Macroeconometrics

Lecture 20 Stochastic Volatility models

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A simple Stochastic Volatility model

Unconditional moments

Conditional heteroskedasticity of the All Ordinaries Index

References:

Jacquier, Polson, Rossi (1994) Bayesian Analysis of Stochastic Volatility Models, Appendix A, Journal of Business & Economic Statistics

Materials:

A zip file `L20 mcxs.zip` for the reproduction of the results

Stochastic Volatility models

Stochastic Volatility models provide a flexible way of modeling and forecasting conditional variances of time series

Estimation of SV models for heteroskedastic data improved precision of the estimation

Many recent empirical studies provide evidence that conditional heteroskedasticity modelled with SV process is the most important extension of macroeconomic models improving the forecast precision to a largest extent

Bayesian estimation via the Gibbs sampler using the simulation smoother and an auxiliary mixture is efficient and computationally fast

Objectives.

- To introduce a basic model for conditional heteroskedasticity
- To investigate the properties of the data that the model can capture
- To familiarise with the estimation outcomes and basic interpretations

Learning outcomes.

- Specifying stationary and non-stationary volatility processes
- Understanding of the benefits of modeling heteroskedasticity
- Analysing a log-normally distributed random process

A simple Stochastic Volatility model

A simple Stochastic Volatility model

$$
\begin{aligned}
y_t &= \sqrt{h_t}\epsilon_t \\
\log h_t &= \mu_0 + \alpha \log h_{t-1} + \sigma_v v_t \\
\epsilon_t &\sim \mathcal{N}(0, 1) \\
v_t &\sim \mathcal{N}(0, 1)
\end{aligned}
$$

y_t – a zero mean real-valued random variable

h_t – a conditional variance of y_t

$\log h_t$ – log of conditional variance of y_t

μ_0 , α, σ_v – parameters of the model

ϵ_t, v_t – error terms of the measurement and state equation respectively

A simple Stochastic Volatility model

Predictive density of $\log h_t$

$$
\begin{aligned}
\log h_t &= \mu_0 + \alpha \log h_{t-1} + \sigma_v v_t \\
v_t &\sim \mathcal{N}(0, 1) \\
&\downarrow \\
\log h_t | \log h_{t-1}, \mu_0, \alpha, \sigma_v &\sim \mathcal{N}(\mu_0 + \alpha \log h_{t-1}, \sigma_v^2)
\end{aligned}
$$

A simple Stochastic Volatility model

Unconditional moments of $\log h_t$

$$\begin{aligned}
\mathbb{E}[\log h_t] &= \mathbb{E}[\mu_0 + \alpha \log h_{t-1} + \sigma_v v_t] \\
&= \mu_0 + \alpha \mathbb{E}[\log h_{t-1}] + \sigma_v \mathbb{E}[v_t] \\
&= \mu_0 + \alpha \mathbb{E}[\log h_{t-1}]
\end{aligned}$$

Assume stationarity: $|\alpha| < 1$, and $\mathbb{E}[\log h_t] = \mathbb{E}[\log h_{t-1}]$

$$\begin{aligned}
(1-\alpha)\mathbb{E}[\log h_t] &= \mu_0 \\
\mathbb{E}[\log h_t] &= \frac{\mu_0}{1-\alpha}
\end{aligned}$$

A simple Stochastic Volatility model

Unconditional moments of $\log h_t$

$$\begin{aligned}\mathbb{V}ar[\log h_t] &= \mathbb{V}ar[\mu_0 + \alpha \log h_{t-1} + \sigma_v v_t] \\ &= \alpha^2 \mathbb{V}ar[\log h_{t-1}] + \sigma_v^2 \mathbb{V}ar[v_t] \\ &= \alpha^2 \mathbb{V}ar[\log h_{t-1}] + \sigma_v^2 1\end{aligned}$$

Assume stationarity $|\alpha| < 1$, and $\mathbb{V}ar[\log h_t] = \mathbb{V}ar[\log h_{t-1}]$

$$\begin{aligned}(1-\alpha^2)\mathbb{V}ar[\log h_t] &= \sigma_v^2 \\ \mathbb{V}ar[\log h_t] &= \frac{\sigma_v^2}{1-\alpha^2}\end{aligned}$$

