Lecture 34: Sorting and Algorithmic Bounds

11/16/2020

Sorting

- Sorting is a foundational problem
  - Useful for putting things in order
  - But can be used to solve other tasks, sometimes in non-trivial ways
    - Sorting improves duplicate finding from a naive N^2 to N log N
    - Sorting improves 3SUM from a naive N^3 to N^2
  - There are many ways to sort an array, each with its own interesting tradeoffs and algorithmic features
- Today we'll discuss the fundamental nature of the sorting problem itself: How hard is it to sort?

Sorts Summary

	Memory	# Compares	Notes	Stable?
Heapsort	$\Theta(1)$	$\Theta(N \log N)$	Bad caching (61C)	No
Insertion	$\Theta(1)$	$\Theta(N^2)$	Best for almost sorted and N < 15	Yes
Mergesort	$\Theta(N)$	$\Theta(N \log N)$	Fastest stable sort	Yes
Quicksort LTHS	$\Theta(\log N)$	$\Theta(N \log N)$ expected	Fastest sort	No

This is due to the cost of tracking recursive calls by the computer, and is also an "expected" amount. The difference between log N and constant memory is trivial.

You can create a stable Quicksort. However, using unstable partitioning schemes (like Hoare partitioning) and using randomness to avoid bad pivots tend to yield better runtimes.

Math Problems out of Nowhere

A Math Problem out of Nowhere

- Consider the functions N! and (N/2)^(N/2)
- Is N! \in Omega((N/2)^(N/2))?
  - Can experiment and find that the above statement is true

Another Math Problem

- Given that N! > (N/2)^(N/2), show that log(N!) \in Omega(N log N)
  - We have that N! > (N/2)^(N/2)
  - Taking the log of both sides, we can show the above statement

- In other words, log(N!) grows at least as quickly as N log N

Last Math Problem

- Show that N log N \in Omega(log(N!))
  - log(N!) = log(N) + log(N-1) + log(N-2) + ... + log(1)
  - N log N = log(N) + log(N) + log(N) + ... + log(N)
  - Hence, N log N \in Omega(log(N!))

Omega and Theta

- Given:
  - log(N!) \in Omega(N log N)
  - N log N \in Omega(log(N!))
- We can conclude that:
  - log(N!) \in Theta(N log N) AND
  - N log N \in Theta(log(N!))

Theoretical Bounds on Sorting

Sorting

- We have shown several sorts to require Theta(N log N) worst case time
  - Can we build a better sorting algorithm?
- Let the ultimate comparison sort (TUCS) be the asymptotically fastest possible comparison sorting algorithm, possibly yet to be discovered, and let R(N) be its worst case runtime in Theta notation
  - Comparison sort means that it uses the compareTo method in Java to make decisions
  - Worst case rn-time of TUCS, R(N), is O(N log N)
    - We already have algorithms that take Theta(N log N) worst case
  - Worst case run-time of TUCS, R(N) is Omega(1)
    - Obvious: Problem doesn't get easier than N
    - Can we make a stronger statement than Omega(1)?
  - Worst case run-time of TUCS, R(N), is also Omega(N)
    - Have to at least look at every item
  - But, with a clever argument, we can see that the lower bound will turn out to be Omega(N log N)
    - This lower bound means that across the infinite space of all possible ideas that any human might ever have for sorting using sequential comparisons, NONE has a worst case runtime that is better than N log N

The Game of Puppy, Cat, Dog

- Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale

a < b	b < c	[redacted]	Which is which?
Yes	Yes		a: puppy, b: cat, c: dog (sorted order: abc)
No	No		c: puppy, b: cat, a: dog (sorted order: cba)
Yes	No		

