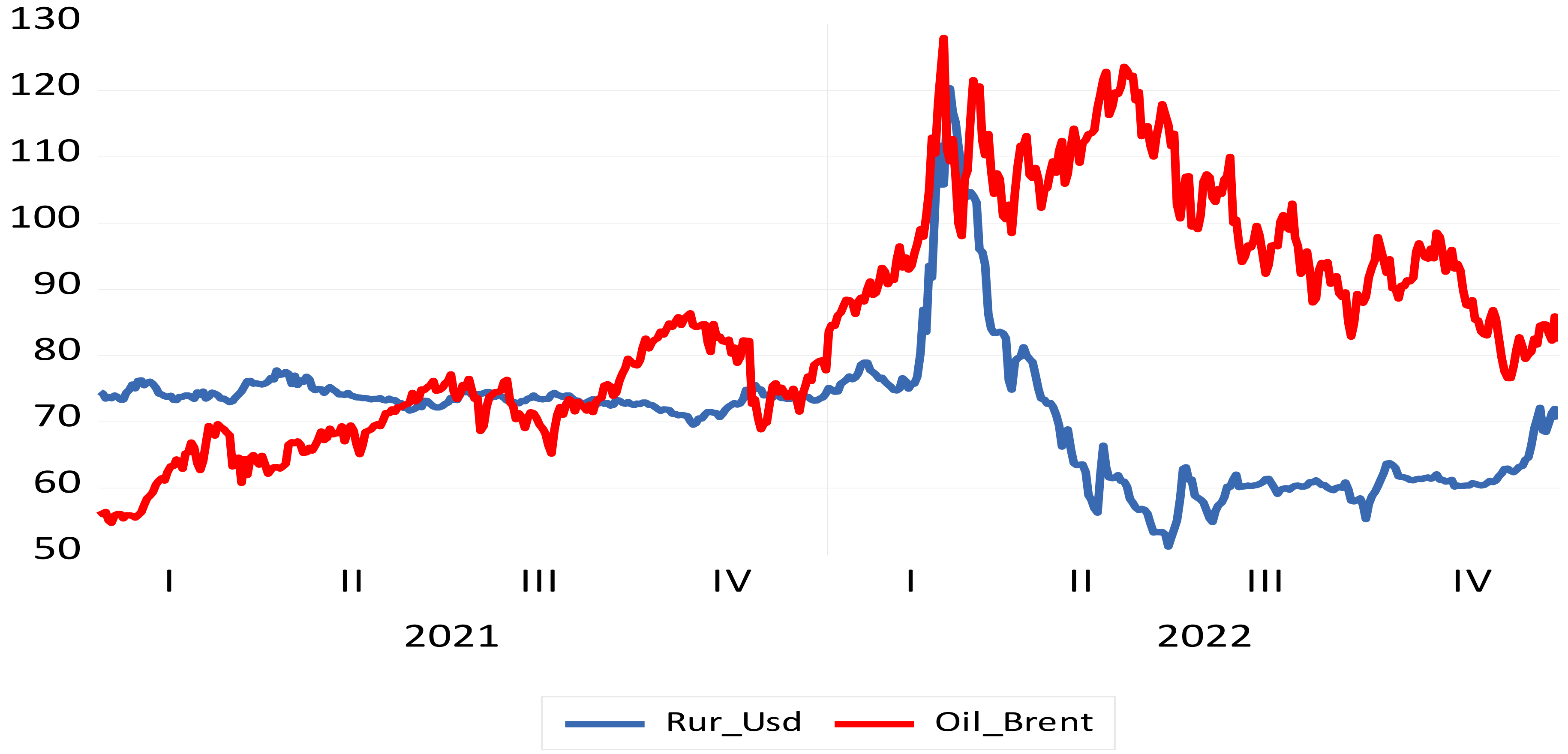
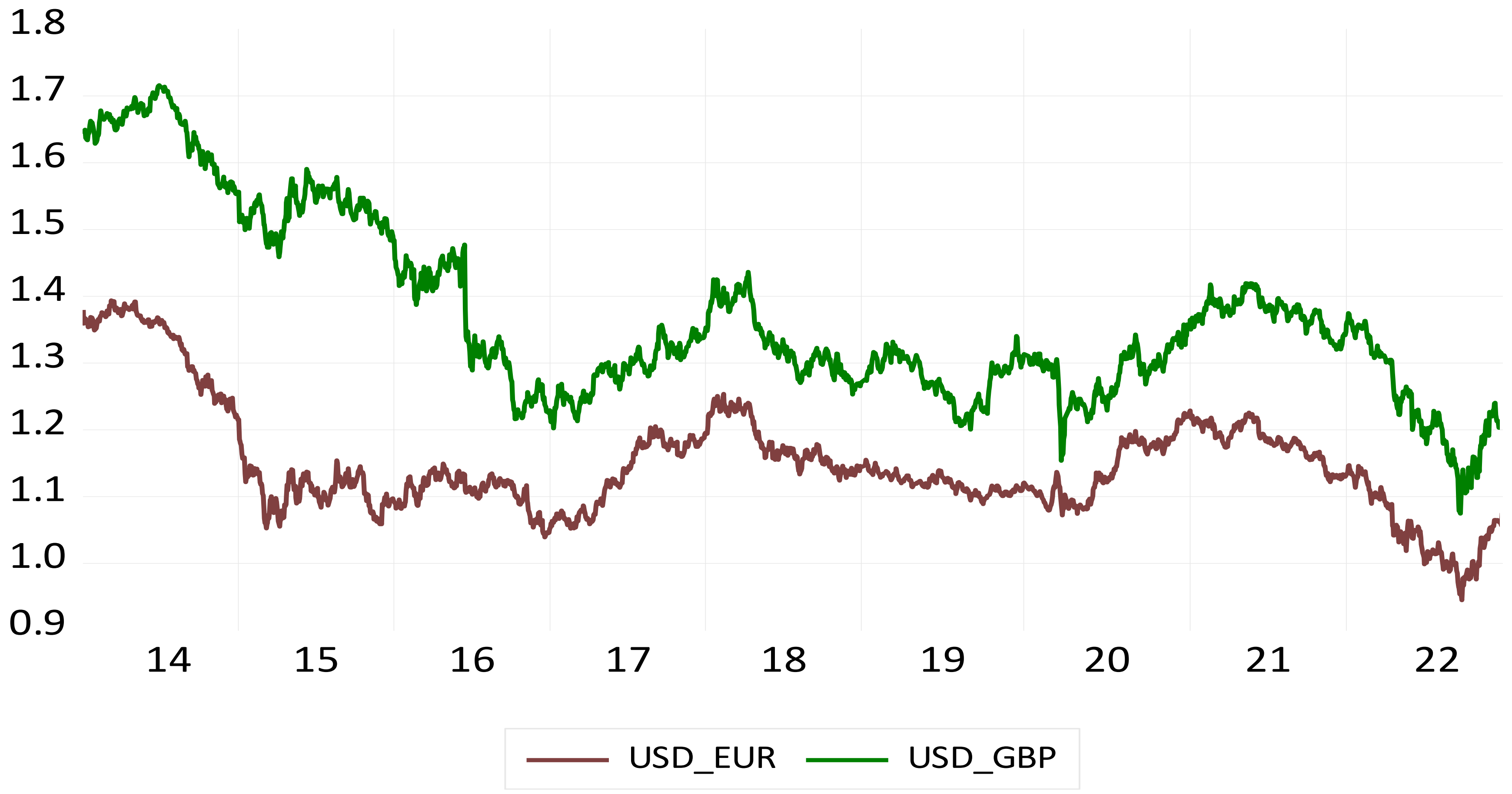
Elements of Econometrics.
Lecture 16.
Modeling with Time Series Data.
Part 1.

