SOS Optimization For Region Of Attraction Analysis

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Lyapunov Functions

$$\dot{x} = f(x) \quad 0 = f(0)$$

Suppose we can find a function *V(x)* and constants c_1, c_2 > 0 such that the following holds:

$$\begin{aligned} V(x) - c_1 x^T x \geq 0 \\ V(0) = 0 \\ \nabla V(x)^T f(x) + c_2 x^T x \leq 0 \end{aligned} \quad \Rightarrow \quad \dot{x} = f(x)$$

is asymptotically stable!

Lyapunov Functions

Linear Systems

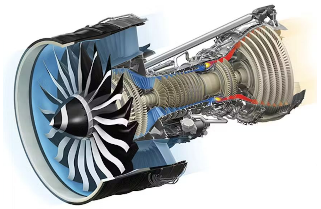
$$\dot{x} = Ax$$

$$\begin{aligned} &\text{find } P \succ 0 \\ &\text{ s.t. } AP + PA \prec 0 \end{aligned}$$

$$V(x) = x^T P x$$

Jet Engine

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$



(See [2] for derivation)

Searching For Lyapunov Functions

$$
\begin{aligned}
\text{find } & V \in \Omega \\
\text{s.t. } & V(x) - \epsilon x^T x \geq 0 \quad \forall x \\
& V(0) = 0 \\
& \nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x
\end{aligned}
$$

Problem: Impossible to solve in its most general case!! [2]

Searching For Lyapunov Functions

$$\begin{aligned} \text{find } & V \in \Omega \\ \text{s.t. } & V(x) - \epsilon x^T x \geq 0 \quad \forall x \\ & V(0) = 0 \\ & \nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x \end{aligned}$$

Restrictions

1. Restrict class of systems to polynomial systems: f(x) is polynomial
2. Restrict search space of V(x) to polynomial functions

Searching For Lyapunov Functions

$$\begin{aligned} \text{find } & V \in \Omega \\ \text{s.t. } & V(x) - \epsilon x^T x \geq 0 \quad \forall x \\ & V(0) = 0 \\ & \nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x \end{aligned}$$

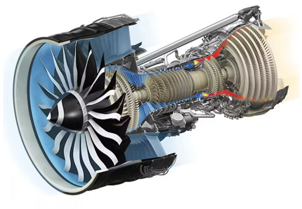
Equivalent to checking the non-negativity or non-positivity of polynomials!

Polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d$$

$$f(x, y) = 1 + x + xy + x^3 y + xy^4$$

$$f(x, y) = \begin{bmatrix} -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\ 3x - y \end{bmatrix}$$



Monomial: Polynomial with a single term

Sum of Squares (SOS) Polynomials

A polynomial *p(x)* in R[x] is a **Sums-of-Squares (SOS)** if there exists polynomials $g_i(x)$ in R[x] s.t.

$$p(x) = \sum_{i=1}^{k} g_i(x)^2$$

Equivalently, we write that $p \in \Sigma_s$

SOS Polynomials

Let $Z_d(x)$ be a basis of all the monomials of x of degree less than d.

Ex. for $x = [x_1\ x_2]^T$

$$Z_2(x) = \begin{bmatrix}1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2\end{bmatrix}^T$$

Suppose $p(x)$ is a polynomial of degree $2d$. Then:

$$p \in \Sigma_S \iff \exists M \succeq 0 \text{ s.t } p(x) = Z_d(x)^T M Z_d(x)$$

[2]

Suppose *p(x)* is a polynomial of degree *2d.* Then:

$$p \in \Sigma_S \iff \exists M \succeq 0 \text{ s.t } p(x) = Z_d(x)^T M Z_d(x)$$

[2]

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \in \Sigma_S \ ?$$

Check SOS Condition…

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

$$= (2xy + y^2)^2 + (2x^2 + xy + 3y^2)^2$$

Suppose *p(x)* is a polynomial of degree *2d.* Then:

