Heteroscedasticity

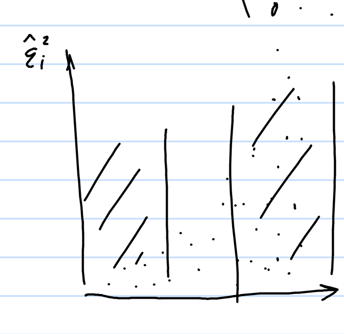
$y_i = \alpha + \beta X_i + \varepsilon_i$

$E(\varepsilon_i) = 0$

$\begin{cases} E(\varepsilon_i^2) \neq \sigma_\varepsilon^2 & \text{Heteroscedasticity} \\ E(\varepsilon_i \varepsilon_j) \neq 0,\ i \neq j & \text{Autocorrelation} \end{cases}$

$E(\varepsilon'\varepsilon) = \sigma_\varepsilon^2 \mathbb{1}_n$

$$E(\varepsilon'\varepsilon) = \begin{pmatrix} \sigma^2 x_1 & \rho & 0 & \cdots & 0 \\ \rho & \ddots & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & \ddots & \rho \\ 0 & \cdots & 0 & \rho & \sigma^2 x_n \end{pmatrix}$$



$y_i = \alpha + \beta \cdot W_i + \varepsilon_i + \gamma \cdot \hat{G}_i +$

Ex. 1 $\quad y_i = \beta X_i + \varepsilon_i$

$E(\varepsilon_i) = 0$

$\rightarrow E(\varepsilon_i^2) = a x_i^2$

$E(\varepsilon_i \varepsilon_j) = 0$

$\sum X_i^2 = n$

a) $\hat{\beta}_{OLS}$, $Var(\hat{\beta}_{OLS})$

b) $\hat{\beta}_{WLS}$, $Var(\hat{\beta}_{WLS})$

c) $Var(\hat{\beta}_{OLS}) \overset{?}{>} Var(\hat{\beta}_{WLS})$

$y_i = \beta X_i + \varepsilon_i$

$RSS = \sum \hat{\varepsilon}_i^2 = \sum (y_i - \hat{y})^2 = \sum (y_i - \hat{\beta} x_i)^2 \rightarrow \min\limits_{\hat{\beta}}$

$\frac{\partial RSS}{\partial \hat{\beta}} = -2 \sum x_i (y_i - \hat{\beta} x_i) = 0$

$\sum x_i y_i = \hat{\beta} \cdot \sum x_i^2$

$\hat{\beta}_{OLS} = \frac{\sum x_i y_i}{\sum x_i^2}$

$E(\hat{\beta}_{OLS}) = E\left(\frac{\sum x_i(\beta x_i + \varepsilon_i)}{\sum x_i^2}\right) =$

$\frac{\sum(\beta x_i^2 + x_i E(\varepsilon_i)^{=0})}{\sum x_i^2} = \beta$

$Var(\hat{\beta}_{OLS}) = E((\hat{\beta} - E\hat{\beta})^2) \ominus$

$\left\{\begin{array}{l} Var(X) = E(X^2) - E^2(X) \\ Cov(X, Y) = E((X - E(X))(Y - E(Y))) \\ Cov(X, Y) = E(XY) - E(X)E(Y) \end{array}\right\}$

$\ominus\ E\left(\frac{\sum x_i y_i}{\sum x_i^2} - \beta\right)^2 = E\left(\frac{\sum x_i(\beta x_i + \varepsilon_i)}{\sum x_i^2} - \beta\right)^2 =$

$E\left(\cancel{\beta \frac{\sum x_i^2}{\sum x_i^2}} + \frac{\sum x_i \varepsilon_i}{\sum x_i^2} \cancel{- \beta}\right)^2 =$

$= \left(\frac{1}{\sum x_i^2}\right)^2 E\left(\sum x_i \varepsilon_i\right)^2 =$

$\left\{ \left(\sum x_i\right)^2 = \sum x_i^2 + \sum_{i \neq j} x_i x_j \right\}$

$= \left(\frac{1}{\sum x_i^2}\right)^2 E\left(\sum x_i^2 \varepsilon_i^2 + \sum x_i x_j \varepsilon_i \varepsilon_j\right)$

$= \frac{1}{(\sum x_i^2)^2} \left(\sum x_i^2 E(\varepsilon_i^2)^{= a x_i^2} + \sum x_i x_j E(\varepsilon_i \varepsilon_j)^{0}\right) =$

$= \frac{a \sum x_i^4}{(\sum x_i^2)^2} = \frac{a \sum x_i^4}{n^2}$

b) $y_i = \beta x_i + \varepsilon_i \,|\, x_i \quad \varepsilon_i \sim N(0, a x_i^2)$

$\frac{y_i}{x_i} = \beta + \frac{\varepsilon_i}{x_i} \qquad \frac{\varepsilon_i}{x_i} = u_i \sim N(0, a)$

$\hat{\beta}_{WLS} = \frac{\sum y_i / x_i}{n}$

$E(\hat{\beta}_{WLS}) = \frac{1}{n} E\left(\sum \frac{\beta x_i + \varepsilon_i}{x_i}\right) =$

$\frac{1}{n}\left(\sum \beta + \sum \frac{E\varepsilon_i = 0}{x_i}\right) = \frac{\beta n}{n} = \beta$

