big\downarrow & & \Big\downarrow d \\ E_r^{p,q} & == & \frac{\{\xi\in F^{p}K^{p+q} \,|\, d\xi\in F^{p+r}K^{p+q+1}\}}{d\left(F^{p-r+1}K^{p+q-1}\right)+F^{p+1}K^{p+q}} \\ d_r\Big\downarrow & & \Big\downarrow d \\ E_r^{p+r,q-r+1} & == & \frac{\{\xi\in F^{p+r}K^{p+q+1} \,|\, d\xi\in F^{p+2r}K^{p+q+2}\}}{d\left(F^{p+1}K^{p+q}\right)+F^{p+r+1}K^{p+q+1}} \end{array}$$

where

$$E_{r+1}^{p,q} = \frac{\big\{\xi \in F^pK^{p+q} \;\big|\; d\xi \in F^{p+r+1}K^{p+q+1}\big\}}{d\big(F^{p-r}K^{p+q-1}\big) + F^{p+1}K^{p+q}}$$

□

Example 4.4.

$$E_2^{p,q} = \frac{\big\{\xi \in F^pK^{p+q} \;\big|\; d\xi \in F^{p+2}K^{p+q+1}\big\}}{\big\{d\big(F^{p-1}K^{p+q-1}\big) + F^{p+1}K^{p+q}\big\} \bigcap \text{Numerator}}.$$

[6]We put $F^0 = F^{\bullet\leq 0}$.

4.5. **Double complexes and the Grothendieck spectral sequences.** Again, for simplicity of notation, we will assume non-negative indices. Consider a (bounded) double complex

$$K^{\bullet,\bullet} = \bigoplus_{p,q\geq 0} K^{p,q},\ d : K^{p,q} \to K^{p+1,q},\ \delta : K^{p,q} \to K^{p,q+1},$$

with

$$d^2 = \delta^2 = 0,\ d\delta + \delta d = 0.$$

We can form the associated single complex

$$\mathbf{s}K^n := \bigoplus_{p+q=n} K^{p,q},\ D = d + \delta,$$

where we observe that

$$D^2 = d^2 + \delta^2 + d\delta + \delta d = 0.$$

The complex $(\mathbf{s}K^\bullet, D)$ has two descending filtrations, viz.,

$$'F^\nu \mathbf{s}K^n := \bigoplus_{p+q=n, p\geq\nu} K^{p,q}$$

$$''F^\nu \mathbf{s}K^n := \bigoplus_{p+q=n, q\geq\nu} K^{p,q}$$

This automatically leads to two spectral sequences:

$$'E_r \Rightarrow H_D^{p+q}(\mathbf{s}K^\bullet)$$

$$''E_r \Rightarrow H_D^{p+q}(\mathbf{s}K^\bullet)$$

Note that

$$'E_1^{p,q} = H_D^{p+q}(Gr^p_{'F}\mathbf{s}K^\bullet) = H^q_\delta(K^{p,\bullet}),$$

$$''E_1^{p,q} = H_D^{p+q}(Gr^p_{''F}\mathbf{s}K^\bullet) = H^q_d(K^{\bullet,p}).$$

Note that

$$D = d + \delta = \begin{cases} d & \text{on } 'E_1 \\ \delta & \text{on } ''E_1 \end{cases},$$

hence

$$d_1 = d : H^q_\delta(K^{p,\bullet}) = {'E_1^{p,q}} \to {'E_1^{p+1,q}} = H^q_\delta(K^{p+1,\bullet})$$

$$d_1 = \delta : H^q_d(K^{\bullet,p}) = {''E_1^{p,q}} \to {''E_1^{p+1,q}} = H^q_d(K^{\bullet,p+1})$$

Therefore

$$'E_2^{p,q} = H^p_d\big(H^q_\delta(K^{\bullet,\bullet})\big),$$

$$''E_2^{p,q} = H^p_\delta\big(H^q_d(K^{\bullet,\bullet})\big).$$

Example 4.6. Let $\mathcal{E}^k_X$ be the sheaf of germs of C^∞ complex-valued forms on X, and $E^k_X := H^0(X, \mathcal{E}^k_X)$. One has a Hodge decomposition

$$\mathcal{E}^k_X = \bigoplus_{p+q=k} \mathcal{E}^{p,q}_X, \quad E^k_X = \bigoplus_{p+q=k} E^{p,q}_X.$$

The complex $(E^\bullet_X, d)$ is filtered by subcomplexes $(F^pE^\bullet_X, d)$, $p \geq 0$, where

$$F^pE^k_X = \bigoplus_{i+j=k, i\geq p} E^{i,j}_X,\ D := d = \partial + \bar\partial.$$

$[d^2=0$, hence by type, $\partial^2=\bar\partial^2=\partial\bar\partial+\bar\partial\partial=0.]$ Explicitly,

$$\begin{array}{c} F^pE_X^\bullet : 0 \to \cdots \to 0 \to F^pE_X^p \xrightarrow{d} \cdots \xrightarrow{d} F^pE_X^{2d} \\ \bigcap \\ E_X^\bullet : E_X^0 \xrightarrow{d} \cdots \xrightarrow{d} E_X^{p-1} \xrightarrow{d} E_X^p \xrightarrow{d} \cdots \xrightarrow{d} E_X^{2d} \end{array}$$

The Hodge to de Rham spectral sequence is given by

$$E_1^{p,q} := H^{p+q}(Gr_F^pE_X^\bullet) \Rightarrow H_{\mathrm{DR}}^{p+q}(X).$$

But

$$H^{p+q}(Gr_F^pE_X^\bullet) = H_{\bar\partial}^q(E_X^{p,\bullet}) =: H^{p,q}(X).$$

Degeneration at E_1 is a result of the equivalence of Laplacians ($\frac{\Delta_d}{2}=\Delta_{\bar\partial}=\Delta_\partial$).

Example 4.7. This example is intended for those familiar with the definition of Bloch's higher Chow groups (see subsection 6.10). One can think of smooth complex quasiprojective variety U of the form $U=X\backslash Y$, where recall $X/\mathbb{C}$ is a smooth projective variety of dimension d, $Y=Y_1\cup\cdots\cup Y_N\subset X$ a NCD with smooth components. For an integer $t\geq 0$, put $Y^{[t]}$ = disjoint union of t-fold intersections of the various components of Y, with corresponding coskeleton $Y^{[\bullet]}$. We also put $Y^{(t)}$ to be the union of t-fold intersections of the various components of Y, where we observe that $Y^{[t]}$ is the 'canonical' desingularization of $Y^{(t)}$. Here we put $Y^{[0]}=X$, $Y^{[1]}=\coprod_1^N Y_j$, and so on. (Note that $Y^{(0)}=X$, $Y^{(1)}=Y$, and so on.) One has a simplicial complex $Y^{[\bullet]}\to Y$, viz.,

$$\cdots \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} Y^{[2]} \begin{smallmatrix}\to\\ \to\end{smallmatrix} Y^{[1]} \to Y,$$

where descent arguments enables on to compute homology of Y in terms of $Y^{[\bullet]}$. One can think of the arrows as defining (alternating) Gysin maps on homology. Corresponding to $Y^{[\bullet]}\to Y^{[0]}:=X$ is a third quadrant double complex

$$E_0^{i,j}(r) := z^{r+i}(Y^{[-i]},-j),\ i,\ j\leq 0;$$

$$\begin{array}{ccc} E_0^{i,j+1} & & \\ \Big\uparrow \partial & & \\ E_0^{i,j} & \xrightarrow{\mathrm{Gy}} & E_0^{i+1,j} \end{array}$$

whose differentials are $d_0:=\partial$ vertically and $d_1:=\mathrm{Gy}$ (= "alternating" Gysin) horizontally. Corresponding to this double complex are associated first and second Grothendieck spectral sequences of the corresponding single complex $\mathbf{s}E(r)^\bullet$ with $D=\partial\pm\mathrm{Gy}$, which have E_2-terms:

$${}'E_2^{p,q} := H_{\mathrm{Gy}}^p\big(H_\partial^q(E_0^{\bullet,\bullet}(r))\big)$$

$${}''E_2^{p,q} := H_\partial^p(H_{\mathrm{Gy}}^q\big(E_0^{\bullet,\bullet}(r))\big)$$

The second spectral sequence, together with a quasi-isomorphism (due to Bloch [Blo1]):

$$\frac{z^\bullet(X,*)}{z_Y^\bullet(X,*)} \xrightarrow{\text{Restriction}} z^\bullet(X\backslash Y,*),$$

shows that

$$H^{-m}(\mathbf{s}E(r)^\bullet) = {}''E_2^{0,-m} = \mathrm{CH}^r(X\backslash Y,m).$$

The first spectral sequence gives:

$$E_1^{i,j} = \mathrm{CH}^{r+i}(Y^{[-i]}, -j)$$

$$E_2^{i,j} = \frac{\ker\big(\mathrm{Gy} : \mathrm{CH}^{r+i}(Y^{[-i]}, -j) \to \mathrm{CH}^{r+i+1}(Y^{[-i+1]}, -j)\big)}{\mathrm{Gy}\big(\mathrm{CH}^{r+i-1}(Y^{[i-1]}, -j)\big)}$$

Then we have an increasing "weight" filtration

$$\mathrm{Image}\big(\mathrm{CH}^r(X, m) \to \mathrm{CH}^r(X\backslash Y, m)\big)$$

$=: \underline{\mathrm{CH}}^r(X\backslash Y, m) = W_{-m}\mathrm{CH}^r(X\backslash Y, m) \subset \cdots \subset W_0\mathrm{CH}^r(X\backslash Y, m) = \mathrm{CH}^r(X\backslash Y, m).$
Let $\{\xi\} \in W_\ell \mathrm{CH}^r(X\backslash Y, m)$. Then

$$\widetilde{\partial}_R^{\ell+m}(\xi) \in \mathrm{Image}\big(\mathrm{CH}^{r-\ell-m}(Y^{[\ell+m]}, -\ell)$$

$$\to \mathrm{CH}^{r-\ell-m}(Y^{(\ell+m)}\backslash Y^{(\ell+m+1)}, -\ell)\big).$$

In general $E_\infty^{-\ell-m,\ell} =$

$$Gr_W^\ell \mathrm{CH}^r(X\backslash Y, m) \hookrightarrow \begin{Bmatrix} \text{A subquotient of} \\ \mathrm{CH}^{r-\ell-m}(Y^{[\ell+m]}, -\ell) \end{Bmatrix}.$$

$[\ell = -m, ..., 0]$

4.8. **Hypercohomology.** Let $(\mathcal{S}^{\bullet \geq 0}, d)$ be a (bounded) complex of sheaves on X. One has a Cech double complex

$$\big(C^\bullet(\mathcal{U}, \mathcal{S}^\bullet),\ d,\ \delta\big),$$

where $\mathcal{U}$ is an open cover of X. The k-th hypercohomology is given by the k-th total cohomology of the associated single complex

$$\big(M^\bullet := \oplus_{i+j=\bullet} C^i(\mathcal{U}, \mathcal{S}^j),\ D = d \pm \delta\big),$$

viz.,

$$\mathbb{H}^k(\mathcal{S}^\bullet) := \lim_{\overrightarrow{\mathcal{U}}} H^k(M^\bullet).$$

Associated to the double complex are two filtered subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences abutting to $\mathbb{H}^k(\mathcal{S}^\bullet)$ (where $p + q = k$):

$${}'E_2^{p,q} := H_\delta^p(X, \mathcal{H}_d^q(\mathcal{S}^\bullet))$$

$${}''E_2^{p,q} := H_d^p(H_\delta^q(X, \mathcal{S}^\bullet))$$

The first spectral sequence shows that quasi-isomorphic complexes yield the same hypercohomology. The second spectral sequence is generally used for calculations.

Alternate take. Two complexes of sheaves $\mathcal{K}_1^\bullet$, $\mathcal{K}_2^\bullet$ are said to be quasi-isomorphic if there is a morphism $h : \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : \mathcal{H}^\bullet(\mathcal{K}_1^\bullet) \xrightarrow{\sim} \mathcal{H}^\bullet(\mathcal{K}_2^\bullet)$. Take a complex of acyclic sheaves $(\mathcal{K}^\bullet, d)$ (viz., $H^{i>0}(X, \mathcal{K}^j) = 0$ for all j) quasi-isomorphic to $\mathcal{S}^\bullet$. Then from the second spectral sequence,

$$\mathbb{H}^i(\mathcal{S}^\bullet) := H^i\big(\Gamma(\mathcal{K}^\bullet)\big),$$

where in this situation we define $\Gamma(\mathcal{K}^\bullet) := \Gamma(X, \mathcal{K}^\bullet) := H^0(X, \mathcal{K}^\bullet)$. For example if $\mathcal{L}^{\bullet,\bullet}$ is an [double complex] acyclic resolution of $\mathcal{S}^\bullet$, then the associated single complex $\mathcal{K}^\bullet = \oplus_{i+j=\bullet}\mathcal{L}^{i,j}$ is acyclic and quasi-isomorphic to $\mathcal{S}^\bullet$.[7]

Example 4.9. Let $(\Omega^\bullet_X, d)$, $(\mathcal{E}^\bullet_X, d)$ be complexes of sheaves of holomorphic and $\mathbb{C}$-valued C^∞ forms respectively. By the holomorphic and C^∞ Poincaré lemmas, one has quasi-isomorphisms:

$$(\mathbb{C} \to 0 \to \cdots) \xrightarrow{\approx} (\Omega^\bullet_X, d) \xrightarrow{\approx} (\mathcal{E}^\bullet_X, d),$$

where the latter two are Hodge filtered. The first spectral sequence of hypercohomology shows that[8]

$$H^k(X, \mathbb{C}) \simeq \mathbb{H}^k(\mathbb{C} \to 0 \to \cdots) \simeq \mathbb{H}^k((F^p)\Omega^\bullet_X) \simeq \mathbb{H}^k((F^p)\mathcal{E}^\bullet_X).$$

The second spectral sequence of hypercohomology applied to the latter term, using the known acyclicity of $\mathcal{E}^\bullet_X$, yields

$$\mathbb{H}^k(F^p\mathcal{E}^\bullet_X) \simeq \frac{\ker d : F^pE^k_X \to F^pE^k_X}{dF^pE^{k-1}_X} \simeq F^pH^k_{\mathrm{DR}}(X),$$

where the latter isomorphism is due to the degeneration at E_1 of the Hodge to de Rham spectral sequence.[9] The same story holds for currents.

Example 4.10. *Leray spectral sequence.* This is one of the most versatile spectral sequences in the literature. First some business about a push-forward. Let $f : X \to Y$ be a continuous map of 'nice' spaces, and $\mathcal{F}$ a sheaf on X. The push-forward $f_*\mathcal{F}$ (or direct image sheaf) is the sheaf on Y given by

$$U \subset Y \text{ open } \mapsto f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Assume given a flasque resolution of $\mathcal{F}$, viz.,

$$0 \to \mathcal{F} \to \mathcal{A}^\bullet.$$

Note that $f_*\mathcal{A}$ is flasque for any flasque sheaf $\mathcal{A}$ on X. Furthermore because of flasqueness,

$$\mathbb{H}^i(f_*\mathcal{A}^\bullet) = H^i\big(\Gamma(Y, f_*\mathcal{A}^\bullet)\big) = H^i\big(\Gamma(X, \mathcal{A}^\bullet)\big) \simeq H^i(X, \mathcal{F}).$$

The E_2-term of one of the Grothendieck spectral sequences associated to $\mathbb{H}^i(f_*\mathcal{A}^\bullet)$ is again, via flasqueness:

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(f_*\mathcal{A}^\bullet)) = H^p(X, R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

---

[7] We apply the Grothendieck spectral sequences on the level of sheaves. Let's write $D = d + \delta : \mathbf{s}^\bullet(\mathcal{L}) \to \mathbf{s}^{\bullet+1}(\mathcal{L})$, $d : \mathcal{L}^{i,\bullet} \to \mathcal{L}^{i+1,\bullet}$ and $\delta : \mathcal{L}^{\bullet,j} \to \mathcal{L}^{\bullet,j+1}$, and where we can assume for simplicity that $d\delta + \delta d = 0$, i.e, $D^2 = 0$. Since $\mathcal{L}^{i,\bullet}$ is a resolution of $\mathcal{S}^i$, it follows that

$$\mathcal{H}^q_\delta(\mathcal{L}^{i,\bullet}) := \begin{cases} \mathcal{S}^i & \text{if q} = 0 \\ 0 & \text{if q} > 0 \end{cases},$$

and so $(\mathcal{S}^\bullet, d)$ is quasi-isomorphic to $(\mathbf{s}^\bullet(\mathcal{L}), D)$.

[8] Alternatively, and for any sheaf $\mathcal{S}$, $\mathbb{H}^k\big(\mathcal{S} \to 0 \to 0 \to \cdots\big) \simeq H^k(X, \mathcal{S})$. For if $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{S}$. Then $\mathcal{I}^\bullet$ is quasi-isomorphic to $\mathcal{S} \to 0 \to 0 \to \cdots$, and hence $\mathbb{H}^k\big(\mathcal{S} \to 0 \to 0 \to \cdots\big) := H^k\big(\Gamma(\mathcal{I}^\bullet)\big) =: H^k(X, \mathcal{S})$.

[9] Note that the second spectral sequence of hypercohomology gives ${}''E_2^{p,q} = H^p_d(H^q_\delta(X, \Omega^\bullet_X)) \simeq H^q(X, \Omega^p_X)$ by the Hodge theorem.

Keep in mind that $R^q f_* \mathcal{F}$, called the Leray cohomology sheaf, is really the sheaf associated to the presheaf:

$$U \subset Y \text{ open } \mapsto H^q(f^{-1}(U), \mathcal{F}).$$

Two key applications come to mind:

• Suppose $f : X \to S$ is a smooth and proper morphism of smooth quasiprojective varieties over $\mathbb{C}$. Then it is well known, by Deligne (see [GH]), that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q}),$$

degenerates at E_2. In the case where $X = S \times Y$, this yields the Künneth formula:

$$H^i(S \times Y, \mathbb{Q}) \simeq \bigoplus_{p+q=i} H^p(S, \mathbb{Q}) \otimes H^q(Y, \mathbb{Q}).$$

• Let X be a smooth projective variety. One has a morphism of sites, $f : X_{an} \to X_{\mathrm{Zar}}$, i.e. a continuous map where X_{an} is the analytic "classical" topology on X as a complex manifold. Let $\mathcal{F}$ be an algebraic sheaf defined on X (such as a coherent sheaf of algebraic $\mathcal{O}_X$-modules, with a corresponding analytic sheaf $\mathcal{F}_{an}$). Then we have a spectral sequence:

$$H^p_{\mathrm{Zar}}\big(X, \mathcal{H}^q(\mathcal{F})\big) \Rightarrow H^{p+q}(X, \mathcal{F}_{an}).$$

Example 4.11. *MV cohomology.* Let $Y = Y_1 \cup \cdots \cup Y_N$ be a normal crossing divisor (with smooth components) on a projective algebraic manifold X. For $I = \{i_0, ..., i_q\} \subset \{1, ..., N\}$, put $Y_I = Y_{i_0} \cap \cdots \cap Y_{i_q}$, and $|I| = q + 1$. Set

$$Y^{[q]} := \coprod_{|I|=q+1} Y_I.$$

Note that $Y^{[q]}$ is smooth projective. Also let

$$A^{r,s}(Y) = r - \text{forms on } Y^{[s]}.$$

Exterior differentiation gives

$$d : A^{r,s}(Y) \to A^{r+1,s}(Y).$$

One also has

$$\delta : A^{r,s}(Y) \to A^{r,s+1}(Y),$$

given by the formula

$$\delta\omega(i_0, ..., i_{s+1}) = \sum_j (-1)^j \omega(i_0, ..., \hat{i}_j, ..., i_{s+1})\big|_{Y_{(i_0,\dots,i_{s+1})}}.$$

This turns

$$A^{\bullet,\bullet}(Y) := \bigoplus_{r,s} A^{r,s}(Y),$$

into a double complex, with associated single complex

$$\big(\mathbf{s}A^\bullet(Y), D := d + \delta\big).$$

Proposition 4.12. $H^i(Y, \mathbb{C}) = H^i_D\big(\mathbf{s}A^\bullet(Y)\big)$.

Proof. Form the double complex of fine (hence acyclic) sheaves $\mathcal{A}_Y^{r,s}$ on Y given by $U \subset Y$ open, $\Gamma(U, \mathcal{A}_Y^{r,s}) = A^{r,s}(U)$. All we have to do is show that the associated single complex of acyclic sheaves $(\mathbf{s}\mathcal{A}_Y^\bullet, D)$ is a resolution of the constant sheaf $\mathbb{C}$ on Y. But for $f \in \mathcal{A}_Y^{0,0}$, $Df = 0 \Leftrightarrow df = \delta f = 0$. Hence $f \in \mathbb{C}$ is constant. We need to prove the Poincaré lemma for $(\mathbf{s}\mathcal{A}_Y^\bullet, D)$. Taking hypercohomology of the double complex $\mathcal{A}_Y^{\bullet,\bullet}$ leads to

$$\mathcal{E}_2^{p,q} = \mathcal{H}_\delta^q \mathcal{H}_d^p(\mathcal{A}_Y^{\bullet,\bullet}).$$

By the usual Poincaré lemma for forms, $\mathcal{E}_2^{p,q} = 0$ for $p > 0$, and hence $\mathcal{E}_2 = \mathcal{E}_\infty$; moreover

$$\mathcal{H}_D^q(\mathbf{s}\mathcal{A}^\bullet) = \mathcal{E}_2^{0,q}.$$

But $\mathcal{E}_2^{0,q}$ is the q-th cohomology of the complex of sheaves

$$\cdots \to \mathbb{C}_{Y^{[q]}} \xrightarrow{\delta} \mathbb{C}_{Y^{[q+1]}} \to \cdots,$$

where $\mathbb{C}_W$ is the constant sheaf $\mathbb{C}$ on W. This is the same thing as the cohomology of a simplicial complex, and so

$$\mathcal{E}_2^{0,q} = \begin{cases} 0 & \text{if } q > 0 \\ \mathbb{C}_Y & \text{if } q = 0 \end{cases},$$

and we are done. $\square$

Now put

$$W_m A^{\bullet,\bullet}(Y) = \bigoplus_{s \le m} A^{\bullet,s}(Y),$$
$$F^p A^{\bullet,\bullet}(Y) = \bigoplus_{r,s} F^p A^{r,s}(Y).$$

Then one can easily show that $\{W_m\}$ and $\{F^p\}$ define a MHS on $H^\bullet(Y)$.

4.13. **Kähler differentials and arithmetic de Rham cohomology.** Let $S \supset R$ be commutative rings. Then $\Omega^1_{S/R}$ = free S-module generated by symbols $\{ds \mid s \in S\}$, and divided out by the submodule generated by expressions of the form

(1) $d(s + s') - ds - ds'$ for $s, s' \in S$,

(2) $d(ss') - sds' - s'ds$ for $s, s' \in S$, and (3) dr for $r \in R$.

Example 4.14. $\Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}} = \Omega^1_{\mathbb{C}/\mathbb{Q}}$. For if $\xi \in \overline{\mathbb{Q}}$, and if $p(t) \in \mathbb{Q}[t]$ is the minimum polynomial of ξ, then $p'(\xi)d\xi = dp(\xi) = 0$. By minimality, $p'(\xi) \neq 0$, hence $d\xi = 0$. Thus $d\overline{\mathbb{Q}} = 0$. In the same spirit, if $\{\alpha_\lambda\}_{\lambda \in I}$ is a transcendence base for $\mathbb{C}/\mathbb{Q}$, then $\{d\alpha_\lambda\}_{\lambda \in I}$ is a $\mathbb{C}$-basis of $\Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}}$. $\Omega^1_{\mathbb{C}/\mathbb{Q}}$ is called the space of absolute Kähler differentials.

Now set $\Omega^p_{S/R} = \bigwedge^p \Omega^1_{S/R}$, $\quad \Omega^0_{S/R} = S$. One has $d : \Omega^p_{S/R} \to \Omega^{p+1}_{S/R}$ characterized by $d\Big(gdf_1 \wedge \cdots \wedge df_p\Big) = dg \wedge df_1 \wedge \cdots \wedge df_p$ and which satisfies $d^2 = 0$. Hence we arrive at a complex $\big(\Omega^\bullet_{S/R}, d\big)$.

Sheafifying everything. Let X be a smooth projective variety defined over a field K, and $k \subset K$ a subfield. Define a presheaf:

$$U \subset X \text{ Zariski open} \quad \mapsto \quad \Omega^p_{X(K)/k}(U) = \Omega^p_{\mathcal{O}_X(U)/k}.$$

We define the arithmetic de Rham cohomology:

$$H^i_{\mathrm{DR}}(X(K)/k) := \mathbb{H}^i(\Omega^\bullet_{X(K)/k}).$$

Remark 4.15. (i) If X is defined over K, and L/K a field extension, then $\Omega^\bullet_{X(L)/L} \simeq \Omega^\bullet_{X(K)/K} \otimes_K L$, and hence we arrive at the base change:

$$H^i_{\mathrm{DR}}(X(L)/L) \simeq H^i_{\mathrm{DR}}(X(K)/K) \otimes_K L.$$

(ii) X defined over $\mathbb{C}$. Then by GAGA:

$$H^i_{\mathrm{DR}}(X(\mathbb{C})/\mathbb{C}) \simeq \mathbb{H}^i(\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}) \simeq H^i_{\mathrm{sing}}(X, \mathbb{C}).$$

4.16. **Arithmetic Gauss-Manin connection.** Again, X smooth projective and defined over K. Put

$$\mathrm{Filt}^m\Omega^p_{X(K)/k} := \mathrm{Im}\bigg(\Omega^m_{K/k} \otimes_K \Omega^{p-m}_{X(K)/k} \to \Omega^p_{X(K)/k}\bigg).$$

Thus:

$$\mathrm{Gr}^m\Omega^p_{X(K)/k} \simeq \Omega^m_{K/k} \otimes_K \Omega^{p-m}_{X(K)/K}.$$

Moreover:

$$0 \to \mathrm{Gr}^{m+1}\Omega^\bullet_{X(K)/k} \to \mathrm{Gr}^{m,m+2}\Omega^\bullet_{X(K)/k} \to \mathrm{Gr}^m\Omega^\bullet_{X(K)/k} \to 0,$$

translates to

$$0 \to \Omega^{m+1}_{K/k} \otimes \Omega^\bullet_{X(K)/K}[-1] \to \mathrm{Gr}^{m,m+2}\Omega^\bullet_{X(K)/k} \to \Omega^m_{K/k} \otimes \Omega^\bullet_{X(K)/K} \to 0.$$

Taking hypercohomology, we get a natural connecting map:

$$\nabla_{X(K)/k} : \Omega^m_{K/k} \otimes H^i_{\mathrm{DR}}(X(K)/K) \to \Omega^{m+1}_{K/k} \otimes H^i_{\mathrm{DR}}(X(K)/K),$$

called the arithmetic Gauss-Manin Connection. Note that $\Omega^\bullet_{X(K)/k}$ is a filtered complex using Filt^m. The spectral sequence computing $H^\bullet_{\mathrm{DR}}(X(K)/K)$ has E_1 term

$$E_1^{p,q} = \Omega^p_{K/k} \otimes \mathbb{H}^q(\Omega^\bullet_{X(K)/K}),$$

with $d_1 = \nabla_{X(K)/k}$. This is really a Leray spectral sequence, which by Deligne, is known to degenerate E_2. Thus $\nabla^2 = 0$ (viz., flat connection). Furthermore, by considering the Hodge filtered hypercohomology,[10] and repeating the above, one can argue that

$$\nabla\bigg(\Omega^m_{K/k} \otimes F^pH^i_{\mathrm{DR}}(X(K)/K)\bigg) \subset \Omega^{m+1}_{K/k} \otimes F^{p-1}H^i_{\mathrm{DR}}(X(K)/K),$$

which can be interpreted as an arithmetic version of Griffiths transversality.

[10]Namely, replace $\Omega^\bullet_{X(K)/K}$ by $F^p\Omega^\bullet_{X(K)/K}$. Then

$$F^pH^\bullet_{\mathrm{DR}}(X(K)/K) = \mathbb{H}^\bullet(F^p\Omega^\bullet_{X(K)/K}).$$

4.17. **Geometric Gauss-Manin connection.** Let $\rho : \mathcal{X} \to S$ be a smooth proper family of complex varieties. Recall the flat VB $R^i\rho_*\mathbb{C}$ with fiber $H^i(\mathcal{X}_s, \mathbb{C})$. One has the geometric Gauss-Manin

$$\nabla_{\mathcal{X}_s,\mathrm{Geom}} = \partial \otimes 1 : \mathcal{O}_S \otimes H^i(\mathcal{X}_s, \mathbb{C}) \to \Omega^1_{S/\mathbb{C}} \otimes H^i(\mathcal{X}_s, \mathbb{C}).$$

[Here we consider $s_0 \in \Delta \subset S$ a polydisk, $\rho^{-1}(\Delta) \simeq_{C^\infty} \Delta \times \mathcal{X}_{s_0}$, $\nabla_{\mathcal{X}_s,\mathrm{Geom}}$-flat classes are the pullback of $H^i(\mathcal{X}_{s_0}, \mathbb{C})$ from $Pr_2 : \Delta \times \mathcal{X}_{s_0} \to \mathcal{X}_{s_0}$.] Now assume given a smooth projective X defined over $K \subset \mathbb{C}$. We can assume that

$$K = \overline{\mathbb{Q}}(\alpha_1, \dots, \alpha_N) \simeq \mathrm{Quot}\big(\overline{\mathbb{Q}}[x_1, \dots, x_N]/I\big), \quad \text{where}$$

I = ideal of polynomial relations among $\alpha_1, \dots, \alpha_N/\overline{\mathbb{Q}}$. Let $S \subset \mathbb{C}^N$ be the

variety over $\overline{\mathbb{Q}}$ defined by I. $s_0 \in S \leftrightarrow (\alpha_1, \dots, \alpha_N)$, $s \in S \leftrightarrow (\alpha_1(s), \dots, \alpha_N(s))$, $K(s) = \overline{\mathbb{Q}}((\alpha_1(s), \dots, \alpha_N(s))$. Now let's write $X = V(F_1, \dots, F_M)$, F_i homogeneous in $K[y_0, \dots, y_r]$, $X \subset \mathbb{P}^r$. For each $s \in S$ we get $F_i(s)$ by replacing $(\alpha_1, \dots, \alpha_N)$ by their values corresponding to s. $\mathcal{X}_s := V(F_1(s), \dots, F_M(s))$. We get a family

$$\begin{array}{ccc} \mathcal{X} & \subset & \mathbb{P}^r \times S \\ \downarrow & & \\ S & & \end{array}$$

[Warning: One may have to shrink S to get a smooth family.] If $\phi \in \mathbb{H}^i(\Omega^\bullet_{X(K)/K})$, we get $\phi(s) \in \mathbb{H}^i(\Omega^\bullet_{\mathcal{X}_s(K(s))/K(s)}) \otimes_{K(s)} \mathbb{C} = \mathbb{H}^i(\Omega^\bullet_{\mathcal{X}_s(\mathbb{C})/\mathbb{C}})$. Then:

$$\nabla_{\mathcal{X}(K)/\overline{\mathbb{Q}}}\phi = \nabla_{\mathcal{X},\mathrm{Geom}}(\phi(s)), \quad \text{at } s = s_0.$$

This uses $\Omega^1_{S/\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \Omega^1_{S(\mathbb{C})/\mathbb{C}}$. Note that $\Omega^1_{S/\overline{\mathbb{Q}}} = \Omega^1_{S/\mathbb{Q}}$. Further, since $\Omega^1_{S/\overline{\mathbb{Q}}} = \bigoplus_i \mathcal{O}_S(S)dx_i/\{df \mid f \in I\}$, we identify $\Omega^1_{S/\mathbb{Q}}\big|_{s_0}$ with its image (induced by evaluation at s_0),

$$\Omega^1_{S/\mathbb{Q}}\big|_{s_0} \hookrightarrow \Omega^1_{\mathbb{C}/\mathbb{Q}}, \quad dx_i \mapsto d\alpha_i.$$

4.18. **The arithmetic cycle class map.** Let $K \subset \mathbb{C}$ be a subfield containing $\overline{\mathbb{Q}}$, and consider our smooth projective X defined over K. Correspondingly, let $\mathcal{K}^M_{r,X} = \mathcal{O}^\times_X \otimes \cdots \otimes \mathcal{O}^\times_X / \langle \tau_1 \otimes \cdots \otimes \tau_i \otimes \cdots \otimes \tau_j \otimes \cdots \otimes \tau_r \mid \tau_i + \tau_j = 1,\ i \neq j \rangle$ be the Milnor sheaf of X (this will be discuss in greater detail in Appendix 4.22). Then by the results of Gabber, Müller-Stach, Elbaz-Vincent and Kerz, (see [EV-MS], [MS2], [Ke]),

$$\mathrm{CH}^r(X) \simeq H^r_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) = \mathbb{H}^r(\mathcal{K}^M_{r,X} \to 0 \to 0 \to \cdots),$$

where we recall that $\mathrm{CH}^r(X)$ is the classical Chow group. This isomorphism will be explained in more detail in Appendices 4.22 and 6.25. The natural map $\mathcal{K}^M_{r,X} \to \Omega^{\bullet \geq r}_{X(K)/\overline{\mathbb{Q}}}[r]$ given by

$$\{f_1, \dots, f_r\} \mapsto \bigwedge_j d\log f_j, \ \ f_j \in \mathcal{O}^\times_X,$$

factors through a morphism of complexes

$$\big(\mathcal{K}^M_{r,X} \to 0 \to 0 \to \cdots\big) \to \Omega^{\bullet \geq r}_{X(K)/\overline{\mathbb{Q}}}[r],$$

and hence determines a morphism

$$c_{r,K} : \mathrm{CH}^r(X/_K) \to \mathbb{H}^{2r}(\Omega^{\bullet\geq r}_{X(K)/\overline{\mathbb{Q}}}),$$

called the arithmetic cycle class map. Now recall

$$\nabla : \Omega^m_{K/k} \otimes H^i_{\mathrm{DR}}(X(K)/K) \to \Omega^{m+1}_{K/k} \otimes H^i_{\mathrm{DR}}(X(K)/K),$$

and that the Leray spectral sequence computing the cohomology $H^\bullet_{\mathrm{DR}}(X(K)/K)$ has E_1 term

$$E_1^{p,q} = \Omega^p_{K/k} \otimes \mathbb{H}^q(\Omega^\bullet_{X(K)/K}),$$

with $d_1 = \nabla := \nabla_{X(K)/k}$, which by Deligne, is known to degenerate E_2.

