Elements of Econometrics. Lecture 18.
Autocorrelation. Part 1.

FCS, 2022-2023

ASSUMPTIONS FOR MODEL C: REGRESSIONS WITH TIME SERIES DATA

ASSUMPTIONS FOR MODEL C

…

C.6 The values of the disturbance term have independent distributions:

u_t **is distributed independently of** $u_{t'}$ **for** $t' \neq t$

…

The assumption C.6 (Gauss-Markov 3 condition) is usually supposed to satisfy in Cross Sections Data Models, while it may be violated for Time Series.

The consequences of C.6 violation for OLS are similar to those of heteroscedasticity. In general, the regression coefficients remain unbiased, but OLS is inefficient because one can find an alternative regression technique that yields estimators with smaller variances.

The other main consequence is that autocorrelation causes the standard errors to be estimated wrongly, often being biased downwards. Finally, although in general OLS estimates are unbiased, there is an important special case where they are biased (lagged dependent variable as a regressor).

RESIDUAL AUTOCORRELATION

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

First-order autoregressive autocorrelation: AR(1)

$$u_t = \rho u_{t-1} + \varepsilon_t$$

p's order autoregressive autocorrelation: AR(p)

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \rho_3 u_{t-3} + \ldots + \rho_p u_{t-p} + \varepsilon_t$$

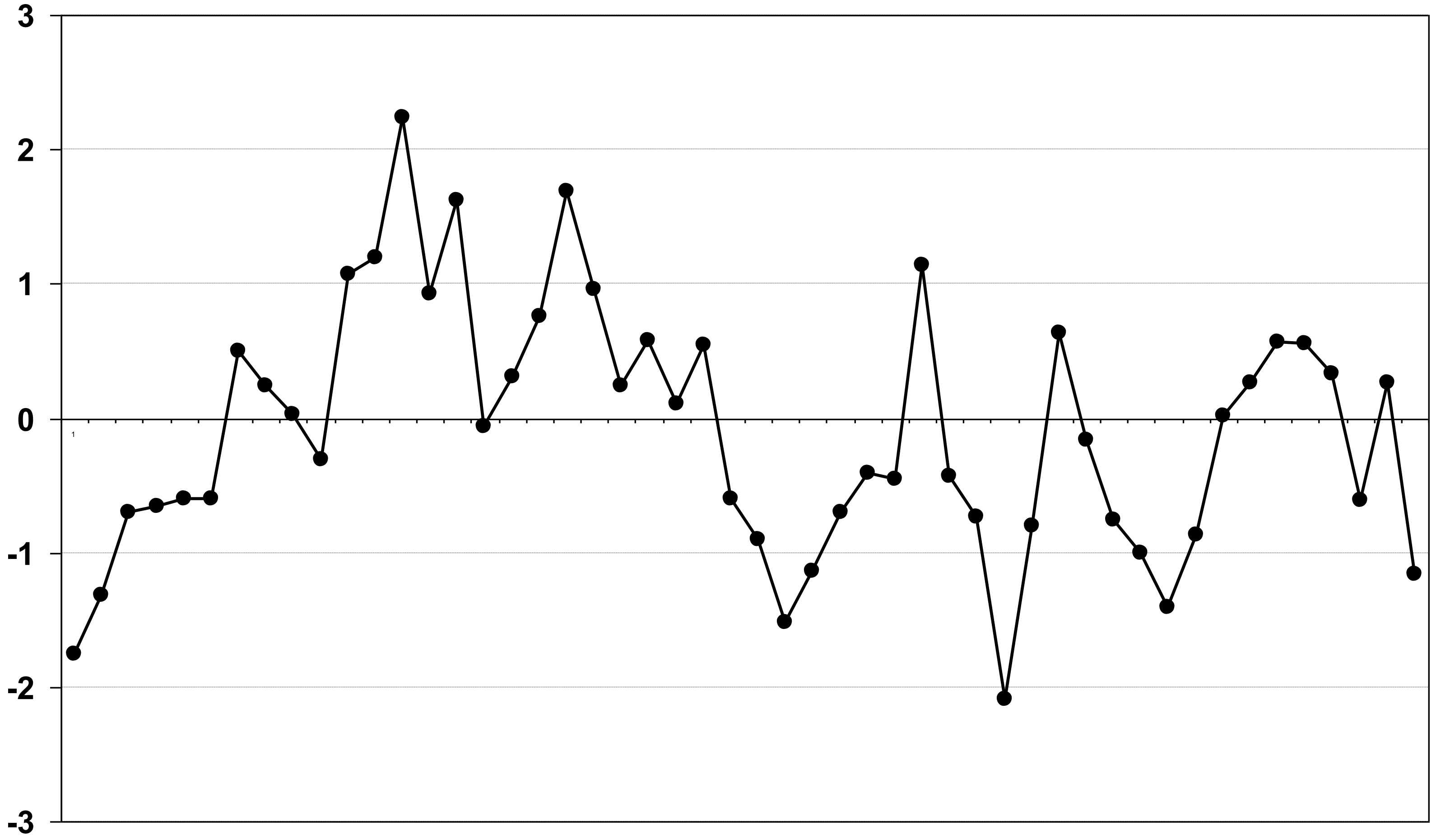
q's order moving average autocorrelation: MA(q)

$$u_t = \lambda_0 \varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2} + \ldots + \lambda_q \varepsilon_{t-q}$$

ARMA(p,q)

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \ldots + \rho_p u_{t-p} + \varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2} + \ldots + \lambda_q \varepsilon_{t-q}$$

