Macroeconometrics

Lecture 13 SVARs: Bayesian estimation I

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Estimating models with exclusion restrictions

Estimating models with sign restrictions

Useful readings:

Rubio-Ramírez, Waggoner & Zha (2010) Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference, Review of Economic Studies

Materials:

An R file `L13 mcxs.R` for the reproduction of the example for Algorithm 1 and 2

Objectives.

- To present general estimation algorithms of SVAR models with exclusion or sign restrictions
- To work with procedures taking Bayesian estimation of VARs as a starting point
- To introduce the identification of structural shocks using sign restrictions

Learning outcomes.

- Understanding the rotations of the structural system
- Generating random draws of rotation matrices
- Sampling random draws of parameters with appropriate restrictions

Bayesian VARs

$$p\left(A, \Sigma | Y, X\right) = p(A|Y, X, \Sigma) p\left(\Sigma | Y, X\right)$$

$$p(A|Y, X, \Sigma) = \mathcal{MN}_{K \times N}\left(\overline{A}, \Sigma, \overline{V}\right)$$
$$p(\Sigma|Y, X) = \mathcal{IW}_N\left(\overline{S}, \overline{\nu}\right)$$

$$\overline{V} = \left(X'X + \underline{V}^{-1}\right)^{-1}$$
$$\overline{A} = \overline{V}\left(X'Y + \underline{V}^{-1}\underline{A}\right)$$
$$\overline{\nu} = T + \underline{\nu}$$
$$\overline{S} = \underline{S} + Y'Y + \underline{A}'\underline{V}^{-1}\underline{A} - \overline{A}'\overline{V}^{-1}\overline{A}$$

Bayesian Structural VARs

$$B_0 y_t = b_0 + B_1 y_{t-1} + \cdots + B_p y_{t-p} + u_t$$

The concept for the sampling algorithm

Sample draws from the posterior distribution of (A, Σ) to get

$$\left\{A^{(s)}, \Sigma^{(s)}\right\}_{s=1}^{S}$$

Compute draws from the posterior distribution of a triangular SVAR system

$$\tilde{B}_0^{(s)} = \text{chol}\left(\left(\Sigma^{(s)}\right)^{-1}\right) \qquad \tilde{B}_+^{(s)} = \tilde{B}_0^{(s)} A^{(s)}$$

Compute or sample an orthogonal matrix $Q^{(s)}$ that is consistent with the restrictions

Compute draws of parameter matrices with desired restrictions from the target posterior distribution

$$B_0^{(s)} = Q^{(s)} \tilde{B}_0^{(s)} \qquad B_+^{(s)} = Q^{(s)} \tilde{B}_+^{(s)}$$

Estimating models with **exclusion restrictions**

Identification of models with exclusion restrictions

$$QB_0y_t = Qb_0 + QB_1y_{t-1} + \cdots + QB_py_{t-p} + Qu_t$$

All of the structural VARs are identified up to a rotation matrix.

SVARs identified with exclusion restrictions are identified to a special case of a rotation matrix, that is, a diagonal matrix with each of the diagonal elements equal to ± 1

$$Q = D$$

Individual equations and the structural shocks are identified up to a sign.

See more on normalization as a solution to this problem

Estimating models with exclusion restrictions

Algorithm 1 described below transforms any SF parameters $(\tilde{B}_+, \tilde{B}_0)$ to parameters such that the restrictions of interests hold. These parameters are denoted by (B_+, B_0).

Algorithm 1 works for exactly identified models, that is, the restrictions of interest to be imposed on the system must exactly identify the model. The appropriate conditions should be verified.

Algorithm 1 is applicable to any parameters $(\tilde{B}_+, \tilde{B}_0)$, e.g.:

Maximum likelihood estimates

Bootstrapped parameters sampled from their empirical distribution in an appropriate bootstrap procedure

Posterior draws in Bayesian inference

Estimating models with exclusion restrictions

Let $(\tilde{B}_+, \tilde{B}_0)$ be any value of the structural parameters.

Algorithm 1.

