# Macroeconometrics

## Lecture 11 Structural Vector Autoregressions

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## Structural VARs

## Identification problem

## Identification of the monetary policy shock using exclusion restrictions

## Identification using exclusion restrictions

## Other ways of identifying structural shocks

Compulsory readings:

Kilian & Lütkepohl (2017) Chapter 8: Identification by Short-Run Restrictions, Structural Vector Autoregressive Analysis

Useful readings:

Rubio-Ramírez, Waggoner & Zha (2010) Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference, Review of Economic Studies

**Modeling Effects of Monetary Policy**


11 Structural Vector Autoregressions
12 Structural VAR tools
13 Structural VARs: Bayesian estimation I
14 Structural VARs: Bayesian estimation II
15 Modeling effects of monetary policy

**Objectives.**

- To introduce SVARs – a basic tool of empirical analyses
- To analyse the identification of the monetary policy shock
- To present the identification problem of SVARs

**Learning outcomes.**

- Understanding various forms of SVAR models
- Checking the identification of SVARs with exclusion restrictions
- Working with rotation and orthogonal matrices

# Structural VARs

# Structural VARs

$$B_0 y_t = b_0 + B_1 y_{t-1} + \dots + B_p y_{t-p} + u_t$$
$$u_t | Y_{t-1} \sim iid(\mathbf{0}_N, I_N)$$

$B_0$ – $N \times N$ matrix of contemporaneous relationships also called structural matrix

It captures contemporaneous relationships between variables

$u_t$ – $N \times 1$ vector of conditionally on $Y_{t-1}$ orthogonal or independent structural shocks

Isolating these shocks allows us to identify dynamic effects of uncorrelated shocks on variables $y_t$

**Structural Form (SF) model**

The SVAR above is called a structural form model

# Structural VARs

Premultiply the SVAR equation by $B_0^{-1}$

$$y_t = B_0^{-1} b_0 + B_0^{-1} B_1 y_{t-1} + \cdots + B_0^{-1} B_p y_{t-p} + B_0^{-1} u_t$$

to obtain a model in a form that uses the autoregressive parameters of the VAR

$$y_t = \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_0^{-1} u_t$$

and a different formulation of the SF model

$$y_t = \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B u_t$$

# Structural VARs

$$y_t = \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B u_t$$
$$u_t | Y_{t-1} \sim iid(\mathbf{0}_N, I_N)$$

$B = B_0^{-1}$ – contemporaneous effects matrix

It captures contemporaneous effects of shocks on variables $y_t$

$A_i = B_0^{-1} B_i$ – autoregressive slope coefficients for $i = 1, \ldots, p$

$\mu_0 = B_0^{-1} b_0$ – a constant term

# Structural VARs

**Reduced Form (RF) representation**

$$
\begin{aligned}
y_t &= \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + \epsilon_t \\
\epsilon_t | Y_{t-1} &\sim iid(\mathbf{0}_N, \Sigma)
\end{aligned}
$$

Either of the SF models lead to the same RF representation through various equivalence transformations

$$
\begin{aligned}
\epsilon_t &= B u_t = B_0^{-1} u_t \\
B_0 \epsilon_t &= u_t \\
\Sigma &= BB' = B_0^{-1} B_0^{-1\prime}
\end{aligned}
$$

These SF models have the same value of the likelihood function

# Structural VARs

**Observational Equivalence**

Structural models that lead to exactly the same value of the likelihood function are called observationally equivalent

$$L(B_+, B_0|Y, X) = L(A, B|Y, X) = L(A, \Sigma|Y, X)$$

$$\underset{(N\times K)}{B_+} = \begin{bmatrix} b_0 & B_1 & \dots & B_p \end{bmatrix}$$
$$\underset{(N\times K)}{A} = \begin{bmatrix} \mu_0 & A_1 & \dots & A_p \end{bmatrix}$$

# Identification problem

# Identification problem

**Estimation**

To estimate an SF model utilize the information from an easy to estimate RF model and the parameter transformations

Given $B_0$ it is straightforward to compute autoregressive parameters by

$$B_i = B_0 A_i \text{ for } i = 1, \ldots, p$$
$$b_0 = B_0 \mu_0$$

Estimation of the structural matrix relies on the system of equations

$$\boldsymbol{\Sigma} = B_0^{-1} B_0^{-1\prime}$$

# Identification problem

**Problem 1. Insufficient information**

$$\Sigma = B_0^{-1} B_0^{-1\prime}$$

$\Sigma$ is a symmetric matrix and has $N(N+1)/2$ unique elements

– number of equations

$B_0$ has $N^2$ elements: the system has $N^2$ unknowns

$B_0$ and $u_t$ are not identified

# Identification problem

**Problem 2. Identification up to a rotation matrix**

Let $\tilde{B}_0 = QB_0$ where $Q$ is an $N \times N$ orthogonal matrix such that $Q'Q = I_N$

$$
\begin{aligned}
\Sigma &= \tilde{B}_0^{-1}\tilde{B}_0^{-1\prime} \\
&= (QB_0)^{-1}(QB_0)^{-1\prime} \\
&= B_0^{-1}Q^{-1}Q^{-1\prime}B_0^{-1\prime} \\
&= B_0^{-1}(Q'Q)^{-1}B_0^{-1\prime} \\
&= B_0^{-1}B_0^{-1\prime}
\end{aligned}
$$