A simple Stochastic Volatility model

Unconditional distribution of $\log h_t$

$$\log h_t \sim \mathcal{N}\left(\frac{\mu_0}{1-\alpha}, \frac{\sigma_v^2}{1-\alpha^2}\right)$$

Unconditional distribution of h_t

$$h_t \sim \log\mathcal{N}\left(\frac{\mu_0}{1-\alpha}, \frac{\sigma_v^2}{1-\alpha^2}\right)$$

Unconditional moments

Log-normal distribution

Let x be a positive real-valued random variable the log of which is normally distributed. Then, x is log-normally distributed.

$$\log x \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

$$x \sim \log \mathcal{N}\left(\mu, \sigma^2\right)$$

μ – a mean parameter of the log-normal distribution
σ^2 – a variance parameter of the log-normal distribution
$\frac{1}{x}$ – the Jacobian of the transformation

Density function.

$$\frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp\left\{-\frac{1}{2} \frac{(\log x - \mu)^2}{\sigma^2}\right\}$$

Moments. All of the moments exist and can be computed using:

$$\mathbb{E}[x^n] = \exp\left\{n\mu + \frac{n^2\sigma^2}{2}\right\}$$

Unconditional moments

Unconditional moments of y_t

$$\mathbb{E}[y_t] = \mathbb{E}\left[\sqrt{h_t}\epsilon_t\right] \overset{LIE}{=} \mathbb{E}\left[\sqrt{h_t}\mathbb{E}_{t-1}[\epsilon_t]\right] = 0$$

$$\begin{aligned}\mathbb{E}\left[y_t^2\right] &= \mathbb{E}\left[h_t\epsilon_t^2\right] = \mathbb{E}\left[h_t\right]\mathbb{E}\left[\epsilon_t^2\right] = \mathbb{E}\left[h_t\right]1 \\ &= \exp\left\{\frac{\mu_0}{1-\alpha} + \frac{\sigma_v^2}{2\left(1-\alpha^2\right)}\right\}\end{aligned}$$

$$\mathbb{E}\left[y_t^3\right] = \mathbb{E}\left[h_t^{\frac{3}{2}}\epsilon_t^3\right] = \mathbb{E}\left[h_t^{\frac{3}{2}}\right]\mathbb{E}\left[\epsilon_t^3\right] = 0$$

$$\begin{aligned}\mathbb{E}\left[y_t^4\right] &= \mathbb{E}\left[h_t^2\epsilon_t^4\right] = 3\mathbb{E}\left[h_t^2\right] \\ &= 3\exp\left\{\frac{2\mu_0}{1-\alpha} + \frac{2\sigma_v^2}{1-\alpha^2}\right\}\end{aligned}$$

Unconditional moments

Unconditional moments of y_t

$$Kurtosis[y_t] = \frac{\mathbb{E}\left[y_t^4\right]}{\mathbb{E}\left[y_t^2\right]^2} = \frac{3\exp\left\{\frac{2\mu_0}{1-\alpha} + \frac{2\sigma_v^2}{1-\alpha^2}\right\}}{\exp\left\{\frac{2\mu_0}{1-\alpha} + \frac{2\sigma_v^2}{2(1-\alpha^2)}\right\}}$$

$$= 3\exp\left\{\frac{\sigma_v^2}{1-\alpha^2}\right\} > 3$$

$$\mathbb{E}\left[y_t^2 y_{t-s}^2\right] = \mathbb{E}\left[h_t h_{t-s}\right] = \exp\left\{\frac{2\mu_0}{1-\alpha} + \frac{(1-\alpha^s)\sigma_v^2}{1-\alpha^2}\right\}$$

For the derivation see Jacquier, Polson, Rossi (1994, JBES, Appendix A)

A simple Stochastic Volatility model

An alternative notation.

