MIT Thesis Template in Overleaf

by

Tim Beaver

Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

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Abstract

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Acknowledgments

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Chapter 1

Introduction

In this thesis, I will provide a complete treatment of occluder-based imaging. The thesis is split into two main sections. The first section will be focus on the question of designing *optimal* occluders. The natural application of this kind of research is generally within context of designer mask-based cameras. The second section will be about methods for exploiting *accidental cameras*, which means using occlusion provided by objects that happen to be in the world to image hidden scenes.

Although there is obviously a close relationship between these two areas of research, and I will use a common framework for analyzing both they are nevertheless distinct, and which one the reader is interested in will depend on the application they are interested in.

I will begin my thesis by describing in detail the analytic framework that I use for both, and introducing the assumptions I use and the optical model that underlies them. My hope is that even without much or any background in optics, all my readers will have no trouble understanding the model and assumptions I use, and my mathematically-savvy readers will have no trouble understanding the conclusions that follow from them.

I highly recommend that *any* reader begins by reading the chapter that follows before proceeding to either of the two main sections (on optimal occluders and on accidental cameras). Without this background, I expect that it will be difficult to understand what I'm saying.

Chapter 2

The Model and Assumptions

2.1 Ray Optics and BRDFs

Throughout my thesis, unless I say otherwise, I'll be using the *ray optics model* of light. This means that, as a convenient abstraction, I'll be assuming that light moves in a straight line through the air, can be bent when it hits a different light-propagating medium (like a lens), and may be absorbed or reflected by the materials it hits (like a wall). Moreover, I'll be generally assuming that light intensity is additive, meaning that two rays of light hitting the same point will generate an intensity equal to the intensity that would be generated the sum of each individual ray. This corresponds to assuming that the light we care about is incoherent and as such won't interfere with itself—a reasonable assumption when the light in question is coming from the sun or from a commercial electric light. Finally, using the ray optics model means that I'll be ignoring the effects of diffraction, which is reasonable when modeling the behavior of visible-wavelength incoherent light hitting macroscopic structures. If this paragraph was hard to understand, don't worry—just take it to mean that light travels in straight lines, bounces off stuff it hits, and generally does what you intuitively think it should.

What happens when the light hits an opaque surface, according to this model? Some of it will be absorbed, and some reflected. How much of it is reflected, and in what directions, is described by the *bidirectional reflectance distribution function*, or

BRDF, of the surface. The BRDF is a function from *incident*, or incoming, angle of the light to the outgoing angle of the light. It's best explained with two example BRDFs, which happen to describe the many of the surfaces we care about.

The first example BRDF is called the *specular* BRDF, and surfaces that have this BRDF are called specular surfaces. The typical example of a specular surface is a mirror. The specular BRDF takes the incident light and flips it across the surface normal. How can we describe this mathematically? We can describe it using a function of two arguments, the first being the angle of the incident light, θ_{in} and the second being the angle of the outgoing light, θ_{out}. Each angle is given from the surface normal. [1]Here, the specular BRDF we want is:

$$f_{\text{specular}}(\theta_{\text{in}}, \theta_{\text{out}}) = \begin{cases} \rho & \text{if} \theta_{\text{in}} = \pi - \theta_{\text{out}} \\ 0 & \text{otherwise} \end{cases}$$

The ρ here is a constant that determines the overall brightness of the surface in question—how much of the light is actually reflected rather than absorbed or transmitted. For example, for a mirror, ρ might be almost 1, meaning that almost all the light is reflected, but for a window where most of the light is transmitted rather than reflected, ρ might be much smaller, like 0.01 or 0.001.

The second example BRDF is called the *Lambertian* BRDF, after the 18th-century physicist Johann Heinrich Lambert. Surfaces that have this BRDF are often called *Lambertian*, *matte*, or *diffuse* surfaces, and I will use these terms interchangeably in this thesis. Intuitively, this BRDF takes the incoming light and scatters it "equally in all directions." Formally, here is the BRDF in question:

$$f_{\text{Lambertian}}(\theta_{\text{in}}, \theta_{\text{out}}) = \rho \cos(\theta_{\text{out}})$$

[1]If you're already familiar with the concept of BRDFs, you might be confused by this, since BRDFs are often described as functions of four real values. This confusion comes from the fact that for a 2D surface that lives in three dimensions, the angle of a light-ray from that surface is given by a 2D angle, which requires two real numbers to describe (one for the azimuth angle and one for the zenith angle). This is a detail that becomes unimportant if you treat each of the two arguments I'm describing as 2D angles, with equality between 2D angles achieved when both their azimuth angle and their zenith angle match. Here, to keep things simple, I'm implicitly assuming a 1D surface that lives in a 2D world, so angles are described by a single real number.

Once again ρ is a constant that determines the overall surface brightess. Note that the Lambertian BRDF is completely independent of the angle of the incident light—it scatters the light it reflects in exactly the same way no matter where the light came from.[2]

Now at this point, a careful reader may object: why did I claim that Lambertian surfaces scatter light "equally in all directions," when in reality, they scatter light in directions proportionally to that direction's cosine? Indeed, this is a major source of confusion when it comes to Lambertian surfaces. Google "Lambertian surface" or "Lambertian BRDF" and you will find about half your results defining it as I do, and half of them defining it instead as a perfectly constant function, depending neither on θ_{in} nor θ_{out}. This is an important confusion to resolve; I hope now to convince you beyond a doubt that my definition is the right one, and that the alternate definition of a Lambertian surface, while the objects it describes might exist in principle, in practice I've never seen one—whereas the objects my definition describes are all over the place. The walls and ceiling of the room you're sitting in, the paper you may be reading these words on, the clothes you're reading: all of these are nearly perfectly described by the Lambertian BRDF as I have defined it.

