On the Theory of Imaging

August 29, 2017

This writeup discusses the relative theoretical merits of different occluders and their limitations.

1 General Setup

In this section, we describe the setup of the experiment. For the time being, we posit a one-dimensional scene and a one-dimensional observation plane, with a one-dimensional "intermediate frame" halfway in between. We presume that the observer sees the observation plane and knows the form of the intermediate frame, and must infer the scene. See Fig. 1 for a picture of what the scene's setup looks like.

Once we've specified what is in the intermediate frame, we've implied a *transfer matrix*, which we call A. A describes how what is in the scene, x, determines what is on the observation plane, y. We use x and y as vectors representing the amount of intensity at each point in the scene and the observation plane, respectively. In a noiseless setting, we would be able to write:

$$Ax = y$$

.

There are a few possible different scenarios worth discussing, each one corresponding to a different material in the intermediate frame. For example, the intermediate frame could be a simple occlusion pattern, or some sort of pattern of glass and mirrors, or something else altogether. We will consider each of these cases in turn.

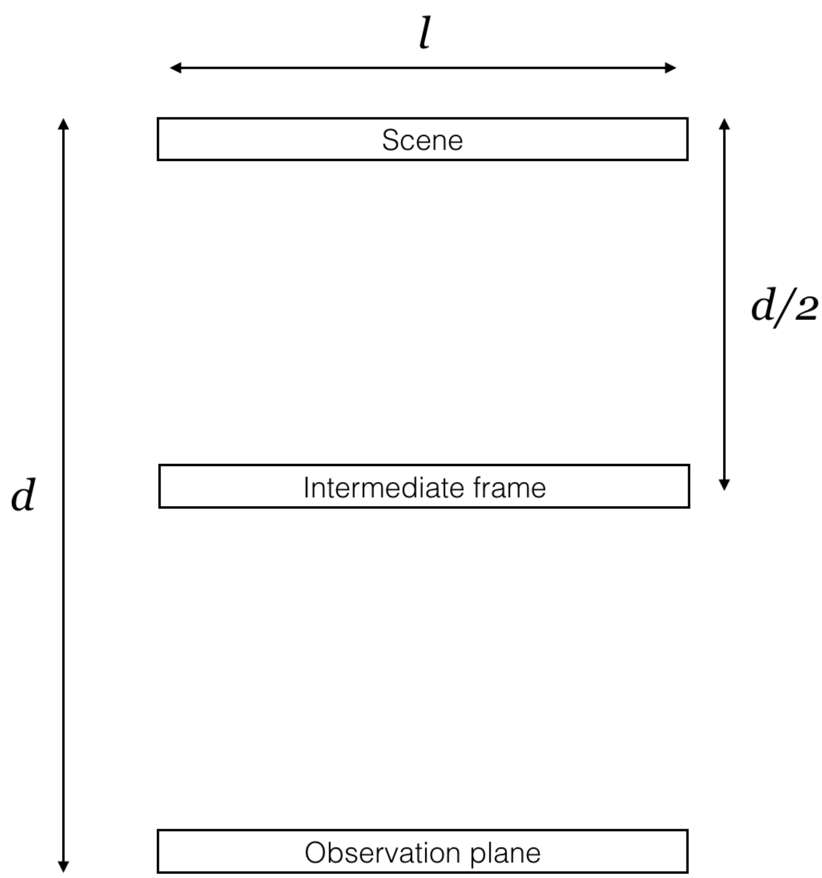

Figure 1: The setup of the scene.

2 Occlusion-based patterns

2.1 The Setup

If the intermediate frame contains a simple occluder—meaning a pattern of material that is somewhere between completely opaque and completely transparent at each point, that does not change the direction of the incoming light—then that implies a lot of constraints on the form of the transfer matrix.

In order to explore the nature of these constraints, let us begin by parametrizing the scene. Let the scene x have size l, and let it be divided into n patches of size l/n. Moreover, let the observation plane y have size l as well, and let it be divided into m patches of size l/m.

Each of the x_i has some intensity given by the native intensity of the scene. (Naturally, the smaller the size of each of the x_i, the smaller this intensity should become.) Due to the light from the scene reaching the observation plane, each of the y_j gains some intensity as well. For these x_i and y_j, we take integer i and j with $0 \leq i < n$ and $0 \leq j < m$. We presume there is also a source of light from outside the scene, which hits all parts of the observation plane equally and does nothing but interfere with our measurements.

We can describe the intermediate frame by its *permittivity function*. Each part of the intermediate frame has some permittivity $p(a)$ between 0 and 1: 0 for completely opaque, and 1 for completely transparent.

Note that because we have posited that the intermediate frame contains only a simple occluder, if we want to measure the intensity contribution to a part of the observation plane y_j from a part of the scene x_i, we need only consider the straight-line path that the light would take from x_i to y_j. This path passes through one part of the intermediate frame on its way there.

