1) Equation (teorethic) $y_i = \alpha + \beta x_i + \varepsilon_i$

2) Assumptions: - deterministic / stochastic

$E(\varepsilon_i) = 0$ $\quad E(\varepsilon_i | X) = 0$

- classic lin. regression

$E(\varepsilon_i) = 0$ $\quad E(\varepsilon_i^2) = const$ $\quad E(\varepsilon_i \varepsilon_j) = 0$

$E(\varepsilon_i^2) = Var(\varepsilon_i) = E(\varepsilon_i^2) - E^2(\varepsilon_i)$

$E(\varepsilon_i \varepsilon_j) = Cov(\varepsilon_i, \varepsilon_j)$

3) Method (OLS, WLS(GLS), IV; MLE)

4) Properties

$Cov(X,Y) / Var(X)$

$$\hat{\beta} \overset{||}{=} \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2} = \beta + \Sigma \alpha_i \cdot \varepsilon_i \qquad \alpha_i = \frac{x_i - \bar{x}}{\Sigma(x_i - \bar{x})^2}$$

$$\bar{\varepsilon} \overset{||}{=} \Sigma(Y_i - Y) \quad 0$$

$$\overset{||}{=} \frac{\Sigma(x_i - \bar{x})(\beta x_i + \varepsilon_i - \beta \bar{x} - \bar{\varepsilon})}{\Sigma(x_i - \bar{x})^2} = \frac{\beta \Sigma(x_i - \bar{x})(x_i - \bar{x}) + \Sigma(x_i - \bar{x})\varepsilon_i - \Sigma(x_i - \bar{x})\bar{\varepsilon}}{\Sigma(x_i - \bar{x})^2} =$$

$$= \beta + \frac{\Sigma(x_i - \bar{x})\varepsilon_i}{\Sigma(x_i - \bar{x})^2} = \beta + \Sigma \underbrace{\left(\frac{x_i - \bar{x}}{\Sigma(x_i - \bar{x})^2}\right)}_{a_i} \varepsilon_i$$

Problem 1.

a) $\Sigma a_i = 0$ $\qquad$ $\delta \sim \delta$

b) $\Sigma a_i^2 = \frac{1}{\Sigma(x_i - \bar{x})^2}$

$$\Sigma \left(\frac{x_i - \bar{x}}{\Sigma(x_i - \bar{x})^2}\right)^2 = \frac{\Sigma(x_i - \bar{x})^2}{\left(\Sigma(x_i - \bar{x})^2\right)^2} = \frac{1}{\Sigma(x_i - \bar{x})^2}$$

c) $\Sigma a_i x_i = 1$

$$\Sigma\left(\frac{x_i - \bar{x}}{\Sigma(x_i - \bar{x})^2} x_i\right) = \frac{1}{\Sigma(x_i - \bar{x})^2}\left(\Sigma(x_i - \bar{x})x_i - \Sigma(x_i - \bar{x})\bar{x}\right) = 1$$

$$\Sigma(x_i - \bar{x})x_i - \Sigma(x_i - \bar{x})\bar{x} \;\overset{||}{=}\; \Sigma\left((x_i - \bar{x})x_i - (x_i - \bar{x})\bar{x}\right) \;\overset{||}{=}\; \Sigma(x_i - \bar{x})^2$$

Problem 2

$$\hat{\beta} = \beta + \sum a_i \varepsilon_i$$

$$E(\hat{\beta}) = \beta + E\left(\sum a_i \varepsilon_i\right) = \beta + \sum \overset{0}{\overset{\shortparallel}{E(\varepsilon_i)}} \cdot a_i = \beta$$

Problem 3 + 4

$$\hat{\beta} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} = \frac{\mathrm{Cov}(X, \alpha + \beta X + \varepsilon)}{\mathrm{Var}(X)} =$$

$$\frac{\overset{0}{\overset{\shortparallel}{\mathrm{cov}(X,\alpha)}}}{\mathrm{Var}(X)} + \beta + \frac{\mathrm{Cov}(X,\varepsilon)}{\mathrm{Var}(X)}$$

$$E(\hat{\beta}) = \beta + E\left(\frac{\mathrm{Cov}(X,\varepsilon)}{\mathrm{Var}(X)}\right) =$$

$$\beta + \frac{1}{\mathrm{Var}(X)} \mathrm{Cov}(\overset{X}{\overset{\shortparallel}{EX}}, \overset{0}{\overset{\shortparallel}{E\varepsilon}}) = \beta$$