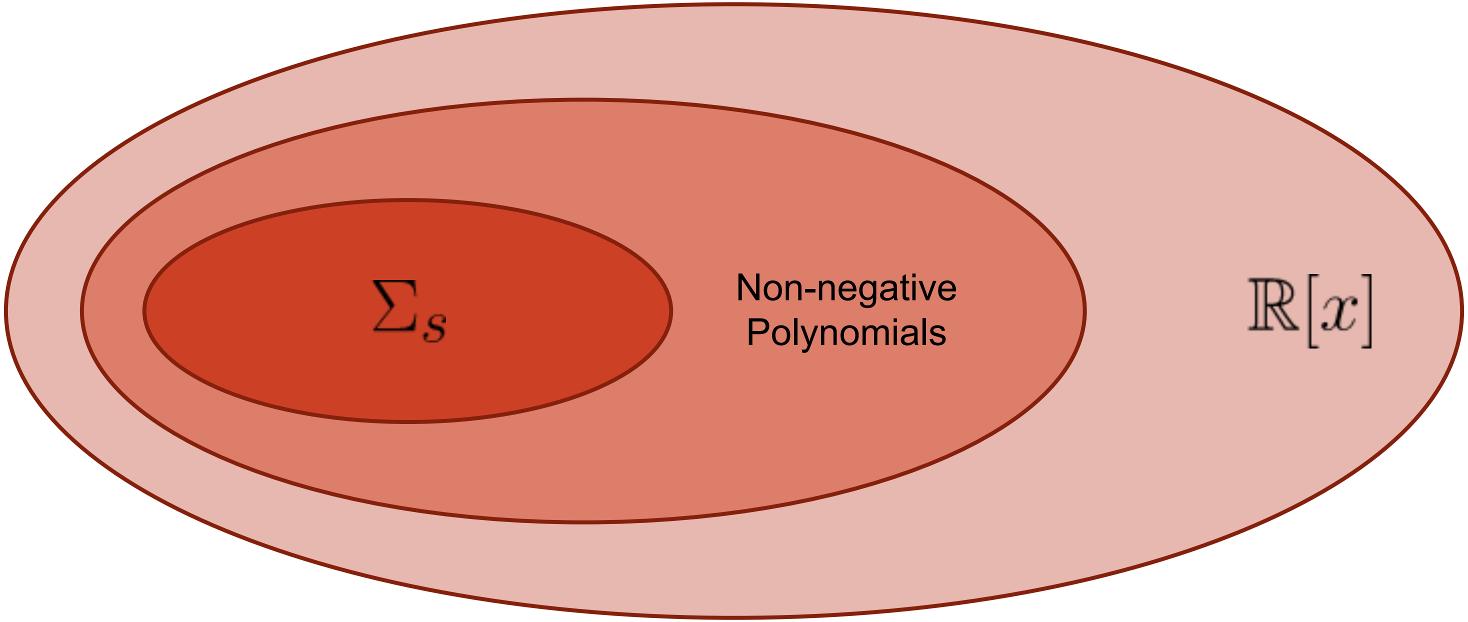
$$p \in \Sigma_S \iff \exists M \succeq 0 \text{ s.t } p(x) = Z_d(x)^T M Z_d(x)$$

[2]

$$x^4y^2+x^2+y^4+z^6-3x^2y^2z^2 \geq 0 \quad \forall x, y, z \quad \text{but...}$$

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \notin \Sigma_S$$

SOS Polynomials



Searching For Lyapunov Functions

$$\begin{aligned} &\text{find } V \in \Omega \\ \text{s.t. } &\boxed{V(x) - \epsilon x^T x \geq 0 \quad \forall x} \\ &V(0) = 0 \\ &\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x \end{aligned}$$

Equivalent to checking the non-negativity or non-positivity of polynomials!

SOS Constraints at LMIs

$$V(x) - \epsilon x^T x \geq 0 \quad \Longleftarrow \quad V(x) - \epsilon x^T x \in \Sigma_S$$

$$\Longleftrightarrow \quad \boxed{\begin{array}{c} V(x) - \epsilon x^T x = Z_d(x)^T M Z_d(x) \\ M \succ 0 \end{array}} \quad \textbf{LMI}$$

Key Takeaway: If a non-negativity constraint is affine in a polynomial decision variable, a sufficient condition can be written as a **LMI** [2].