Premultiplying the SF model by an orthogonal matrix $Q$ does not change the value of the likelihood function – it leads to an observationally equivalent representation

SF models are often identified up to an orthogonal matrix that is a rotation matrix

# Identification problem

**Problem 2. Identification up to a rotation matrix**

Premultiplying the SF model by a rotation matrix $Q$ leads to observationally equivalent SF representation

$$L(QB_+, QB_0|Y, X) = L(A, BQ'|Y, X) = L(A, \Sigma|Y, X)$$

SF models are identified up to a rotation matrix

Various ways of identifying SVARs set the type of the rotation matrix

# Orthogonal matrix

Let $\mathcal{O}(N)$ denote a set of $N \times N$ orthogonal matrices such that $Q \in \mathcal{O}(N)$

**Properties.**

$$
\begin{aligned}
QQ' &= Q'Q = I_N \\
Q_{[n\cdot]}Q'_{[n\cdot]} &= Q'_{[\cdot n]}Q_{[\cdot n]} = 1 \\
Q' &= Q^{-1} \\
\det(Q) &= \pm 1
\end{aligned}
$$

# Rotation matrix

**Definition.**

A square matrix Q of order $N$ is a rotation matrix if for given $r, s : r < s < N$

$$
\begin{aligned}
Q_{rr} &= Q_{ss} = \cos(x) \\
Q_{ii} &= 1 \text{ for } i = 1, \ldots, N \text{ and } i \neq r, s \\
Q_{sr} &= -\sin(x) \\
Q_{rs} &= \sin(x)
\end{aligned}
$$

and all other elements are zero. Other rotations are obtained by multiplying a sequence of rotation matrices.

**Examples of rotation matrices.**

$$
\begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \qquad \begin{bmatrix} \cos(x) & 0 & -\sin(x) \\ 0 & 1 & 0 \\ \sin(x) & 0 & \cos(x) \end{bmatrix}
$$

$D$ – a diagonal matrix with $\pm 1$ on the main diagonal

$P$ – a permutation matrix with a single 1 in each column and row and zeros elsewhere

# Identification of the monetary policy shock using exclusion restrictions

# Identification of the monetary policy shock

$$\Sigma = B_0^{-1} B_0^{-1\prime}$$

At least $N(N-1)/2$ restrictions on $B_0$ are needed to identify the system

Impose exclusion (zero) restrictions to

- obtain the identification of the system: shocks $u_t$ and matrix $B_0$
- assign shocks economic interpretation

# Identification of the monetary policy shock

**Monetary policy shock.**
is often defined...

- as an unanticipated part of the monetary policy
- as an orthogonal shock to the monetary policy instrument
- as an orthogonal shock to the short-run nominal interest rate $i_t$
- through a Taylor's rule type relationship to the output gap $\tilde{y}_t$ and inflation's deviation from its target value $\pi_t$ in which all of the variables are treated as endogenous

$$i_t = r^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t + u_t^{(mp)}$$

  $r^n$ is a natural rate of interest
- through a Taylor's rule using real output $rgdp_t$ and prices $p_t$

# Identification of the monetary policy shock

To represent identifying restrictions consider a simplified system

$$B_0 y_t = u_t$$

**Monetary policy shock.**

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} p_t \\ rgdp_t \\ i_t \\ m_t \end{bmatrix} = \begin{bmatrix} u_{1.t}^{(as)} \\ u_{2.t}^{(ad)} \\ u_{3.t}^{(mp)} \\ u_{4.t}^{(md)} \end{bmatrix}$$

$(as)$ – aggregate supply shock $(ad)$ – aggregate demand shock
$(mp)$ – monetary policy shock $(md)$ – money demand shock

Shocks can be given economic interpretations thanks to the structure imposed on the model in the form of zero restrictions

# Identification using exclusion restrictions

Based on Rubio-Ramírez, Waggoner & Zha (2010)
The material in this section presumes normalized systems

# Identification using exclusion restrictions

**Definitions.**

A parameter point $(B_+, B_0)$ is globally identified if and only if there is no other parameter point that is observationally equivalent.

A parameter point $(B_+, B_0)$ is locally identified if and only if there is an open neighbourhood about $(B_+, B_0)$ containing no other observationally equivalent parameter point.

A parameter point $(B_+, B_0)$ is partially identified that is the $n$th equation is globally identified at the parameter point $(B_+, B_0)$ if and only if there does not exist another observationally equivalent parameter point $(\tilde{B}_+, \tilde{B}_0)$ such that $B_{+[n\cdot]} \neq \tilde{B}_{+[n\cdot]}$ and $B_{0[n\cdot]} \neq \tilde{B}_{0[n\cdot]}$, where $X_{[n\cdot]}$ is the $n$th row of matrix $X$.