Likewise, the analogous Leray spectral sequence associated

$$F^r H^{2r}_{\mathrm{DR}}(X(K)/\overline{\mathbb{Q}})) := \mathbb{H}^{2r}(\Omega^{\bullet\geq r}_{X(K)/\overline{\mathbb{Q}}})$$

degenerates at E_2, and defines a corresponding Leray filtration $F^\nu_L \mathbb{H}^{2r}(\Omega^{\bullet\geq r}_{X(K)/\overline{\mathbb{Q}}})$. The associated graded $Gr^\nu_{F_L}$ is given by $\nabla J^{r,\nu}(X(K)/\overline{\mathbb{Q}})$, which is defined as the cohomology of:

$$\Omega^{\nu-1}_{K/\overline{\mathbb{Q}}} \otimes F^{r-\nu+1} H^{2r-\nu}_{\mathrm{DR}}(X(K)/\overline{\mathbb{Q}}) \xrightarrow{\nabla} \Omega^{\nu}_{K/\overline{\mathbb{Q}}} \otimes F^{r-\nu} H^{2r-\nu}_{\mathrm{DR}}(X(K)/\overline{\mathbb{Q}})$$
$$\xrightarrow{\nabla} \Omega^{\nu+1}_{K/\overline{\mathbb{Q}}} \otimes F^{r-\nu-1} H^{2r-\nu}_{\mathrm{DR}}(X(K)/\overline{\mathbb{Q}}).$$

4.19. **De Rham and Mumford-Griffiths invariants.** Let us now assume $K = \mathbb{C}$, in this subsection. We have two sequences:

$$\Omega^{\nu-1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}) \xrightarrow{\nabla} \Omega^{\nu}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}) \xrightarrow{\nabla} \Omega^{\nu+1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}). \tag{2}$$

$$\Omega^{\nu-1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu+1} H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}) \xrightarrow{\nabla} \Omega^{\nu}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu} H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}) \xrightarrow{\nabla} \Omega^{\nu+1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu-1} H^{2r-\nu}_{\mathrm{DR}}(X(\mathbb{C})/\overline{\mathbb{Q}}). \tag{3}$$

Definition 4.20 (See [L-S]). *The space of de Rham invariants, $\nabla DR^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}})$, is given by the cohomology of (2). The space of Mumford-Griffiths invariants, $\nabla J^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}})$, is given by the cohomology of (3).*

Remark 4.21. (1) One has a natural map

$$\nabla J^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}}) \to \nabla DR^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}}),$$

which need not be injective.

(2) Later we will work with a candidate Bloch-Beilinson filtration $\{F^\nu \mathrm{CH}^r(X/\mathbb{C};\mathbb{Q})\}_{\nu\geq 0}$ and show that there is a commutative diagram of maps:

$$\begin{array}{ccc} Gr^\nu_F \mathrm{CH}^r(X;\mathbb{Q}) & \to & \nabla J^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}}) \\ & \searrow & \downarrow \\ & & \nabla DR^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}}) \end{array}$$

Let $\nabla J^{r,\nu}_{\text{alg}}(X(\mathbb{C})/\overline{\mathbb{Q}})$, $\nabla DR^{r,\nu}_{\text{alg}}(X(\mathbb{C})/\overline{\mathbb{Q}})$ be the respective images of $Gr^{\nu}_F \mathrm{CH}^r(X;\mathbb{Q})$. The works of [L-S] and [MSa1] show that under the natural map in (1),

$$\nabla J^{r,\nu}_{\text{alg}}(X(\mathbb{C})/\overline{\mathbb{Q}}) = \nabla DR^{r,\nu}_{\text{alg}}(X(\mathbb{C})/\overline{\mathbb{Q}}). \tag{4}$$

Not only can $\nabla J^{r,\nu}_{\text{alg}}(X(\mathbb{C})/\overline{\mathbb{Q}})$ be highly non-trivial, but the kernel of

$$Gr^{\nu}_F \mathrm{CH}^r(X;\mathbb{Q}) \to \nabla J^{r,\nu}(X(\mathbb{C})/\overline{\mathbb{Q}}),$$

as well ([L-S]). The work of M. Saito [MSa1] generalizes (4) to the situation of higher Chow groups (to be discussed later).

4.22. **Appendix: Milnor K-theory.** This section provides some of the foundations for the subsequent sections. In the first part of this section, we follow closely the treatment of Milnor K-theory provided in [B-T], which allows us to provide an abridged definition of the higher Chow groups $\mathrm{CH}^r(X,m)$, for $0 \le m \le 2$. The reader with pressing obligations who prefers to work with concrete examples may skip this section, without losing sight of the main ideas presented in this paper.

Let $\mathbb{F}$ be a field, with multiplicative group $\mathbb{F}^\times$, and put $T(\mathbb{F}^\times) = \coprod_{n\ge 0} T^n(\mathbb{F}^\times)$, the tensor product of the $\mathbb{Z}$-module $\mathbb{F}^\times$. Here $T^0(\mathbb{F}^\times) := \mathbb{Z}$, $\mathbb{F}^\times = T^1(\mathbb{F}^\times)$, $a \mapsto [a]$. If $a \ne 0,1$, set $r_a = [a]\otimes[1-a] \in T^2(\mathbb{F}^\times)$. The two-sided ideal R generated by r_a is graded, and we put:

$$K^M_\bullet \mathbb{F} = \frac{T(\mathbb{F}^\times)}{R} = \coprod_{n\ge 0} K^M_n \mathbb{F}, \quad \text{(Milnor } K-\text{theory)}.$$

For example, $K_0(\mathbb{F}) = \mathbb{Z}$, $K_1(\mathbb{F}) = \mathbb{F}^\times$, and $K^M_2(\mathbb{F})$ is the abelian group generated by symbols $\{a,b\}$, subject to the Steinberg relations:

$$\{a_1a_2, b\} = \{a_1,b\}\{a_2,b\}$$
$$\{a, 1-a\} = 1, \text{ for } a \ne 0,1$$
$$\{a,b\} = \{b,a\}^{-1}$$

Furthermore, one can also show that:

$$\{a,a\} = \{-1,a\} = \{a,a^{-1}\} = \{a^{-1},a\}, \text{and } \{a,-a\} = 1. \tag{2.1}$$

Quite generally, one can argue that $K^M_n(\mathbb{F})$ is generated $\{a_1,\dots,a_n\}$, $a_1,\dots,a_n \in \mathbb{F}^\times$, subject to:

$$\text{(i)} \qquad (a_1,\dots,a_n) \mapsto \{a_1,\dots,a_n\},$$

is a multilinear function from $\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times \to K^M_n(\mathbb{F})$,

$$\text{(ii)} \qquad \{a_1,\dots,a_n\} = 0,$$

if $a_i + a_{i+1} = 1$ for some $i < n$.

Next, let us assume given a field $\mathbb{F}$ with discrete valuation $\nu : \mathbb{F}^\times \to \mathbb{Z}$, with corresponding discrete valuation ring $\mathrm{O}_{\mathbb{F}} := \{a \in \mathbb{F} \mid \nu(a) \ge 0\}$, and residue field $\mathbf{k}(\nu)$. One has maps $T : K^M_\bullet(\mathbb{F}) \to K^M_{\bullet-1}(\mathbf{k}(\nu))$. Choose $\pi \in \mathbb{F}^\times$ such that $\nu(\pi) = 1$, and note that $\mathbb{F}^\times = \mathrm{O}^\times_{\mathbb{F}} \cdot \pi^{\mathbb{Z}}$. For example, if we write $a = a_0\pi^i$, $b = b_0\pi^j \in K^M_1(\mathbb{F})$, then $T(a) = i \in \mathbb{Z} = K^M_0(\mathbf{k}(\nu))$ and

$$T\{a,b\} = (-1)^{ij}\,\frac{a^j}{b^i} \in \mathbf{k}(\nu)^\times = K^M_1(\mathbf{k}(\nu)) \quad \text{(Tame symbol)}.$$

The Gersten-Milnor complex. The reader may find [MS2] particularly useful regarding the discussion in this part. Let $\mathcal{O}_X$ be the sheaf of regular functions on X, with sheaf of units $\mathcal{O}_X^\times$. As in [Ka], we put

$$\mathcal{K}_{r,X}^M := \big(\underbrace{\mathcal{O}_X^\times \otimes \cdots \otimes \mathcal{O}_X^\times}_{r \text{ times}}\big)/\mathcal{J}, \quad \text{(Milnor sheaf)},$$

where $\mathcal{J}$ is the subsheaf of the tensor product generated by sections of the form:

$$\big\{\tau_1 \otimes \cdots \otimes \tau_r \bigm| \tau_i + \tau_j = 1, \quad \text{for some } i \text{ and } j,\ i \neq j\big\}.$$

For example, $\mathcal{K}_{1,X}^M = \mathcal{O}_X^\times$. Introduce the Gersten-Milnor complex (a flasque resolution of $\mathcal{K}_{r,X}^M$, see [EV-MS], [Ke]):

$$\mathcal{K}_{r,X}^M \to K_r^M(\mathbb{C}(X)) \to \bigoplus_{\mathrm{cd}_X Z=1} i_{Z,*}K_{k-1}^M(\mathbb{C}(Z)) \to \cdots \to$$

$$\bigoplus_{\mathrm{cd}_X Z=r-2} i_{Z,*}K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z=r-1} i_{Z,*}K_1^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z=r} i_{Z,*}K_0^M(\mathbb{C}(Z)) \to 0.$$

We have

$$K_0^M(\mathbb{C}(Z)) = \mathbb{Z}, \qquad K_1^M(\mathbb{C}(Z)) = \mathbb{C}(Z)^\times,$$
$$K_2^M(\mathbb{C}(Z)) = \big\{\text{symbols } \{f,g\} \ / \text{ Steinberg relations}\big\}.$$

The last three terms of this complex (again taking global sections over X) then are:

$$\bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbb{C}(Z)^\times \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X Z=r} \mathbb{Z} \to 0$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z=r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D=r-1} K_1^M(\mathbb{C}(D)),$$

defined earlier.

Definition 4.23. For $0 \leq m \leq 2$,

$$\mathrm{CH}^r(X,m) = H_{\mathrm{Zar}}^{r-m}(X, \mathcal{K}_{r,X}^M).$$

Example 4.24.

$$\mathrm{CH}^1(X,1) \simeq H_{\mathrm{Zar}}^0(X, \mathcal{K}_{1,X}^M) \simeq H_{\mathrm{Zar}}^0(X, \mathcal{O}_X^\times) \simeq \mathbb{C}^\times.$$

Remark 4.25. The higher Chow groups $\mathrm{CH}^r(W,m)$ were introduced in [Blo1], and are defined for any non-negative integers r and m, and quasi-projective variety W over a field k. The formula in Definition 4.23 is only for smooth varieties X, and is proven as a theorem in Appendix 6.25.

5. **Deligne cohomology**

5.1. **Deligne cohomology I.** Let $\mathbb{A} \subset \mathbb{R}$ be a subring and $r \geq 0$ an integer. We recall the Tate twist $\mathbb{A}(r) = (2\pi \mathrm{i})^r \cdot \mathbb{A}$, and declare $\mathbb{A}(r)$ a pure $\mathbb{A}$-Hodge structure of weight $-2r$ and of (pure) Hodge type $(-r, -r)$. We introduce the Deligne complex $\mathbb{A}_{\mathcal{D}}(r)$:

$$\mathbb{A}(r) \to \underbrace{\mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{r-1}}_{=:\Omega_X^{\bullet < r}}.$$

Definition 5.2. Deligne cohomology[11] is given by the hypercohomology:

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) := \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r)).$$

Example 5.3. When $\mathbb{A} = \mathbb{Z}$, we have a quasi-isomorphism

$$\mathbb{Z}_{\mathcal{D}}(1) \approx \mathcal{O}_X^\times[-1],$$

hence

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq H^1(X, \mathcal{O}_X^\times) =: \mathrm{Pic}(X) \simeq \mathrm{CH}^1(X).$$

$$H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}_X^\times) \simeq \mathbb{C}^\times \simeq H^0_{\mathrm{Zar}}(X, \mathcal{K}_{1,X}) =: \mathrm{CH}^1(X, 1),$$

where the latter is Bloch's higher Chow group.

Example 5.4. Alternate take on $H^1_{\mathcal{D}}(X, \mathbb{Z}(1))$. Look at the Cech double complex:

$$\begin{array}{ccc} \mathcal{C}^0(\mathcal{U}, \mathbb{Z}(1)) & \to & \mathcal{C}^0(\mathcal{U}, \mathcal{O}_X) \\ \delta \big\downarrow & & \big\downarrow \delta \\ \mathcal{C}^1(\mathcal{U}, \mathbb{Z}(1)) & \to & \mathcal{C}^1(\mathcal{U}, \mathcal{O}_X) \end{array}$$

So a class in $H^1_{\mathcal{D}}(X, \mathbb{Z}(1))$ is represented (after a suitable refinement) by $(\lambda := \{\lambda_{\alpha\beta}\}, f := \{f_\gamma\}) \in \big(\Gamma(U_\alpha \cap U_\beta, \mathbb{Z}(1)), \Gamma(U_\gamma, \mathcal{O}_X)\big)$, with $f_\beta - f_\alpha =: \delta(f)_{\alpha\beta} = \lambda_{\alpha\beta}$. Note that $\exp(f) \in H^0(X, \mathcal{O}_X^\times)$ determines the isomorphism $H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}_X^\times) \simeq \mathbb{C}^\times$.

Definition 5.5. The product structure on Deligne cohomology

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(i)) \otimes H^l_{\mathcal{D}}(X, \mathbb{Z}(j)) \to H^{k+l}_{\mathcal{D}}(X, \mathbb{Z}(i+j)),$$

is induced by the multiplication of complexes $\mu : \mathbb{Z}_{\mathcal{D}}(i) \otimes \mathbb{Z}_{\mathcal{D}}(j) \to \mathbb{Z}_{\mathcal{D}}(i+j)$ defined by

$$\mu(x, y) := \begin{cases} x \cdot y, & \text{if } \deg x = 0, \\ x \wedge dy, & \text{if } \deg x > 0 \text{ and } \deg y = j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.6. For example, this product structure implies that

$$H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \cup H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) = \{0\} \subset H^2_{\mathcal{D}}(X, \mathbb{Z}(2)).$$

Alternate take. Let $h : (A^\bullet, d) \to (B^\bullet, d)$ be a morphism of complexes. We define

$$\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big)$$

[11]This definition applies to *any* complex manifold, not just projective algebraic X. It is the definition of *analytic* Deligne cohomology.

by the formula

$$\big[\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big)\big]^q := A^{q+1}\oplus B^q,\quad \delta(a,b) = (-da, h(a)+db).$$

Example 5.7. $\mathrm{Cone}\big(\mathbb{A}(r)\oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega^\bullet\big)[-1]$ is given by:

$$\mathbb{A}(r)\to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-2} \xrightarrow{(0,d)} \big(\Omega_X^r\oplus\Omega_X^{r-1}\big)$$
$$\xrightarrow{\delta} \big(\Omega_X^{r+1}\oplus\Omega_X^r\big) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \big(\Omega_X^d\oplus\Omega_X^{d-1}\big)\to\Omega_X^d$$

Using the holomorphic Poincaré lemma, one can show that the natural map

$$\mathbb{A}_{\mathcal{D}}(r)\to \mathrm{Cone}\big(\mathbb{A}(r)\oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet\big)[-1],$$

is a quasi-isomorphism.[12] Thus

$$H^k_{\mathcal{D}}(X,\mathbb{A}(r)) \simeq \mathbb{H}^r\big(\mathrm{Cone}\big(\mathbb{A}(r)\oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet\big)[-1]\big).$$

Let $\mathcal{D}_X^\bullet$ be the sheaf of currents acting on C^∞ compactly supported $(2d-\bullet)$-forms. Further, put $\mathcal{D}_X^{p,q}$ to be the sheaf of currents acting on C^∞ compactly supported$(d-p,d-q)$- forms. One has a decomposition

$$\mathcal{D}_X^\bullet = \bigoplus_{p+q=\bullet}\mathcal{D}_X^{p,q},$$

with a morphism of complexes $\mathcal{E}_X^\bullet \hookrightarrow \mathcal{D}_X^\bullet$ and with $\mathcal{E}_X^{p,q}\hookrightarrow \mathcal{D}_X^{p,q}$, compactible with both ∂ and $\bar\partial$. Likewise, let $\mathcal{C}_X^\bullet = \mathcal{C}_{2d-\bullet,X}(\mathbb{A}(r))$ be the sheaf of (Borel-Moore) chains of real codimension $\bullet$. This is known to be a soft sheaf. Identifying the constant sheaf $\mathbb{A}(r)$ with the complex $\mathbb{A}(r)\to 0\to\cdots\to 0$, we have quasi-isomorphisms

$$\mathbb{A}(r)\xrightarrow{\approx}\mathcal{C}_X^\bullet(\mathbb{A}(r)),\quad \Omega_X^\bullet\xrightarrow{\approx}\mathcal{E}_X^\bullet,\quad \mathcal{E}_X^\bullet\xrightarrow{\approx}\mathcal{D}_X^\bullet$$

where the latter two quasi-isomorphims are (Hodge) filtered. As the sheaves on the RHS are all known to be acyclic, we deduce:

$$H^k_{\mathcal{D}}(X,\mathbb{A}(r)) \simeq H^k\big(\mathrm{Cone}\big(\mathcal{C}_X^\bullet(X,\mathbb{A}(r))\oplus F^r\mathcal{D}_X^\bullet(X) \xrightarrow{\epsilon - l} \mathcal{D}_X^\bullet(X)\big)[-1]\big).$$

Note that

$$\mathbb{H}^k(F^p\Omega_X^\bullet)\simeq \mathbb{H}^k(F^p\mathcal{E}_X^\bullet)\simeq F^pH^k_{\mathrm{DR}}(X),\ \text{hence}\ \mathbb{H}^k(\Omega_X^{\bullet<p})\simeq \frac{H^k_{\mathrm{DR}}(X)}{F^pH^k_{\mathrm{DR}}(X)}.$$

From the short exact sequence:

$$0\to\Omega_X^{\bullet<r}[-1]\to\mathbb{A}_{\mathcal{D}}(r)\to\mathbb{A}(r)\to 0,$$

together with Hodge theory, we arrive at the short exact sequence:[13]

---

[12] $(a,b)\in\Omega_X^r\oplus\Omega_X^{r-1}\overset{\delta}{\mapsto}(-da,db-a)\in\Omega_X^{r+1}\oplus\Omega_X^r$. $\delta(a,b)=(0,0)\Leftrightarrow da=0\ \&\ a=db\Leftrightarrow a=db$. Therefore $\ker\delta/\mathrm{Im}(0,d)\simeq\Omega^{r-1}/d\Omega_X^{r-2}=\mathcal{H}^{r-1}(\mathbb{A}_{\mathcal{D}}(r))$. Next, for $j\geq 1$, $(a,b)\in\Omega_X^{r+j}\oplus\Omega_X^{r+j-1}$, $\delta(a,b)=0\Leftrightarrow(a,b)=\delta(-b,0)$.

[13] *Alternate take.* $\mathcal{D}_X^\bullet(X))[-1]$ is a subcomplex of $\mathrm{Cone}\big(C_X^\bullet(X,\mathbb{A}(r))\oplus F^r\mathcal{D}_X^\bullet(X)\xrightarrow{\epsilon-l}\mathcal{D}_X^\bullet(X)\big)[-1]$. Hence the cone complex description of $H^i_{\mathcal{D}}(X,\mathbb{A}(r))\simeq$

$$H^i\big(\mathrm{Cone}\big(C_X^\bullet(X,\mathbb{A}(r))\oplus F^r\mathcal{D}_X^\bullet(X)\xrightarrow{\epsilon-l}\mathcal{D}_X^\bullet(X)\big)[-1]\big),$$

yields the exact sequence:

$$\cdots\to H^{i-1}(X,\mathbb{A}(r))\oplus F^rH^{i-1}(X,\mathbb{C})\to H^{i-1}(X,\mathbb{C}) \tag{5}$$
$$\to H^i_{\mathcal{D}}(X,\mathbb{A}(r))\to H^i(X,\mathbb{A}(r))\oplus F^rH^i(X,\mathbb{C})\to\cdots$$

$$0 \to \frac{H^{i-1}(X,\mathbb{C})}{H^{i-1}(X,\mathbb{A}(r)) + F^rH^{i-1}(X,\mathbb{C})} \to H^i_{\mathcal{D}}(X,\mathbb{A}(r))$$
$$\to H^i(X,\mathbb{A}(r)) \cap F^rH^i(X,\mathbb{C}) \to 0,$$

which is the same thing as

$$0 \to J_{\mathbb{A}}(H^{i-1}(X,\mathbb{Q}(r)) \to H^i_{\mathcal{D}}(X,\mathbb{A}(r)) \to \Gamma_{\mathbb{A}}(H^i(X,\mathbb{Q}(r)) \to 0.$$

In particular, if $(\mathbb{A}, i, r) = (\mathbb{Z}, 2r-m, r)$, we arrive at the short exact sequence:

$$0 \to J(H^{2r-m-1}(X,\mathbb{Z}(r))) \to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to \Gamma(H^{2r-m}(X,\mathbb{Z}(r))) \to 0.$$

Note that for $m \geq 1$, $\Gamma(H^{2r-m}(X,\mathbb{Z}(r))) = H^{2r-m}_{\mathrm{tor}}(X,\mathbb{Z}(r))$ is the torsion subgroup. Further, observe that

$$J(H^{2r-m-1}(X,\mathbb{Z}(r))) \simeq \frac{H^{2r-m-1}(X,\mathbb{C})}{F^rH^{2r-m-1}(X,\mathbb{C}) + H^{2r-m-1}(X,\mathbb{Z}(r))}$$
$$\simeq \frac{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})^\vee}{H_{2d-2r+m+1}(X,\mathbb{Z}(d-r))}.$$

Example 5.8. If $\mathbb{A} = \mathbb{R}$ and $i = 2r-m$, where $m \geq 1$, then $H^i_{\mathrm{tor}}(X,\mathbb{R}(r)) = 0$; moreover if we set

$$\pi_{r-1} : \mathbb{C} = \mathbb{R}(r) \oplus \mathbb{R}(r-1) \to \mathbb{R}(r-1)$$

to be the projection, then we have the isomorphisms:

$$H^{2r-m}_{\mathcal{D}}(X,\mathbb{R}(r)) \simeq \frac{H^{2r-m-1}(X,\mathbb{C})}{F^rH^{2r-m-1}(X,\mathbb{C}) + H^{2r-m-1}(X,\mathbb{R}(r))}$$
$$\underset{\simeq}{\xrightarrow{\pi_{r-1}}} \frac{H^{2r-m-1}\big(X,\mathbb{R}(r-1)\big)}{\pi_{r-1}\big(F^rH^{2r-m-1}(X,\mathbb{C})\big)}$$

For example:

$$H^{2r-1}_{\mathcal{D}}(X,\mathbb{R}(r)) \simeq \frac{H^{2r-2}(X,\mathbb{C})}{F^rH^{2r-2}(X,\mathbb{C}) + H^{2r-2}(X,\mathbb{R}(r))}$$
$$\underset{\simeq}{\xrightarrow{\pi_{r-1}}} H^{r-1,r-1}(X,\mathbb{R}) \otimes \mathbb{R}(r-1)$$
$$=: H^{r-1,r-1}(X,\mathbb{R}(r-1)) \simeq \Big\{H^{d-r+1,d-r+1}(X,\mathbb{R}(d-r+1))\Big\}^\vee.$$

5.9. **Real Deligne cohomology I.** Let X be a smooth projective variety defined over $\mathbb{R}$, with corresponding complex space $X(\mathbb{C}) = X_{\mathbb{R}} \times \mathbb{C}$. One has the notion of algebraic Kähler differentials defining a complex $\Omega^\bullet_{X/\mathbb{R}}$. One defines (in the Zariski topology) $F^pH^i_{DR}(X/\mathbb{R}) := \mathbb{H}^i(\Omega^{\bullet\geq p}_{X/\mathbb{R}})$. One has $\Omega^\bullet_{X/\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \Omega^\bullet_{X(\mathbb{C})/\mathbb{C}}$ and accordingly

$$F^pH^i_{\mathrm{DR}}(X/\mathbb{R}) \otimes \mathbb{C} = \mathbb{H}^i(\Omega^{\bullet\geq p}_{X/\mathbb{R}}) \otimes \mathbb{C} = \mathbb{H}^i(\Omega^{\bullet\geq p}_{X(\mathbb{C})/\mathbb{C}}) = F^pH^i_{\mathrm{DR}}(X(\mathbb{C}),\mathbb{C}),$$

where the latter term isomorphism is due to GAGA, and is defined in the analytic topology. Complex conjugation on $\mathbb{C}$ in $F^pH^i_{\mathrm{DR}}(X/\mathbb{R}) \otimes \mathbb{C}$ induces the so-called DR-conjugation on $F^pH^i_{\mathrm{DR}}(X(\mathbb{C}),\mathbb{C})$, which preserves the Hodge filtration (this is not the same thing as the usual complex conjugation on $H^i(X,\mathbb{C})$ induced from $\mathbb{C}$

in $H^i(X,\mathbb{R})\otimes\mathbb{C}$ in the classical topology). DR acts by conjugating the coefficients of a holomorphic form on X. It is the same thing as the involution defined on $X(\mathbb{C})$ by complex conjugation, *together* with the action of complex conjugation on the $\mathbb{C}$-valued C^∞ forms $E^\bullet_X = E^\bullet_{X,\mathbb{R}}\otimes\mathbb{C}$. This leads to the notion of real Deligne cohomology $H^{2r-m}_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(r)) := H^{2r-m}_{\mathcal{D}}(X(\mathbb{C}),\mathbb{R}(r))^{\mathrm{DR}}$. Two example situations come to mind:

• Let $X/\mathbb{C}$ be a smooth complex variety, and consider $X\to\mathrm{Spec}(\mathbb{C})\to\mathrm{Spec}(\mathbb{R})$. Then $X_{\mathbb{R}}\times_{\mathbb{R}}\mathbb{C} = X(\mathbb{C})\coprod X(\mathbb{C})$, and where σ permutes the factors. Correspondingly, $H^{2r-m}_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(r)) = H^{2r-m}_{\mathcal{D}}(X(\mathbb{C}),\mathbb{R}(r))$.

• Let $K\subset\mathbb{C}$ be a number field and $X=\mathrm{Spec}(K)$. We view $X/\mathbb{Q}$ via $\mathrm{Spec}(K)\to\mathrm{Spec}(\mathbb{Q})$, induced via the inclusion $\mathbb{Q}\subset K$. Then

$$X_{\mathbb{C}} = X_{\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{C} = \coprod_{\sigma\in\hom_{\mathbb{Q}}(K,\mathbb{C})}\mathrm{Spec}(\mathbb{C}).$$

So for example

$$H^1_{\mathcal{D}}(X/\mathbb{R},\mathbb{R}(r)) = \Big[\bigoplus_{\sigma\in\hom_{\mathbb{Q}}(K,\mathbb{C})}\mathbb{R}(r-1)\Big]^{\mathrm{DR}}$$

$$= \Big[\bigoplus_{\sigma\in\hom_{\mathbb{Q}}(K,\mathbb{C})}\mathbb{R}(r-1)\Big]^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \begin{cases}\mathbb{R}^{r_1+r_2} & \text{if } r\geq 1 \text{ is odd}\\ \mathbb{R}^{r_2} & \text{if } r\geq 2 \text{ is even}\end{cases},$$

where r_1 (resp. r_2) denotes the number of real (resp. pairs of complex conjugatations) embeddings of K in $\mathbb{C}$.

5.10. **Deligne-Beilinson cohomology, II.** The reader may find the sources [EV] and [Ja1] particularly useful. The definition of Deligne cohomology in the previous section works well for projective algebraic manifolds, but not so good for smooth open $U\subset X$. First of all, the naive Hodge filtration on U, viz., $\Omega^{\bullet\geq r}_U$ is the *wrong* choice. For example, if W is a Stein manifold, then $H^q(W,\Omega^i_W)=0$ for all i and where $q\geq 1$. This tells us, via the Grothendieck spectral sequences associated to hypercohomology, that

$$H^j(W,\mathbb{C})\simeq\frac{\Gamma\big(W,\Omega^j_W\big)_{d-\mathrm{closed}}}{d\Gamma\big(\Omega^{j-1}_W\big)}.$$

[Note: If W is a smooth affine variety, then by Grothendieck, one can use algebraic differential forms.] We hardly expect $H^j(W,\mathbb{C}) = F^jH^j(W,\mathbb{C})$ to be the case in general. Secondly, analytic Deligne cohomology fails to take into consideration the underlying algebraic structure of U. For instance $H^1_{\mathcal{D}}(U,\mathbb{Z}(1)) = H^0(U,\mathcal{O}^\times_U)$, i.e. the non-zero analytic functions on X. It would be preferable to recover the non-zero algebraic functions on U instead! Beilinson's remedy is to incorporate Deligne's logarithmic complex into the picture. By a standard reduction, we may asume that $j: U = X\backslash Y\hookrightarrow X$, where Y is a NCD[14] with smooth components. We define $\Omega^\bullet_X\langle Y\rangle$ to be the de Rham complex of meromorphic forms on X, holomorphic on U, with at most logarithmic poles along Y. One has a filtered complex

$$F^r\Omega^\bullet_X\langle Y\rangle = \Omega^{\bullet\geq r}_X\langle Y\rangle,$$

[14] Y is a normal crossing divisor, which in local analytic coordinates $(z_1,...,z_d)$ on X, Y is given by $z_1\cdots z_\ell = 0$, and so $\Omega^1_X\langle Y\rangle$ has local frame $\big\{dz_1/z_1,...,dz_\ell/z_\ell, dz_{\ell+1},...,dz_d\big\}$.

with Hodge to de Rham spectral sequence degenerating at E_1. This gives

$$F^rH^i(U,\mathbb{C}) = \mathbb{H}^i(F^r\Omega_X^\bullet\langle Y\rangle) \subset \mathbb{H}^i(\Omega_X^\bullet\langle Y\rangle) = H^i(U,\mathbb{C}),$$

and defines the correct Hodge filtration. The weight filtration is characterized in terms of differentials with residues along $Y^{[\bullet]}$.

Definition 5.11. Deligne-Beilinson cohomology is given by

$$H^i_{\mathcal{D}}(U,\mathbb{A}(r)) := \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r)),$$

where

$$\mathbb{A}_{\mathcal{D}}(r) := \mathrm{Cone}\big(Rj_*\mathbb{A}(r)\bigoplus F^r\Omega_X^\bullet\langle Y\rangle \xrightarrow{\epsilon-l} Rj_*\Omega_U^\bullet\big)[-1].$$

One has a short exact sequence:

$$0\to \frac{H^{i-1}(U,\mathbb{C})}{F^rH^{i-1}(U,\mathbb{C})+H^{i-1}(U,\mathbb{A}(r))} \to H^i_{\mathcal{D}}(U,\mathbb{A}(r)) \to F^r\bigcap H^i(U,\mathbb{A}(r))\to 0.$$

Here ϵ and l are the natural maps obtained after a choice of (the direct image of) injective resolutions of $\mathbb{A}(r)$ and $\Omega_U^\bullet$. One shows that this is independent of the good compactifications of U. We would like a more earthly description of $H^i_{\mathcal{D}}(U,\mathbb{A}(r))$. First observe that there are filtered quasi-isomorphisms

$$(F^r,\Omega_X^\bullet\langle Y\rangle) \hookrightarrow (F^r,\mathcal{E}_X^\bullet\langle Y\rangle) \hookrightarrow (F^r,\mathcal{D}_X^\bullet\langle Y\rangle), \tag{6}$$

where

$$F^r\mathcal{D}_X^\bullet\langle Y\rangle \;=\; \{F^r\Omega_X^\bullet\langle Y\rangle\}\otimes_{\Omega_X^\bullet}\mathcal{D}_X^\bullet.$$

To see this, one uses the argument in [Ja1]. By a spectral sequence argument,[15] it is enough to show that the associated graded pieces in (6) are quasi-isomorphic, viz.,

$$\Omega_X^r\langle Y\rangle \approx \Omega_X^r\langle Y\rangle\otimes_{\mathcal{O}_X}\mathcal{E}_X^{0,\bullet} \approx \Omega_X^r\langle Y\rangle\otimes_{\mathcal{O}_X}\mathcal{D}_X^{0,\bullet},$$

where the differential is now $1\otimes\overline{\partial}$. One now applies the $\overline{\partial}$ lemma together with the flatness of $\Omega_X^r\langle Y\rangle$ over $\mathcal{O}_X$, and using $\mathcal{O}_X$ as $\overline{\partial}$-linear. According to [Ki], $\mathcal{D}_X^\bullet\langle Y\rangle$ admits the interpretation of the space of currents acting on those (compactly supported) forms on X which "vanish holomorphically" on Y. Let $\mathcal{C}^i(X,\mathbb{A}(r))$ be the chains of real codimension i in X, and $\mathcal{C}^i_Y(X,\mathbb{A}(r))$ the subspace of chains supported on Y. Put

$$\mathcal{C}^i(X,Y,\mathbb{A}(r)) := \frac{\mathcal{C}^i(X,\mathbb{A}(r))}{\mathcal{C}^i_Y(X,\mathbb{A}(r))}.$$

One has a map of complexes:

$$\big(\mathcal{C}^\bullet(X,Y,\mathbb{A}(r)),d\big) \to \big(\mathcal{D}_X^\bullet\langle Y\rangle(X),d\big),$$

which induces a quasi-isomorphism

$$\mathcal{C}^\bullet(X,Y,\mathbb{A}(r))\otimes\mathbb{C} \to \mathcal{D}_X^\bullet\langle Y\rangle(X).$$

Definition 5.12. Deligne-Beilinson cohomology is given by

$$H^i_{\mathcal{D}}(U,\mathbb{A}(r)) := H^i\big(\mathrm{Cone}\big(\mathcal{C}^\bullet(X,Y,\mathbb{A}(r))\bigoplus F^r\mathcal{D}_X^\bullet\langle Y\rangle(X) \xrightarrow{\epsilon-l} (\mathcal{D}_X^\bullet\langle Y\rangle(X)\big)[-1]\big).$$

[15] Use Proposition 4.3, specifically the E_1-terms, and replace $K^\bullet$ by the stalks of the relevant sheaf complexes.

