

Macroeconometrics

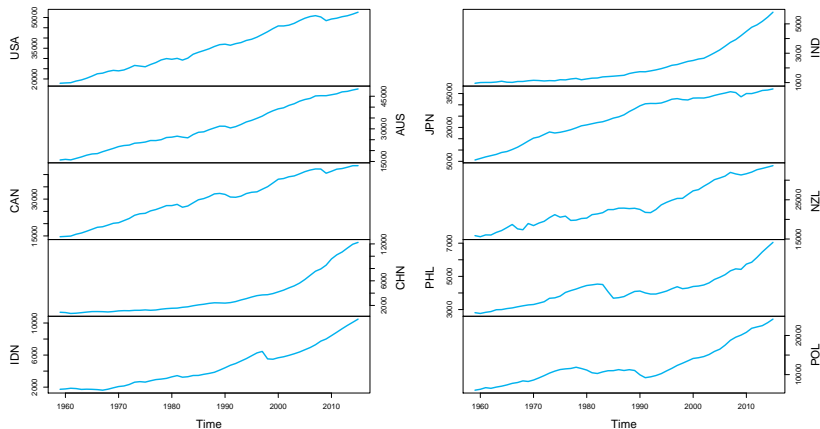
Lecture 23 Less than 2°C warming by 2100 unlikely – partial reproduction

Topics in Climate Change
Forecasting CO₂ Emissions for the 21st Century

Tomasz Woźniak

The data, the model, and prior assumptions

The data: GDP data

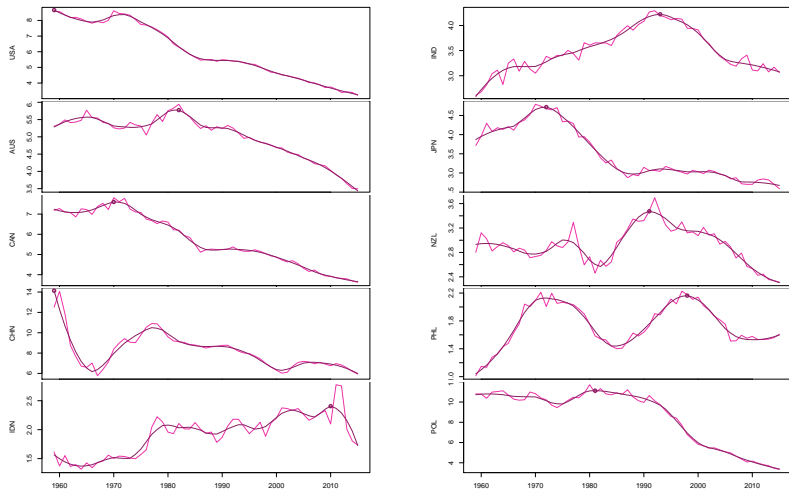


Annual data from 1959–2015 ($T = 56$ of differentiated series, $N = 10$)

Model applied to logarithms of the original data

Source: data files of Raftery et al. (2017)

The data: carbon intensity



The data, loess smoothed values, and cut-off dates

Model and prior assumptions

Model for the frontier economy – prior distributions.

$$F_t = F_{t-1} + \gamma + \gamma_{pre1973}\mathcal{I}(t \leq 1973) + \epsilon_t^{(f)}$$
$$\epsilon_t^{(f)} \sim \mathcal{N}(0, \sigma_f^2)$$

$$\gamma \sim \mathcal{U}[0, 1]$$

$$\gamma_{pre1973} \sim \mathcal{U}[-0.1, 0.1]$$

$$\sigma_f^2 \sim \mathcal{IG}2(\hat{s}^2, 3)$$

Model and prior assumptions

Model for other economies – prior distributions.

$$(F_t - G_{c.t}) = \phi_c(F_{t-1} - G_{c.t-1}) + \epsilon_{c.t}^{(g)} \\ \epsilon_{c.t}^{(g)} \sim \mathcal{N}(0, \sigma_{g.c}^2)$$

$$\phi_c | \mu_\phi, \sigma_\phi^2 \sim \mathcal{TN}_{[0,1]}(\mu_\phi, \sigma_\phi^2)$$

$$\mu_\phi \sim \mathcal{U}[0, 1]$$

$$\sigma_\phi^2 \sim \mathcal{U}[0, 1]$$

$$\sigma_{g.c}^2 | \underline{s} \sim \mathcal{IG2}(\underline{s}, 3)$$

$$\underline{s} \sim \mathcal{G}(1, 1)$$

Model and prior assumptions

Model for carbon intensity.