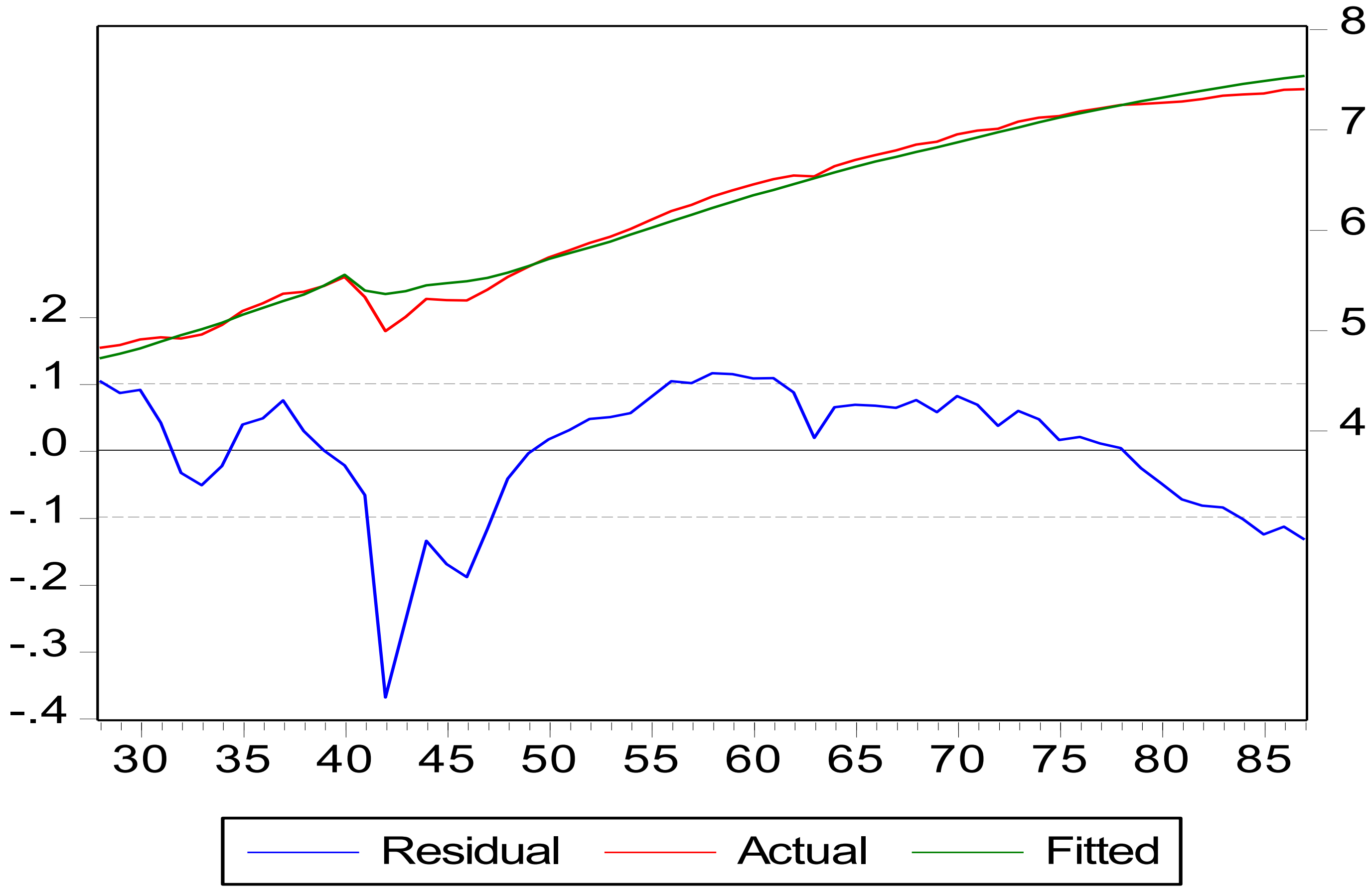
AUTOCORRELATION



$$u_t = 0.6u_{t-1} + \varepsilon_t$$

With ρ equal to 0.6, it is obvious that *u* is subject to positive autocorrelation. Positive values tend to be followed by positive ones and negative values by negative ones.