Example 5.13. Let us compute $H^1_{\mathcal{D}}(U,\mathbb{Z}(1))$. First of all $\{\xi\} \in H^1_{\mathcal{D}}(U,\mathbb{Z}(1))$ is represented by a D-closed triple:

$$\xi = (a,b,c) \in \big(\mathcal{C}^1(X,Y,\mathbb{Z}(1)) \bigoplus F^1\mathcal{D}^1_X\langle Y\rangle(X) \bigoplus \mathcal{D}^0_X\langle Y\rangle(X)\big),$$

where $da = 0$, $db = 0$ and $a - b = dc$. Note that $\overline{\partial}$-regularity implies that $b \in \Omega^1_X\langle Y\rangle(X)_{d-\text{closed}}$. Let $\hat{\Omega}^1_U$ be the sheaf d-closed holomorphic 1-forms on U, and let's make the identification $\mathbb{C}^\times = \mathbb{C}/\mathbb{Z}(1)$. From the short exact sequence:

$$0 \to \mathbb{C}^\times \to \mathcal{O}^\times_U \xrightarrow{d\log} \hat{\Omega}^1_U \to 0,$$

and the relation $a - b = dc$, it follows that

$$b \in \ker\big(H^0(U,\hat{\Omega}^1_U) \to H^1(U,\mathbb{C}^\times),$$

and hence $b = d\log f$ for some $f \in \mathcal{O}^\times_U(U)$. Since $b \in \Omega^1_X\langle Y\rangle(X)$, it follows that $f \in \mathcal{O}^\times_{U,\text{alg}}(U)$. Thus in Deligne cohomology[16]

$$\{\xi\} = \big(2\pi\mathrm{i}T_{f^{-1}[-\infty,0]}, d\log f, T_{\log f}\big). \tag{7}$$

Corollary 5.14.

$$\mathrm{cl}_{1,1} : CH^1(U,1) := \mathcal{O}^\times_{U,\text{alg}}(U) \xrightarrow{\sim} H^1_{\mathcal{D}}(U,\mathbb{Z}(1)),$$

is an isomorphism.

Recall that there is a short exact sequence:

$$0 \to \frac{H^0(U,\mathbb{C})}{F^1H^0(U,\mathbb{C}) + H^0(U,\mathbb{Z}(1))} \to H^1_{\mathcal{D}}(U,\mathbb{Z}(1)) \to \underbrace{F^1 \bigcap H^1(U,\mathbb{Z}(1))}_{=:\Gamma\left(H^1(U,\mathbb{Z}(1))\right)} \to 0.$$

We then deduce the short exact sequence:

$$0 \to \mathbb{C}^\times \to \mathrm{CH}^1(U,1) \xrightarrow{d\log} \Gamma\big(H^1(U,\mathbb{Z}(1))\big) \to 0.$$

Remark 5.15. Let $U/\mathbb{C}$ be a smooth quasi-projective variety. If $H^\bullet_{\mathcal{D},an}(U,\mathbb{Z}(\bullet))$ denotes that analytic Deligne cohomology, then we know that $H^2_{\mathcal{D},an}(U,\mathbb{Z}(1)) \simeq H^1(U,\mathcal{O}^\times_U)$, the holomorphic isomorphism classes of holomorphic line bundles over U. For Deligne-Beilinson cohomology, and using the fact that $H^1(U,\mathbb{Z}(1)) = W_0H^1(U,\mathbb{Z}(1))$, it follows that there is a short exact sequence:

$$0 \to J\big(H^1(U,\mathbb{Z}(1))\big) \to H^2_{\mathcal{D}}(U,\mathbb{Z}(1)) \xrightarrow{\alpha} F^1 \cap H^2(U,\mathbb{Z}(1)) \to 0,$$

but in general

$$\Gamma\big(H^2(U,\mathbb{Z}(1))\big) = F^0 \cap W_0H^2(U,\mathbb{Z}(1)) \subsetneq F^1 \cap H^2(U,\mathbb{Z}(1)).$$

To remedy this, let us put $H^2_{\mathcal{H}}(U,\mathbb{Z}(1)) = \alpha^{-1}\big(\Gamma\big(H^2(U,\mathbb{Z}(1))\big)\big)$. Then $H^2_{\mathcal{H}}(U,\mathbb{Z}(1))$ amounts to a special instance of absolute Hodge cohomology. Now recall from Bloch's formula that $\mathrm{CH}^1(U) = H^1(U,\mathcal{O}^\times_{U,\text{alg}})$. One can easily deduce the following:

[16]For compactly supported $\omega \in E^{2d-1}_{U,c}$, and $f \in \mathcal{O}^\times_U(U)$,

$$\int_U \frac{df}{f} \wedge \omega = \int_U d\big(\log f \wedge \omega\big) - \int_{U\backslash f^{-1}[-\infty,0]} \log f \wedge d\omega = 2\pi\mathrm{i}\int_{f^{-1}[-\infty,0]} \omega + dT_{\log f}(\omega),$$

where we use the principal branch of log.

Proposition 5.16. *Let $U/\mathbb{C}$ be a smooth quasi-projective variety. Then:*
$$CH^1(U) \simeq H^2_{\mathcal{H}}(U,\mathbb{Z}(1)).$$

Proof. There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{CH}^1_{\mathrm{hom}}(U) & \to & \mathrm{CH}^1(U) & \to & \frac{\mathrm{CH}^1(U)}{\mathrm{CH}^1_{\mathrm{hom}}(U)} & \to & 0 \\
 & & \Phi_1 \downarrow & & \mathrm{cl}_1 \downarrow & & \downarrow \wr & & \\
0 & \to & J\big(H^1(U,\mathbb{Z}(1))\big) & \to & H^2_{\mathcal{H}}(U,\mathbb{Z}(1)) & \to & \Gamma\big(H^2(U,\mathbb{Z}(1))\big) & \to & 0
\end{array}$$

It suffices to show that Φ_1 is an isomorphism. Let $\overline{U}$ be a smooth projective compactification of U. We may assume that $D := \overline{U}\backslash U$ is a divisor. With regard to the short exact sequence
$$0 \to H^1(\overline{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1)) \to H^2_D(\overline{U},\mathbb{Z}(1))^\circ \to 0,$$
it is clear that $J\big(H^2_D(\overline{U},\mathbb{Z}(1))^\circ\big) = 0$, and hence the following diagram finishes the proof:

$$\begin{array}{ccccccc}
\mathrm{CH}^1_D(\overline{U})^\circ & \to & \mathrm{CH}^1_{\mathrm{hom}}(\overline{U}) & \to & \mathrm{CH}^1_{\mathrm{hom}}(U) & \to & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \Phi_1 & & \\
\Gamma H^2_D(\overline{U},\mathbb{Z}(1))^\circ & \to & J\big(H^1(\overline{U},\mathbb{Z}(1))\big) & \to & J\big(H^1(U,\mathbb{Z}(1))\big) & \to & 0
\end{array}$$

□

5.17. **Real Deligne cohomology II.** The definition of real Deligne cohomology here will coincide with the definition in the literature for real varieties, using the structure map $X \to \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$, for any smooth complex variety X. Again, let U be a smooth variety over $\mathbb{C}$. We set $\mathcal{E}^{p,q}_Y$ = sheaf of $C^\infty - (p,q)$ forms on Y. Now assume given a good compactification X of U, i.e. where $Y \stackrel{\mathrm{def}}{=} X\backslash U$ is a normal crossing divisor. We recall $F^rE^\bullet_{X^\infty}\langle Y\rangle \;=\; \{F^r\Omega^\bullet_X\langle Y\rangle\}\otimes_{\Omega^\bullet_X}\mathcal{E}^\bullet_X$. Note that $F^r\Omega^\bullet_X\langle Y\rangle$ is the r^{th} filtration of the complex of C^∞-forms $\Omega^\bullet_X\langle Y\rangle\otimes_{\Omega^\bullet_X}\mathcal{E}^\bullet_X$, where the grading in the tensor product is given by the sum of the degrees in $\Omega^\bullet_X\langle Y\rangle$ and $\mathcal{E}^\bullet_X$, and the differential is given in the standard way in the tensor product of chain complexes: $d(\alpha\otimes\beta) = d\alpha\otimes\beta \;+\; (-1)^{\deg\,\alpha}\alpha\otimes d\beta$.

We introduce a corresponding *real* complex introduced in [BU]. Let $Y \subset X$ be given in local coordinates $(z_1,\dots,z_m)$ (on X) by $\prod_{i=1}^k z_i = 0$, and put $q : U \hookrightarrow X$. Let $\mathcal{E}_X$ be the sheaf of germs of C^∞ $\mathbb{C}$-valued functions on X, and $\mathcal{E}^*_X$ the $\mathcal{E}_X$-algebra of differential forms. Define $\mathcal{E}^*_X\langle Y\rangle$ to be the sub-$\mathcal{E}_X$-algebra of $q_*\mathcal{E}^*_U$ generated by
$$\log z_i\overline{z}_i,\ \frac{dz_i}{z_i},\ \frac{d\overline{z}_i}{\overline{z}_i},\ \text{for } i\in[1,k] \text{ and } dz_i,\ d\overline{z}_i,\ \text{for } i\notin[1,k].$$
The corresponding subsheaf of real forms is denoted by $\mathcal{E}^*_{X,\mathbb{R}}\langle Y\rangle$. Also, if $Z\subset Y$ is a subvariety, we define $\Sigma_Z\mathcal{E}^*_U$ to be the sheaf complex of forms which pullback to zero on $Z - Z_{\mathrm{sing}}$. There is a morphism of complexes from $\mathcal{E}^*_X\langle Y\rangle$ to a sheaf complex of distributions on $\Sigma_Y\mathcal{E}^*_X$ ([BU]). $H^*(U,\mathbb{C})$ can be computed in terms of the cohomology of $\mathcal{E}^*_X\langle Y\rangle$ [BU].

Proposition 5.18 ([Lew8])**.**

$$H^i_{\mathcal{D}}(U,\mathbb{R}(p)) \simeq H^i(\mathrm{Cone}\{F^p\mathcal{E}^{\cdot}_X\langle Y\rangle(X) \xrightarrow{-\pi_{p-1}} \mathcal{E}^{\cdot}_{X,\mathbb{R}}\langle Y\rangle(p-1)(X)\}[-1]).$$

The product structure is given in [EV] by the table below:

	f_q	s_q
f_p	$f_p \wedge f_q$	$(-1)^{\deg f_p}\pi_p(f_p)\wedge s_q$
s_p	$s_p \wedge \pi_q(f_q)$	0

Remark 5.19. In §6, we consider, for U/k smooth and quasi-projective over $k \subseteq \mathbb{C}$, the cycle class map

$$\mathrm{CH}^r(U,m) \to H^{2r-m}_{\mathcal{H}}(U(\mathbb{C}),\mathbb{Z}(r)),$$

where $H^{2r-m}_{\mathcal{H}}(U,\mathbb{Z}(r))$ is Beilinson's absolute Hodge cohomology, and fits in the short exact sequence

$$J(H^{2r-m-1}(U(\mathbb{C}),\mathbb{Z}(r))) \hookrightarrow H^{2r-m}_{\mathcal{H}}(U(\mathbb{C}),\mathbb{Z}(r)) \twoheadrightarrow \Gamma(H^{2r-m}(U(\mathbb{C}),\mathbb{Z}(r))).$$

We define $\mathrm{CH}^r_{\mathrm{hom}}(U,m)$ to be the kernel of the composite

$$\mathrm{CH}^r(U,m) \to H^{2r-m}_{\mathcal{H}}(U(\mathbb{C}),\mathbb{Z}(r)) \to \Gamma(H^{2r-m}(U(\mathbb{C}),\mathbb{Z}(r))).$$

In the event that $U = X$ is smooth and projective,

$$H^{2r-m}_{\mathcal{H}}(X(\mathbb{C}),\mathbb{Z}(r)) = H^{2r-m}_{\mathcal{D}}(X(\mathbb{C}),\mathbb{Z}(r)).$$

In this case we are interested in the corresponding induced Abel-Jacobi map

$$AJ_X : \mathrm{CH}^r_{\mathrm{hom}}(X,m) \to J(H^{2r-m-1}(X(\mathbb{C}),\mathbb{Z}(r))).$$

Even in the situation where U is not complete, the weight filtered spectral sequence in Example 4.7 essentially reduces everything to the smooth projective case [K-L].

6. **Cycle class maps**

Recall $X/\mathbb{C}$ smooth projective, $\dim X = d$. For $0 \le r \le d$, put $z^r(X)$ ($= z_{d-r}(X)$) = free abelian group generated by subvarieties of codim r (= dim $d-r$) in X. There is the fundamental class map (described later) $z^r(X) \to H^{2r}(X,\mathbb{Z})$ whose kernel is denoted by $z^r_{\mathrm{hom}}(X)$. More precisely, the target space and map requires some twisting, viz.,

$$z^r(X) \to H^{2r}(X,\mathbb{Z}(r)).$$

To explain the role of twisting here, we illustrate this with three case scenarios.

• Let $f : Y \to X$ be a morphism of smooth projective varieties, where $\dim Y = \dim X - 1$. One has a commutative diagram of cycle class maps:

$$\begin{array}{ccc} z^{r-1}(Y) & \to & H^{2(r-1)}(Y,\mathbb{Z}(r-1)) \\ f_*\Big\downarrow & & \Big\downarrow f_* \\ z^r(X) & \to & H^{2r}(X,\mathbb{Z}(r)) \end{array}$$

Thus from the perspective of (mixed) Hodge theory, this diagram is "natural", as the right hand vertical arrow is a morphism of (M)HS.

• Let $U/\mathbb{C}$ be a smooth quasi-projective variety of dimension d, and $Y \subset U$ a closed algebraic subset. Using the twisted Poincaré duality theory formalism in this situation (see [Ja2] (p. 82, p. 92)), Poincaré duality gives us an isomorphism of MHS:

$$H^i_Y(U,\mathbb{Z}(j)) \simeq H_{2d-i}(Y,\mathbb{Z}(d-j)) := H_{2d-i}(Y,\mathbb{Z})(j-d),$$

where $H_i(Y,\mathbb{Z}) := H_i^{BM}(Y,\mathbb{Z})$ is Borel-Moore homology.[17] For example if $U = Y = X$ is smooth projective, then $H^i(X,\mathbb{Z}(j))$ is a pure HS of weight $i-2j$, and $H_a(X,\mathbb{Z}(b)) := H_a(X,\mathbb{Z})(-b)$ is known to be a pure HS of weight $2b-a$, hence $H_{2d-i}(Y,\mathbb{Z}(d-j))$ has weight $2(d-j)-(2d-i) = i-2j$. Thus

$$H^i(X,\mathbb{Z}(j)) \simeq H_{2d-i}(X,\mathbb{Z}(d-j)), \tag{8}$$

is an isomorphism of HS.

Remark 6.1. Although tempting, from a "purist "point of view, it would be a mistake to interpret $H_a(X,\mathbb{Z}(b)) = H_a(X,\mathbb{Z})(b)$. This would imply that the Poincaré duality isomorphism would not preserve weights, and hence not an isomorphism of HS.

• Let $\mathcal{O}_X$ be the sheaf of analytic functions on X. Recall the exponential short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp(2\pi \mathrm{i}\cdot(-))} \mathcal{O}_X^\times \to 0,$$

[17]We remind the reader that for singular homology $H_*^{sing}(U,\mathbb{Z})$ and ignoring twists, Poincaré duality gives the isomorphism $H^i_c(U,\mathbb{Z}) \simeq H^{sing}_{2d-i}(U,\mathbb{Z})$, where $H^i_c(U,\mathbb{Z})$ is cohomology with compact support; whereas $H^i(U,\mathbb{Z}) \simeq H^{BM}_{2d-i}(U,\mathbb{Z})$.

where $\mathcal{O}_X^\times \subset \mathcal{O}_X$ is the sheaf of units. Recall that $H^1(X, \mathcal{O}_X^\times) \simeq \mathrm{CH}^1(X)$, and hence there is an induced Chern class map $\mathrm{CH}^1(X) \to H^2(X, \mathbb{Z})$. But this is not so natural as there is no canonical choice of i. Instead, one considers

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0,$$

and accordingly the induced cycle class map $\mathrm{CH}^1(X) \to H^2(X, \mathbb{Z}(1))$.

6.2. **Classical case.** We recall that

$$H^r_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) = \mathrm{CH}^r(X).$$

The fundamental class map:

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \to H^{2r}_{\mathrm{DR}}(X, \mathbb{C}) \simeq H^{2d-2r}_{\mathrm{DR}}(X, \mathbb{C})^\vee,$$

can be defined in a number of equivalent ways:

(i) (See [E-P].) The $d\log$ map $\mathcal{K}^M_{r,X} \to \Omega^r_X$, $\{f_1, ..., f_r\} \mapsto \bigwedge_j d\log f_j$, induces a morphism of complexes in the Zariski topology $\{\mathcal{K}^M_{r,X} \to 0\} \to \Omega^{\bullet \geq r}_X[r]$, and thus using GAGA,

$$\mathrm{CH}^r(X) = H^r_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) = \mathbb{H}^r\big(\{\mathcal{K}^M_{r,X} \to 0\}\big) \to \mathbb{H}^r\big(\Omega^{\bullet \geq r}_X[r]\big)$$
$$= \mathbb{H}^{2r}(\Omega^{\bullet \geq r}_X) = F^r H^{2r}_{\mathrm{DR}}(X, \mathbb{C}).$$

(ii) (See [GH].) Let $V \subset X$ be a subvariety of codimension r in X, and $\{w\} \in H^{2d-2r}_{\mathrm{DR}}(X, \mathbb{C})$. Define

$$\mathrm{cl}_r(V)(w) = \frac{1}{(2\pi \mathrm{i})^{d-r}} \delta_V := \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_{V^*} w,$$

and extend to $\mathrm{CH}^r(X)$ by linearity, where $V^* = V \backslash V_{\mathrm{sing}}$. Note that $\dim_{\mathbb{R}} V = 2d - 2r$. The easiest way to show that cl_r is well-defined (finite volume, closed current) is to first pass to a desingularization of V above, and apply a Stokes' theorem argument.

One way to connected (i) and (ii) is as follows. If we write Γ for $H^0(X, -)$, then there is a diagram that commutes up to sign:

$$\begin{array}{ccccc}
\Gamma K^M_r(\mathbb{C}(X)) & \to & \Gamma \bigoplus_{\mathrm{cd}_X Y = 1} K^M_{r-1}(\mathbb{C}(Y)) & \to \cdots \to & \Gamma \bigoplus_{\mathrm{cd}_X V = r} K^M_0(\mathbb{C}(X)) \\
\int_X \frac{d\log_r}{(2\pi \mathrm{i})^d} \Big\downarrow & & \int_Y \frac{d\log_{r-1}}{(2\pi \mathrm{i})^{d-1}} \Big\downarrow & \cdots & \int_V \frac{d\log_0}{(2\pi \mathrm{i})^{d-r}} \Big\downarrow \\
\Gamma F^r \mathcal{D}^r_X & \xrightarrow{d} & \Gamma F^r \mathcal{D}^{r+1}_X & \xrightarrow{d} \cdots \xrightarrow{d} & \Gamma F^r \mathcal{D}^{2r}_X
\end{array}$$

where

$$d\log_r\big(\{f_1, ..., f_r\}\big) = \bigwedge_{j=1}^r d\log f_j, \quad \int_V \frac{d\log_0}{(2\pi \mathrm{i})^{d-r}} = \frac{1}{(2\pi \mathrm{i})^{d-r}} \delta_V.$$

From the aforementioned filtered quasi-isomorphism $\Omega^\bullet_X \hookrightarrow \mathcal{D}^\bullet_X$, the prescriptions in (i) and (ii) can be seen as almost tautologies.

(iii) Thirdly one has a fundamental class generator $\{V\} \in H_{2d-2r}(V, \mathbb{Z}(d-r)) \simeq H^{2r}_V(X, \mathbb{Z}(r)) \to H_{2d-2r}(X, \mathbb{Z}((d-r)) \simeq H^{2r}(X, \mathbb{Z}(r))$.

In summary we have

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \to \Gamma(H^{2r}(X, \mathbb{Z}(r))).$$

This map fails to be surjective in general for $r > 1$ (see [Lew1]).

Conjecture 6.3 (Hodge$_{\mathbb{Q}}$)**.**

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \otimes \mathbb{Q} \to \Gamma(H^{2r}(X, \mathbb{Q}(r))),$$

is surjective.

Next, the Abel-Jacobi map:

$$\Phi_r : \mathrm{CH}^r_{\mathrm{hom}}(X) \to J(H^{2r-1}(X, \mathbb{Z}(r))),$$

is defined as follows. Recall that

$$J(H^{2r-1}(X, \mathbb{Z}(r))) \simeq \frac{F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})^{\vee}}{H_{2d-2r+1}(X, \mathbb{Z}(d-r))},$$

is a compact complex torus, called the Griffiths jacobian.

Prescription for Φ_r: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$. Then $\xi = \partial\zeta$ bounds a $2d - 2r + 1$ real dimensional chain ζ in X. Let $\{w\} \in F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})$. Define:

$$\Phi_r(\xi)(\{w\}) = \frac{1}{(2\pi\mathrm{i})^{d-r}} \int_{\zeta} w \quad \text{(modulo periods)}.$$

That Φ_r is well-defined follows from the fact that $F^{\ell}H^i(X, \mathbb{C})$ depends only on the complex structure of X, namely

$$F^{\ell}H^i(X, \mathbb{C}) \simeq \frac{F^{\ell}E^i_{X,d-\mathrm{closed}}}{d\big(F^{\ell}E^{i-1}_X\big)},$$

where we recall that E^i_X are the C^{∞} complex-valued i-forms on X.

Alternate take for Φ_r: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$. First observe that $H^{2r-1}_{|\xi|}(X, \mathbb{Z}) \simeq H_{2d-2r+1}(|\xi|, \mathbb{Z}) = 0$ as $\dim_{\mathbb{R}} |\xi| = 2d - 2r$. Secondly there is a fundamental class map $\xi \mapsto \{\xi\} \in H_{2d-2r}(|\xi|, \mathbb{Z}(d-r)) \simeq H^{2r}_{|\xi|}(X, \mathbb{Z}(r))$ (Poincaré duality). Further, since ξ is nulhomologous, we have by duality

$$[\xi] \in H^{2r}_{|\xi|}(X, \mathbb{Z}(r))^{\circ} := \ker\big(H^{2r}_{|\xi|}(X, \mathbb{Z}(r)) \to H^{2r}(X, \mathbb{Z}(r))\big).$$

Hence ξ determines a morphism of MHS, $\mathbb{Z}(0) \to H^{2r}_{|\xi|}(X, \mathbb{Z}(r))^{\circ}$. From the short exact sequence of MHS

$$0 \to H^{2r-1}(X, \mathbb{Z}(r)) \to H^{2r-1}(X \backslash |\xi|, \mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X, \mathbb{Z}(r))^{\circ} \to 0,$$

we can pullback via this morphism to obtain another short exact sequence of MHS,

$$0 \to H^{2r-1}(X, \mathbb{Z}(r)) \to E \to \mathbb{Z}(0) \to 0.$$

Then $\Phi_r(\xi) := \{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(0), H^{2r-1}(X, \mathbb{Z}(r))\big)$. This class $\{E\}$ is easy to calculate in $J(H^{2r-1}(X, \mathbb{Z}(r))))$, in terms of a membrane integral. Note that via duality,

$$E \subset H^{2r-1}(X \backslash |\xi|, \mathbb{Z}(r)) \simeq H_{2d-2r+1}(X, |\xi|, \mathbb{Z}(d-r)),$$

and that if ζ is a real $2d-2r+1$ chain such that $\partial\zeta = \xi$ on X, then $\{\zeta\} \in H_{2d-2r+1}(X, |\xi|, \mathbb{Z})$. One can show that the class $x \in W_0E$ corresponding to the current

$$\frac{1}{(2\pi\mathrm{i})^{d-r}} \int_{\zeta},$$

maps to $1 \in \mathbb{Z}(0)$. Now choose $y \in F^0W_0E_{\mathbb{C}}$ also mapping to $1 \in \mathbb{Z}(0)$. By Hodge type alone, the current corresponding to $x-y$ in the Poincaré dual description of $J^r(X)$ is the same as for $x = \frac{1}{(2\pi\mathrm{i})^{d-r}} \int_{\zeta}$, which is precisely the classical description of the Griffiths Abel-Jacobi map.

Example 6.4. We define the cycle class map $\mathrm{cl}_{r,0} : \mathrm{CH}^r(X) = \mathrm{CH}^r(X,0) \to H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r))$. Recall the short exact sequence:

$$0 \to J(H^{2r-1}(X,\mathbb{Z}(r))) \to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to \Gamma(H^{2r}(X,\mathbb{Z}(r))) \to 0,$$

which was derived from the LES:

$$\cdots \to H^{2r-1}(X,\mathbb{Z}(r)) \oplus F^rH^{2r-1}(X,\mathbb{C}) \to H^{2r-1}(X,\mathbb{C})$$

$$\to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to H^{2r}(X,\mathbb{Z}(r)) \oplus F^rH^{2r}(X,\mathbb{C}) \xrightarrow{x-y} H^{2r}(X,\mathbb{C}) \to \cdots$$

Let $\xi \in \mathrm{CH}^r(X)$ with support $|\xi|$. One has a similar LES:

$$\cdots \to H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^rH^{2r-1}_{|\xi|}(X,\mathbb{C}) \to H^{2r-1}_{|\xi|}(X,\mathbb{C})$$

$$\to H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^rH^{2r}_{|\xi|}(X,\mathbb{C}) \xrightarrow{x-y} H^{2r}_{|\xi|}(X,\mathbb{C}) \to \cdots$$

Via Poincaré duality, one has cycle class maps

$$\xi \mapsto \left[(2\pi\mathrm{i})^{r-d}\big(\{\xi\}, \delta_{\xi}\big)\right] \in \ker\left(H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^rH^{2r}_{|\xi|}(X,\mathbb{C}) \to H^{2r}_{|\xi|}(X,\mathbb{C})\right);$$

moreover recall that $H^{2r-1}_{|\xi|}(X,\mathbb{C}) = 0$ (weak purity: via Poincaré duality, just use $H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \simeq H_{2d-2r+1}(|\xi|, \mathbb{Z}(d-r)) = 0$ using $2d-2r+1 > \dim_{\mathbb{R}} |\xi|$). Thus we have a class $[\xi] \in H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r))$. Now use the forgetful map

$$H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)),$$

to define $\mathrm{cl}_r(\xi) \in H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r))$. From the injection

$$H^{2r}_{\mathcal{D},|\xi|}(X,\mathbb{Z}(r)) \hookrightarrow H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \oplus F^rH^{2r}_{|\xi|}(X,\mathbb{C}),$$

and the aforementioned forgetful map, in terms of the cone complex, $\mathrm{cl}_r(\xi)$ is represented by $\big((2\pi\mathrm{i})^{r-d}\{\xi\}, (2\pi\mathrm{i})^{r-d}\delta_{\xi}, 0\big)$. If $\xi \sim_{\mathrm{hom}} 0$, then $\xi = \partial\zeta$, $(2\pi\mathrm{i})^{r-d}\delta_{\xi} = dS$ for some $S \in F^r\mathcal{D}^{2r-1}(X)$. So

$$D((2\pi\mathrm{i})^{r-d}\zeta, S, 0) + \big((2\pi\mathrm{i})^{r-d}\{\xi\}, (2\pi\mathrm{i})^{r-d}\delta_{\xi}, 0\big) = \left(0, 0, (2\pi\mathrm{i})^{r-d}\int_{\zeta} - S\right).$$

For $\omega \in F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})$,

$$(2\pi\mathrm{i})^{r-d}\int_{\zeta}\omega \; - \; S(\omega) = \frac{1}{(2\pi\mathrm{i})^{d-r}}\int_{\zeta}\omega, \tag{9}$$

by Hodge type. This is the Griffiths Abel-Jacobi map.

Both maps (cl_r, Φ_r) can be combined to give

$$\mathrm{cl}_{r,0} : \mathrm{CH}^r(X) = \mathrm{CH}^r(X,0) \to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)),$$

with commutative diagram:

$$\begin{array}{ccccccc}
0 \to & \mathrm{CH}^r_{\mathrm{hom}}(X) & \to \mathrm{CH}^r(X) \to & \frac{\mathrm{CH}^r(X)}{\mathrm{CH}^r_{\mathrm{hom}}(X)} & \to 0 \\
 & \Phi_r \downarrow & \mathrm{cl}_{r,0} \downarrow & \mathrm{cl}_r \downarrow & \\
0 \to & J(H^{2r-1}(X,\mathbb{Z}(r))) & \to H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) \to & \Gamma(H^{2r}(X,\mathbb{Z}(r))) & \to 0.
\end{array}$$

Remark 6.5. (i) Recall that

$$\mathrm{cl}_{1,0} : \mathrm{CH}^1(X) \xrightarrow{\sim} H^2_{\mathcal{D}}(X,\mathbb{Z}(1)),$$

is an isomorphism. However, $\mathrm{cl}_{r,0}$ is need not be injective nor surjective for $r > 1$.

(ii) $\mathrm{cl}_{r,0}$ is a special case of the Bloch cycle class map [Blo2], $\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))$; an abridged version discussed in subsection 6.18.

6.6. **Generalized cycles, I.** The basic idea is this:

$$\mathrm{CH}^r(X) = \mathrm{Coker}\Big(\bigoplus_{\mathrm{cd}_X V = r-1} \mathbb{C}(V)^\times \xrightarrow{\mathrm{div}} z^r(X)\Big).$$

In the context of Milnor K-theory, this is just

$$\underbrace{\Big(\to \cdots \bigoplus_{\mathrm{cd}_X V=r-2} K_2^M(\mathbb{C}(V))\Big) \xrightarrow{\mathrm{Tame}}}_{\text{building a complex on the left}} \bigoplus_{\mathrm{cd}_X V=r-1} K_1^M(\mathbb{C}(V)) \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X V=r} K_0^M(\mathbb{C}(V)).$$

Recall the Gersten-Milnor resolution of a sheaf of Milnor K-groups on X, which leads to a complex whose last three terms and corresponding homologies (norm/graph maps, indicated at $\updownarrow$) for $0 \le m \le 2$ are:

$$\begin{array}{ccccc}
\bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbb{C}(Z)) & \overset{T}{\to} & \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbb{C}(Z)^\times & \overset{\mathrm{div}}{\to} & \bigoplus_{\mathrm{cd}_X Z=r} \mathbb{Z} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathrm{CH}^r(X,2) & & \mathrm{CH}^r(X,1) & & \mathrm{CH}^r(X,0)
\end{array} \tag{10}$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map. The Tame symbol map

$$T : \bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X D=r-1} K_1^M(\mathbb{C}(D)),$$

is defined as follows. First $K_2^M(\mathbb{C}(Z))$ is generated by symbols $\{f,g\}$, $f,\ g \in \mathbb{C}(Z)^\times$.

For $f,\ g \in \mathbb{C}(Z)^\times$,

$$T\big(\{f,g\}\big) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D,$$

where $\big(\cdots\big)_D$ means restriction to the generic point of D, and ν_D represents order of a zero or pole along an irreducible divisor $D \subset Z$.

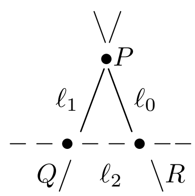
Example 6.7. Taking cohomology of the complex in (10), we have:

(i) $\mathrm{CH}^r(X,0) = z^r(X)/z^r_{\mathrm{rat}}(X) =: \mathrm{CH}^r(X)$.

(ii) $\mathrm{CH}^r(X,1)$ is represented by classes of the form $\xi = \sum_j (f_j, D_j)$, where $\mathrm{codim}_X D_j = r-1$, $f_j \in \mathbb{C}(D_j)^\times$, and $\sum \mathrm{div}(f_j) = 0$; modulo the image of the Tame symbol.

(iii) $\mathrm{CH}^r(X,2)$ is represented by classes in the kernel of the Tame symbol; modulo the image of a higher Tame symbol.

Example 6.8. (i) $X = \mathbb{P}^2$, with homogeneous coordinates $[z_0, z_1, z_2]$. $\mathbb{P}^1 = \ell_j := V(z_j)$, $j = 0,1,2$. Let $P = [0,0,1] = \ell_0 \cap \ell_1$, $Q = [1,0,0] = \ell_1 \cap \ell_2$, $R = [0,1,0] = \ell_0 \cap \ell_2$. Introduce $f_j \in \mathbb{C}(\ell_j)^\times$, where $(f_0) = P - R$, $(f_1) = Q - P$, $(f_2) = R - Q$. Explicitly, $f_0 = z_1/z_2$, $f_1 = z_2/z_0$ and $f_2 = z_0/z_1$. Then $\xi := \sum_{j=0}^2 (f_j, \ell_j) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ represents a higher Chow cycle.



This cycle turns out to be nonzero.[18] Consider the line $\mathbb{P}^1_0 := V(z_0+z_1+z_2) \subset \mathbb{P}^2$, and set $q_j = \mathbb{P}^1_0 \cap \ell_j$, $j = 0,1,2$. Then $q_0 = [0,1,-1]$, $q_1 = [1,0,-1]$, $q_2 = [1,-1,0]$, and accordingly $f_j(q_j) = -1$. These Chow groups are known to satisfy a projective bundle formula (see [Blo1], p. 269) which implies that

$$\mathrm{CH}^2(\mathbb{P}^2, 1) \simeq \{\mathbb{P}^1\} \otimes \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}), 1),$$

$$\mathrm{CH}^2(\mathbb{P}^1_0, 1) \simeq \{\mathbb{P}^1 \cap \mathbb{P}^1_0\}_{\mathbb{P}^2} \otimes \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}), 1),$$

where $\mathbb{P}^2 \to \mathrm{Spec}(\mathbb{C})$, and $\mathbb{P}^1_0 \to \mathrm{Spec}(\mathbb{C})$ are the structure maps, and $\mathbb{P}^1 \subset \mathbb{P}^2$ is a choice of line. It is well-known that $\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}), 1) = \mathbb{C}^\times$ ([Blo1], see Example 4.24 above), and thus via restriction we have the isomorphisms:

$$\mathrm{CH}^2(\mathbb{P}^2, 1) \simeq \mathrm{CH}^2(\mathbb{P}^1_0, 1) \simeq \mathbb{C}^\times;$$

moreover under this isomorphism,

$$\xi \mapsto \prod_{j=0}^{2} f_j(q_j) = -1 \in \mathbb{C}^\times.$$

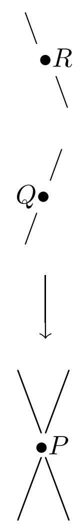
Hence $\xi \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ is a nonzero 2-torsion class.[19]

(ii) Again $X = \mathbb{P}^2$. Let $C \subset X$ be the nodal rational curve given by $z_2^2 z_0 = z_1^3 + z_0 z_1^2$ (In affine coordinates $(x,y) = (z_1/z_0, z_2/z_0) \in \mathbb{C}^2$, C is given by $y^2 = x^3 + x^2$). Let $\tilde{C} \simeq \mathbb{P}^1$ be the normalization of C, with morphism $\pi : \tilde{C} \to C$. Put $P = (0,0) \in C$ (node) and let $R + Q = \pi^{-1}(P)$. Choose $f \in \mathbb{C}(\tilde{C})^\times = \mathbb{C}(C)^\times$, such

[18] A special thanks to Rob de Jeu for supplying us this idea.