$$\tau_{c,t} = \eta(t - \bar{t}) + \beta\tau_{c,t-1} - \delta_c + \rho \frac{\sigma_c}{\sigma_{g.c}} \epsilon_{c,t}^{(g)} + \epsilon_{c,t}$$

$$\epsilon_{c,t} \sim \mathcal{N}(0, \sigma_c^2)$$

$$\eta \sim \mathcal{N}(0.1, 0.01)$$

$$\beta \sim \mathcal{U}[0, 1]$$

$$\rho \sim \mathcal{U}[-1, 1]$$

$$\delta_c | \mu_\delta, \sigma_\delta^2 \sim \mathcal{N}(\mu_\delta, \sigma_\delta^2)$$

$$\mu_\delta \sim \mathcal{N}(0, 1)$$

$$\sigma_\delta^2 \sim \text{IG2}(1, 1)$$

$$\sigma_c^2 | \underline{s}_\sigma \sim \text{IG2}(\underline{s}_\sigma, 3)$$

$$\underline{s}_\sigma \sim \mathcal{G}(1, 1)$$

Matrix notation and Gibbs sampler

Metropolis-Hastings sampler

an MCMC method for sampling from the posterior distribution of parameters θ

requires only an ordinate of the kernel of the posterior density

$$k(\theta) = L(\theta, \mathbf{y})p(\theta)$$

relies on the specification of a candidate drawing density

$$\theta^* \sim q(\theta)$$

accept the candidate draw θ^* with probability

$$\min \left\{ 1, \frac{k(\theta^*)q(\theta^*)}{k(\theta^{(s-1)})q(\theta^{(s-1)})} \right\}$$

Gibbs sampler is its special case with acceptance probability 1

Metropolis-Hastings sampler

Due to a non-standard form of dependence in equation

$$\tau_{c.t} = \eta(t - \bar{t}) + \beta\tau_{c.t-1} - \delta_c + \rho \frac{\sigma_c}{\sigma_{g.c}} \epsilon_{c.t}^{(g)} + \epsilon_{c.t}$$

the full conditional posterior distributions are non-standard

Estimation strategy: derive the full conditional posterior densities for all of the parameters as if $\frac{\sigma_c}{\sigma_{g.c}} \epsilon_{c.t}^{(g)}$ was a fixed regressor and use these densities as candidate drawing densities. Accept or reject the candidate draws with appropriate probabilities.

Uniform prior distribution

Uniform prior distributions help to impose restrictions

Density function of a uniform distribution $\mathcal{U}(a, b)$ does not depend on the random variable and is equal to $(b - a)^{-1}$

Full conditional posterior distribution for a parameter $\theta \sim \mathcal{U}(a, b)$ is a truncated density, for instance:

$$L(\theta|\mathbf{y}) = \mathcal{N}(\tilde{\theta}, \tilde{V}_\theta)$$

$$\theta \sim \mathcal{U}(a, b)$$

\downarrow

$$p(\theta|\mathbf{y}, \dots) = \mathcal{N}(\tilde{\theta}, \tilde{V}_\theta) \mathcal{I}(\theta \in [a, b])$$

Hierarchical prior distributions: normal

$$\theta | \mu_\theta, \sigma_\theta^2 \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$$

$$\mu_\theta \sim \mathcal{N}(\mu_\mu, \sigma_\mu^2)$$

$$\sigma_\theta^2 \sim \text{IG2}(s_\theta, \nu_\theta)$$

↓

$$\mu_\theta | \theta, \sigma_\theta^2, \mu_\mu, \sigma_\mu^2 \sim \mathcal{N}\left((\sigma_\theta^{-2} + \sigma_\mu^{-2})^{-1}(\sigma_\theta^{-2}\theta + \sigma_\mu^{-2}\mu_\mu), (\sigma_\theta^{-2} + \sigma_\mu^{-2})^{-1}\right)$$

$$\sigma_\theta^2 | \theta, \mu_\theta, s_\theta, \nu_\theta \sim \text{IG2}\left(s_\theta + (\theta - \mu_\theta)^2, \nu_\theta + 1\right)$$

Hierarchical prior distributions: inverse gamma 2

$$\sigma^2 | \underline{s} \sim \mathcal{IG}2(\underline{s}, \nu) \propto \underline{s}^{\frac{\nu}{2}} (\sigma^2)^{-\frac{\nu+2}{2}} \exp \left\{ -\frac{1}{2} \frac{\underline{s}}{\sigma^2} \right\}$$

$$\underline{s} \sim \mathcal{G}(s, a) \propto \underline{s}^{a-1} \exp \left\{ -\frac{\underline{s}}{s} \right\}$$

↓

$$\underline{s} | \sigma^2 \sim \mathcal{G} \left(\left(s^{-1} + 0.5 \sigma^{-2} \right)^{-1}, \frac{\nu}{2} + a \right)$$

Model for the frontier economy: matrix notation.

