Macroeconometrics: Test 1

Examples of solutions prepared by Tomasz (in this note I explain a bit more than was required in the test)

Exercise 1. (2.5 points) Consider the following autoregression for a random variable y_t with the scalar parameters α and σ^2 and normally distributed error term u_t:

$$y_t = \alpha y_{t-2} + u_t \tag{1}$$
$$u_t | y_{t-2} \sim iid\mathcal{N}\left(0, \sigma^2\right) \tag{2}$$

- Derive autocorrelations at lags 0, 1, 2, 3, and 4 implied by this model. Show your workings. State the assumptions you are applying to get your result.
- Given the derived autocorrelations, comment in two sentences about what memory patterns this model implies about the data generated by the corresponding data-generating process.

Proposed solution. The process in eq (1) has a zero constant term and, thus, $E[y_t] = 0$. Multiply both sides by y_{t-s} and apply the expectation:

$$E[y_t y_{t-s}] = E[(\alpha y_{t-2} + u_t) y_{t-s}] \tag{3}$$
$$= \alpha E[y_{t-2} y_{t-s}] + E[u_t y_{t-s}] \tag{4}$$
$$\gamma_s = \alpha \gamma_{s-2} + E[u_t y_{t-s}] \tag{5}$$

Assume stationarity to facilitate $\gamma_s = \gamma_{-s}$ (It's OK to bring in an assumption if that leads to a solution). Write out eq (5) for $s = 0, 1, 2, 3, 4$:

$$s = 0 \tag{6}$$
$$\gamma_0 = \alpha \gamma_{-2} + E[u_t y_t] \tag{7}$$
$$\gamma_0 = \alpha \gamma_2 + \sigma^2 \tag{8}$$

$$s > 1 \tag{9}$$
$$\gamma_s = \alpha \gamma_{s-2} + E[u_t y_{t-s}] \tag{10}$$
$$\gamma_s = \alpha \gamma_{s-2} \tag{11}$$

$$s = 1 \tag{12}$$
$$\gamma_1 = \alpha \gamma_1 \tag{13}$$
$$\gamma_1 = 0 \tag{14}$$

$$s = 2 \tag{15}$$
$$\gamma_2 = \alpha \gamma_0 \tag{16}$$
$$\Downarrow \tag{17}$$
$$\gamma_0 = \frac{\sigma^2}{1 - \alpha^2} \tag{18}$$
$$\gamma_2 = \alpha \frac{\sigma^2}{1 - \alpha^2} \tag{19}$$

$$s = 3 \tag{20}$$

$$\gamma_3 = \alpha\gamma_1 = 0 \tag{21}$$

$$s = 4 \tag{22}$$

$$\gamma_4 = \alpha\gamma_2 = \alpha^2 \frac{\sigma^2}{1-\alpha^2} \tag{23}$$

$$\tag{24}$$

This leads to the autocorrelations:

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1 \tag{25}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = 0 \tag{26}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \alpha \tag{27}$$

$$\rho_3 = \frac{\gamma_3}{\gamma_0} = 0 \tag{28}$$

$$\rho_4 = \frac{\gamma_4}{\gamma_0} = \alpha^2 \tag{29}$$

This process has a specific memory pattern where there is no memory at odd lags, that is, zero autocorrelations at odd lags, and memory decaying exponentially at rate α at even lags.

Exercise 2. (2.5 points) Consider the autoregression from **Exercise 1** applied to T observations on variable y.

- Write out the model in a matrix notation.
- State the distribution of the error term vector explicitly.
- State the predictive density implied by the model for the dependent variable vector given the explanatory variables.
- Write out the likelihood function for the model.

a pdf of the multivariate normal distribution for an N-random vector X with mean μ and covariance Σ

$$X \sim \mathcal{N}_N(\mu, \Sigma) = (2\pi)^{-\frac{N}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu)\right\} \tag{30}$$

Proposed solution. Define $T-2\times 1$ column-vectors:

$$Y = \begin{bmatrix} y_3 \\ \vdots \\ y_T \end{bmatrix}, \qquad X = \begin{bmatrix} y_1 \\ \vdots \\ y_{T-2} \end{bmatrix}, \qquad U = \begin{bmatrix} u_3 \\ \vdots \\ u_T \end{bmatrix} \tag{31}$$

and write out the model as:

$$Y = \alpha X + U \tag{32}$$

where the error term vector follow a $T-2$-variate normal distribution

$$U \mid X \sim \mathcal{N}_{T-2}\left(\mathbf{0}_{T-2}, \sigma^2 I_{T-2}\right) \tag{33}$$

Since Y is a linear transformation of a normal vector U, according to eq (32), it is also normal:

$$Y \mid X \sim \mathcal{N}_{T-2}\left(\alpha X, \sigma^2 I_{T-2}\right) \tag{34}$$

The latter equation is the predictive density of Y given X, and parameters α and σ^2 (neglected in the notation above), that determines the form of the likelihood function:

$$L\left(Y \mid X, \alpha, \sigma^2\right) = (2\pi)^{-\frac{T-2}{2}} \left(\sigma^2\right)^{-\frac{T-2}{2}} \exp\left\{-\frac{1}{2\sigma^2}(Y-\alpha X)'(Y-\alpha X)\right\} \tag{35}$$

Exercise 3. (2.5 points) Consider the autoregression from **Exercise 1** represented in the matrix notation in your answer to **Exercise 2** Assume the following prior distribution for the parameter α:

$$\alpha \mid \underline{\alpha}, \underline{\sigma}_\alpha^2 \sim \mathcal{N}\left(\underline{\alpha}, \underline{\sigma}_\alpha^2\right) \tag{36}$$

where the hyper-parameters $\underline{\alpha}$ and $\underline{\sigma}_\alpha^2$ are assumed to be known.

