1) Equation (teoriotical) $y_i = \alpha + \beta x_i + \varepsilon_i$

2) Assumptions : - <u>deterministic</u> / stochastic

$E(\varepsilon_i)$ $\quad$ $E(\varepsilon_i | X)$

- classic linear reg. assumptions

$$\left[E(\varepsilon_i) = 0 \qquad \underset{\displaystyle Var(\varepsilon_i^2) = E(\varepsilon_i^2) - E^2(\varepsilon_i)}{\underset{\|}{E(\varepsilon_i^2) = \sigma^2}} \qquad \underset{\displaystyle \underset{\|}{Cov(\varepsilon_i \varepsilon_j)}}{\underset{\|}{E(\varepsilon_i \varepsilon_j) = 0}} \right]$$

$$Cov(\varepsilon_i \varepsilon_j) = E(\varepsilon_i \varepsilon_j) - E(\varepsilon_i) \cdot E(\varepsilon_j)$$

3) Method : OLS (WLS, IV, MLE)

4) Properties

$$\beta \sum (x_i - \bar{x})^2 + \sum (x_i - \bar{x})\varepsilon_i - \bar{\varepsilon} \sum_i (x_i - \bar{x})$$

$$\overset{\parallel}{} \qquad\qquad\qquad\qquad\qquad \overset{\parallel}{0}$$

$$\sum (x_i - \bar{x})(\beta x_i + \varepsilon_i - \beta \bar{x}_i - \bar{\varepsilon}_i)$$

$$\parallel$$

$$\hat{\beta} = \frac{Cov(x,y)}{Var(x)} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \beta + \frac{\sum (x_i - \bar{x})\varepsilon_i}{\sum (x_i - \bar{x})^2} =$$

$$a_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \qquad = \beta \sum \left(\frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \right) \varepsilon_i =$$

$$= \beta + \sum a_i \varepsilon_i$$

Problem 1. Properties of a_i

a) $\sum a_i = 0$ $\qquad$ d~f

b) $\sum a_i^2 = \frac{1}{\sum (x_i - \bar{x})^2}$

$$\sum \left(\frac{x - \bar{x}}{\sum (x_i - \bar{x})^2} \right)^2 = \frac{\sum (x_i - \bar{x})^2}{\left(\sum (x_i - \bar{x})^2\right)^2} = \frac{1}{\sum (x_i - \bar{x})^2}$$

c) $\sum a_i \cdot x_i = 1$

$$\sum \frac{(x_i - \bar{x})x_i}{\sum (x_i - \bar{x})^2} = \frac{1}{\sum (x_i - \bar{x})^2} \left(\sum (x_i - \bar{x})x_i - \underbrace{\sum (x_i - \bar{x})\bar{x}}_{=0} \right) = 1$$

Problem 2. $\hat{\beta} = \beta + \Sigma a_i \varepsilon_i$

$$E(\hat{\beta}) = \beta + E(\Sigma a_i \varepsilon_i) =$$

$$\Sigma a_i \underbrace{E(\varepsilon_i)}_{=0} = 0$$

Problem 3.

$$\hat{\beta} = \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(X, \alpha + \beta X + \varepsilon)}{Var(X)} =$$

$$\frac{\overset{\to 0}{Cov(X,\alpha)} + Cov(X, \beta X) + Cov(X, \varepsilon)}{Var(X)} = \beta + \frac{Cov(X,\varepsilon)}{Var(X)}$$

$$E(\hat{\beta}) = \beta + E\left(\frac{Cov(X,\varepsilon)}{Var(X)}\right) =$$

$$\beta + \frac{Cov(X, E\varepsilon)}{Var(X)} = \beta$$

Problem 3. $y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$

$$\| b - Ax \|_2^2$$

$$\Downarrow$$

$$A^T A x = A^T b$$ - normal equation

$\hat{\varepsilon}$ normal to span(A)

(orthogonal

$$\| y - X \hat{\beta} \|_2^2$$

$$X' X \beta = X^T y$$ - normal equation

$$y_i = \alpha + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$$

$$y = \hat{\alpha} + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$$

$\mathrm{Cov}(x_1, -)$; $\mathrm{Cov}(x_2, -)$:

$$\begin{cases} \hat{\beta}_1 \mathrm{Var}(x_1) + \hat{\beta}_2 \mathrm{Cov}(x_1, x_2) = \mathrm{Cov}(x_1, y) \\ \hat{\beta}_1 \mathrm{Cov}(x_1, x_2) + \hat{\beta}_2 \mathrm{Var}(x_2) = \mathrm{Cov}(x_2, y) \end{cases}$$

$$\begin{cases} \hat{\beta}_1 \, Var(X_1) + \hat{\beta}_2 \, Cov(X_1, X_2) = Cov(X_1, y) \\ \hat{\beta}_1 \, Cov(X_1, X_2) + \hat{\beta}_2 \, Var(X_2) = Cov(X_2, y) \end{cases}$$

$$\left[\begin{array}{cc|c} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, y) \\ Cov(X_1, X_2) & Var(X_2) & Cov(X_2, y) \end{array}\right]$$

$$\hat{\beta}_1 = \frac{\Delta_1}{\Delta} = \frac{Cov(X_1, y)\, Var(X_2) - Cov(X_1, X_2)\, Cov(X_2, y)}{Var(X_1) \cdot Var(X_2) - [Cov(X_1, X_2)]^2}$$

$$\hat{\beta}_2 = \frac{\Delta_2}{\Delta}$$

$$\hat{\beta} = (X'X)^{-1} X'y$$

F-tests for linear restrictions

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$$

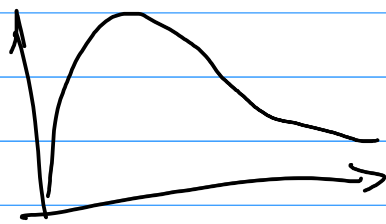
H_0: $\beta_1 = \beta_2 = \beta_3 = 0 \Rightarrow y_R$

H_a: one of restrictions isn't valid $\Rightarrow y_{UR}$

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR}/(n-k)} \overset{H_0}{\sim} F(q, n-k)$$

q: # of restrictions

k: # est. coefficient



$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1-R^2_{UR})/(n-k)}$$

$RSS_R \overset{?}{>} RSS_{UR}$

F-tests for linear restrictions

$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$

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# of restrictions

F - test
Goodness-of-fit

$H_0: \beta_2 = \cdots = \beta_k = 0 \qquad y_i = \beta_1 + \varepsilon_i$

$H_a: \exists i : \beta_i \neq 0$

$$F = \frac{(TSS - RSS)/k-1}{RSS/n-k} = \frac{ESS/k-1}{RSS/n-k}$$

$k = 1 \Rightarrow F = t^2$

$$R^2 = 1 - \frac{RSS}{TSS}$$

$$R^2_{adj.} = \bar{R}^2 = 1 - \frac{RSS/(n-k)}{TSS/(n-1)}$$