# Macroeconometrics: Test 2 sample questions

**Exercise 1.** Consider the following autoregression for a detrended random variable $y_t$ with the scalar parameters $\alpha \in (-1, 1)$ and $\sigma^2$, autocorrelated error term $\epsilon_t$, and normally distributed error term $u_t$:

$$y_t = \beta_0 + \beta_1 t + \epsilon_t \tag{1}$$
$$\epsilon_t = \alpha \epsilon_{t-2} + u_t \tag{2}$$
$$u_t | Y_{t-1} \sim iid\mathcal{N}\left(0, \sigma^2\right) \tag{3}$$

where $Y_{t-1}$ collects all the observations on variable $y$ up to time $t-1$.

- Compute the unconditional expected value of $y_t$, denoted by $E[y_t]$.
- Derive autocorrelations at lags 0, 1, and 2 implied by this model. Show your workings. State the assumptions you are applying to get your result.
- Comment in a sentence about the memory patterns this model implies about the data.

**Exercise 2.** Consider the following stationary VAR(1) model for an $N$-vector $\mathbf{y}_t$ with $N \times N$ parameter matrices $\mathbf{A}$ and $\mathbf{\Sigma}$ denoting the autoregressive and covariance matrices respectively and normally distributed error term $\mathbf{u}_t$:

$$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{u}_t \tag{4}$$
$$\mathbf{u}_t \sim \mathcal{N}_N(\mathbf{0}_N, \mathbf{\Sigma}) \tag{5}$$

- Derive autocovariance matrices at lags 0, 1, and 2 implied by this model. Show your workings. State the assumptions you are applying to get your result.
- Comment in a sentence about the memory patterns this model implies about the data.

**Exercise 3.** Consider the following autoregression with $p$ lags for a scalar random variable $y_t$ with the constant term $\mu_0$, autoregressive parameters $\alpha_i$ for $i = 1, \dots, p$, and variance $\sigma^2$:

$$y_t = \mu_0 + \alpha_1 y_{t-1} + \dots + \alpha_p y_{t-p} + u_t \tag{6}$$
$$u_t \mid Y_{t-1} \sim \mathcal{N}\left(0, \sigma^2\right) \tag{7}$$

- Write out the model in a matrix notation.
- State the distribution of the error term vector explicitly.
- State the predictive density implied by the model for the dependent variable vector given the explanatory variables.
- Write out the likelihood function for the model.

**a pdf of the multivariate normal distribution** for an $N$-random vector $\mathbf{X}$ with mean $\boldsymbol{\mu}$ and covariance $\mathbf{\Sigma}$

$$\mathbf{X} \sim \mathcal{N}_N\left(\boldsymbol{\mu}, \mathbf{\Sigma}\right) = (2\pi)^{-\frac{N}{2}} \det(\mathbf{\Sigma})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\mathbf{X} - \boldsymbol{\mu}\right)' \mathbf{\Sigma}^{-1}\left(\mathbf{X} - \boldsymbol{\mu}\right)\right\} \tag{8}$$

**Exercise 4.** Consider the autoregression from **Exercise 3** represented in the matrix notation. Assume the following prior distribution for the $p+1 \times 1$ vector parameter $\boldsymbol{\alpha} = (\mu_0, \alpha_1, \ldots, \alpha_p)'$:

$$\boldsymbol{\alpha} \mid \underline{\boldsymbol{\alpha}}, \underline{\sigma}_{\alpha}^2 \sim \mathcal{N}_{p+1}\left(\underline{\boldsymbol{\alpha}}, \underline{\sigma}_{\alpha}^2 \mathbf{I}_{p+1}\right) \tag{9}$$

where $\mathbf{I}_K$ is the identity matrix of order $K$, and $\underline{\boldsymbol{\alpha}}$ is the $(p+1) \times 1$ vector of the prior mean and the scalar hyper-parameter $\underline{\sigma}_{\alpha}^2$ is the prior variance.

- Derive the full-conditional posterior distribution of the parameter vector $\boldsymbol{\alpha}$ given data, as well as parameter $\sigma^2$ and hyper-parameters $\underline{\boldsymbol{\alpha}}$ and $\underline{\sigma}_{\alpha}^2$, denoted by $p\left(\boldsymbol{\alpha} \mid data, \sigma^2, \underline{\boldsymbol{\alpha}}, \underline{\sigma}_{\alpha}^2\right)$. Show your workings.

**Exercise 5.** Consider the autoregression from **Exercise 3** represented in the matrix notation. Assume the following prior distribution for the scalar parameter $\sigma^2$:

$$\sigma^2 \mid \underline{s}, \underline{\nu} \sim \mathcal{IG}2(\underline{s}, \underline{\nu}) \tag{10}$$

where $\underline{s}$ and $\underline{\nu}$ are positive scalar hyper-parameters of the scale and shape respectively.

- Derive the full-conditional posterior distribution of the parameter $\sigma^2$ given data, as well as parameter $\boldsymbol{\alpha}$ and hyper-parameters $\underline{s}$ and $\underline{\nu}$, denoted by $p\left(\sigma^2 \mid data, \boldsymbol{\alpha}, \underline{s}, \underline{\nu}\right)$. Show your workings.