FCS, 2022-2023

Time Series Data Example: RuR/USD Exchange Rate and Oil Price (Brent), 01.2021-12.2022, daily



Time Series Data Example: USD/EUR and USD/GBP Exchange Rates, 01.2014-12.2022, daily



Time Series Data Example: Price of Bitcoin in USD, 01.2017-01.2023, daily

Time Series Data Specifics Essential for Regression Models

- Assumption B.2 is irrelevant (the observations do not look as being taken randomly from fixed populations): B.2 to be replaced by another assumption
- There may be regularities in the time series and in their relationships: trends, autocorrelations in variables and disturbance terms, lags (fixed or distributed); to be identified and dealt with
- Some regularities in the data (nonstationarity) may lead to estimation of spurious regressions: the data/model has to be transformed to provide desirable estimators' properties

ASSUMPTIONS FOR MODEL C: REGRESSIONS WITH TIME SERIES DATA

ASSUMPTIONS FOR MODEL C

C.1 The model is linear in parameters and correctly specified

$$Y = \beta_1 + \beta_2 X_2 + \ldots + \beta_k X_k + u$$

C.2 The time series for the regressors are (at most) weakly persistent

C.3 There does not exist an exact linear relationship among the regressors

C.4 The disturbance term has zero expectation

C.5 The disturbance term is homoscedastic

C.6 The values of the disturbance term have independent distributions:
u_t is distributed independently of $u_{t'}$ for $t' \neq t$

C.7 The disturbance term is distributed independently of the regressors:
u_t is distributed independently of $X_{jt'}$ for all t' (including t) and j

C.8 The disturbance term has a normal distribution

Assumptions C.1, C.3, C.4, C.5, C.6, C.7 and C.8, and the consequences of their violations are the same as those for Model B. Weak persistency (or weak non-stationarity), C.2: see the next slide.

Stationary Stochastic Processes

Stationarity (strong stationarity) of a stochastic process X_t is observed if the joint distribution of $X_{t1}, X_{t2}, \ldots, X_{tm}$ is identical to the joint distribution of $X_{t1+t}, X_{t2+t}, \ldots, X_{tm+t}$ for any $m, t, t_1, \ldots, t_m$.

A stochastic process is weakly stationary (or covariance stationary) if $E(X_t)$ is constant, $Var(X_t)$ is constant, and for any $t, s \geq 1$, $Cov(X_t, X_{t+s})$ depends only on s and not on t

If for a weakly stationary process $Cov(X_t, X_{t+s}) \rightarrow 0$ as $s \rightarrow \infty$, the process is called weakly dependent (or weakly persistent)

ASSUMPTIONS FOR MODEL C

C.7 The disturbance term is distributed independently of the regressors

u_t is distributed independently of $X_{jt'}$ for all t' (including t) and j

(1) The disturbance term in any observation is distributed independently of the values of the regressors in the same observation, and

(2) The disturbance term in any observation is distributed independently of the values of the regressors in the other observations.

Assumption C.7, like its counterpart Assumption B.7, is essential for both the unbiasedness and the consistency of OLS estimators. Both parts are required for unbiasedness. However only the first part is required for consistency (as a necessary, but not sufficient, condition).

UNBIASEDNESS

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (X_i - \bar{X}) u_i}{\sum (X_i - \bar{X})^2}$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \sum a_i u_i \qquad a_i = \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$$

$$\mathrm{E}\left(b_2^{\mathrm{OLS}}\right) = \beta_2 + \sum \mathrm{E}(a_i u_i)$$

$$= \beta_2 + \sum \mathrm{E}(a_i)\,\mathrm{E}(u_i)$$

$$= \beta_2 + \sum \mathrm{E}(a_i) \times 0 = \beta_2$$

It is required that u_i distributed independently of a_i. a_i is a function of *all* of the X values in the sample, not just X_i. So Part (1) of Assumption C.7, that u_i is distributed independently of X_i, is not enough.

ASSUMPTION C.7

CROSS-SECTIONAL DATA:
We assume u_j and X_i are independent ($i \neq j$).
The main issue is whether u_i is independent of X_i.

TIME SERIES DATA:

$$If\ \ Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t$$

$$Y_{t+1} = \beta_1 + \beta_2 Y_t + u_{t+1}$$

The disturbance term u_t is automatically correlated with the explanatory variable Y_t in the next observation.

$$b_2^{\mathrm{OLS}} = \beta_2 + \frac{\sum (X_i - \bar{X}) u_i}{\sum (X_i - \bar{X})^2}$$

$$b_2^{\mathrm{OLS}} = \beta_2 + \sum a_i u_i$$

$$a_i = \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$$

$$E\left(b_2^{\mathrm{OLS}}\right) = \beta_2 + \sum E(a_i u_i)$$

$$= \beta_2 + \sum E(a_i) E(u_i)$$

$$= \beta_2 + \sum E(a_i) \times 0 = \beta_2$$

Let Y_{t-1} be a regressor for Y_t. As a consequence u_i is not independent of a_i and so we cannot write $E(a_i u_i) = E(a_i)E(u_i)$. It follows that the OLS slope coefficient will in general be biased.

