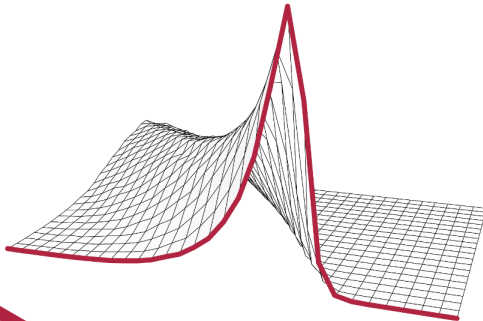


mcxs



Macroeconometrics

Lecture 5 Understanding unit-rooters

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Concepts

Autoregressions

Random walk processes

Inference

A helicopter tour

Useful readings:

Sims, Uhlig (1991) Understanding Unit Rooters: A Helicopter Tour,
Econometrica

Materials:

An R file 05 mcxs.R, 06 mcxs.R for the reproduction the results
Shiny interactive apps: 05shiny-ar1.R and 05shiny-ar2.R

Concepts

Strict stationarity

A time series y_t is said to be strictly stationary if the joint distribution of $(y_{t_1}, \dots, y_{t_k})$ is identical to that of $(y_{t_1+s}, \dots, y_{t_k+s})$ for all t , where k is an arbitrary positive integer and (t_1, \dots, t_k) is a collection of k positive integers:

$$p(y_{t_1}, \dots, y_{t_k}) = p(y_{t_1+s}, \dots, y_{t_k+s}).$$

Covariance stationarity

A time series y_t is said to be covariance or weakly stationary if both the mean of y_t and the covariance between y_t and y_{t-s} are time invariant, where s is an arbitrary integer:

$$\mathbb{E}[y_t] = \mu,$$

$$\mathbb{Cov}[y_t, y_{t-s}] = \gamma_s,$$

where:

$$\gamma_0 = \mathbb{Var}[y_t]$$

$$\gamma_{-s} = \gamma_s$$

Unconditional moments

Expected value:

$$\mathbb{E}[y_t] = \int_{\mathbb{R}} y_t p(y_t) dy_t$$

Covariance matrix.

$$\mathbb{V}\text{ar}[y_t] = \mathbb{E} \left[(y_t - \mathbb{E}[y_t]) (y_t - \mathbb{E}[y_t])' \right]$$

Autocovariance.

$$\mathbb{C}\text{ov}[y_t, y_{t-s}] = \mathbb{E} \left[(y_t - \mathbb{E}[y_t]) (y_{t-s} - \mathbb{E}[y_{t-s}])' \right]$$

Sample moments

Sample mean.

$$\bar{y} = \frac{1}{T} \sum_{t=1}^T y_t$$

Sample covariance matrix.

$$\hat{\gamma}_0 = \frac{1}{T} \sum_{t=1}^T [(y_t - \bar{y})(y_t - \bar{y})']$$

Sample autocovariance.

$$\hat{\gamma}_s = \frac{1}{T-s} \sum_{t=s+1}^T [(y_t - \bar{y})(y_{t-s} - \bar{y})']$$

Autocorrelation

Autocorrelation function - ACF.

$$\rho_s = \text{Corr}[y_{n,t}, y_{n,t-s}] = \frac{\mathbb{E}[(y_{n,t} - \mathbb{E}[y_{n,t}]) (y_{n,t-s} - \mathbb{E}[y_{n,t-s}])]}{\sqrt{\text{Var}[y_{n,t}] \text{Var}[y_{n,t-s}]}}$$

Sample autocorrelation.

$$\hat{\rho}_s = \frac{\sum_{t=s+1}^T [(y_{n,t} - \bar{y}_n)(y_{n,t-s} - \bar{y}_n)]}{\sqrt{\left[\sum_{t=s+1}^T (y_{n,t} - \bar{y}_n)^2 \right] \left[\sum_{t=1}^{T-s} (y_{n,t-s} - \bar{y}_{n,s})^2 \right]}}$$

Assume covariance stationarity:

$$\rho_s = \rho_{-s} = \frac{\gamma_s}{\gamma_0}$$

White noise process

A stochastic process ϵ_t is called a white noise if $\{\epsilon_t\}$ is an i.i.d. sequence:

$$\mathbb{E}[\epsilon_t] = 0$$

$$\mathbb{V}\text{ar}[\epsilon_t] = \sigma_\epsilon^2 < \infty$$

$$\text{Cov}[\epsilon_t, \epsilon_{t-s}] = 0 \quad \forall s \neq 0$$

Denoted by

$$\epsilon_t \sim iid(0, \sigma_\epsilon^2)$$

Gaussian white noise process.

$$\epsilon_t \sim iid\mathcal{N}(0, \sigma_\epsilon^2)$$

Autoregressions

Autoregressions

AR(p) model.