# Identification using exclusion restrictions

**General form of restrictions.**

$$\mathbf{R}_n f\left(B_+, B_0\right) e_n = \mathbf{0}_{R\times 1} \qquad \text{for } n = 1, \dots, N$$

$f\left(B_+, B_0\right)$ – $R \times N$ matrix of functions of parameters to be restricted, e.g.:

$f\left(B_+, B_0\right) = B_0'$ – restrictions on contemporaneous relationships

$f\left(B_+, B_0\right) = B_0^{-1}$ – restrictions on contemporaneous effects

$\mathbf{R}_n$ – $R \times R$ matrix with ones and zeros such that $\operatorname{rank}\left(\mathbf{R}_n\right) = r_n$

Assume that $r_1 \geq r_2 \geq \cdots \geq r_N$

$e_n$ – the $n$th column of $I_N$

# Identification using exclusion restrictions

**Example.**
Consider the restrictions on the second row of $B_0$ from slide 21

$$
\begin{aligned}
f\left(B_+, B_0\right) &= B_0' \\
e_2 &= (0, 1, 0, 0)' \\
\mathbf{R}_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&\downarrow \\
\mathbf{R}_n B_0' e_n &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \\ b_{24} \end{bmatrix} = \begin{bmatrix} b_{23} \\ b_{24} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

# Identification using exclusion restrictions

**Conditions for** $f(B_+, B_0)$

**admissible** $f(QB_+, QB_0) = f(B_+, B_0)\, Q'$
**continuously differentiable** $\operatorname{rank}[f'(B_+, B_0)] = RN$
**strongly regular** see Rubio-Ramírez, Waggoner & Zha (2010)

# Identification using exclusion restrictions

**Rank conditions.**

The identification results are stated as rank conditions for matrix:

$$\underset{(R+n)\times N}{\mathbf{M}_n[X]} = \begin{bmatrix} & \mathbf{R}_n X \\ I_n & \mathbf{0}_{n\times(N-n)} \end{bmatrix} \quad \text{for } n = 1, \ldots, N$$

# Identification using exclusion restrictions

The results below are the most useful for non-recursive identification patterns

**Results.**

Consider parameter point $(B_+, B_0)$ with imposed zero restrictions. If $\mathbf{M}_n [f(B_+, B_0)]$ is of rank $N$ for $n = 1, \ldots, N$, then the SVAR is globally identified at the parameter point $(B_+, B_0)$.

Consider parameter point $(B_+, B_0)$ with imposed zero restrictions. If $\mathbf{M}_i [f(B_+, B_0)]$ is of rank $N$ for $i = 1, \ldots, n$, then the $n$th row of the SVAR is globally identified at the parameter point $(B_+, B_0)$.

# Identification using exclusion restrictions

**Example.**

Consider the restrictions on $B_0$ from slide 21

$$
\begin{array}{ccccc}
n = & 1 & 2 & 3 & 4 \\[1em]
\mathbf{R}_n = &
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\[1em]
\mathbf{M}_n(B_0') = &
\begin{bmatrix} 0 & b_{22} & b_{32} & b_{42} \\ 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} &
I_4 \\[1em]
\operatorname{rk}(\mathbf{M}_n(B_0')) = & 4 & 4 & 4 & 4
\end{array}
$$

The model is globally identified

# Identification using exclusion restrictions

**Exact identification.**

The results below provide simplified analysis for triangular identification patterns

**Definition.**

The SVAR with zero restrictions is exactly identified if and only if, for almost any RF parameter point $(A, \Sigma)$, there exists a unique structural parameter point $(B_+, B_0)$ such that $\left(B_0^{-1} B_+, B_0^{-1} B_0^{-1\prime}\right) = (A, \Sigma)$

**Rank condition.**

The SVAR with zero restrictions is exactly identified if and only if $r_n = N - n$ for $n = 1, \ldots, N$.

# Identification using exclusion restrictions

**Example.**
Consider the restrictions on $B_0$ from slide 21

$$
\begin{array}{ccccc}
n = & 1 & 2 & 3 & 4 \\
\mathbf{R}_n = & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{rank}(\mathbf{R}_n) = & 3 & 2 & 1 & 0
\end{array}
$$

The model is exactly identified

# Other ways of identifying structural shocks

# Other ways of identifying structural shocks

- flexible exclusion restrictions, e.g., on long-run relationships
- sign restrictions
    - on contemporaneous effects
    - flexible sign restrictions
    - narrative sign restrictions
- using zero and sign restrictions
- using prior distributions
- using non-normal error terms
- using heteroskedastic error terms
- using instrumental variables
- using high-frequency data
- using narrative sign restrictions

# Structural Vector Autoregressions

**Structural models** rely on economic theory that provides additional identifying information

**Rank conditions** provide necessary and sufficient conditions for global identification of SVARs with zero restrictions

**Simple conditions** guarantee global identification of triangular systems