$$F = X_F \gamma + \epsilon^{(f)}$$
$$\epsilon^{(f)} \sim \mathcal{N}_T(\mathbf{0}_T, \sigma_f^2 I_T)$$

$$\gamma \sim \mathcal{U}[0, 1]$$
$$\gamma_{pre1973} \sim \mathcal{U}[-0.1, 0.1]$$
$$\sigma_f^2 \sim \mathcal{IG}2(\hat{s}^2, 3)$$

$$F_{56 \times 1} = \begin{bmatrix} F_2 - F_1 \\ \vdots \\ F_T - F_{T-1} \end{bmatrix}, \quad X_F_{56 \times 2} = \begin{bmatrix} I_{14} & I_{14} \\ I_{42} & \mathbf{0}_{42} \end{bmatrix}, \quad \epsilon^{(f)}_{56 \times 1} = \begin{bmatrix} \epsilon_1^{(f)} \\ \vdots \\ \epsilon_T^{(f)} \end{bmatrix}, \quad \gamma_{2 \times 1} = \begin{bmatrix} \gamma \\ \gamma_{pre1973} \end{bmatrix},$$

Model for the frontier economy: MCMC sampler.

$$\gamma|F, X_F, \sigma_f^2 \sim \mathcal{N}_2(\bar{\gamma}, \bar{V}_\gamma) \mathcal{I}\left(\begin{array}{l} \gamma \in [0, 1] \\ \gamma_{pre1973} \in [-.1, .1] \end{array}\right)$$

$$\begin{aligned}\bar{V}_\gamma &= (\sigma_f^{-2} X_F' X_F)^{-1} \\ \bar{\gamma} &= (X_F' X_F)^{-1} X_F' F\end{aligned}$$

$$\sigma_f^2|F, X_F, \gamma \sim \mathcal{IG2}(\bar{s}_f, \bar{\nu}_f)$$

$$\begin{aligned}\bar{s}_f &= \hat{s}^2 + (F - X_F \gamma)'(F - X_F \gamma) \\ \bar{\nu}_f &= T + 3\end{aligned}$$

Model for other economies – matrix notation.

$$G = X_G \phi + \epsilon^{(g)}$$

$$\epsilon^{(g)} \sim \mathcal{N}_{(N-1)T}(\mathbf{0}_{(N-1)T}, \Sigma_G)$$

$$\Sigma_G = \text{diag}(\sigma_{g.2}^2, \dots, \sigma_{g.N}^2) \otimes I_T$$

$$\phi | \mu_\phi, \sigma_\phi^2 \sim \mathcal{N}_{N-1}(\mu_\phi I_{N-1}, \sigma_\phi^2 I_{N-1})$$

$$\mu_\phi \sim \mathcal{U}[0, 1]$$

$$\sigma_\phi^2 \sim \mathcal{U}[0, 1]$$

$$\sigma_{g.c}^2 | \underline{s} \sim \mathcal{IG}2(\underline{s}, 3)$$

$$\underline{s} \sim \mathcal{G}(1, 1)$$

Model for other economies – matrix notation.

$$G_{(N-1)T \times 1} = \begin{bmatrix} F_2 - G_{2,1} \\ \vdots \\ F_T - G_{2,T} \\ \vdots \\ F_2 - G_{N,2} \\ \vdots \\ F_T - G_{N,T} \end{bmatrix}, \quad \epsilon_{(N-1)T \times 1}^{(g)} = \begin{bmatrix} \epsilon_{2,2}^{(g)} \\ \vdots \\ \epsilon_{2,T}^{(g)} \\ \vdots \\ \epsilon_{N,2}^{(g)} \\ \vdots \\ \epsilon_{N,T}^{(g)} \end{bmatrix}, \quad \phi_{N-1 \times 1} = \begin{bmatrix} \phi_2 \\ \vdots \\ \phi_N \end{bmatrix}$$

$$X_G_{(N-1)T \times (N-1)} = \begin{bmatrix} F_1 - G_{2,1} & \dots & 0 \\ \vdots & & \vdots \\ F_{T-1} - G_{2,T-1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & F_1 - G_{N,1} \\ \vdots & & \vdots \\ 0 & \dots & F_{T-1} - G_{N,T-1} \end{bmatrix}$$

Model for other economies – MCMC sampler.