[19] Matt Kerr informed me of an alternate and slick approach to this example via Milnor K-theory (next section). Namely one needs to add $\mathrm{Tame}\{z_1/z_0, z_2/z_0\}$ to ξ to get the 2-torsion class $(-1, \ell_0)$, which is the same as ξ in $\mathrm{CH}^2(\mathbb{P}^2, 1)$.

that $(f)_{\tilde{C}} = R - Q$. Then $(f)_C = 0$ and hence $(f, C) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ defines a higher Chow cycle.



Exercise 6.9. Show that $(f, C) = 0 \in \mathrm{CH}^2(\mathbb{P}^2, 1)$.

6.10. **Generalized cycles, II.** The higher Chow groups were invented by S. Bloch ([Blo1]) (and independently by S. Landsberg). Let $W/\mathbb{C}$ a quasiprojective variety. Put $z^r(W)$ = free abelian group generated by subvarieties of codimension r (= $\dim W - r$) in W. Consider the m-simplex:

$$\Delta^m = \mathrm{Spec}\left\{\frac{\mathbb{C}[t_0, \dots, t_m]}{\left(1 - \sum_{j=0}^m t_j\right)}\right\} \simeq \mathbb{C}^m.$$

We set $z^r(W, m) =$

$$\left\{\xi \in z^r(W \times \Delta^m) \;\middle|\; \xi \text{ meets all faces } \{t_{i_1} = \cdots = t_{i_\ell} = 0,\ \ell \geq 1\} \text{ properly}\right\}.$$

Note that $z^r(W, 0) = z^r(W)$. Now set $\partial_j : z^r(W, m) \to z^r(W, m-1)$, the restriction to j-th face given by $t_j = 0$. The boundary map

$$\partial := \sum_{j=0}^m (-1)^j \partial_j : z^r(W, m) \to z^r(W, m-1),$$

satisfies $\partial^2 = 0$.

Definition 6.11. $\mathrm{CH}^\bullet(W, \bullet)$ = homology of $\{z^\bullet(W, \bullet), \delta\}$. We now put $\mathrm{CH}^r(W) := \mathrm{CH}^r(W, 0)$.

Alternate take: Cubical version. Let $\square^m := (\mathbb{P}^1 \backslash \{1\})^m$ with coordinates z_i and 2^m codimension one faces obtained by setting $z_i = 0, \infty$, and boundary maps

$$\partial = \sum (-1)^{i-1}(\partial_i^0 - \partial_i^\infty),$$

where $\partial_i^0,\ \partial_i^\infty$ denote the restriction maps to the faces $z_i = 0,\ z_i = \infty$ respectively. The rest of the definition is completely analogous except that one has to divide out degenerate cycles. It is known that both complexes are quasi-isomorphic.

Example 6.12 ("Totaro (pre-)cycle")**.**

$$X = \mathrm{Pt},\ a \in \mathbb{C}^\times\backslash\{1\},\ V_2(a) := \big\{(t, 1-t, 1-at^{-1})\ \big|\ t \in \mathbb{P}^1\big\} \bigcap \square^3.$$

One computes

$$\partial V_2(a) = \left\{\begin{matrix} [(1,\infty) - (\infty,1)] \\ \\ -[(1,1-a) - (\infty,1)] \\ \\ +[(a,1-a) - (0,1)] \end{matrix}\right\} \bigcap \square^2 = (a, 1-a)$$

6.13. **Cycle class map (generic point).** We provide a description of the cycle class map for higher Chow cycles that are obtained as "graphs of maps". Let $V \subset X$ be a subvariety, $v = \dim V$, and $U_V = V\backslash V_{\rm sing}$. We need the following multiplication table, pertaining to

$$\bigcup_\alpha :\ \mathbb{A}_{\mathcal{D}}(p) \otimes \mathbb{A}_{\mathcal{D}}(q) \to \mathbb{A}_{\mathcal{D}}(p+q),$$

for any given $\alpha \in \mathbb{R}$. Here $\mathbb{A}_{\mathcal{D}}(p)$ is defined in terms of a cone complex (see [EV]):

	a_q	f_q	ω_q
a_p	$a_p \cdot a_q$	0	$(1-\alpha)\cdot a_p \cdot \omega_q$
f_p	0	$f_p\ f_q$	$(-1)^{\deg f_p}\cdot \alpha \cdot f_p \cdot \omega_q$
ω_p	$\alpha \cdot \omega_p \cdot a_q$	$(1-\alpha)\cdot \omega_p\ f_q$	0

Evidently, up to homotopy, $\bigcup_\alpha$ is independent of $\alpha \in \mathbb{R}$. Now put $\bigcup = \bigcup_{\alpha=0}$. Now let $V \subset X$ be a subvariety of dimension v, $U_V := V\backslash V_{\rm sing}$, and $\{f_1, ..., f_m\} \subset \mathcal{O}^\times_{U_V}(U_V)$. Put $T_{f_j} := 2\pi \mathrm{i} \int_{f_j^{-1}[-\infty,0]}$, $\Omega_{f_j} := \int_{U_V} d\log f_j$, $R_{f_j} = T_{\log f_j}$. We have the following

Proposition 6.14. *(i) For each $j = 1, ..., m$, the triple $(T_{f_j}, \Omega_{f_j}, R_{f_j})$ defines a class*

$$\big\{(T_{f_j}, \Omega_{f_j}, R_{f_j})\big\} \in H^{\mathcal{D}}_{2v-1}(U_V, \mathbb{Z}(v-1)) \overset{PD}{\simeq} H^1_{\mathcal{D}}(U_V, \mathbb{Z}(1)).$$

(ii) Via the cup product:

$$\bigcup_{j=1}^m \big\{(T_{f_j},\ \Omega_{f_j},\ R_{f_j})\big\} = \big\{(T_f,\ \Omega_f,\ R_f)\big\},$$

where $f = (f_1, \ldots, f_m)$,

$$T_f(\omega) = (2\pi \mathrm{i})^m \int_{(f_1\times\cdots\times f_m)^{-1}[-\infty,0]^m} \omega, \quad , \Omega_f(\omega) = \int_V \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega,$$

and where

$$R_f(\omega) \ =\ \bigg[\int_{U_V\backslash f_1^{-1}[-\infty,0]} (\log f_1)\frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_m}{f_m} \wedge \omega$$

$$+ (-2\pi \mathrm{i}) \int_{f_1^{-1}[-\infty,0]\backslash(f_1\times f_2)^{-1}[-\infty,0]^2} (\log f_2)\frac{df_3}{f_3}\ \cdots\ \frac{df_m}{f_m} \wedge \omega$$

$$+\cdots+(-2\pi\mathrm{i})^{m-1}\int_{(f_1\times\cdots\times f_{m-1})^{-1}[-\infty,0]^{m-1}\setminus(f_1\times\cdots\times f_m)^{-1}[-\infty,0]^m}(\log f_m)\omega\Bigg].$$

Proof. Part (i) was discussed in (7), and part (ii) uses the multiplication table above, the cone complex description of Deligne homology together with Poincaré duality, and induction on m. □

Remark 6.15. As a consequence of part (ii) above, we have the Deligne homology relation

$$\Omega_f = T_f + d[R_f], \tag{11}$$

as currents acting on forms that are compactly supported on U_V. Using induction, the proof of Proposition 6.14(i) can be extended to act on forms on V:

Proposition 6.16. *Consider (dominant) morphisms $f_1,\dots,f_m$ from V to $\mathbb{P}^1$, in general position and put*

$$R_{\partial f} = \sum_{j=1}^{m}(-1)^{j-1}R_{\{f_1,\dots,\hat{f}_j,\dots,f_m\}}\Big|_{(f_j)}.$$

Then

$$\Omega_f = T_f + d[R_f] \pm (2\pi\mathrm{i})R_{\partial f}.$$

Evidently, these are the same formulas that arise in the explicit description of the cycle class map $\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m)\to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))$,[20] where in this case $\mathrm{codim}_X V = r-m$, equivalently $v = d-r+m$. So in this case the graph of $(f_1,..,f_m) : V\to\square^m$, defines a cycle in $z^r(X,m)\subset z^r(X\times\square^m)$. This doesn't describe all cycles in $z^r(X,m)$. First of all, on $\square^m$, the coordinate functions $(z_1,...,z_m)$ define $T_{\square^m},\Omega_{\square^m},R_{\square^m}$. For $W\subset z^r(X,m)$ an irreducible subvariety in "general" position, one can define operators via projections $\pi_1 : W\to X$, $\pi_2 : W\to\square^m$. Put $T_W = \pi_{1,*}\pi_2^*T_{\square^m},\Omega_W = \pi_{1,*}\pi_2^*\Omega_{\square^m}, R_W = \pi_{1,*}\pi_2^*R_{\square^m}$. Then from [KLM], and extending by linearity, and for $\xi\in z^r(X,m)$, with $W = |\xi|$, $\mathrm{cl}_{r,m}(\xi) = (2\pi\mathrm{i})^{d-r+m}\big(T_\xi,\Omega_\xi,R_\xi\big)$ induces the Bloch map in [Blo2]. Replacing the m-cube $\square^m$ by the m-simplex Δ^m yields similar operators $T_\xi = \pi_{1,*}\pi_2^*T_{\Delta^m},\Omega_\xi = \pi_{1,*}\pi_2^*\Omega_{\Delta^m}, R_\xi = \pi_{1,*}\pi_2^*R_{\Delta^m}$. The description of these operators appear in [BKLP].

Example 6.17. Suppose we are given a higher Chow cycle $\xi = \sum_\alpha(f_\alpha,V_\alpha)\in\mathrm{CH}^r_{\mathrm{hom}}(X,1)$. Note that $\mathrm{codim}_X V_\alpha = r-1$. Then by Hodge theory, we have $\sum_\alpha\Omega_{f_\alpha} = dS$, where by Poincaré duality $S\in F^r$ acts as the zero current on $F^{d-r+1}H^{2d-2r+2}(X)$. Note that $\gamma := \sum_\alpha\gamma_\alpha$ bounds a chain ζ, thus $T_\xi := \sum_\alpha T_{f_\alpha} = -2\pi\mathrm{i}d\delta_\zeta$. Taking the coboundary, viz.,

$$\delta(-\delta_\zeta,S,0) = (-T_\xi,-\sum_\alpha\Omega_\alpha,-2\pi\mathrm{i}\delta_\zeta - S),$$

this leads us to

$$(T_\xi,\sum_\alpha\Omega_{f_\alpha},\sum_\alpha R_{f_\alpha})\sim(0,\,0,\sum_\alpha R_{f_\alpha}-2\pi\mathrm{i}\delta_\zeta - S)$$

[20] At this point, we can clarify why Betti cohomology fails to detect higher cycle classes , for there is an exact sequence

$$0\to J(H^{2r-m-1}(X,\mathbb{Z}(r)))\to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Z}(r))\to H^{2r-m}_{\mathrm{tor}}(X,\mathbb{Z}(r))\to 0,$$

for $m\geq 1$.

in Deligne homology. By applying Poincaré duality, this leads us to Levine's formula for the regulator on K_1, induced by

$$\omega \in F^{d-r+1}E_X^{2d-2r+2}$$
$$\mapsto \frac{1}{(2\pi\mathrm{i})^{d-r+1}}\left(\sum_{\alpha}\int_{V_\alpha\backslash f_\alpha^{-1}[-\infty,0]}(\log f_\alpha)\omega \ - 2\pi\mathrm{i}\int_\zeta \omega\right).$$

[Note: In Levine's formula (op. cit.), the $-2\pi\mathrm{i}\int_\zeta\omega$ is replaced by $+2\pi\mathrm{i}\int_\zeta\omega$. This is because he is using the branch of the logarithm with imaginary part $\in (0,2\pi)$. Also, we have used the homological version of the Tate twist, which includes the factor $(2\pi\mathrm{i})^m$.] Here we also give an intuitive approach to the map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to H^{2r-1}_{\mathcal{D}}(X,\mathbb{Z}(r)),$$

i.e., the map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to \frac{F^{d-r+1}H^{2d-2r+2}(X,\mathbb{C})^\vee}{H_{2d-2r+2}(X,\mathbb{Z}(d-r))}.$$

Assume given a cycle $\xi = \sum_{i=1}^N (f_i, Z_i)$ representing a class in $\mathrm{CH}^r_{\mathrm{hom}}(X,1)$. Then via a proper modification, we can view $f_i : Z_i \to \mathbb{P}^1$ as a morphism, and consider the $2d-2r+1$-chain $\gamma_i = f_i^{-1}([-\infty,0])$. Then $\sum_{i=1}^N \mathrm{div}(f_i) = 0$ implies that $\gamma := \sum_{i=1}^N \gamma_i$ defines a $2d-2r+1$-cycle. Since ξ is null-homologous, it is easy to show that γ bounds some real dimensional $2d-2r+2$-chain ζ in X, viz., $\partial\zeta = \gamma$. For $\omega \in F^{d-r+1}H^{2d-2r+2}(X,\mathbb{C})$, the current defining $\mathrm{cl}_{r,1}(\xi)$ is given by:

$$\mathrm{cl}_{r,1}(\xi)(\omega) \ = \ \frac{1}{(2\pi\mathrm{i})^{d-r+1}}\left[\sum_{i=1}^N \int_{Z_i\backslash\gamma_i}\omega\log f_i - \quad 2\pi\mathrm{i}\int_\zeta\omega\right],$$

where we choose the principal branch of the log function. One can easily check that the current defined above is d-closed. Namely, if we write $\omega = d\eta$ for some $\eta \in F^{d-r+1}E_X^{2d-2r}$, then by a Stokes' theorem argument, both integrals above contribute to "periods" which cancel. The details of this argument can be found in [G-L].

Using the description of real Deligne cohomology given above, and the regulator formula, we arrive at the formula for the real regulator $r_{r,1} : \mathrm{CH}^r(X,1) \to H^{2r-1}_{\mathcal{D}}(X,\mathbb{R}(r)) = H^{r-1,r-1}(X,\mathbb{R}((r-1)) \simeq H^{d-r+1,d-r+1}(X,\mathbb{R}(d-r+1))^\vee$. Namely:

$$r_{r,1}(\xi)(\omega) = \frac{1}{(2\pi\mathrm{i})^{d-r+1}}\sum_j\int_{Z_j}\omega\log|f_j|.$$

6.18. **An abridged version of the Bloch map.** Let $H^i(-,j)$ be any "good" cohomology theory with weights (satisfying the homotopy axiom, weak purity, etc.), and $X/\mathbb{C}$ smooth quasi-projective. One has the diagram below with exact rows and columns (in particular the injectivity of $\kappa_1,\ \kappa_4$ follows from the Künneth formula[21]

[21] For Deligne cohomology, this works in this restricted situation. Indeed, a five-lemma argument yields $H^i_{\mathcal{D}}(X\times\partial\Delta^m,\mathbb{Q}(r)) \simeq H^i_{\mathcal{D}}(X,\mathbb{Q}(r))\otimes H^0_{\mathcal{D}}(\partial\Delta^m,\mathbb{Q})\bigoplus H^{i-m+1}_{\mathcal{D}}(X,\mathbb{Q}(r))\otimes H^{m-1}_{\mathcal{D}}(\partial\Delta^m,\mathbb{Q})$, together with $H^\bullet_{\mathcal{D}}(\partial\Delta^m,\mathbb{Q})) = H^\bullet(\partial\Delta^m,\mathbb{Q}))$ by purity of HS and the definition of the Deligne complex. [The only special case is $m=2$. But $H^0(\partial\Delta^m,\mathbb{Q})/F^0 = 0$.] Note that $H^{m-1}(\partial\Delta^m,\mathbb{Q})$ actually has weight zero. One simple reason for this is that $H^{m-1}(\partial\Delta^m,\mathbb{Q}) \simeq H^{m-1}(\partial\square^m,\mathbb{Q})$, the latter of which is dual to $H^{m+1}_{\partial\square^m}(\square^m,\mathbb{Q}(m)) \simeq H^m(\square^m\backslash\partial\square^m,\mathbb{Q}(m))$ The

and the homotopy property, and λ_1, λ_2 are injective by weak purity. (Here we identify $\partial\Delta^m$ with its support $|\partial\Delta^m|$, which homologically is identified with the $m-1$ sphere S^{m-1}, $|\partial\xi|$ means $|\xi| \cap |\partial\Delta^m|$, and the first column is a split short

---

latter is generated by $\bigwedge_{j=1}^m d\log z_j$, which lies in the image of $H^m([\mathbb{P}^1]^{\times m}\backslash\partial[\mathbb{P}^1]^{\times m}, \mathbb{Q}(m)) \to H^m(\square^m\backslash\partial\square^m, \mathbb{Q}(m))$. But $H^m([\mathbb{P}^1]^{\times m}\backslash\partial[\mathbb{P}^1]^{\times m}, \mathbb{Q}(m)) = H^1(\mathbb{C}^\times, \mathbb{Q}(1))^{\otimes m}$ has pure weight 0, and where "$\partial\mathbb{P}^1$" $:= \{0, \infty\}$. Alternate take: Set $Y := \partial\Delta^m$, and $\pi : Y^{[j]} \to Y$ for all j. Note that $Y^{[m]} = \emptyset$. Consider the simplicial complex

$$Y^{[1]} \overset{\leftarrow}{\leftarrow} Y^{[2]} \cdots \begin{smallmatrix}\leftarrow\\ \vdots\\ \leftarrow\end{smallmatrix} Y^{[m-1]}.$$

Applying π_* to the sheaf diagram

$$\begin{array}{ccccccc}
\Omega^0_{Y^{[1]}} & \xrightarrow{\mathrm{Gy}^*} & \Omega^0_{Y^{[2]}} & \xrightarrow{\mathrm{Gy}^*} & \cdots & \xrightarrow{\mathrm{Gy}^*} & \Omega^0_{Y^{[m-1]}} \\
d\downarrow & & d\downarrow & & & & d\downarrow \\
\vdots & & & & & \vdots & \\
\vdots & & \vdots & & \vdots & & \\
\vdots & \vdots & & \vdots & & & \\
\Omega^{m-2}_{Y^{[1]}} & \xrightarrow{\mathrm{Gy}^*} & \Omega^{m-2}_{Y^{[2]}} & & & & \\
d\downarrow & & & & & & \\
\Omega^{m-1}_{Y^{[1]}} & & & & & &
\end{array}$$

it follows from the holomorphic Poincaré lemma and the cohomology of a simplicial complex that the total complex of sheaves (after applying π_*) is quasi-isomorphic to $\mathbb{C}_Y \to 0 \to 0 \to \cdots$. This is on the *sheaf* level. In particular $Y^{[m-1]}$ is a finite set of points, and exactness on the sheaf level involves behaviour at a given point. Taking cohomology of the global sections of the total complex gives us $\Gamma(\Omega^0_{Y^{[m-1]}})/\mathrm{Gy}^*\Gamma(\Omega^0_{Y^{[m-2]}}) \simeq \mathbb{C}$, which is also of weight 0.

exact sequence by the Künneth formula and the homotopy property.)
(12)

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
\downarrow & & & & & & \downarrow \\
H^{2r-1}(X,r) & & & & & & H^{2r}(X,r) \\
|| & & A & \mapsto & \{\xi\} & & || \\
H^{2r-1}(X\times\Delta^m,r) & \xhookrightarrow{\lambda_1} & H^{2r-1}(X\times\Delta^m\backslash|\xi|,r) & \xrightarrow{\lambda_3} & H^{2r}_{|\xi|}(X\times\Delta^m,r) & \xrightarrow{\lambda_5} & H^{2r}(X\times\Delta^m,r) \\
\kappa_1\Big\downarrow & & \kappa_2\Big\downarrow & & \kappa_3\Big\downarrow & & \kappa_4\Big\downarrow \\
H^{2r-1}(X\times\partial\Delta^m,r) & \xhookrightarrow{\lambda_2} & H^{2r-1}(X\times\partial\Delta^m\backslash|\partial\xi|,r) & \xrightarrow{\lambda_4} & H^{2r}_{|\partial\xi|}(X\times\partial\Delta^m,r) & \xrightarrow{\lambda_6} & H^{2r}(X\times\partial\Delta^m,r) \\
\kappa_0\Big\downarrow & & & & & & \\
H^{2r-m}(X,r) & & & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

Consider the ith face map $\partial_i : z^r(X,m) \to z^r(X,m-1)$, and recall the normalized description of Bloch's higher Chow groups ([Blo1]).

$$\mathrm{CH}^r(X,m) = \left\{\frac{\bigcap_{i=0}^m \ker\partial_i}{\partial_{m+1}\left(\bigcap_{i=0}^m \ker\partial_i\right)}\right\}.$$

Now start with

$$\xi \in \bigcap_{i=0}^m \ker\partial_i,$$

with cycle class $\{\xi\} \in H^{2r}_{|\xi|}(X\times\Delta^m,r)$. Then $\kappa_3(\{\xi\}) = 0$, hence $\lambda_5(\{\xi\}) = 0$, and so $\{\xi\} = \lambda_3(A)$, for some A. Note that the ambiguity of the choice of A will be quotiented out by κ_0 in the end. By commutativity of the above diagram, $\kappa_2(A) = \lambda_2(B)$ for a unique B dependent on A, which amounts to saying a unique choice of B modulo $H^{2r-1}(X,r)$, dependent on $\{\xi\}$.

Definition 6.19. *The abridged cycle class map is given by*

$$[\xi] := \kappa_0(B) \in H^{2r-m}(X,r).$$

Lemma 6.20. *If*

$$\xi \in \partial_{m+1}\left(\bigcap_{i=0}^m \ker\partial_i\right),$$

then $[\xi] = 0$.

Proof. We give a proof in the case where X is smooth and projective, albeit the story should hold for a general smooth quasi-projective X using a residue argument. Let $\xi' \in \bigcap_{i=0}^{m} \ker \partial_i$ be given such that $\partial_{m+1}\xi' = \xi$. The homotopy property gives us an exact sequence

$$0 \to H^{2r-1}(X,r) \to H^{2r-1}(X \times \Delta^{m+1}\backslash|\xi'|,r) \to H^{2r}_{|\xi'|}(X \times \Delta^{m+1},r) \to H^{2r}(X,r),$$

and hence $\{\xi'\} \in H^{2r}_{|\xi'|}(X \times \Delta^{m+1},r)$ maps to $0 \in H^{2r}(X,r)$, using the isomorphism

$$H^{2r}(X,r) \simeq H^{2r}(X \times \Delta^{m+1},r) \xrightarrow{\partial_{m+1}\ \sim} H^{2r}(X \times \Delta^{m},r) \simeq H^{2r}(X,r),$$

and the fact that $\{\xi = \partial_{m+1}\xi'\} \in H^{2r}_{|\xi|}(X \times \Delta^{m},r)$ maps to $0 \in H^{2r}(X \times \Delta^m,r)$. Thus $\{\xi'\}$ is in the image of $A' \in H^{2r-1}(X \times \Delta^{m+1}\backslash|\xi'|,r)$, for some A' unique up to an element of $H^{2r-1}(X,r) \simeq H^{2r-1}(X \times \Delta^{m+1},r)$. Note that $A' \mapsto B_0' \in H^{2r-1}(X \times \partial\Delta^{m+1}\backslash|\partial\xi'|,r) = H^{2r-1}(X \times \partial\Delta^{m+1}\backslash|\xi|,r)$, and that under ∂_{m+1}, $B_0' \mapsto \kappa_2(A) \in \text{Image}\big(H^{2r-1}(X \times \partial\Delta^m,r) \to H^{2r-1}(X \times \partial\Delta^m\backslash|\partial\xi|,r)\big)$, which is the weight -1 part of $H^{2r-1}(X \times \partial\Delta^m\backslash|\partial\xi|,r)$. From Hodge theory, $\kappa_2(A)$ is in the image of some $B'' \in W_{-1}H^{2r-1}(X \times \partial\Delta^{m+1}\backslash|\partial\xi'|,r) = \text{Image}\big(H^{2r-1}(X \times \partial\Delta^{m+1},r) \to H^{2r-1}(X \times \partial\Delta^{m+1}\backslash|\partial\xi'|,r)\big)$, using the exact sequence,

$$H^{2r-1}(X \times \partial\Delta^{m+1},r) \to H^{2r-1}(X \times \partial\Delta^{m+1}\backslash|\partial\xi'|,r)\big) \to H^{2r}_{|\partial\xi'|}(X \times \partial\Delta^{m+1},r),$$

where the latter term is pure Tate of weight 0. Now use the fact that the compose

$$H^{2r-1}(X \times \partial\Delta^{m+1},r) \xrightarrow{\partial_{m+1}} H^{2r-1}(X \times \partial\Delta^{m},r) \to H^{2r-m}(X,r),$$

is zero. □

Proposition 6.21. *The induced map*

$$[\ ] : CH^r(X,m) \to H^{2r-m}(X,r)$$

agrees with the Bloch cycle class map.

Proof. Well, assuming no mistakes in the above construction, what else could it be? We can easily check this in the cases of singular, de Rham and Deligne cohomologies (the latter using some results from our KLM paper). Observe that the Künneth induced inclusion $H^{2r-m}(X,r) \subset H^{2r-1}(X \times \partial\Delta^m,r)$ identifies $H^{2r-m}(X,r)$ with $H^{2r-m}(X,r) \otimes H^{m-1}(\partial\Delta^m,0)$. Accordingly, the dual $H^{m-1}(\partial\Delta^m,0)^\vee$ is given by $H^{m+1}_{\partial\Delta^m}(\Delta^m,m) \simeq H^m(\Delta^m\backslash\partial\Delta^m,m)$. Recall the triple

$$\big(T_{\Delta^m(\mathbb{R}^+)}, \Omega_{\Delta^m}, R_{\Delta^m}\big),^{22}$$

and the relation

$$d[R_{\Delta^m}] = T_{\Delta^m(\mathbb{R}^+)} - \Omega_{\Delta^m}\ \ \pm 2\pi\mathrm{i}R_{\partial\Delta^m}$$

Then if $H^i(X,j)$ is singular, de Rham (resp.), Deligne (resp.), then up to Tate twist, the generator for $H^{m+1}_{\partial\Delta^m}(\Delta^m,m)$ is precisely $d[T_{\Delta^m(\mathbb{R}^+)}]$, $d[\Omega_{\Delta^m}]$ and accordingly $\big(d[T_{\Delta^m(\mathbb{R}^+)}], d[\Omega_{\Delta^m}], \pm 2\pi\mathrm{i}R_{\partial\Delta^m}\big)$ respectively. The deal then is that to arrive at $[\xi] = \kappa_0(B)$, we take the cup product of B with the generator(s) above, and project to X. This situation is well described in [KLM]. That process will simply be denoted by $\wedge$ say. For instance, in the case of singular cohomology, and ignoring twists, we have $H^{2r-1}(X \times \Delta^m\backslash|\xi|) \simeq H_{2d+2m-2r+1}(X \times \Delta^m, |\xi|)$, and A corresponds to a membrane ζ for which $\partial\zeta = \xi$. Then $\kappa_2(A) \wedge d[T_{\Delta^m(\mathbb{R}^+)}]$ amounts to

[22] Here we write $T_{\Delta^m(\mathbb{R}^+)}$ for T_{Δ^m}, to emphasize that we are dealing with the real m-simplex.

the same calculation as $\zeta \wedge d[T_{\Delta^m(\mathbb{R}^+)}]$, which up to coboundary, is the same thing as $\partial\zeta \wedge T_{\Delta^m(\mathbb{R}^+)} = \xi \wedge T_{\Delta^m(\mathbb{R}^+)}$. Similar story for de Rham cohomology.

For Deligne cohomology, we do the following. In this case $\{\xi\} = (T_\xi, \delta_\xi, 0)$, and $A = (T_\zeta, S, K)$ where $\partial\zeta = \xi,\ dS = \delta_\xi$, where S lies in the correct Hodge filter, and $dK = T_\zeta - S$. Everything now reduces to the calculation of

$$(T_\zeta, S, K) \wedge \big(d[T_{\Delta^m(\mathbb{R}^+)}], d[\Omega_{\Delta^m}], \pm 2\pi \mathrm{i} R_{\partial\Delta^m}\big) =$$

$$\big(T_\zeta \wedge d[T_{\Delta^m(\mathbb{R}^+)}], S \wedge d[\Omega_{\Delta^m}], T_\zeta \wedge (\pm 2\pi\mathrm{i}) R_{\partial\Delta^m} + K \wedge d[\Omega_{\Delta^m}]\big).$$

Note that

$$T_\zeta \wedge (\pm 2\pi\mathrm{i}) R_{\partial\Delta^m} + K \wedge d[\Omega_{\Delta^m}] \equiv \delta_\xi \wedge R_{\Delta^m} + T_\zeta \wedge T_{\Delta^m(\mathbb{R}^+)} - S \wedge \Omega_{\Delta^m},$$

and where $\equiv$ means the same "up to coboundary". Now let us suppose that $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X, m)$. Then $dT_0 = T_\zeta \wedge d[T_{\Delta^m(\mathbb{R}^+)}]$ for some integral current T_0 (and a similar story for $S \wedge d[\Omega_{\Delta^m}]$ being a Hodge filtered coboundary, which plays no role in the AJ map). But $d(T_0 \pm T_\zeta \wedge T_{\Delta^m(\mathbb{R}^+)}) = \delta_\xi \wedge T_{\Delta^m(\mathbb{R}^+)}$ is equivalent to a membrane for which $\delta_\xi \wedge T_{\Delta^m(\mathbb{R}^+)}$ bounds. Thus the AJ map is precisely the formula we propose. On the other hand, $H^{m-1}(\partial\Delta^m)$ is naturally identified with $H^{m-1}(\partial\Box^m)$, and the same constructions leads to the KLM version of the AJ map, which is Bloch's construction. Thus the simplicial version of the AJ map coincides naturally with the KLM version, viz., Bloch's map. $\Box$

6.22. **[Beilinson-] Rigidity.** Recall $X/\mathbb{C}$ is smooth projective.

Theorem 6.23 ([Be2])**.** $\mathrm{cl}_{r,m} : CH^r(X, m) \to H^{2r-m}_{\mathcal{D}}(X, \mathbb{Q}(r))$ *has countable image for* $m \geq 2$.

Proof. Let $k = \overline{k}$ be an algebraically closed subfield of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$ such that $X/\mathbb{C} = X_k \times \mathbb{C}$. It suffices to show that $\mathrm{cl}_{r,m}(\mathrm{CH}^r(X_k, m)) = \mathrm{cl}_{r,m}(\mathrm{CH}^r(X/\mathbb{C}, m))$, as k is countable. Let $\xi \in \mathrm{CH}^r(X/\mathbb{C}, m)$. There exists a smooth quasiprojective variety S/k and spread cycle $\tilde{\xi} \in \mathrm{CH}^r(S \times_k X, m)$ such that wrt a suitable embedding $k(\eta_S) = k(S) \hookrightarrow \mathbb{C}$, $\tilde{\xi}_{\eta_S} = \xi$. This embedding corresponds to a point $p \in S(\mathbb{C})$. (Thus $\tilde{\xi}_p = \xi$.) Choose $q \in S(k) \neq \emptyset$ (as $k = \overline{k}$). It suffices to show that $\mathrm{cl}_{r,m}(\xi = \tilde{\xi}_p) = \mathrm{cl}_{r,m}(\tilde{\xi}_q)$. Choose any smooth connected affine curve $C \subset S(\mathbb{C})$ passing through p and q. Obviously

$$(\tilde{\xi}_C)|_q = \tilde{\xi}_q,$$

is defined over k in $\mathrm{CH}^r(X_k, m)$. There is an exact sequence

$$\to H^{2r-m-1}(C \times X, \mathbb{C}) \to H^{2r-m}_{\mathcal{D}}(C \times X, \mathbb{Q}(r)) \to \Gamma\big(H^{2r-m}(C \times X, \mathbb{Q}(r))\big) \to 0.$$

Since $H^2(C) = 0$ (as C is affine), and $H^0(C)$ is pure, it follows that

$$\Gamma\big(H^{2r-m}(C \times X, \mathbb{Q}(r))\big) = \Gamma\big(H^1(C, \mathbb{Q}) \otimes H^{2r-m-1}(X, \mathbb{Q})(r)\big) = 0,$$

since $\mathrm{wt}(H^1(C, \mathbb{Q})) \leq 2$. Furthermore $\mathrm{cl}_{r,m}(\tilde{\xi}) \in H^{2r-m}_{\mathcal{D}}(C \times X, \mathbb{Q}(r)) \mapsto 0 = \Gamma\big(H^{2r-m}(C \times X, \mathbb{Q}(r))\big)$. Thus $\mathrm{cl}_{r,m}(\tilde{\xi})$ lifts to a class in $H^{2r-m-1}(C \times X, \mathbb{C})$. But for $t \in S(\mathbb{C})$ the restriction

$$H^{2r-m-1}(C \times X, \mathbb{C}) \to H^{2r-m-1}(\{t\} \times X, \mathbb{C}) = H^{2r-m-1}(X, \mathbb{C}),$$

factors through $H^0(C, \mathbb{C}) \otimes H^{2r-m-1}(X, \mathbb{C})$ by the Künneth formula (as $H^\bullet(C) \to H^\bullet(\{t\})$ is zero for $\bullet > 0$). Now consider $t \in \{p, q\}$. $\Box$

Corollary 6.24 (See [MS1]).

$$\mathrm{cl}_{r.1} : CH^r(X,1) \to \frac{H_{\mathcal{D}}^{2r-1}(X,\mathbb{Q}(r))}{H^{r-1,r-1}(X,\mathbb{Q}(r-1))\otimes\mathbb{C}/\mathbb{Q}(1))},$$

has countable image.