ASSUMPTION C.7: CONSISTENCY

$$\boldsymbol{Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t}$$

$$b_2^{\mathrm{OLS}} = \frac{\sum (Y_{t-1} - \bar{Y}_{t-1})(Y_t - \bar{Y})}{\sum (Y_{t-1} - \bar{Y}_{t-1})^2} = \beta_2 + \frac{\sum (Y_{t-1} - \bar{Y}_{t-1})(u_t - \bar{u})}{\sum (Y_{t-1} - \bar{Y}_{t-1})^2}$$

$$\operatorname{plim}\left(\frac{\sum (Y_{t-1} - \bar{Y}_{t-1})(u_t - \bar{u})}{\sum (Y_{t-1} - \bar{Y}_{t-1})^2}\right) = \operatorname{plim}\left(\frac{\frac{1}{n}\sum (Y_{t-1} - \bar{Y}_{t-1})(u_t - \bar{u})}{\frac{1}{n}\sum (Y_{t-1} - \bar{Y}_{t-1})^2}\right)$$

$$= \frac{\operatorname{plim}\frac{1}{n}\sum (Y_{t-1} - \bar{Y}_{t-1})(u_t - \bar{u})}{\operatorname{plim}\frac{1}{n}\sum (Y_{t-1} - \bar{Y}_{t-1})^2} = \frac{\sigma_{Y_{t-1},u_t}}{\sigma^2_{Y_{t-1}}} = \frac{0}{\sigma^2_{Y_{t-1}}} = 0$$

If Part (1) of Assumption C.7 is valid, the covariance between u_t and Y_{t-1} is zero. We suppose that Part(1) is valid because Y_{t-1} is determined before u_t is generated.

TIME SERIES MODELS: STATIC MODELS AND MODELS WITH LAGS

$$LGHOUS_t = \beta_1 + \beta_2 LGDPI_t + \beta_3 LGPRHOUS_t + u_t$$

Alternative dynamic specifications, housing services
(*LGHOUS* is the dependent variable, USA, 1959-2003)

Variable	(1)	(2)	(3)	(4)	(5)
LGDPI	1.03 (0.01)	–	–	0.33 (0.15)	0.29 (0.14)
LGDPI(–1)	–	1.01 (0.01)	–	0.68 (0.15)	0.22 (0.20)
LGDPI(–2)	–	–	0.98 (0.01)	–	0.49 (0.13)
LGPRHOUS	–0.48 (0.04)	–	–	–0.09 (0.17)	–0.28 (0.17)
LGPRHOUS(–1)	–	–0.43 (0.04)	–	–0.36 (0.17)	0.23 (0.30)
LGPRHOUS(–2)	–	–	–0.38 (0.04)	–	–0.38 (0.18)
R^2	0.9985	0.9989	0.9988	0.9990	0.9993

For housing expenditure we expect long lags, but direct inclusion of lagged explanatory variables leads to severe multicollinearity.

TIME SERIES MODELS:

STATIC MODELS AND MODELS WITH LAGS

Estimates of long-run income and price elasticities					
Specification	(1)	(2)	(3)	(4)	(5)
Sum of income elasticities	1.03	1.01	0.98	1.01	1.00
Sum of price elasticities	–0.48	–0.43	–0.38	–0.45	–0.43

$$Y_t = \beta_1 + \beta_2 X_t + \beta_3 X_{t-1} + \beta_4 X_{t-2} + u_t$$

$$\bar{Y} = \beta_1 + \beta_2 \bar{X} + \beta_3 \bar{X} + \beta_4 \bar{X} = \beta_1 + (\beta_2 + \beta_3 + \beta_4)\bar{X}$$

$$Y_t = \beta_1 + (\beta_2 + \beta_3 + \beta_4) X_t - \beta_3 (X_t - X_{t-1}) - \beta_4 (X_t - X_{t-2}) + u_t$$

The estimate of the coefficient of X_t will be the sum of the point estimates of β_2, β_3, and β_4 and shows the long-run effect. Since X_t is unlikely highly correlated with ($X_t - X_{t-1}$) or ($X_t - X_{t-2}$), there should not be a problem of severe multicollinearity. The estimate of β_2 shows the short run effect.

