Problem 57: Square root convergents.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

First four iterations:

$$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1 + \frac{1}{\frac{5}{2}} = 1 + \frac{2}{5}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 1 + \cfrac{1}{2 + \cfrac{2}{5}} = 1 + \frac{5}{12}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = 1 + \frac{1}{2 + \frac{5}{12}} = 1 + \frac{12}{29}$$

Note that each iteration is of the form $1 + \frac{1}{2 + (\text{previous iteration} - 1)}$.

Problem 64: Odd period square roots. Every square root can be written as

$$\sqrt{n} = \text{floor}(\sqrt{n}) + (\sqrt{n} - \text{floor}(\sqrt{n})).$$

Since $(\sqrt{n} - \text{floor}(\sqrt{n})) < 1$, we can repeat this on $1/(\sqrt{n} - \text{floor}(\sqrt{n}))$ to build a continued fraction. A cycle is reached once a repeated fraction is equal to $1/(\sqrt{n} - \text{floor}(\sqrt{n}))$. This is the simplest solution, but can run into precision errors. Expanding explicitly,

$$\sqrt{n} = a + (\sqrt{n} - a) = a + \cfrac{1}{\cfrac{1}{\sqrt{n} - a} \times \cfrac{\sqrt{n} + a}{\sqrt{n} + a}} = a + \cfrac{1}{\cfrac{\sqrt{n} + a}{n - a^2}} = a + \cfrac{1}{b + \left(\cfrac{\sqrt{n} + a}{n - a^2} - b \right)}$$

where $a = \text{floor}(\sqrt{n})$ and $b = \text{floor}\left(\frac{\sqrt{n} + a}{n - a^2} \right)$. Continuing, the term in parentheses becomes the fraction

$$\frac{\sqrt{n} + a - b(n - a^2)}{n - a^2} = \cfrac{1}{\cfrac{\text{int1}}{\sqrt{n} - \text{int2}} \times \cfrac{\sqrt{n} + \text{int2}}{\sqrt{n} + \text{int2}}} = \cfrac{1}{c + \left(\cfrac{\text{float}}{\text{int3}} - c \right)}.$$

This general structure works, but float and int3 quickly become very large, even though float/int3 < 1. A similar but better solution is as follows. Let a_i denote the floors and l_i, m_i be integers. Then

$$\sqrt{n} = a_0 + (\sqrt{n} - a_0) = a_0 + \cfrac{1}{a_1 + \cfrac{\sqrt{n} + a_0 - a_1(n - a_0^2)}{n - a_0^2}} = a_0 + \cfrac{1}{a_1 + \cfrac{\sqrt{n} + l_1}{m_1}}$$

where $m_1 = n - a_0^2$, $a_1 = \text{floor}((\sqrt{n}+a_0)/m_1)$, and $l_1 = a_0 - a_1(n - a_0^2)$. Iterating once more,

$$a_1 + \frac{\sqrt{n}+l_1}{m_1} = a_1 + \cfrac{1}{a_2 + \cfrac{m_1(\sqrt{n}-l_1) - a_2(n-l_1^2)}{n-l_1^2}} = \cfrac{1}{a_2 + \cfrac{\sqrt{n}+l_2}{m_2}}$$

where $m_2 = (n-l_1^2)/m_1$, $a_2 = \text{floor}((\sqrt{n}-l_1)/m_2)$, and $l_2 = -l_1 - a_2(n-l_1^2)/m_1$. It's straightforward to deduce the general recurrence relation for $i = 0, 1, 2, \ldots$,

$$\begin{aligned} m_{i+1} &= (n - l_i^2)/m_i \\ a_{i+1} &= \text{floor}((\sqrt{n} - l_i)/m_{i+1}) \\ l_{i+1} &= -l_i - a_{i+1}(n - l_i^2)/m_i \end{aligned}$$

with $m_0 = 1, a_0 = \text{floor}(\sqrt{n}), l_0 = -a_0$. A cycle is reached once $m_i, a_i, l_i = m_1, a_1, l_1$, $i > 1$.

Problem 65: Convergents of *e*. We can't use the same algorithm as Problem 64 because l_i, m_i are no longer guaranteed to be integers. Instead we have the following. Let $x \in \mathbb{R}$. Then

$$x = a_0 + (x - a_0) = a_0 + \cfrac{1}{\cfrac{1}{x-a_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{1 - a_1(x-a_0)}{x-a_0}}.$$

Iterating twice more,

$$a_1 + \frac{1-a_1(x-a_0)}{x-a_0} = a_1 + \cfrac{1}{\cfrac{x-a_0}{1-a_1(x-a_0)}} = a_1 + \cfrac{1}{a_2 + \cfrac{x-a_0-a_2[1-a_1(x-a_0)]}{1-a_1(x-a_0)}}$$

$$\begin{aligned} a_2 + \frac{x-a_0-a_2[1-a_1(x-a_0)]}{1-a_1(x-a_0)} &= a_2 + \cfrac{1}{\cfrac{1-a_1(x-a_0)}{x-a_0-a_2[1-a_1(x-a_0)]}} \\ &= a_2 + \cfrac{1}{a_3 \cfrac{1-a_1(x-a_0)-a_3\{x-a_0-a_2[1-a_1(x-a_0)]\}}{x-a_0-a_2[1-a_1(x-a_0)]}}, \end{aligned}$$

we can begin to see a pattern. Let k_i, l_i, n_i, m_i be integers at each iteration, $i = 0, 1, 2, \ldots$, with $a_0 = \text{floor}(x)$, $k_0 = 1$, $l_0 = -a_0$, $m_0 = 0$, $n_0 = 1$, i.e.

