THE j-FUNCTION

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Abstract. The j-function is a fundamental modular function with profound connections to various areas of mathematics. We begin by expressing the j-function in terms of Eisenstein series and demonstrating its role as the "simplest" non-constant modular function. We also introduce elliptic curves and their relation to lattices and make use of a theorem of complex multiplication to show the intriguing property that $j(\frac{1+\sqrt{-163}}{2})$ is an integer. Using this and the Fourier expansion of the j-function, we explain why $e^{\pi\sqrt{163}}$ and some other similar expressions are remarkably close to integers. Finally, we briefly show the rapidly converging Chudnovsky formula for π, which also relies on the j-function and complex multiplication.

This paper assumes some familiarity with abstract algebra and complex analysis. While we try to present the necessary background to appreciate the interesting results, the later sections on complex multiplication and its applications are technically demanding. We try to give key insights without delving too deeply into the extensive theoretical framework.

1. Modular Functions

We begin by introducing the modular group and the concept of modular functions, which provides the context within which the j-function operates.

Definition 1.1. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the multiplicative group of 2×2 matrices over $\mathbb{Z}$ with determinant 1:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane $\mathbb{H} = \tau \in \mathbb{C} : \Im(\tau) > 0$ such that for some matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and some $\tau \in \mathbb{H}$, we have

$$A\tau = \frac{a\tau + b}{c\tau + d}.$$

One can also check that $A\tau \in \mathbb{H}$.

To understand modular functions, we first need to introduce the concept of a lattice in the complex plane.

Definition 1.2. Given two complex numbers ω_1 and ω_2 with ratio $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, we define a lattice $L = [\omega_1, \omega_2]$ as the set of all linear combinations $m\omega_1 + n\omega_2$, where m and n are integers.

Geometrically, a lattice can be pictured as a parallelogram tiling of the complex plane with periods ω_1 and ω_2. The ratio of ω_1 and ω_2 being nonreal guarantees that the periods are $\mathbb{R}$-linearly independent.

Modular functions are functions from lattices to the complex numbers. However, we are more interested in the ratio of the lattice periods than their actual values. We define $\tau = \frac{\omega_2}{\omega_1}$,

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noting that lattices with the same ratio τ are homothetic and considered equivalent. This leads to our definition of a modular function:

Definition 1.3. A *modular function* is a meromorphic function (a ratio of two holomorphic functions) $f : \mathbb{H} \to \mathbb{C}$ invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. That is, for any matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$f(\alpha\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$$

for every $\tau \in \mathbb{H}$.

Remark 1.4. Modular functions are actually a special case of *modular forms*, which are alone quite interesting. Modular forms $f(\tau)$ of weight k satisfy $f(\frac{a\tau+b}{b\tau+d}) = (c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, so modular functions are just modular forms of weight 0.

Also closely related to modular functions are *elliptic functions* and *elliptic curves*. An elliptic function f is a meromorphic function of z such that $f(z) = f(z+\omega_1) = f(z+\omega_2)$, and elliptic curves will be introduced in Section 5, which will also lead to another way of viewing modular functions. While elliptic functions are functions on z assuming a particular lattice, or elliptic curve, modular functions can be viewed as functions on the space of lattices, or the space of elliptic curves.

However, for now, we are not concerned with elliptic curves or elliptic functions on z, but instead on functions of τ, the ratio of the pair of periods ω_1 and ω_2.

2. Eisenstein Series

Definition 2.1. For integers $n \geq 3$, the *Eisenstein series* of weight n is given by

$$G_n = \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{1}{(a+b\tau)^n}.$$

We now present a proof that Eisenstein series are absolutely convergent.

Lemma 2.2. *Einsenstein series of even weight converge absolutely.*

Proof. We will reduce this to showing the result for just $\tau = i$ with some bounding of $|a+b\tau|$. Specifically, for each $\tau \in \mathbb{H}$, we first show that there is a positive real number c_τ such that

$$c_\tau \geq \left|\frac{a+bi}{a+b\tau}\right|$$

for all integers a and b.

For $b = 0$, this is true if we use any $c_\tau \geq 1$. Since the upper half plane does not include the real axis, $a + b\tau \neq 0$. Now, consider $b \neq 0$. We have

$$\left|\frac{a+bi}{a+b\tau}\right| = \left|\frac{i+a/b}{\tau+a/b}\right|.$$

As the ratio a/b approaches $\pm\infty$, the expression $\left|\frac{i+a/b}{\tau+a/b}\right|$ approaches 1. Hence the expression is bounded for each τ by some constant c_τ.

