Lecture 17: B-Trees (2-3, 2-3-4 Trees)

10/5/2020

BST Tree Height

BST Tree Height

- Trees range from best-case "bushy" to worst-case "spindly"
    - Height varies dramatically among the two
        - Theta(log N) for bushy vs Theta(N) for spindly
- Performance of operations on spindly trees can be just as bad as a linked list!
- A worst case (spindly tree) has a height that grows exactly linearly - Theta(N)
- A best case (bushy tree) has a tree height that grows exactly logarithmically - Theta(log N)

The Usefulness of Big O

- Big O is a useful idea:
    - Allows us to make simple blanket statements, e.g. can just say "binary search is O(log N)" instead of "binary search is Theta(log N) in the worst case"
    - Sometimes don't know the exact runtime, so use O to give an upper bound
        - Example: Runtime for finding shortest route that goes to all world cities is O(2^N). There might be a faster way, but nobody knows one yet
    - Easier to write proofs for Big O than Big Theta, e.g. finding runtime of mergesort, you can round up the number of items to the next power of 2. A little beyond the scope of the course.

Height, Depth, and Performance

Height and Depth

- Height and average depth are important properties of BSTs
    - The **"depth" of a node** is how far it is from the root
        - The root has depth 0
    - The **"height" of a tree** is the depth of its deepest leaf
    - The **"average depth"** of a tree is the average depth of a tree's nodes

Height, Depth, and Runtime

- Height and average depth determine runtimes for BST operations
    - The **"height"** of a tree determines the worst case runtime to find a node
    - The **"average depth"** determines the average case runtime to find a node

Important Question: What about real world BSTs?

- BSTs have:
    - Worst case Theta(N) height
    - Best case Theta(log N) height

- One way to approximate real world BSTs is to consider randomized BSTs
- **Nice Property**. Random trees have Theta(log N) average depth and height
  - In other words: Random trees are bushy, not spindly

Randomized Trees: Mathematical Analysis

- **Average Depth**. If N distinct keys are inserted into a BST, the expected average depth is ~ 2 ln N
  - Thus, average runtime for contains operation is Theta(log N) on a tree built with random inserts
- **Tree Height**. If N distinct keys are inserted in random order, expected tree height is ~ 4.311 ln N
  - Thus, worst case runtime for contains operation is Theta(log N) on a tree built with random inserts
- BSTs have:
  - Worst case Theta(N) height
  - Best case Theta(log N) height
  - Theta(log N) height if constructed via random inserts
- In real world applications we expect both insertion and deletion
  - Can show that random trees including deletion are still Theta(log N) height