$$y_t = \mu_0 + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + \epsilon_t$$
$$\epsilon_t | y_{t-1}, \dots, y_{t-p} \sim iid(0, \sigma_\epsilon^2)$$

Exogeneity assumption in time series.

$$\mathbb{E}[\epsilon_t | y_{t-1}, \dots, y_{t-p}] = 0$$

Autoregressions

AR(p) model: alternative notations.

$$y_t = \mu_0 + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + \epsilon_t$$

$$y_t - \alpha_1 y_{t-1} - \cdots - \alpha_p y_{t-p} = \mu_0 + \epsilon_t$$

$$(1 - \alpha_1 L - \cdots - \alpha_p L^p) y_t = \mu_0 + \epsilon_t$$

$$\alpha(L) y_t = \mu_0 + \epsilon_t$$

L denotes the lag operator such that $L^s y_t = y_{t-s}$ and $L^s c = c$

AR(p) model: matrix notation.

$$Y = X\beta + E$$

$$\underset{(p+1 \times 1)}{\beta} = \begin{bmatrix} \mu_0 \\ \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \quad \underset{(T \times 1)}{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \quad \underset{(p+1 \times 1)}{x'_t} = \begin{bmatrix} 1 \\ y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix} \quad \underset{(T \times p+1)}{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} \quad \underset{(T \times 1)}{E} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_T \end{bmatrix}$$

Autoregressions: stationarity

An $AR(p)$ model is called stationary if:

$$\alpha(z) = 0 \quad \text{for complex numbers } z \text{ with } |z| > 1,$$

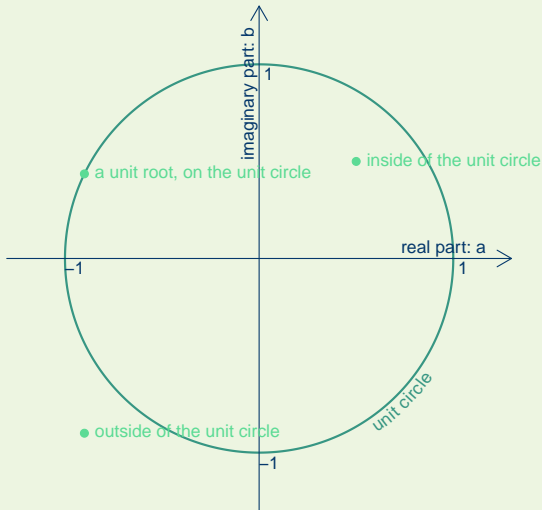
that is:

$$1 - \alpha_1 z - \cdots - \alpha_p z^p = 0 \quad \text{for } |z| > 1 \text{ and } z \in \mathbb{C}.$$

Modulus of a complex number $z = a + ib$ is:

$$|z| = \sqrt{a^2 + b^2}$$

Unit circle



The unit circle is given by equation: $1^2 = a^2 + b^2$

Autoregressions: invertibility

A stationary AR(p) model has an MA(∞) representation:

$$\alpha(L)y_t = \mu_0 + \epsilon_t$$

$$\begin{aligned}y_t &= \alpha(L)^{-1}\mu_0 + \alpha(L)^{-1}\epsilon_t \\&= \mu + \phi(L)\epsilon_t \\&= \mu + \epsilon_t + \phi_1\epsilon_{t-1} + \phi_2\epsilon_{t-2} + \dots\end{aligned}$$

where:

$$\phi_j = \sum_{i=1}^j \phi_{j-i}\alpha_i,$$

for $j = 0, 1, 2, \dots$, and $\phi_0 = 1$, and $\alpha_i = 0$ for $i > p$.

Autoregressions: invertibility

Consider a stationary AR(1) model:

$$y_t = \mu_0 + \alpha_1 y_{t-1} + \epsilon_t$$

$$y_t - \alpha_1 y_{t-1} = \mu_0 + \epsilon_t$$

$$(1 - \alpha_1 L)y_t = \mu_0 + \epsilon_t$$

It has an MA(∞) representation:

$$\begin{aligned} y_t &= \frac{\mu_0}{1 - \alpha_1} + \epsilon_t + \alpha_1 \epsilon_{t-1} + \alpha_1^2 \epsilon_{t-2} + \alpha_1^3 \epsilon_{t-3} + \dots \\ &= \mu + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_2 \epsilon_{t-2} + \dots \end{aligned}$$

Inverting polynomial of order 1:

$$(1 - az)^{-1} = 1 + \sum_{i=1}^{\infty} a^i z^i$$

Autoregressions: unconditional moments

Unconditional mean.