2.2 The Transfer Matrix

What fraction A_{ij} of the light emitted from x_i, then, reaches y_j? We assume each patch in the scene emits light uniformly across the semi-circle pointed directly downward. If we say that the patch x_i has total light intensity I_{x_i}, and size l/n, then each little piece dx of the patch x_i has light density $I_{x_i}n/l$. If we then define the vertical offset between x_i and y_j to be h, and take x_{li} and y_{lj} to be the left boundaries of the patches x_i and y_j, respectively, the amount of light L_{ij} emitted by x_i that reaches y_j is given by:

$$L_{ij} = \int_{x_{li}}^{x_{li}+\frac{l}{n}} \int_{y_{lj}}^{y_{lj}+\frac{l}{m}} I_{x_i} \frac{n}{l} \frac{d \cdot t(\frac{x+y}{2})}{\pi(d^2 + (y-x)^2)} dydx \tag{1}$$

So we can get the fraction A_{ij} of the light emitted from x_i that reaches y_j simply by dividing by I_{x_i}.

$$A_{ij} = \int_{x_{li}}^{x_{li}+\frac{l}{n}} \int_{y_{lj}}^{y_{lj}+\frac{l}{m}} \frac{n}{l} \frac{d \cdot t(\frac{x+y}{2})}{\pi(d^2 + (y-x)^2)} dydx \tag{2}$$

This is how we can determine the entries of the transfer matrix A.

2.3 The Mutual Information Upper Bound

Suppose we have a Gaussian channel, with x being a vector of random variables whose entries are drawn i.i.d. from a Gaussian distribution with mean 0 and variance 1. Suppose that we have $y = Ax + \eta$, where the noise vector η is also drawn i.i.d from a Gaussian distribution, this one with mean 0 and variance σ^2. What is the mutual information between x and y? In other words, how many bits of information to we learn about x when we learn y (or vice-versa)?

We know from information theory that in this case, the mutual information $I(x, y)$ is given by the following expression:

$$I(x, y) = \log(\det(\frac{AA^T}{\sigma^2} + I_n)), \tag{3}$$

where I_n is the $n \times n$ identity matrix. This expression is useful for two purposes. The first is that by calculating this expression for various specific A matrices, we can compare the reconstruction quality of different occluders under varying levels of SNR. The second is that we can use this expression, combined with known bounds on the determinants of matrices, to bound the overall mutual information possible with *any* occluder.

Let's talk about doing that now. Recall from the previous subsection that we have:

$$A_{ij} = \int_{x_{li}}^{x_{li}+\frac{l}{n}} \int_{y_{lj}}^{y_{lj}+\frac{l}{m}} \frac{dp(\frac{x+y}{2})}{\pi(d^2 + (y-x)^2)} dydx \tag{4}$$

$$0 \leq p(a) \leq 1 \tag{5}$$

Using Eqs. **??** and **??**, we can write an upper bound on A_{ij}. Because $(y-x)^2$ is guaranteed to be positive, we can write:

$$A_{ij} \leq \int_{x_{li}}^{x_{li}+\frac{l}{n}} \int_{y_{lj}}^{y_{lj}+\frac{l}{m}} \frac{1}{\pi h} dy dx \tag{6}$$

$$A_{ij} \leq \frac{l^2}{\pi n m h} \tag{7}$$

This bound approaches tightness in the absence of occlusion when h is very large.

This bound we get from the fact that the denominator in the exact expression for F_{ij} can be no smaller than d^2. The bound is tight when $i + j = n - 1$. Moreover, the coefficient in front of m above shows up so often in this problem that it deserves its own variable. We define:

$$c_f = \frac{l^2}{\pi n h} \tag{8}$$

This implies:

$$A_i j \leq \frac{c_f}{m} \forall i, j \tag{9}$$

The amount of light hitting each part of the observation plane y_i is also subject to an amount of noise whose variance σ^2 is equal to the intensity of the interference light hitting y_i. The total intensity of interference light is given by w; because the interference light is generated by a Poisson process, it can be well-approximated by drawing from a normal distribution with mean w and variance w.

What about the distribution of the light hitting each part of the observation plane y_i? In the same vein, we can approximate that as drawing from a normal distribution with mean $\frac{w}{m}$ and variance $\frac{w}{m}$.

Thus, to recap, we have:

$\vec{y} = A\vec{x} + \eta$

Where x is a length-n vector describing the scene, y is a length-m vector describing the observation plane, A is the $m \times n$ transfer matrix and explains how illumination from the scene gets reflected onto the observation plane. η is the length-m noise vector, each element of which is normally distributed with mean and variance $\frac{w}{m}$.