$$y_t = \exp\left\{\frac{1}{2}h_t\right\}\epsilon_t$$

$$h_t = \mu_0 + \alpha h_{t-1} + \sigma_v v_t$$

$$\epsilon_t \sim \mathcal{N}(0,1)$$

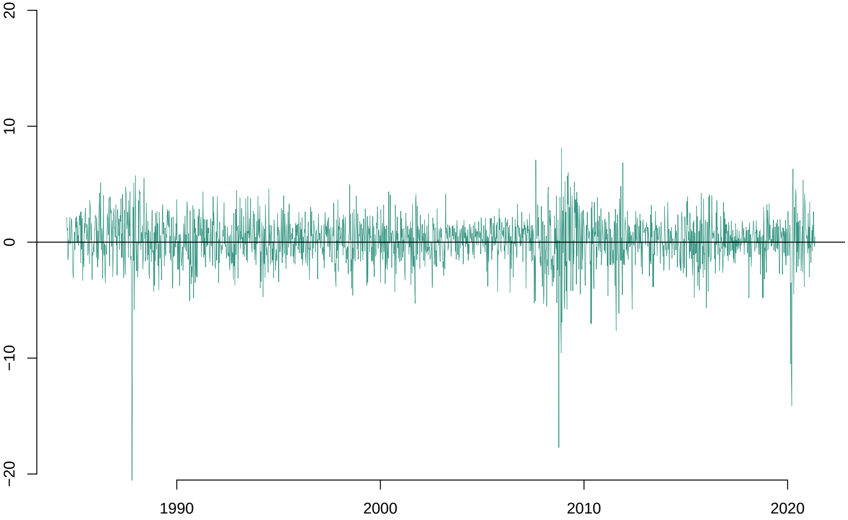
$$v_t \sim \mathcal{N}(0,1)$$

h_t – log of conditional variance of y_t

$\sigma^2_{y.t} = \exp\{h_t\}$ – a conditional variance of y_t

Conditional heteroskedasticity
of the All Ordinaries Index

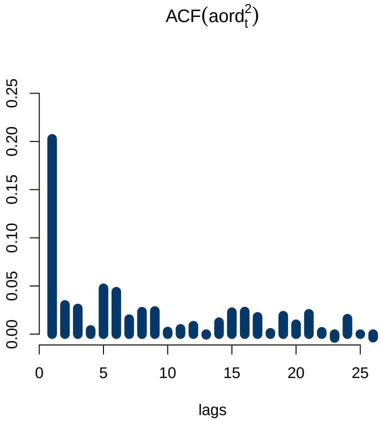
Conditional heteroskedasticity of the All Ordinaries Index

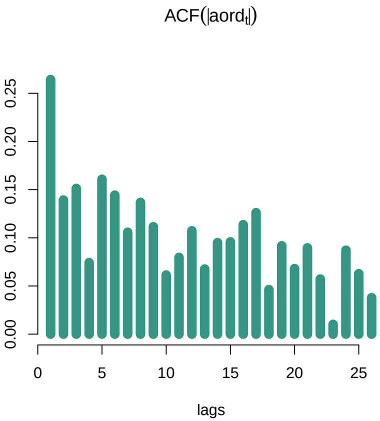


Weekly log-returns in pp. from August 1984 to May 2021
Source: Yahoo Finance ($T = 1919$)

The series clearly exhibits volatility clustering

Conditional heteroskedasticity of the All Ordinaries Index





The heteroskedasticity of the data seems to be persistent

Conditional heteroskedasticity of the All Ordinaries Index

SV model.

$$y_t = \exp\left\{\frac{1}{2}h_t\right\}\epsilon_t$$

$$h_t = h_{t-1} + \sigma_v v_t$$

$$\epsilon_t \sim \mathcal{N}(0,1)$$

$$v_t \sim \mathcal{N}(0,1)$$

SV–AR model.