Now, we can examine a term of the Eisenstein series, $\frac{1}{(a+b\tau)^n}$. Using our earlier bound, we have

$$\left|\frac{1}{(a+b\tau)^n}\right| \leq \frac{c_\tau^n}{|a+bi|^n} = \frac{1}{(a^2+b^2)^{n/2}}c_\tau^n.$$

All that is left is to prove that this lattice sum is convergent for $n > 2$, which we do using the integral test:

$$\begin{aligned}
G_n &\leq c_\tau^n \cdot \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{1}{(a^2+b^2)^{n/2}} \\
&\leq c_\tau^n \iint_{x^2+y^2\geq 1} \frac{1}{(\sqrt{x^2+y^2})^n}\,dx\,dy \\
&= c_\tau^n \int_{r=1}^{\infty}\int_{\theta=0}^{\theta=2\pi} \frac{1}{r^n}\cdot r\,d\theta\,dr \\
&= c_\tau^n \int_1^{\infty} \frac{2\pi}{r^{n-1}}\,dr,
\end{aligned}$$

which converges for $n > 2$.

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We now present some basic facts about Eisenstein series.

Proposition 2.3. *We have*

(i) $G_n(\tau) = 0$ *for all odd* $n \geq 3$.
(ii) $G_n(\tau+1) = G_n(\tau)$.
(iii) $G_n(-1/\tau) = \tau^k G_n(\tau)$.
(iv)

$$G_n\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_n(\tau)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Proof. (i) For odd n,

$$\frac{1}{(a+b\tau)^n} + \frac{1}{(-a-b\tau)^n} = 0.$$

If we sum this over all lattice points except $(0,0)$, we can group each term (a,b) with its inverse $(-a,-b)$. Since every term has an inverse and the sum of each pair cancels out as above, the total sum is 0.

(ii) Plugging in $\tau+1$,

$$G_n(\tau+1) = \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{1}{(a+b(\tau+1))^n} = \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{1}{((a+b)+b\tau)^n}.$$

Now, using the change of variables $c = a+b$, we have

$$G_n(\tau+1) = \sum_{\substack{(c,b)\in\mathbb{Z}^2 \\ (c,b)\neq(0,0)}} \frac{1}{(c+b\tau)^n} = G_n(\tau).$$

(iii) Plugging in $-1/\tau$,

$$G_n(-1/\tau) = \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{1}{(a+b(-1/\tau))^n} = \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{1}{(a-b/\tau)^n}$$
$$= \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{\tau^n}{(a\tau-b)^n} = \tau^n \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{1}{(b+a\tau)^n} = \tau^n G_n(\tau).$$

(iv) Notice that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A full proof of this can be found in [8]. Since these two matrices correspond to the two operations from parts b and c, we need only prove the result for those two group actions, which we have already done. Thus G_n is a modular form of weight n.

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3. The j-function

We are now ready to give the definition of the j-function and then show some interesting properties of the j-function, especially related to modular functions.

Definition 3.1. Define the *modular invariants* g_2 and g_3 to be $g_2(\tau) = 60G_4$ and $g_3(\tau) = 140G_6$.

Definition 3.2. The j-function $j : \mathbb{H} \to \mathbb{C}$ is defined by

$$j(\tau) = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = \frac{(12 g_2(\tau))^3}{\Delta(\tau)}.$$

It can be shown that the denominator $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, known as the *modular discriminant*, is nonzero for $\tau \in \mathbb{H}$. Now, we show that j is a modular function, and see how it actually classifies all modular functions.