Proof. As in the above proof, we work over a curve C. Let $\overline{C}$ be the smooth projective closure of C. Again since $H^2(C)=0$ (as C affine) and $H^0(C)$ is pure, it follows that

$$\begin{aligned}\Gamma\big(H^{2r-1}(C\times X,\mathbb{Q}(r))\big) &= \Gamma\big(H^1(C,\mathbb{Q}(1))\otimes H^{2r-2}(X,\mathbb{Q}(r-1))\big)\\ &= \Gamma\big(H^1(C,\mathbb{Q}(1))\big)\otimes H^{r-1,r-1}(X,\mathbb{Q}(r-1)),\end{aligned}$$

as

$$\Gamma\big(H^1(C,\mathbb{Q}(1))\big)\hookrightarrow Gr_W^0 H^1(C,\mathbb{Q}(1)),$$

is pure of type $(0,0)$. Next we have the short exact sequence

$$\begin{aligned}H^1(\overline{C},\mathbb{Q}(1))\otimes H^{r-1,r-1}(X,\mathbb{Q}(r-1)) &\hookrightarrow H^1(C,\mathbb{Q}(1))\otimes H^{r-1,r-1}(X,\mathbb{Q}(r-1))\\ &\twoheadrightarrow Gr_W^0 H^1(C,\mathbb{Q}(1))\otimes H^{r-1,r-1}(X,\mathbb{Q}(r-1)).\end{aligned}$$

Since $H^{r-1,r-1}(X,\mathbb{Q}(r-1)$ is purely Tate $(\oplus\mathbb{Q}(0))$, we can assume without loss of generality that $H^{r-1,r-1}(X,\mathbb{Q}(r-1)=\mathbb{Q}(0)$. And so we arrive at

$$0\to H^1(\overline{C},\mathbb{Q}(1))\to H^1(C,\mathbb{Q}(1))\to Gr_W^0 H^1(C,\mathbb{Q}(1))\to 0.$$

Obviously a class in $Gr_W^0 H^1(C,\mathbb{Q}(1))$ determines a class in $J\big(H^1(\overline{C},\mathbb{Q}(1))\big)$ via the $\mathrm{Ext}^{\bullet}_{\mathrm{MHS}}(\mathbb{Q}(0),-)$ LES. In particular,

$$\Gamma\big(H^1(C,\mathbb{Q}(1))\big)\mapsto 0\in J\big(H^1(\overline{C},\mathbb{Q}(1))\big).$$

Let ξ_0 be the zero-cycle supported on $\overline{C}\backslash C$ with $[\xi_0]\in\Gamma\big(H^1(C,\mathbb{Q}(1))\big)$ corresponding to $\mathrm{cl}_{r,1}(\tilde{\xi})$. Since $\overline{C}$ is a curve, $\xi_0\sim_{\mathrm{rat}} 0$ on $\overline{C}$. Thus $\exists f\in\mathbb{C}(C)^{\times}$ with $f\in\mathcal{O}_C^{\times}(C)$, such that $\mathrm{div}(f)=\xi_0$. There is a product

$$H^{r-1,r-1}(X,\mathbb{Q}(r-1))\times H^1_{\mathcal{D}}(C,\mathbb{Q}(1))\to H^{2r-1}_{\mathcal{D}}(C\times X,\mathbb{Q}(r)).$$

We have shown (via a snake lemma argument) that the cokernel is dominated by

$$\frac{H^{2r-2}(C\times X,\mathbb{C})}{F^r+H^{r-1,r-1}(X,\mathbb{Q}(r-1))\otimes(\mathbb{C}/\mathbb{Q}(1))+H^{2r-2}(C\times X,\mathbb{Q}(r))}.$$

The rest is clear. □

6.25. **Appendix: Local to global issues.** Let W/k be a quasi-projective variety over a field k. The Chow groups $\mathrm{CH}^r(W,m)$ are natural equipped with a coniveau filtration $N^p\mathrm{CH}^r(W,m)$ of cycles supported on $Y\times\Delta^m$, where $\mathrm{codim}_W Y\geq p$. The local to global spectral sequence machinery in [BO] is aimed at computing the coniveau graded pieces of $\mathrm{CH}^r(W,m)$, when W is regular, which we will now assume. Let $\mathcal{CH}^p(r)$ be the sheafified Chow groups in the Zariski topology on W. By Bloch [Blo1], there exists a (flasque) resolution of $\mathcal{CH}^p(r)$:

$$\begin{aligned}(13)\qquad 0\to\mathcal{CH}^p(r)\to\mathrm{CH}^p(k(W),r)\to &\bigoplus_{\mathrm{cd}_W Z=1} i_{Z,*}\mathrm{CH}^{p-1}(k(Z),r-1)\\ \to &\bigoplus_{\mathrm{cd}_W Z=2} i_{Z,*}\mathrm{CH}^{p-2}(k(Z),r-2)\to\cdots\end{aligned}$$

Thus

$$H^{\bullet>\min(p,r)}_{\mathrm{Zar}}(W,\mathcal{CH}^p(r)) = 0. \tag{14}$$

By the works of Suslin, Nesterenko and Kerz, $\mathcal{CH}^r(r) = \mathcal{K}^M_{r,W}$ and the resolution in (13) becomes:

$$0 \to \mathcal{K}^M_{r,W} \to K^M_r(k(W)) \to \bigoplus_{\mathrm{cd}_W Z=1} i_{Z,*}K^M_{r-1}(k(Z)) \to \bigoplus_{\mathrm{cd}_W Z=2} i_{Z,*}K^M_{r-2}(k(Z)) \to \cdots$$

$$\to \bigoplus_{\mathrm{cd}_W Z=r-2} i_{Z,*}K^M_2(k(Z)) \to \bigoplus_{\mathrm{cd}_W Z=r-1} i_{Z,*}K^M_1(k(Z)) \to \bigoplus_{\mathrm{cd}_W Z=r} i_{Z,*}K^M_0(k(Z)) \to 0.$$

An immediate consequence is a Milnor analogue of the famous Bloch-Quillen formula:

Theorem 6.26.

$$CH^r(W) \simeq H^r_{\mathrm{Zar}}(W,\mathcal{K}^M_{r,W}).$$

From [Blo1], [BO], there is a local to global spectral sequence:

$$E^{a,b}_2(r) := H^a_{\mathrm{Zar}}(W,\mathcal{CH}^r(2r-b)) \Rightarrow \mathrm{CH}^r(W,2r-a-b).$$

We have:

Proposition 6.27. *[See* [MS2]*] For W/k regular,*

$$CH^r(W,m) \simeq H^{r-m}_{\mathrm{Zar}}(W,\mathcal{K}^M_{r,W}),\ \text{for } 0 \le m \le 2.$$

Proof. Observe that $0 \le 2r-a-b = m \le 2$, and by (14), $E^{a,b}_2(r) = 0$ for $a > \min\{r, 2r-b\}$, hence we require $a \le \min\{r,2r-b\}$. Next, we already know that the proposition is true for $m = 0$, hence we need only consider the cases $m = 1, 2$. We first whow that $E^{a,b}_2(r) = 0$, unless $(a,b) = (r-m,r)$ for the given cases of m. In calculating $E^{a,b}_2(r)$, we are led from (13) to consider terms of the form $\mathrm{CH}^{r-a}(\mathbb{F},m)$ for a field $\mathbb{F}$. But for dimension reasons alone $\mathrm{CH}^{r-a}(\mathbb{F},m) = 0$ if $r-a > m$, i.e. if $a < r-m$. Next, since $m > 0$, it is clear that $\mathrm{CH}^0(\mathbb{F},m) = 0$, hence $a < r$. Also if $a = r-m$, then from $2r-a-b = m$, we arrive at $b = r$, i.e. $(a,b) = (r-m,r)$. Thus we can assume that $r-m < a < r$, which cannot happen if $m = 1$. So let us assume that $m = 2$. By the same reasoning, we are reduced to $r-2 < a < r$, i.e. $a = r-1$. From $2r-a-b = m = 2$, we arrive at $b = r-1$. Then $E^{r-1,r-1}_2(r)$ leads us to terms of the form $\mathrm{CH}^1(\mathbb{F},2) = 0$ by [Blo1](p. 269). Thus the only remaining terms are $E^{r-m,r}_2(r)$ for $m = 1,2$. the sequence

$$E^{r-m-2,r+1}_2(r) \xrightarrow{d_2} E^{r-m,r}_2(r) \xrightarrow{d_2} E^{r-m+2,r-1}_2(r).$$

With regard to $E^{r-m+2,r-1}_2(r)$, we have $r \le a := r-m+2 \le r+1$, for $m = 1,2$. But we cannot have $a = r+1 > r$, hence $a = r$. The computation of $E^{r,r-1}_2(r)$ will involve terms of the for $\mathrm{CH}^0(\mathbb{F},1) = 0$, hence $E^{r,r-1}_2(r) = 0$. Next regarding $E^{r-m-2,r+1}_2(r)$, we have $r-4 \le a = r-m-2 \le r-3$ for $m = 1,2$. The computation of $E^{r-m-2,r+1}_2(r)$ will then lead to terms of the form $\mathrm{CH}^3(\mathbb{F},2) = 0 = \mathrm{CH}^4(\mathbb{F},3)$, the vanishing due to dimension reasons alone. Thus $E^{r-m,r}_2(r) = E^{r-m,r}_3(r)$, and by a similar iterated argument, $E^{r-m,r}_2(r) = E^{r-m,r}_\infty(r)$. Thus for $0 \le m \le 2$, $H^{r-m}_{\mathrm{Zar}}(W,\mathcal{K}^M_{r,W}) = E^{r-m,r}_\infty(r) = \mathrm{CH}^r(W,m)$. $\square$

Remark 6.28. For all $m \geq 0$, $H^{r-m}_{\mathrm{Zar}}(W, \mathcal{K}^M_{r,W})$ is the E_2 term of an E_∞ term describing the graded coniveau $r-m$ term of $\mathrm{CH}^r(X,m)$. All other terms relate to higher codimension.

6.29. **Appendix: Abel-Jacobi map.** Let $X/\mathbb{C}$ be smooth and projective. From [KLM], the Abel-Jacobi map

$$\Phi_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X,m) \to J(H^{2r-m-1}(X,\mathbb{Z}(r))) \simeq \frac{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})^\vee}{H_{2d-2r+m+1}(X,\mathbb{Z}(d-r)))},$$

is given as follows. Consider the cubical description of $\mathrm{CH}^r(X,m)$. Let $\{\xi\} \in \mathrm{CH}^r_{\mathrm{hom}}(X,m)$ with corresponding R_ξ. Observe that $\partial\xi = 0$, hence $R_{\partial\xi} = 0$. Further, let $\pi_1 : |\xi| \to X$, $\pi_2 : |\xi| \to \square^m$ be the obvious projections. One has $\gamma := \{\pi_2^{-1}[-\infty,0]^m\} \cap \xi = \partial\zeta$, and the AJ map is given by up to Tate twist by $R_W + (-2\pi \mathrm{i})^m \int_\zeta$, where $\partial\zeta = \gamma$. More explicitly, using the cubical complex, the formula for the AJ map is:

$$\begin{aligned}(15)\qquad \Phi_{r,m}(\xi) = \frac{1}{(2\pi\mathrm{i})^{d-r+m}}\Bigg[&\int_{\xi\backslash\{\xi\cap\pi_2^{-1}([-\infty,0]\times\square^{m-1})\}} \pi_2^*\big((\log z_1)d\log z_2\wedge\cdots\wedge d\log z_m\big)\wedge\pi_1^*(\omega)\\ &-(2\pi\mathrm{i})\int_{\{\xi\cap\pi_2^{-1}([-\infty,0]\times\square^{m-1})\}\backslash\{\xi\cap\pi_2^*([-\infty,0]^2\times\square^{m-2})\}} \pi_2^*\big((\log z_2)d\log z_3\wedge\cdots\wedge d\log z_m\big)\wedge\pi_1^*(\omega)\\ &+\cdots+(-2\pi\mathrm{i})^{m-1}\int_{\{\xi\cap\pi_2^{-1}([-\infty,0]^{m-1}\times\square^1)\}\backslash\{\xi\cap\pi_2^{-1}([-\infty,0]^m)\}} \pi_2^*(\log z_m)\wedge\pi_1^*(\omega)\\ &+\left\{(-2\pi\mathrm{i})^m\int_\zeta \pi_1^*(\omega)\right\}\Bigg],\end{aligned}$$

where the latter term is a membrane integral.

Example 6.30 ([KLM])**.** Let $a,\ b \in \mathbb{C}^\times\backslash\{1\}$, and put:

$$V(a) = \left\{\left(1-\frac{a}{t},1-t,t\right)\,\middle|\, t\in\mathbb{P}^1\right\}\bigcap\square^3,\ \ W(a) = \left\{\left(1-\frac{b}{t},t,1-t\right)\,\middle|\, t\in\mathbb{P}^1\right\}\bigcap\square^3,$$

and note that

$$\partial V(a) = (1-a,a), \quad \partial W(b) = (b,1-b).$$

Therefore

$$\xi_a := V(a) - W(1-a) \in \mathrm{CH}^2(\mathrm{Pt},3).$$

The value $\Phi_{2,3}(\xi_a) \in \mathbb{C}/\mathbb{Z}(2)$ is easy to compute:

$$(16)\qquad \Phi_{2,3}(\xi_a) = \mathrm{Li}_2(a) + \mathrm{Li}_2(1-a) + \log a \log(1-a),$$

where Li_2 is the dilogarithm, and log has the principal branch. By Beilinson rigidity[23]

$$\Phi_{2,3}(\xi_a) = \lim_{a\to 0}\Phi_{2,3}(\xi_a) = \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \in \mathbb{C}/\mathbb{Z}(2),$$

which is a torsion class. Note that if we insist on $a \in \mathbb{Q}$, then ξ_a is a generator of $\mathrm{CH}^2(\mathrm{Spec}(\mathbb{Q}),3) \simeq \mathbb{Z}/24\mathbb{Z}$.

[23]Alternatively, the RHS of (16) is independent of a by differentiation in a.

7. Examples of regulators and normal functions

The reader is encouraged to consult a number of works due to Bloch, Beilinson, Goncharov, Levine, Esnault and many others, for various earlier incarnations of regulator type currents for higher Chow cycles. A complete description of the Bloch regulator in terms of polylogarithmic type currents for complex varieties can be found in [KLM], [K-L], and [BKLP].

7.1. **Case $m = 0$ and CY threefolds.** Suppose that $\xi \in \mathrm{CH}^r(X)$ is given and that $Y \subset X$ is a smooth hypersurface. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^r(X) & \to & \mathrm{CH}^r(Y) \\ \Big\downarrow & & \Big\downarrow \\ H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) & \to & H^{2r}_{\mathcal{D}}(Y, \mathbb{Z}(r)); \end{array}$$

Further, if we assume that the restriction $\xi_Y \in \mathrm{CH}_{\mathrm{hom}}(Y)$ is null-homologous, then $\mathrm{cl}_{r,0}(\xi) \in H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) \mapsto J(H^{2r-1}(Y, \mathbb{Z}(r))) \subset H^{2r}_{\mathcal{D}}(Y, \mathbb{Z}(r))$. Next, if $Y = X_0 \in \{X_t\}_{t\in S}$ is a family of smooth hypersurfaces of X, then such a ξ determines a holomorphically varying map $\nu_\xi(t) \in J(H^{2r-1}(X_t, \mathbb{Z}(r)))$, called a normal function. The class $\mathrm{cl}_r(\xi) = \delta(\nu_\xi) \in \Gamma(H^{2r}(X, \mathbb{Z}(r)))$ is called the topological invariant of ν_ξ, i.e. ν_ξ determines $\mathrm{cl}_r(\xi)$. In [K-L], these ideas are extended in complete generality to the situation of the higher Chow groups, where the notion of "arithmetic normal functions" are introduced.

This next result is a consequence of the work of Griffiths (see [Gr], as well as §14 of [Lew1]).

Theorem 7.2. *If $F^{r-1}H^{2r-1}(X, \mathbb{C}) \cap H^{2r-1}(X, \mathbb{Q}(r)) = 0$, then there is an induced map*

$$\underline{\Phi}_r : \mathrm{Griff}^r(X; \mathbb{Q}) \to J(H^{2r-1}(X, \mathbb{Q}(r))).$$

In particular $\Phi_r\big(CH^r_{\mathrm{alg}}(X)\big) = 0 \in J(H^{2r-1}(X, \mathbb{Q}(r)))$. This is the case for a general CY threefold with $r = 2$.

Example 7.3 (Griffiths' famous example ([Gr]))**.** Let:

$$X = V(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5) \subset \mathbb{P}^5$$

be the Fermat quintic fourfold. Consider these 3 copies of $\mathbb{P}^2 \subset X$:

$$L_1 := V(z_0 + z_1, z_2 + z_3, z_4 + z_5),$$

$$L_2 := V(z_0 + \xi z_2, z_2 + \xi z_3, z_4 + z_5),$$

$$L_3 := V(z_0 + \xi z_1, z_2 + \xi z_3, z_4 + \xi z_5).$$

where ξ is a primitive 5-th root of unity. Then $L_3 \bullet (L_1 - L_2) = 1 \neq 0$, hence $\xi := [L_1 - L_2]$ is a non-zero class in $H^{2,2}(X, \mathbb{Z}(2))$. Further, if $\{X_t\}_{t\in U\subset\mathbb{P}^1}$ is a general pencil of smooth hyperplane sections of X, and if $t \in U$, then it is well known that $\xi_t \in \mathrm{CH}^2_{\mathrm{hom}}(X_t)$ by a theorem of Lefschetz. Since $\delta(\nu_\xi) = [L_1 - L_2] \neq 0$, it follows that $\nu_\xi(t)$ is non-zero for most $t \in U$. Therefore for general $t \in U$, $\mathrm{Griff}^2(X_t)$ contains an infinite cyclic group by Theorem 7.2. The upshot is that if:

$$Y = V\bigg(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \Big(\sum_{j=0}^{4} a_j z_j\Big)^5\bigg) \subset \mathbb{P}^4,$$

for general $a_0, \ldots, a_4 \in \mathbb{C}$, then $\mathrm{Griff}^2(Y) \neq 0$ contains an infinite cyclic subgroup. H. Clemens [Cl], was the first to show that the Griffiths group of a general quintic threefold in $\mathbb{P}^4$ is (countably) infinite dimensional, when tensored over $\mathbb{Q}$. Later it was shown by C. Voisin [Vo], that the same holds for general CY threefolds.

Theorem 7.4 (See [B-S], [G], [G-S], [Gr], [Cl], [Vo]). *Let $X \subset \mathbb{P}^4$ be a (smooth) threefold of degree d. If $d \leq 4$, then $\Phi_2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \xrightarrow{\sim} J^2(X)$ is an isomorphism. Now assume that X is general. If $d \geq 6$ then $Im\big(\Phi_2\big)$ is torsion. If $d = 5$, then $Im\big(\Phi_2\big) \otimes \mathbb{Q}$ is countably infinite dimensional.*

Theorem 7.5 ([Vo]). *If X is a general Calabi-Yau threefold, then $Im\big(\Phi_2\big)$ is countably infinite dimensional, when tensored over $\mathbb{Q}$. In particular, since $\Phi_2(\mathrm{CH}^2_{\mathrm{alg}}(X)) = 0$, it follows that $Griff^2(X;\mathbb{Q})$ is (countably) infinite dimensional over $\mathbb{Q}$.*

7.6. **Case $m = 1$ and $K3$ surfaces.** Recall the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z = r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D = r-1} K_1^M(\mathbb{C}(D)).$$

Then:

$$\mathrm{CH}^r(X,1) = H^{r-1}_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) \simeq \left\{ \frac{\sum_j (f_j, D_j) \ \big| \ \sum_j \mathrm{div}(f_j) = 0}{T\big(\Gamma(\bigoplus_{\mathrm{codim}_X Z = r-2} K_2^M(\mathbb{C}(Z)))\big)} \right\}.$$

We recall:

Definition 7.7. The subgroup of $\mathrm{CH}^r(X,1)$ in the image of

$$\mathbb{C}^\times \otimes \mathrm{CH}^{r-1}(X) = \mathrm{CH}^1(X,1) \otimes \mathrm{CH}^{r-1}(X,0) \xrightarrow{\cup} \mathrm{CH}^r(X,1),$$

is called the subgroup of decomposables $\mathrm{CH}^r_{\mathrm{dec}}(X,1) \subset \mathrm{CH}^r(X,1)$. The space of indecomposables is given by

$$\mathrm{CH}^r_{\mathrm{ind}}(X,1) := \frac{\mathrm{CH}^r(X,1)}{\mathrm{CH}^r_{\mathrm{dec}}(X,1)}.$$

Example 7.8. Suppose that X is a surface. Then we have

$$\mathrm{cl}_{2,1} : \mathrm{CH}^2_{\mathrm{hom}}(X,1) \to \frac{\{H^{2,0}(X) \oplus H^{1,1}(X)\}^\vee}{H_2(X,\mathbb{Z})}.$$

The corresponding transcendental regulator is defined to be

$$\Phi_{2,1} : \mathrm{CH}^2_{\mathrm{hom}}(X,1) \to \frac{H^{2,0}(X)^\vee}{H_2(X,\mathbb{Z})},$$

$$\Phi_{2,1}(\xi)(\omega) = \int_\zeta \omega.$$

and real regulator

$$r_{2,1} : \mathrm{CH}^2(X,1) \to H^{1,1}(X,\mathbb{R}(1))^\vee \simeq H^{1,1}(X,\mathbb{R}(1)),$$

$$r_{2,1}(\xi)(\omega) = \frac{1}{2\pi \mathrm{i}} \sum_j \int_{Z_j} \log|f_j|\omega.$$

There is an induced map

$$\underline{r}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to H^{1,1}_{\mathrm{tr}}(X,\mathbb{R}(1)).$$

If X is a $K3$ surface, then $\mathrm{CH}^2_{\mathrm{hom}}(X,1) = \mathrm{CH}^2(X,1)$, hence there is an induced map

$$\underline{\Phi}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to \frac{H^{2,0}(X)^\vee}{H_2(X,\mathbb{Z})}.$$

Theorem 7.9. (i) *(*[MS1],[C-L3],[P],[C-S]*) Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. If $d \le 3$, then $r_{2,1} : \mathrm{CH}^2(X,1) \to H^{1,1}(X,\mathbb{R}(1))$ is surjective; moreover $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) = 0$. Now assume that X is general. If $d \ge 5$, then $Im(r_{2,1})$ is "trivial", i.e. its image in the transcendental part of $H^{1,1}(X,\mathbb{R}(1))$ is zero.*

(ii) *[Hodge-$\mathcal{D}$-conjecture for $K3$ surfaces (*[C-L2]*)] Let X be a general member of a universal family of projective $K3$ surfaces, in the sense of the real analytic topology. Then*

$$r_{2,1} : CH^2(X,1) \otimes \mathbb{R} \to H^{1,1}(X,\mathbb{R}(1)),$$

is surjective.

(iii) *(*[CDKL]*) Let $X/\mathbb{C}$ be a general algebraic $K3$ surface. Then the transcendental regulator $\Phi_{2,1}$ is non-trivial. Quite generally, if X is a general member of a general subvariety of dimension $20-\ell$, describing a family of $K3$ surfaces with general member of Picard rank ℓ, with $\ell < 20$, then $\Phi_{2,1}$ is non-trivial.*

Remark 7.10. (i) Regarding part (iii) of Theorem 7.9, one can ask whether $\Phi_{2,1}$ can be non-trivial for those $K3$ surfaces X with Picard rank 20, (which are rigid and therefore defined over $\overline{\mathbb{Q}}$)? In [CDKL], some evidence is provided in support of this.

(ii) One of the key ingredients in the proof of the above theorem is the existence of plenty of nodal rational curves on a general $K3$ surface. Indeed, there is the following result:

Theorem 7.11 ([C-L1])**.** *For a general $K3$ surface, the union of rational curves on X is a dense subset in the analytic topology.*

Remark 7.12. It is well known that for an elliptic curve E defined over an algebraically closed subfield $k \subset \mathbb{C}$, the torsion subgroup $E_{\mathrm{tor}}(\mathbb{C}) \subset E(k)$. An analogous result holds for rational curves on a $K3$ surface. Quite generally, the following result which may be common knowledge among experts, seems worthwhile mentioning:

Proposition 7.13. *Assume given $X/\mathbb{C}$ a smooth projective surface with $\mathrm{p_g}(X) := \dim H^{2,0}(X) > 0$. If we write $X/\mathbb{C} = X_k \times_k \mathbb{C}$, viz., $X/\mathbb{C}$ obtained by base change from a smooth projective surface X_k defined over an algebraically closed subfield $k \subset \mathbb{C}$, and if $C \subset X/\mathbb{C}$ is a rational curve, then C is likewise defined over k.*

Proof. By a standard spread argument, there is a smooth projective variety S/k of dimension ≥ 0, and a k-family $\mathcal{C} \twoheadrightarrow S$ of rational curves containing C as a general member, with embedding h:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{h} & S \times_k X & \xrightarrow{Pr_X} & X \\ & {\scriptstyle Pr_S}\searrow & & \swarrow & \\ & & S & & \end{array}$$

Since $\mathrm{p_g}(X) > 0$, there are only at most a countable number of rational curves on $X/\mathbb{C}$, and hence $Pr_X(h(\mathcal{C})) = Pr_X(h(Pr_S^{-1}(t)))$ for any $t \in S(\mathbb{C})$. Now use the fact that $S(k) \neq \emptyset$. $\square$

Now suppose that X is a $K3$ surface defined over $\overline{\mathbb{Q}}$. Let $\Sigma \subset X$ be the union of all rational curves on X. Then Σ is defined over $\overline{\mathbb{Q}}$.

Question 7.14. Is $X(\overline{\mathbb{Q}}) \subset \Sigma(\overline{\mathbb{Q}})$?

An affirmative answer to this question would not only imply that Σ is dense in $X(\mathbb{C})$ in the usual topology, but this would also provide a nontrivial instance of the Bloch-Beilinson conjecture on the injectivity of Abel-Jacobi maps for smooth projective varieties defined over $\overline{\mathbb{Q}}$. More specifically, by an application of the connectedness part of Bertini's theorem, Σ is connected, hence $\mathrm{CH}^2_{\mathrm{hom}}(X/\overline{\mathbb{Q}}) = 0$.

7.15. **Torsion indecomposables.** The story about torsion indecomposable classes takes an interesting turn from the geometric story presented in Theorem 7.9(i). The situation is this, and for the moment let X be any projective algebraic manifold. An elementary consequence of the Merkurjev-Suslin theorem implies the following:

Theorem 7.16 (See [dJ-L])**.** *The kernel of the Abel-Jacobi map*

$$\underline{AJ}_X : \frac{CH^2_{\mathrm{hom}}(X,1)}{CH^2_{\mathrm{dec}}(X,1)} \to J\biggl(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbb{Z}(2))}\biggr),$$

is uniquely divisible. This implies that $\underline{AJ}_X$ is injective on torsion indecomposables $\left\{\frac{CH^2_{\mathrm{hom}}(X,1)}{CH^2_{\mathrm{dec}}(X,1)}\right\}_{\mathrm{tor}}$.

(Here we remind the reader that since we are working integrally, we have an inclusion that for torsion reasons, need not be an equality:

$$\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} \subseteq \frac{\mathrm{CH}^2(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} =: \mathrm{CH}^2_{\mathrm{ind}}(X,1).)$$

On the other hand, one has the torsion subgroup $\bigl\{\mathrm{CH}^2_{\mathrm{ind}}(X,1)\bigr\}_{\mathrm{tor}}$. Put

$$H^2_{\mathrm{tr}}(X,\mathbb{Q}(2)/\mathbb{Z}(2)) = \mathrm{Cokernel}\bigl(H^2_{\mathrm{alg}}(X,\mathbb{Q}(2)/\mathbb{Z}(2)) \to H^2(X,\mathbb{Q}(2)/\mathbb{Z}(2))\bigr).$$

Theorem 7.17 ([KaB])**.** *There is an identification*

$$\bigl\{CH^2_{\mathrm{ind}}(X,1)\bigr\}_{\mathrm{tor}} \xrightarrow{\sim} H^2_{\mathrm{tr}}(X,\mathbb{Q}(2)/\mathbb{Z}(2)).$$

In light of these two theorems, one expects that

$$\underline{AJ}_X : \left\{\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)}\right\}_{\mathrm{tor}} \xrightarrow{\sim} \left\{J\biggl(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbb{Z}(2))}\biggr)\right\}_{\mathrm{tor}}.$$

For example, suppose that X is a $K3$ surface of Picard rank 20. Then $E := J\biggl(\frac{H^2(X,\mathbb{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbb{Z}(2))}\biggr)$ is an elliptic curve defined over a number field. In this case one expects the identification

$$\bigl\{\mathrm{CH}^2_{\mathrm{ind}}(X,1)\bigr\}_{\mathrm{tor}} \xrightarrow{\sim} \bigl\{E(\overline{\mathbb{Q}})\bigr\}_{\mathrm{tor}}.$$

7.18. **Case $m = 2$ and elliptic curves.** *Regulator examples on* $\mathrm{CH}^2(X,2)$. Let X be a compact Riemann surface. In [Lew2] there is constructed a real regulator

$$r : \mathrm{CH}^2(X,2) \to H^1(X,\mathbb{R}(1)), \tag{17}$$

given by

$$\begin{aligned}\omega \in H^1(X,\mathbb{R}) \simeq H^1(X,\mathbb{R}(1))^\vee \mapsto \int_X \Big[\log|f| d\log|g| - \log|g| d\log|f|\Big]\wedge\omega \\ = 2\int_X \log|f| d\log|g| \wedge \omega, \ \text{(by a Stokes}' \text{ theorem argument)}.\end{aligned} \tag{18}$$

Alternatively, up to a twist, and real isomorphism, this is the same as the real part of the regulator $\mathrm{cl}_{2,2}$, viz.,

$$r_{2,2}(\omega) = \frac{1}{2\pi}\int_X \big[\log|f| d\arg g - \log|g| d\arg f\big)\big]\wedge\omega, \tag{19}$$

where the formula for:
(20)

$$\mathrm{cl}_{2,2} : \mathrm{CH}^2(X,2) \to H^2_{\mathcal{D}}(X,\mathbb{Z}(2)) \simeq H^1(X,\mathbb{C}/\mathbb{Z}(2)) \simeq \frac{H^1(X,\mathbb{C})}{H_1(X,\mathbb{Z}(-1))} = \frac{H^1(X,\mathbb{C})}{H_1(X,\mathbb{Z})(1)},$$

(for $\omega \in H^1(X,\mathbb{C})$), which can be found for example in [KLM] (or worked out using the results of §6), is induced, up to a factor[24] of $(2\pi\mathrm{i})^{-1}$, by:

$$\{f,g\} \quad \mapsto \tag{21}$$

$$\left[\int_{X\backslash f^{-1}[-\infty,0]} \log f d\log g\wedge\omega \quad -2\pi\mathrm{i}\int_{f^{-1}[-\infty,0]\backslash(f\times g)^{-1}[-\infty,0]^2} \log g\wedge\omega + (2\pi\mathrm{i})^2\int_\zeta \omega\right],$$

where if we assume for the moment that $T\{f,g\}) = 0$, then ζ is a real membrane with $\partial\zeta = (f\times g)^{-1}[-\infty,0]^2$. Otherwise if $T\{f,g\} \neq 0$, we are then dealing with a situation where $\{f,g\}$ is replaced by a given $\prod_\alpha\{f_\alpha,g_\alpha\}$, where

$$T\bigg(\prod_\alpha\{f_\alpha,g_\alpha\}\bigg) = \sum_\alpha T\{f_\alpha,g_\alpha\} = 0,$$

and accordingly arrive at a corresponding ζ. Note that (21) is really the current written in the slang form:

$$\big[\log f d\log g - 2\pi\mathrm{i}\log g\delta_{f^{-1}(\mathbb{R}^-)} + (2\pi\mathrm{i})^2\delta_\zeta\big] =: \tilde{R}.$$

To connect formulas (19) and (21), one takes the imaginary part of $\tilde{R}$ (consistent with $\mathbb{C}/\mathbb{Z}(2) \twoheadrightarrow \mathbb{C}/\mathbb{R}(2) \simeq \mathbb{R}(1)$). This gives us

$$\mathrm{Im}(\tilde{R}) = \big[\log|f| d\arg g + \arg f d\log|g| - 2\pi\log|g|\delta_{f^{-1}(\mathbb{R}^-)}\big].$$

Now add the coboundary current $d\big[\log|g|\arg f\big]$ and apply a Stokes' theorem argument.[25]

[24] The decision to consider the factor $(2\pi\mathrm{i})^{-1}$ is somewhat "political", as reflected in the remark on page 2 of [KLM]. From a cohomological point of view, one works with $\mathbb{Z}(2)$ coefficient periods, whereas homologically, is it with $\mathbb{Z}(1)$ coefficients. This is neatly illustrated via the Poincaré duality isomorphism in (20).

[25] Alternatively, taking $\mathrm{Re}\big((2\pi\mathrm{i})^{-1}\tilde{R}\big)$ gives the formula in (19), viz., with the factor $(2\pi)^{-1}$, right on the nose.

