Macroeconometrics

Lecture 18 Unobserved Component Model Extensions

Tomasz Woźniak

Department of Economics
University of Melbourne

Simulation smoother for band matrix

Hierarchical priors for variances

Autoregressive slopes from a stationary region

Correlated trend and cycle

Deterministic or stochastic trend

Time-varying drift

Objectives.

- To achieve flexibility in them model specification
- To extend the model by crucial features
- To estimate the extended models

Learning outcomes.

- Modeling flexibility reflecting data properties
- Constructing new estimation algorithms
- Understanding how great UC models can be

A simple UC-AR model

UC-AR(p) model.

$$y_t = \tau_t + \epsilon_t$$

$$\tau_t = \mu + \tau_{t-1} + \eta_t$$

$$\epsilon_t = \alpha_1 \epsilon_{t-1} + \cdots + \alpha_p \epsilon_{t-p} + e_t$$

$$\begin{bmatrix} \eta_t \\ e_t \end{bmatrix} \Bigg| Y_{t-1} \sim ii\mathcal{N}\left(\mathbf{0}_2, \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}\right)$$

A simple UC-AR model

Prior distributions.

$$p(\tau, \epsilon, \alpha, \beta, \sigma) = p\left(\tau|\beta, \sigma_\eta^2\right) p(\beta) p\left(\sigma_\eta^2\right) p\left(\epsilon|\alpha, \sigma_e^2\right) p(\alpha) p\left(\sigma_e^2\right)$$

$$\begin{aligned}
\tau|\beta, \sigma_\eta^2 &\sim \mathcal{N}\left(H^{-1} X_\tau \beta, \sigma_\eta^2 (H'H)^{-1}\right) \\
\beta &\sim \mathcal{N}_2\left(\underline{\beta}, \underline{V}_\beta\right) \\
\sigma_\eta^2 &\sim \mathcal{IG}2\left(\underline{s}, \underline{\nu}\right)
\end{aligned}$$

$$\begin{aligned}
\epsilon|\alpha, \sigma_e^2 &\sim \mathcal{N}\left(\mathbf{0}_T, \sigma_e^2 (H_\alpha' H_\alpha)^{-1}\right) \\
\alpha &\sim \mathcal{N}_p\left(\underline{\alpha}, \underline{V}_\alpha\right) \mathcal{I}(\alpha \in A) \\
\sigma_e^2 &\sim \mathcal{IG}2\left(\underline{s}, \underline{\nu}\right)
\end{aligned}$$

Simulation smoother for band matrix

Simulation smoother for band matrix

When the autoregressive lag order is greater than $p \geq 2$ the precision matrix of the full conditional posterior distributions for τ and ϵ is not tridiagonal but band.

Use functions `bandchol`, `forwardsolve` and `backsolve` from package `mgcv`

```
library(mgcv)

N             = dim(D)[1]
D.L           = t(bandchol(D))
x             = rnorm(N)

a             = forwardsolve(D.L, b)
draw          = backsolve(t(D.L), a + x)
```

Hierarchical priors for variances

Hierarchical priors for variances

Priors might drive the posterior results.

Extensive latent structure of UC models might make the dependence of the posterior results on prior assumptions strong.

Choosing arbitrary values of the prior distribution parameters might drive the definition, estimates, and the interpretation of the trend and cyclical components.

A solution.

Extend the hierarchy of the prior distributions, specify a prior distribution for $\underline{s}$, and estimate it.

The hierarchical prior for the model's variances is given by

$$p\left(\sigma_\eta^2 \mid \underline{s}\right) p\left(\sigma_e^2 \mid \underline{s}\right) p\left(\underline{s}\right)$$

Hierarchical priors for variances

The prior.

The conditionally-conjugate prior for $\underline{s}$ is the gamma distribution.

$$\underline{s} \sim \mathcal{G}(s, a) \quad \propto \quad \underline{s}^{a-1} \exp\left\{-\frac{\underline{s}}{s}\right\}$$

$$E[\underline{s}] = as$$

$$Var[\underline{s}] = as^2$$

The hierarchy.

$$\sigma_\eta^2|\underline{s} \sim \mathcal{IG}2(\underline{s}, \underline{\nu}) \qquad \propto \underline{s}^{\frac{\underline{\nu}}{2}} \left(\sigma_\eta^2\right)^{-\frac{\underline{\nu}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}}{\sigma_\eta^2}\right\}$$

$$\sigma_e^2|\underline{s} \sim \mathcal{IG}2(\underline{s}, \underline{\nu}) \qquad \propto \underline{s}^{\frac{\underline{\nu}}{2}} \left(\sigma_e^2\right)^{-\frac{\underline{\nu}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}}{\sigma_e^2}\right\}$$

$$\underline{s} \sim \mathcal{G}(s, a) \qquad \propto \underline{s}^{a-1} \exp\left\{-\frac{\underline{s}}{s}\right\}$$

Hierarchical priors for variances

Full conditional posterior distribution.