Step 1 Set $n = 1$

Step 2 Form matrix

$$\tilde{\mathbf{R}}_n = \begin{bmatrix} \mathbf{R}_n f(\tilde{B}_+, \tilde{B}_0) \\ q_1 \\ \vdots \\ q_{n-1} \end{bmatrix}$$

If $n = 1$, then $\tilde{\mathbf{R}}_1 = \mathbf{R}_1 f(\tilde{B}_+, \tilde{B}_0)$

Step 3 Compute vector $q_n = \tilde{\mathbf{R}}_{n\perp}$ such that $\tilde{\mathbf{R}}_n q_n = 0$
where $X_\perp$ is the orthogonal complement of matrix X

Step 4 If $n = N$, stop. If not, set $n = n + 1$ and go to **Step 2**

Return $Q = \begin{bmatrix} q_1' & \dots & q_N' \end{bmatrix}'$ $\quad B_+ = Q\tilde{B}_+ \quad B_0 = Q\tilde{B}_0$
Parameters (B_+, B_0) are such that the restrictions hold.

Estimating models with exclusion restrictions

Orthogonal complement matrix.

To compute the orthogonal complement matrix of an $M \times N$ matrix X where $M > N$

Compute the QR decomposition of matrix X where Q is an orthogonal matrix

Return the last $M - N$ columns of matrix Q as and $(M - N) \times N$ matrix $X_{\perp}$

```
orthogonal.complement.matrix = function(x){
  N       = dim(x)
  tmp     = qr.Q(qr(x, tol = 1e-10),complete=TRUE)
  out     = as.matrix(tmp[,(N[2]+1):N[1]])
  return(out)
}
```

Estimating models with exclusion restrictions: example

Restrictions for IRFs on horizons 0 and ∞ for a model with $p = 1$

$$f(B_+, B_0) = \begin{bmatrix} \Theta_0 \\ \Theta_\infty \end{bmatrix} = \begin{bmatrix} B_0^{-1} \\ (B_0 - B_1)^{-1} \end{bmatrix} = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \\ 0 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Which requires setting

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{R}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These matrices are of ranks $r_1 = 2$, $r_2 = 1$, and $r_3 = 0$ respectively.
The model is exactly identified.

It suffices to consider matrices with non-zero rows

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Estimating models with exclusion restrictions: example

Let the estimated RF parameters (A, Σ) be:

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 \\ -1.25 & 0.25 & 0 \\ -1 & 0 & 0.5 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 4.25 & 2.5 \\ 1 & 2.5 & 3 \end{bmatrix}$$

Compute initial values of SF parameters $\tilde{B}_0 = \text{chol}(\Sigma)^{-1\prime}$ and $\tilde{B}_1 = \tilde{B}_0 A$

$$\tilde{B}_1 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.75 & 0 & 0 \\ -0.75 & -0.5 & 0.5 \end{bmatrix} \qquad \tilde{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 0.5 & 0 \\ -0.75 & -0.5 & 1 \end{bmatrix}$$

These are the estimates of parameters that maximize the likelihood function or are drawn from the posterior distribution, however, they are subject to a likelihood invariant transformation by premultiplying by a rotation matrix that that will impose zero restrictions on appropriate elements.

Estimating models with exclusion restrictions: example

Construct function $f(\tilde{B}_+, \tilde{B}_0)$:

$$f(\tilde{B}_+, \tilde{B}_0) = \begin{bmatrix} \tilde{B}_0^{-1} \\ (\tilde{B}_0 - \tilde{B}_1)^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

And proceed to **Algorithm 1.**

Estimating models with exclusion restrictions: example

Iteration: $n = 1$

$$\tilde{\mathbf{R}}_1 = \mathbf{R}_1 f(\tilde{B}_+, \tilde{B}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{\perp} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

The vector above is the first row of rotation matrix Q that will rotate $(\tilde{B}_+, \tilde{B}_0)$ assigning it the correct restrictions.

Estimating models with exclusion restrictions: example

Iteration: $n = 2$

$$\tilde{\mathbf{R}}_2 = \begin{bmatrix} \mathbf{R}_2 f(\tilde{B}_+, \tilde{B}_0) \\ q_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\perp} = \begin{bmatrix} -0.7071068 & 0.7071068 & 0 \end{bmatrix}$$

Iteration: $n = 3$

$$\tilde{\mathbf{R}}_3 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -0.7071068 & 0.7071068 & 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 & 0 & 1 \\ -0.7071068 & 0.7071068 & 0 \end{bmatrix}_{\perp} = \begin{bmatrix} -0.7071068 & -0.7071068 & 0 \end{bmatrix}$$

Estimating models with exclusion restrictions: example

Return parameter matrices:

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -0.7071068 & 0.7071068 & 0 \\ -0.7071068 & -0.7071068 & 0 \end{bmatrix}$$

$$B_0 = Q\tilde{B}_0 = \begin{bmatrix} -0.75 & -0.5 & 1 \\ -0.884 & 0.354 & 0 \\ -0.53 & -0.354 & 0 \end{bmatrix}$$

$$B_1 = Q\tilde{B}_1 = \begin{bmatrix} -0.75 & -0.5 & 0.5 \\ -0.884 & -0.354 & 0 \\ 0.177 & -0.354 & 0 \end{bmatrix}$$

Estimating models with exclusion restrictions: example

Verify IRFs:

$$\Theta_0 = B_0^{-1} = \begin{bmatrix} 0 & -0.707 & -0.707 \\ 0 & 1.061 & -1.768 \\ 1 & 0 & -1.414 \end{bmatrix}$$

$$\Theta_\infty = (B_0 - B_1)^{-1} = \begin{bmatrix} 0 & 0 & -1.414 \\ 0 & 1.414 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Estimating models with **sign restrictions**

Estimating models with sign restrictions

Sign restrictions.

$$\mathbf{R}_n f\left(B_+, B_0\right) e_n > \mathbf{0}_{R\times 1} \quad \text{for } n = 1, \ldots, N$$

Provide identification of the structural shocks without the need to impose strict exclusion restrictions that might be controversial

Are motivated by economic theory and empirical stylized facts

Set identify the model, that is, for any set of sign restrictions, given a parameter point (B_+, B_0) that satisfies such restrictions, there always exists an orthogonal matrix Q, arbitrarily close to an identity matrix, such that (QB_+, QB_0) will also satisfy the sign restrictions.

Set identification implies that there is a non-empty set of orthogonal matrices $Q \in \mathbb{O} \subset \mathcal{O}(N)$ that satisfy the sign restrictions.

Estimation procedure has to efficiently exploit set $\mathbb{O}$

Estimating models with sign restrictions

Sign restrictions: Example 1

Uhlig (2005) What are the effects of monetary policy on output? Results from an agnostic identification procedure, Journal of Monetary Economics

Variables in y_t

$rgdp_t$ – real GDP, tr_t – total reserves, p_t – GDP price deflator, nbr_t – non-borrowed reserves, cpi_t – commodity price index, FFR_t – federal funds rate

Sign restrictions for the monetary policy shock

A monetary policy impulse vector is an impulse vector u so that the impulse responses to u of prices and non-borrowed reserves are not positive and the impulse responses for the federal funds rate are not negative, all at horizons $i = 0, 1, \ldots, h$.

Estimating models with sign restrictions

Sign restrictions: Example 2

Canova, Paustian (2011) Business cycle measurement with some theory, Journal of Monetary Economics

Sign restrictions

$$\begin{bmatrix} i_t \\ rw_t \\ \pi_t \\ rgdp_t \\ hw_t \end{bmatrix} = \begin{bmatrix} + & + & + & - & * \\ - & + & - & + & * \\ + & - & + & - & * \\ - & - & + & + & * \\ - & - & + & - & * \end{bmatrix} \begin{bmatrix} u_t^{markup} \\ u_t^{monetary} \\ u_t^{taste} \\ u_t^{technology} \\ u_t^{measurement} \end{bmatrix}$$

i_t – interest rate, rw_t – real wage, π_t – inflation, $rgdp_t$ – real output, hw_t – hours worked

Useful distribution: Haar

Definition.

Haar distribution is a uniform distribution over the space of orthogonal matrices $\mathcal{O}(N)$

Random number generator.

Let X be an $N \times N$ random matrix with each element having an independent standard normal distribution. Let $X = QR$ be the QR decomposition of X with the diagonal of R normalized to be positive. The random matrix Q is orthogonal and is a draw from the uniform distribution over $\mathcal{O}(N)$.

Estimating models with sign restrictions

Algorithm 2 described below transforms any SF parameters $(\tilde{B}_+, \tilde{B}_0)$ to parameters such that the restrictions of interests hold. These parameters are denoted by (B_+, B_0).

Algorithm 2 works for set identified models with the sign restrictions.

Algorithm 2 is applicable to any parameters $(\tilde{B}_+, \tilde{B}_0)$, e.g.:

Bootstrapped parameters, that is, parameters sampled from their empirical distribution in an appropriate bootstrap procedure

Posterior draws in Bayesian inference

Algorithm 2 is not designed for the MLE. Apply all of the recommendations from
Fry, Pagan (2011) Sign Restrictions in Structural Vector Autoregressions: A Critical Review, *Journal of Economic Literature*

Estimating models with sign restrictions

Let $(\tilde{B}_+, \tilde{B}_0)$ be any value of the structural parameters.