Here's where the confusion comes from. Find a sheet of paper. Lay it flat against a desk, then look at it from a few different angles. No matter what angle you look at it from, it looks equally bright.

At first, this seems it argues for the alternate definition of a Lambertian surface. After all, if you see the same amount of light coming from the sheet of paper no matter what direction you observe it from, doesn't that it mean that the amount of light it transmits is equal in all directions—that is, it doesn't depend on θ_{out}? In fact, no. Consider: depending on what angle you are looking at the sheet of paper from, the paper will take up a larger or smaller part of your field of vision. Look at the paper head-on, and it takes up a relatively large part of your field of vision; look at the paper from a very glancing angle, however, and it takes up just a sliver. And yet,

[2] For a 2D surface living in a 3D world, the Lambertian surface BRDF is exactly the same, but the cosine is of the outgoing zenith angle, and the outgoing azimuth angle doesn't matter.

no matter what the angle you observe it from is, you can still see the entire sheet of paper.

What's the upshot of this? What this means is that when you are looking at the sheet of paper from a very glancing angle, you are actually getting less total light from the paper, since the amount of light per amount-of-your-field-of-vision (sometimes called a "steradian") remains fixed, but the amount of your field of vision filled by the paper has decreased. And in fact, it has decreased by a factor of $\cos(\theta)$, where θ is your angle from the paper's normal. This is where the factor of $\cos(\theta)$ in the definition of the Lambertian BRDF comes from. Indeed, if the Lambertian BRDF sent light truly equally in all directions, as you looked at the paper from an increasingly glancing angle, the paper would appear to get *brighter*, in order to keep the total amount of light you were receiving from the paper constant. Some objects behave the opposite way, like backlit LCD screens; if you tilt them away from you, their apparent brightness will usually decrease (depending on the screen), which means that their BRDF is attenuated *faster* than $\cos(\theta_{\text{out}})$. But no object I've ever seen has a BRDF that is attenuated *slower* than $\cos(\theta_{\text{out}})$.

Most real-world surfaces lie on a spectrum somewhere between Lambertian and specular. This isn't to say that most real-world BRDFs are a linear combination of the Lambertian and specular BRDFs; an example of an object that *does* behave that way is a dirty or smudged mirror. But most objects aren't like that; they may look "shiny" or "glossy," but they don't give you a sharp-but-faint reflection, like a dirty mirror might. Rather, many glossy objects have BRDFs that send some light in all directions, but more light in directions where the outgoing angle be relatively close to the incident angle reflected across the surface normal. These BRDFs are often modeled using the "Phong" model, after the model described by Bui Tuong Phong in his PhD thesis. According to the Phong model, the extra light in the reflected directions (also called the "specular highlights") fall off polynomially with the the dot product of the outgoing angle with the reflected incident angle. The degree of that polynomial depends on the how shiny or dull the surfaces, with higher-degree polynomials yielding a smaller and more focused specular highlight.

In this thesis, the main focus will be on Lambertian surfaces. When I do consider the possibility of specular of Phong surfaces, I'll go into more detail about what exactly the BRDF model I'm using is at that time. So for the time being, let's consider what can happen using the simplest model that nevertheless describes much of reality very well: a 2D world of Lambertian surfaces.

2.2 The Far-Field Assumption

2.2.1 A point light source and a nearby surface

Let's suppose we live in a 2D world of Lambertian surfaces and diffuse light sources. (When I say a "diffuse" light source, I mean that the light source scatters light equally in all directions. Confusingly, this isn't quite the same thing as a "diffuse" surface—which is another way of saying a Lambertian surface, which actually doesn't quite scatter light equally in all directions, as I explained in the previous section—but that's how these terms are used.) Consider a point light source suspended at $(0, y_p)$, with a Lambertian surface at $y = 0$ (see Figure **??**). What pattern of illumination can we expect to see on the surface?

The way we proceed with this analysis is to discretize the surface into many small chunks, and then to consider what fraction of the light radiating out of the point light source is hitting any single given small chunk of the surface. We assume each chunk is small enough that its luminance is constant across the chunk. Asking what fraction of light radiating out of the point light source hitting any given chunk is equivalent to asking what angle over the light source is subtended by that chunk, and then dividing that angle by 2π.

Supposing that the chunk extends from $(x_c, 0)$ to $(x_c + dx, 0)$, trigonometry tells us that θ_c, the angle subtended by the chunk, is given by:

$$\theta_c = \tan^{-1}\left(\frac{x_c + dx}{y_p}\right) - \tan^{-1}\left(\frac{x_c}{y_p}\right)$$