The Gibbs sampler has to be extended by one more step:

$$\underline{s}|y, \sigma_\eta^2, \sigma_e^2 \sim \mathcal{G}\left(\left(s^{-1} + 0.5\left(\sigma_\eta^{-2} + \sigma_e^{-2}\right)\right)^{-1}, \underline{\nu} + a\right)$$

Autoregressive slopes from a stationary region

Autoregressive slopes from a stationary region

Sampling autoregressive slope parameters α from the stationary region $\alpha \in A$ might be non-trivial

Begin with a simple case of $p = 1$ with $\alpha_1 \in (-1, 1)$

Assume a truncated normal prior

$$\alpha_1 \sim \mathcal{N}\left(\underline{\alpha}_1, \underline{V}_\alpha\right)\mathcal{I}(\alpha_1 \in (-1, 1))$$

Sample it from a univariate truncated normal distribution

$$p\left(\alpha | y, \epsilon, \sigma_e^2\right) = \mathcal{N}_p\left(\overline{\alpha}, \overline{V}_\alpha\right)\mathcal{I}(\alpha \in (-1, 1))$$

$$\overline{V}_\alpha = \left[\sigma_e^{-2} X_\epsilon' X_\epsilon + \underline{V}_\alpha^{-1}\right]^{-1}$$

$$\overline{\alpha} = \overline{V}_\alpha \left[\sigma_e^{-2} X_\epsilon' \epsilon + \underline{V}_\alpha^{-1} \underline{\alpha}\right]$$

Using function `RcppTN::rtn`

Autoregressive slopes from a stationary region

A recommended strategy for the case of $p \geq 2$ is to not impose the restrictions in the sampler.

The benefits are

No tedious sampler taking forever to run when the true probability of $\alpha \in A$ is not equal to 1

If is not such that $\alpha \in A$ then y_t has more than one unit root and the trend and cycle are not identified in a simple UC model. Need to refine the model anyways.

Correlated trend and cycle

Correlated trend and cycle

When the autoregressive lag order is greater than $p \geq 2$ the covariance between shocks η_t and e_t is identified and can be estimated.

Appropriate parameterisation of a model makes the estimation fast and simple.

Error term specification.

$$\begin{bmatrix}\eta_t\\ e_t\end{bmatrix}\Bigg| Y_{t-1} \sim ii\mathcal{N}\left(\mathbf{0}_2, \Sigma\right)$$

$$\Sigma = \begin{bmatrix}\sigma_\eta^2 & \rho\sigma_\eta\sigma_e\\ & \sigma_e^2\end{bmatrix}$$

ρ – correlation between the shocks

Correlated trend and cycle

The implementation is based on the decomposition of the joint bivariate normal distribution into conditional and marginal distributions.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ & \sigma_2^2 \end{bmatrix}\right)$$

$$\downarrow$$

$$x_1 \mid x_2 \sim \mathcal{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}x_2, \left(1-\rho^2\right)\sigma_1^2\right)$$

$$x_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$$

Correlated trend and cycle

The model in its original form

$$y = \tau + \epsilon \tag{1}$$

$$H\tau = X_\tau \beta + \eta \tag{2}$$

$$H_\alpha \epsilon = e \tag{3}$$

$$\begin{bmatrix} \eta \\ e \end{bmatrix} \Bigg| y \sim \mathcal{N}_{2T}\left(\mathbf{0}_{2T}, \begin{bmatrix} \sigma_\eta^2 & \rho\sigma_\eta\sigma_e \\ & \sigma_e^2 \end{bmatrix} \otimes I_T\right) \tag{4 \& 5}$$

is rewritten as

$$H\tau = X_\tau \beta + \rho\frac{\sigma_\eta}{\sigma_e} e + \eta \tag{2}$$

$$H_\alpha \epsilon = e \tag{3}$$

$$\eta \mid e \sim \mathcal{N}_T\left(\mathbf{0}_T, \left(1 - \rho^2\right)\sigma_\eta^2 I_T\right) \tag{4}$$

$$e \sim \mathcal{N}_T\left(\mathbf{0}_T, \sigma_e^2 I_T\right) \tag{5}$$

Correlated trend and cycle

The modifications include defining new matrices and parameters

$$\tilde{X}_\tau = \begin{bmatrix} \imath_t & e_{1.T} & e \end{bmatrix} \qquad \tilde{\beta} = \begin{bmatrix} \mu & \tau_0 & \gamma \end{bmatrix}'$$

$$\gamma = \rho \frac{\sigma_\eta}{\sigma_e} \qquad \tilde{\sigma}_\eta^2 = \left(1 - \rho^2\right) \sigma_\eta^2$$