7.19. **Constructing $K_2(X)$ classes on elliptic curves X.** We consider the following trick due to Bloch ([Blo3]). Let X be an elliptic curve and assume given $f,\ g \in \mathbb{C}(X)^\times$ such that $\Sigma := |\mathrm{div}(f)| \cup |\mathrm{div}(g)|$ are points of order N in $\mathrm{Pic}(X)$. Then

$$T\big(\{f,g\}^N\big) \in \coprod \mathbb{C}^\times \quad \text{and} \ \mapsto 0 \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times.$$

A clarification. This uses the Weil reciprocity theorem. Let X be a compact Riemann surface, $f, g \in \mathbb{C}(X)^\times$, and for $p \in X$, write

$$T_p\{f,g\} = (-1)^{\nu_p(g)\nu_p(f)} \left(\frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}\right)\bigg|_p \in \mathbb{C}^\times.$$

Note that for $p \notin |\mathrm{div}(f)| \bigcup |\mathrm{div}(g)|$, we have $T_p\{f,g\} = 1$. Thus we can write $T\{f,g\} = \sum_{p\in X} T_p\{f,g\}$. Weil reciprocity says that $\prod_{p\in X} T_p\{f,g\} = 1$. Let us rewrite this as follows. If we write $T\{f,g\} = \sum_{j=1}^M (c_j, p_j)$, where $p_j \in X$ and $c_j \in \mathbb{C}^\times$, then $\prod_{j=1}^M c_j = 1$. Now fix $p \in X$ and let us suppose that $Np_j \sim_{\mathrm{rat}} Np$ for all j. Thus there exists $h_j \in \mathbb{C}(X)^\times$ such that $(h_j) = Np_j - Np$. Then $T\{h_j, c_j\} = (c_j^N, p) + (c_j^{-N}, p_j)$. The result is that

$$T\big(\{f,g\}^N\{h_1,c_1\}\cdots\{h_M,c_M\}\big) = \prod_{j=1}^M (c_j^N, p) = (1,p) = 0.$$

Thus there exists $\{h_i\} \in \mathbb{C}(X)^\times$ and $\{c_i\} \in \mathbb{C}^\times$ such that $\{f,g\}^N \prod\{h_i,c_i\} \in \mathrm{CH}^2(X,2)$. Note that the terms $\{h_i,c_i\}$ do not contribute to the regulator value by the formula in (18) above. Clearly this construction takes advantage of the existence of a dense subset of torsion points on X. Bloch (*op. cit.*) shows that the real regulator is nontrivial for general elliptic curves, and indeed A. Collino ([Co1]) shows that the regulator image of $\mathrm{CH}^2(X,2)$ for a general elliptic curve X is infinite dimensional (over $\mathbb{Q}$). Actually it is pretty easy to see why $r_{2,2}$ is non-trivial for a general elliptic curve:

Theorem 7.20 (Hodge-$\mathcal{D}$-conjecture for elliptic curves)**.** *If X is a general elliptic curve in the real analytic Zariski topology, then $r_{2,2}$ is surjective.*

Proof. Let X be an elliptic curve given in affine coordinates by the equation $y^2 = h(x)$, where $h(x)$ is a cubic polynomial with distinct roots. A basis for $H^1(X,\mathbb{R})$ is given by

$$\omega_1 := \frac{dx}{y} + \frac{d\overline{x}}{\overline{y}} \quad ; \quad \omega_2 := \mathrm{i}\left(\frac{dx}{y} - \frac{d\overline{x}}{\overline{y}}\right).$$

Next, we consider

$$f_1 := y + \mathrm{i}x \quad ; \quad f_2 = y + x \quad ; \quad g_1 = g_2 = x.$$

We claim that for general X,

$$\det \begin{bmatrix} \int_X \log|f_1| d\log|g_1| \wedge \omega_1 & \int_X \log|f_1| d\log|g_1| \wedge \omega_2 \\ \int_X \log|f_2| d\log|g_2| \wedge \omega_1 & \int_X \log|f_2| d\log|g_2| \wedge \omega_2 \end{bmatrix} \neq 0. \tag{22}$$

Now let us first assume that X is given for which (22) holds, and note that the rational functions f_1, f_2, g_1, g_2 can each be expressed in the form L_1/L_2, where L_j are homogeneous linear polynomials in the homogeneous coordinates of $\mathbb{P}^2$ (and where $X \subset \mathbb{P}^2$). Since X has a dense subset of torsion points X_{tor}, and by Abel's theorem, one can find $\tilde{L}_j$ "close" to L_j, $j = 1, 2$, such that $\tilde{L}_j \cap X \subset X_{\rm tor}$. Thus up to $\mathbb{C}^\times$ multiple, $\tilde{L}_1/\tilde{L}_2$ is "close" to L_1/L_2. Hence one can find $\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2$ for which

$$\left\{\big|\mathrm{div}(\tilde{f}_1)\big|\bigcup\big|\mathrm{div}(\tilde{f}_2)\big|\bigcup\big|\mathrm{div}(\tilde{g}_1)\big|\bigcup\big|\mathrm{div}(\tilde{g}_2)\big|\right\} \subset X_{\rm tor}, \tag{23}$$

and that by continuity considerations,

$$\det\begin{bmatrix}\int_X \log|\tilde{f}_1| d\log|\tilde{g}_1|\wedge\omega_1 & \int_X \log|\tilde{f}_1| d\log|\tilde{g}_1|\wedge\omega_2 \\ \\ \int_X \log|\tilde{f}_2| d\log|\tilde{g}_2|\wedge\omega_1 & \int_X \log|\tilde{f}_2| d\log|\tilde{g}_2|\wedge\omega_2\end{bmatrix} \neq 0. \tag{24}$$

Thus one can complete $\{\tilde{f}_1, \tilde{g}_1\}$, $\{\tilde{f}_2, \tilde{g}_2\}$ to classes $\xi_1,\ \xi_2 \in \mathrm{CH}^2(X, 2)$, for which

$$\det\begin{bmatrix} r_{2,2}(\xi_1)(\omega_1) & r_{2,2}(\xi_1)(\omega_2) \\ \\ r_{2,2}(\xi_2)(\omega_1) & r_{2,2}(\xi_2)(\omega_2)\end{bmatrix} \neq 0, \tag{25}$$

and so modulo the claim in (22), we are done. We sketch a proof of the claim. With regard to $dV = d\mathrm{Re}(x)\wedge d\mathrm{Im}(x)$:

$$\frac{d\log|x|\wedge\omega_1}{2} = \frac{1}{4}\left(\frac{1}{x\overline{y}} - \frac{1}{\overline{x}y}\right)dx\wedge d\overline{x} = \frac{\mathrm{Im}(\overline{x}y)}{|x|^2|y|^2}dV \tag{26}$$

$$\frac{d\log|x|\wedge\omega_2}{2} = -\frac{\mathrm{i}}{4}\left(\frac{1}{x\overline{y}} + \frac{1}{\overline{x}y}\right)dx\wedge d\overline{x} = -\frac{\mathrm{Re}(\overline{x}y)}{|x|^2|y|^2}dV \tag{27}$$

Now let us degenerate X to the rational elliptic curve X_0 given by $y^2 = x^3$. Note that X_0 is given parametrically by $(x, y) = (z^2, z^3)$, $z \in \mathbb{C}$. Thus $\overline{x}y = |z|^4 z$, and up to a real positive multiplicative constant times the standard volume element on $\mathbb{C}$, which we will denote by dV_0, (26) and (27) become:

$$d\log|x|\wedge\omega_1 = \frac{\mathrm{Im}(z)}{|z|^4}dV_0 \quad ; \quad d\log|x|\wedge\omega_2 = -\frac{\mathrm{Re}(z)}{|z|^4}dV_0. \tag{28}$$

Let $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ be the upper half plane. Now one has the following <u>formal</u> calculations after degenerating to X_0, and using symmetry arguments:

$$\int_{X_0} \log|f_1| d\log|g_1|\wedge\omega_1 = \int_{\mathbb{C}} \log|z^3 + \mathrm{i}z^2|\frac{\mathrm{Im}(z)}{|z|^4}dV_0 \tag{29}$$

$$= \int_{\mathbb{C}} \log|z+\mathrm{i}|\frac{\mathrm{Im}(z)}{|z|^4}dV_0 = \int_{\mathbf{H}} \log\left|\frac{z+\mathrm{i}}{\overline{z}+\mathrm{i}}\right|\frac{\mathrm{Im}(z)}{|z|^4}dV_0 \mapsto +\infty,$$

using the fact

$$\left|\frac{z+\mathrm{i}}{\overline{z}+\mathrm{i}}\right| > 1 \Leftrightarrow\ \mathrm{Im}(z) > 0.$$

$$\int_{X_0} \log|f_2| d\log|g_2|\wedge\omega_1 = \int_{\mathbb{C}} \log|z+1|\frac{\mathrm{Im}(z)}{|z|^4}dV_0 = 0. \tag{30}$$

For the remaining two formal calculations, put $w = \mathrm{i}z$, and note that $\mathrm{Re}(z) = \mathrm{Im}(w)$, and that $|z+1| = |w+\mathrm{i}|$. Then

$$\int_{X_0} \log|f_2| d\log|g_2|\wedge\omega_2 = -\int_{\mathbb{C}} \log|z+1|\frac{\mathrm{Re}(z)}{|z|^4}dV_0 \tag{31}$$

$$= -\int_{\mathbb{C}} \log|w+\mathrm{i}|\frac{\mathrm{Im}(w)}{|w|^4}dV_0 = -\int_{\mathbf{H}} \log\left|\frac{z+\mathrm{i}}{\overline{z}+\mathrm{i}}\right|\frac{\mathrm{Im}(z)}{|z|^4}dV_0 \mapsto -\infty.$$

$$\int_{X_0} \log|f_1| d\log|g_1| \wedge \omega_2 = -\int_{\mathbb{C}} \log|z+\mathrm{i}|\frac{\mathrm{Re}(z)}{|z|^4}dV_0 = 0. \tag{32}$$

Note that two of these integrals blow up over the singular point $z = 0$ of the singular curve X_0, as expected. By using the Lebesgue theory of integration, we can make the calculations in (29)–(32) more precise. First, by using the projection $(x, y) \mapsto x$, we have a double covering $X \to \mathbb{P}^1$. Thus for $f, g \in \mathbb{C}(X)$, and $\omega = \omega_1$ or $\omega = \omega_2$, we can express $\int_X \log|f| d\log|g| \wedge \omega$ as the integral of some Lebesgue integrable function $H(x)$ over $\mathbb{P}^1$. Next, by converting to polar coordinates, viz. $x = \mathrm{e}^{\mathrm{i}t}$, we can Fubini integrate in $t \in [0, 2\pi]$ and $r \in [0, \infty]$. Let $h(r)$ be the result of integrating $H(x)$ with respect to t over $[0, 2\pi]$. As X degenerates to X_0, we can construct a sequence $\{h_n(r)\}$ which limits to $h_\infty(r)$ over X_0. In the cases of (29)–(32), we have that $h_\infty(r)$ is either zero, nonnegative, or nonpositive. By using the the standard Lebesgue integral limit theorems, we arrive at the claim in (22), and hence the theorem. □

For curves X of genus $g > 1$, the problem of constructing classes in $\mathrm{CH}^2(X, 2)$ seems to be related to the fact that under the Abel-Jacobi mapping $\Phi : X \to J^1(X), \quad p \mapsto \{p - p_0\}$, the inverse image of the torsion subgroup, $\Phi^{-1}(J^1(X)_{\mathrm{tor}})$, is finite, this being the import of the Mumford-Manin theorem (see [Ra] for a proof). Indeed as explained in [Lew2] (as well as in [Lew3]), one can prove a weak version of the Mumford-Manin theorem based on the fact that for a general curve X of genus $g > 1$, the image of the regulator map $\mathrm{cl}_{2,2} : \mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ is torsion (A. Collino [Co1]). Collino's approach (*op. cit.*) uses infinitesimal methods. The reader should also consult [GG] for similar refined results in this direction. For the benefit of the reader, we will provide an ad hoc explanation as to why this is the case (in "Observation 1" below). In order to do so, we must first digress and consider the following setting.

Assume given a dominant morphism $\overline{\rho} : \overline{X} \to \overline{C}$ of smooth complex projective varieties, where $\overline{X}$ is a surface and $\overline{C}$ is a curve. Let $C \subset \overline{C}$ be an affine open subset over which $\overline{\rho}$ is smooth, and $\Sigma := \overline{C} \backslash C$, $X = \overline{\rho}^{-1}(C)$ and $\rho = \overline{\rho}\big|_X : X \to C$. For $t \in \Sigma$, we will assume that the singular set of X_t is a single node. Next, we will assume given a class $\{\xi\} \in \mathrm{CH}^2(X, 2)$. In particular $\partial\xi = 0$ on X (here ∂ is the same thing as the Tame symbol). Note that ξ is is given by a product of symbols of the form $\{f, g\}$, where $f, g \in \mathbb{C}(X)^\times$. However, since $\mathbb{C}(X) = \mathbb{C}(\overline{X})$, one can also think of ξ as defined on $\overline{X}$ (call it $\overline{\xi}$) with $\partial\overline{\xi}$ supported on $\overline{X}_\Sigma := \overline{\rho}^{-1}(\Sigma)$. Now for $t \in \Sigma$, the contribution ("residue") of $\partial\overline{\xi}$ gives rise to a class in $\mathrm{CH}^1(X_t, 1)$. If X_t were smooth, then $\mathrm{CH}^1(X_t, 1) = \mathbb{C}^\times$; but here we are assuming that X_t has a single node $P \in X_t$ as singularity. Under the desingularization $\sigma : \widetilde{X}_t \to X_t$, let $\{Q, R\} = \sigma^{-1}(P)$. Next if $Q - R \in \mathrm{CH}^1_{\mathrm{tor}}(\widetilde{X}_t)$, then for some integer N, $N \cdot (Q - R) = \mathrm{div}(f)$ for some $f \in \mathbb{C}(\widetilde{X}_t)^\times$. But on X_t, $\mathrm{div}(f) = 0$, and hence $\mathbb{C}^\times \subsetneq \mathrm{CH}^1(X_t, 1)$. The upshot is that if $\partial\overline{\xi}$ contributes to a nonzero element of $\mathrm{CH}^1(X_t, 1)/\mathbb{C}^\times$ for some $t \in \Sigma$, then via a residue calculation and a calculation of the MHS $H^2(X, \mathbb{Q}(2))$, the current $d\log\xi$ (induced by $\{f, g\} \mapsto d\log f \wedge d\log g$) will contribute to a nonzero class in $[d\log\xi] \in \Gamma H^2(X, \mathbb{Q}(2))$. *The converse statement also holds:* if $Q - R \notin \mathrm{CH}^1_{\mathrm{tor}}(\widetilde{X}_t)$, for all such $t \in \Sigma$, then $[d\log\xi] = 0 \in \Gamma H^2(X, \mathbb{Q}(2))$. Next, the Leray spectral sequence associated to ρ (which by Deligne, degenerates at E_2, see [GH] (p. 466)), together with the fact that since C is an affine curve (hence $H^2(C, R^0\rho_*\mathbb{Q}(2)) = 0$), yields the short exact sequence of MHS:

$$0 \to H^1(C, R^1\rho_*\mathbb{Q}(2)) \to H^2(X, \mathbb{Q}(2)) \to H^0(C, R^2\rho_*\mathbb{Q}(2)) \to 0.$$

Note that $\Gamma H^0(C, R^2\rho_*\mathbb{Q}(2)) = 0$ as $H^0(C, R^2\rho_*\mathbb{Q}(2))$ is of pure weight -2. Hence $\Gamma H^2(X, \mathbb{Q}(2)) = \Gamma H^1(C, R^1\rho_*\mathbb{Q}(2))$. On the other hand, for $t \in C$, ξ restricts to a class $\xi_t \in \mathrm{CH}^2(X_t, 2)$, and hence we have a normal function $\nu_\xi : C \to \coprod_{t\in C} J\big(H^1(X_t, \mathbb{Z}(2))\big)$, whose topological invariant is the aforementioned class $[d\log\xi] \in \Gamma H^1(C, R^1\rho_*\mathbb{Q}(2))$, and which we will now denote it by $\delta(\nu_\xi) := [d\log\xi]$. It is a general fact that there is a short exact sequence:

$$(33) \qquad 0 \to J\big(H^0(C, R^1\rho_*\mathbb{Q}(2))\big) \to \left\{\begin{matrix}\text{Normal}\\ \text{functions}\end{matrix}\right\}_{\mathbb{Q}} \xrightarrow{\delta} \Gamma H^1(C, R^1\rho_*\mathbb{Q}(2)) \to 0,$$

where $\{\cdots\}_{\mathbb{Q}}$ means with respect to $\mathbb{Q}$-periods. We will explain this in more detail below, but comment in passing that the technical details can be found in [K-L]. If $\delta(\nu_\xi) = 0$, then $\nu_\xi \in J\big(H^0(C, R^1\rho_*\mathbb{Q}(2))\big)$, i.e. belongs to the fixed part of a corresponding variation of Hodge structure. The situation is not unlike what occurs in the short exact sequence involving Deligne cohomology, and the nature of this argument is completely analogous to that in Example 7.3. We can frame this discussion in more precise terms. One has a cycle class map $\mathrm{cl}_{2,2} : \mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$, (Deligne-Beilinson cohomology); moreover by a weight argument, there is a short exact sequence:

$$(34) \qquad 0 \to J\big(H^1(X, \mathbb{Z}(2))\big) \to H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \to \Gamma H^2(X, \mathbb{Z}(2)) \to 0.$$

For $t \in C$, X_t is a smooth curve. Then for such t, $H^2_{\mathcal{D}}(X_t, \mathbb{Z}(2)) = J\big(H^1(X_t, \mathbb{Z}(2))\big)$, and accordingly the map

$$t \in C \mapsto \mathrm{cl}_{2,2}(\xi_t) \in J\big(H^1(X_t, \mathbb{Z}(2))\big),$$

is our normal function ν_ξ; moreover the image of ξ via the composite

$$\mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbb{Q}(2)) \to \Gamma H^2(X, \mathbb{Q}(2)) = \Gamma H^1(C, R^1\rho_*\mathbb{Q}(2)),$$

is precisely $\delta(\nu_\xi)$. Finally to explain (33) more precisely, we observe that there is a short exact sequence:

$$0 \to H^1(C, R^0\rho_*\mathbb{Q}(2))) \to H^1(X, \mathbb{Q}(2)) \to H^0(C, R^1\rho_*\mathbb{Q}(2)) \to 0.$$

But $\Gamma H^0(C, R^1\rho_*\mathbb{Q}(2)) = 0$, hence we arrive at the short exact sequence:

$$0 \to J\big(H^1(C, R^0\rho_*\mathbb{Q}(2)))\big) \to J\big(H^1(X, \mathbb{Q}(2))\big) \to J\big(H^0(C, R^1\rho_*\mathbb{Q}(2))\big) \to 0.$$

This together with (33) and $(34)_{\mathbb{Q}}$ leads to the identification:

$$\left\{\begin{matrix}\text{Normal}\\ \text{functions}\end{matrix}\right\}_{\mathbb{Q}} \simeq \frac{H^2_{\mathcal{D}}(X, \mathbb{Q}(2))}{J\big(H^1(C, R^0\rho_*\mathbb{Q}(2)))\big)}.$$

Now having discussed the relationship between a cycle class $\xi \in \mathrm{CH}^2(X, 2)$, the associated normal function ν_ξ, and the topological invariant $\delta(\nu_\xi) \in \Gamma H^2(X, \mathbb{Q}(2))$ and how it is related to the "torsion" nature of the nodal singularities of the singular fibers $\{X_t\}_{t\in\Sigma}$, we are led to consider two divergent observations:

Observation 1. Suppose that X_0 is a general curve of genus $g > 1$. By general, we can assume that X_0 is a very general member of a pencil of curves $\{X_t\}_{t\in\mathbb{P}^1}$, defining a smooth surface $\overline{X}_{\mathbb{P}^1} := \coprod_{t\in\mathbb{P}^1} X_t \to \mathbb{P}^1$, whose singular fibers are Lefschetz, i.e.

admit a single ordinary node. Let $\xi_0 \in \mathrm{CH}^2(X_0, 2)$. After a suitable base extension $\overline{C} \to \mathbb{P}^1$, for some smooth projective curve $\overline{C}$, and corresponding $\overline{X} := \overline{C} \times_{\mathbb{P}^1} \overline{X}_{\mathbb{P}^1}$, with setting as in the above discussion, ξ_0 will then spread to a class $\xi \in \mathrm{CH}^2(X, 2)$, in a general family $\rho : X \to C$, where ρ is smooth and proper over an affine curve C. Granted that the singular fibers over $\Sigma \subset \overline{C}$ are not necessarily nodes (as $\overline{C} \to \mathbb{P}^1$ may ramify over the singular points), a similar line of reasoning as the nodal situation will occur, based on a parallel situation encountered in [C-L3]. So for simplicity, let us assume that for each $t \in \Sigma$, that X_t is Lefschetz. Since $g(X_t) \geq 2$ for $t \in C$, it follows that for $t \in \Sigma$, $g(X_t) \geq 1$.

Proposition 7.21. *If X_0 is sufficiently general, then one can arrange for the following to hold:*

(i) $H^0(C, R^1\rho_*\mathbb{Q}(2)) = 0$.

(ii) *For every $t \in \Sigma$, the corresponding $Q - R$ is nontorsion in $CH^1(\widetilde{X}_t)$.*

Proof. Although we won't prove this, it goes without mentioning that (i) is a standard result in the deformation theory of curves and corresponding VHS. For (ii), one considers via deformation, a family of nodal curves of genus at least 1, together with an argument of Baire type using the fact that the torsion points on a curve of genus $g \geq 1$ is at most countable. $\square$

It follows that such a ξ would define a normal function for which $\delta(\nu_\xi) = 0 \in \Gamma H^1(C, R^1\rho_*\mathbb{Q}(2))$, and so $\nu_\xi \in J\big(H^0(C, R^1\rho_*\mathbb{Q}(2))\big) = 0$. This leads to $\mathrm{cl}_{2,2}(\xi_t) = 0 \in H^2_{\mathcal{D}}(X_t, \mathbb{Q}(2))$ for very general $t \in C$, and hence $\mathrm{cl}_{2,2}(\xi_0)$ is torsion as a class in $H^2_{\mathcal{D}}(X_0, \mathbb{Z}(2))$.

Observation 2. Consider an elliptic surface $\overline{\rho} : \overline{X} \to \overline{C}$. The singular fibers X_Σ are unions of rational curves. If for some $t \in \Sigma$, X_t is nodal with node $P \in X_t$, then on $\widetilde{X}_t$, $Q - R \sim_{\mathrm{rat}} 0$, hence $\mathrm{CH}^1(X_t, 1)/\mathbb{C}^\times \neq 0$, and the possibility of a class $\xi \in \mathrm{CH}^2(X, 2)$, with nontrivial value $[d\log\xi] \in \Gamma H^2(X, \mathbb{Q}(2))$ arises. Assuming this is the case, then ν_ξ is nontrivial, and hence for general X_t, $\mathrm{cl}_{2,2}(\xi_t)$ is a nontorsion class (using a Baire category argument). This will be illustrated in Theorem 7.23 below, but as a preliminary warm-up, consider the nodal curve $D = \overline{V(y^2 - x^3 - x^2)} \subset \mathbb{P}^2$, with singular point $P = (0,0)$. By making the substitution $(x, y) = (x, ux)$, we end up with the desingularization $\sigma : \tilde{D} := \overline{V(u^2 = x + 1)} \to D$, and where $\sigma^{-1}(P) = \{Q = (0,1), R = (0,-1)\}$ in (x,u)-coordinates. Let

$$f = \frac{u+1}{u-1} = \frac{y+x}{y-x}.$$

Then viewing $f \in \mathbb{C}(\tilde{D})$, $\mathrm{div}_{\tilde{D}}(f) = R - Q$, and viewing $f \in \mathbb{C}(D) = \mathbb{C}(\tilde{D})$, $\mathrm{div}_D(f) = 0$. We apply this to the following.

Example 7.22. Let $\pi : X \to \mathbb{P}^1$ be the elliptic surface defined by

$$y^2 = x^3 + x^2 + t =: h(x),$$

and let $\Sigma \subset \mathbb{P}^1$ be the singular set of π. One shows that

$$\Sigma = \left\{0, \infty, \frac{-4}{27}\right\},$$

furthermore X_0, $X_{\frac{-4}{27}}$ are nodal curves, and X_∞ is a simply-connected tree of $\mathbb{P}^1$'s. We then have:

Theorem 7.23. *Let* $U = X\backslash\big\{X_0, X_{\frac{-4}{27}}, X_\infty\big\}$. *Then*

$$\Gamma\big(H^2(U, \mathbb{Q}(2))\big) \simeq \mathbb{Q}^2;$$

moreover it is generated by $[d\log(\xi_1)], [d\log(\xi_2)]$, *where*

$$\xi_1 = \left\{\frac{(y-x)^3}{8}, \frac{(y+x)^3}{8}\right\}\left\{\frac{y+x}{y-x}, t\right\}^3,$$

$$\xi_2 = \left\{\frac{(\mathrm{i}y+x+2/3)^3}{8}, \frac{(\mathrm{i}y-x-2/3)^3}{8}\right\}\left\{\frac{\mathrm{i}y+x+2/3}{\mathrm{i}y-x-2/3}, -t-4/27\right\}^3,$$

are classes in $CH^2(U, 2; \mathbb{Q})$.[26]

Now choose a class $\xi \in \mathrm{CH}^2(U, 2)$ such that $[d\log\xi] \neq 0 \in \Gamma\big(H^2(U, \mathbb{Q}(2))\big)$. Thus for general $t \in \mathbb{P}^1$, $\mathrm{cl}_{2,2}(\xi_t)$ is nontorsion.

Remark 7.24. The contents of this remark are taken from [A-L]. There is another way of showing that for general $t \in \mathbb{P}^1$, $\mathrm{cl}_{2,2}(\xi_t)$ is nontorsion. It is based on an idea that goes back to Collino and Bloch [Co1], and it has also been exploited by Kerr [GGK]. Take a small disk Δ centred at $t = 0$, and let's use say $\xi := \xi_2$. Notice that ξ pullback to X_t for any $t \in \Delta$, as X_t is a local complete intersection. Now for $t \in \Delta^*$, $\xi_t \in \mathrm{CH}^2(X_t, 2)$ and evidently there is an induced class $\xi_0 \in \mathrm{CH}^2(\mathrm{Spec}(\mathbb{C}), 3)$. Firstly, $X_0 = V(y^2 = x^3 - x^2)$ is a nodal rational curve with parameterization, induced by $u \in \mathbb{C} \mapsto (x, y) := (u^2 - 1, u(u^2 - 1))$. Consider

$$\omega_0 = -2\frac{dx}{y} = \frac{-4du}{u^2-1}.$$

Let us consider a change of coordinates:

$$u = \frac{z-1}{z+1},\ du = \frac{2dz}{(z+1)^2},\ u^2 - 1 = \frac{-4z}{(z+1)^2},\ \omega_0 = \frac{dz}{z}.$$

Thus for an obvious choice of ω_t,

$$\lim_{t\to 0} \omega_t = \omega_0.$$

Over the node $u = \pm 1 \mapsto (0, 0)$, lies the corresponding singularities of ω_0, viz., $z \in \{0, \infty\} \subset \square^1$. Note that ξ_0 is given by a parametric curve in $\mathrm{Spec}(\mathbb{C}) \times \square^3$, parameterized by $z \in \mathbb{P}^1 \xrightarrow{\approx} X_0$. Indeed we have the following:

Proposition 7.25. *As a class in* $CH^2(\mathrm{Spec}(\mathbb{C}), 3)$, *we have*

$$\xi_0 = \left\{\frac{x+y}{y-x}, \frac{(\mathrm{i}y+x+2/3)^3}{8}, \frac{(\mathrm{i}y-x-2/3)^3}{8}\right\}\left\{\frac{x+y}{y-x}, \frac{\mathrm{i}y+x+2/3}{\mathrm{i}y-x-2/3}, -4/27\right\}^3,$$

$$= \left\{\frac{x+y}{y-x}, \frac{(\mathrm{i}y+x+2/3)}{2}, \frac{(\mathrm{i}y-x-2/3)}{2}\right\}^9\left\{\frac{x+y}{y-x}, \frac{\mathrm{i}y+x+2/3}{\mathrm{i}y-x-2/3}, -4/27\right\}^3.$$

Now using the explicit description of ω_0, one can show the following:

Proposition 7.26.

$$\lim_{t\to 0} \mathrm{cl}_{2,2}(\xi_t) = \mathrm{cl}_{2,3}(\xi_0) \in H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{C}), \mathbb{Q}(2)) \simeq \mathbb{C}/\mathbb{Q}(2),$$

is non-torsion. In particular, it takes a non-zero value in $\mathrm{i}\mathbb{R}$.

---

[26] M. Asakura informed me of his work in [A], which includes this theorem as a special case. Further he provides an upper bound for the rank of the $d\log$ image for variants of the family in Example 7.22.

7.27. **Appendix: Horizontality issues of cycle induced normal functions.** Let $\rho : \mathcal{X} \to \mathcal{S}$ be a smooth and proper family of quasi-projective varieties defined over a subfield $\mathbb{C}$. Consider a normal function, which can be regarded as a holomorphic cross-section

$$\nu : \mathcal{S} \to \mathcal{O}_{\mathcal{S}}\Big(\coprod_{t\in\mathcal{S}} J(H^{2r-m-1}(\mathcal{X}_t, \mathbb{Q}(r)))\Big),$$

where the latter term is a complex manifold. In terms of sheaves, define the sheaf of germs of normal functions $\mathcal{I}$ by the short exact sequence

$$0 \to R^{2r-m-1}\rho_*\mathbb{Z}(r) \to \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r}_{\mathcal{X}/\mathcal{S}} \to \mathcal{I} \to 0.$$

Thus $\nu \in \Gamma(\mathcal{I})$. Note that

$$\mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r}_{\mathcal{X}/\mathcal{S}} = \frac{\mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet}_{\mathcal{X}/\mathcal{S}})}{\mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet\geq r}_{\mathcal{X}/\mathcal{S}})}.$$

Further, note that $\nabla\Gamma(R^{2r-m-1}\rho_*\mathbb{C}) = 0$, hence by Griffiths transversality, there is an induced map

$$\nabla_J : \Gamma(\mathcal{I}) \to \Gamma\big(\Omega^1_{\mathcal{S}} \otimes \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r-1}_{\mathcal{X}}\big).$$

Definition 7.28. ν is said to be horizontal if $\nabla_J\nu = 0$.

Here is an efficient proof of horizontality in the event that ν arises from a higher algebraic cycle. Let $S \subset \mathcal{S}$ be a polydisk, and correspondingly $\mathcal{X}/S$ is a smooth projective family, and recall the analytic Deligne complex $0 \to \mathbb{Z}(r) \to \Omega^{\bullet<r}_{\mathcal{X}}$, which leads to an exact sequence $\mathbb{H}^{2r-m-1}(\Omega^{\bullet<r}_{\mathcal{X}}) \to H^{2r-m}_{\mathcal{D}}(\mathcal{X}, \mathbb{Z}(r)) \to H^{2r-m}(\mathcal{X}, \mathbb{Z}(r))$. We consider a relatively null-homologous cycle on $\mathrm{CH}^r(\mathcal{X}/S, m)$, which will map to zero in $H^{2r-m}(\mathcal{X}, \mathbb{Z}(r))$ (as S is a polydisk). Hence the induced normal function has a lift in $\mathbb{H}^{2r-m-1}(\Omega^{\bullet<r}_{\mathcal{X}})$, which is all we shall need. The Leray spectral sequence for $\mathcal{X}/S$ gives us an edge map $\mathbb{H}^{2r-m-1}(\Omega^{\bullet<r}_{\mathcal{X}}) \to H^0(S, \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r}_{\mathcal{X}})$. One has a filtering of the complex $\mathcal{L}^\nu\Omega^{\bullet<r}_{\mathcal{X}} :=$ Image $\big(\rho^*\Omega^\nu_S \otimes \Omega^{\bullet<r-\nu}_{\mathcal{X}} \to \Omega^{\bullet<r}_{\mathcal{X}}\big)$, with $Gr^\nu_{\mathcal{L}} = \rho^*\Omega^\nu_S \otimes \Omega^{\bullet<r-\nu}_{\mathcal{X}/S} \simeq \Omega^\nu_S \otimes \Omega^{\bullet<r-\nu}_{\mathcal{X}/S}$. There is a spectral sequence computing $\mathbb{R}^{p+q}\rho_*\Omega^{\bullet<r}_{\mathcal{X}}$ with $\mathcal{E}^{p,q}_1 = \mathbb{R}^{p+q}Gr^p_{\mathcal{L}} = \Omega^p_S \otimes \mathbb{R}^q\rho_*\Omega^{\bullet<r-p}_{\mathcal{X}/S}$. So we have the composite

$$H^0(S, \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r}_{\mathcal{X}}) \to H^0(S, \mathcal{E}^{0,2r-m-1}_1) \xrightarrow{d_1} H^0(S, \mathcal{E}^{1,2r-m-1}_1),$$

which must be zero by spectral sequence degeneration, using the fact that

$$\mathcal{E}^{0,2r-m-1}_\infty \subset \ker\big(d_1 : \mathcal{E}^{0,2r-m-1}_1 \to \mathcal{E}^{1,2r-m-1}_1\big).$$

But $H^0(S, \mathcal{E}^{0,2r-m-1}_1) \xrightarrow{d_1} H^0(S, \mathcal{E}^{1,2r-m-1}_1)$, is precisely the Gauss-Manin connection

$$H^0(S, \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r}_{\mathcal{X}/S}) \xrightarrow{\nabla} H^0(S, \Omega^1_S \otimes \mathbb{R}^{2r-m-1}\rho_*\Omega^{\bullet<r-1}_{\mathcal{X}/S}).$$

8. **Arithmetic normal functions**

8.1. **Bloch-Beilinson filtration.** Consider fields $k \subset K \subset \mathbb{C}$, where K/k is finitely generated. We consider the filtration constructed in [Lew5] in the case $m = 0$, and more generally for $m \geq 0$ in [A]. We recall:

Theorem 8.2. *Let X/K be smooth projective of dimension d. Then for all r, there is a filtration, depending on $k \subset K$,*

$$CH^r(X_K, m; \mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \\ \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots,$$

which satisfies the following

(i) $F^1 = CH^r_{\mathrm{hom}}(X_K, m; \mathbb{Q})$.

(ii) $F^2 \subset \ker AJ \otimes \mathbb{Q} : CH^r_{\mathrm{hom}}(X_K; , m; \mathbb{Q}) \to J\big(H^{2r-m-1}(X_K(\mathbb{C}), \mathbb{Q}(r))\big)$.

(iii)[27] $F^{\nu_1} CH^{r_1}(X, m_1; \mathbb{Q}) \bullet F^{\nu_2} CH^{r_2}(X, m_2; \mathbb{Q}) \subset F^{\nu_1+\nu_2} CH^{r_1+r_2}(X, m_1 + m_2; \mathbb{Q})$, *where $\bullet$ is the intersection product.*

(iv) *F^ν is preserved under the action of correspondences between smooth projective varieties over K.*

(v) *Let $\mathrm{Gr}^\nu_F := F^\nu / F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d} [\Delta_X(p, q)] \in H^{2d}(X \times X, \mathbb{Q}(d)))$ are algebraic and defined over K. Then*

$$\Delta_X(2d - 2r + \ell + m, 2r - \ell - m)_*\big|_{\mathrm{Gr}^\nu_F CH^r(X,m;\mathbb{Q})} = \delta_{\ell,\nu} \cdot \mathrm{Identity}.$$

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that Gr^ν_F factors through the Grothendieck motive.]