RESIDUAL AUTOCORRELATION: EXAMPLE 1

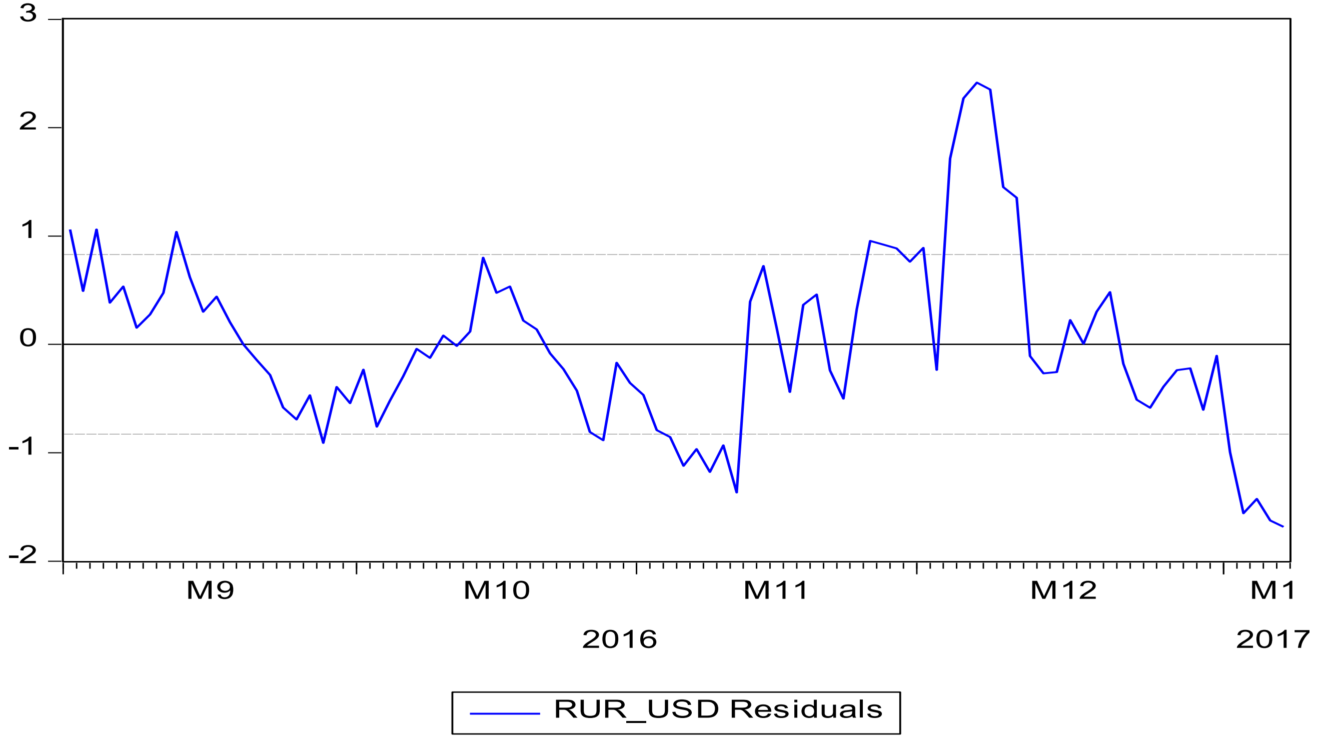


The plot of the residuals of regression of Log(GNP) on Log(K) and Log(L), USSR, 1928-1987 (Cobb-Douglas Production Function). If the disturbance term is subject to autocorrelation, then the residuals are subject to a similar pattern of autocorrelation.

RESIDUAL AUTOCORRELATION: EXAMPLE 2 (RuR/USD exchange rate, 01/09/2016 – 14/01/2017

$$RUR\_USD_t = \beta_1 + \beta_2 OIL\_BRENT_{t-1} + \beta_3 OIL\_BRENT_{t-2} + u_t$$

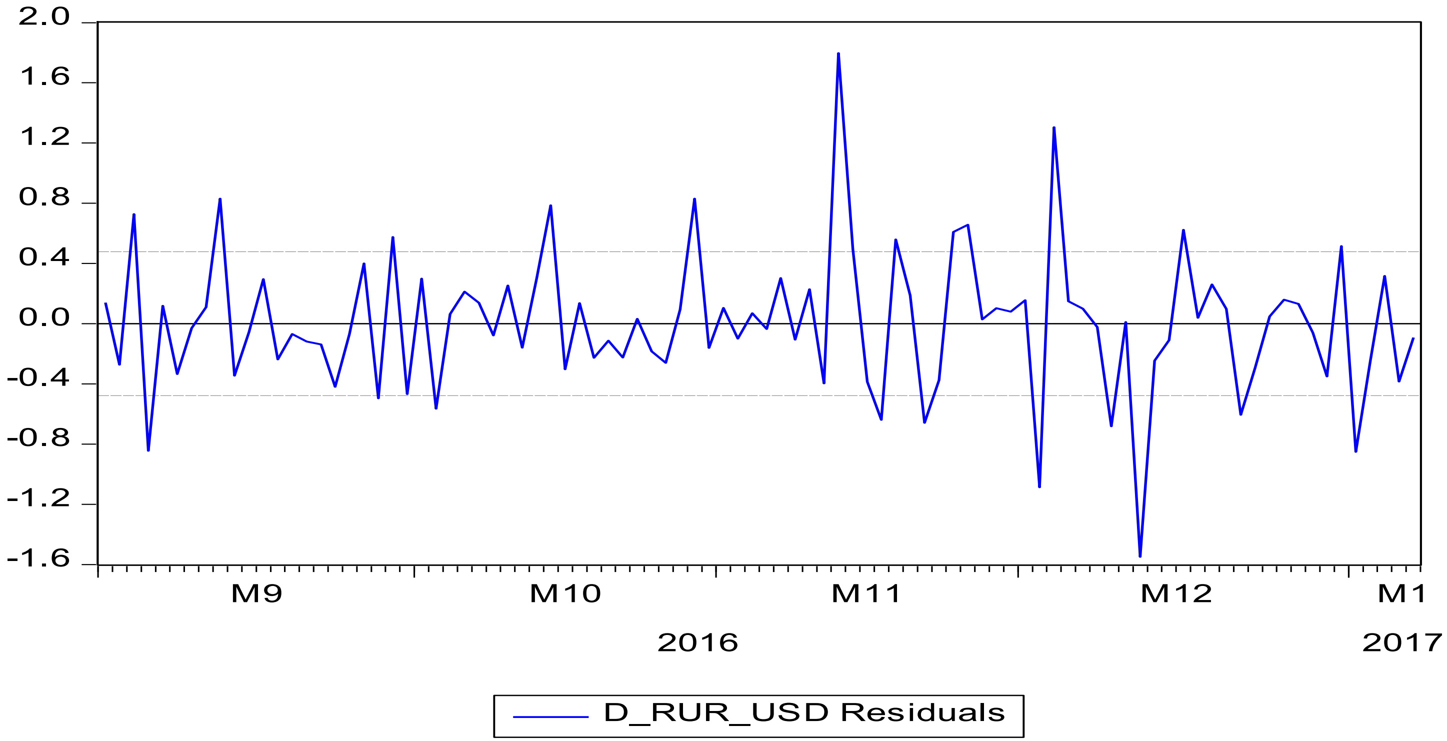
$$\hat{\beta}_1 = 82.8; \quad \hat{\beta}_2 = -0.194; \quad \hat{\beta}_3 = -0.197; \quad R^2 = 0.74; \quad d = 0.37.$$