$$\phi|G, X_G, \sigma_g^2, \mu_\phi, \sigma_\phi^2 \sim \mathcal{N}_{N-1}(\bar{\phi}, \bar{V}_\phi) \mathcal{I}(\phi_c \in [0, 1])$$

$$\bar{V}_\phi = \left(X_G' \Sigma_G^{-1} X_G + \sigma_\phi^{-2} I_{N-1} \right)^{-1}$$

$$\bar{\phi} = \bar{V}_\phi \left(X_G' \Sigma_G^{-1} G + \sigma_\phi^{-2} \mu_\phi I_{N-1} \right)$$

$$\sigma_{g.c}^2 | G, X_G, \underline{\varepsilon} \sim \mathcal{IG2}(\bar{\varepsilon}_{g.c}, \bar{\nu}_{g.c})$$

$$\bar{\varepsilon}_{g.c} = \underline{\varepsilon} + (G - X_G \phi)' (G - X_G \phi)$$

$$\bar{\nu}_{g.c} = T + 3$$

$$\mu_\phi | \phi, \sigma_\phi^2 \sim \mathcal{N}_{N-1}(\bar{\mu}_\phi, \bar{V}_{\mu_\phi}) \mathcal{I}(\mu_\phi \in [0, 1])$$

$$\bar{V}_{\mu_\phi} = \sigma_\phi^2 / (N - 1)$$

$$\bar{\mu}_\phi = \bar{V}_{\mu_\phi} \sigma_\phi^{-2} I_{N-1}' \phi$$

$$\sigma_\phi^2 | \phi, \mu_\phi \sim \mathcal{IG2}(\bar{\varepsilon}_\phi, N - 3) \mathcal{I}(\sigma_\phi^2 \in [0, 1])$$

$$\bar{\varepsilon}_\phi = (\phi - \mu_\phi I_{N-1})' (\phi - \mu_\phi I_{N-1})$$

$$\underline{\varepsilon} | \sigma_g^2 \sim \mathcal{G} \left(\left(1 + .5 \sum_c \sigma_{g.c}^{-2} \right)^{-1}, 1.5(N - 1) + 1 \right)$$

Model for carbon intensity – matrix notation.

$$\tau = X_{\tau}\beta_{\tau} + \epsilon$$

$$\epsilon \sim \mathcal{N}_{(\sum_{c=1}^N T_c)}(\mathbf{0}, \Sigma)$$

$$\beta_{\tau} | \mu_{\delta}, \sigma_{\delta}^2 \sim \mathcal{N}_{N+3} \left(\underline{\mu}_{\beta}, \underline{V}_{\beta} \right) \mathcal{I} \left(\begin{array}{l} \beta \in [0, 1] \\ \rho \in [-1, 1] \end{array} \right)$$

$$\mu_{\delta} \sim \mathcal{N}(0, 1)$$

$$\sigma_{\delta}^2 \sim \text{IG2}(1, 1)$$

$$\sigma_c^2 | \underline{s}_{\sigma} \sim \text{IG2}(\underline{s}_{\sigma}, 3)$$

$$\underline{s}_{\sigma} \sim \mathcal{G}(1, 1)$$

$$\underline{\mu}_{\beta} = \begin{bmatrix} 0.1 \\ 0 \\ \mu_{\delta}/N \\ 0 \end{bmatrix}, \quad \underline{V}_{\beta}^{-1} = \begin{bmatrix} 100 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \sigma_{\delta}^{-2}/N & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2/T_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_N^2/T_N \end{bmatrix}$$

Model for carbon intensity – matrix notation.

$$(\sum_{c=1}^N T_c) \times 1 = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_N \end{bmatrix}, \quad (\sum_{c=1}^N \epsilon_c) \times 1 = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_N \end{bmatrix}, \quad \beta_{\tau} = \begin{bmatrix} \eta \\ \beta \\ \delta_1 \\ \vdots \\ \delta_N \\ \rho \end{bmatrix},$$

$$X_{\tau} = \begin{bmatrix} trend_1 & \tau_{1,t-1} & -I_{T_1} & \mathbf{0} & \dots & \mathbf{0} & \frac{\sigma_1}{\sigma_f} \epsilon^{(f)} \\ trend_1 & \tau_{1,t-1} & \mathbf{0} & -I_{T_2} & \dots & \mathbf{0} & \frac{\sigma_2}{\sigma_{g,2}} \epsilon_2^{(g)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ trend_N & \tau_{N,t-1} & \mathbf{0} & \mathbf{0} & \dots & -I_{T_N} & \frac{\sigma_N}{\sigma_{g,2}} \epsilon_N^{(g)} \end{bmatrix},$$

Model for carbon intensity – MCMC sampler.