... and rewriting the model

$$H\tau = \tilde{X}_\tau \tilde{\beta} + \eta \tag{2}$$

$$\eta \mid e \sim \mathcal{N}_T\left(\mathbf{0}_T, \tilde{\sigma}_\eta^2 I_T\right) \tag{4}$$

The interpretable parameters of the model can be retrieved by

$$\sigma_\eta^2 = \tilde{\sigma}_\eta^2 + \gamma^2 \sigma_e^2 \qquad \rho = \gamma \frac{\sigma_e}{\sqrt{\tilde{\sigma}_\eta^2 + \gamma^2 \sigma_e^2}}$$

The Gibbs sampler stays the same after replacing $\tilde{X}_\tau$ for X_τ, $\tilde{\beta}$ for β, and $\tilde{\sigma}_\eta^2$ for σ_η^2.

Deterministic or stochastic trend

Deterministic or stochastic trend

Autoregressive model obtained by imposing $\sigma_\eta^2 = 0$

$$\begin{aligned} y_t - \mu t &= \epsilon_t \\ \alpha_p(L)\epsilon_t &= e_t \\ e_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_e^2\right) \end{aligned}$$

Alternative hypotheses.

$\sigma_\eta^2 \neq 0$ – trend-cycle decomposition of unit-root non-stationary series with on unit root

$\sigma_\eta^2 = 0$ – unit-root stationary cycle around a deterministic trend

Deterministic or stochastic trend

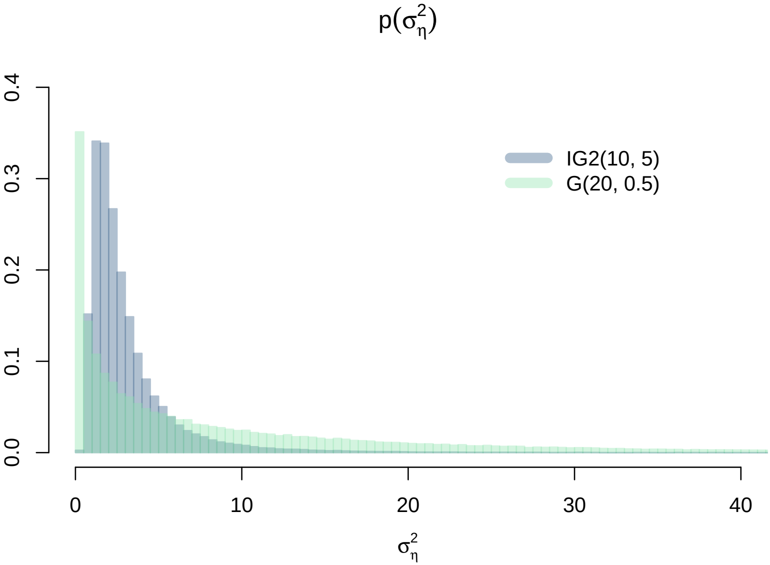
Gamma prior for σ_η^2.

The possibility of of shrinking the value of variance to zero is facilitated by a gamma prior the shape parameter set to $\frac{1}{2}$

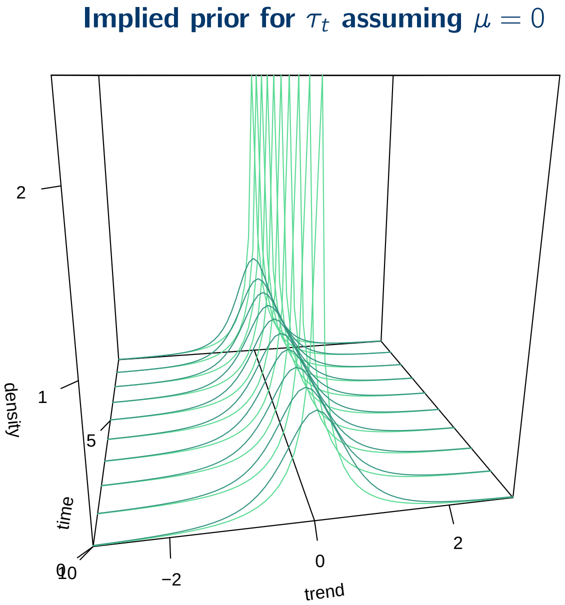
$$\sigma_\eta^2 \mid \underline{s} \sim \mathcal{G}\left(2\underline{s}, \frac{1}{2}\right)$$

where $E[\sigma_\eta^2] = \underline{s}$ and $\underline{s}$ is a hyper-parameter to be specified

Deterministic or stochastic trend



Deterministic or stochastic trend



Deterministic or stochastic trend

Full conditional posterior density.