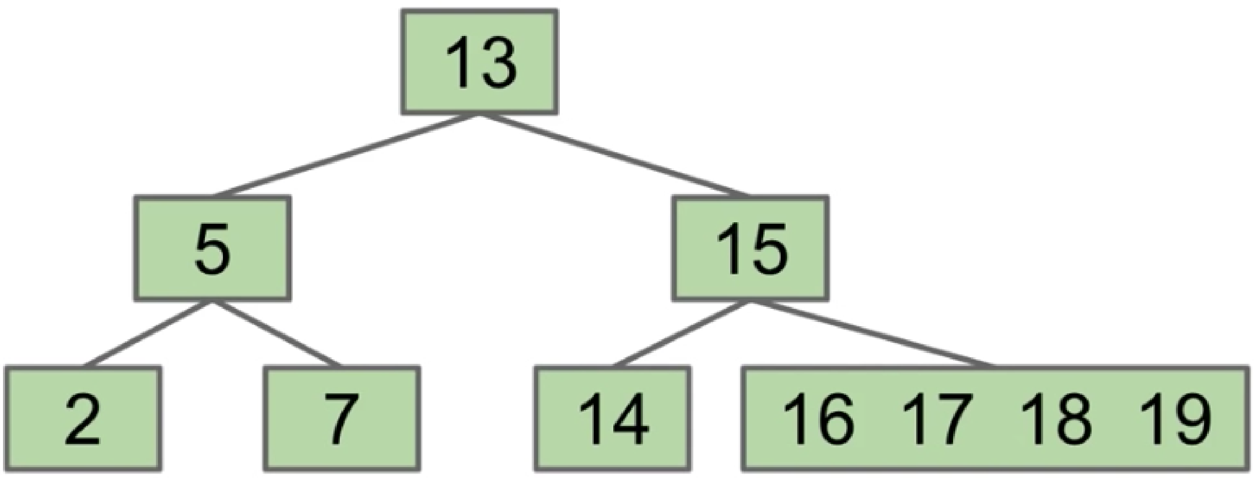
Good News and Bad News

- Good news: BSTs have great performance if we insert items randomly
  - Performance is Theta(log N) per operation
- Bad news: We can't always insert our items in a random order
  - Data comes in over time, don't have all at once

B-trees / 2-3 trees / 2-3-4 trees

Avoiding Imbalance through Overstuffing

- The problem is adding new leaves at the bottom
- Crazy idea: never add new leaves at the bottom
  - Tree can never get imbalanced
- Avoid new leaves by "overstuffing" the leaf nodes
  - "Overstuffed tree" always has balanced height, because leaf depths never change
- Overstuffed trees are a logically consistent but very weird data structure