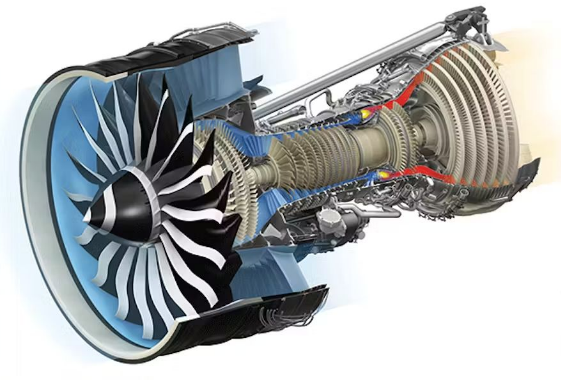
SOS Optimization

$$\begin{aligned} &\text{find } V \in \Omega \\ &\text{s.t. } V(x) - \epsilon x^T x \boxed{\in \Sigma_S} \\ &\quad\;\; \nabla V(x)^T f(x) + \gamma x^T x \boxed{\in \Sigma_S} \\ &\quad\;\; V(0) = 0 \end{aligned}$$

Searching for a fixed-degree polynomial Lyapunov function for a polynomial system is an LMI

Returning to our jet engine…

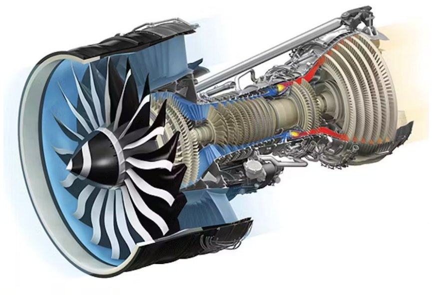
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x, y) = \begin{bmatrix} -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\ 3x - y \end{bmatrix}$$



(See [2] for derivation)