(vi) *Let $D^r(X) := \bigcap_\nu F^\nu$, and $k = \overline{\mathbb{Q}}$. If the Bloch-Beilinson conjecture on the injectivity of the Abel-Jacobi map ($\otimes\mathbb{Q}$) holds for smooth quasi-projective varieties defined over $\overline{\mathbb{Q}}$ (see §4 of* [K-L]*), then $D^r(X) = 0$.*

It is instructive to briefly explain how this filtration comes about. For X/K smooth projective, one can find $\mathcal{S}/k$ such that $k(\mathcal{S})$ is identified with K. One can then spread out X/K to a family $\rho : \mathcal{X} \to \mathcal{S}$ (called a k-spread), where ρ is a smooth and proper morphism of smooth quasiprojective varieties over k. Let η be the generic point of $\mathcal{S}$, (hence $k(\mathcal{S}) = k(\eta)$ and K corresponds to an embedding $k(\eta) \overset{\approx}{\hookrightarrow} K \subset \mathbb{C}$), with X_K identified with the generic fiber of ρ. Via M. Saito's notion of polarizable mixed Hodge modules (see [A], as well as §4 of [K-L]), there is a decreasing filtration $\mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X}/k, m; \mathbb{Q})$, with the property that $\mathrm{Gr}^\nu_{\mathcal{F}} \mathrm{CH}^r(\mathcal{X}/k, m; \mathbb{Q}) \hookrightarrow E^{\nu, 2r-\nu-m}_\infty(\rho)$, where $E^{\nu,2r-\nu-m}_\infty(\rho)$ is the ν-th graded piece of a Leray filtration associated to ρ. More specifically, there is a cycle class map

$$\mathrm{CH}^r(\mathcal{X}, m; \mathbb{Q}) \to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\mathcal{X})}(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)),$$

where $\mathrm{MHM}(\mathcal{X})$ is the category of polarizable mixed Hodge modules on $\mathcal{X}$. There is the Leray spectral sequence associated to ρ:

$$E^{p,q}_2 = \mathrm{Ext}^p_{\mathrm{MHM}(\mathcal{S})}(\mathbb{Q}_{\mathcal{S}}(0), R^q\rho_*\mathbb{Q}_{\mathcal{X}}(r)) \Rightarrow \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\mathcal{X})}(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)), \\ p + q = 2r - m,$$

[27]Not specifically stated in [A], although true for the higher Chow groups.

which degenerates at E_2 since ρ is smooth and proper. Then we put

$$E_\infty^{\nu,2r-\nu-m}(\rho) := E_2^{\nu,2r-\nu-m} = \mathrm{Ext}^\nu_{\mathrm{MHM}(\mathcal{S})}(\mathbb{Q}_{\mathcal{S}}(0), R^{2r-\nu-m}\rho_*\mathbb{Q}_{\mathfrak{X}}(r)).$$

Applying the Leray spectral sequence to the structure map $\mathcal{S} \to \mathrm{Spec}(k)$, together with $\mathrm{MHM}(\mathrm{Spec}(k)) = \mathrm{MHS}$, we see that the term $E_\infty^{\nu,2r-\nu-m}(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu-m}(\rho) \to E_\infty^{\nu,2r-\nu-m}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu-m}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu-m}(\rho) = \Gamma\big(H^\nu(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big), \tag{35}$$

$$\underline{E}_\infty^{\nu,2r-\nu-m}(\rho) = \frac{J\big(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)}{\Gamma\big(Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)}$$
$$\subset J\big(H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big),$$

where the latter inclusion is a result of the short exact sequence:

$$W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))$$
$$\twoheadrightarrow Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)),$$

and where the image

$$\Gamma\big(Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big) \to J\big(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big), \tag{36}$$

is desrcibed as follows. Let $y \in \Gamma\big(Gr_W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)$, and choose

$$x \in W^0 H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r)),\ x_{\mathbb{C}} \in F^0W^0H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{C}),$$

which map to y under the surjection $W_0 \to Gr_W^0$. Then the image of y in (36) is given by the image of $x - x_{\mathbb{C}}$ in $J\big(W_{-1}H^{\nu-1}(\mathcal{S}(\mathbb{C}), R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)$. Under the identification of K with $k(\eta)$, one then has (by definition)

$$F^\nu \mathrm{CH}^r(X_K, m; \mathbb{Q}) = \lim_{\substack{\to \\ U\subset\mathcal{S}/k}} \mathcal{F}^\nu \mathrm{CH}^r(\mathfrak{X}_U/\overline{\mathbb{Q}}, m; \mathbb{Q}), \quad \mathfrak{X}_U := \rho^{-1}(U).$$

We put

$$E_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) = \lim_{\substack{\longrightarrow \\ U\subset\mathcal{S}/k}} E_\infty^{\nu,2r-\nu-m}(\rho)$$

and the same definition for $\underline{E}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}})$ and $\underline{\underline{E}}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}})$. Specifically,

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) = \Gamma\big(H^\nu(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big),$$
$$\underline{E}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) = J\big(W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)/\Gamma(Gr_W^0).$$

We have a short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) \to E_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}) \to 0,$$

and an injection:

$$Gr_F^\nu \mathrm{CH}^r(X_K, m; \mathbb{Q}) \hookrightarrow E_\infty^{\nu,2r-\nu-m}(\eta_{\mathcal{S}}).$$

(ii) We then define

$$F^\nu \mathrm{CH}^r(X_{\mathbb{C}}, m; \mathbb{Q}) = \lim_{\substack{\longrightarrow \\ k\subset K\subset\mathbb{C}}} F^\nu \mathrm{CH}^r(X_K, m; \mathbb{Q}),$$

which becomes a candidate BB filtration on $\mathrm{CH}^r(X_{\mathbb{C}}, m; \mathbb{Q})$ in the case $k = \overline{\mathbb{Q}}$.

8.3. **Approach via Lewis.** Consider a $\overline{\mathbb{Q}}$-spread $\rho : \mathcal{X} \to \mathcal{S}$, where ρ is smooth and proper. Let η be the generic point of $\mathcal{S}$, and put $K := \overline{\mathbb{Q}}(\eta)$. Write $X_K := \mathcal{X}_\eta$. From [Lew5] we introduced a decreasing filtration $\mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, with the property that $Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \hookrightarrow E_\infty^{\nu,2r-\nu}(\rho)$, where $E_\infty^{\nu,2r-\nu}(\rho)$ is the ν-th graded piece of the Leray filtration on the lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ of Beilinson's absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ associated to ρ. That lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \subset H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ is given by the image $H^{2r}_{\mathcal{H}}(\overline{\mathcal{X}},\mathbb{Q}(r)) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, where $\overline{\mathcal{X}}$ is a smooth compactification of $\mathcal{X}$. There is a cycle class map $\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) := \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, which is conjecturally injective under the Bloch-Beilinson conjecture assumption, using the fact that there is a short exact sequence:

$$0 \to J(H^{2r-1}(\mathcal{X},\mathbb{Q}(r))) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \to \Gamma(H^{2r}(\mathcal{X},\mathbb{Q}(r))) \to 0.$$

(Injectivity would imply $D^r(X) = 0$.) Regardless of whether or not injectivity holds, the filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ is given by the pullback of the Leray filtration on $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ to $\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$. It is proven in [Lew5] that the term $E_\infty^{\nu,2r-\nu}(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho) \to E_\infty^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) = \Gamma(H^\nu(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))),$$
$$\underline{E}_\infty^{\nu,2r-\nu}(\rho) = \frac{J(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}{\Gamma(Gr^0_W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))}$$
$$\subset J(H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))).$$

[Here the latter inclusion is a result of the short exact sequence:
$W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow Gr^0_W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))$.]
One then has (by definition)

$$F^\nu\mathrm{CH}^r(X_K;\mathbb{Q}) = \varinjlim_{U\subset\mathcal{S}/\overline{\mathbb{Q}}} \mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U)$$

$$F^\nu\mathrm{CH}^r(X_\mathbb{C};\mathbb{Q}) = \varinjlim_{K\subset\mathbb{C}} F^\nu\mathrm{CH}^r(X_K;\mathbb{Q})$$

Further, since direct limits preserve exactness,

$$Gr^\nu_F\mathrm{CH}^r(X_K;\mathbb{Q}) = \varinjlim_{U\subset\mathcal{S}/\overline{\mathbb{Q}}} Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}),$$

$$Gr^\nu_F\mathrm{CH}^r(X_\mathbb{C};\mathbb{Q}) = \varinjlim_{K\subset\mathbb{C}} Gr^\nu_F\mathrm{CH}^r(X_K;\mathbb{Q})$$

8.4. **(Generalized) normal functions.** Let us now assume that with regard to the smooth and proper map $\rho : \mathcal{X} \to \mathcal{S}$ over a subfield $k \subset \mathbb{C}$, and after possibly shrinking $\mathcal{S}$, that $\mathcal{S}$ is affine, with $K = k(\mathcal{S})$. Let $V \subset \mathcal{S}(\mathbb{C})$ be smooth, irreducible, closed subvariety of dimension $\nu - 1$ (note that $\mathcal{S}$ affine $\Rightarrow$ V affine). One has a commutative square

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X}(\mathbb{C}) \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S}(\mathbb{C}), \end{array}$$

and a commutative diagram

$$\begin{array}{ccccccccc}
 & & \xi\in & Gr^{\nu}_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X},m;\mathbb{Q}) & \mapsto & Gr^{\nu}_{F}\mathrm{CH}^r(X_K;\mathbb{Q}) & & & \\
 & & & \downarrow & & & & & \\
0 & \to & \underline{E}^{\nu,2r-\nu-m}_{\infty}(\rho) & \to & E^{\nu,2r-\nu-m}_{\infty}(\rho) & \to & \underline{\underline{E}}^{\nu,2r-\nu-m}_{\infty}(\rho) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{E}^{\nu,2r-\nu-m}_{\infty}(\rho_V) & \to & E^{\nu,2r-\nu-m}_{\infty}(\rho_V) & \to & \underline{\underline{E}}^{\nu,2r-\nu-m}_{\infty}(\rho_V) & \to & 0 \\
 & & & & & & || & & \\
 & & & & & & 0 & &
\end{array}$$

where $\underline{\underline{E}}^{\nu,2r-\nu-m}_{\infty}(\rho_V) = 0$ follows from the weak Lefschetz theorem for locally constant systems over affine varieties (see for example [Ar], and the references cited there). Thus for any $\xi \in Gr^{\nu}_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X};\mathbb{Q})$, we have a "normal function" η_ξ with the property that for any such smooth irreducible closed $V \subset S(\mathbb{C})$ of dimension $\nu-1$, we have a value $\eta_\xi(V) \in \underline{E}^{\nu,2r-\nu}_{\infty}(\rho_V)$. Here we think of V as a point on a suitable open subset of the Chow variety of dimension $\nu-1$ subvarieties of $\mathcal{S}(\mathbb{C})$ and η_ξ defined on that subset. For example if $\nu = 1$, then we recover the classical notion of normal functions.

Definition 8.5. η_ξ is called an arithmetic normal function.

Example 8.6. If $\mathcal{S}$ is affine of dimension $\nu-1$. Then in this case $V = \mathcal{S}$, and $\xi \in Gr^{\nu}_{\mathcal{F}}\mathrm{CH}^r(\mathfrak{X},m;\mathbb{Q})$ induces a "single point" normal function

$$\eta_\xi(V) = \eta_\xi(\mathcal{S}) \in J(H^{\nu-1}(\mathcal{S}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))).$$

Now let $\xi \in \mathcal{F}^{\nu}\mathrm{CH}^r(\mathfrak{X},m;\mathbb{Q})$ be given, and let $[\xi] \in \underline{\underline{E}}^{\nu,2r-\nu-m}_{\infty}(\rho)$ be its image via the composite

$$\mathcal{F}^{\nu}\mathrm{CH}^r(\mathfrak{X},m;\mathbb{Q}) \to E^{\nu,2r-\nu-m}_{\infty}(\rho) \to \underline{\underline{E}}^{\nu,2r-\nu-m}_{\infty}(\rho).$$

Theorem 8.7 (see [K-L])**.** *The class $[\xi]$ depends only on η_ξ, and is called the topological invariant of η_ξ.*

Remark 8.8. Let us assume for example that $m = 0$, $k = \overline{\mathbb{Q}}$ and $\xi \in \mathcal{F}^{\nu}\mathrm{CH}^r(\mathfrak{X};\mathbb{Q})$. As conjectured in [L-S] and proven in [MSa1], one has a map $\underline{\underline{E}}^{\nu,2r-\nu}_{\infty}(\rho) \hookrightarrow \nabla J^{r,\nu}(\mathfrak{X}/\mathcal{S}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, which captures the [generalized] Griffiths infinitesimal invariant of normal functions. If we shrink $\mathcal{S}$ to the generic point $\eta_{\mathcal{S}} \in \mathcal{S}/\overline{\mathbb{Q}}$, then the map $\xi \mapsto [\xi] \to \underline{\underline{E}}^{r,2r-\nu}_{\infty}(\eta_{\mathcal{S}})$ factors through the analogous Leray spectral sequence term $\nabla J^{r,\nu}(X_K/\overline{\mathbb{Q}}) \hookrightarrow J^{r,\nu}(X_{\mathbb{C}}/\overline{\mathbb{Q}}) \hookrightarrow \underline{\underline{E}}^{r,2r-\nu}_{\infty}(\eta_{\mathcal{S}})$. This story completely generalizes to all $m \geq 0$.

Note that it is rather clear from this that

$$F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \ker\big(AJ : \mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m-1}(X,\mathbb{Q}(r)).$$

Question 8.9. (i) Can one characterize the BB filtration in terms of arithmetic normal functions?

(ii) By choosing V sufficiently general, can one characterize the BB filtration in terms of the corresponding Abel-Jacobi map for a fixed general variety? E.g. we know that $F^1\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q})$ and

$$F^2\mathrm{CH}^r(X;\mathbb{Q}) \subset \mathrm{CH}^r_{AJ}(X;\mathbb{Q}) := \ker AJ_X : \mathrm{CH}^r_{\mathrm{hom}}(X;\mathbb{Q}) \to J^r(X)_{\mathbb{Q}}.$$

Is it the case that $F^2\mathrm{CH}^r(X;\mathbb{Q}) = \mathrm{CH}^r_{AJ}(X;\mathbb{Q})$?

(ii)$'$ What about the zero (or torsion) locus of such normal functions. I.e., are they sensitive to the field of definition of algebraic cycles?

Remark 8.10. $\bullet_1$ Special cases of Question 8.9 are worked out in [K-L]. Further, if both X and S are defined over k, with $\mathcal{X} = S \times X$, with $\rho = \mathrm{Pr}_1$, then the answer is yes, as shown in [Lew7].

$\bullet_2$ In the case where $\nu = 1$, (ii) and (ii)$'$ can be shown to be equivalent. (See for example [Lew6].)

9. **The Beilinson-Hodge conjectures**

To accurately explain the motivation in this section, we indulge a moment of the reader's patience by beginning on a somewhat technical note, with the promise of a subsequent smoother ride. In his fundamental work on absolute Hodge cohomology $H^\bullet_{\mathcal{H}}$, Beilinson [Be2](Conj. 6)) formulates a version of the Hodge conjecture in the $\mathcal{H}$-cohomology of a smooth scheme $X/\mathbb{C}$. Specifically,

The cycle class map (interpreted here through the lens of higher Chow cycles [Be1]*), to absolute Hodge cohomology,*

$$\mathrm{cl}^{\mathcal{H}}_{r,m} : CH^r(X,m;\mathbb{Q}) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)), \tag{37}$$

has dense image.

To explain this conjecture, we first observe that there is a natural topology on $H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r))$ obtained via the short exact sequence involving (extension) classes of MHS:

$$0 \to J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)) \to \Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big) \to 0,$$

where the torus $J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big)$ is a connected component of $H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r))$. Thus the conjecture in (37) says that the Betti cycle class map

$$\mathrm{CH}^r(X,m;\mathbb{Q}) \to \Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big), \tag{38}$$

is surjective, and that the induced Abel-Jacobi map

$$\mathrm{CH}^r_{\mathrm{hom}}(X,m;\mathbb{Q}) \to J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big), \tag{39}$$

is dense in the (non-separated!) topology. In the case $m = 0$, this is equivalent to the surjectivity of the map in (38), which amounts to the statement of the Hodge conjecture. This can be seen more clearly in the case $X = \overline{X}$ complete and $m = 0$.

To "unravel" the map in (39), it can be shown that there is an exact sequence,

$$\frac{W_0H^{2r-m-1}(X,\mathbb{R}(r))}{\begin{Bmatrix} F^0 \bigcap W_0H^{2r-m-1}(X,\mathbb{R}(r)) \\ + W_0H^{2r-m-1}(X,\mathbb{Q}(r)) \end{Bmatrix}} \hookrightarrow H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{R}(r)), \tag{40}$$

which evidently becomes a short exact sequence in the event that $X = \overline{X}$ is complete and $m \geq 1$. Then the statement in (39) translates to the the surjectivity of the real regulator,

$$(41) \qquad R_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X,m) \otimes \mathbb{R} \to \frac{W_0 H^{2r-m-1}(X, \mathbb{R}(r-1))}{\pi_{r-1}\big(F^0 \cap W_0 H^{2r-m-1}(X,\mathbb{C})\big)},$$

where $\pi_{r-1} : \mathbb{C} = \mathbb{R}(r-1) \oplus \mathbb{R}(r) \to \mathbb{R}(r-1)$, is the canonical projection. (We observe that in the case $m = 0$, the RHS of (41) is zero by a "trivial" weight argument, and if additionally $X = \overline{X}$, then the first term of (40) can be identified with $J(H^{2r-1}(X, \mathbb{Q}(r)))$.)

As subsequent history would dictate, the conjectured density of the map in (37) turned out to be too optimistic on two counts (for $m \geq 1$):

1) The anticipated surjectivity of the map in (38) fails for some $X/\mathbb{C}$ (first observed in [Ja1](Cor. 9.11)).

2) There are counterexamples to the surjectivity of $R_{r,m}$, first observed by M. Nori, S. Müller-Stach and M. Green, (see [MS1]). (Also see [C-L1], [C-L2].) These counterexamples involve the case where $X = \overline{X}$ is complete. This amounts to counterexamples to Beilinson's Hodge-$\mathcal{D}$-conjecture for real varieties $\overline{X}$ ([Be1](1.10); also see [Ja2](3.11)), where one views $\overline{X}/\mathbb{C}$ as a 'real variety' via the structure maps $\overline{X} \to \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$.

Remark 9.1. (i) Regarding 2), there are non-trivial instances where it is indeed true, as worked out in [C-L3].

(ii) Both situations in 1) and 2) above occur even in the case $m = 1$.

Regarding (37), and in light of the conjectures of Bloch and Beilinson on the injectivity (modulo torsion) of the Abel-Jacobi map for smooth projective varieties defined over number fields, we state the following:

Conjecture 9.2. Let $X/\overline{\mathbb{Q}}$ be a smooth quasi-projective variety. Then

$$\mathrm{cl}^{\mathcal{H}}_{r,m} : \mathrm{CH}^r(X/\overline{\mathbb{Q}}, m; \mathbb{Q}) \to H^{2r-m}_{\mathcal{H}}(X(\mathbb{C}), \mathbb{Q}(r)),$$

is injective with dense image.

Although more modest in formulation than the aforementioned conjecture in (37), Conjecture 9.2 serves as a strong endorsement of Beilinson's visionary approach on this subject. Further, we explain how a suitable tinkering of the aforementioned surjectivity statement in (38) above withstands critical examination.

For any 'good' cohomology theory, $H^a(-, b)$, and recalling X being smooth quasi-projective, Bloch ([Blo2]) has constructed a cycle class map $\mathrm{CH}^r(X,m) \to H^{2r-m}(X, r)$. In terms of singular cohomology, this gives us a map

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to \Gamma H^{2r-m}(X, \mathbb{Z}(r)), \quad \xi \mapsto (2\pi \mathrm{i})^r \big[\mathrm{Pr}_X\big(\xi \cap \{X \times \Delta^m(\mathbb{R}_+)\}\big)\big],$$

$[\cdots]$ representing Poincaré duality, and likewise,

$$\mathrm{cl}^{\lim}_{r,m} : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)), m) \to \Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Z}(r)),$$

as well as $\mathbb{Q}$-analogues:

$$\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \Gamma H^{2r-m}(X, \mathbb{Q}(r)),$$

$$\mathrm{cl}^{\lim}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)), m; \mathbb{Q}) \to \Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r)).$$

For motivational purposes, we recall the following situation discussed in [dJ-L]: Fix $m \geq 0$. Consider these 3 statements:

(S1) $\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \Gamma H^{2r-m}(X, \mathbb{Q}(r))$ is surjective for all smooth projective $X/\mathbb{C}$ and all r.

(S2) $\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \Gamma H^{2r-m}(X, \mathbb{Q}(r))$ is surjective for all smooth quasi-projective $X/\mathbb{C}$ and all r.

(S3) $\mathrm{cl}^{\lim}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(X)), m; \mathbb{Q}) \to \Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r))$ is surjective for all varieties $X/\mathbb{C}$ and all r. (Hence for dimension reasons, $\Gamma H^{2r-m}(\mathbb{C}(X), \mathbb{Q}(r)) = 0$ for $r > m$).

Proposition 9.3 (Hodge conjecture (HC))**.** *For $m = 0$,*

$$\text{(S1)} \Leftrightarrow \text{(S2)} \Leftrightarrow \text{(S3)};$$

moreover this is equivalent to saying that

$$\Gamma\big(H^{2r}(\mathbb{C}(X), \mathbb{Q}(r))\big) = 0,$$

for all X and all $r > 0$.

Proof. This is really the import of Deligne's mixed Hodge theory ([De]) and localization sequences. Let us show for example that statement (S1) is equivalent to $\Gamma\big(H^{2r}(\mathbb{C}(X), \mathbb{Q}(r))\big) = 0$, for all X and all $r > 0$. Note that if $\xi \in \mathrm{CH}^r(X)$ is given, with $r > 0$, then ξ vanishes on $X \backslash |\xi| \neq \emptyset$. Thus it is clear that (S1) $\Rightarrow \Gamma\big(H^{2r}(\mathbb{C}(X), \mathbb{Q}(r))\big) = 0$. Conversely, let $\xi \in \Gamma H^{2r}(X, \mathbb{Q}(r))$, where $X/\mathbb{C}$ is smooth and projective. Then $\Gamma\big(H^{2r}(\mathbb{C}(X), \mathbb{Q}(r))\big) = 0$ implies that $\xi \mapsto 0 \in H^{2r}(X \backslash Y, \mathbb{Q}(r))$ for some closed codimension one $Y \subset X$. Thus we have $\xi \in \mathrm{Image}\big(H^{2r}_Y(X, \mathbb{Q}(r)) \to H^{2r}(X, \mathbb{Q}(r))\big)$. Since $H^{2r}(X, \mathbb{Q}(r))$ is a pure Hodge structure, it follows from a weight argument ([De]) that $\xi \in \mathrm{Image}\big(\Gamma H^{2r-2}(\tilde{Y}, \mathbb{Q}(r-1)) \to H^{2r}(X, \mathbb{Q}(r))\big)$, where $\tilde{Y} \to Y$ is a desingularization of Y. If $r - 1 = 0$, then ξ is algebraic. Otherwise $r - 1 > 0$, so we now use $\Gamma H^{2(r-1)}(\mathbb{C}(\tilde{Y}), \mathbb{Q}(r-1)) = 0$ and repeat the argument. □

In the case $m > 0$, it turns out that (S1), (S2) and (S3) are independent statements. For instance, by a weight argument $\Gamma H^{2r-m}(X, \mathbb{Q}(r)) = 0$ for X smooth projective, hence (S1) trivially holds. It turns out that (S2) fails, but (S3) is probably true.

There are two conjectural ingredients we want to consider, namely the HC, and a generalized version of a conjecture originally due to Bloch and independently Beilinson.

Conjecture 9.4 (BBC)**.** Let $\overline{X}/\overline{\mathbb{Q}}$ be smooth projective. Then the Abel-Jacobi map

$$\mathrm{CH}^r_{\mathrm{hom}}(\overline{X}/\overline{\mathbb{Q}}, m; \mathbb{Q}) \to J\big(H^{2r-m-1}(\overline{X}(\mathbb{C}), \mathbb{Q}(r))\big),$$

is injective.

Remark 9.5. If one assumes the HC, then $\overline{X}$ in the above conjecture can be replaced by a smooth quasi-projective $X/\overline{\mathbb{Q}}$. The proof of this can be found in [K-L], which involves a weight filtered spectral sequence calculation.

9.6. **A key observation.** For this subsection, let $k \subset \mathbb{C}$ be a subfield, $\overline{X}/k$ smooth projective, and $Y \subset \overline{X}/k$ a proper subvariety. Let
$$H_Y^{2r}(\overline{X}, \mathbb{Q}(r))^\circ = \ker\big(H_Y^{2r}(\overline{X}, \mathbb{Q}(r)) \to H^{2r}(\overline{X}, \mathbb{Q}(r))\big).$$
From the short exact sequence:
$$0 \to \frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{H_Y^{2r-1}(\overline{X}, \mathbb{Q}(r))} \to H^{2r-1}(\overline{X}\backslash Y, \mathbb{Q}(r)) \to H_Y^{2r}(\overline{X}, \mathbb{Q}(r))^\circ \to 0,$$
we arrive at:
$$\begin{array}{ccccc} \mathrm{CH}^r(\overline{X}\backslash Y, 1; \mathbb{Q}) & \xrightarrow{\partial} & \mathrm{CH}_Y^r(\overline{X}; \mathbb{Q})^\circ & \xrightarrow{\beta} & \mathrm{CH}^r_{\mathrm{hom}}(\overline{X}; \mathbb{Q}) \\ \mathrm{cl}_{r,1,\mathbb{Q}}\Big\downarrow & & \lambda\Big\downarrow & & \Big\downarrow \underline{AJ} \\ \Gamma H^{2r-1}(\overline{X}\backslash Y, \mathbb{Q}(r)) & \hookrightarrow & \Gamma H_Y^{2r}(\overline{X}, \mathbb{Q}(r))^\circ & \to & J\big(\frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{H_Y^{2r-1}(\overline{X}, \mathbb{Q}(r))}\big) \end{array}$$
Here $\underline{AJ}$ is the Abel-Jacobi map. Observe that the HC implies that λ is onto. Indeed, via Poincaré duality, the surjectivity of λ is implied by a homological version of the HC for singular varieties, which is in fact equivalent to the HC (cohomological version) for smooth projective varieties (see [Ja1]). In general, λ onto together with the snake lemma implies that
$$\frac{\ker \underline{AJ}\big|_{\mathrm{Im}(\beta)}}{\beta(\ker \lambda)} \simeq \mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}).$$
Further, the HC implies:
$$\mathrm{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) \simeq \frac{\beta(\ker \lambda) + \ker AJ\big|_{\mathrm{Im}(\beta)}}{\beta(\ker \lambda)}, \tag{42}$$
where $AJ : \mathrm{CH}^r_{\mathrm{hom}}(\overline{X}, \mathbb{Q}) \to J\big(H^{2r-1}(\overline{X}, \mathbb{Q}(r))\big)$. To see this, let us first assume for notational simplicity that Y has pure codimension $\ell \geq 1$ in $\overline{X}$. Let $\tilde{Y} \to Y$ be a desingularization. By a weight argument ([De]), the images:
$$\mathrm{Gy} : H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)) \to H^{2r-1}(\overline{X}, \mathbb{Q}(r)),$$
$$H_Y^{2r-1}(\overline{X}, \mathbb{Q}(r)) \to H^{2r-1}(\overline{X}, \mathbb{Q}(r)),$$
coincide. By semi-simplicity considerations, we can write
$$H^{2r-1}(\overline{X}, \mathbb{Q}(r)) = \mathrm{Im\ Gy} \oplus \{\mathrm{Im\ Gy}\}^\perp,$$
$$H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)) = \{\ker \mathrm{Gy}\}^\perp \oplus \ker \mathrm{Gy}.$$
By the HC, the composite
$$H^{2r-1}(\overline{X}, \mathbb{Q}(r)) \xrightarrow{\mathrm{Pr}_1} \mathrm{Im\ Gy} \xrightarrow{\{\mathrm{Gy}|_{\{\ker \mathrm{Gy}\}^\perp}\}^{-1} \simeq} \{\ker \mathrm{Gy}\}^\perp$$
$$\simeq \{\ker \mathrm{Gy}\}^\perp \oplus \{0\} \hookrightarrow H^{2r-1-2\ell}(\tilde{Y}, \mathbb{Q}(r-\ell)),$$
is induced by some $w \in \mathrm{CH}^{d-\ell}(\overline{X} \times \tilde{Y}; \mathbb{Q})$, for which $\mathrm{Gy} \circ [w]_* = \mathrm{Pr}_1$. Now let $\xi \in \mathrm{Im}(\beta) \subset \mathrm{CH}^r_{\mathrm{hom}}(\overline{X})$ be given such that $\underline{AJ}(\xi) = 0$. Then by semi-simplicity considerations, $AJ(\xi) \in J(\mathrm{Im(Gy)}) \subset J\big(H^{2r-1}(\overline{X}, \mathbb{Q}(r))\big)$. But $w_*(\xi) \in \mathrm{CH}^{r-\ell}_{\mathrm{hom}}(\tilde{Y}; \mathbb{Q})$, which maps to some $\xi_0 \in \ker(\lambda) \subset \mathrm{CH}_Y^r(\overline{X}; \mathbb{Q})^\circ$ for which $AJ(\xi - \beta(\xi_0)) = 0$. The isomorphism in (42) then follows.

Remark 9.7. We now consider two extreme scenarios (the latter is due to Jannsen [Ja1]):

• $k \subseteq \overline{\mathbb{Q}}$. Then BBC + HC $\Rightarrow \operatorname{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) = 0$.

• $k = \mathbb{C}$ and $\mathrm{cd}_{\overline{X}} Y = r \Rightarrow \lambda$ is an isomorphism and $H_Y^{2r-1}(\overline{X}, \mathbb{Q}(r)) = 0$. Hence $\operatorname{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}) = \ker AJ\big|_{\mathrm{Im}(\beta)}$, which need not be zero (Mumford - See [Lew1](Ch. 15)).

Next, assuming $k = \mathbb{C}$ and taking the limit over all codimension one subvarieties $Y \subset \overline{X}$, and assuming the HC, we arrive at

$$\operatorname{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}^{\lim}) \simeq \frac{N^1\mathrm{CH}^r(\overline{X};\mathbb{Q}) + \ker AJ}{N^1\mathrm{CH}^r(\overline{X};\mathbb{Q})}, \tag{43}$$

where $N^p\mathrm{CH}^r(\overline{X})$ are those cycles that are homologous to zero on subvarieties of codimension p in $\overline{X}$. Jannsen, et al. (see [Ja3]) raised the important question as to whether the RHS in (43) is always zero, and in the case $r = 2$, this would imply that $\ker AJ \subset \mathrm{CH}^2_{\mathrm{alg}}(\overline{X};\mathbb{Q})$, where $\mathrm{CH}^r_{\mathrm{alg}}(\overline{X}) \subset \mathrm{CH}^r(\overline{X})$ is the subgroup of cycles that are algebraically equivalent to zero. Thus there is a reasonable expectation that the RHS in (43) is always zero. Further, and for a connection between the vanishing of $\operatorname{cok}(\mathrm{cl}_{r,1,\mathbb{Q}}^{\lim})$ for all $\overline{X}$ and the field of definition of the zero locus of a cycle induced normal function, the reader can consult [Lew3].

9.8. **Formulation of the conjectures.** For general r and m and smooth quasi-projective $X/\mathbb{C}$, the approach to studying

$$\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \Gamma H^{2r-m}(X, \mathbb{Q}(r)),$$

involves a weight filtered spectral sequence associated to the simplicial complex $Y^{[\bullet]} \to Y \subset \overline{X}$, where $\overline{X}$ is a good compactification of X with NCD Y.

Theorem 9.9 (Vague form, see [dJ-L](Thm 4.9))**.** *The obstruction to the surjectivity of* $\mathrm{cl}_{r,m,\mathbb{Q}}$ *is "explained" in terms of kernels of (higher) Abel-Jacobi maps.*

Remark 9.10. One can show that the BBC + HC implies that (S2) holds for all $X/\overline{\mathbb{Q}}$ ([MSa2]; also see [K-L] as well as [dJ-L] for different proofs). Remark 9.7 gives one instance of this (viz., the case $m = 1$). Further, Remark 9.7 also gives a counterexample to (S2) in the case $m = 1$.

Quite generally, we have the following picture for any smooth quasi-projective variety $X/\mathbb{C}$ and corresponding cycle class map

$$\mathrm{cl}_{r,m,\mathbb{Q}} : \mathrm{CH}^r(X, m; \mathbb{Q}) \to \Gamma H^{2r-m}(X, \mathbb{Q}(r)).$$

$$\begin{array}{llllll}
 & r & & & & \\
 & \uparrow & \text{(III)} & & \nearrow & \\
d & \bullet - - - - & & \nearrow & & \\
 & | & \text{(I)} \quad \nearrow & & & \\
 & | & \quad \nearrow & \text{(II)} & & \\
 & | \nearrow & & & & \\
 & - - - - - - - - - - - \to m & & & &
\end{array}$$

Region (I): $r \le d$, $m > 0$ and $r > m$.
Region (II): $r < m$.
Region (III): $r > m$ and $r > d$.

The map $\mathrm{cl}_{r,m,\mathbb{Q}}$ is trivially surjective in regions (II) and (III), since in those cases $\Gamma H^{2r-m}(X,\mathbb{Q}(r)) = 0$. In general surjectivity fails in region (I). Note that the r-axis corresponds to the classical Hodge conjecture.

Theorem 9.11 ([dJ-L](Thm 5.1))**.** *Assume given (r,m,d) in region I above. Then there exists a smooth quasi-projective variety $X/\mathbb{C}$ of dimension d such that $\mathrm{cl}_{r,m,\mathbb{Q}}$ fails to be surjective.*

In summary, and outside of the situation involving varieties over number fields, we see that the following situation emerges:

Conjecture 9.12. (i) $\mathrm{cl}_{m,m,\mathbb{Q}}$ is surjective for all m.

(ii) $\mathrm{cl}^{\lim}_{r,m,\mathbb{Q}}$ is surjective for all r and m.

Remark 9.13. (1) The statement in (i) is generally referred to as the Beilinson-Hodge conjecture (see [A-K], [A-S], [Sa]). It seems more natural to also refer to it as the Beilinson-Milnor-Hodge conjecture.

(2) The present form of (ii) is indicative that there are numerator conditions in the definition of $\mathrm{CH}^r(X,m)$ for $m>0$, that can either be removed after passing to the generic point, or the Hodge theory of the target becomes trivial.

9.14. **Integral formulations and the Milnor regulator.** Let X be a smooth quasi-projective variety. This section concerns the integrally defined map $\mathrm{cl}^{\lim}_{m,m} = d\log$:

$$\mathrm{cl}^{\lim}_{m,m} : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \to \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big),$$
$$\{f_1,...,f_m\} \mapsto \bigwedge_1^m d\log f_j,$$

the formula being induced by $d\log : K_m^M(\mathbb{C}(X)) \to \Omega^m_{\mathbb{C}(X)/\mathbb{C}} =: \Omega^m_{\mathrm{Spec}(\mathbb{C}(X))/\mathbb{C}}$, vis-à-vis the composite:

$$d\log : \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \simeq K_m^M(\mathbb{C}(X)) = H^0_{\mathrm{Zar}}(\mathrm{Spec}(\mathbb{C}(X)),\mathcal{K}^M_{m,\mathrm{Spec}(\mathbb{C}(X))})$$
$$= \mathbb{H}^m_{\mathrm{Zar}}(\mathcal{K}^M_{m,\mathrm{Spec}(\mathbb{C}(X))}[-m]) \to \mathbb{H}^m_{\mathrm{Zar}}(\Omega^{\bullet\geq m}_{\mathrm{Spec}(\mathbb{C}(X))/\mathbb{C}}) =: F^m H^m_{\mathrm{DR}}(\mathbb{C}(X),\mathbb{C}).$$

Alternatively, the formula is a rather obvious consequence of the commutative diagram:

$$\begin{array}{ccc}
\bigotimes_{j=1}^m \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1) & \overset{\cup}{\twoheadrightarrow} & \mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m) \\
\bigotimes_{j=1}^m d\log f_j \Big\downarrow & & \Big\downarrow \bigwedge_{j=1}^m d\log f_j \\
\bigotimes_{j=1}^m \Gamma H^1(\mathbb{C}(X),\mathbb{Z}(1)) & \overset{\cup}{\to} & \Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m))
\end{array}$$

It turns out that $H^m(\mathbb{C}(X),\mathbb{Z}(m))$ is torsion-free (see Theorem 9.31 below). Thus

$$\Gamma H^m(\mathbb{C}(X),\mathbb{Z}(m)) = H^m(\mathbb{C}(X),\mathbb{Z}(m)) \cap F^m H^m(\mathbb{C}(X),\mathbb{C}),$$

makes sense.