What happens as we consider increasingly smaller and smaller chunks dx? The definition of the derivative tells us that $\lim_{dx\to 0}\theta_c = dx \cdot \frac{d}{dx_c}(\tan^{-1}(\frac{x_c+dx}{y_p})) = dx(y_p/(x^2+y_p^2))$. Thus the luminance of a chunk on the surface, assuming that the point source had a luminance of 1, would be $dx(y_p/(2\pi(x^2+y_p^2)))$.[3]We can say that the continuous illumination function of the surface $I(x)$ is the following:

$$I(x) = \frac{y_p}{2\pi(x^2+y_p^2)}$$

This simple formula captures a lot of interesting phenomena. Consider for instance that we take $x = 0$, meaning we consider the illumination only of the closest point on the surface to the point source. The formula tells us then that the illumination of that point goes as $1/y_p$, meaning that it scales inversely with that point's distance from the point source. Now consider fixing $y_p = 1$ and varying x. This gives us a illumination pattern that scales with $1/(1+x^2)$, a nice "hump" pattern. The closer the surface is to the point source (meaning a smaller y), the narrower the hump will be. (See Fig. **??**) Also note that no matter what y_p is, we have:

$$\int_{-\infty}^{\infty} \frac{y_p}{2\pi(x^2+y_p^2)} dx = 1/2$$

It stands to reason that this is true, because no matter how far the surface is from the point source, if the surface is infinitely broad, exactly half the light from the point source will hit the surface. Additionally, for reference, I'll provide here the illumination function for the equivalent situation in three dimensions: a point source of luminance 1 suspended at $(0, 0, z_p)$, and a plane at $z = 0$. Then, the illumination function $I(x, y)$ can be derived in much the same way as in the two-dimensional case. This function is given by:

[3]Readers who are unfamiliar with terms like "luminance" may be confused by a subtle distinction between what I mean by "luminance" versus "illumination pattern" or "brightness." When I talk about the "luminance" of something, I'm referring to the absolute amount of light that thing emits. In contrast, when I use the term "brightness" or "illumination pattern," I'm referring to the light emission *density* of that thing. When you look at a surface, that surface's apparent brightness is proportional to how much light it emits per area of your vision it occupies. So whereas the *luminance* of a small surface chunk in the example given above would be $dx(y/(2\pi(x^2+y^2)))$, to get the *brightness* of that same surface chunk we'd want to divide by its area; hence its brightness would be $y/(2\pi(x^2+y^2))$. This makes sense: the apparent brightness of a surface shouldn't depend on how finely you choose to discretize it!

$$I(x,y) = \frac{z_p}{4\pi(x^2 + y^2 + z_p^2)^{3/2}}$$

In any case, the important thing to note at this point is that, as shown in Figure **??**, the illumination pattern becomes flatter and broader the further the point source is from the surface. This phemonenon is what we rely on when we make the "far-field assumption." The far-field assumption is that the assumption that the contribution of a point light source to a faraway surface is approximately constant across that surface. As you can see, this assumption holds as long as the size of the surface in question is much smaller than the distance of the point source to the surface; that is, if, for all relevant values of x, $x^2 \ll y_p^2$, then it follows that $I(x)$ holds a constant value of approximately $1/(2\pi y_p)$ ($1/(4\pi z_p^2)$ in three dimensions), assuming the point source has a luminance of 1.

Because of the quadratic dependence on x and y_p in Eq. **??**, the far-field assumption yields a reasonable approximation even when the difference between x and y_p isn't enormous; for example, if you hold a diffuse light source three meters away from the center of a flat surface two meters in diameter, the brightness of that surface won't vary by more than about 16% (compare $1/9^{3/2}$ to $1/10^{3/2}$). The far-field assumption gets relied on very heavily, both in my research and in work by others, and admittedly the reason for that isn't that it's always a hugely robust assumption to real-world situations (after all, depending on the application, sometimes 16% can matter a lot!). The reason, rather, is that it's an extremely *convenient* assumption. For the time being I'll leave it at that, but in later sections we will see that tolerating the far-field assumption allows us to solve quite a few different optics problems in closed form, or reduce them to easy rather than difficult problems of linear algebra. When I can I will extend my analysis to cases where the far-field assumption cannot be made.

2.3 The Standard Setup

In this section, I will briefly describe what I call "the standard setup," and introduce some terminology that I will use throughout the dissertation. The simplest version of the standard setup is shown in Fig **??**: three parallel frames in flatland, with the "intermediate frame" halfway in between the scene and the observation plane. The presumption is that the observation is a known quantity, and we'd like to infer what's in the scene. Depending on the details of the problem, the intermediate frame may also be a known quantity, or its form may be unknown. In any case, we'd like to see how much we're able to infer about the scene from the observation thanks to (or despite!) the presence of the intermediate frame.

The term "intermediate frame" is left deliberately vague. In an ordinary camera, the intermediate frame would be a lens. In most of this dissertation, I'll be considering intermediate frames that don't directly focus the light from the scene like a lens would, but partially occlude the scene. In principle, there are any number of other realistic intermediate frames.

Of course, there are many other ways to relax the standard setup to make it richer or more realistic. The intermediate frame need not be halfway in between the observation plane; the three frames need not be parallel to each other; the scene need not be planar. And, of course, the real world isn't flatland! But the standard setup is a great starting point for any optical analysis. Colloquially, the form that analysis might take is that an intermediate frame is better for imaging with if, given its presence, the observation tells us more about the scene.

2.3.1 The Transfer Matrix

Another critical concept in my dissertation is the *transfer matrix.* The transfer matrix is a matrix that describes the action of an intermediate frame on the scene to create the observation. To be more precise, suppose we approximate the scene by a vector $\vec{x}$, where each entry of that vector gives the illumination of a single chunk of the scene. Suppose that we approximate the observation plane in the same way with a vector

$\vec{y}$. Then, the transfer matrix, A, will be whichever matrix satisfies $\vec{y} = A\vec{x}$ for all possible pairs $(\vec{x}, \vec{y})$.