Proposition 3.3. *The j-function is a modular function.*

Proof. It turns out that the j-function is holomorphic on the upper half plane. We will not prove this in this paper, but we recommend [1] for a proof using the uniform convergence of the modular invariants. Now, the main property of modular functions is the invariance under actions of $\mathrm{SL}_2(\mathbb{Z})$. To prove this part, we use Proposition 2.3. Consider some arbitrary $\dfrac{a\tau+b}{c\tau+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Applying our previous results,

$$j\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1728((c\tau+d)^4 g_2(\tau))^3}{((c\tau+d)^4 g_2(\tau))^3 - 27((c\tau+d)^6 g_3(\tau))^2} = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = j(\tau).$$

The modular forms of weight 12 in the numerator and denominator cancel, thus we get that the j-function is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

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Theorem 3.4. *The set of modular functions is the same as the set of rational functions of $j(\tau)$.*

Proof. This is equivalent to proving that all rational functions of $j(\tau)$ are modular functions, and all modular functions are rational functions of $j(\tau)$. However, this first part is trivial because for any function $f(\tau) = \frac{P(j(\tau))}{Q(j(\tau))}$, f is meromorphic and $f(A\tau) = f(\tau)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$.

To show that all modular functions are rational functions of $j(\tau)$, we can first manipulate some arbitrary modular function $f(\tau)$ to get rid of its poles. Notice that $j(\tau)-j(\tau_0)$ has a zero at τ_0. Multiplying this function by f at its poles with the correct multiplicity will effectively get rid of all of the finite number of poles of f in the fundamental domain. Specifically, let each pole τ_k in the fundamental domain of j have multiplicity m_k. The function

$$g(\tau) = f(\tau) \prod_k (j(\tau) - j(\tau_k))^{m_k}$$

has no poles on the fundamental domain, so it has no poles on all of $\mathbb{H}$. Thus, $g(\tau)$ is holomorphic on the upper half plane and $g(\tau)$ can be written as

$$g(\tau) = a_{-n}q^{-n} + a_{-n+1}q^{-n+1} + \cdots .$$

By Theorem 4.4, the q-expansion of $j(\tau)$ only has one negative power of q, namely q^{-1}. This means it is possible to find a polynomial $P(j(\tau))$ such that $h(\tau) = g(\tau) - P(j(\tau))$ does not have any terms in its q-expansion with nonpositive power of $q = e^{2\pi i\tau}$. Therefore, $h(i\infty) = \lim_{\Im(\tau)\to\infty} f(\tau) = 0$. It can be shown further that $h(\mathbb{H}\cup\{\infty\})$ is compact [8], which would imply by the maximum modulus principle that h is constant, so $h(\tau) = 0$ for all $\mathbb{H}\cup\{\infty\}$. Therefore, $g(\tau)$ is a polynomial in $j(\tau)$ and so $f(\tau)$ is a rational function in $j(\tau)$. ■

This theorem shows how the j-function characterizes every modular function. Also, if we restrict the functions in consideration to holomorphic modular functions, one can now also easily show that the set of holomorphic modular functions is the same as $\mathbb{C}[j]$, the set of polynomials in $j(\tau)$.

Let $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ be the quotient of the upper half plane by the action of $\mathrm{SL}_2(\mathbb{Z})$. We now introduce a mapping property of the j-function.

Definition 3.5. A *fundamental domain* of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ is an open connected subset of the upper half plane with the property that no two elements are $\mathrm{SL}_2(\mathbb{Z})$ equivalent, and every point in $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$ equivalent to some point in the closure of the fundamental domain.

Such a fundamental domain with set

$$R_\Gamma = \left\{ z \in H : |z| > 1,\ |\Re(z)| < \frac{1}{2} \right\}$$

is shown on the next page in Figure 1.

Theorem 3.6. *The j-function is a bijection from any fundamental domain of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$.*

Proof. To prove injectivity, suppose $j(\tau) = j(\tau')$. Now, we pick some λ such that

$$\lambda^4 = \frac{g_2(\tau)}{g_2'(\tau)}.$$

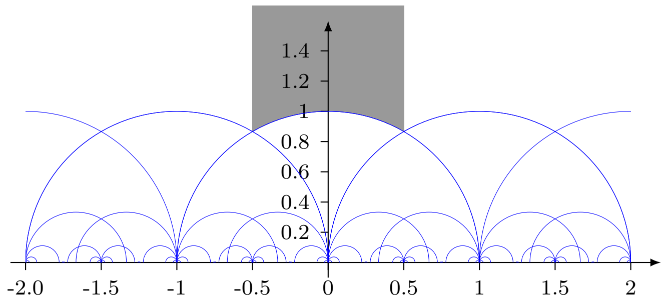

Figure 1. The canonical fundamental domain of the modular group, denoted R_Γ [1].