- Derive the full-conditional posterior distribution of the parameter α given data, as well as parameter σ^2 and hyper-parameters $\underline{\alpha}$ and $\underline{\sigma}_\alpha^2$, denoted by $p\left(\alpha \mid data, \sigma^2, \underline{\alpha}, \underline{\sigma}_\alpha^2\right)$. Show your workings.

Proposed solution. The kernel of the prior distribution is given by:

$$\exp\left\{-\frac{1}{2}(\alpha-\underline{\alpha})'\underline{\sigma}_\alpha^{-2}(\alpha-\underline{\alpha})\right\} \tag{37}$$

Bayes rule for the full-conditional posterior is given by:

$$\begin{aligned}
&p\left(\alpha \mid Y, X, \sigma^2, \underline{\alpha}, \underline{\sigma}_\alpha^2\right) \propto L\left(Y \mid X, \alpha, \sigma^2\right) p\left(\alpha \mid \underline{\alpha}, \underline{\sigma}_\alpha^2\right) &(38)\\
&= \exp\left\{-\frac{1}{2}(Y-\alpha X)'\sigma^{-2}(Y-\alpha X)\right\}\exp\left\{-\frac{1}{2}(\alpha-\underline{\alpha})'\underline{\sigma}_\alpha^{-2}(\alpha-\underline{\alpha})\right\} &(39)\\
&= \exp\left\{-\frac{1}{2}\left[\alpha'(X'\sigma^{-2}X)\alpha - 2\alpha'X'\sigma^{-2}Y + \dots + \alpha'\underline{\sigma}_\alpha^{-2}\alpha - 2\alpha'\underline{\sigma}_\alpha^{-2}\underline{\alpha} + \dots\right]\right\} &(40)\\
&= \exp\left\{-\frac{1}{2}\left[\alpha'(X'\sigma^{-2}X + \underline{\sigma}_\alpha^{-2})\alpha - 2\alpha'(X'\sigma^{-2}Y + \underline{\sigma}_\alpha^{-2}\underline{\alpha}) + \dots\right]\right\} &(41)
\end{aligned}$$

Let $\overline{\sigma}_\alpha^2 = (X'\sigma^{-2}X + \underline{\sigma}_\alpha^{-2})^{-1}$. Rewrite the kernel as:

$$\exp\left\{-\frac{1}{2}\left[\alpha'\overline{\sigma}_\alpha^{-2}\alpha - 2\alpha'\overline{\sigma}_\alpha^{-2}\overline{\sigma}_\alpha^{2}(X'\sigma^{-2}Y + \underline{\sigma}_\alpha^{-2}\underline{\alpha}) + \dots\right]\right\} \tag{42}$$

Let $\overline{\alpha} = \overline{\sigma}_\alpha^2(X'\sigma^{-2}Y + \underline{\sigma}_\alpha^{-2}\underline{\alpha})$. Then...

$$\exp\left\{-\frac{1}{2}\left[\alpha'\overline{\sigma}_\alpha^{-2}\alpha - 2\alpha'\overline{\sigma}_\alpha^{-2}\overline{\alpha} + \dots\right]\right\} \tag{43}$$

in which I recognise the kernel of the following normal distribution:

$$\begin{aligned}
\alpha \mid Y, X, \sigma^2, \underline{\alpha}, \underline{\sigma}_\alpha^2 &\sim \mathcal{N}\left(\overline{\alpha}, \overline{\sigma}_\alpha^2\right) &(44)\\
\overline{\sigma}_\alpha^2 &= (X'\sigma^{-2}X + \underline{\sigma}_\alpha^{-2})^{-1} &(45)\\
\overline{\alpha} &= \overline{\sigma}_\alpha^2(X'\sigma^{-2}Y + \underline{\sigma}_\alpha^{-2}\underline{\alpha}) &(46)
\end{aligned}$$

Exercise 4. (2.5 points) Consider the prior distribution for the autoregressive parameter assumed in **Exercise 3** in the context of applying the model to fortnightly (bi-weekly) data on cash rate target (Australian interest rate).

- Propose the values for the hyper-parameters $\underline{\alpha}$ and $\underline{\sigma}_{\alpha}^2$ that would reflect a weak presumption that the time series is unit-root non-stationary.
- Write two sentences to make a case for the proposed values for each of the hyper-parameters.

Proposed solution. Rewrite the model in a lag polynomial form:

$$(1 - \alpha L^2)y_t = u_t \tag{47}$$

Use the lag polynomial to construct a characteristic polynomial in z:

$$1 - \alpha z^2 = 0 \tag{48}$$

Consider a unit root $z = 1$. The value of α implied by the unit root is:

$$1 - \alpha 1^2 = 0 \tag{49}$$

$$\alpha = 1 \tag{50}$$

Note: all of the above is not required as long as you state that:

The value of the parameter α that implies the unit root in y_t is 1. Therefore, to reflect the presumption that the time series is unit-root non-stationary in the prior specification we set $\underline{\alpha} = 1$.

To incorporate a weak conviction that the series is unit-root non-stationary in the prior distribution, I set a fairly large value of the prior shrinkage hyper-parameter $\underline{\sigma}_{\alpha}^2 = 4$. As long as the prior distribution is centred at the unit-root non-stationary process, this value of the shrinkage implies e.g. that the stationarity region $\alpha \in (-1, 1)$ is covered by the 68% confidence region of the prior distribution (the mean $\pm$ one standard deviation).