Class 2. Simple Linear Regression. OLS.
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Problem 1. Properties of residuals

1) $\bar{e} = 0$
2) $\sum X_i e_i = 0$
3) $\sum \hat{Y}_i e_i = 0$
4) $\bar{\hat{Y}} = \bar{Y}$

Problem 2. Decomposition of TSS
- Show that for OLS estimation $Y_i = b_1 + b_2 X_i + e_i$ of simple linear regression $Y_i = \beta_1 + \beta_2 X_i + u_i$ is always true $TSS = ESS + RSS$.
- Show that this property is equivalent to $\mathrm{Var}(Y_i) = \mathrm{Var}(\hat{Y}_i) + \mathrm{Var}(e_i)$.
- Prove the latter formula not using decomposition of TSS.

Problem 3. Meaning of R^2.
Derive from here that for OLS estimation $Y_i = b_1 + b_2 X_i + e_i$ of simple linear regression $Y_i = \beta_1 + \beta_2 X_i + u_i$ is always true $R^2 = \dfrac{ESS}{TSS} = 1 - \dfrac{RSS}{TSS}$.

Problem 4. Show that for simple linear regression OLS is equivalent to the problem of maximization of R^2.

Problem 5. Are equalities from Problems 2-4 hold for OLS estimation $Y_i = bX_i + e_i$ of simple linear regression without constant term $Y_i = \beta X_i + u_i$?

Problem 6.. Show that for OLS estimation $Y_i = b_1 + b_2 X_i + e_i$ of simple linear regression $Y_i = \beta_1 + \beta_2 X_i + u_i$ always $\mathrm{Cov}(X, e) = 0$.

Problem 7.. Under the same conditions is the property $\mathrm{Cov}(\hat{Y}, e) = 0$ always true (where $\hat{Y}_i = b_1 + b_2 X_i$)?

Problem 8.. Derive alternative expression for the OLS estimator of the slope coefficient in simple linear regression $\hat{\beta}_2 = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$ using formulas from the lecture 2.

Problem 9.. Derive alternative expression for the OLS estimator of the slope coefficient in simple linear regression $\hat{\beta}_2 = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$ NOT using formulas from the lecture, based directly on the OLS principle.

Problem 10.. Prove formulas $\mathrm{Cov}(X, e) = 0$ and $\mathrm{Cov}(\hat{Y}, e) = 0$ using only properties of covariance and formula $\hat{\beta}_2 = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$ for the regression coefficient.

Problem 11. OLS estimation of the regression without constant.
Let the regression be

$$Y_t = \beta X_t + u_t;\ t = 1, 2, ..., T$$

where $E(u_t) = 0$; $E(u_t^2) = \sigma^2$ and $E(u_s u_t) = 0$ if $s \neq t$.
Obtain the ordinary least squares (OLS) estimator of β (derive necessary and sufficient conditions for the minima).

Problem 12. Demeaning.

For the OLS estimation of simple linear regression $Y_i = \beta_1 + \beta_2 X_i + u_i$ derive formulas for estimation of regression coefficients using transformed values of the independent $x_i = X_i - \overline{X}$ and dependent $y_i = Y_i - \overline{Y}$ variables (demeaning)?

Problem 13. Demeaning continued.
Compare the result in the Problem 9 with the answer to the Problem 8 and explain why the formulas for the regression coefficients are nearly the same.