How do we know that such a matrix even exists for all intermediate frames? Well, if we accept the assumptions implicit the ray-optics model described in Sec. 2.1—that is, we ignore the effects of diffraction and assume that light is incoherent—then what you observe should be a linear function of the presence or absence of light sources. That means that we call what you see if light a is on $f(a)$, and what you see if light b is on $f(b)$, then what you see when both lights are on, $f(a+b)$, should be the sum of what you saw in either case, $f(a) + f(b)$. You can try this at home! If you have a room which is perfectly dark when the lightswitch is off, see what the room looks like when you turn the lights are on versus when you turn on a lamp or flashlight. The brightness of every part of your room when both the lightswitch and lamp are on should roughly correspond to the sum of how bright there were when each were on individually.[4]The fact that this works in most real-world settings tells us that the assumptions of the ray-optics model aren't leading us too far astray.

Because we're assuming that combining light sources behaves linearly, most real-world objects we can put in between a scene and an observation plane should be representable by a transfer matrix A. But what do matrices like these actually look like? Well, they can look like a variety of different things; Figure **??** shows the action of a variety of example intermediate frames, with an example transfer matrix corresponding to one of them.

So suppose we have a scene x, an observation y, and a transfer matrix A that represents how the intermediate frame distorts the scene to produce the observation. If $y = Ax$, and we know A and y, what transfer matrices A are best? In the absence of noise, any full-rank transfer matrix A should allow us in principle to perfectly

[4]If you do this, what you see may not "feel" like it actually correspond to the sum of the two room brightnesses. For example, if you have two lights that both illuminate your room equally well, turning both on may not once may not feel like it's giving you a room that's "twice as bright." Make no mistake, though—that's not the fault of the ray optics model, that's the fault of your lying eyes! "Perceived brightness" is a bit of a slippery concept, but it isn't linear in the actual amount of light hitting your retina. In the same way that a 70 dB sound (vacuum cleaner) doesn't sound like it's "half as loud" as an 80 dB sound (garbage disposal), 100 lumens doesn't look "half as bright" as 200 lumens.

reconstruct $x = A^{-1}y$. That makes the question of which transfer matrix not very interesting—it's a multi-way tie between all full-rank transfer matrices, which make up the vast majority of possible transfer matrices.

2.3.2 Noise

Of course, it's unrealistic to expect no noise. Every real-world imaging setting will have at least some noise, and in any case it's the presence of noise that makes the problem interesting, and lets us distinguish between better and worse transfer matrices, even if both matrices have the same rank.

Adding noise, our new equation becomes:

$$y = Ax + \eta$$

where η is another vector representing random noise. Now that we have introduced a random variable into our equation, we will need to provide a probability distribution not only for η (to describe how the noise is distributed) but also for x.

Let's start by discussing the probability distribution of the scene vector, x. The simplest model to begin with is to have each entry of the scene be independent and identically distributed (IID), and drawn from a Gaussian. For example, suppose that each entry of the scene vector x was independently drawn from a Gaussian with a mean of μ and a standard deviation of σ. To describe this situation, we can write:

$$x \sim \mathcal{N}(\mu I, \sigma^2)$$

Before we can proceed with this model, there are a few problems for us to worry about. The first is possible negative entries. Real scenes don't cast negative light! To solve this problem, we take $\mu \gg \sigma$. That way, the probability of negative entries will be vanishingly small. A vanishingly small chance of a negative entry is good enough for us; it means that our model's distance from the real world due to this issue (where negative entries are impossible) is also vanishingly small.

The next problem is subtler: recall that x is a discrete vector, but it is meant to represent a continuous scene of fixed size. We haven't yet talked about the number of entries in x, which we'll call n. The variable n controls how finely we discretize the scene x. Ideally, choosing n to be larger will mean that our discrete representation of the true continuous scene will be more faithful (though perhaps at a computational cost). And we might also hope that once n gets large enough, that's a close enough to the real scene that increasing it further won't make the model noticeably better. That's not such an unrealistic expectation; after all, if you're reading these words on a laptop screen, you're probably looking at a discrete array of a couple thousand by a thousand pixels, and that's plenty enough to give you the impression of a "continuous image" on your screen. Tripling the number of pixels on your laptop without increasing the size of your screen probably won't improve your impression of how "continuous" your screen looks by much, unless you're very good at noticing this kind of thing.

We'll talk more about exactly what we mean by this concept later (i.e, how finely we need to discretize the scene before we consider that to be "good enough"). For the time being, though, we have a more serious problem. The problem is this: varying n should give us representations of the true, continuous scene that are varyingly faithful. However, choosing a different value of n shouldn't qualitatively change what the scene looks like. It should always give us the closest discrete approximation possible to the true, continuous scene.

So first, we need to make sure that the total luminance of the scene (in other words, the total amount of light the scene emits) doesn't depend on n. But we said earlier that each entry of x was IID with a mean of μ, so at the moment the total luminance of the scene is $n\mu$. This means that μ is going to have to depend on n; in particular, we'll say that $\mu = J/n$, where J is a constant that represents the total luminance of the scene.