RESIDUAL AUTOCORRELATION: EXAMPLE 2 (RuR/USD exchange rate, 01/09/2016 – 14/01/2017

$$DRUR\_USD_t = \beta_1 + \beta_2 DOIL\_BRENT_{t-1} + \beta_3 DOIL\_BRENT_{t-2} + u_t$$

$$\hat{\beta}_2 = -0.15; \quad \hat{\beta}_3 = -0.33; \quad R^2 = 0.17; \quad d = 2.19.$$



Assumptioins C.6 and C.7

C.6 u_t **is distributed independently of** $u_{t'}$ **for any** $t' \neq t$

C.7. u_t **is distributed independently of the regressors** $X_{jt'}$ **for all** t' **(including** t**) and** j

C.7.1 The disturbance term is distributed independently of the regressors in the same observation

C.7.2. The disturbance term is distributed independently of the regressors in the other observations.

$$Y_t = \beta_1 + \beta_2 X_t + u_t \qquad a_t = \frac{X_t - \overline{X}}{\sum_{s=1}^{T}\left(X_s - \overline{X}\right)^2}$$

$$\hat{\beta}_2 = \beta_2 + \sum_{t=1}^{T} a_t u_t$$

If Assumption C.7 is satisfied, a_t **and** u_t **are distributed independently and we can write the expectation of** $\hat{\beta}_2$ **as shown (no reference to C.6):**

$$\mathrm{E}\left(\hat{\beta}_2\right) = \beta_2 + \mathrm{E}\left(\sum_{t=1}^{T} a_t u_t\right) = \beta_2 + \sum_{t=1}^{T} \mathrm{E}(a_t u_t) = \beta_2 + \sum_{t=1}^{T} \mathrm{E}(a_t)\mathrm{E}(u_t) = \beta_2$$

ASSUMPTIONS C.6 and C.7

If lagged dependent variable is one of the regressors, and C.6 is satisfied,

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + u_t \qquad\qquad Y_{t-1} = \beta_1 + \beta_2 Y_{t-2} + u_{t-1}$$

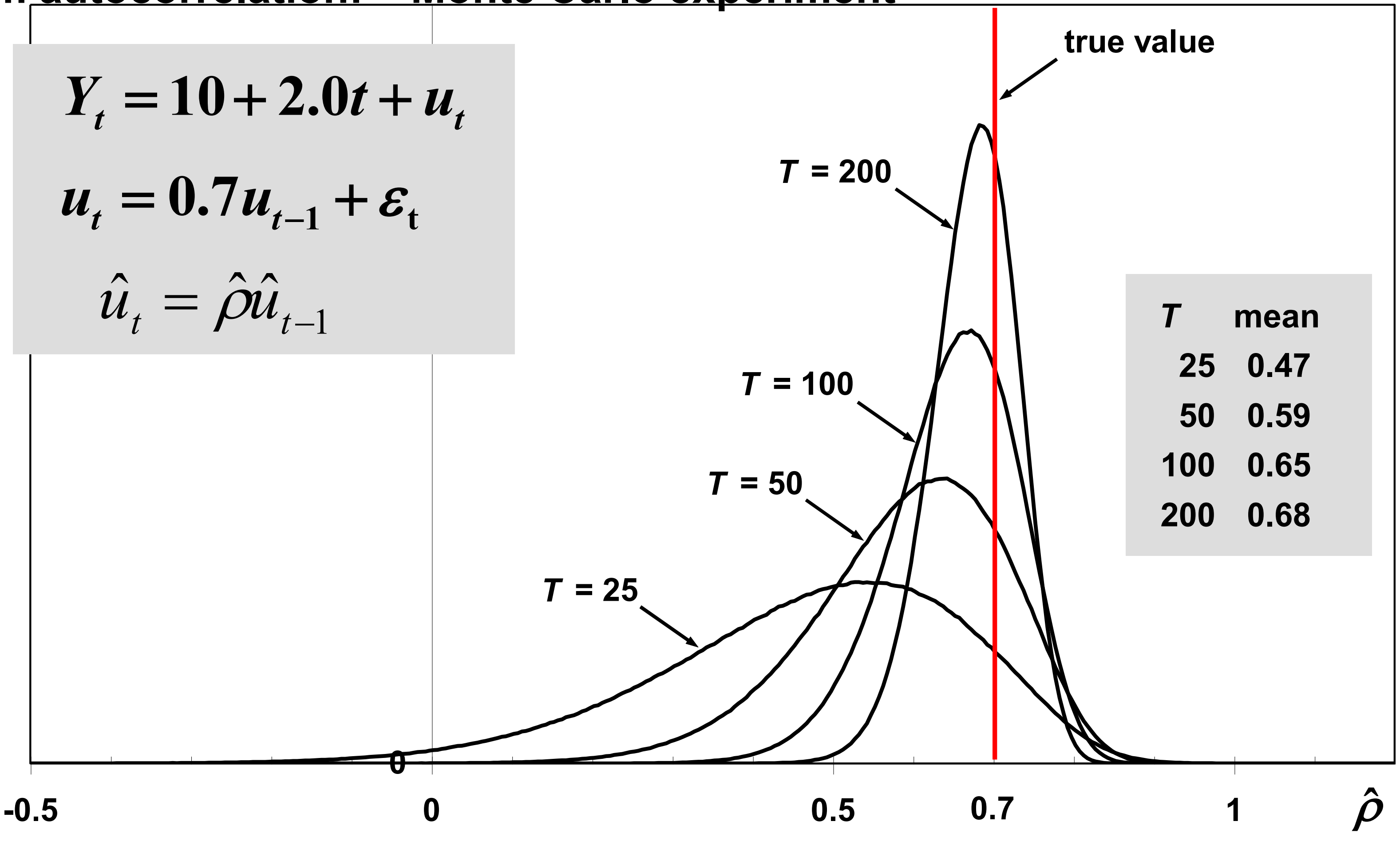
Then the disturbance term u_t is correlated with the explanatory variable Y_t in the next observation, and C.7.2 is violated. The OLS estimators are biased but consistent:

$$p\lim b_2^{\text{OLS}} = \beta_2 + \frac{\operatorname{plim} \frac{1}{n}\sum\left(Y_{t-1} - \overline{Y}_{t-1}\right)\left(u_t - \overline{u}\right)}{\operatorname{plim} \frac{1}{n}\sum\left(Y_{t-1} - \overline{Y}_{t-1}\right)^2} = \beta_2 + \frac{\sigma_{Y_{t-1},u_t}}{\sigma^2_{Y_{t-1}}} = \beta_2 + \frac{0}{\sigma^2_{Y_{t-1}}} = \beta_2$$

But if C.6 is violated $(u_t = \rho u_{t-1} + \varepsilon_t)$, then C.7.1 is violated, and the OLS estimators are biased and inconsistent since $\sigma_{Y_{t-1},u_t} \neq 0$.

TESTS FOR AUTOCORRELATION

Simple autoregression of the residuals for testing the disturbance term autocorrelation: Monte Carlo experiment



$$Y_t = 10 + 2.0t + u_t$$

$$u_t = 0.7u_{t-1} + \varepsilon_t$$

$$\hat{u}_t = \hat{\rho}\hat{u}_{t-1}$$

T	mean
25	0.47
50	0.59
100	0.65
200	0.68

When $\hat{u}_t$ is regressed on $\hat{u}_{t-1}$, the distribution of the estimator of ρ is left skewed and heavily biased downwards for *T* = 25. The mean of the distribution is 0.47. However, as the sample size increases, the downwards bias diminishes and it is clear that it is converging on 0.7 as the sample becomes large.

TESTS FOR AUTOCORRELATION

Breusch–Godfrey test

$$\boldsymbol{Y_t = \beta_1 + \sum_{j=2}^{k} \beta_j X_{jt} + u_t}$$

$$\hat{u}_t = \gamma_1 + \sum_{j=2}^{k} \gamma_j X_{jt} + \rho \hat{u}_{t-1}$$

Test statistic: nR^2, distributed as $\chi^2(1)$ when testing for first-order autocorrelation

Alternatively, simple t test on coefficient of $\hat{\boldsymbol{u}}_{\boldsymbol{t-1}}$, again with asymptotic validity.

Several asymptotically-equivalent versions of the test have been proposed. The most popular involves the computation of the Lagrange multiplier statistic nR^2 when the residuals regression is fitted, n being the actual number of observations in the regression.

TESTS FOR AUTOCORRELATION

Breusch–Godfrey test for higher order autocorrelation

$$\boldsymbol{Y_t = \beta_1 + \sum_{j=2}^{k} \beta_j X_{jt} + u_t}$$

$$\hat{u}_t = \gamma_1 + \sum_{j=2}^{k} \gamma_j X_{jt} + \sum_{s=1}^{q} \rho_s \hat{u}_{t-s}$$

Test statistic: nR^2, distributed as $\chi^2(q)$, valid also for MA(q) autocorrelation

For the lagrange multiplier version of the test, the test statistic remains nR^2 (with n smaller than before, the inclusion of the additional lagged residuals leading to a further loss of initial observations).

The t test version becomes an F test comparing RSS for the residuals regression with RSS for the same specification without the residual terms. Again, the test is valid only asymptotically.

TESTS FOR AUTOCORRELATION

Durbin–Watson test

$$d = \frac{\sum_{t=2}^{T} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{T} \hat{u}_t^2}$$