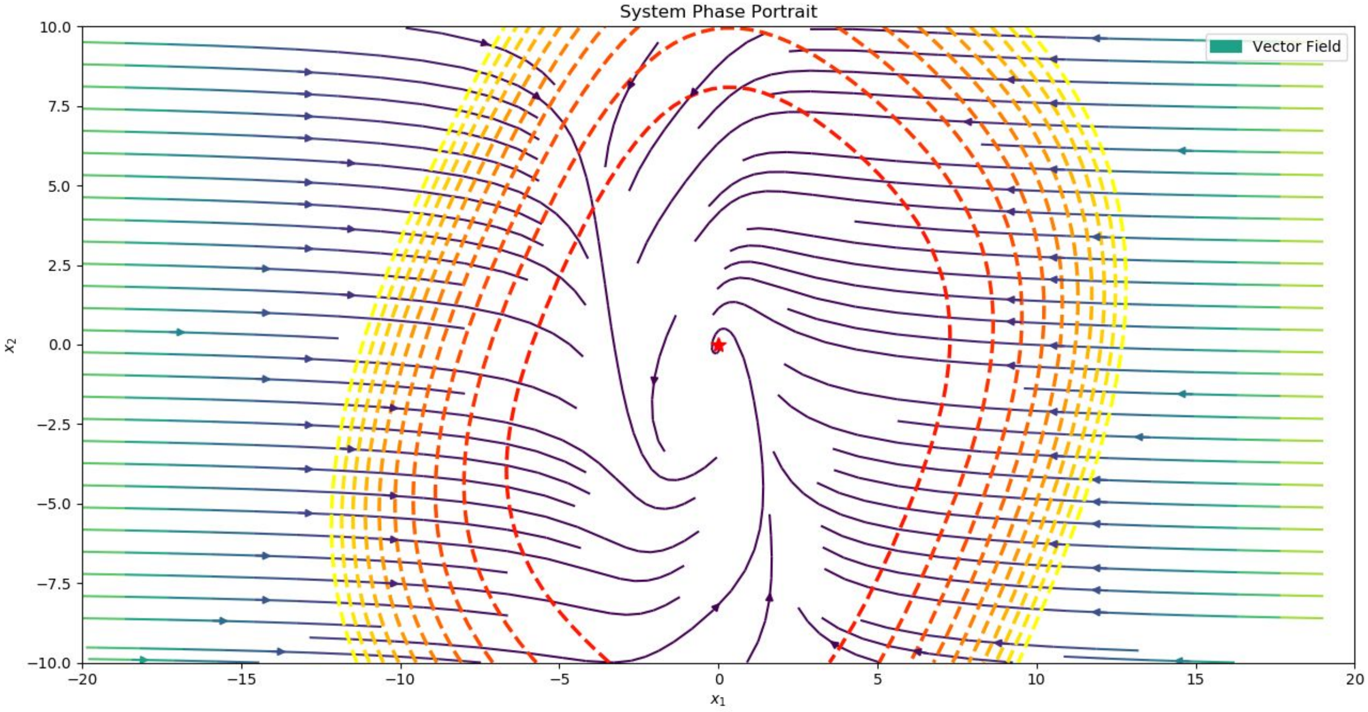
Jet Engine: SOS Program

$$
\begin{aligned}
\text{find } & V(x) \\
\text{s.t. } & V(x) \in \Sigma_S \\
& -\nabla V(x)^T f(x) \in \Sigma_S \\
& V(0) = 0,\ V([1,0]) = 1
\end{aligned}
$$



$$
V(x) = -0.560127 x_0 x_1 - 0.171134 x_0 x_1^2 + 0.933423 x_0^2 x_1 + 0.208393 x_0^2 x_1^2 - 0.130605 x_0^3 x_1 + 1.294490 x_0^2 - 0.436473 x_0^3 + 0.141984 x_0^4 + 0.671026 x_1^2 + 0.243786 x_1^3 + 0.030216 x_1^4
$$

Jet Engine: Lyapunov Function



Motivation: Regions of Attraction (ROA)

1. We may not need the system to be globally stable.

$$V(x) \geq 0,\ \dot{V}(x) \leq 0 \quad \forall x \geq 0$$

2. Many systems of interest **are not** globally stable. We want to identify ROAs where the system is **provably** stable.

Motivation: Regions of Attraction

S-Procedure

We desire a tool to check the positivity of a polynomial on a given set

$$V(x) \geq 0,\ \dot{V}(x) \leq 0 \quad \forall x \in \mathcal{R}$$

Boils down to checking the non-negativity polynomials **on a set**

Given a polynomial p(x) and a polynomial of vectors g(x), then:

$$\begin{aligned} &\text{find } \lambda(x) \in \Sigma_S \\ &\text{s.t. } p(x) + \lambda^T(x) g(x) \in \Sigma_S \end{aligned} \implies p(x) \geq 0 \quad \forall x \in \{x | g(x) \leq 0\}$$

[1]

S-Procedure

Given a polynomial p(x) and a polynomial of vectors g(x), then:

$$\begin{aligned} &\text{find } \lambda(x) \in \Sigma_S \\ &\text{s.t. } p(x) + \lambda^T(x) g(x) \in \Sigma_S \end{aligned} \implies p(x) \geq 0 \quad \forall x \in \{x | g(x) \leq 0\}$$

[1]

When $x \in \{x|g(x) \leq 0\}$:

$$\begin{aligned} p(x) + \lambda^T(x)g(x) \in \Sigma_S &\implies p(x) + \lambda^T(x)g(x) \geq 0 \\ &\implies p(x) \geq -\lambda^T(x)g(x) \geq 0 \end{aligned}$$

Region of Attraction (ROA)

$$\mathcal{R} = \{x | g(x) \leq 0\}$$

R is a region of attraction (ROA) if

1. *R* is an invariant set. $\text{Let } \mathcal{R} = \{x | V(x) \leq \rho\} \implies g(x) = V(x) - \rho$

