ORDER INVARIANT GRAPHS AND FINITE INCOMPLETENESS

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EXTENDED ABSTRACT

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ABSTRACT. Every order invariant graph on $[Q]^{<\infty}$ has a free $E \supseteq \text{ush}(E)$ reducing $[\cup E \cup N]^{<\infty}$. Every order invariant graph on $[Q]^{\leq k}$ has a free $\{x_1,...,x_r,\text{ush}(x_1),...,\text{ush}(x_r)\}$, each $\{x_1,...x_{(8kni)!}\}$ reducing $[x_1 \cup ... \cup x_i \cup \{0,...,n\}]^{\leq k}$. The second statement finitely approximates the first. The proofs with ush removed are unremarkable. The proofs of the full statements use standard large cardinal hypotheses, and we show that there is no proof in ZFC (assuming ZFC is consistent). The second statement is explicitly Π^0_2. The complexity of the free set can be exponentially bounded, resulting in an explicitly Π^0_1 form.

1. PRELIMINARIES

DEFINITION 1.1. We use i,j,k,n,m,r,s,t for positive integers unless stated otherwise. Q is the set of all rational numbers. $[X]^{<\infty}$ is the set of all finite subsets of X. $[X]^{\leq k}$ is the set of all subsets of X with at most k elements. For finite x, |x| is the number of elements of x. ush(x) is the upper shift of x, which is obtained by adding 1 to all nonnegative elements of x. ush(E) = {ush(x): x $\in$ E}. The complexity of $x \in Q^k$ is the least n such that x can be written with numerators and denominators of magnitude $\leq$ n. The complexity of $X \subseteq Q^{\leq k}$ is the least n such that every element of X has complexity $\leq$ n (0 for X = $\emptyset$ and ∞ for infinite X).

DEFINITION 1.2. A graph is a pair (V,E), where $E \subseteq V^2$ is irreflexive and symmetric. V is the set of vertices and E is the edge relation. X is free if and only if $X \subseteq V$ and no two elements of X are related by E. Let G be a graph on $[Q]^{<\infty}$ ($[Q]^{<k}$). A set of vertices X reduces a set of vertices Y if and only if for all $y \in Y$, either $y \in X$ or there is an $x \in X$ such that (x E y and $\max(x) \leq \max(y)$). Here we take $\max(\emptyset) = 0$.

DEFINITION 1.3. Let $(x,y),(z,w) \in [Q]^{<\infty} \times [Q]^{<\infty}$. We say that (x,y) and (z,w) are order equivalent if and only if
i. $|x| = |z| \wedge |y| = |w|$.
ii. for all $1 \leq i \leq |x|$ and $1 \leq j \leq |y|$, the i-th element of x is less than the j-th element of y if and only if the i-th element of z if less than the j-th element of w.

DEFINITION 1.4. An order invariant graph on $[Q]^{<\infty}$ ($[Q]^{\leq k}$) is a graph on $[Q]^{<\infty}$ ($[Q]^{\leq k}$), where E is the union of cosets of order equivalence on $[Q]^{<\infty} \times [Q]^{<\infty}$ ($[Q]^{\leq k} \times [Q]^{\leq k}$).

Note that there are only finitely many order invariant graphs on $[Q]^{\leq k}$ for fixed k. However, there are continuumly many order invariant graphs on $[Q]^{<\infty}$.

DEFINITION 1.5. A formal system is 1-consistent if and only if for any given TM, if it proves that TM halts at the empty input tape, then TM actually halts at the empty input tape. A Π^0_1 sentence is a sentence asserting that a given TM never halts at the empty input. A Π^0_2 sentence is a sentence asserting that a given TM always halts at every finite input.

We now introduce the standard large cardinal hypotheses that we use.

DEFINITION 1.6. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP abbreviates "stationary Ramsey property".