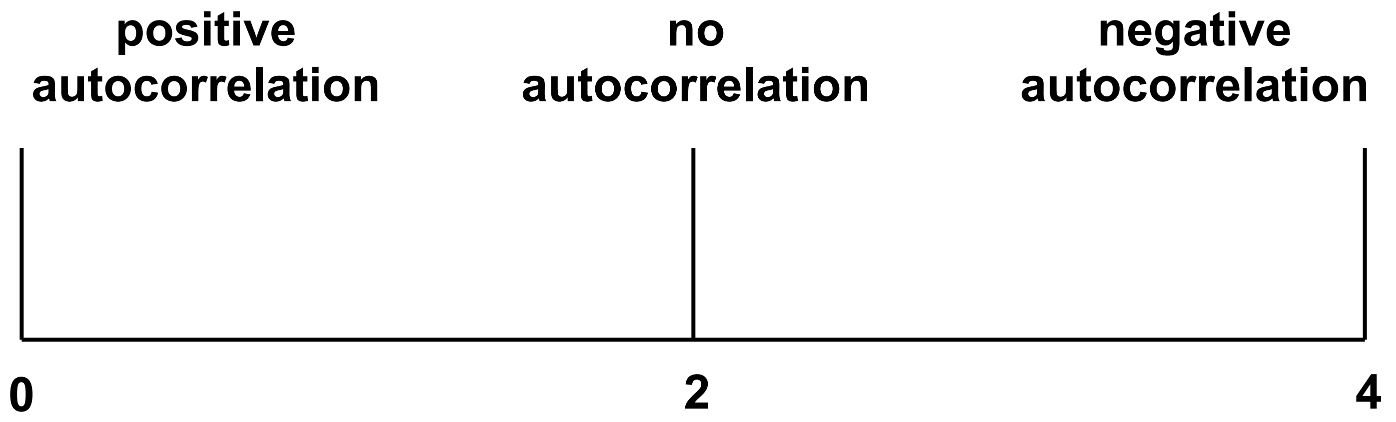
In large samples $d \to 2 - 2\rho$

The first major test to be used for the detection of autocorrelation was the Durbin–Watson test for AR(1) autocorrelation based on the Durbin–Watson *d* statistic calculated from the residuals.

It can be shown that in large samples *d* tends to 2 – 2ρ, where ρ is the parameter in the AR(1) relationship $u_t = \rho u_{t-1} + \varepsilon_t$.

TESTS FOR AUTOCORRELATION

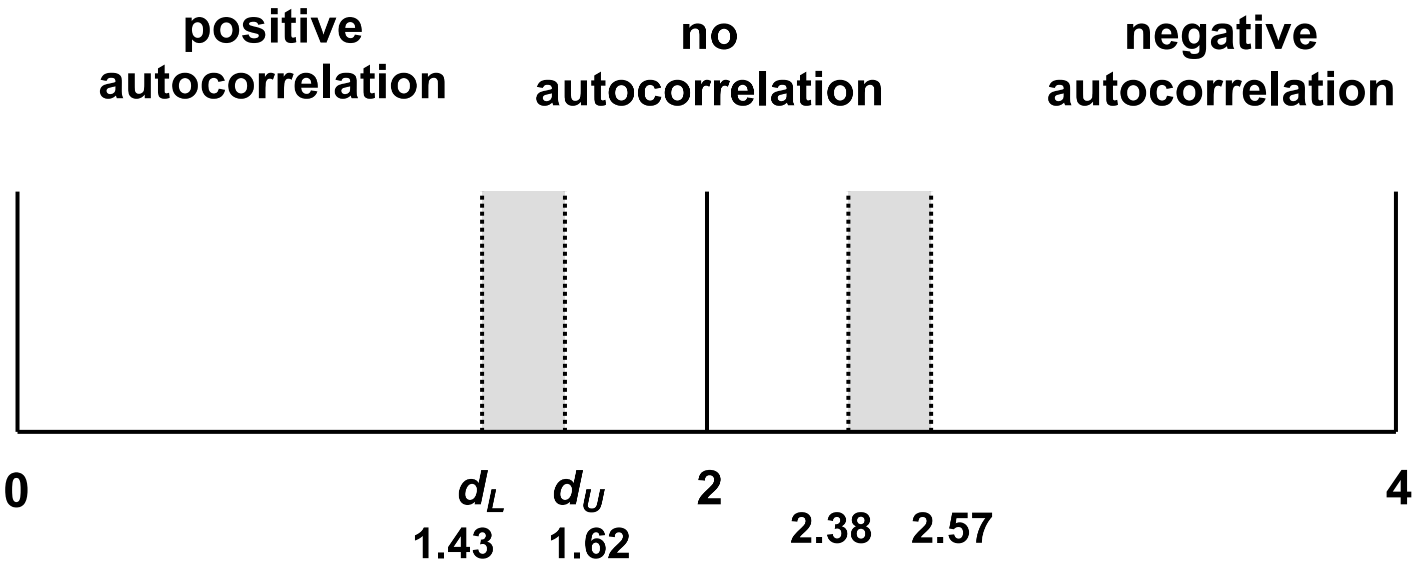
Durbin–Watson test



In large samples	$d \to 2 - 2\rho$
No autocorrelation	$d \to 2$
Severe positive autocorrelation	$d \to 0$
Severe negative autocorrelation	$d \to 4$

The critical values, at any significance level, depend on the number of observations in the sample, the number of parameters estimated, and also on the actual values of explanatory variables.

