# 1 Introduction

One common approach to controlling robotic manipulators (known colloquially as "robot arms") is to pick a pose of the end effector and reverse-engineer the joint angles required to reach this pose. This is known as the **inverse kinematics** problem. For an arm with $n$ rotating (also known as revolute) joints, one can model the desired end effector position and orientation by $x \in \mathbb{R}^3 \times SO(3)$ and the angles of the joints by $\theta \in \mathbb{T}^n$ [1]. The inverse kinematics problem requires solving the equation $x = f(\theta)$ for $\theta$, where $f$ is a function arising from the physics of the system [2].

Unfortunately, there always exist joint angles where the relationship $x = f(\theta)$ breaks down. These points are called **singular configurations**. If the dimension of the workspace is $k > 0$, the singular configurations are characterized by the set

$$S = \left\{ \theta_0 \in \mathbb{T}^n \mid \operatorname{rank} \left\{ \frac{\partial f}{\partial \theta}(\theta_0) \right\} < k \right\}$$

**Example 1.** The 2D planar manipulator with two revolute joints of fixed lengths $r_1 > 0$ and $r_2 > 0$ (Figure 1), with end-effector orientation equal to that of the second joint, has position $(x, y) \in \mathbb{R}^2$ and joint angles $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$. The relation connecting these two is given by (1).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \cos(\theta_1) + r_2 \cos(\theta_1 + \theta_2) \\ r_1 \sin(\theta_1) + r_2 \sin(\theta_1 + \theta_2) \end{bmatrix} =: f(\theta) \tag{1}$$

The Jacobian has determinant

$$\left| \frac{\partial f}{\partial \theta} \right| = r_1 r_2 \sin(\theta_2)$$

which means the singular configurations are $S = \{(\theta_1, \theta_2) \in \mathbb{T}^2 \mid \theta_2 = \pm\pi\}$; that is, the singular configurations of this robot arm are exactly when the second joint is colinear with the first joint.

Singularities cause various issues when controlling robot arms, as the controllers will attempt to apply infinite torque to move the arm a small amount. For this reason, it is preferable to avoid singular configurations when possible. In particular, when generating a trajectory for the end-effector, a good control mechanism should avoid choosing joint positions which land "close to" the singular configurations.

Previous researchers have tried to solve this problem using extra limbs [2] and velocity constraints [3], among other approaches involving kinematic models of the robots. The authors of "Topology and the Robot Arm" offer a different approach: they ask whether avoiding kinematic singularities is possible at all, and prove when it can be done using the topology of the robot's configuration space [4].

This report will cover the relevant background required to understand "Topology and the Robot Arm", and will summarize the results of the paper.

# 2 Relevant Background

This section covers definitions, notation, and results that are required to fully comprehend [4]. It is assumed that anyone reading this report has a foundational understanding of topology, as taught in the University of Toronto's MAT327 course.

**Definition 1** (Fiber bundles [5])**.** Let $E$, $X$, and $F \subset E$ be topological spaces, with $X$ connected. Let $f : E \to X$ be a continous surjective function. We say $f$ is a **locally trivial fibration** or a **fiber bundle with fiber** $F$ and write $F \to E \xrightarrow{f} X$ when

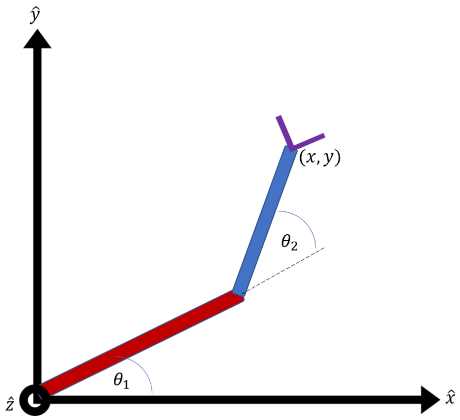


Figure 1: A 2D planar robot arm with fixed lengths and a non-rotating end-effector.