Assume stationarity.

$$\begin{aligned}\mu = \mathbb{E}[y_t] &= \mathbb{E}[\alpha(1)^{-1}\mu_0 + \alpha(L)^{-1}\epsilon_t] \\ &= \frac{\mu_0}{\alpha(1)} + \alpha(L)^{-1}\mathbb{E}[\epsilon_t] \\ &= \frac{\mu_0}{\alpha(1)} + \alpha(L)^{-1}\mathbb{E}[\mathbb{E}[\epsilon_t|y_{t-1}, \dots, y_{t-p}]]\end{aligned}$$

$$\mu = \frac{\mu_0}{1 - \alpha_1 - \dots - \alpha_p}$$

given that: $\alpha_1 + \dots + \alpha_p \neq 1$

The Law of Iterated Expectations.

$$\mathbb{E}[\epsilon_t] \stackrel{LIE}{=} \mathbb{E}[\mathbb{E}[\epsilon_t|y_{t-1}, \dots, y_{t-p}]] \stackrel{\text{exogeneity}}{=} \mathbb{E}[0] = 0$$

Autoregressions: autocorrelations

AR(1) model.

Step 1: Use $\mu_0 = \mu(1 - \alpha_1)$, to write the model as:

$$y_t - \mu = \alpha_1(y_{t-1} - \mu) + \epsilon_t$$

Step 2: Multiply by $y_{t-s} - \mu$:

$$(y_{t-s} - \mu)(y_t - \mu) = \alpha_1(y_{t-s} - \mu)(y_{t-1} - \mu) + (y_{t-s} - \mu)\epsilon_t$$

Step 3: Take the expectations:

$$\gamma_s = \alpha_1\gamma_{s-1} + \mathbb{E}[(y_{t-s} - \mu)\epsilon_t]$$

Step 4: Write equation above for $s = 0, 1, \dots$ and solve for $\gamma_0, \gamma_1, \dots$

Solution:

$$\gamma_0 = \sigma_\epsilon^2 / (1 - \alpha_1^2)$$

$$\rho_0 = 1$$

$$\gamma_s = \alpha_1\gamma_{s-1}$$

$$\rho_s = \alpha_1\rho_{s-1}$$

for $s > 0$

Autoregressions: autocorrelations

AR(2) model.

Step 1: Use $\mu_0 = \mu(1 - \alpha_1 - \alpha_2)$, to write the model as:

$$y_t - \mu = \alpha_1(y_{t-1} - \mu) + \alpha_2(y_{t-2} - \mu) + \epsilon_t$$

Step 2: Multiply by $y_{t-s} - \mu$:

$$\begin{aligned} (y_{t-s} - \mu)(y_t - \mu) &= \\ \alpha_1(y_{t-s} - \mu)(y_{t-1} - \mu) &+ \alpha_2(y_{t-s} - \mu)(y_{t-2} - \mu) + (y_{t-s} - \mu)\epsilon_t \end{aligned}$$

Step 3: Take the expectations:

$$\gamma_s = \alpha_1\gamma_{s-1} + \alpha_2\gamma_{s-2} + \mathbb{E}[(y_{t-s} - \mu)\epsilon_t]$$

Step 4: Write the equation above for $s = 0, 1, \dots$ and solve for autocovariances.

Autoregressions: autocorrelations

Step 4: Solve the set of equations for γ_0 , γ_1 and γ_2 .

$$\gamma_0 = \alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \sigma_u^2$$

$$\gamma_1 = \alpha_1 \gamma_0 + \alpha_2 \gamma_1$$

$$\gamma_2 = \alpha_1 \gamma_1 + \alpha_2 \gamma_0$$

Solution:

$$\gamma_0 = \frac{1 - \alpha_2}{(1 - \alpha_2^2)(1 - \alpha_2) - \alpha_1^2(1 + \alpha_2)} \sigma_\epsilon^2 \quad \rho_0 = 1$$

$$\gamma_1 = \frac{\alpha_1}{(1 - \alpha_2^2)(1 - \alpha_2) - \alpha_1^2(1 + \alpha_2)} \sigma_\epsilon^2 \quad \rho_1 = \alpha_1 / (1 - \alpha_2)$$

$$\gamma_2 = \frac{\alpha_1^2 + \alpha_2(1 - \alpha_2)}{(1 - \alpha_2^2)(1 - \alpha_2) - \alpha_1^2(1 + \alpha_2)} \sigma_\epsilon^2 \quad \rho_2 = \frac{\alpha_1^2 - \alpha_2^2 + \alpha_2}{1 - \alpha_2}$$

$$\gamma_s = \alpha_1 \gamma_{s-1} + \alpha_2 \gamma_{s-2} \quad \rho_s = \alpha_1 \rho_{s-1} + \alpha_2 \rho_{s-2}$$

for $s \geq 1$

Autoregressions: autocorrelations

Shiny interactive apps.

shiny is an R package to build interactive apps in R. They are html pages with the computational engine of R behind it.