$Var(\hat{\beta}_{WLS}) = E((\hat{\beta} - E\hat{\beta})^2) =$

$E\left(\frac{\sum y_i / x_i}{n} - \beta\right)^2 =$

$E\left(\frac{1}{n} \sum \left(\cancel{\frac{\beta x_i}{x_i}} + \frac{\varepsilon_i}{x_i}\right) \cancel{- \beta}\right)^2 =$

$E\left(\frac{1}{n} \sum \frac{\varepsilon_i}{x_i}\right)^2 = \frac{1}{n^2} E\left(\sum \frac{\varepsilon_i^2}{x_i^2} + \sum \frac{\varepsilon_i \varepsilon_j}{x_i x_j}\right) =$

$= \frac{1}{n^2} \sum \frac{E(\varepsilon_i^2)^{= a x_i^2}}{x_i^2} + \frac{1}{n^2} \sum \frac{E(\varepsilon_i \varepsilon_j)^{= 0}}{x_i x_j} =$

$= \frac{1}{n^2} \sum a = \frac{a}{n}$

c) $\frac{Var(\hat{\beta}_{OLS})}{Var(\hat{\beta}_{WLS})} = \frac{a \sum x_i^4 / n^2}{a/n} = \frac{\sum x_i^4}{n} \geq 1$

$\sum x_i^2 = n$

$\sum (x_i^2 \cdot t - 1)^2 \geq 0 \quad \forall t$

$\frac{D}{4} = n^2 - n \sum x_i^4 \leq 0$

$\sum x_i^4 \geq n$

$Var(\varepsilon_i) = a x_i^2$

$y_i = \beta_0 + \beta_1 x_{1i} + \ldots + \beta_k x_{ki} + \varepsilon_i \,\Big|\, : \sqrt{f(\cdot)}$

$Var(\varepsilon_i) = a f(x_1, \ldots, x_k)$

Unfeasible WLS

Goldfeld-Quandt Test

Ass.: $\sigma_{\varepsilon_i}^2 \sim X_i$

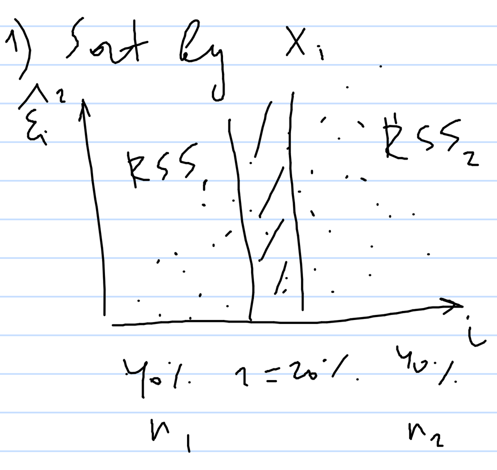
$\varepsilon_i \sim N(0, \sigma_{\varepsilon_i}^2)$

Works with small samples

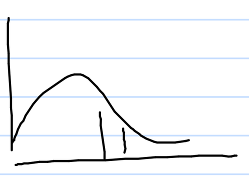
H_0: ε_i - homosc.

H_a: ε_i - heterosc. (prop. X_i)

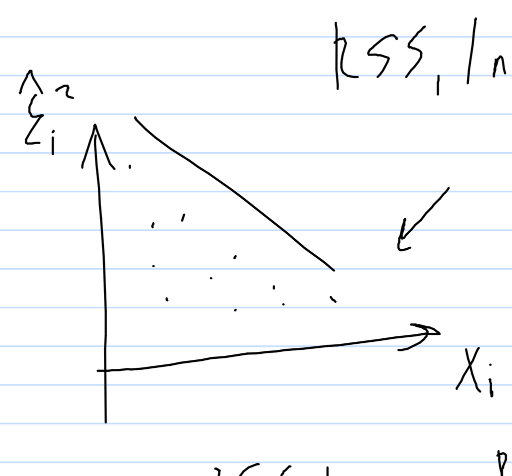
1) Sort by X_i



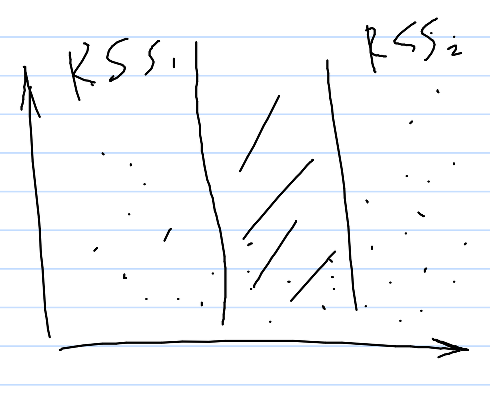
2) Drop 20% middle obs.



3) $F = \frac{RSS_2 / n_2 - k}{RSS_1 / n_1 - k} \sim F(n_2 - k; n_1 - k)$



F^{-1}



White Test

No assumptions about $\sigma_{\varepsilon_i}^2$

Asymptotic

H_0: ε_i — homoscedasticity

H_a: ε_i — heteroscedasticity

aux: $\hat{\varepsilon}_i^2 \,|\, X_i, X_i^2, X_i X_j \Rightarrow R^2_{aux}$

$R^2_{aux}\, n \sim \chi^2_p \qquad p$ - # reg in aux