Conjecture 9.15. $\mathrm{cl}^{\lim}_{m,m}$ is surjective.

Indeed the surjectivity of $\mathrm{cl}^{\lim}_{m,m}$ will be shown to be equivalent to the corresponding surjectivity statement for $\mathrm{cl}^{\lim}_{m,m,\mathbb{Q}}$ (Corollary 9.33).

9.16. **Key example I.** For $X/\mathbb{C}$ smooth quasi-projective, we first consider the case $r = m = 1$:

$$\mathrm{cl}_{1,1} : \mathrm{CH}^1(X,1) \to H^1(X,\mathbb{Z}(1)).$$

We recall that

$$\mathrm{CH}^1(X,1) = \mathcal{O}_X^\times(X) \simeq H^1_{\mathcal{D}}(X,\mathbb{Z}(1)),$$

where the latter is Deligne-Beilinson cohomology. This leads to a commutative diagram of short exact sequences:

$$\begin{array}{ccccc} H^0(X,\mathbb{C}/\mathbb{Z}(1)) & \hookrightarrow & H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) & \twoheadrightarrow & \Gamma H^1(X,\mathbb{Z}(1)) \\ \| \wr & & \| \wr & & \| \wr \\ \mathbb{C}^\times & \hookrightarrow & \mathrm{CH}^1(X,1) & \xrightarrow{\mathrm{cl}_{1,1}} & \Gamma H^1(X,\mathbb{Z}(1)) \end{array}$$

Corollary 9.17. (i) $\mathrm{cl}_{1,1}$, $d\log$ *are surjective.*

(ii) $\ker(d\log)$, $\ker(\mathrm{cl}_{1,1})$ *are divisible.*

Thus for any integer $M \neq 0$,

$$\frac{\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1)}{M\cdot \mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)),1)} \simeq \frac{\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)}{M\cdot\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)}.$$

Next, there is an exact sequence:

$$0 \to \frac{H^1(X,\mathbb{Z}(1))}{\Gamma\big(H^1(X,\mathbb{Z}(1))\big)} \to \frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C})} \to \mathrm{CH}^1(X).$$

Further,

$$\frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C})} \simeq \mathbb{H}^1(\Omega^{\bullet<1}_{\overline{X}}\langle Y\rangle) \simeq H^1(\overline{X},\mathcal{O}_{\overline{X}}).$$

Letting X shrink, and using $\mathrm{CH}^1(\mathrm{Spec}(\mathbb{C}(X)) = 0$, we arrive at

$$\frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{\Gamma\big(H^1(\mathbb{C}(X),\mathbb{Z}(1))\big)} \simeq H^1(\overline{X},\mathcal{O}_{\overline{X}}),$$

which is (uniquely) divisible. The upshot is that

$$\frac{\mathrm{CH}^1(\mathbb{C}(X),1)}{M\cdot\mathrm{CH}^1(\mathbb{C}(X),1)} \simeq \frac{H^1(\mathbb{C}(X),\mathbb{Z}(1))}{M\cdot H^1(\mathbb{C}(X),\mathbb{Z}(1))} \simeq H^1\Big(\mathbb{C}(X),\frac{\mathbb{Z}(1))}{M\cdot\mathbb{Z}(1)}\Big),$$

where the latter uses $H^2(\mathbb{C}(X),\mathbb{Z})$ torsion-free, which can be deduced from the Merkurjev-Suslin theorem (or more generally the Bloch-Kato theorem, see Theorem 9.31 below), or the Lefschetz $(1,1)$ theorem (see [dJ-L]). Indeed, the following result will suffice:

Lemma 9.18. *Let $V/\mathbb{C}$ be smooth projective and let $Y \subset V$ be a pure codimension one subset such that the image $H^2_Y(V,\mathbb{Z}) \to H^2(V,\mathbb{Z})$ is precisely the Neron-Severi group $NS(V) = H^2_{\mathrm{alg}}(V,\mathbb{Z})$ of V. Let $U = V\backslash Y$. Then $H^2(U,\mathbb{Z})$ is torsion-free.*

Proof. By the Lefschetz $(1,1)$ theorem, $H^2(V,\mathbb{Z})/\mathrm{NS}(V)$ is torsion-free; moreover there is an exact sequence

$$0 \to \frac{H^2(V,\mathbb{Z})}{\mathrm{NS}(V)} \to H^2(U,\mathbb{Z}) \to H^3_Y(V,\mathbb{Z}).$$

It suffices to show that $H^3_Y(V,\mathbb{Z})$ is torsion-free. Let $d = \dim V$. Then by Poincaré duality, $H^3_Y(V,\mathbb{Z}) \simeq H_{2d-3}(Y,\mathbb{Z})$. Let $U_Y = Y \backslash Y_{\mathrm{sing}}$. For dimension reasons alone, one has an injection $H_{2d-3}(Y,\mathbb{Z}) \hookrightarrow H_{2d-3}(U_Y,\mathbb{Z}) \simeq H^1(U_Y,\mathbb{Z})$. The latter term is well known to be torsion-free.

$\square$

9.19. **Alternate take via a morphism of sites.** To see this in another context, let us work in the étale topology on a smooth quasi-projective variety $V/\mathbb{C}$, and recall the sheaves $\mathbb{G}_m$, μ_M on V, where for $U \to V$ étale, $\mathbb{G}_m(U) = \Gamma(U,\mathcal{O}_U^\times)$, $\mu_M(U) = \{\xi \in \Gamma(U,\mathcal{O}_U) \mid \xi^M = 1\}$, and the corresponding Kummer short exact sequence:

$$0 \to \mu_M \to \mathbb{G}_m \xrightarrow{x\mapsto x^M} \mathbb{G}_m \to 0$$

This gives us

$$0 \to \frac{\Gamma(V,\mathcal{O}_V^\times)}{M} \to H^1_{et}(V,\mu_M) \to \mathrm{Pic}(V). \tag{44}$$

Likewise in the analytic topology, we have the exponential short exact sequence,

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_{V,an} \to \mathcal{O}_{V,an}^\times \to 0,$$

which yields the (de Rham interpreted) connecting homomorphism:

$$\delta := d\log : \Gamma(V,\mathcal{O}_{V,an}^\times) \to H^1(V,\mathbb{Z}(1)) \to H^1(V,\mathcal{O}_{V,an}). \tag{45}$$

Both sequences (44) and (45) have this is common:

1) $V = \mathrm{Spec}(\mathbb{C}(X)) \Rightarrow \mathrm{Pic}(V) = 0$ (Hilbert 90). Hence

$$H^1_{\mathrm{et}}(\mathbb{C}(X),\mu_M) \simeq \mathbb{C}(X)^\times/[\mathbb{C}(X)^\times]^M,$$

where $[\mathbb{C}(X)^\times]^M = \{x^M \mid x \in \mathbb{C}(X)^\times\}$.

2) V Stein (e.g. affine) $\Rightarrow H^1(V,\mathcal{O}_{V,an}) = 0$.

We have this commutative diagram:

$$\begin{array}{ccccc}
\frac{\Gamma(V.\mathcal{O}_V^\times)}{M} & \to & H^1_{\mathrm{et}}(V,\mu_M) & \to & \mathrm{Pic}(V) \\
\big\downarrow & & \big\downarrow\wr & & \\
\frac{\Gamma(V.\mathcal{O}_{V,an}^\times)}{M} & \to & H^1(V,\mathbb{Z}(1)/M\mathbb{Z}(1)) & & \\
\big\uparrow & & \big\uparrow & & \\
\Gamma(V,\mathcal{O}_{V,an}^\times) & \to & H^1(V,\mathbb{Z}(1)) & \to & H^1(V,\mathcal{O}_{V,an}),
\end{array}$$

where the isomorphism $H^1_{\text{et}}(V,\mu_M) \to H^1(V,\mathbb{Z}(1)/M\mathbb{Z}(1))$ is induced by the morphism of sites (étale to analytic, see [Mi]). At the generic point, viz., $V = \text{Spec}(\mathbb{C}(X))$, we arrive at:

$$\begin{array}{ccc} H^1_{\text{et}}(\mathbb{C}(X),\mu_M) & \simeq & H^1(\mathbb{C}(X),\mathbb{Z}(1)/M\mathbb{Z}(1)) \\ \wr| & & \wr| \\ \mathbb{C}(X)^\times/[\mathbb{C}(X)^\times]^M & \simeq & \text{CH}^1(\text{Spec}(\mathbb{C}(X),1)/M \end{array}$$

9.20. **Key example II.** The following is essentially Lemma 3.1 in [Ja2], adapted to our situation.

Proposition 9.21. *Let $\overline{X}/\mathbb{C}$ be a smooth projective variety, and $Y \subset \overline{X}$ a proper closed algebraic subset. Then the cycle class map*

$$CH^2_Y(\overline{X},1) \to H^3_{\mathcal{D},Y}(\overline{X},\mathbb{Z}(2)),$$

is an isomorphism.

Proof. Set $U = Y\backslash Y_{\text{sing}}$. For dimension reasons, we need only consider the case where Y has pure codimension one. So $d_Y := \dim Y = d-1$. Likewise, for the same reasons, we can assume Y_{sing} has pure codimension two in $\overline{X}$. Note that by definition, $\text{CH}^2_Y(\overline{X},1) = \text{CH}^1(Y,1)$. Further, by Poincaré duality, one has a composite:

$$H^3_{\mathcal{D},Y}(\overline{X},\mathbb{Z}(2)) \xrightarrow{\sim} H^{\mathcal{D}}_{2d_Y-1}(Y,\mathbb{Z}(d_Y-1)) \to H^{\mathcal{D}}_{2d_Y-1}(U,\mathbb{Z}(d_Y-1)) \xrightarrow{\sim} H^1_{\mathcal{D}}(U,\mathbb{Z}(1)).$$

Since $\dim Y_{\text{sing}} = d-2$ (hence $H_{2d-3}(Y_{\text{sing}}) = 0$), one easily checks that

$$H^{\mathcal{D}}_{2d-4}(Y_{\text{sing}},\mathbb{Z}(d-2)) \simeq H_{2d-4}(Y_{\text{sing}},\mathbb{Z}(d-2)) \simeq \text{CH}^0(Y_{\text{sing}}),$$

hence

$$\text{CH}^0(Y_{\text{sing}}) \to H^{\mathcal{D}}_{2d-4}(Y_{\text{sing}},\mathbb{Z}(d-2)),$$

is an isomorphism. We have $\text{CH}^0(Y_{\text{sing}},1) = 0$ and $H^{\mathcal{D}}_{2d_Y-1}(Y_{\text{sing}},\mathbb{Z}(d_Y-1)) = 0$ (for dimension reasons). By localization, we arrive at the commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & \text{CH}^1(Y,1) & \to & \text{CH}^1(U,1) & \to & \text{CH}^0(Y_{\text{sing}}) \\ & & \| & & \downarrow\wr & & | \\ & & \text{CH}^2_Y(\overline{X},1) & & \mathcal{O}^\times_U(U) & & | \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \to & H^3_{\mathcal{D},Y}(\overline{X},\mathbb{Z}(2)) & \to & H^1_{\mathcal{D}}(U,\mathbb{Z}(1)) & \to & H^{\mathcal{D}}_{2d-4}(Y_{\text{sing}},\mathbb{Z}(d-2)) \end{array}$$

Now apply the five-lemma. □

Using Proposition 9.21, we deduce:

Proposition 9.22 ([dJ-L](Cor. 6.5))**.**

$$\frac{\Gamma H^2(\mathbb{C}(\overline{X}),\mathbb{Z}(2))}{d\log CH^2(\text{Spec}(\mathbb{C}(\overline{X})),2)} \simeq \ker\left[\frac{CH^2_{\text{hom}}(\overline{X},1)}{CH^2_{\text{dec}}(\overline{X},1)} \xrightarrow{\underline{AJ}} J\left(\frac{H^2(\overline{X},\mathbb{Z}(2))}{H^2_{\text{alg}}(\overline{X},\mathbb{Z}(2))}\right)\right].$$

Remark 9.23. The Merkurjev-Suslin theorem implies that the former (hence latter) term is uniquely divisible (see Theorem 9.31). Thus $\underline{AJ}$ is injective on torsion.

Proof. Let $U := \overline{X}\backslash Y$ be given such that $H^2(U,\mathbb{Z}(2))$ is torsion-free. There is a diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & \mathrm{CH}^2_{Y,\mathrm{dec}}(\overline{X},1) & & \\ & & \downarrow & & \\ \mathrm{CH}^2(U,2) & \to & \mathrm{CH}^2_Y(\overline{X},1)^\circ & \to & \mathrm{CH}^2_{\mathrm{hom}}(\overline{X},1) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \Gamma H^2(U,\mathbb{Z}(2)) & \hookrightarrow & \Gamma H^3_Y(\overline{X},\mathbb{Z}(2))^\circ & \to & J\big(\frac{H^2(\overline{X},\mathbb{Z}(2))}{H^2_Y(\overline{X},\mathbb{Z}(2))}\big) \end{array}$$

Now apply the snake lemma and shrink U. □

In light of Remark 9.23, we mention the following:

Theorem 9.24 (Bruno Kahn [KaB])**.**

$$\left\{\frac{CH^2(\overline{X},1)}{CH^2_{\mathrm{dec}}(\overline{X},1)}\right\}_{\mathrm{tor}} \simeq H^2_{\mathrm{tr}}(\overline{X},\mathbb{Q}/\mathbb{Z}(2)),$$

where

$$H^2_{\mathrm{tr}}(\overline{X},\mathbb{Z}) = \mathrm{cok}\big(H^2_{\mathrm{alg}}(\overline{X},\mathbb{Z}) \to H^2(\overline{X},\mathbb{Z})\big).$$

One then anticipates via the Abel-Jacobi map $\underline{AJ}$ that

$$\left\{\frac{\mathrm{CH}^2_{\mathrm{hom}}(\overline{X},1)}{\mathrm{CH}^2_{\mathrm{dec}}(\overline{X},1)}\right\}_{\mathrm{tor}} \simeq J\big(H^2_{\mathrm{tr}}(\overline{X},\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{tor}}.$$

Another application of the Merkurjev-Suslin theorem (see paragraph 9.27 below) gives:

Theorem 9.25 (Asakura. See [dJ-L](Thm 11.1))**.** *Let U be a smooth complex variety. Then the cokernel of*

$$\mathrm{cl}_{2,2} : CH^2(U,2) \to H^2(U,\mathbb{Z}(2))$$

is torsion-free.

As an immediate consequence, we have:

Corollary 9.26 ([dJ-L](Cor. 11.2))**.** *The following statements are equivalent:*

- $\mathrm{cl}_{2,2} : CH^2(U,2) \to \Gamma H^2(U,\mathbb{Z}(2))$ *is surjective.*
- $\mathrm{cl}_{2,2,\mathbb{Q}} : CH^2(U,2;\mathbb{Q}) \to \Gamma H^2(U,\mathbb{Q}(2))$ *is surjective.*

9.27. **Bloch-Kato theorem and its consequences.** Let $\mathbb{F}$ be a field with multiplicative group $\mathbb{F}^\times \subset \mathbb{F}$. Consider the graded tensor algebra

$$T(\mathbb{F}) := \bigoplus_{m=0}^{\infty}\{\mathbb{F}^\times\}^{\otimes_{\mathbb{Z}} m} = \mathbb{Z}\oplus\mathbb{F}^\times\oplus\cdots,$$

and let $R(\mathbb{F})$ be the graded 2-sided ideal generated by

$$\big\{\tau\otimes(1-\tau)\ \big|\ \tau\in\mathbb{F}^\times\backslash\{1\}\big\}.$$

Recall that the Milnor K-theory of $\mathbb{F}$ is given by

$$K_{\bullet}^{M}(\mathbb{F}) := T(\mathbb{F})/R(\mathbb{F}) = \bigoplus_{m=0}^{\infty} K_m^M(\mathbb{F}).$$

Further, recall that $K_m^M(\mathbb{F}) \simeq \mathrm{CH}^m(\mathrm{Spec}(\mathbb{F}), m)$, (Nesterenko/Suslin (1990), Totaro (1992)).

Theorem 9.28 (Bloch-Kato theorem)**.** *For a prime $M \neq char(\mathbb{F})$, the norm-residue map*

$$K_m^M(\mathbb{F})/M \to H_{\mathrm{et}}^m(\mathbb{F}, \mu_M^{\otimes m}),$$

is an isomorphism.

Remark 9.29. (i) The proof of the Bloch-Kato theorem is due to V. Voevodsky and M. Rost (see [We]).

(ii) The case $m = 1$ is due to the aforementioned Kummer short exact sequence and Hilbert 90.

(iii) The case $M = 2$ is the former Milnor conjecture, proven by V. Voevodsky.

(iv) The case $m = 2$ is the Merkurjev-Suslin theorem.

In our situation, the Bloch-Kato theorem translates to saying,

Theorem 9.30. *The map*

$$\frac{CH^m(\mathrm{Spec}(\mathbb{C}(X)), m)}{M \cdot CH^m(\mathrm{Spec}(\mathbb{C}(X)), m)} \to H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m))}{M \cdot \mathbb{Z}(m)}\right),$$

is an isomorphism, for any integer $M \neq 0$.

We have the following:

Theorem 9.31 ([dJ-L](Thm 7.7))**.** (i) *$\forall i$, $H_{\mathrm{tor}}^i(\mathbb{C}(X), \mathbb{Z}) = 0$. In particular, the torsion subgroup of $H^i(\overline{X}, \mathbb{Z})$, where $\overline{X}$ is a smooth projective completion of X, is supported in codimension one, generalizing the case $i = 2$ from the Lefschetz $(1, 1)$ theorem.*

(ii) $\ker(\lim \mathrm{cl}_{m,m})$ *is divisible.*

(iii) *The groups:*

$$\frac{H^m(\mathbb{C}(X), \mathbb{Z}(m))}{\Gamma\big(H^m(\mathbb{C}(X), \mathbb{Z}(m))\big)}, \quad \frac{\Gamma\big(H^m(\mathbb{C}(X), \mathbb{Z}(m))\big)}{\mathrm{Image}(\mathrm{cl}_{m,m})},$$

are uniquely divisible.

Remark 9.32. Although part (i) of Theorem 9.31 is known among some experts, the knowledge of this fact doesn't seem to be universal.

Proof. First observe that the map in Theorem 9.30 is given by the composite

$$\begin{array}{ccc} \frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)), m)}{M} & \to & H^m\left(\mathbb{C}(X), \frac{\mathbb{Z}(m)}{M}\right) \\ \searrow & & \nearrow \\ & \frac{H^m(\mathbb{C}(X), \mathbb{Z}(m))}{M} & \end{array}$$

Notice that the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\times M} \mathbb{Z} \to \mathbb{Z}/M\mathbb{Z} \to 0,$$

induces the short exact sequence:

$$0 \to \frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M} \to H^m\left(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{M}\right) \to H^{m+1}_{M-\mathrm{tor}}(\mathbb{C}(X),\mathbb{Z}(m)) \to 0.$$

By Theorem 9.30, it follows that $H^{m+1}_{M-\mathrm{tor}}(\mathbb{C}(X),\mathbb{Z}(m)) = 0$, thus proving part (i). Next observe that

$$\Gamma\big(H^m(\mathbb{C}(X),m)\big) = F^m H^m(\mathbb{C}(X),\mathbb{C}) \cap H^m(\mathbb{C}(X),\mathbb{Z}(m)),$$

and hence

$$\Gamma\big(H^m(\mathbb{C}(X),m)\big) \cap M \cdot H^m(\mathbb{C}(X),\mathbb{Z}(m)) = M\cdot\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big).$$

By Theorem 9.30, we have the commutative diagram of isomorphisms:

$$\begin{array}{ccc}
\frac{\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)}{M\cdot\mathrm{CH}^m(\mathrm{Spec}(\mathbb{C}(X)),m)} & \simeq & H^m\big(\mathbb{C}(X),\frac{\mathbb{Z}(m)}{M\cdot\mathbb{Z}(m)}\big) \\
\wr\| & & \\
\frac{\mathrm{Image}(\mathrm{cl}_{m,m})}{M\cdot\mathrm{Image}(\mathrm{cl}_{m,m})} & & \wr\uparrow \\
\wr\| & & \\
\frac{\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big)}{M\cdot\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big)} & \simeq & \frac{H^m(\mathbb{C}(X),\mathbb{Z}(m))}{M\cdot H^m(\mathbb{C}(X),\mathbb{Z}(m))},
\end{array}$$

for which parts (ii) and (iii) easily follow. $\square$

Corollary 9.33 ([dJ-L](Cor. 7.8))**.**

$$\frac{\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big)}{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m})} = \frac{\Gamma\big(H^m(\mathbb{C}(X),\mathbb{Q}(m))\big)}{\mathrm{Image}(\mathrm{cl}^{\lim}_{m,m,\mathbb{Q}})}.$$

In particular

$$d\log : CH^m(\mathrm{Spec}(\mathbb{C}(X)),m) \to \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Z}(m))\big),$$

is surjective iff $d\log\otimes\mathbb{Q}$:

$$CH^m(\mathrm{Spec}(\mathbb{C}(X)),m;\mathbb{Q}) \to \Gamma\big(H^m(\mathbb{C}(X),\mathbb{Q}(m))\big),$$

is surjective.

References

[Ar] Arapura, D. *The Leray spectral sequence is motivic,* Invent. Math. **160** (2005), no. 3, 567-589.

[A-K] Arapura, D., Kumar, M. *Beilinson-Hodge cycles on semiabelian varieties.* Math. Res. Lett. **16**, (2009), no. 4, 557–562.

[A-L] Angel, P. L., Lewis, J. D., *et al,* A going-up theorem for K-theory and normal functions. Manuscript in progress.

[A] Asakura, M. *Motives and algebraic de Rham cohomology,* in: The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the CRM Summer School, June 7-19, 1998, Banff, Alberta, Canada (Editors: B. Gordon, J. Lewis, S. Müller-Stach, S. Saito and N. Yui), NATO Science Series **548** (2000), Kluwer Academic Publishers.

______ *On* $d\log$ *image of* K_2 *of elliptic surface minus singular fibers,* preprint, 2006, `arXiv:math/0511190v4`.

[A-S] Asakura, M., Saito, S. *Beilinson's Hodge conjecture with coefficients,* in Algebraic Cycles and Motives, Vol. 2, Edited by J. Nagel and C. Peters, LNS **344**, London Math. Soc., (2007), 3-37.

[B-T] Bass, H. and Tate, J. *The Milnor ring of a global field, in Algebraic* K*-theory II*, Lecture Notes in Math. **342**, Springer-Verlag, 1972, 349–446.

[Be1] Beilinson, A. *Notes on absolute Hodge cohomology*, in Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Contemporary Mathematics **55**, part 1, AMS, Providence 1986, 35-68.

[Be2] Beilinson, *Higher regulators and values of L-functions,* J. Sov. Math. **30** (1985), 2036-2070.

[B-S] Bloch, S. and Srinivas, V. *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105** (1983), 1235–1253.

[Blo1] Bloch, S. *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[Blo2] ———, *Algebraic cycles and the Beilinson conjectures,* Contemporary Math. Vol. **58**, Part I, (1986), 65-79.

[Blo3] Bloch, S., Lectures on algebraic cycles, Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980. 182 pp.

[BO] Bloch, S., Ogus, A. *Gersten's conjecture and the homology of schemes,* Annales scientifiques de l'É.N.S. 4^e série, tome **7**, no. 2 (1974), 181-201.

[BKLP] Burgos (Appendix), Kerr, Lewis, Patrick. Simplicial Abel-Jacobi maps and reciprocity laws. In progress.

[BU] Burgos, J. *A C^∞ logarithmic Dolbeault complex,* Compositio Math. **92** (1994), no. 1, 61-86.

[Ca] Carlson, J. *Extension of mixed Hodge structures.* In Journées de Géométrie Algébrique d'Angers 1979, Sijhoff and Nordhoff (1980), 107-127.

[C-L1] Chen, X., Lewis, J. D. *Density of rational curves on $K3$ surfaces.* Math. Ann. **356** (2013), no. 1, 331-354.

[C-L2] ———, *The Hodge$-\mathcal{D}-$conjecture for $K3$ and Abelian surfaces*, J. Algebraic Geom. **14** (2005), no. 2, 213-240.

[C-L3] ———, *Noether-Lefschetz for K_1 of a certain class of surfaces,* Bol. Soc. Mat. Mexicana (3) **10** (2004), no. 1, 29-41.

[CDKL] Chen, X., Doran, C., Kerr, M., Lewis, J. D. *Higher normal functions, derivatives of normal functions, and elliptic fibrations.* To appear in Crelle.

[Cl] Clemens, C. H. *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Publ. I.H.E.S. **58** (1983), 19–38.

[Co1] Collino, A. *Griffiths' infinitesimal invariant and higher K-theory on hyperelliptic jacobians*, J. Algebraic Geometry **6** (1997), 393–415.

[Co2] ———, *Indecomposable motivic cohomology classes on quartic surfaces and on cubic fourfolds*, Algebraic K-theory and its applications (Trieste, 1997), World Sci. Publishing, River Edge, NJ, 1999, 370–402.

[CF] Collino, A., Fakhruddin, N. *Indecomposable higher Chow cycles on Jacobians,* Math. Z. **240** (2002), no. 1, 111-139.

[C-S] Coombes, K., Srinivas, V. *A remark on K_1 of an algebraic surface*, Math. Ann. **265** (1983), 335–342.

[dJ-L] de Jeu, R., Lewis, J. D. (with an appendix by M. Asakura), *Beilinson's Hodge conjecture for smooth varieties.* J. K-Theory **11** (2013), no. 2, 243-282.

[dJ-L-P] de Jeu, R., Lewis, J. D., Patel, D., *A relative version of the Beilinson-Hodge conjecture,* to appear in Recent advances in Hodge theory: Period domains, algebraic cycles, and arithmetic M. Kerr and G. Pearlstein, Eds., Cambridge University Press. 18 pages.

[De] Deligne, P. *Théorie de Hodge, II, III,* Inst. Hautes Études Sci. Publ. Math. No. **40** (1971), 5-57; No. **44** (1974), 5-77.

[E-P] Esnault, H., Paranjape, K. H. *Remarks on absolute de Rham and absolute Hodge cycles,* C. R. Acad. Sci. Paris, t. **319**, Serie I, (1994), 67-72.

[EV] Esnault, H. and Viehweg, E. *Deligne-Beilinson cohomology*, in Beilinson's Conjectures on Special Values of L-Functions, (Rapoport, Schappacher, Schneider, eds.), Perspect. Math. **4**, Academic Press, 1988, 43–91.

[EV-MS] Elbaz-Vincent, P., Müller-Stach, S. *Milnor K-theory of rings, higher Chow groups and applications*, Invent. Math. **148** (2002), 177–206.

[G-L] Gordon, B. B., Lewis, J. D. *Indecomposable higher Chow cycles.* In The arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), 193-224, NATO Sci. Ser. C Math. Phys. Sci., **548**, Kluwer Acad. Publ., Dordrecht, 2000.

[GG] Green, M. and Griffiths, P. *The regulator map for a general curve.* Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), 117-127, Contemp. Math., **312**, Amer. Math. Soc., Providence, RI, 2002.

[GH] Griffiths, P. and Harris, J. *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[G] Green, M. *Griffiths' infinitesimal invariant and the Abel-Jacobi map*, J. Differential Geom. **29** (1989), 545–555.

[Gr] Griffiths, P. A. *On the periods of certain rational integrals: I and II*, Annals of Math. **90** (1969), 460–541.

[GGK] M. Green, P. Griffiths, M. Kerr, *Néron models and limits of Abel-Jacobi mappings,* Compos. Math. **146** (2010), no. 2, 288 - 366.

[G-S] Green, M., Müller-Stach, S. *Algebraic cycles on a general complete intersection of high multi-degree of a smooth projective variety*, Comp. Math. **100** (3), (1996) 305–309.

[Ja1] Jannsen, U. *Deligne cohomology, Hodge-$\mathcal{D}$-conjecture, and motives*, in Beilinson's Conjectures on Special Values of L-Functions, (Rapoport, Schappacher, Schneider, eds.), Perspect. Math. **4**, Academic Press, 1988, 305–372.

[Ja2] ———, Mixed Motives and Algebraic K-Theory, Lecture Notes in Math. **1000** (1990), Springer-Verlag, Berlin.

[Ja3] ———, *Equivalence relations on algebraic cycles,* in: The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the CRM Summer School, June 7-19, 1998, Banff, Alberta, Canada (Editors: B. Gordon, J. Lewis, S. Müller-Stach, S. Saito and N. Yui), NATO Science Series **548** (2000), 225-260, Kluwer Academic Publishers.

[KaB] Kahn, B. *Groupe de Brauer et* $(2,1)$*-cycles indecomposables.* Preprint 2011.

[Ka] Kato, K. *Milnor K-theory and the Chow group of zero cycles*, Contemporary Mathematics, Part I, **55** 1986, 241–253.

[KLM] Kerr, M., Lewis, J. D., Müller-Stach, S. *The Abel-Jacobi map for higher Chow groups,* Compos. Math. **142** (2006), no. 2, 374-396.

[K-L] Kerr, M., Lewis, J. D. *The Abel-Jacobi map for higher Chow groups, II,* Invent. Math. **170** (2007), no. 2, 355-420.

[Ke] Kerz, M. *The Gersten conjecture for Milnor K-theory,* Invent. Math. **175**, no. 1, (2009), 1-33.

[Ki] King, J. *Log complexes of currents and functorial properties of the Abel-Jacobi map*, Duke Math. J. **50** (1983), no. 1, 1-53.

[Lev] Levine, M. *Localization on singular varieties*, Invent. Math. **31** (1988), 423–464.

[Lew1] Lewis, J. D. A Survey of the Hodge Conjecture. Second edition. Appendix B by B. Brent Gordon. CRM Monograph Series, **10**. American Mathematical Society, Providence, RI, 1999. xvi+368 pp.

[Lew2] ———, *Real regulators on Milnor complexes*, K-Theory **25** (2002), no. 3, 277-298.

[Lew3] ———, *Regulators of Chow cycles on Calabi-Yau varieties.* In Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), 87-117, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.

[Lew4] ———, *Lectures on algebraic cycles*, Bol. Soc. Mat. Mexicana (3) **7** (2001), no. 2, 137-192.

[Lew5] ———, *A filtration on the Chow groups of a complex projective variety,* Compositio Math. **128** (2001), no. 3, 299-322.

[Lew6] ———, *Abel-Jacobi equivalence and a variant of the Beilinson-Hodge conjecture,* J. Math. Sci. Univ. Tokyo **17** (2010), 179-199.

[Lew7] ———, *Arithmetic normal functions and filtrations on Chow groups,* Proc. Amer. Math. Soc. **140** (2012), no. 8, 2663-2670.

[Lew8] ———, *A note on indecomposable motivic cohomology classes,* J. Reine Angew. Math. **485** (1997), 161-172.

[Lew9] ———, *Cycles on varieties over subfields of $\mathbb{C}$ and cubic equivalence,* Motives and algebraic cycles, 233-247, Fields Inst. Commun., **56**, Amer. Math. Soc., Providence, RI, 2009.

[Lew10] ———, *Hodge type conjectures and the Bloch-Kato theorem.* in Hodge theory, complex geometry, and representation theory, 235-258, Contemp. Math., **608**, Amer. Math. Soc., Providence, RI, 2014.

[Lew11] ———, *Transcendental methods in the study of algebraic cycles with a special emphasis on Calabi-Yau varieties,* in Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, 29-69, Fields Inst. Commun., **67**, Springer, New York, 2013.

[L-S] Lewis, J. D., Saito, S. *Algebraic cycles and Mumford-Griffiths invariants,* Amer. J. Math. **129** (2007), no. 6, 1449-1499.

[Mi] Milne, J. Étale Cohomology, Princeton Mathematical Series, **33**, Princeton University Press, Princeton, N.J., 1980.

[MS1] Müller-Stach, S. *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geometry **6** (1997), 513–543.

[MS2] ———, *Algebraic cycle complexes*, in Proceedings of the NATO Advanced Study Institute on the Arithmetic and Geometry of Algebraic Cycles **548**, (Lewis, Yui, Gordon, Müller-Stach, S. Saito, eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, 285–305.

[Ra] Raynaud, M. *Courbes sur une variété abélienne et points de torsion*, Invent. Math. **71** (1983), 207–233.

[MSa1] Saito, M., *Direct image of logarithmic complexes and infinitesimal invariants of cycles,* Algebraic cycles and motives. Vol. 2, 304-318, London Math. Soc. Lecture Note Ser., **344**, Cambridge Univ. Press, Cambridge, 2007.

[MSa2] ———, *Hodge-type conjecture for higher Chow groups,* Pure and Applied Mathematics Quarterly Volume **5**, Number 3, (2009), 947-976.

[P] Paranjape, K. *Cohomological and cycle-theoretic connectivity,* Ann. of Math. (2) **139** (1994), no. 3, 641-660.

[Sa] Saito, S. *Beilinson's Hodge and Tate conjectures,* in Transcendental Aspects of Algebraic Cycles, Edited by S. Müller-Stach and C. Peters, LNS **313**, London Math. Soc., (2004), 276–290.

[St] Steenbrink, J. H. M. *A summary of mixed Hodge theory*, Motives (Seattle, WA, 1991), 31-41, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[SJK-L] Kang, S-J., Lewis, J. D. *Beilinson's Hodge conjecture for K_1 revisited,* Cycles, motives and Shimura varieties, 197-215, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2010.

[Vo] Voisin, C. *The Griffiths group of a general Calabi-Yau threefold is not finitely generated*, Duke Math. J. **102** (1) (2000), 151–186.

[We] Weibel, C. *The norm residue isomorphism theorem,* Topology **2**, (2009), 346-372.

632 Central Academic Building, University of Alberta, Edmonton, Alberta T6G 2G1, CANADA

E-mail address: lewisjd@ualberta.ca