Shiny apps for AR-model implied autocorrelations.

Install package shiny

Open in RStudio files shiny-ar1.R and shiny-ar2.R

Run the app by pressing **Run App** button.

Discover the patterns in autocorrelations that AR(1) and AR(2) models can capture.

Random walk processes

Random walk process

A time series $\{y_t\}$ is called a random walk if:

$$y_t = y_{t-1} + \epsilon_t$$

$$\epsilon_t \sim iid(0, \sigma_\epsilon^2)$$

Different representation:

$$y_t = y_{t-1} + \epsilon_t$$

$$= \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \dots$$

$$= \sum_{i=0}^{\infty} \epsilon_{t-i}$$

Random walk process

Random walk process with initial value.

Let y_0 be a real number denoting the starting value of the process, then the random walk process can be written as:

$$y_t = y_0 + \sum_{i=0}^{t-1} \epsilon_{t-i}$$

Random walk process

Consider a random walk process with initial value:

$$y_t = y_0 + \sum_{i=0}^{t-1} \epsilon_{t-i}$$

Moments.

$$\mathbb{E}[y_t] = \mathbb{E}[y_0 + \sum_{i=0}^{t-1} \epsilon_{t-i}] = y_0$$

$$\mathbb{V}\text{ar}[y_t] = \sum_{i=0}^{t-1} \mathbb{E}[\epsilon_{t-i}^2] = t\sigma_\epsilon^2$$

$$\rho_s = \frac{\gamma_s}{\sqrt{\gamma_0} \sqrt{\mathbb{V}\text{ar}[y_{t-s}]}} = \frac{t-s}{\sqrt{t} \sqrt{t-s}} = \sqrt{\frac{t-s}{t}}$$

Random walk process

Consider a random walk process:

$$y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \dots$$

Moments.

$$\mathbb{E}[y_t] = \mathbb{E}[\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \dots] = 0$$

$$\text{Var}[y_t] = \mathbb{E}[\epsilon_t^2] + \mathbb{E}[\epsilon_{t-1}^2] + \dots = \lim_{t \rightarrow \infty} t\sigma_\epsilon^2 = \infty$$

$$\rho_s = \lim_{t \rightarrow \infty} \sqrt{\frac{t-s}{t}} = 1 \quad \forall s = 0, 1, \dots$$

Random walk process

$$y_t = y_{t-1} + \epsilon_t$$

Forecasting.

$$y_{T+h|T} = \mathbb{E}[y_{T+h}|y_T, y_{T-1}, \dots] = y_T$$

$$\begin{aligned}\sigma_{T+h|T}^2 &= \mathbb{E}[(y_{T+h} - y_{T+h|T})^2 | \mathbf{y}_T] \\ &= \mathbb{E}[(\epsilon_{T+1} + \dots + \epsilon_{T+h})^2 | \mathbf{y}_T] \\ &= h\sigma_\epsilon^2\end{aligned}$$

Random walk with drift

A random walk with drift process.

$$y_t = \mu_0 + y_{t-1} + \epsilon_t = y_0 + t\mu_0 + \sum_{i=1}^t \epsilon_i$$

$$\epsilon_t \sim iid(0, \sigma_\epsilon^2)$$

$$\mu_0 = \mathbb{E}[y_t - y_{t-1}] \quad - \text{a time trend}$$

Moments.

$$\begin{aligned}\mathbb{E}[y_t] &= \mathbb{E}[y_0 + t\mu_0 + \epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1] \\ &= y_0 + t\mu_0\end{aligned}$$

$$\mathbb{V}\text{ar}[y_t] = \mathbb{E}[(\epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1)^2] = t\sigma_\epsilon^2$$

Random walk with drift

$$y_t = \mu_0 + y_{t-1} + \epsilon_t = y_0 + t\mu_0 + \sum_{i=1}^t \epsilon_i,$$

Forecasting.

$$\begin{aligned} y_{T+h|T} = \mathbb{E}[y_{T+h}|\mathbf{y}_T] &= \mathbb{E}[\epsilon_{T+h} + \epsilon_{T+h-1} + \cdots + \epsilon_{T+1} + h\mu_0 + y_T|\mathbf{y}_T] \\ &= h\mu_0 + y_T \end{aligned}$$

$$\begin{aligned} \sigma_{T+h|T}^2 &= \mathbb{E}[(y_{T+h} - y_{T+h|T})^2|\mathbf{y}_T] \\ &= h\sigma_\epsilon^2 \end{aligned}$$