$$x = a_0 + (x - a_0) = a_0 + \frac{x-a_0}{1} = a_0 + \frac{k_0 x + l_0}{m_0 x + n_0}.$$

Iterating,

$$\begin{aligned} a_0 + \frac{k_0x+l_0}{m_0x+n_0} &= a_0 + \cfrac{1}{\cfrac{m_0x+n_0}{k_0x+l_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{m_0x+n_0-a_1(k_0x+l_0)}{k_0x+l_0}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{(m_0-a_1k_0)x+(n_0-a_1l_0)}{k_0x+l_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{k_1x+l_1}{m_1x+n_1}} \end{aligned}$$

with $a_1 = \text{floor}[(m_0 x + n_0)/(k_0 x + l_0)]$, $k_1 = m_0 - a_1 k_0$, $l_1 = n_0 - a_1 l_0$, $m_1 = k_0$, and $n_1 = l_0$. The general recurrence relation is

$$
\begin{aligned}
a_{i+1} &= \text{floor}[(m_i x + n_i)/(k_i x + l_i)] \\
k_{i+1} &= k_1 = m_i - a_{i+1} k_i \\
l_{i+1} &= n_i - a_{i+1} l_i \\
m_{i+1} &= k_i \\
n_{i+1} &= l_i
\end{aligned}
$$

$i = 0, 1, 2, \ldots$, with base $a_0 = \text{floor}(x)$, $k_0 = 1$, $l_0 = -a_0$, $m_0 = 0$, $n_0 = 1$. Unfortunately this runs into numerical issues because both the numerator and denominator are floats. Instead, I brute force the calclulation using the given information $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots]$.

Problem 94: Almost equilateral triangles. Let the triangle base have length $m_\pm = n \pm 1$, with remaining sides of length n. The area

$$A = \frac{m}{2}\sqrt{n^2 - \left(\frac{m}{2}\right)^2}$$

can only be integer if the height $h = \sqrt{n^2 - (\frac{m}{2})^2}$ is rational. For bases $m_\pm$,

$$h_\pm^2 = n^2 - \left(\frac{m_\pm}{2}\right)^2 = \frac{1}{4}(3n^2 \mp 2n - 1)$$

Rather than looping through all $n = 2, 3, 4, \ldots$, and checking if each $h_\pm$ is rational, we can convert the above to a quadratic Diophantine equation (e.g. Problem 66), obtain the fundamental integer solutions, and recursively compute other solutions; the resultant search space of possible (n, h) pairs will be drastically reduced. After re-arranging and completing the square, we can express

$$\left(\frac{3n \mp 1}{2}\right)^2 - 3h^2 = 1,$$

which is Pell's equation with $x_\pm = \frac{3n \mp 1}{2}$, $y = h$, and $D = 3$. The fundamental solution (x_1, y_1) can be found via continued fractions (see `p66.py`), which can then recursively generate additional solutions via

$$
\begin{aligned}
x_{k+1} &= x_1 x_k + D y_1 y_k \\
y_{k+1} &= x_1 y_k + y_1 x_k.
\end{aligned}
$$

Converting back to (n, h) is simple:

$$n = \frac{1}{3}\left(2x_\pm \pm 1\right), \quad h = y.$$

The solution search terminates once $n > \frac{p_{\max}}{3} + 1$, or

$$x_\pm > \frac{1}{2}\left(p_{\max} + 3 \mp 1\right).$$

Thus after obtaining all (x_i, y_i), we convert to (n, h) and test if n and the triangle area A are integer. (Note that it is possible that n is not an integer; it is x and y that are guaranteed to be integer.)

Problem 100: Arranged probability. Let the total number of discs be $N = n_B + n_R$. We seek integers n_B, N such that

$$P(BB) = \left(\frac{n_B}{N}\right)\left(\frac{n_B - 1}{N - 1}\right) = \frac{1}{2}.$$

After some algebra, this can be expressed as the negative Pell equation

$$x^2 - Dy^2 = -1 \qquad \text{where} \quad D = 2, \quad x = 2N - 1, \quad \text{and} \quad y = 2n_B - 1.$$

Since the continued fraction of $\sqrt{D} = \sqrt{2}$ has odd period (equal to 1), the fundamental solution (x_1, y_1) can be obtained just like the positive Pell equation via continued fraction convergents (see `p66.py`). Other solutions are obtained recursively from

$$\begin{aligned} x_{k+1} &= 3x_k + 4y_k \\ y_{k+1} &= 2x_k + 3y_k \end{aligned}$$

up until $N > N_{\max}$ or $(x+1)/2 > N_{\max}$.