We have

$$\frac{1728 g_2(\tau')^3}{g_2(\tau')^3 - 27 g_3(\tau')^2} = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}.$$

Simplifying,

$$\lambda^{12} = \left(\frac{g_3(\tau)}{g_3(\tau')}\right)^2.$$

Therefore, such a λ exists so that $g_2(\tau') = \lambda^{-4} g_2(\tau)$ and $g_3(\tau') = \lambda^{-6} g_3(\tau)$, so τ and τ' must be $\mathrm{SL}_2(\mathbb{Z})$ equivalent.

We can also show that the j-function is surjective to the complex plane. Since $j(\tau)$ is holomorphic, the image $j(\mathbb{H})$ is an open set by the open mapping theorem. Now, we show that the image is also closed. Let there be some sequence $\tau_1, \tau_2, \ldots$ with each τ_k in the fundamental domain so that $j(\tau_1), j(\tau_2), \ldots$ converges to some $w \in \mathbb{C}$. We must show that w is in the image of j. First, if the imaginary parts of the τ_k's were unbounded, then there would be some subsequence of the $j(\tau_k)$'s that approached ∞ since $j(i\infty) = \infty$, which is a contradiction. Therefore, the imaginary parts of the τ_k's must be bounded above by some value A. Each of the τ_k's must lie in the compact subset of $\mathbb{H}$,

$$R'_\Gamma = \left\{ z \in H : |z| > 1,\ |\Re(z)| < \frac{1}{2}, \Im(z) \le A \right\}.$$

This means there is some subsequence of the τ_k's converging to $\tau \in R'_\Gamma$. By the continuity of j, we must have $j(\tau) = w$, and so the image of j is closed.

Since the only nonempty clopen set in $\mathbb{C}$ is $\mathbb{C}$ itself, $j(\mathbb{H}) = \mathbb{C}$, so j is bijective from $\mathbb{H}$ to $\mathbb{C}$, up to the action of $\mathrm{SL}_2(\mathbb{Z})$. Equivalently, the j-function forms a bijection between any fundamental domain of the modular group to the complex plane.

$\blacksquare$

4. Fourier expansions

Since $j(\tau + 1) = j(\tau)$, the j-function is a periodic function, so it has a Fourier series expansion. We now calculate the Fourier expansions, known as q-expansions, for the modular invariants and the j-function, as they will be helpful in showing some results later. First, we start with a useful lemma. Define $q = e^{2\pi i \tau}$.

Lemma 4.1. *We have the following Fourier series expansions:*

$$\sum_{a=-\infty}^{\infty} \frac{1}{(a+b\tau)^4} = \frac{8\pi^4}{3} \sum_{k=1}^{\infty} k^3 q^{kb}$$

$$\sum_{a=-\infty}^{\infty} \frac{1}{(a+b\tau)^6} = -\frac{8\pi^6}{15} \sum_{k=1}^{\infty} k^5 q^{kb}.$$

Proof. We start by computing decompositions for certain trigonometric expressions that may at first may seem unrelated. For $z \in \mathbb{C} \setminus \mathbb{Z}$, if we integrate $f(w) = \frac{\pi \cot(\pi w)}{(w+z)^2}$ over a large circular contour, we obtain the identity,

$$\frac{1}{\sin^2(\pi z)} = \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

Then, if we integrate this series term by term, we get

$$-\frac{1}{\pi \tan(\pi z)} = \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} -\frac{1}{z+n}.$$

Rearranging,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

Note that we write the expression in this form rather than the nicer-looking but incorrect expression,

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n},$$

to fix convergence issues. Now, again using $q = e^{2\pi i \tau}$, we have

$$\begin{aligned} \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau-n} + \frac{1}{\tau+n} \right) &= \pi \cot(\pi \tau) \\ &= \pi \frac{(e^{\pi i \tau} + e^{-\pi i \tau})/2}{(e^{\pi i \tau} - e^{-\pi i \tau})/2i} = \pi i \left(\frac{q+1}{q-1} \right) \\ &= \pi i \left(1 + \frac{2}{q-1} \right) = \pi i \left(1 - 2 \sum_{k=0}^{\infty} q^k \right) \\ &= -\pi i \left(1 + 2 \sum_{k=1}^{\infty} q^k \right). \end{aligned}$$