Trend-stationary series vs. random walk with drift

$$y_t = \beta_0 + \beta_1 t + u_t$$

$$u_t \sim iid(0, \sigma_u^2)$$

$$E[y_t] = \beta_0 + \beta_1 t$$

$$Var[y_t] = \sigma_u^2$$

$$\rho_I = 0$$

$$y_t = y_0 + \mu_0 t + u_t + \dots + u_1$$

$$u_t \sim iid(0, \sigma_u^2)$$

$$E[y_t] = y_0 + \mu_0 t$$

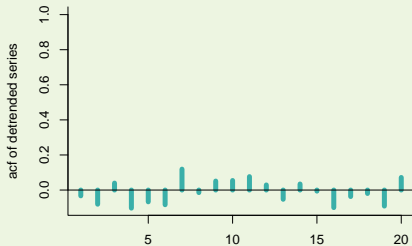
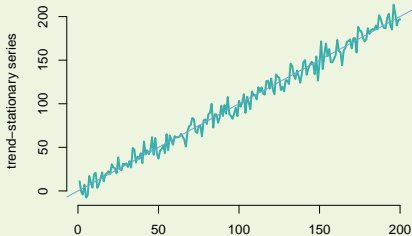
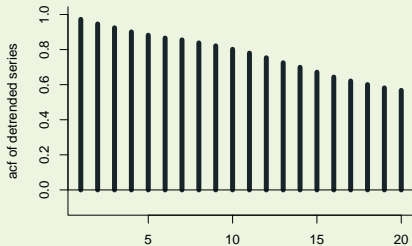
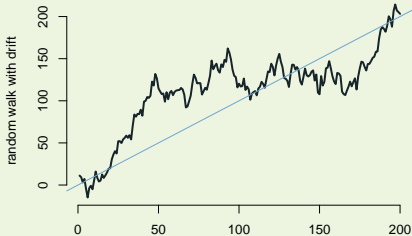
$$Var[y_t] = t\sigma_u^2$$

$$\rho_I = \sqrt{\frac{t-1}{t}}$$

Trend-stationary series is characterised by random deviations from a time trend. It is unit-root stationary

Random walk with drift accumulates shocks over time around a time trend. It is unit-root non-stationary

Trend-stationary series vs. random walk with drift



Both series are generated using the same simulated shocks.

Inference

Random walk process

Data generating process: $y_t = y_{t-1} + \epsilon_t$

Estimated model.

$$y_t = \alpha_1 y_{t-1} + \epsilon_t$$
$$\hat{\alpha}_1 = \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2}$$

Asymptotic distribution.

$$T(\hat{\alpha}_1 - 1) \xrightarrow{d} \frac{B(1)^2 - 1}{2 \int_0^1 B(s)^2 ds}$$

T – convergence rate implying superconsistency

$B(s)$ – Brownian motion

$B(1)^2 - 1 = \chi_1^2$ – a χ^2 distribution

$2 \int_0^1 B(s)^2 ds$ – non-normal distribution

Random walk process

Data generating process: $y_t = y_{t-1} + \epsilon_t$

Estimated model.

$$y_t = \alpha_1 y_{t-1} + \epsilon_t$$
$$\epsilon_t | y_{t-1} \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

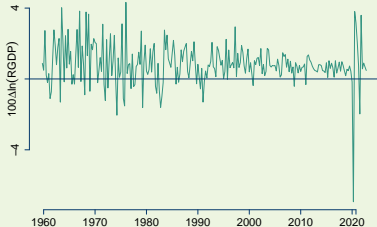
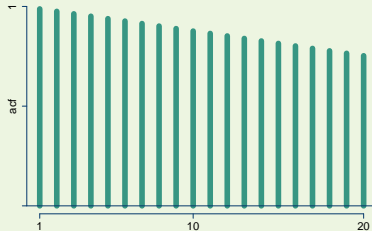
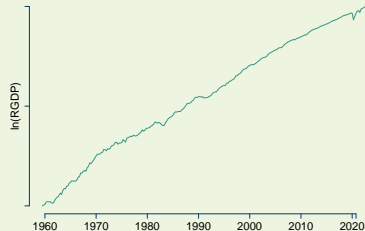
$$\hat{\alpha}_1 = \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2}$$

Likelihood function.