Our difficulties don't stop there, though. If each entry of x is IID and drawn from a Gaussian, then if we want the scene to be qualitatively the same independent of n, σ must also depend on n. In this case, what we mean by "qualitatively the same" is a little bit fuzzy, but we might make it formal by asking that scenes with $n = k$

be drawn from the same distribution as scenes with $n = 2k$ that are then "pixelated" by a factor of 2. (To "pixelate" a vector by a factor of k is to average groups of k contiguous pixels together—for example, if we pixelated the vector [1,2,3,4,5,6] by a factor of 2, we would be left with the vector [1.5, 3.5, 5.5].) By this definition, we would need σ to actually *increase* linearly with n in order to have the finer-discretized scenes be less-pixelated versions of the coarser ones! And this is a problem because we said that $\mu \gg \sigma$, and that needs to be true for all values of n, which will be be impossible if σ grows with n and μ shrinks with n. If this is hard to follow, see Figure **??** for an illustration of this issue.

So what's the solution? The solution is for our model of the scene to include correlations between nearby pixels. We can do this by supposing that the covariance matrix of the scene includes off-diagonal elements:

$$x \sim \mathcal{N}(Q, \sigma^2)$$

The covariance matrix Q captures the correlations between nearby pixels. In real scenes, the closer together two pixels are, the more correlated they'll become. Our model should be faithful to this as well—and in doing so, we will simultaneously create the situation we wanted before, in which scenes with different values of n look qualitatively similar to each other, just at different levels of fidelity.

To make things concrete, I'll provide an example of a scene covariance matrix, which I'll call the *exponential-decay prior*. Recall that an IID covariance matrix would just be a multiple of the identity, $Q = \frac{\theta}{n}I$, where θ is a constant in n. The exponential-decay prior is given by $\mathbf{Q} = \mathbf{F}_n^* \mathbf{D}^\star \mathbf{F}_n$, where $\mathbf{F}_n$ is the normalized DFT matrix of size n and $\mathbf{D}^\star$ is a diagonal matrix with the following entries: $d_1 = 1$, $d_i^\star = d_{n-i+1}^\star = \frac{\theta}{n}\beta^{\frac{i-1}{\lceil (n-1)/2 \rceil}}, i = 2, \ldots, \lceil (n+1)/2 \rceil$, for some frequency decay rate parameter $0 < \beta < 1$. A lower β implies a more strongly correlated scene.

That's a lot to take in at once. Here's the summary of what the above means, in English: let's say the scene has size 1. Then two pixels that are distance 1 from each other—that is, all the way across the scene from each other—will have a correlation of

β. Naturally, each pixel will also have a correlation of 1 with itself—that is, with pixels that are at distance 0 from it. We interpolate between these two points (correlation β at distance 1, correlation 1 at distance 0) using an exponential function.

It will be easier to make sense of all this if you look at Figure **??** to see for yourself what these covariance matrices look like, and what scenes generated from them look like.

2.3.3 Noise

Now that we have a model of the probability distribution over scenes, we need a model for the probability distribution over the noise. This crucially depends on what application it is we care about. We'll try and make the noise model general enough to apply well to all cases.

We distinguish between two different types of noise.

(Thermal noise): This includes noise sources that are independent of the contribution to the measurements due to the scene of interest. That means that the thing causing the noise isn't light coming from the scene; it's light coming from somewhere else (i.e. "glare"). We model it as additive Gaussian with variance W/n, where W denotes the constant net noise power and each pixel absorbs power proportional to its size, giving rise to the $1/n$ factor.

(Shot noise): This includes measurement noise that depends on the contribution due to the scene of interest. This results in additive Gaussian noise of variance $\rho \cdot \frac{J}{n}$ (proportional to the net power of light that goes through the aperture).

Thermal noise should be more important in passive non-line-of-sight imaging applications using accidental cameras (in other words, the applications described in Chapter 3) because in that application, the bulk of the light reaching the observation will generally come from sources that aren't the scene of interest, like overhead lighting or sunlight.

Shot noise should be more important in designer-camera applications using coded apertures (in other words, the applications most relevant to the considerations described in Chapter 2) because in those applications, the camera will presumably be designed in such a way as to prevent glare.

Both of these sources of noise are in reality captured by a Poisson distribution, since that's the way light behaves in real-world scenarios. Fortunately, the limit of a Poisson distribution as the number of photons gets very large is a Gaussian, which is much more easily modeled. This approximation is extremely close to reality in the applications we're interested in, which are passive-imaging applications with plenty of light. In super-low-light applications, however, such as active-imaging scenarios where the scene is in perfect darkness except for light introduced by the experimenter through a laser, this modeling issue can become important.

Now that we've established the noise model, we can finally write down the equation relating the scene to the observation.

$$y = Ax + \eta$$

$$x \sim \mathcal{N}(\mu\mathbf{1}, Q)$$

$$\eta \sim \mathcal{N}(0, (W + \rho \cdot J)/n)$$

We'll leave Q ambiguous for now, and talk about it on a case-by-case basis depending on the application. $\mu = J/n$ as described earlier. J is the total radiance of the scene, whereas W is a parameter that describes the level of glare (the scene-independent noise).

2.3.4 Mutual Information

Mutual information. The mutual information (MI) between the measurements $y_j, j \in [n]$ and the unknowns $f_i, i \in [1, n]$ of the imaging problem is given as $\mathcal{I} = \log\det\left(\frac{1}{\sigma^2}\widetilde{\mathbf{A}}\mathbf{Q}\widetilde{\mathbf{A}}^T + \mathbf{I}\right)$, where the noise variance $\sigma^2 = (W + \rho \cdot J)/n$.

