1) Equations (teorethical) $y_i = \alpha + \beta x_i + \varepsilon_i$

2) Assumptions

- stochastic / deterministic

$E(\varepsilon_i | X) = 0 \qquad E(\varepsilon_i) = 0$

- classic lin. regression

$E(\varepsilon_i) = 0 \qquad \underset{Var(\varepsilon_i^2)}{E(\varepsilon_i^2) = \sigma} \qquad \underset{\substack{cov(\varepsilon_i, \varepsilon_j) \\ i = j}}{E(\varepsilon_i \varepsilon_j) = 0}$

3) Method (OLS; WLS; IV; MLE)

4) Properties

Problem 1.

$$\hat{\beta} = \frac{cov(X,Y)}{Var(X)} = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2} = \qquad a_i = \frac{X_i - \bar{X}}{\Sigma(X - \bar{X})^2}$$

$$= \beta + \Sigma a_i \cdot \varepsilon_i$$

$$E(\hat{\beta}) = \beta + \Sigma a_i \overset{0}{E(\varepsilon_i)} = \beta$$

$$\hat{\beta} = \frac{\Sigma(X_i - \bar{X})(\cancel{\alpha} + \beta X_i + \varepsilon_i - \cancel{\alpha} - \beta\bar{X} - \bar{\varepsilon})}{\Sigma(X_i - \bar{X})^2} = \qquad \frac{\Sigma X_i}{n} = \bar{X}$$

$$= \frac{\beta \Sigma(X_i - \bar{X}) \cdot (X_i - \bar{X}) + \Sigma(X_i - \bar{X}) \cdot \varepsilon_i - \bar{\varepsilon}\overset{0}{\Sigma(X_i - \bar{X})}}{\Sigma(X_i - \bar{X})^2} =$$

$$= \beta + \frac{\sum (X_i - \bar{X}) \varepsilon_i}{\sum (X_i - \bar{X})^2} = \beta + \sum \underbrace{\left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right)}_{= a_i} \varepsilon_i$$

Properties: $\quad a_i = \dfrac{X_i - \bar{X}}{\sum (X - \bar{X})^2}$

1) $\sum a_i = 0$ $\qquad\qquad \delta \sim \delta$

2) $\sum a_i^2 = \dfrac{1}{\sum (X_i - \bar{X})^2}$

$$\sum \left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right)^2 = \frac{\sum (X_i - \bar{X})^2}{\left(\sum (X_i - \bar{X})^2 \right)^2} = \frac{1}{\sum (X_i - \bar{X})^2}$$

3) $\sum a_i X_i = 1$

$$\sum \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} X_i = \frac{1}{\sum (X_i - \bar{X})^2} \left(\sum (X_i - \bar{X}) X_i - \sum (X_i - \bar{X}) \bar{X} \right) = 1$$

Problem 3

$$\hat{\beta} = \frac{Cov(X, Y)}{Var(X)} = \frac{Cov(X, \alpha + \beta X + \varepsilon)}{Var(X)} =$$

$$\frac{\overset{=0}{Cov(X, \alpha)} + Cov(X, \beta X) + Cov(X, \varepsilon)}{Var(X)} =$$

$$\beta + \frac{Cov(X, \varepsilon)}{Var(X)}$$

$$E(\hat{\beta}) = \beta + E\left(\frac{Cov(x,\varepsilon)}{Var(x)}\right) = \beta + \frac{1}{Var(X)} \cdot Cov(\overset{X}{\overset{\shortparallel}{EX}}, \overset{0}{\overset{\shortparallel}{E\varepsilon}}) = \beta$$

P5.

Normal Equations

$$\|b - Ax\|_2^2 \to \min_x$$

$A^T A x = A^T b$ — normal equation

residuals normal (orthogonal) to span(A)

$X^T X \hat{\beta} = X^T y$ — normal equation (LR)

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

$$y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i \qquad \frac{\partial RSS}{\partial \alpha} = 0$$

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 \qquad \frac{\partial RSS}{\partial \beta_1} = 0$$

$Cov(x_1, -)$, $Cov(x_2, -)$

$$\begin{cases} \hat{\beta}_1 Var(x_1) + \hat{\beta}_2 Cov(x_1, x_2) = Cov(x_1, y) \\ \hat{\beta}_1 Cov(x_1, x_2) + \hat{\beta}_2 Var(x_2) = Cov(x_2, y) \end{cases}$$

$$X^T X \hat{\beta} \qquad X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad X^T X = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$\begin{cases} \hat{\beta}_1 \, Var(x_1) + \hat{\beta}_2 \, Cov(x_1, x_2) = Cov(x_1, y) \\ \hat{\beta}_1 \, Cov(x_1, x_2) + \hat{\beta}_2 \, Var(x_2) = Cov(x_2, y) \end{cases}$$

$$\left[\begin{array}{cc|c} Var(x_1) & Cov(x_1, x_2) & Cov(x_1, y) \\ Cov(x_1, x_2) & Var(x_2) & Cov(x_2, y) \end{array}\right]$$

$$\hat{\beta}_1 = \frac{\Delta_1}{\Delta} = \frac{Cov(x_1, y)\, Var(x_2) - cov(x_2, y) \cdot cov(x_1, x_2)}{Var(x_1, x_2) - \left(Cov(x_1, x_2)\right)^2}$$

$$\hat{\beta}_2 = \frac{\Delta_2}{\Delta} = -//-$$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

$$R^2_{adj} = 1 - \frac{RSS / n-k}{TSS / n-1} \qquad RSS_R > RSS_{UR}$$