$$\alpha_1 \sim \mathcal{N}(\hat{\alpha}_1, \sigma_\epsilon^2 (X'X)^{-1})$$

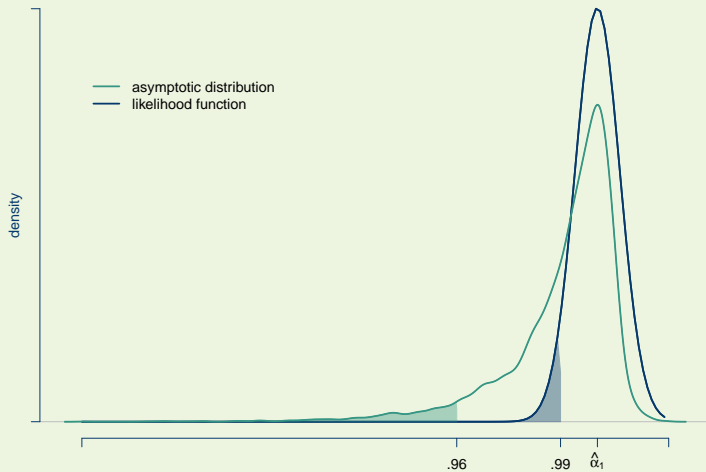
Illustration for Australian real GDP

Australian real GDP



Based on real GDP data for period Q3 1959 – Q4 2022
Data source: Australian Macro Database www.ausmacrodata.org

Australian real GDP



A helicopter tour

A helicopter tour: simulation setup

Data generating process.

$$y_t = \rho y_{t-1} + \epsilon_t \quad \text{and} \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2 = 1)$$

$\rho \in \{.8, .81, .82, \dots, 1.09, 1.1\}$ – a grid of 31 values for ρ

$$y_0 = 0$$

Matrix notation.

$$\mathbf{R}y = E \quad \text{and} \quad E \sim \mathcal{N}(\mathbf{0}_T, I_T)$$

$y = \mathbf{R}^{-1}E$ – generate data from AR(1) *DGP*

$$\underset{(T \times T)}{\mathbf{R}} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -\rho & 1 & 0 & \dots & 0 & 0 \\ 0 & -\rho & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\rho & 1 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{T-1} \\ y_T \end{bmatrix}$$

A helicopter tour: simulation setup

Estimated parameter.

$$\hat{\rho}_{ML} = (X'X)^{-1}X'Y$$

$$Y' = \begin{bmatrix} y_2 & y_3 & \dots & y_T \end{bmatrix}$$

$$X' = \begin{bmatrix} y_1 & y_2 & \dots & y_{T-1} \end{bmatrix}$$

Generate the graph.

Generate $S = 50,000$ times $T = 100$ random numbers from $\mathcal{N}(0, 1)$

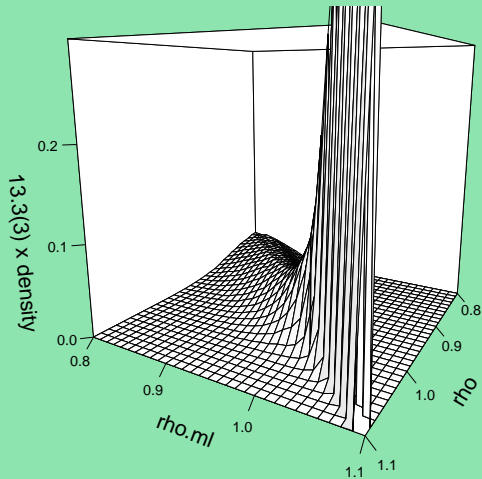
Create For 50,000 $T \times 1$ normal vectors and for 31 values of ρ from the grid compute y according to the DGP

Estimate $\hat{\rho}_{ML}$ for each of the generated y vector

Compute histograms of $\hat{\rho}_{ML}$ for each of grid values ρ using bins:
 $(-\infty, .795], (.795, .805], (.805, .815], \dots, (1.095, 1.105], , (1.105, \infty]$

Plot the joint distribution of $\hat{\rho}_{ML}$ and ρ

A helicopter tour



A helicopter tour: conditional distributions

$$p(\hat{\rho}_{ML}|\rho = 1)$$

Is a simulated distribution of MLE of ρ given that the data is generated using a fixed value of $\rho = 1$

Is equivalent to the asymptotic distribution of $\hat{\rho}_{ML}$ with $T \rightarrow \infty$

Is used to compute the critical values for unit root tests

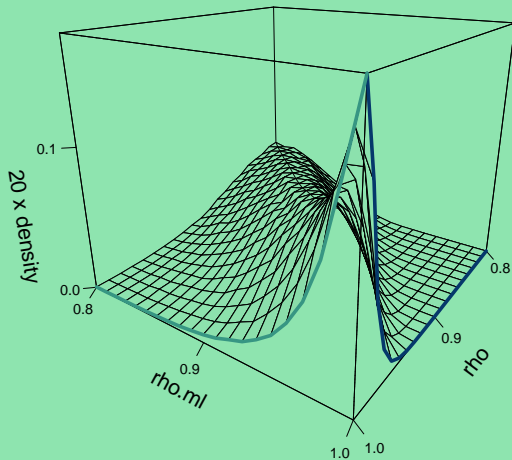
$$p(\rho|\hat{\rho}_{ML} = 1)$$

Is equivalent to the posterior distribution of ρ given that data y is such that $\hat{\rho}_{ML} = 1$ and σ_ϵ^2 is fixed to 1