The gamma prior implies a generalised inverse Gaussian full conditional posterior distribution for σ_η^2

$$p\left(\sigma_\eta^2 \mid y, \tau, \beta\right) = \mathcal{GIG}\left(\lambda, \chi, \psi\right)$$

$$\lambda = -\frac{T-1}{2}$$

$$\chi = (X_\tau\beta - H\tau)'(X_\tau\beta - H\tau)$$

$$\psi = \frac{2}{\underline{s}}$$

Sample from this distribution using function `GIGrvg::rgig`

Time-varying drift

Time-varying drift

Make the drift parameter change over time

$$\tau_t = \mu_t + \tau_{t-1} + \eta_t$$
$$\mu_t = \mu_{t-1} + m_t$$

$$m_t | Y_{t-1} \sim \mathcal{N}\left(0, \sigma_m^2\right)$$
$$\sigma_m^2 \mid \underline{s} \sim \mathcal{IG}2(\underline{s}, \underline{\nu})$$

The time-varying intercept parameter μ_t follows a Gaussian random walk process with initial value μ_0

Unit-root non-stationarity of μ_t require the series y_t to have two unit roots

This might be a perfect model for *CPI prices*

Time-varying drift

Define $T \times 1$ matrices $\mu = \begin{bmatrix} \mu_1 & \dots & \mu_T \end{bmatrix}'$ and $m = \begin{bmatrix} m_1 & \dots & m_T \end{bmatrix}'$

Rewrite the trend equation

$$H\tau = \mu + e_{1.T}\tau_0 + \eta$$

Specify equations for μ

$$\begin{aligned} H\mu &= \mu_0 e_{1.T} + m \\ \mu &= \mu_0 \imath_T + H^{-1} m \\ m &\sim \mathcal{N}\left(\mathbf{0}_T, \sigma_m^2 I_T\right) \end{aligned}$$

Time-varying drift

Sample τ from a multivariate normal distribution using the simulation smoother

$$p\left(\tau | y, \alpha, \beta, \sigma\right) = \mathcal{N}_T\left(\overline{\tau}, \overline{V}\right)$$

$$\begin{aligned}
\overline{V} &= \left[\sigma_e^{-2} H_\alpha' H_\alpha + \sigma_\eta^{-2} H' H\right]^{-1} \\
\overline{\tau} &= \overline{V}\left[\sigma_e^{-2} H_\alpha' H_\alpha y + \sigma_\eta^{-2} H'(\mu + \tau_0 e_{1.T})\right]
\end{aligned}$$

Time-varying drift

Sample μ from a multivariate normal distribution using the simulation smoother

$$p\left(\mu | y, \tau, \tau_0, \mu_0, \sigma_\eta^2, \sigma_m^2\right) = \mathcal{N}_T\left(\overline{\mu}, \overline{V}_\mu\right)$$

$$\begin{aligned}
\overline{V}_\mu &= \left[\sigma_\eta^{-1} I_T + \sigma_m^{-1} H' H\right]^{-1} \\
\overline{\mu} &= \overline{V}_\mu \left[\sigma_\eta^{-1}(H\tau - e_{1.T}\tau_0) + \sigma_m^{-1} e_{1.T}\mu_0\right]
\end{aligned}$$

Time-varying drift

Sample μ_0 from a normal distribution

$$p\left(\mu_0|y,\mu,\sigma_m^2\right) = \mathcal{N}\left(\overline{\mu}_0, \overline{V}_{\mu.0}\right)$$

$$\overline{V}_{\mu.0} = \left[\sigma_m^{-2} + \underline{V}_{\mu}^{-1}\right]^{-1}$$

$$\overline{\mu}_0 = \overline{V}_{\mu.0}\left[\frac{\mu_1}{\sigma_m^2} + \frac{\underline{\mu}_0}{\underline{V}_{\mu}}\right]$$

... and σ_m^2 from the inverse gamma 2

$$p\left(\sigma_m^2|y,\mu,\mu_0,\underline{s}\right) = \mathcal{IG}2\left(\overline{s}_m, \overline{\nu}_m\right)$$

$$\overline{s}_m = \underline{s} + (H\mu - e_{1.T}\mu_0)'(H\mu - e_{1.T}\mu_0)$$

$$\overline{\nu}_m = \underline{\nu} + T$$

Other extensions and applications of state space models:

- handling seasonality in time series
- dynamic factor models capturing the variability of many variables with a few factors
- time-varying parameter models
- Stochastic Volatility

Unobserved Component Model Extensions

UC models are used to introduce to state-space modeling

Every single extension might make the model suitable for the data

Their various combination result in a flexible setting applicable to a range of economic and financial data