TESTS FOR AUTOCORRELATION



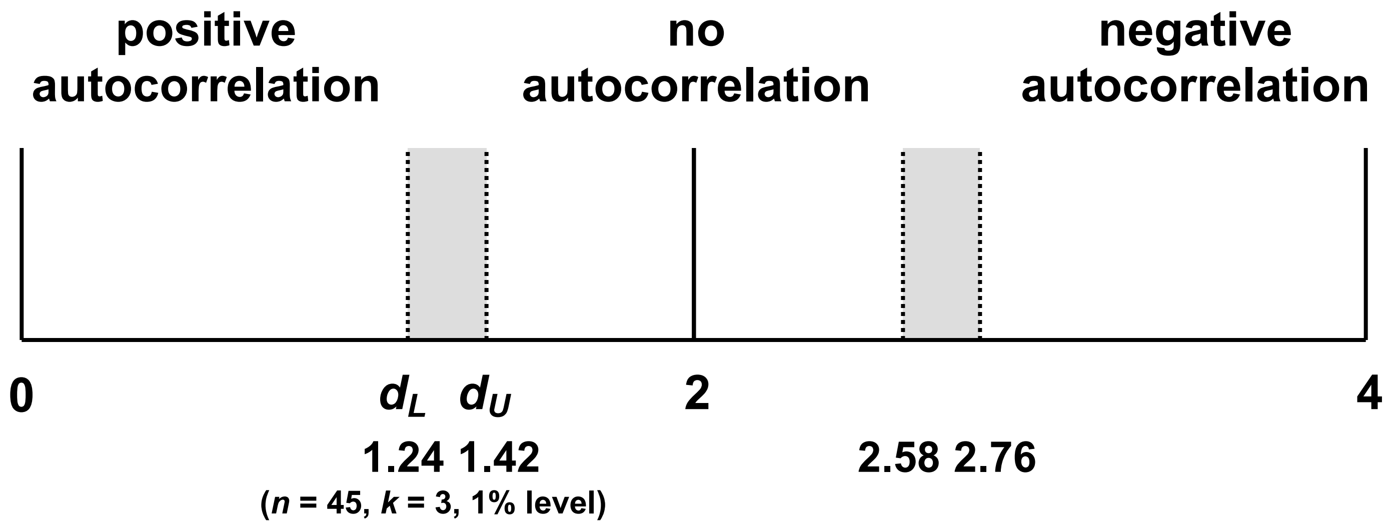
(n = 45, k = 3, 5% level)

Durbin and Watson determined upper and lower bounds, d_U and d_L, for the critical values, and these are presented in standard tables. If d lies between d_L and d_U, we cannot tell whether it is above or below the critical value and so the test is indeterminate. Here the bounds for n=45; k=3; 5% significance level are given: d_L=1.43; d_U=1.62; the bounds for negative autocorrelation (symmetric with respect to 2) are 2.38 and 2.37 respectively.

If d < 1.43, we reject the null hypothesis and conclude that there is positive autocorrelation. If 1.62 < d < 2.38, we do not reject the null hypothesis of no autocorrelation. If 1.43 < d < 1.62, the test is indeterminate and we do not come to any conclusion.

TESTS FOR AUTOCORRELATION

Durbin–Watson test



The Durbin-Watson test is valid only when all the explanatory variables are deterministic. This is in practice a serious limitation since usually interactions and dynamics in a system of equations cause Assumption C.7 part (2) to be violated. In particular, if the lagged dependent variable is used as a regressor, the statistic is biased towards 2 and therefore will tend to under-reject the null hypothesis.

The test is restricted by AR(1) autocorrelation.

The model should include the constant.

But the test is applicable for finite samples and is provided by standard packages.

TESTS FOR AUTOCORRELATION

Durbin's *h* test

$$h = \hat{\rho}\sqrt{\frac{n}{1-ns^2_{\hat{\beta}_{Y(-1)}}}}$$

$$d \rightarrow 2-2\rho$$

$$\hat{\rho} = 1 - 0.5d$$

Durbin proposed *h* test, appropriate for the detection of AR(1) autocorrelation where the use of the lagged dependent variable as a regressor made the original Durbin–Watson test inapplicable. $\hat{\rho}$ is an estimate of *r* in the AR(1) process, $s^2_{\hat{\beta}_{Y(-1)}}$ is an estimate of the variance of the coefficient of the lagged dependent variable, and *n* is the number of observations in the regression.
Problem with h test appears when $ns^2_{\hat{\beta}_{Y(-1)}}$ is greater than, or close to 1.

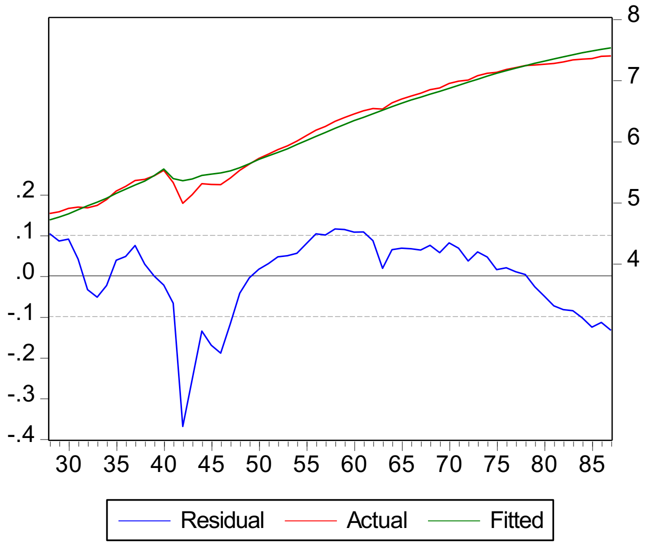
Darbin *h* statistics has asymptotically standardized normal distribution under the *H_0* hypothesis of no autocorrelation of *u_t*.

TESTS FOR AUTOCORRELATION: APPLICATION

(Cobb-Douglas Production Function, USSR, 1928-1987

Dependent Variable: Log(GNP)		Method: Least Squares		
Sample: 1928 1987 USSR		Included observations: 60		
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.324	0.486	0.667	0.508
Log(K)	0.661	0.041	15.98	0.000
Log(L)	0.218	0.146	1.499	0.139
R-squared	0.987		Mean dependent var	6.187
S.D. dependent var	0.872		S.E. of regression	0.100
Sum squared resid	0.565		F-statistic	2232.78
Durbin-Watson stat	0.299		Prob(F-statistic)	0.0000

The output shown in the table gives the result of a logarithmic regression of GNP on capital K and labour L. The residuals seem to be highly (positively) autocorrelated.
Durbin-Watson test: for k=3; n=60; at 1% level d_L=1.51; d_U=1.65. Since d=0.299<1.51, the H_0 of no first order autocorrelation is rejected.