$$\beta_\tau | \tau, X_\tau, \sigma_c^2, \mu_\delta, \sigma_\delta^2 \sim \mathcal{N}_{N+3}(\bar{\mu}_\beta, \bar{V}_\beta) \mathcal{I}\left(\begin{array}{l} \beta \in [0, 1] \\ \rho \in [-1, 1] \end{array}\right)$$

$$\bar{V}_\beta = (X'_\tau \Sigma^{-1} X_\tau + \underline{V}_\beta^{-1})^{-1}$$

$$\bar{\mu}_\beta = \bar{V}_\beta (X'_\tau \Sigma^{-1} \tau + \underline{V}_\beta^{-1} \underline{\mu}_\beta)$$

$$\sigma_c^2 | \tau, X_\tau, \beta_\tau \underline{\sigma}_\sigma \sim \mathcal{IG}2(\bar{s}_c, \bar{\nu}_c)$$

$$\bar{s}_c = \underline{\sigma}_\sigma + (\tau - X_\tau \beta_\tau)'_{[(T_{c-1}+1):T_c]} (\tau - X_\tau \beta_\tau)_{[(T_{c-1}+1):T_c]}$$

$$\bar{\nu}_c = T_c + 3$$

$$\mu_\delta | \delta, \sigma_\delta^2 \sim \mathcal{N}(\bar{V}_{\mu_\delta} \sigma_\delta^{-2} I'_n \delta, \bar{V}_{\mu_\delta}), \quad \bar{V}_{\mu_\delta} = [\sigma_\delta^{-2} N + 1]^{-1}$$

$$\sigma_\delta^2 | \delta, \mu_\delta \sim \mathcal{IG}2(1 + (\delta - \mu_\delta I_N)'(\delta - \mu_\delta I_N), N + 1)$$

$$\underline{\sigma}_\sigma | \sigma^2 \sim \mathcal{G}\left(\left(1 + \sum_c \sigma_c^2\right)^{-1}, 1.5N + 1\right)$$

Model for the frontier economy

$$F_t = F_{t-1} + \gamma + \gamma_{pre1973}\mathcal{I}(t \leq 1973) + \epsilon_t^{(f)}, \quad \epsilon_t^{(f)} \sim \mathcal{N}(0, \sigma_f^2)$$

$$\gamma \sim \mathcal{U}[0, 1], \quad \gamma_{pre1973} \sim \mathcal{U}[-0.1, 0.1], \quad \sigma_f^2 \sim \mathcal{IG}2(\hat{s}^2, 3)$$

θ	γ	$\gamma_{pre1973}$	σ_f
$E[\theta \mathbf{y}]$	0.016	0.012	0.019
$sd[\theta \mathbf{y}]$	0.003	0.006	0.002

Model for other economies

$$(F_t - G_{c,t}) = \phi_c(F_{t-1} - G_{c,t-1}) + \epsilon_{c,t}^{(g)}, \quad \epsilon_{c,t}^{(g)} \sim \mathcal{N}(0, \sigma_{g,c}^2)$$

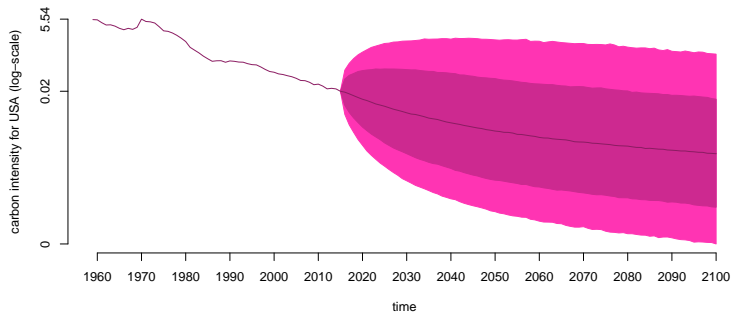
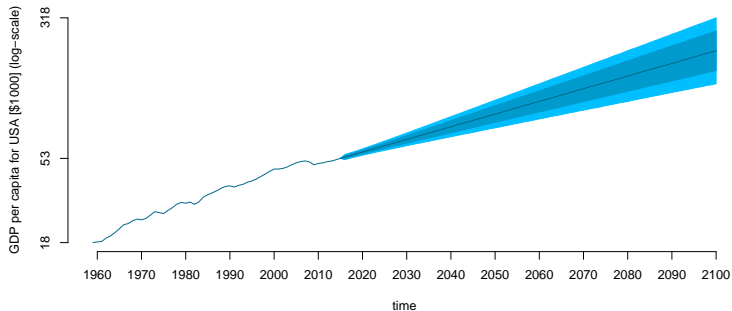
$$\phi_c | \mu_\phi, \sigma_\phi \sim \mathcal{TN}_{[0,1]}(\mu_\phi, \sigma_\phi^2), \quad \mu_\phi \sim \mathcal{U}[0, 1], \quad \sigma_\phi \sim \mathcal{U}[0, 1]$$

$$\sigma_{g,c}^2 | \underline{s} \sim \mathcal{IG2}(\underline{s}, 3), \quad \underline{s} \sim \mathcal{G}(1, 1)$$