2. Lyapunov conditions hold on *R*

$$\left.\begin{aligned} &V(x) - \epsilon x^T x \geq 0 \quad \forall x \in \mathcal{R} \\ &\nabla V(x)^T f(x) + \epsilon x^T x \leq 0 \quad \forall x \in \mathcal{R} \\ &V(0) = 0 \end{aligned}\right\} \text{S-Procedure}$$

Certifying ROA via SOS Optimization

Given a candidate V(x), find the largest sublevel ROA.

$$
\begin{aligned}
&\max \rho \\
&\text{s.t.} \; -\nabla V(x)^T f(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\
&\qquad \lambda(x) \in \Sigma_S
\end{aligned}
$$

Solve via line search (iterative feasibility problems) [2].

Certifying ROA via SOS Optimization

1. Find the closed loop dynamics f(x)

2. Propose a candidate lyapunov function V(x)

3. Solve the following optimization problem via line search.

$$
\begin{aligned}
&\max \rho \\
&\text{s.t.} \; -\nabla V(x)^T f(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\
&\qquad \lambda(x) \in \Sigma_S
\end{aligned}
$$

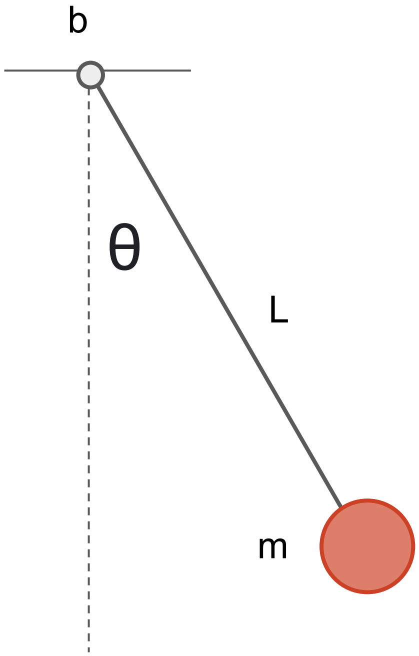
V(x) < ρ* is the largest certifiable ROA given V(x)

Inverted Pendulum: Step 1

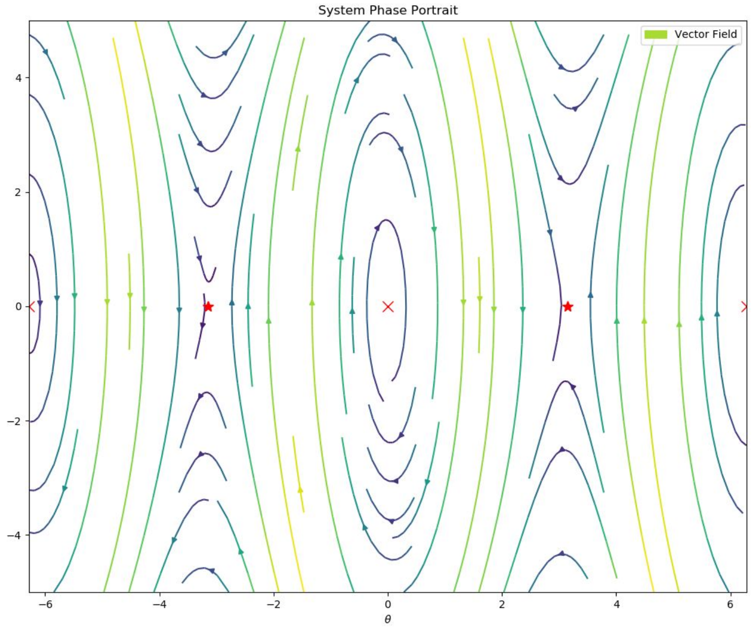
$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