2.4 High SNR, IID scene, Occluding mask

In this section, I will go into great detail about a regime that appears unphysical, but gives us a lot of insight into a variety of different important regimes. It also has important mathematical implications.

Consider the mutual information equation from the previous section:

$\mathcal{I} = \log\det\left(\frac{1}{\sigma^2}\widetilde{\mathbf{A}}\mathbf{Q}\widetilde{\mathbf{A}}^T + \mathbf{I}\right)$

Suppose we make the following assumptions:

1. The scene is IID, $Q = kI$.

2. The SNR is very good, $\sigma \ll k$, so we can ignore the identity term.

3. The intermediate frame only occludes light or lets it through; it doesn't redirect the light.

4. The far-field assumption applies. This point and the previous one let us assume that the transfer matrix $\widetilde{\mathbf{A}}$ is a multiple of a binary-valued Toeplitz matrix.

Suppose now that we want to find the best possible intermediate frame under these conditions, i.e. we want to find the best possible transfer matrix that satisfies the constraints above.

What transfer matrix maximizes the mutual information? Naturally, it will be whichever transfer matrix maximizes $\log\det\left(\frac{1}{\sigma^2}\widetilde{\mathbf{A}}\mathbf{Q}\widetilde{\mathbf{A}}^T + \mathbf{I}\right)$, which given our assumptions is equivalent to maximizing the determinant of $\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}}$. This corresponds to maximizing the product of the norms of the transfer matrix's singular values, since each eigenvalue of $\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}}$ corresponds to the norm squared of one of the eigenvalues of $\widetilde{\mathbf{A}}$.

If we further assume that $\widetilde{\mathbf{A}}$ is circulant, not just Toeplitz, then that tells us that in fact the eigenvalues and singular values of $\widetilde{\mathbf{A}}$ have the same norm. This is a convenient assumption that doesn't necessarily match reality. Under which real-world conditions will the transfer matrix actually be circulant rather than Toeplitz? This is somewhat unintuitive, but it corresponds to the scenario in which the occluding pattern of the intermediate frame repeats itself once. Once is enough! See Figure **??** for a visual explanation of why this is.

Assuming that the transfer matrix is circulant, not just Toeplitz, is tremendously convenient. It means that we can compute the mutual information between the scene and the observation extremely efficiently, since the eigenvalues λ_i are given by the Fourier transform of the first row of the transfer matrix, for which the time to compute is log-linear in n (as opposed to $O(n^3)$ for a general determinant). And since $|\lambda_i| = |\sigma_i|$, that gives us all the information we need to compute $\log\det(\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^T)$.

But realistically speaking, can we restrict ourselves just to circulant transfer matrices rather than Toeplitz ones? After all, occluders that give rise to that kind of transfer matrix make up only a very small fraction of all possible occluders. The answer is that it depends on what you want, but if you're only concerned with finding the *best* possible occluder, restricting your attention to circulant transfer matrices costs you very little. Here's why.

2.4.1 Hadamard's Bound

The point of this section is to justify the following bound on all $\{0,1\}$ matrices B. If this doesn't interest you, skip to the next section.

$$|\det B| =\leqslant 2^{-n}(n+1)^{(n+1)/2}, \tag{2.1}$$

By a *binary* matrix we mean a matrix whose elements are in one of the sets $S_{01} := \{0,1\}$ or $S_{\pm1} := \{-1,1\}$. It will be clear from the context which of these two cases is being considered. A *binary circulant* is a circulant matrix whose elements are in S_{01} or $S_{\pm1}$.

There is a natural correspondence between the integers $\{0,1,\dots 2^n-1\}$ and the binary circulant matrices of order n. If $N \in \{0,1,\dots,2^n-1\}$ has the representation

$$N = \sum_{j=0}^{n-1} 2^{n-1-j} b_j,$$

so may be written in binary as $b_0 \ldots b_{n-1}$, we associate N with $\mathrm{circ}(a_0, \ldots, a_{n-1})$, where $a_j = b_j$ in the case of S_{01}, and $a_j = 2b_j - 1$ in the case of $S_{\pm 1}$.

The *maximal determinant problem* is concerned with the maximal value of $|\det A|$ for an $n \times n$ binary matrix A. The *Hadamard bound* [?] states that, in the case of binary matrices A over $\{\pm 1\}$, we have

$$|\det A| \leqslant n^{n/2}. \tag{2.2}$$

Moreover, Hadamard's inequality is sharp for infinitely many n, for example powers of two or n of the form $q+1$ where q is a prime power and $q \equiv 3 \pmod 4$ (Paley [?]).

There is a well-known connection between the determinants of $\{0, 1\}$-matrices of order n and $\{\pm 1\}$-matrices of order $n+1$. This implies that an $(n+1) \times (n+1)$ $\{\pm 1\}$-matrix always has determinant divisible by 2^n. See [?] for details. We give an example with $n = 3$, starting with an $n \times n$ binary matrix B and ending with an $(n+1) \times (n+1)$ $\{\pm 1\}$-matrix A, with $\det A = 2^n \det(B)$.

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{double}} \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{\text{border}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[\text{first row}]{\text{subtract}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} = A.$$

The doubling step is the only step where the determinant changes, and there it is multiplied by 2^n.