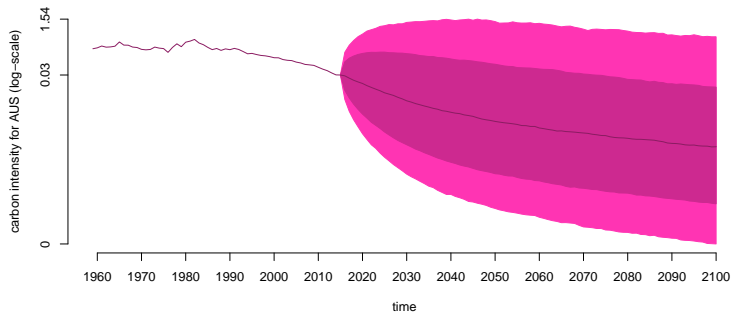
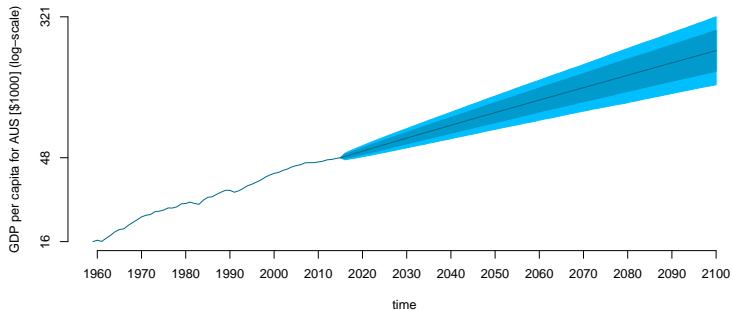
ϕ_c	AUS	CAN	CHN	IDN	IND	JPN	NZL	PHL	POL
$E[\phi_c \mathbf{y}]$	0.98	0.99	0.99	0.99	0.99	0.95	0.99	0.99	0.99
$sd[\phi_c \mathbf{y}]$.013	.007	.003	.003	.002	.008	.005	.001	.004
$\sigma_{g,c}$	AUS	CAN	CHN	IDN	IND	JPN	NZL	PHL	POL
$E[\sigma_{g,c} \mathbf{y}]$	0.02	0.01	0.06	0.05	0.04	0.03	0.03	0.04	0.04
$sd[\sigma_{g,c} \mathbf{y}]$.002	.001	.005	.004	.004	.003	.003	.003	.003
θ	μ_ϕ	σ_ϕ	\underline{s}						
$E[\theta \mathbf{y}]$	0.37	0.56	0.04						
$sd[\theta \mathbf{y}]$.28	.25	.006						

Probabilistic predictions

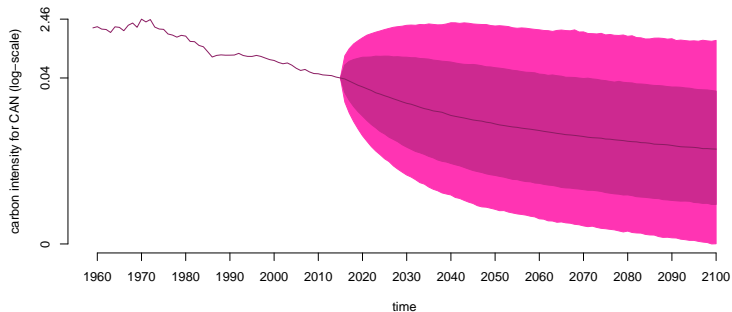
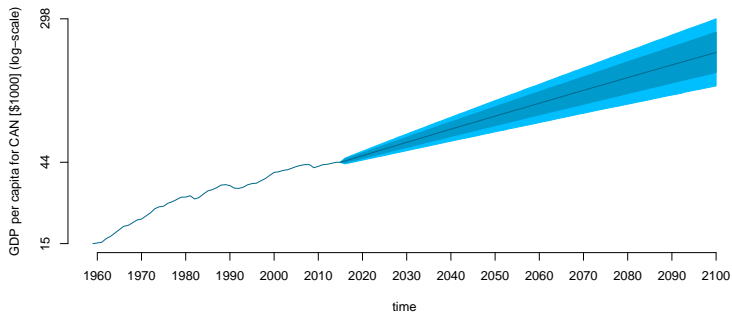
Predictions: USA