$$y_t = \exp\left\{\frac{1}{2}h_t\right\}\epsilon_t$$

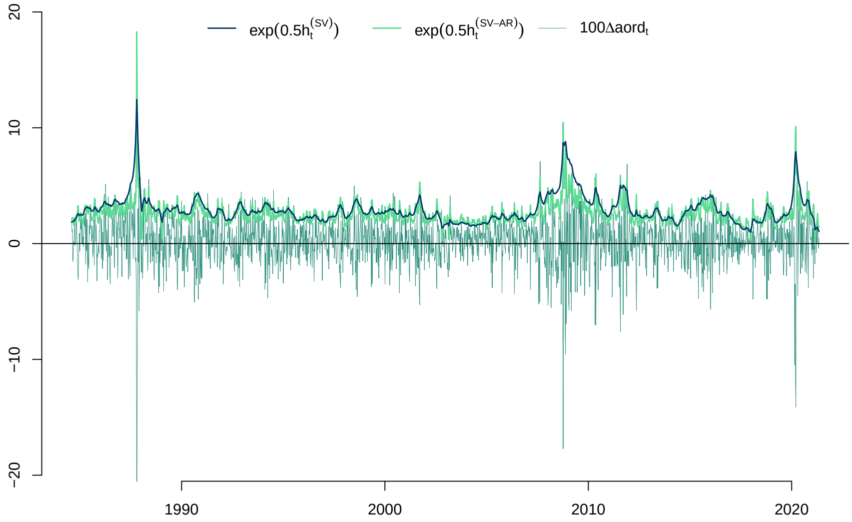
$$h_t = \mu_0 + \alpha h_{t-1} + \sigma_v v_t$$

$$\epsilon_t \sim \mathcal{N}(0,1)$$

$$v_t \sim \mathcal{N}(0,1)$$

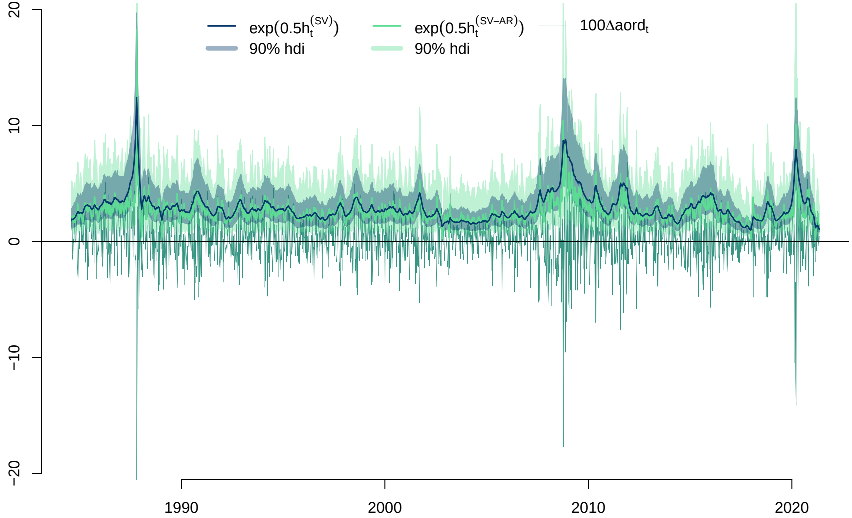
h_0 – initial condition to be estimated

Conditional heteroskedasticity of the All Ordinaries Index



The estimated conditional standard deviations capture
the volatility clustering accurately

Conditional heteroskedasticity of the All Ordinaries Index



Volatility estimates from the SV model seem to be more smooth
Volatility estimates from the SV-AR model seem to be more volitile

Conditional heteroskedasticity of the All Ordinaries Index

Volatility estimates from the SV model seem to be more smooth
Volatility estimates from the SV-AR model seem to be more volitile

Conditional heteroskedasticity of the All Ordinaries Index

Parameter estimation results

	h_0	σ_v^2	μ_0	α
SV model	1.145 (0.549)	0.092 (0.021)		
SV–AR model	0.647 (0.944)	1.029 (0.089)	0.684 (0.238)	0.621 (0.132)

Posterior means and standard deviations (in parentheses) are reported

Stochastic Volatility models

Stochastic Volatility models provide a flexible way of modeling and forecasting conditional variances of time series

Model specification including prior distributions determine the properties of the estimated and forecasted conditional volatility

Scarce data do not include a strong signal to estimate many latent variables

Model comparisons based on hypothesis testing and forecasting performance should decide on which model to use