Prior distribution in this case is set to an improper distribution $p(\rho) \propto 1$, e.g., $p(\rho) = 1$

Posterior distribution is then proportional to
$$\exp\left\{-\frac{1}{2}(\rho - \hat{\rho}_{ML})' X'X(\rho - \hat{\rho}_{ML})\right\}$$

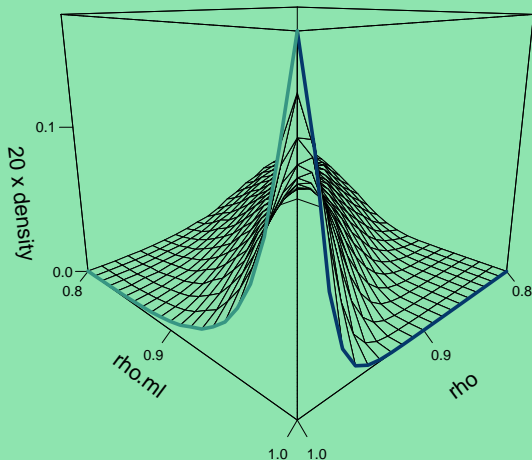
A helicopter tour: joint distribution



$$p(\hat{\rho}_{ML}|\rho = 1)$$

$$p(\rho|\hat{\rho}_{ML} = 1)$$

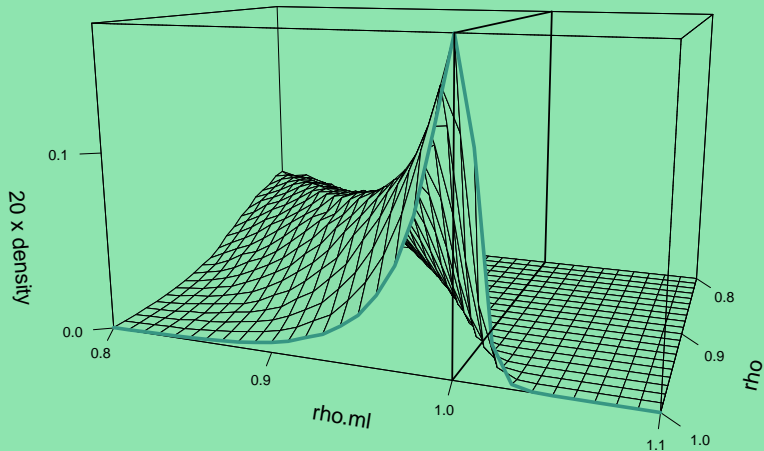
A helicopter tour: asymmetry



$$p(\hat{\rho}_{ML}|\rho = 1)$$

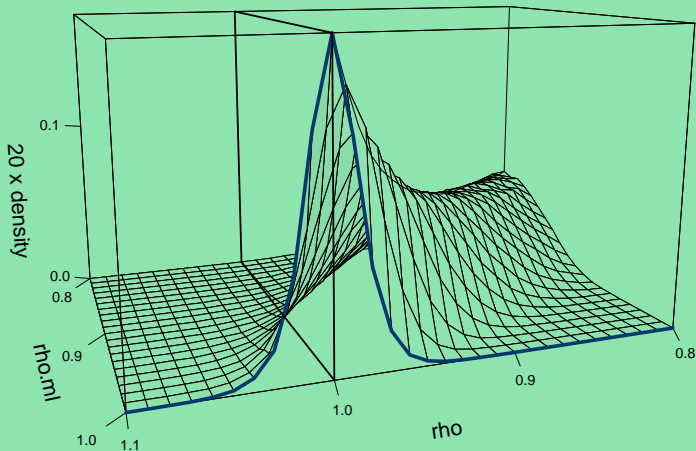
$$p(\rho|\hat{\rho}_{ML} = 1)$$

A helicopter tour: asymptotic distribution



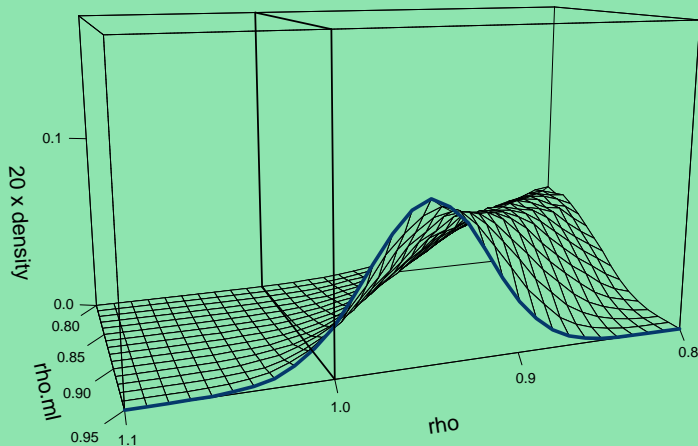
$$p(\hat{\rho}_{ML} | \rho = 1)$$

A helicopter tour: posterior distribution



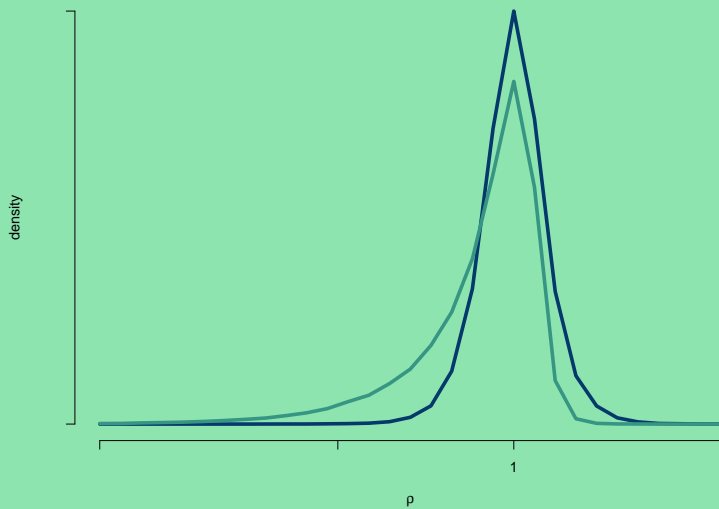
$$p(\rho | \hat{\rho}_{ML} = 1)$$

A helicopter tour: posterior distribution