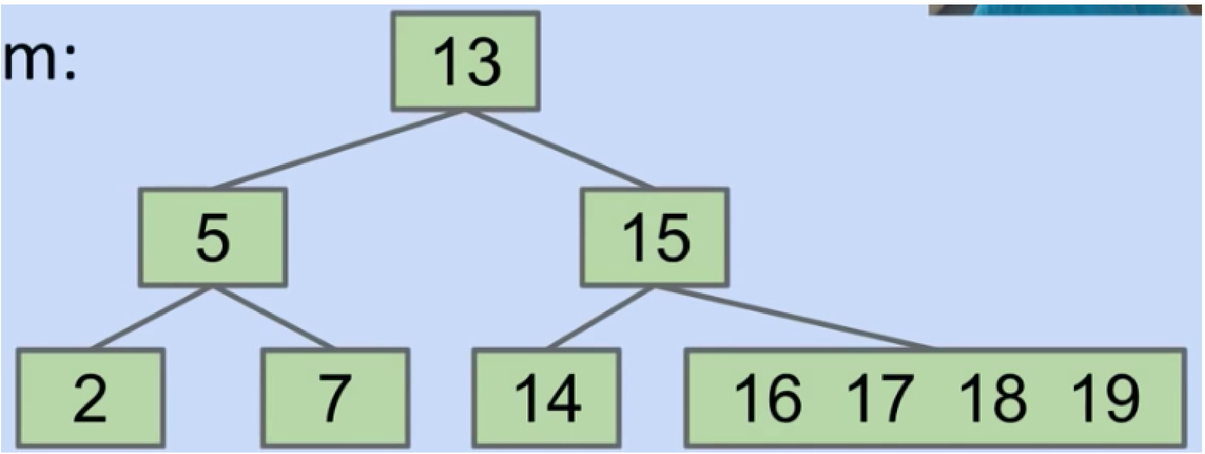


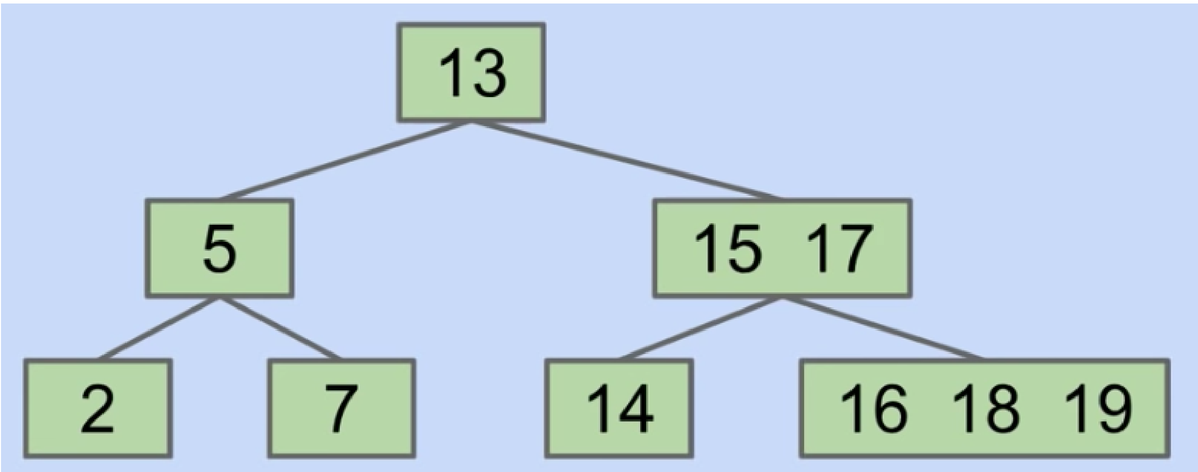
  - contains(18):
    - 18 > 13? Yes, go right

        - 18 > 15? Yes, go right
        - 16 = 18? No
        - 17 = 18? No
        - 18 = 18? Yes! Found it
    - Problem with this idea? Degenerates into linked list

Revising Our Overstuffed Tree Approach: Moving Items Up

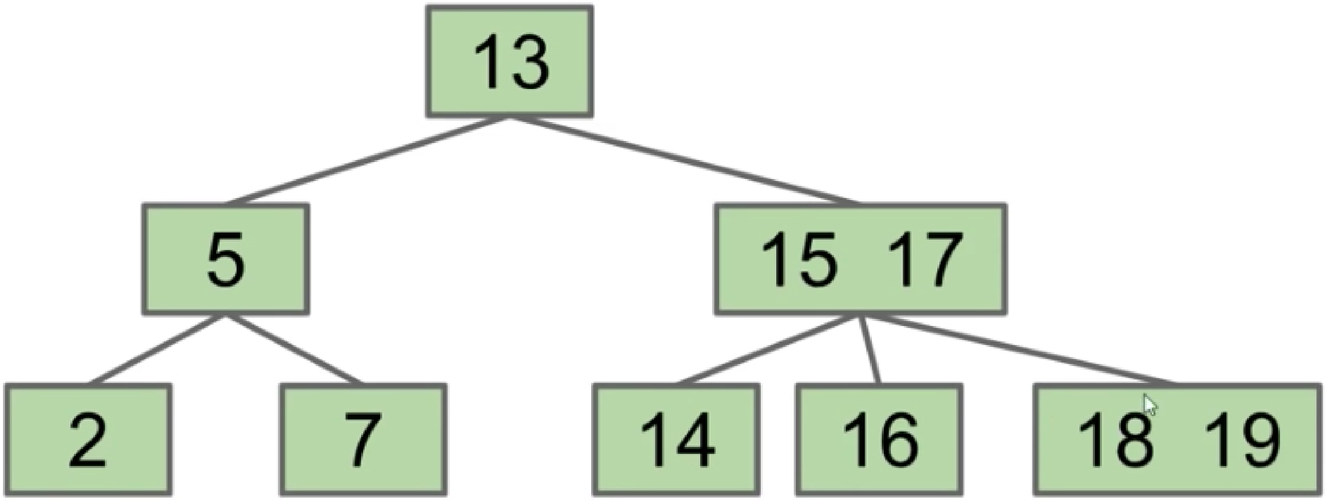
- Height is balanced, but we have a new problem
    - Leaf nodes can get too juicy
- Solution?
    - Set a limit L on the number of items, say L=3
    - If any node has more than L items, give an item to parent
        - Which one? Let's say (arbitrarily) the left-middle
- What's the problem now?
    - 16 is to the right of 17