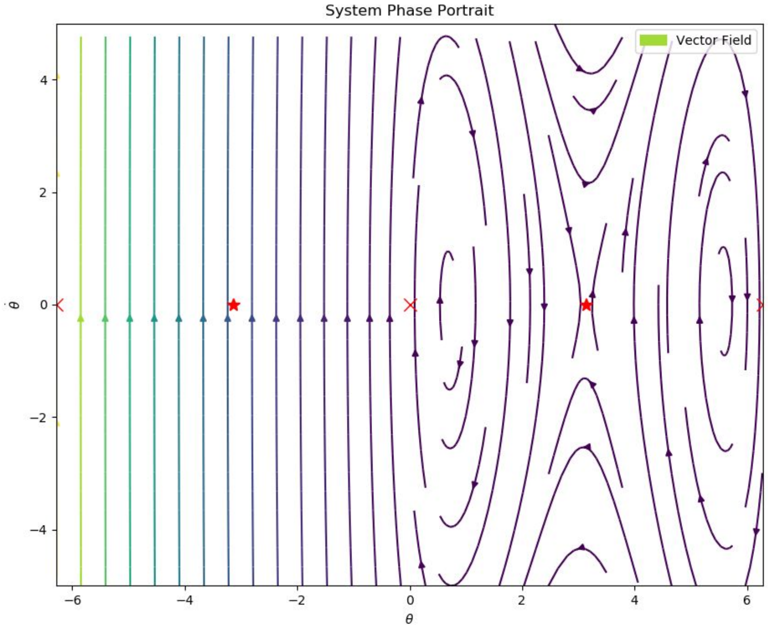
$$u = -K\bar{x}, \quad \bar{x} = \begin{bmatrix} \theta - \pi \\ \dot{\theta} \end{bmatrix}$$

$$\dot{\bar{x}} = \hat{f}(\bar{x}) = \begin{bmatrix} \bar{x}_2 \\ \frac{-K\bar{x} - b\dot{\bar{x}}_2 + mgl(\bar{x}_1 - \frac{\bar{x}_1^3}{3!} + \frac{\bar{x}_1^5}{5!})}{ml^2} \end{bmatrix}$$



Inverted Pendulum: Step 1





Inverted Pendulum: Step 2

$$(K, P) = \mathrm{lqr}(A, B, Q, R)$$

$$V(x) = x^T P x$$

Inverted Pendulum: Step 3

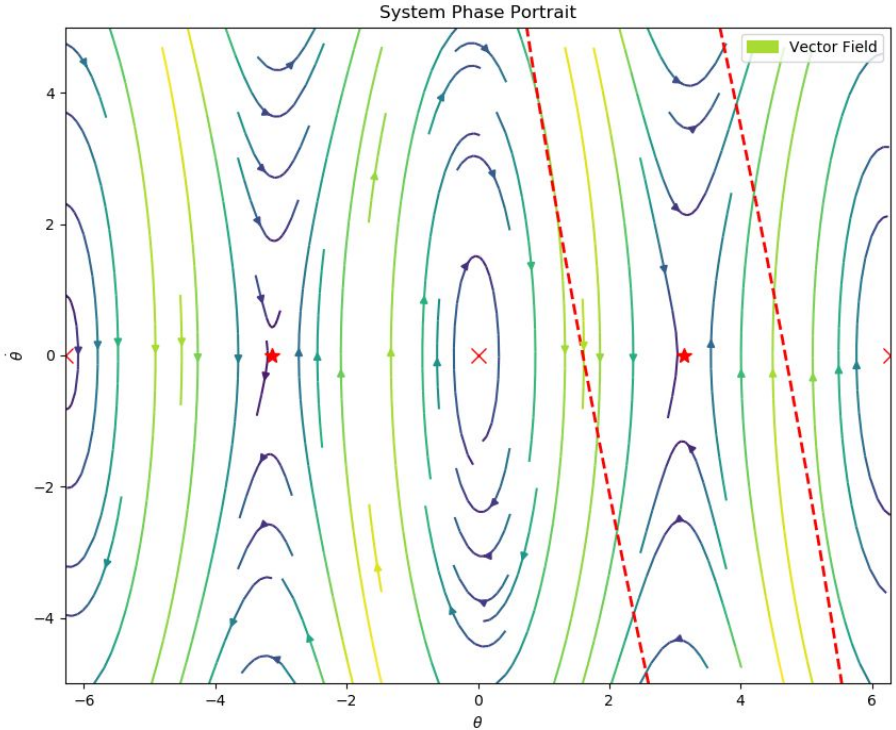
Given:

$$\dot{\bar{x}} = \hat{f}(\bar{x}) = \begin{bmatrix} \bar{x}_2 \\ \frac{-K\bar{x} - b\dot{\bar{x}}_2 + mgl(\bar{x}_1 - \frac{\bar{x}_1^3}{3!} + \frac{\bar{x}_1^5}{5!})}{ml^2} \end{bmatrix} \qquad V(x) = x^T P x$$

Solve:

$$\begin{aligned} &\max \rho \\ &\text{s.t.} \ -\nabla V(x)^T \hat{f}(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\ &\qquad \lambda(x) \in \Sigma_S \end{aligned}$$

Inverted Pendulum Certified ROA



$$x^T P x < \rho$$

Simulations…

Bilinear Search for ROA

Can we simultaneously search for *V(x)*?

$$\begin{aligned} \max_{V(x),\lambda(x),\rho} \quad & \text{volume}(\{x | V(x) < \rho\}) \\ \text{s.t.} \quad & -\nabla V(x)^T f(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\ & \lambda(x) \in \Sigma_S \\ & V(x) \in \Sigma_S \end{aligned}$$

Challenges

- Bilinear in V(x) and λ(x)
- Convex formulation of volume?

Bilinear Search for ROA

1. Given candidate V(x):

$$\max_{\lambda(x),\rho} \rho$$
$$\text{s.t.} \quad -\nabla V(x)^T f(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S$$
$$\lambda(x) \in \Sigma_S$$

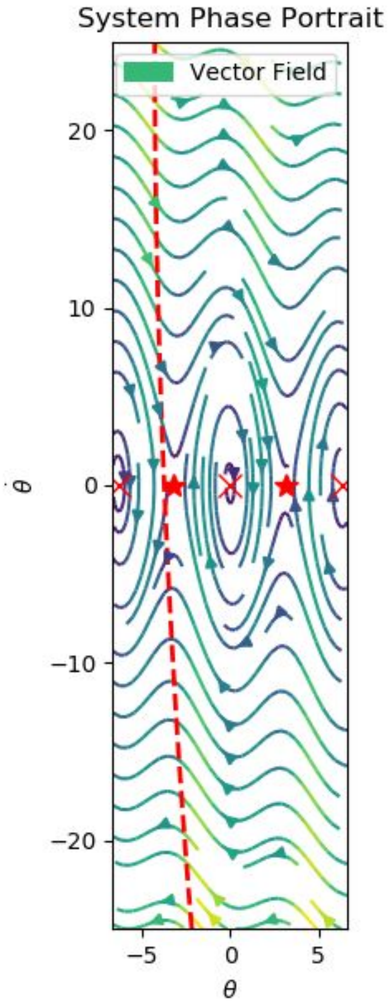
(Same as last SOS program)

2. Given SOS λ(x):

$$\min_{V(x)} \text{trace}(M)$$
$$\text{s.t.} \quad -\nabla V(x)^T f(x) + \lambda(x)(V(x) - 1) \in \Sigma_S$$
$$V(x) = Z_d(x)^T M Z_d(x)$$
$$M \succeq 0$$

- Interactively generates larger ROA certificates for polynomial systems [2]
- Works perfectly for polynomial systems!

Demo…



System Phase Portrait
Vector Field
20
10
0
−10
−20
$\dot{\theta}$
−5
0
5
θ

RL for Large-ROA LQR Controllers

We now have tools (SOS Optimization) for finding large ROA for closed-loop systems.

Can we use the volume of the ROA as a metric of “goodness” in controller design?