Differentiating with respect to τ, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = -2\pi i \sum_{k=1}^{\infty} (2\pi i k) q^k = 4\pi^2 \sum_{k=1}^{\infty} k q^k.$$

Taking higher derivatives, we get the following expressions:

$$-6 \sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^4} = -16\pi^4 \sum_{k=1}^{\infty} k^3 q^k$$

$$-120\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^6}=64\pi^6\sum_{k=1}^{\infty}k^5q^k.$$

Therefore,

$$\sum_{a=-\infty}^{\infty}\frac{1}{(a+b\tau)^4}=\frac{8\pi^4}{3}\sum_{k=1}^{\infty}k^3q^{kb}$$
$$\sum_{a=-\infty}^{\infty}\frac{1}{(a+b\tau)^6}=-\frac{8\pi^6}{15}\sum_{k=1}^{\infty}k^5q^{kb}.$$

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These are not quite the Eisenstein series, but now all that is left to compute the series expansions for them is to sum over all values of b.

Definition 4.2. Denote the *normalized Eisenstein series* E_2, E_4, and E_6 to be

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$
$$E_4(\tau)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}$$
$$E_6(\tau)=1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

Lemma 4.3. *The Fourier expansions of $g_2(\tau)$ and $g_3(\tau)$ are*

$$g_2(\tau)=\frac{4\pi^4}{3}\left(1+240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right)=\frac{4\pi^4}{3}E_4(\tau)$$
$$g_3(\tau)=\frac{8\pi^6}{27}\left(1-504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right)=\frac{8\pi^6}{27}E_6(\tau).$$

Proof. We first compute the series expansions of the general Eisenstein series. We have

$$G_n=\sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}}\frac{1}{(a+b\tau)^n}=\sum_{\substack{a\in\mathbb{Z}\\a\neq0,\,b=0}}\frac{1}{(a+b\tau)^n}+2\sum_{b=1}^{\infty}\sum_{a=-\infty}^{\infty}\frac{1}{(a+b\tau)^n}.$$

For $n=4$, this is

$$G_4=2\sum_{a=1}^{\infty}\frac{1}{a^4}+2\sum_{b=1}^{\infty}\left(\frac{8\pi^4}{3}\sum_{k=1}^{\infty}k^3q^{kb}\right)=2\zeta(4)+\frac{16\pi^4}{3}\sum_{(b,k)\in\mathbb{Z}_+^2}k^3q^{kb}.$$

To rewrite the final sum over the pairs of positive integers (b,k), we can consider each particular value of bk, which we can call m. For a given value of m, the contribution to the above sum of the pairs of positive integers (b,k) with $bk=m$ is just $\sigma_3(m)q^m$ where $\sigma_k=\sum_{d|m}d^k$. Therefore,

$$G_4=2\left(\frac{\pi^4}{90}\right)+\frac{16\pi^4}{3}\sum_{m=1}^{\infty}\sigma_3(m)q^m=\frac{\pi^4}{45}\left(1+240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right).$$

By a similar calculation,

$$G_6 = 2\zeta(6) - \frac{16\pi^6}{15} \sum_{(b,k)\in\mathbb{Z}_+^2} k^5 q^{kb} = \frac{2\pi^6}{945}\left(1 - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m\right).$$

Multiplying these two Eisenstein series by 60 and 140 respectively, we get our desired q-expansions for g_2 and g_3.

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Using the previous lemma combined with our definition of j, we can calculate the first few terms of the Fourier expansion of $j(\tau)$.

Theorem 4.4. *The Fourier expansion of j is*

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

where c_n are integer coefficients and $q = e^{2i\pi\tau}$.

Proof. We can express j in terms of the normalized Eisenstein series. Using the definitions,

$$\begin{aligned} j(\tau) &= 1728\frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)} \\ &= 1728\frac{(4\pi^4/3)^3(E_4(\tau))^3}{(4\pi^4/3)^3(E_4(\tau))^3 - 27(8\pi^6/27)^2(E_6(\tau))^2} \\ &= 1728\frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}. \end{aligned}$$

If we plug in sufficiently many terms into our known Fourier series for the Eisenstein series and simply, we get the first few coefficients of the expansion, as displayed above.