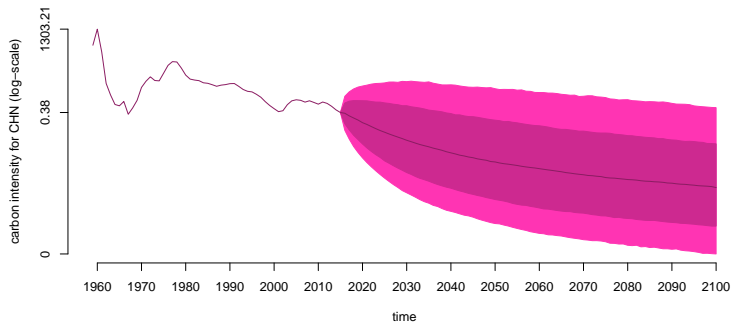
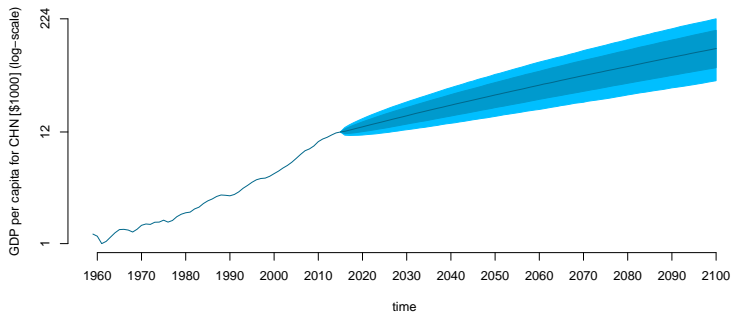
Predictions: Australia



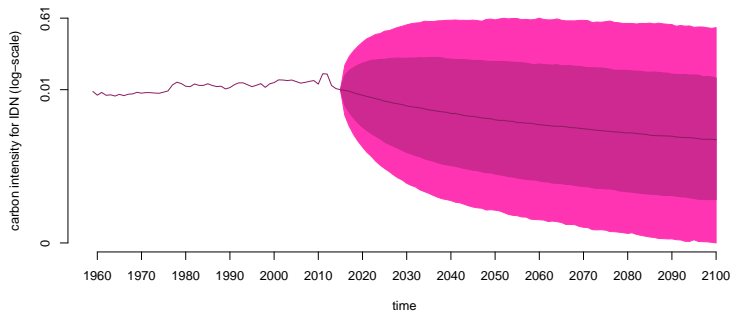
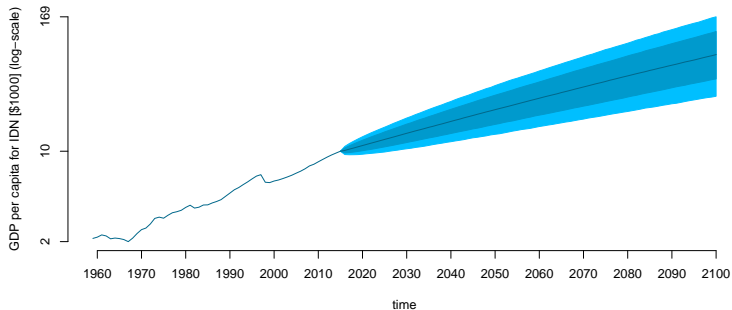
Predictions: Canada



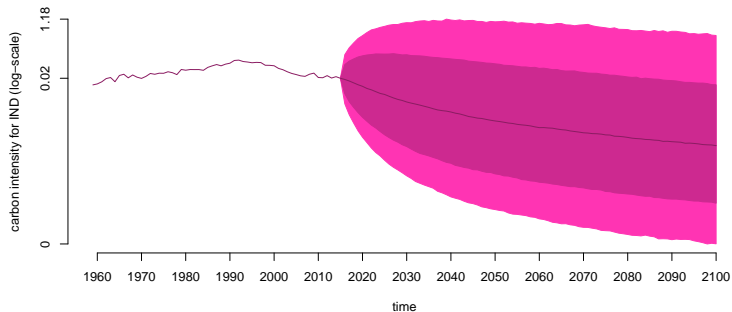
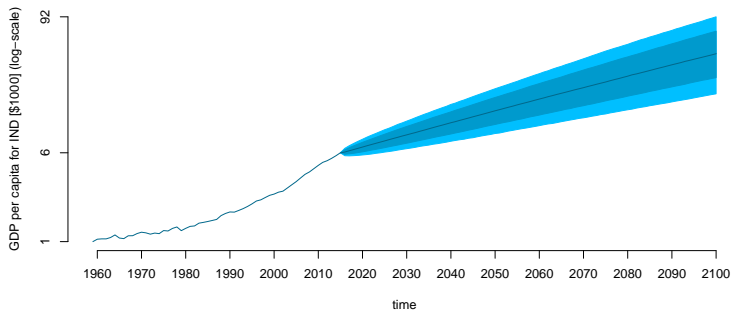
Predictions: China