Thus, Hadamard's bound (2.2) gives the bound

$$|\det B| = 2^{-n} |\det A| \leqslant 2^{-n} (n+1)^{(n+1)/2}, \tag{2.3}$$

which applies for all $\{0, 1\}$-matrices B of order n. We shall refer to both (2.2) and

(2.3) as *Hadamard's inequality*, since it will be clear from the context which inequality is intended.[5]

2.4.2 Binary Circulants Achieving Hadamard's Bound

Now we get to the reason that, if you're only concerned with finding the *best* possible occluder, restricting your attention to circulant transfer matrices costs you very little. The reason is that there are several constructions for binary circulant matrices that achieve Hadamard's bound—this *despite* the fact that Hadamard's bound is general for all binary matrices, not just circulant ones! Please take a moment to reflect on how lucky it is that among the binary circulant matrices, which comprise a tiny subset of all binary matrices that, some of those binary circulant matrices achieve determinants as large as any from *all* binary matrices of the same size. I'll call these remarkable binary circulants "determinant-maximizing binary circulants," or DMBCs.

There are four known constructions for DMBCs. All four of these constructions yield a matrix whose eigenvalues are all equal, save for the first; the first eigenvalue has a value of $(n+1)/2$, and every other eigenvalue has a value of $\sqrt{(}n+1)/2$. The dot product of any pair of rows in a DMBC from one of these four constructions is $(n+1)/4$. In each construction, there are $(n+1)/2$ 1s and $(n-1)/2$ -1s. It follows from that last sentence that if you take one of these constructions and replace all the 0s with -1s to yield a $\{1,-1\}$ matrix, the dot product of any pair of rows will be -1.

Thanks to this last point, DMBCs have a close relationship to Hadamard matrices. Hadamard matrices are $\{1,-1\}$ matrices (not necessarily circulant) for which every pair of rows is orthogonal, that is, the dot product of any pair of rows is 0. (The same is true of pairs of columns.) Any size-n DMBC from one of these four constructions can be easily adapted to create a size-$(n+1)$ Hadamard matrix as follows: replace all the 0s with -1s in the DMBC, and then add a first row and a first column of all -1s. The dot product of the first row with any other row will be 0 (because every other but the first is exactly half 1s and half -1s). The dot product of every row other

[5] In fact, Hadamard in [?] proved a more general inequality than (2.2), and as far as we are aware he never stated (2.3) explicitly. A simple proof of (2.2) is given by Cameron [?].

than the first with any other row other than the first will also be 0 (because the dot product of each pair of rows before adding the extra row and column was -1, and then adding the extra column adds an extra 1 to the dot product). Hence each size-n DMBC yields a size-$(n+1)$ Hadamard matrix.

Hadamard matrices of this kind are known in the literature as "Hadamard matrices with circulant core," for obvious reasons. Now I will go into a little bit more detail about the various constructions.

Theorem 1 (Hadamard circulant core construction). *A Hadamard matrix of order $n+1$ with circulant core of order n exists if*

(1) $n \equiv 3 \pmod 4$ *is a prime;*

(2) $n = p(p+2)$*, where p and $p+2$ are prime;*

(3) $n = 2^k - 1$*, where k is a positive integer; or*

(4) $n = 4k^2 + 27$*, where k is a positive integer and n is a prime.*

Proof. Case (1) is due to Paley [**?**]; case (2) is due to Stanton and Sprott [**?**] and also Whiteman [**?**]; case (3) is due to Singer [**?**]; and case (4) is due to Hall [**?**, Theorem 2.2]. □

Hall [**?**, p. 980] remarks that case (4) is subsumed by case (1), since $4k^2 + 27 \equiv 3 \pmod 4$, but we mention case (4) since Hall's construction is different from that of Paley.

We do not know if the list given by Theorem 1 is exhaustive. The computational results given in Tables **??**–**??** show that, for $1 \leqslant n \leqslant 52$, only those n given by Theorem 1 can provide a Hadamard matrix of order $n+1$ with a circulant core. Also, a circulant $\{0,1\}$-matrix of order $n \leqslant 52$ can achieve the upper bound (**??**) if and only if $n \leqslant 4$ or n satisfies condition (1), (2) or (3) of Theorem 1.

This gives us the four known constructions for DMBCs. Note that the fourth construction is completely redundant with the first, since any prime n such that

$n = 4k^2 + 27$ where k is a positive integer is guaranteed to also be prime and congruent to 3 mod 4! It is only considered a separate construction because it yields an additional DMBC beyond the one given by the first construction (that isn't a trivial transformation). The fourth construction is therefore of no additional practical value to us: we can't use it to create a mask that images at a given level of resolution that we couldn't already. It's quite interesting mathematically, but doesn't help us solve an imaging problem.

It's worth giving more detail about the first construction, since it's by far the most common one over the real numbers; there are many more primes congruent to 3 mod 4 than there are powers of 2 or products of twin primes! Indeed, it's common enough that no matter what level of resolution you need, there will be a reasonably suitable mask at a nearby resolution level, thanks to the first construction.