RL for Large-ROA LQR Controllers

$$\dot{x} = f(x,u) = \begin{bmatrix} -x_1 - 2x^2 \\ x_2 + x_1 x_2 + 2x_2^3 \end{bmatrix} + \begin{bmatrix} u \\ u \end{bmatrix}$$

Objective

$$\begin{aligned} \max_{Q,R,V,\lambda,\rho} \ & \text{volume}(\{x | V(x) < \rho\}) + \text{reward terms} \\ \text{s.t. } & (K, P) = \text{lqr}(A, B, Q, R) \\ & -\nabla V(x)^T f(x, -Kx) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\ & \lambda(x) \in \Sigma_S, \ V(x) \in \Sigma_S \\ & Q \succeq 0, \ R \succ 0 \end{aligned}$$

Weight Perturbation Algorithm

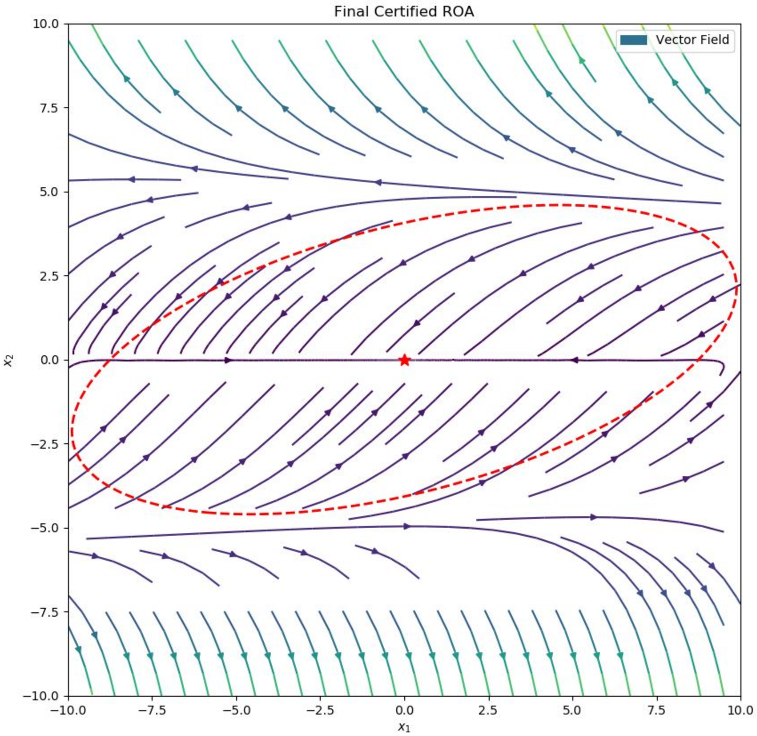
1. Randomly perturb Q and R and resolve the following optimization via bilinear optimization

$$
\begin{aligned}
\max_{V(x),\lambda(x),\rho} \quad & \text{volume}(\{x | V(x) < \rho\}) \\
\text{s.t.} \quad & -\nabla V(x)^T f(x) + \lambda(x)(V(x) - \rho) \in \Sigma_S \\
& \lambda(x) \in \Sigma_S \\
& V(x) \in \Sigma_S
\end{aligned}
$$

2. Update Q and R accordingly (see [2])

3. Repeat

Demo…



$$Q^* = \begin{bmatrix} 0.018 & 0 \\ 0 & 1.59 \end{bmatrix}, \quad R^* = 0.0007$$

$$V(x) = 0.013x_1^2 - 0.026x_1x_2 + 0.06x_2^2 < 1$$

Limitations of SOS Optimization

- Restricted to polynomial systems or the accuracy of the Taylor approximation
- Provides *sufficient,* but not *necessary* certificates of stability
- SOS programs are incredibly nasty and numerically brittle
- RL for LQR requires intensive cost tuning for desirable performance

Conclusion

SOS optimization is a power (convex) tool for searching for Lyapunov functions, certifying global stability, regions of attraction, and controller design

Additional Resources (Citations)

[1] **Prof. Pablo Parrilo's Thesis:** Chapters 4 and 7

[2] **MAE509** by Prof. Matthew M. Peet: Lectures 16-17

[3] **Underactuated Robotics** by Prof. Russ Tedrake: Chapter 9

https://github.com/adamw8/ECE1659/