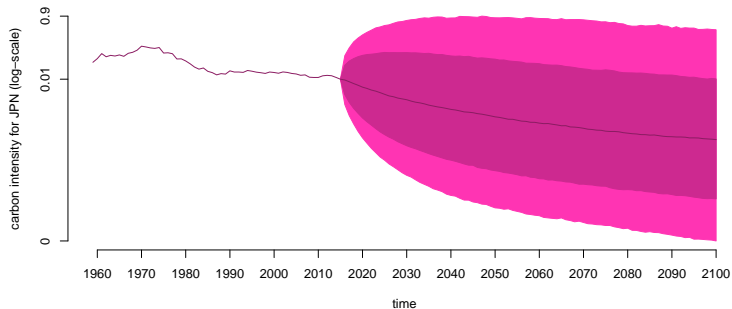
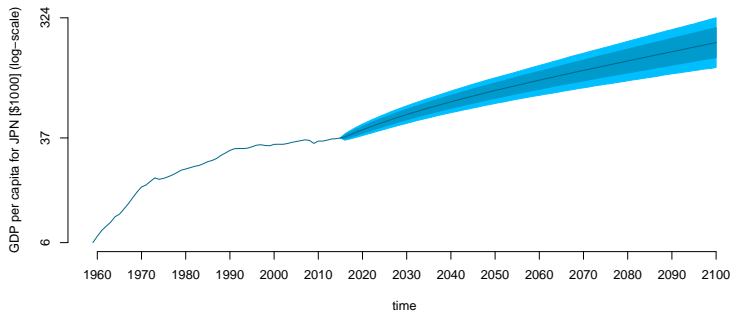
Predictions: Indonesia



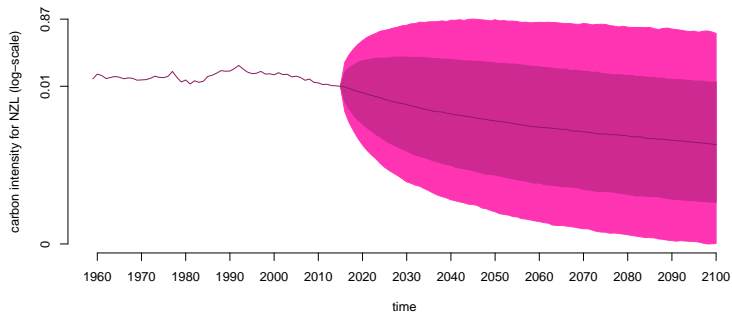
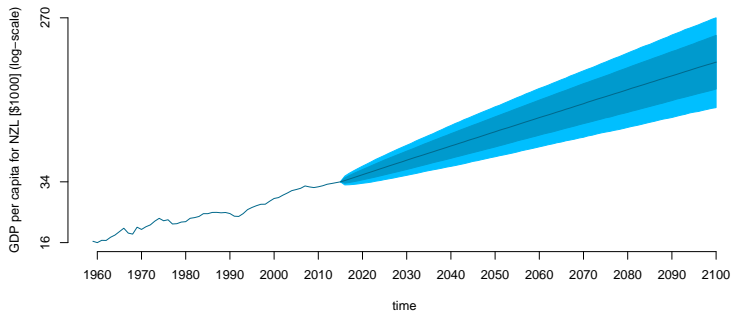
Predictions: India



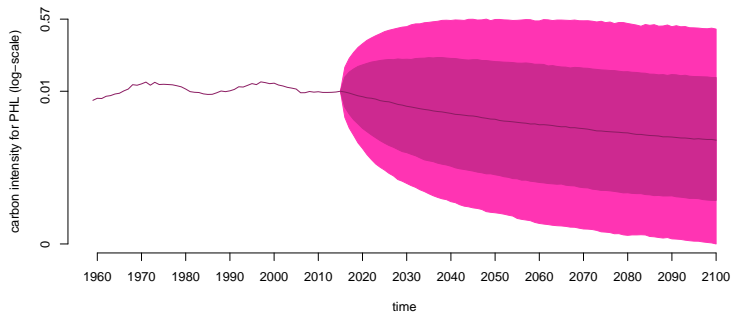
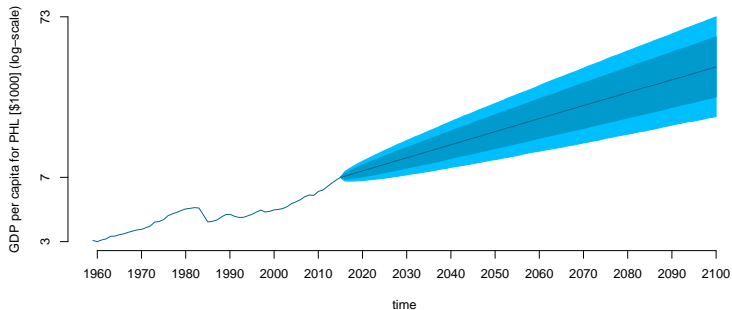
Predictions: Japan



Predictions: New Zealand



Predictions: Philippines



Predictions: Poland