$$p(\rho | \hat{\rho}_{ML} = .95)$$

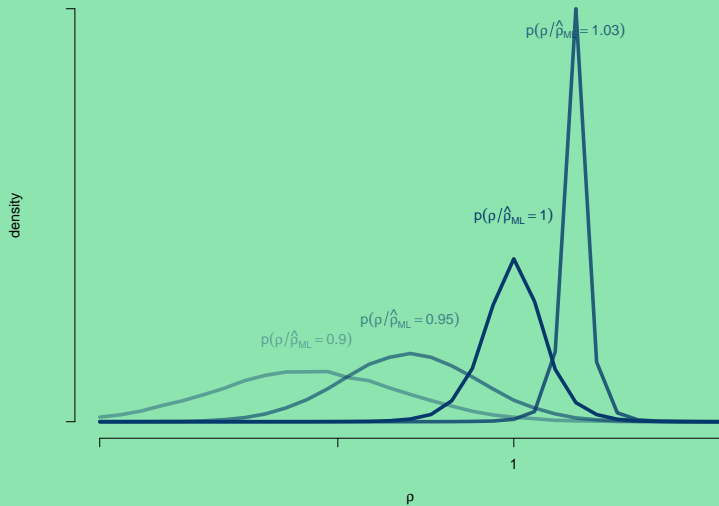
A helicopter tour



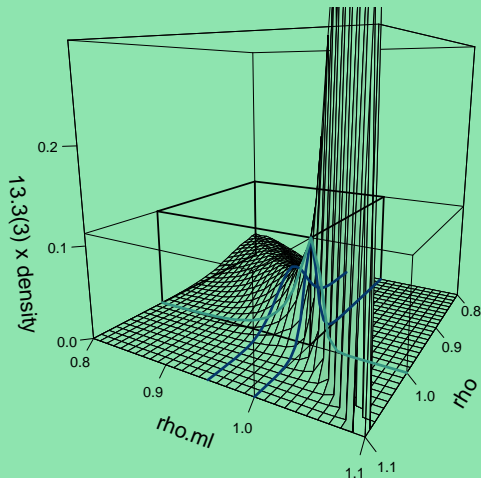
$$p(\hat{\rho}_{ML} | \rho = 1)$$

$$p(\rho | \hat{\rho}_{ML} = 1)$$

A helicopter tour



A helicopter tour: behind the scenes



$$p(\hat{\rho}_{ML} | \rho = 1)$$

$$p(\rho | \hat{\rho}_{ML} = 1)$$

$$p(\rho | \hat{\rho}_{ML} = .95)$$

Random walk process

Autoregressions provide convenient parameterization for modeling decaying patterns in autocorrelations of time series

Random walk process imply long-memory and unit autocorrelations at all lags

Note: seemingly conflicting evidence from the asymptotic distribution and the likelihood function

The posterior distribution is normal

Unit-root inference seems to be the only case when Bayesian and frequentist approaches do not converge asymptotically