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Remark 4.5. An asymptotic formula for the coefficients c_n of this q-expansion as $n \to \infty$ was shown by Petersson [10]:

$$c_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}.$$

Again, the definition and q-expansion coefficients for the j-function seem completely arbitrary at first. However, the only other functions that have this property in Theorem 3.4 of generating all modular functions this way are the nontrivial functions of the form

$$\frac{aj(\tau) + b}{cj(\tau) + d}$$

for complex numbers a, b, c, d. If we further restrict these functions to be holomorphic on $\mathbb{H}$ and have the coefficient of q^{-1} in the Fourier series be 1, then we also get that $a = d = 1$ and $c = 0$, so such a function is uniquely determined up to the addition of a constant. This shows how the j-function is actually, in a way, the most fundamental and simple function satisfying the relations $j(\tau) = j(\tau+1)$ and $j(\tau) = j(-1/\tau)$ (up to the addition of a constant) and that there is also something very special and fundamental to modular functions about these higher coefficients of the q-expansion $196884, 21493760, \ldots$.

It turns out that these coefficients have a very surprising connection with the Monster group, the largest sporadic simple group. McKay first noticed that the coefficient of q, 196884, is 1 greater than the dimension of the lowest irreducible representation of the Monster group, 196883. After more similar apparent numerical coincidences were found and more concrete relations between the Monster group and modular functions were discovered, Conway and Norton formulated the Monstrous Moonshine conjecture [7], that connects these two areas of math. We will not discuss this further as it is a very difficult and deep connection beyond the scope of this paper, but the conjecture was eventually proven by Richard Borcherds, who won the Fields Medal for his solution.

5. Complex Multiplication and Why $e^{\pi\sqrt{163}}$ is a Near-Integer

Hermite [2] first observed the remarkable near-integer property of

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots.$$

This fact is not just a coincidence! Some other examples are

$$e^{\pi\sqrt{67}} = 147197952743.9999987\ldots$$

and

$$e^{\pi\sqrt{43}} = 884736743.9998\ldots.$$

Another unusual part about this is that all three of these examples are just slightly less than integers that end with 744, written in base 10. This number 744 also happens to appear as the constant term of the Fourier expansion for j! Indeed, we will see that this is explained by the j-function. This relies on some advanced math beyond the scope of this paper, but we will try to explain some insights for how this fact comes up.

First, we consider these these numbers $43, 67$, and 163 in the above expressions. These are the three largest *Heegner numbers*.

Definition 5.1. A *Heegner number* is a square-free number d such that the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-d})$ possesses unique prime factorization.

It has been shown that there are exactly nine Heegner numbers:

$$1, 2, 3, 7, 11, 19, 43, 67, 163.$$

The proof that the first few of these are Heegner numbers can be done fairly easily with an extension of the Euclidean algorithm. For the larger numbers like 67 and 163, proving they the unique prime factorization can be done by computing class numbers. Showing that there are no other Heegner numbers is much more difficult and a proof can be found in [12].

We will just examine the largest Heegner number, 163, in the rest of this paper, as it will provide the most interesting results. Similar corresponding properties exist for the other Heegner numbers as we saw above with $e^{\pi\sqrt{67}}$ and $e^{\pi\sqrt{43}}$.

Lemma 5.2. *We have*

$$j\left(\frac{1+\sqrt{-163}}{2}\right) = -640320^3.$$

To prove this lemma, we need to first introduce *elliptic curves* and *complex multiplication.*

Definition 5.3. An *elliptic curve* over the complex numbers $E/\mathbb{C}$ is a curve of the form $y^2 = 4x^3 - g_2x - g_3$ where $g_2, g_3 \in \mathbb{C}$ and $\Delta = g_2^3 - 27g_3^2 \neq 0$.

This definition matches our previous discussion of the j-function, and indeed we denote the j-invariant of such an elliptic curve to be

$$j(E) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} \in \mathbb{C}.$$

Normally, we would use g_2 as a function of τ, or rather of a lattice. The reason we define elliptic curves in this way and with the notation g_2 and g_3 is because they correspond directly with lattices through the *Weierstrass $\wp$-function*.

Definition 5.4. We define the *Weierstrass $\wp$-function* to be

$$\wp(z;L) = \frac{1}{z^2} + \sum_{w\in L\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right).$$

We will refer to this function as just $\wp(z)$ when L is being held constant.