Revising Overstuffed Tree Approach: Node Splitting

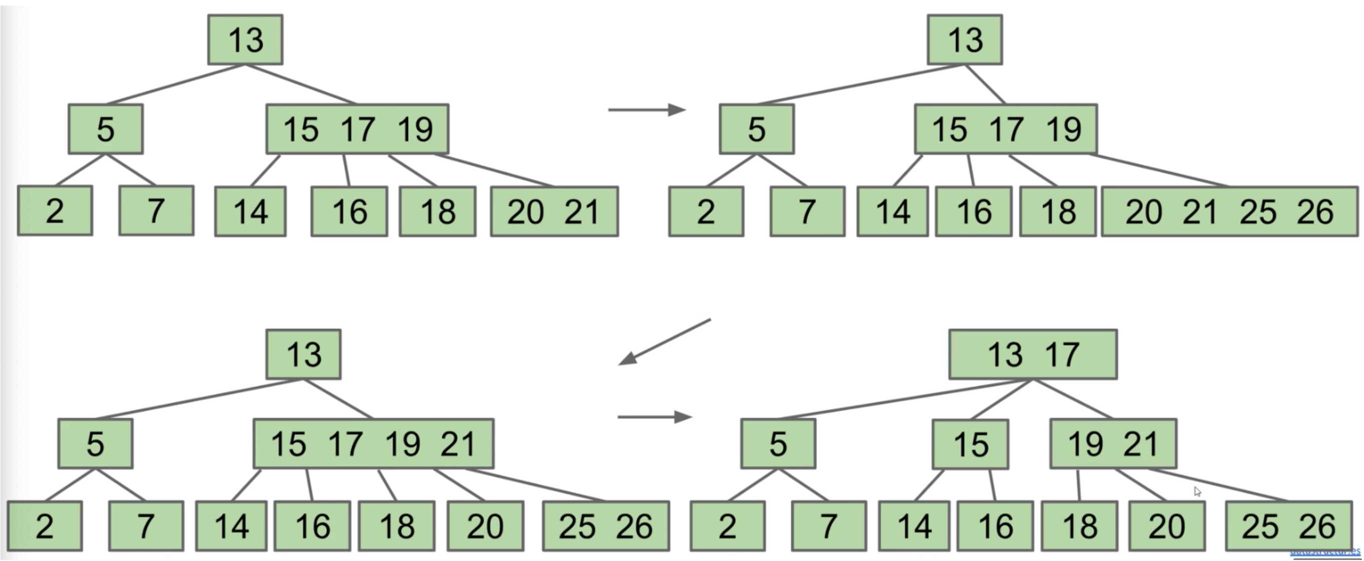
- Solution?
    - Set a limit L on the number of items, say L=3
    - If any node has more than L items, give an item to parent
        - Pulling item out of full node splits it into left and right
        - Parent node now has three children!