DEFINITION 1.7. SRP is the formal system ZFC + $\{(\exists\lambda)(\lambda$ is k-SRP$)\}_k$. SRP^+ is ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-SRP$)$. SRP_k is ZFC + $(\exists\lambda)(\lambda$ is k-SRP$)$. EFA is exponential (or elementary) function arithmetic. See [WIKIa]. WKL_0 is the second main system of reverse mathematics. See [WIKIb].

2. INFINITE FREE SETS

THEOREM 2.1. Every graph on $[Q]^{<\infty}$ has a free E reducing $[\cup E \cup N]^{<\infty}$.

PROPOSITION 2.2. Every order invariant graph on $[Q]^{<\infty}$ has a free $E \supseteq ush(E)$ reducing $[\cup E \cup N]^{<\infty}$.

PROPOSITION 2.3. For all k, every order invariant graph on $[Q]^{\leq k}$ has a free $E \supseteq ush(E)$ reducing $[\cup E \cup N]^{\leq k}$.

3. FINITE FREE SETS

THEOREM 3.1. For all k,n,r, every graph on $[Q]^{\leq k}$ has a free $\{x_1,\ldots,x_r\}$, each $\{x_1,\ldots x_{(8kni)!}\}$ reducing $[x_1 \cup \ldots \cup x_i \cup \{0,\ldots,n\}]^{\leq k}$.

PROPOSITION 3.2. For all k,n,r, every order invariant graph on $[Q]^{\leq k}$ has a free $\{x_1,\ldots,x_r,ush(x_1),\ldots,ush(x_r)\}$, each $\{x_1,\ldots x_{(8kni)!}\}$ reducing $[x_1 \cup \ldots \cup x_i \cup \{0,\ldots,n\}]^{\leq k}$.

THEOREM 3.3. For all k,n,r, every graph on $[Q]^{\leq k}$ has a free $\{x_1,\ldots,x_r\}$ of complexity $\leq (8knr)!$, each $\{x_1,\ldots x_{(8kni)!}\}$ reducing $[x_1 \cup \ldots \cup x_i \cup \{0,\ldots,n\}]^{\leq k}$.

PROPOSITION 3.4. For all k,n,r, every order invariant graph on $[Q]^{\leq k}$ has a free $\{x_1,\ldots,x_r,ush(x_1),\ldots,ush(x_r)\}$ of complexity $\leq (8knr)!$, each $\{x_1,\ldots x_{(8kni)!}\}$ reducing $[x_1 \cup \ldots \cup x_i \cup \{0,\ldots,n\}]^{\leq k}$.

Note that Theorem 3.1, Proposition 3.2 are explicitly $\Pi^0{}_2$, and Theorem 3.3, Proposition 3.4 are explicitly $\Pi^0{}_1$.

THEROEM 3.5. Propositions 2.2, 2.3, 3.2, 3.4 are
i. provable in SRP^+ but not in SRP (assuming SRP is consistent).
ii. unprovable in ZFC (assuming ZFC is consistent).
iii. neither provable nor refutable in SRP (assuming SRP is 1-consistent).
iv. neither provable nor refutable in ZFC (assuming SRP is 1-consistent, although very likely the 1-consistency of ZFC is sufficient).
v. provably equivalent to the consistency of SRP over WKL_0 (EFA suffices for 3.2, 3.4).

THEOREM 3.6. For each fixed k, Propositions 2.3, 3.2, 3.4 are provable in SRP. This is not the case for any SRP_m (assuming SRP is consistent). There exists k,n such that Propositions 2.3, 3.2, 3.4 are unprovable in ZFC (assuming

ZFC is consistent), and neither provable nor refutable in ZFC (assuming SRP is 1-consistent, although very likely the 1-consistency of ZFC is sufficient).

REFERENCES

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[WIKIa] http://en.wikipedia.org/wiki/Elementary_function_arithmetic

[WIKIb] http://en.wikipedia.org/wiki/Reverse_mathematics