We know that in this situation, the mutual information $I(x, y)$ is given by:

$$I(x, y) = \log\det(\frac{A^T A}{\sigma^2} + I_n) \tag{10}$$

$$I(x, y) = \log\det(A^T A \frac{m}{w} + I_n) \tag{11}$$

Because A is an $m \times n$ matrix, we know that the matrix $A^T A$ must be an $n \times n$ matrix. Each entry of $A^T A$ is the dot product of two length-m vectors, each of which has entries that are between 0 and $\frac{c_f}{m}$ (see Eq. **??**). Therefore, each entry of $A^T A$ must be between 0 and $m(\frac{c_f}{m})^2 = \frac{c_f^2}{m}$.

From this, we can infer the following bound on the entries of the overall matrix

$$S = A^T A \frac{m}{w} + I_n \tag{12}$$

$$0 \leq S_{ij} \leq \frac{c_f^2}{w} \forall i, j s.t. i \neq j \tag{13}$$

$$1 \leq S_{ii} \leq 1 + \frac{c_f^2}{w} \tag{14}$$

We can get the mutual information by asking: what's the maximum value that $\log\det(S)$ could take? Brent et al. prove the following upper bound on the determinant of a matrix M whose diagonal entries have absolute value at most δ and whose off-diagonal entries have absolute value at most ϵ:

$$\det(M) \leq ((1+\delta)^2 + (n-1)\epsilon^2)^{\frac{n}{2}} \tag{15}$$

By using this bound, we can prove the following upper bound on the mutual information (note that we're wasting a bit of the bound, since the bound doesn't realize the fact that all of our entries have to be positive; we can probably do even better):

$$I(x, y) \leq \log((1 + \frac{c_f^2}{w})^2 + (n-1)\frac{c_f^4}{w^2})^{\frac{n}{2}} \tag{16}$$

Note that this upper bound is *constant* in m! This tells us that in the limit of higher and higher resolution on the observation plane, there's only so much information we can milk out.

(Also, in case you are worried about the fact that it seems to scale faster than linearly with n, don't be—the c_f term cleverly masks a factor of $1/n$. But scaling with n is unphysical anyway...)

We can get a second upper bound on the mutual information, which will be tighter when $n > m$ (and so adds nothing extra in the limit of large m). We can do this by considering the matrix S', defined as:

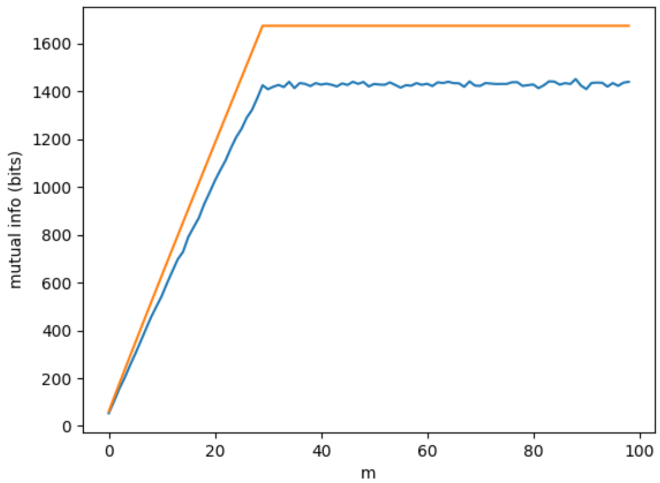

Figure 2: In orange, the upper bound. In blue, the empirical mutual information for a random occluder. $l = 10$, $n = 30$, $m = 100$, $h = 100$, $w = 10^{-20}$.

$$S' = AA^T \frac{m}{w} + I_m \tag{17}$$

We know that $\det(S) = \det(S')$ because $\det(AB + I) = \det(BA + I)$. This will give us another bound. We know that each entry of AA^T is the dot product of two n-length vectors, each entry of which is between 0 and $\frac{c_f}{m}$. Thus, we can write the following bounds on the entries of S':

$$0 \leq S_{ij} \leq \frac{nc_f^2}{mw} \forall i, j s.t. i \neq j \tag{18}$$

$$1 \leq S_{ii} \leq 1 + \frac{nc_f^2}{mw} \tag{19}$$

And the bound on $I(x, y)$ that this implies is:

$$I(x, y) \leq \log((1 + \frac{nc_f^2}{mw})^2 + (n-1)\frac{n^2c_f^4}{m^2w^2})^{\frac{m}{2}} \tag{20}$$

$$I(x, y) = m \log(1 + \frac{l^2}{\pi^2 w d^2 m})$$

$$I(x,y) = \frac{l^2}{\pi^2 w d^2}$$

$$I(x,y) = m \log(\frac{l^2}{\pi^2 w d^2 m})$$

$$\alpha = \frac{n^2 c_f^2}{w}$$

$$I(x,y) \leq \frac{m}{2} \log((1 + \frac{\alpha}{n^2})^2 + (n-1)\frac{\alpha^2}{n^4})$$