Algorithm 2.

Step 1 Draw an independent standard normal $N \times N$ matrix X and let $X = QR$ be the QR decomposition of X with the diagonal of R normalized to be positive.

Step 2 Use matrix Q to compute parameters $B_0 = Q\tilde{B}_0$, $B_+ = Q\tilde{B}_+$ and the corresponding impulse responses that are subject to sign restrictions.

Step 3 If these parameters and impulse responses do not satisfy the sign restrictions, return to **Step 1**

Return parameters (B_+, B_0)

Estimating models with sign restrictions: example

Consider restrictions on IRFs on horizons 0 and 1

$$f(B_+, B_0) = \begin{bmatrix} \Theta_0 \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} B_0^{-1} \\ B_0^{-1} B_1 B_0^{-1} \end{bmatrix} = \begin{bmatrix} - & * & * \\ - & * & * \\ + & * & * \\ - & * & * \\ - & * & * \\ + & * & * \end{bmatrix}$$

Which requires setting

$$\mathbf{R}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $\mathbf{R}_1 f(B_+, B_0) e_1 > 0$ imposes the required restrictions.

Estimating models with sign restrictions: example

Let the estimated RF parameters $(A, \boldsymbol{\Sigma})$ be:

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 \\ -1.25 & 0.25 & 0 \\ -1 & 0 & 0.5 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 4.25 & 2.5 \\ 1 & 2.5 & 3 \end{bmatrix}$$

Compute initial values of SF parameters $\tilde{B}_0 = \mathrm{chol}(\boldsymbol{\Sigma})^{-1\prime}$ and $\tilde{B}_1 = \tilde{B}_0 A$

$$\tilde{B}_1 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.75 & 0 & 0 \\ -0.75 & -0.5 & 0.5 \end{bmatrix} \qquad \tilde{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 0.5 & 0 \\ -0.75 & -0.5 & 1 \end{bmatrix}$$

These are the estimates of parameters that coming from a bootstrap procedure or that are drawn from the posterior distribution, however, they are subject to a likelihood invariant transformation by premultiplying by a rotation matrix that will impose sign restrictions on appropriate elements.

Estimating models with sign restrictions: example

Construct function $f(\tilde{B}_+, \tilde{B}_0)$:

$$f(\tilde{B}_+, \tilde{B}_0) = \begin{bmatrix} \tilde{B}_0^{-1} \\ \tilde{B}_0^{-1}\tilde{B}_1\tilde{B}_0^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 2 & 0 \\ 1 & 1 & 1 \\ 0.75 & 1 & 0 \\ -1.125 & 0.5 & 0 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$$

And proceed to **Algorithm 2.**

Estimating models with sign restrictions: example

After 118 iterations the algorithm returned matrices

$$X = \begin{bmatrix} -0.184 & -0.797 & 1.060 \\ -1.702 & 0.957 & -0.494 \\ 2.354 & -1.295 & 1.084 \end{bmatrix} \qquad Q = \begin{bmatrix} -0.063 & -0.585 & 0.809 \\ -0.998 & 0.052 & -0.040 \\ 0.019 & 0.810 & 0.587 \end{bmatrix}$$

that give

$$B_0 = \begin{bmatrix} -0.523 & -0.697 & 0.809 \\ -0.981 & 0.046 & -0.040 \\ -0.624 & 0.111 & 0.587 \end{bmatrix} \qquad B_1 = \begin{bmatrix} -0.200 & -0.436 & 0.404 \\ -0.508 & -0.479 & -0.020 \\ -1.038 & -0.284 & 0.293 \end{bmatrix}$$

and

$$\Theta_0 = \begin{bmatrix} -0.063 & -0.998 & 0.019 \\ -1.201 & -0.395 & 1.628 \\ 0.161 & -0.986 & 1.415 \end{bmatrix} \qquad \Theta_1 = \begin{bmatrix} -0.632 & -0.696 & 0.823 \\ -0.221 & 1.149 & 0.384 \\ 0.144 & 0.505 & 0.689 \end{bmatrix}$$

Structural VARs: Bayesian estimation I

Algorithms proposed by Rubio-Ramírez, Waggoner & Zha (2010) allow the estimation under a great flexibility in the type of identification patterns for SVARs

Estimation procedures are relatively quick, follow simple algorithms and apply to both frequentist and Bayesian approaches

Computations of IRFs and FEVDs are straightforward.

Model comparison and selection requires alternative procedures.