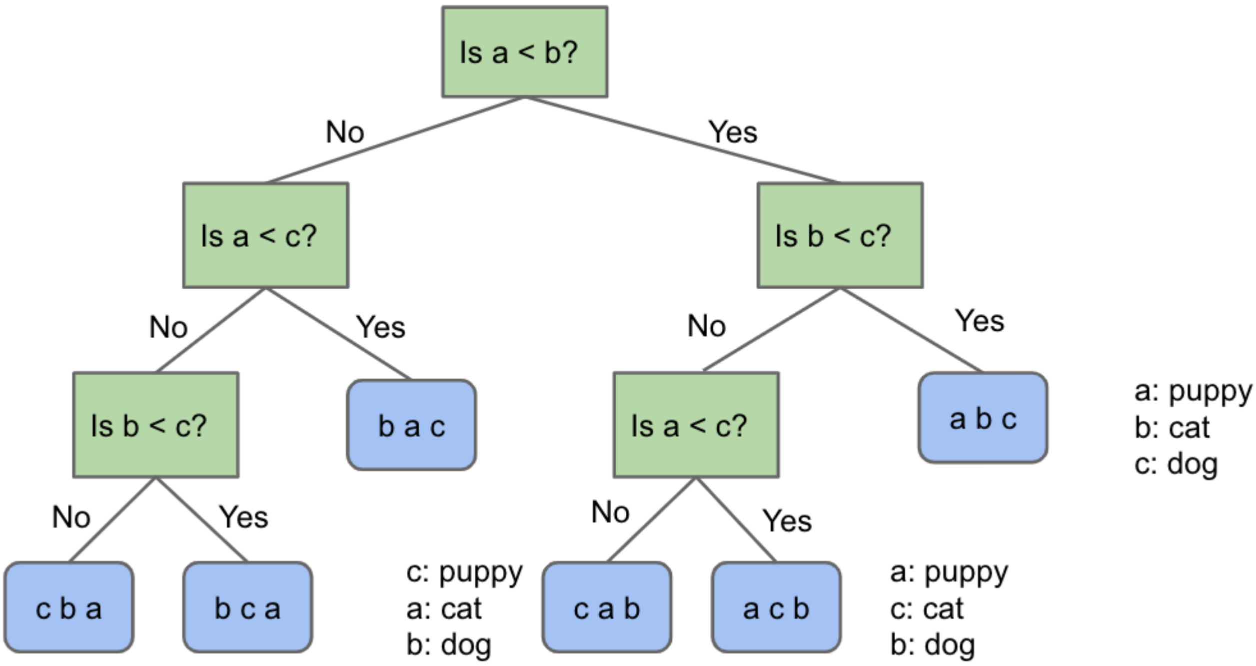
Which is which? How do we resolve the ambiguity?

1. a: puppy, b: cat, c: dog (sorted order: abc)
2. **a: puppy, c: cat, b: dog (sorted order: acb)**
3. **c: puppy, a: cat, b: dog (sorted order: cab)**
4. c: puppy, b: cat, a: dog (sorted order: cba)

- We have to weigh a and c to resolve the final ambiguity

Puppy, Cat, Dog - A Graphical Picture of N = 3

- The full decision tree for puppy, cat, dog:



The Game of Puppy, Cat, Dog

- How many questions would you need to ask to definitely solve the "puppy, cat, dog, walrus" question?
  - If N = 4, we have 4! = 24 permutations of puppy, cat, dog, walrus
  - So we need a binary tree with 24 leaves
    - How many levels minimum? log_2(24) = 4.5, so 5 is the minimum
  - So at least 5 questions

Generalized Puppy, Cat, Dog

- How many questions would you need to ask to definitely solve the generalized "puppy, cat, dog" problem for N items?
  - Give your answer in big Omega notation

- Answer: Omega(log(N!))
    - For N, we have the following argument:
        - Decision tree needs N! leaves
        - So we need log_2(N!) rounded up levels, which is Omega(log(N!))

Generalizing Puppy, Cat, Dog

- Finding an optimal decision tree for the generalized version of puppy, cat, dog is an open problem in mathematics
- Deriving a sequence of yes/no questions to identify puppy, cat, dog is hard. An alternate approach to solving the puppy, cat, dog algorithm
    - Sort the boxes using any generic sorting algorithm
        - Leftmost box is puppy
        - Middle box is cat
        - Right box is dog

Sorting, Puppies, Cats, and Dogs

- A solution to the sorting problem also provides a solution to puppy, cat, dog
    - In other words, puppy, cat, dog **reduces** to sorting
    - Thus, any lower bound on difficulty of puppy, cat, dog must ALSO apply to sorting

Sorting Lower Bound

- We have a lower bound on puppy, cat, dog, namely it takes Omega(log(N!)) comparisons to solve such a puzzle
- Since sorting with comparisons can be used to solve puppy, cat, dog, then sorting also takes Omega(log(N!)) comparisons
- Or in other words:
    - Any sorting algorithm using comparisons, no matter how clever, must use at least k = log_2(N!) compares to find the correct permutation. So even TUCS takes at least log_2(N!) comparisons
    - log_2(N!) is trivially Omega(log(N!)), so TUCS must take Omega(log(N!)) time
- Earlier, we showed the log(N!) \in Theta(N log N), so we have that TUCS is Omega(N log N)
    - **Any comparison based sort requires at least order N log N comparisons**
- Proof summary:
    - Puppy, cat, dog is Omega(log_2(N!)), i.e. requires log_2(N!) comparisons
    - TUCS can solve puppy, cat, dog, and thus takes Omega(log_2(N!)) compares
    - log_2(N!) is Omega(N log N)
        - This was because N! is Omega((N/2)^(N/2))

Optimality

- The punchline:
    - Our best sorts have achieved absolute asymptotic optimality
        - Mathematically impossible to sort using fewer comparisons
        - Note: Randomized quicksort is only probabilistically optimal, but the probability is extremely high for even modest N. So don't worry about quicksort