TIME SERIES MODELS: STATIC MODELS AND MODELS WITH LAGS

$$Y_t = \beta_1 + (\beta_2 + \beta_3 + \beta_4)X_t - \beta_3(X_t - X_{t-1}) - \beta_4(X_t - X_{t-2}) + u_t$$

```
============================================================
Dependent Variable: LGHOUS
Method: Least Squares
Sample(adjusted): 1961 2003
Included observations: 43 after adjusting endpoints
============================================================
     Variable      Coefficient Std. Error t-Statistic  Prob.
============================================================
         C           0.0468    0.134      0.350       0.729
       LGDPI         1.000     0.007      142.96      0.000
        X1          -0.221     0.196     -1.129       0.266
        X2          -0.491     0.134     -3.654       0.001
     LGPRHOUS       -0.425     0.034     -12.67       0.000
        P1          -0.233     0.299     -0.782       0.439
        P2           0.379     0.176      2.155       0.038
============================================================
R-squared            0.999    Mean dependent var 6.399
S.D. dependent var   0.406    Sum squared resid  0.005
S.E. of regression   0.012    Durbin-Watson stat 0.607
F-statistic          8159.9    Prob(F-statistic) 0.000
============================================================
```

The output shows the result of fitting the reparameterized model for housing with two lags (Specification (5) in the table). *X1 = LGDPI – LGDPI*(–1), *X2 = LGDPI – LGDPI*(–2), *P1 = LGPRHOUS – LGPRHOUS*(–1), and *P2 = LGPRHOUS – LGPRHOUS*(–2).

MODELS WITH A LAGGED DEPENDENT VARIABLE

$$Y_t = \beta_1 + \beta_2 X_t + \beta_3 Y_{t-1} + u_t \quad - \quad ADL(1,0)$$

$$\bar{Y} = \beta_1 + \beta_2 \bar{X} + \beta_3 \bar{Y}$$

$$\bar{Y} = \frac{\beta_1}{1-\beta_3} + \frac{\beta_2}{1-\beta_3}\bar{X}$$

$$Y_t = \beta_1 + \beta_2 X_t + \beta_3 X_{t-1} + \ldots + \beta_{q+2} X_{t-q} +$$
$$+ \alpha_3 Y_{t-1} + \alpha_4 Y_{t-2} + \ldots + \alpha_{p+2} Y_{t-p} + u_t \quad - \quad ADL(p,q)$$

One of the possible ways of including dynamics in a model is an autoregressive distributed lag model, ADL(*p*, *q*). Here p is the maximum lag in *Y*, *q* is the maximum lag in *X*.

DISTRIBUTED LAG MODELS

Distributed lag model: $$Y_t = \beta_1 + \beta_2 X_t + \beta_3 X_{t-1} + \ldots + u_t$$

The following problems arise when estimating the β_i parameters:

- Multicollinearity. Lagged values of an explanatory variable can be strongly correlated with each other.

- The observations' set being reduced when the number of lags grows.

- - A large number of parameters is estimated and, therefore, the number of degrees of freedom is reduced.

Solution: to make an assumption concerning the distribution of coefficients with small number of parameters (equivalent to imposing restrictions):

1) Geometric distribution (Koyck distribution) : $$\beta_3 = \beta_2 \cdot \rho; \quad \beta_4 = \beta_2 \cdot \rho^2;$$
where $0 < \rho < 1$

2) Polynomial distribution:
where s are integers. $$\beta_s = \tilde{\beta}_0 + \tilde{\beta}_1 \cdot s + \tilde{\beta}_2 \cdot s^2$$

It is also possible to impose on the parameters restrictions of uniform, linear, "triangular" and other distributions.

GEOMETRICALLY DISTRIBUTED LAG

Geometrically Distributed lag (Koyck model):

$$Y_t = \beta_1 + \beta_2 X_t + \beta_2 \rho X_{t-1} + \beta_2 \rho^2 X_{t-2} + \ldots + u_t$$

Koyck transformation:

$$Y_{t-1} = \beta_1 + \beta_2 X_{t-1} + \beta_2 \rho X_{t-2} + \beta_2 \rho^2 X_{t-3} + \ldots + u_{t-1}$$

$$\Rightarrow Y_t - \rho \cdot Y_{t-1} = \beta_1 (1 - \rho) + \beta_2 \cdot X_t + u_t - \rho \cdot u_{t-1}$$

$$Y_t = \beta_1 (1 - \rho) + \rho \cdot Y_{t-1} + \beta_2 \cdot X_t + v_t$$

If the model is estimated in this form, then the estimates are biased and inconsistent since Y_{t-1} is related with $v_t = u_t - \rho \cdot u_{t-1}$

So the model to be estimated as non-linear regression; lag number increasing stops when the estimates do not change.