Problem 5. Normal equations

$\| b - Ax \|_2^2$

$A^T A x = A^T b$ — normal equation

residuals normal

(orthogonal) to span(A)

$X^T X \hat{\beta} = X^T y$ — normal equation

$y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$

$Cov(X_1, -)$ ; $Cov(X_2, -)$ :

$$\begin{cases} \hat{\beta}_1 Var(X_1) + \hat{\beta}_2 Cov(X_1, X_2) = Cov(X_1, y) \\ \hat{\beta}_1 Cov(X_1, X_2) + \hat{\beta}_2 Var(X_1, X_2) = Cov(X_2, y) \end{cases}$$

$$
\begin{cases}
\hat{\beta}_1 \, Var(X_1) + \hat{\beta}_2 \, Cov(X_1, X_2) = Cov(X_1, y) \\
\hat{\beta}_1 \, Cov(X_1, X_2) + \hat{\beta}_2 \, Var(X_1, X_2) = Cov(X_2, y)
\end{cases}
$$

$\Delta_1$

$$
\left[\begin{array}{cc|c}
Var(X_1) & Cov(X_1, X_2) & Cov(X_1, y) \\
Cov(X_1, X_2) & Var(X_2) & Cov(X_2, y)
\end{array}\right]
$$

$\Delta_2$

$$
\hat{\beta}_1 = \frac{\Delta_1}{\Delta} = \frac{Cov(X_1, y)\,Var(X_2) - Cov(X_2, y)\,Var(X_2)}{Var(X_1)\,Var(X_2) - (Cov(X_1, X_2))^2}
$$

$$
\hat{\beta}_2 = \frac{\Delta_2}{\Delta} = -\,//\,-
$$

$$
\hat{\beta} = (X'X)^{-1} X' y
$$

F_test for linear restriction

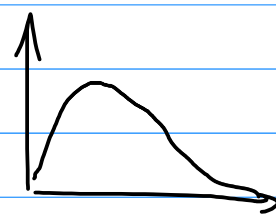
$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{ESS}$$

$H_0: \beta_1 = \beta_2 = \beta_3 = 0 \iff y^R$

$H_a$: one of restrictions is not valid $\Rightarrow y^{UR}$

# lin restr.

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR}/n-k} \sim F(q, n-k)$$



# est. coefficients (UR)

F-test Goodness-of-fit

($\beta_1$ - const)

$H_0: \beta_2 = \dots = \beta_k = 0$

$y^R = \beta_1 + e_i$

$H_a: \exists i \; \beta_i \neq 0$

$$F = \frac{(TSS - RSS)/k-1}{RSS/n-k} = \frac{ESS/k-1}{RSS/n-k} \sim F(k-1, n-k)$$

$k = 2: \quad F = t^2$

P7. $$R^2_{adj} = \bar{R}^2 = 1 - \frac{RSS/n-k}{TSS/n-1}$$

P8. $$\bar{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-k}$$

$$1 - R^2 = \frac{RSS}{TSS}$$

$$\bar{R}^2 = 1 - \frac{(RSS/TSS)/(n-k)}{1/n-1} =$$

$$= 1 - \frac{(1-R^2)/n-k}{1/n-1}$$

pg. $\bar{R}^2 = R^2 - \underbrace{\frac{k-1}{n-k}(1-R^2)}_{} =$

$$= \frac{ESS}{TSS} - \frac{k-1}{n-k} \cdot \frac{ESS}{TSS}$$

$$\bar{R}^2 = 1 - (1-R^2)\frac{n-1}{n-k} =$$

$$= \frac{n-k-(1-R)\ n-1}{n-k} =$$

$$= \ldots$$