Proposition 5.5. *The $\wp$-function satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(L)\wp(z) - g_3(L).$$

Proof. The proof of this theorem just involves computing the Laurent expansion of $\wp$, and then finding a linear combination of these powers of $\wp$ and $\wp'$ that gets rid of the pole at 0. This yields an entire elliptic function, which is constant by Liouville's theorem. The details of the proof can be found in [8]. ■

This differential equation for some lattice L gives us the solutions $(\wp(z), \wp'(z))$ for $z \in \mathbb{C}$ of an elliptic curve $y^2 = 4x^3 - g_2(L)x - g_3(L)$. Note that g_2 and g_3 in this expression have argument L. Therefore, every lattice L can represent a unique elliptic curve in this way, through this mapping using its Weierstrass $\wp$-function. It even turns out that this is a bijection, and every elliptic curve over $\mathbb{C}$ corresponds to a unique lattice, which we will see in the following theorem.

Theorem 5.6 (Uniformization Theorem)**.** *Every elliptic curve $E/\mathbb{C} : y^2 = 4x^3 - g_2x - g_3$ has a unique lattice L such that $g_2 = g_2(L)$ and $g_3 = g_3(L)$.*

Proof. This follows from Theorem 3.5. Consider an arbitrary elliptic curve $y^2 = 4x^3 - g_2x - g_3$. Because the j-function is surjective to the complex plane and $g_2^3 - 27g_3^2 \neq 0$, there must exist some $\tau_L \in \mathbb{H}$ such that

$$j(\tau_L) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = j(E).$$

Therefore, due to the same argument as that in the proof of Theorem 3.5, there must be some λ such that $g_2 = \lambda^{-4}g_2(\tau_L)$ and $g_3 = \lambda^{-6}g_3(\tau_L)$. Thus E corresponds to the lattice $L = \lambda[1, \tau_L]$ since $g_2(L) = g_2$ and $g_3(L) = g_3$. The uniqueness of L follows from the fact that $j(L) = j(L')$ iff L and L' are homothetic. ■

Definition 5.7. An elliptic curve over $\mathbb{C}$ is said to have *complex multiplication* if its *endomorphism ring*,

$$\mathrm{End}(E) = \{\alpha \in \mathbb{C} : \alpha L \subset L\},$$

is larger than the integers.

Since $\mathbb{Z} \subset \mathrm{End}(E)$, this condition is equivalent to the condition, $\mathbb{Z} \neq \mathrm{End}(E)$. The term complex multiplication comes from the idea of the lattice being rotated by multiplication of some nonreal endomorphism $\alpha \notin \mathbb{R}$.

Theorem 5.8. *For an elliptic curve $E/\mathbb{C}$ with complex multiplication, $j(E)$ is an algebraic integer. Furthermore, the degree of the algebraic integer is the same as the class number of the elliptic curve's corresponding quadratic imaginary field.*

This is a difficult theorem connecting the j-function, complex multiplication, and class field theory that we will state without proof. If the reader is interested, multiple proofs can be found in [11]. The key insight is that $j(\tau)$ is invariant under the action of the ideal class group of K. This is because different ideals in the same ideal class correspond to isomorphic elliptic curves. This would then give us Lemma 5.2.

Proof of Lemma 5.2. Consider the quadratic imaginary field $K = \mathbb{Q}(\sqrt{-163})$. This field has class number 1, so its ring of integers $\mathcal{O}_K$ is such that $j(\mathcal{O}_K)$ is an algebraic integer of degree 1, or in other words, an integer. The corresponding lattice here is $\left[1, \frac{1+\sqrt{-163}}{2}\right]$, so expressing this in terms of τ, we have

$$j\left(\frac{1+\sqrt{-163}}{2}\right) \in \mathbb{Z}.$$

All that is left is to compute the value it is equal to. We can easily do this computation by calculating $j\left(\frac{1+\sqrt{-163}}{2}\right)$ with sufficiently many terms of the q-expansion so that the error is less than ± 0.5. Doing this, we find that the precise integer value is $-640320^3 = -262537412640768000$. ■

We now plug in $\tau = \frac{1+\sqrt{-163}}{2}$ into Theorem 4.4, the q-expansion for $j(\tau)$. First calculating q, we get

$$q = e^{2i\pi\tau} = e^{i\pi(1+i\sqrt{163})} = -e^{-\pi\sqrt{163}}.$$

Therefore,

$$\begin{aligned} j\left(\frac{1+\sqrt{-163}}{2}\right) &= \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \\ &= -e^{\pi\sqrt{163}} + 744 - O(e^{-\pi\sqrt{163}}) \\ &= -640320^3. \end{aligned}$$