The first construction, due to Paley, is as follows:

If n is prime and congruent to 3 mod 4:

$$x_i = \begin{cases} 1 & \text{if } \left(\frac{i}{n}\right) = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, $\left(\frac{i}{n}\right)$ is the Legendre symbol. It's equal to 0 is a multiple of n, and is otherwise equal to 1 if i is a quadratic residue (meaning a perfect square) modulo p and -1 if not. It's very easily computed, since by Euler's criterion, we have, for any a and any prime p:

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \mod p$$

Figure **??** shows a few possible occluder frames built using the Paley construction. As you can see, it has a very "random-looking" appearance. Of course, the construction is very much non-random; what gives it that appearance, though, is its non-self-repeating character. All four constructions, in fact, try to repeat themselves as little as possible. That's not surprising from the point of view of wanting to keep the sequence's Fourier spectrum flat, of course. But the way I like to think of it from an imaging point of view is that these sequences try and make the different possible shadows cast by a light source in a variety of different locations as different as possible from each other. A light source at each possible point in the (implicitly planar) scene will yield a different rotation of the occluding sequence, so the best sequences for distinguishing light sources at different locations will be sequences that are orthogonal to their own rotations. See Figure **??** for a visual explanation of this phenomenon.

The Lambertian BRDF is an excellent approximation for the reflectance properties of most objects.

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2.5 Section sample 1

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Praesent pulvinar risus in diam mollis tincidunt. Nam aliquam lacus sed eleifend mattis. Ut at blandit ex, eget molestie massa. Morbi fermentum sit amet mauris ut placerat. Ut in eros pretium, congue felis sit amet, gravida justo. Suspendisse purus leo, posuere sed odio vel, mollis tempus nisi. Aenean est tortor, tincidunt et rhoncus ut, fermentum semper leo [?]. Integer id viverra metus, a blandit neque. In in enim eu ipsum euismod hendrerit.

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2.6 Section sample 2

Phasellus sed elit vehicula, gravida odio in, vulputate quam. Quisque in elit enim. Vivamus finibus justo elit, sed semper turpis aliquam porttitor. Nulla posuere bibendum nunc sit amet consequat. Vivamus commodo lorem sed metus fermentum rhoncus. Etiam porta sodales purus, vel aliquet lacus facilisis et. Etiam ornare velit non dui auctor fermentum. In elit augue, fringilla at lacinia at, facilisis sit amet lectus. Sed et hendrerit ex. Morbi tristique felis a augue egestas commodo. Nulla porttitor ut urna nec dignissim. Fusce ac pharetra risus, id rhoncus ligula. Pellentesque euismod viverra sem, vel porttitor libero blandit quis. Phasellus orci augue, mattis nec dolor ut, cursus mattis quam. Sed tincidunt eu metus sed pulvinar. Ut a nulla at leo semper accumsan efficitur eget leo.

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2.6.1 Subsection sample

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2.6.2 Subsection with list

Ut sollicitudin, lectus eget posuere porttitor, risus dui facilisis risus, a pharetra lacus elit vel eros. Proin fermentum accumsan mauris, quis posuere nisi pharetra scelerisque.

1. Item 1.

2. Item 2.

3. Item 3.

[6] Here is a sample footnote referencing figures B-1 and B-2.

Cras nec ullamcorper mauris. Aliquam erat volutpat. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Suspendisse sed dui ac mi auctor scelerisque. Etiam at semper nisi. Cras nec dolor ac purus feugiat auctor. Nunc eget pulvinar massa.

2.7 Section sample 3

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2.7.1 Another subsection sample

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$$\begin{aligned}
a_i &= a_j + a_k \\
a_i &= 2a_j + a_k \\
a_i &= 4a_j + a_k \\
a_i &= 8a_j + a_k \\
a_i &= a_j - a_k \\
a_i &= a_j \ll m\text{shift}
\end{aligned}$$

instead of the multiplication. For example, you could use:

$$\begin{aligned}
r &= 4s + s \\
r &= r + r
\end{aligned}$$

Or by xx:

$$\begin{aligned} t &= 2s + s \\ r &= 2t + s \\ r &= 8r + t \end{aligned}$$

Cras pharetra ligula nec lectus bibendum, euismod mattis purus cursus. Nullam ut mi molestie purus ultricies lacinia. Phasellus sed orci ac lacus convallis vestibulum. Quisque id nulla ut ipsum finibus vehicula. Curabitur scelerisque erat lobortis, dapibus purus eget, faucibus sapien. Nam enim leo, faucibus id ante sed, fringilla luctus eros. Morbi vulputate, purus at commodo aliquet, turpis dolor sollicitudin libero, id vehicula risus dui sit amet nulla. Sed auctor efficitur urna. Praesent sagittis tellus ac velit vestibulum dignissim. Vivamus justo enim, pellentesque eu posuere id, mattis vitae felis. Aliquam id tincidunt diam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos himenaeos. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas.

Chapter 3

Optimal Occluders

In this chapter, we consider

3.1 The Mutual Information Model of IF Optimality

Throughout this thesis, a recurring theme is that we'd like to be able to measure how useful an occluder (or other intermediate frame) is to do imaging with. The simplest way to do this is to measure the mutual information between the scene and the observation plane, given a particular transfer matrix (which describes how the intermediate frame transforms the scene before it reaches the observation plane) and signal-to-noise ratio.

3.1.1 The Scene Model

For the time being, we will consider a 2D "flatland" model of the world for simplicity. Extensions to the 3D world are simple, and as previously explained, the 2D model is still representative of the real world in that it accurately describes what happens when you integrate over one the two spatial dimensions in your observation.

3.1.2 The Noise Model

Let's assume that the amount of noise

Appendix A

Tables

Table 1: Armadillos

Armadillos	are
our	friends

Appendix B

Figures

Figure B-1: Armadillo slaying lawyer.

Figure B-2: Armadillo eradicating national debt.