GEOMETRICALLY DISTRIBUTED LAG: EXAMPLE, 09.01.2014-14.01.2017, 747 observations

$$RUR\_USD_t = \beta_1 + \beta_2(OIL\_BRENT_{t-2} + $$
$$+ \rho OIL\_BRENT_{t-3} + \rho^2 OIL\_BRENT_{t-4} + \ldots) + u_t$$

$$RUR\_\hat{U}SD_t = 89.57 - 0.51 OIL\_BRENT_{t-2}$$

$$RUR\_\hat{U}SD_t = 89.57 - 0.36 OIL\_BRENT_{t-2} - 0.15 OIL\_BRENT_{t-3}$$

$$RUR\_\hat{U}SD_t = 89.54 - 0.18(OIL\_BRENT_{t-2} + $$
$$+ 0.65 OIL\_BRENT_{t-3} + \ldots + 0.65^8 OIL\_BRENT_{t-10})$$

Short Run Effect = -0.18

Long Run Effect =-0.18(0.65+0.65^2+0.65^3+…)=-0.18/0.35=-0.51

POLYNOMIAL DISTRIBUTED LAG MODEL

Polynomial Distributed lag :

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \ldots + u_t$$

Where $\beta_s = \gamma_0 + \gamma_1 \cdot s + \gamma_2 \cdot s^2 + \ldots + \gamma_m \cdot s^m; \quad s = -1,0,1,2,\ldots$

$$\beta_0 = \gamma_0 - \gamma_1 + \gamma_2 + \ldots; \quad \beta_1 = \gamma_0; \qquad \beta_2 = \gamma_0 + \gamma_1 + \gamma_2 + \ldots + \gamma_m$$

$$\beta_3 = \gamma_0 + 2\gamma_1 + 4\gamma_2 + \ldots + 2^m \gamma_m; \ldots$$

For example, for *n* = 3, *m* = 2 we get:

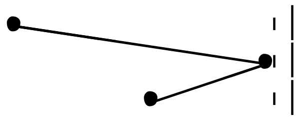
$$Y_t = \alpha + (\gamma_0 - \gamma_1 + \gamma_2) X_t + \gamma_0 X_{t-1} + (\gamma_0 + \gamma_1 + \gamma_2) X_{t-2} +$$
$$(\gamma_0 + 2\gamma_1 + 4\gamma_2) X_{t-3} + u_t = \alpha + \gamma_0 (X_t + X_{t-1} + X_{t-2} + X_{t-3}) +$$
$$\gamma_1 (-X_t + X_{t-2} + 2X_{t-3}) + \gamma_2 (X_t + X_{t-2} + 4X_{t-3}) =$$
$$\alpha + \gamma_0 Z_0 + \gamma_1 Z_1 + \gamma_2 Z_2.$$

$$Estimate \;\; \alpha \; and \; \gamma' s, \quad then \; calculate \;\; \beta' s.$$

POLYNOMIAL DISTRIBUTED LAG MODEL: EXAMPLE. DEPENDENT VARIABLE RUR_USD, REGRESSOR OIL_BRENT(-2). 09.01.2014-14.01.2017, 747 observations, Eviews.

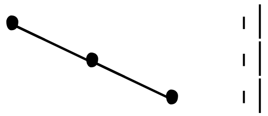
ls rur_usd c pdl(oil_brent(-2),2,2)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	89.56551	0.255745	350.2137	0.0000
PDL01	-0.008316	0.095010	-0.087526	0.9303
PDL02	0.092072	0.052901	1.740470	0.0822
PDL03	-0.244216	0.142521	-1.713537	0.0870

Lag Distribution of...	i	Coefficient	Std. Error	t-Statistic
	0	-0.34460	0.07116	-4.84242
	1	-0.00832	0.09501	-0.08753
	2	-0.16046	0.07108	-2.25761
Sum of Lags		-0.51338	0.00362	-141.876

ls rur_usd c pdl(oil_brent(-2),2,1)

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	89.56151	0.256069	349.7553	0.0000
PDL01	-0.171107	0.001208	-141.6807	0.0000
PDL02	0.092063	0.052970	1.738036	0.0826

Lag Distribution of...	i	Coefficient	Std. Error	t-Statistic
	0	-0.26317	0.05304	-4.96186
	1	-0.17111	0.00121	-141.681
	2	-0.07904	0.05293	-1.49339
Sum of Lags		-0.51332	0.00362	-141.681

Estimation of Polynomial Distributed Lags in Eviews:
ls Y c pdl(X, Number of Lags, Order of Polynomial)