We now see why this value $j(\frac{1+\sqrt{-163}}{2})$ explains the near-integerness of $e^{\pi\sqrt{163}}$. All of the terms in the q-expansion after 744 are very small due to the slow, sub-exponential asymptotic growth of the coefficients, as seen in Remark 4.5. The error is approximately $196884e^{-\pi\sqrt{163}} \approx 7.5 \times 10^{-13}$, thus we have

$$e^{\pi\sqrt{163}} \approx 640320^3 + 744 - 7.5 \times 10^{-13}.$$

6. The Chudnovsky Formula

We now turn our focus to the Chudnovsky formula [6] for π, which is yet another seemingly mysterious and remarkable place where the j-function and 163 show up. While many of the insights needed to understand this are, again, well out of the scope of this expository paper, we'll try to show roughly how the giant constants come up in this formula from the Eisenstein series calculated in section 2.

Theorem 6.1 (Chudnovsky formula)**.** *The constant $1/\pi$ is given by the infinite series*

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{12(-1)^n(6n)!(13591409+545140134n)}{(3n)!(n!)^3(640320^{3n+3/2})}.$$

We have the identities

$$g_2(\tau)=\frac{4}{3}\pi^4E_4(\tau)$$

and

$$g_3(\tau)=\frac{8}{27}\pi^6E_6(\tau).$$

Also define

$$s_2(\tau)=\frac{E_4(\tau)}{E_6(\tau)}\left(E_2(\tau)-\frac{3}{\pi\Im(\tau)}\right).$$

In 1988, the brothers David and Gregory Chudnovsky first published the following identity.

Lemma 6.2. *We have*

$$\frac{1}{2\pi\Im(\tau)}\sqrt{\frac{j(\tau)}{j(\tau)-1728}}=\sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{6n+1-s_2(\tau)}{6j(\tau)^n}.$$

Proof. The proof of this is quite extensive, using the Picard Fuchs differential equation and Clausen's formula for hypergeometric series. We refer the reader to [5]. ∎

Now, to build the most rapid converging series, we can use our special value of τ,

$$\tau=\frac{1+\sqrt{-163}}{2}.$$

By Lemma 5.2, we have $j(\tau)=-640320^3$. Furthermore, it can be shown that

$$\frac{1-s_2(\tau)}{6}=\frac{13591409}{545140134}.$$

A detailed proof of this, again using complex multiplication, can be found in [9]. The number in the denominator comes from the fact that $163(1728+640320^3)=12^2\cdot 545140134^2$.

Proof of Theorem 6.1. Plugging in our values for $j(\tau)$ and $s_2(\tau)$ into the identity, we get

$$\frac{1}{2\pi\Im(\tau)}\sqrt{\frac{j(\tau)}{j(\tau)-1728}}=\sum_{n=0}^{\infty}\left(\frac{(6n)!}{(3n)!(n!)^3}\right)\left(\frac{6n+1-s_2(\tau)}{6j(\tau)^n}\right)$$

$$=\frac{1}{2\pi(\sqrt{163}/2)}\sqrt{\frac{640320^3}{640320^3+1728}}=\sum_{n=0}^{\infty}\left(\frac{(6n)!}{(3n)!(n!)^3}\right)\left(\frac{\frac{13591409}{545140134}+n}{(-640320^3)^n}\right)$$

$$=\frac{(640320)^{3/2}}{12\cdot\pi\cdot 54514013}=\sum_{n=0}^{\infty}\left(\frac{(6n)!}{(3n)!(n!)^3}\right)\left(\frac{\frac{13591409}{545140134}+n}{(-640320)^{3n}}\right)$$

Simplifying,

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\frac{12(-1)^n(6n)!(13591409+545140134n)}{(3n)!(n!)^3(640320^{3n+3/2})}.$$

∎

While we leave out a lot of the details coming from class field theory, hopefully, this gave some understanding of how the j-function and these large constants come up. This formula for π turns out to be extremely rapidly converging compared to other known infinite series or product expansions for π. As of 2021, it has remained the basis for the most efficient algorithm used to compute π for over a decade, holding the current record of 50 trillion digits [13].

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