**a pdf of the inverted gamma 2 distribution** for a positive real-values scalar parameter $\sigma^2$ with scale $s$ and shape $\nu$

$$\sigma^2 \sim \mathcal{IG}2\left(s, \nu\right) = \Gamma\left(\frac{\nu}{2}\right)^{-1}\left(\frac{s}{2}\right)^{\frac{\nu}{2}}\left(\sigma^2\right)^{-\frac{\nu+2}{2}} \exp\left\{-\frac{1}{2}\frac{s}{\sigma^2}\right\} \tag{11}$$

**Exercise 6.** Consider the autoregression from **Exercise 3** represented in the matrix notation and the prior for the parameter vector $\boldsymbol{\alpha}$ from **Exercise 4**. Suppose that you want to estimate the shrinkage hyper-parameter $\underline{\sigma}_{\alpha}^2$ of the prior distribution for parameter $\boldsymbol{\alpha}$. For that purpose, assume the following inverted gamma 2 prior distribution for this hyper-parameter:

$$\underline{\sigma}_{\alpha}^2 \mid \underline{s}_{\sigma}, \underline{\nu}_{\sigma} \sim \mathcal{IG}2(\underline{s}_{\sigma}, \underline{\nu}_{\sigma}) \tag{12}$$

where $\underline{s}_{\sigma}$ and $\underline{\nu}_{\sigma}$ are positive scalar hyper-parameters of the scale and shape respectively.

- Derive the full-conditional posterior distribution of the parameter $\underline{\sigma}_{\alpha}^2$ given parameter $\sigma^2$ and hyper-parameters $\underline{s}$, $\underline{\nu}$, $\underline{s}_{\sigma}$, and $\underline{\nu}_{\sigma}$ denoted by $p\left(\underline{\sigma}_{\alpha}^2 \mid \sigma^2, \underline{s}, \underline{\nu}, \underline{s}_{\sigma}, \underline{\nu}_{\sigma}\right)$. Show your workings.

**Exercise 7.** Consider the following linear regression model for a scalar random variable $y_t$, $K$ explanatory variables denoted by $x_{k.t}$ and with the corresponding regression parameters $\beta_k$ for $k = 1, \dots, K$, and the error term variance $\sigma^2$:

$$y_t = \beta_1 x_{1.t} + \dots + \beta_K x_{K.t} + \epsilon_t \tag{13}$$

$$\epsilon_t \mid x_{1.t}, \dots, x_{K.t} \sim iid\mathcal{N}\left(0, \sigma^2\right) \tag{14}$$

- Write out the model in a matrix notation.
- State the distribution of the error term vector explicitly.
- State the predictive density implied by the model for the dependent variable vector given the explanatory variables.
- Write out the likelihood function for the model.

**Exercise 8.** Consider the linear Gaussian model from **Exercise 7** presented in the matrix notation. Assume a joint zero-mean normal inverted gamma 2 prior distribution for the parameters of the model, a $K$-vector vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_K)'$ and a scalar $\sigma^2$ stated as:

$$\boldsymbol{\beta}, \sigma^2 \mid \underline{\sigma_\beta^2}, \underline{s}, \underline{\nu} \sim \mathcal{NIG}2\left(\mathbf{0}_K, \underline{\sigma_\beta^2}\mathbf{I}_K, \underline{s}, \underline{\nu}\right) \tag{15}$$

where $\mathbf{I}_K$ in the identity matrix of order $K$, and $\underline{\sigma_\beta^2}$, $\underline{s}$, and $\underline{\nu}$ denote positive scalar hyper-parameters.

- Derive the full-conditional posterior distribution of the parameter vector $\boldsymbol{\beta}$ given data, as well as parameter $\sigma^2$ and hyper-parameters $\underline{\sigma_\beta^2}$, $\underline{s}$, and $\underline{\nu}$, denoted by $p\left(\boldsymbol{\beta} \mid data, \sigma^2, \underline{\sigma_\beta^2}, \underline{s}, \underline{\nu}\right)$. Show your workings.

**a pdf of the normal inverted gamma 2 distribution** for an $N$-vector $\mathbf{X}$ and positive real-values scalar parameter $\sigma^2$ with mean parameter $\bar{\boldsymbol{\mu}}$, variance parameter $\boldsymbol{\Sigma}$ scale $s$ and shape $\nu$

$$\Gamma\left(\frac{\nu}{2}\right)^{-1}\left(\frac{s}{2}\right)^{\frac{\nu}{2}}(2\pi)^{-\frac{N}{2}}\det(\boldsymbol{\Sigma})^{-\frac{1}{2}}\left(\sigma^2\right)^{-\frac{\nu+N+2}{2}}\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2}\left[s + (\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]\right\} \tag{16}$$

**Exercise 9.** Consider a linear regression model from **Exercise 7** with the normal inverted gamma 2 joint prior distribution for parameters $\boldsymbol{\beta}$ and $\sigma^2$ as in **Exercise 8**. Suppose that you are interested in estimating the prior shrinkage hyper-parameter $\underline{\sigma_\beta^2}$. For that purpose, assume the following inverted gamma 2 prior distribution for this hyper-parameter:

$$\underline{\sigma_\beta^2} \mid \underline{s}_\sigma, \underline{\nu}_\sigma \sim \mathcal{IG}2(\underline{s}_\sigma, \underline{\nu}_\sigma) \tag{17}$$

where $\underline{s}_\sigma$ and $\underline{\nu}_\sigma$ are positive scalar hyper-parameters of the scale and shape

- Given this setup scrutinise the Gibbs sampler for the parameters $\boldsymbol{\beta}$, $\sigma^2$, and $\underline{\sigma_\beta^2}$. Describe all of the steps of the sampler and make certain that you describe each of its iterations sufficiently to facilitate writing a computer algorithm using your description. Do not derive the full-conditional posterior distributions, but clearly state them making certain that the notation is clear for which parameter a particular distribution is defined and that all of the objects on which you condition the distributions are listed.