Autoregression (applicable for large samples) :

$$\hat{u}_t = 0.853\hat{u}_{t-1}; \;\; s.e. = 0.07$$

Breusch–Godfrey test for the First Order Autocorrelation

Dependent Variable: RESID1   Method: Least Squares
Sample (adjusted): 1929 1987   Included observations: 59 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.516	0.257	-2.007	0.048
Log(L)	0.157	0.077	2.050	0.045
Log(K)	-0.044	0.022	-2.028	0.047
RESID1(-1)	0.874	0.070	12.44	0.000

R-squared	0.738	Mean dependent var	-0.0018
S.D. dependent var	0.098	S.E. of regression	0.0514
Sum squared resid	0.145	F-statistic	51.67
Durbin-Watson stat	1.820	Prob(F-statistic)	0.0000

$$\hat{u}_t = \ldots + 0.87\hat{u}_{t-1} \qquad R^2 = 0.738$$

$$nR^2 = 59 \times 0.738 = 43.54 \qquad \chi^2(1)_{0.1\%} = 10.83$$

The test rejects the H_0 hypothesis of no autocorrelation in favour of the H_1 hypothesis of the first order autocorrelation.

Breusch–Godfrey test for the 3rd Order Autocorrelation

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	47.86	Prob. F(3,54)	0.0000
Obs*R-squared	43.60	Prob. Chi-Square(3)	0.0000

Dependent Variable: RESID
Method: Least Squares Sample: 1928 1987 Included observations: 60

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.418	0.278	-1.507	0.138
Log(K)	-0.042	0.024	-1.728	0.090
Log(L)	0.136	0.084	1.617	0.112
RESID(-1)	0.908	0.135	6.750	0.000
RESID(-2)	-0.167	0.184	-0.906	0.369
RESID(-3)	0.160	0.135	1.179	0.2434
R-squared	0.7267	F-statistic	28.716	
Durbin-Watson stat	1.768	Prob(F-statistic)	0.000	

$$\hat{u}_t = ... + 0.908\hat{u}_{t-1} - 0.167\hat{u}_{t-2} + 0.160\hat{u}_{t-3}$$

$$R^2 = 0.7267$$

$$nR^2 = 60 \times 0.7267 = 43.60$$

$$\chi^2(3)_{0.1\%} = 16.27$$

EViews: in the View – Residual Tests – Serial Correlation LM Test; done automatically.
The H_0 hypothesis is rejected again. EViews assigns zero residual to the first 3 observations, and n=60 is kept. Test is asymptotical, as well as F-test for RESID group.

Model with Lagged Dependent Variable as a Regressor

Dependent Variable: LogGNP		Method: Least Squares		
Sample (adjusted): 1929 1987		Included observations: 59 after adjustments		

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.211	0.352	-0.601	0.550
Log(K)	0.127	0.075	1.681	0.099
Log(L)	0.155	0.104	1.489	0.142
LogGNP(-1)	0.762	0.100	7.633	0.000

R-squared	0.994	Mean dependent var	6.210
S.D. dependent var	0.860	S.E. of regression	0.070
Sum squared resid	0.269	Durbin-Watson stat	1.155

$$h = \hat{\rho}\sqrt{\frac{n}{1 - ns^2_{\hat{\beta}_{Y(-1)}}}} = (1 - 0.5*1.155)\sqrt{\frac{59}{1 - 59(0.1)^2}} = 5.07$$

Under the null hypothesis of no autocorrelation, the *h* statistic asymptotically has a standardized normal distribution, so its value is above the critical value at the 0.1 percent level, 3.29. Hence the first order autocorrelation is still available.

LM test is also (asymptotically) applicable.

AUTOCORRELATION DETECTION: (RuR/USD exchange rate, 01/09/2016 – 14/01/2017, LM TESTS, Eviews, Lag=2,
Residuals represent the dependent variables.

$$RUR\_USD_t = \beta_1 + \beta_2 OIL\_BRENT_{t-1} + \beta_3 OIL\_BRENT_{t-2} + u_t$$

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	79.63484	Prob. F(2,87)	0.0000
Obs*R-squared	59.49904	Prob. Chi-Square(2)	0.0000

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.256862	0.752178	0.341491	0.7336
OIL_BRENT(-1)	0.024110	0.046502	0.518473	0.6054
OIL_BRENT(-2)	-0.029575	0.046747	-0.632664	0.5286
RESID(-1)	0.795594	0.109408	7.271820	0.0000
RESID(-2)	0.037538	0.112535	0.333569	0.7395

$$DRUR\_USD_t = \beta_1 + \beta_2 DOIL\_BRENT_{t-1} + \beta_3 DOIL\_BRENT_{t-2} + u_t$$

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	1.582381	Prob. F(2,87)	0.2113
Obs*R-squared	3.229179	Prob. Chi-Square(2)	0.1990

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.001922	0.049774	0.038608	0.9693
D_OIL_BRENT(-1)	-0.016265	0.044079	-0.368994	0.7130
D_OIL_BRENT(-2)	-0.003860	0.043690	-0.088358	0.9298
RESID(-1)	-0.195045	0.109674	-1.778415	0.0788
RESID(-2)	-0.036749	0.110783	-0.331718	0.7409