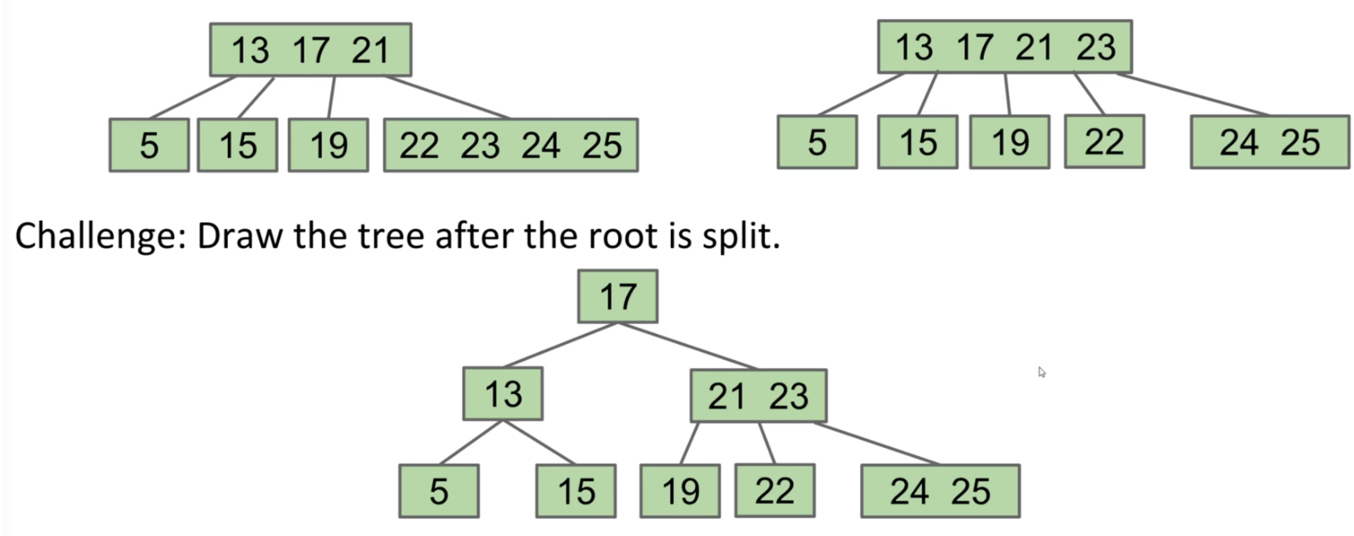
- This is a logically consistent and not so weird data structure
  - Contains(18):
    - 18 > 13, so go right
    - 18 > 15, so compare vs. 17
    - 18 > 17, so go right
- Examining a node costs us O(L) compares, but that's OK since L is constant
- What if a non=leaf node gets too full? Can we split that?

add: Chain Reaction

- Suppose we add 25, 26:



What Happens if the root is too full?



Perfect Balance

- Observation: Splitting-trees have perfect balance
  - If we split the root, every node gets pushed down by exactly one level
  - If we split a leaf or internal node, the height doesn't change
- All operations have guaranteed O(log N) time

THe Real Name for Splitting Trees is "B Trees"

- B-trees of order L=3 (like we used today) are also called a 2-3-4 tree or a 2-4 tree
  - "2-3-4" refers to the number of children that a node can have
- B-trees of order L=2 are also called a 2-3 tree
- B-Trees are most popular in two specific contexts:
  - Small L(L=2 or L=3)
    - Used as a conceptually simple balanced search tree
  - L is very large (say thousands)
    - Used in practice for databases and file systems

B-tree Bushiness Invariants

Exercise

- No matter the insertion order you choose, resulting B-Tree is always bushy!
  - May vary in height a little bit, but overall guaranteed to be bushy

B-Tree Invariants

- Because of the way B-Trees are constructed, we get two nice invariants
  - All leaves must be the same distance from the source
  - A non-leaf node with k items must have exactly k+1 children
- These invariants guarantee that our tree will be bushy

B-Tree Runtime Analysis

Height of a B-Tree with Limit L

- L: Max number of items per node
- Height: Between ~log_{L+1}(N) and ~log_2(N)
    - Largest possible height is all non-leaf nodes have 1 item
    - Smallest possible height is all nodes have L items
    - Overall height is therefore Theta(log N)

Runtime for `contains`

- Runtime or contains:
    - Worst case number of nodes to inspect: H + 1
    - Worst case number of items to inspect per node: L
    - Overall runtime: O(HL)
- Since H = Theta(log N), overall runtime is O(L log N)
    - Since L is a constant, runtime is therefore O(log N)
- Bottom line: contains and add are both O(log N)

Summary

Summary

- BSTs have best case height Theta(log N) and worst case height Theta(N)
    - Big O is not the same thing as worst case
- B-Trees are a modification of the binary search tree that avoids Theta(N) worst case
    - Nodes may contain between 1 and L items
    - `contains` works almost exactly like a normal BST
    - `add` works by adding items to existing leaf nodes
        - If nodes are too full, they split
    - Resulting tree has perfect balance. Runtime for operations is O(log N)
    - Have not discussed deletion
    - Have not discussed how splitting works if L > 3
    - B-trees are more complex, but they can efficiently handle ANY insertion order