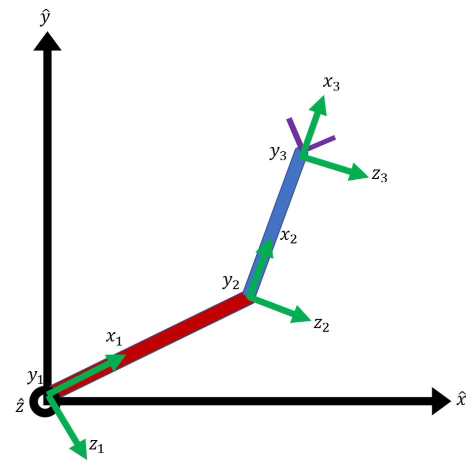


Figure 2: The planar robot arm with coordinate frames attached to each pivot $y_i$.

1. $f^{-1}(\{x_0\}) = F \, \forall x_0 \in X$
2. Around each $x \in X$ there is an open neighbourhood $U_x \subset X$ and a homeomorphism $\psi_x : f^{-1}(U_x) \to U_x \times F$ so that $f|_{f^{-1}(U_x)} = p \circ \psi_x$ (where $p$ is the projection of $U_x \times F$ onto $U_x$)

Note that it is common to say that $E$ itself is the fiber bundle over $X$ if $E = X \times F$, since the natural projection $\pi : X \times F \to X$ is a fiber bundle.

**Definition 2** (Vector bundles [5]). Let $V \to E \xrightarrow{f} E$ be a fiber bundle and $V$ an $n$-dimensional vector space. We say $f$ is a **vector bundle** if $\psi_x$ satisfies $\psi_x|_{x_1} : f^{-1}(x_1) \to x_1 \times V$ is a linear isomorphism for any $x_1 \in U_x$.

**Definition 3** (Cross-sections [5]). Let $F \to E \xrightarrow{f} X$ be a fiber bundle. A **cross-section** is a continuous map $\sigma : X \to E$ such that $f \circ \sigma = \mathrm{id}_X$.

**Definition 4** (Bundle maps [5]). Let $F_1 \to E_1 \xrightarrow{f_1} X_1$ and $F_2 \to E_2 \xrightarrow{f_2} X_2$ be fiber bundles. A continuous function $\phi : E_1 \to E_2$ is a **bundle map** if there is a continuous function $g : X_1 \to X_2$ so that $g \circ f_1 = f_2 \circ \phi$.

**Definition 5** (Pullback [5]). Let $F \to E \xrightarrow{f} X$ be a fiber bundle. Let $Y$ be a topological space and $g : Y \to X$ a continuous function. The **pullback bundle over** $Y$ is $g^\star(E) := \{(y, e) \in Y \times E \mid g(y) = f(e)\}$. The natural projection $\pi_Y : g^\star(E) \to Y$ with $(y, e) \mapsto y$ is a fiber bundle with fiber $F$.

**Definition 6** (Manifolds [6]). A topological space $M$ is a manifold of dimension $n$ if, around any $p \in M$, there exists an open neighbourhood $U \subset M$ which is homeomorphic to an open subset of $\mathbb{R}^n$.

**Definition 7** (Tangent Bundles [7]). Let $M$ be a manifold and $m \in M$. The set $T_mM$ consisting of tangents of curves at $m$ is called the **tangent space** of $M$ at $m$. The set $TM := \{(m, v) | m \in M, \, v \in T_mM\}$ is called the **tangent bundle** of $M$.

**Example 2.** A mechanical system can be modelled by a configuration manifold $\mathcal{Q} = \mathbb{R}^n \times (\mathbb{S}^1)^m$. At each $q \in \mathcal{Q}$, the velocity of the system lies in $T_q\mathcal{Q}$, which is an $n+m$-dimensional vector space. The tangent bundle $T\mathcal{Q} := \{(q, v) \mid q \in \mathcal{Q}, v \in T_q\mathcal{Q}\}$ has the natural projection $\pi : T\mathcal{Q} \to \mathcal{Q}$ given by $\pi(q, v) = q$. This is a vector bundle, because the fibers $\pi^{-1}(q) = T_q\mathcal{Q}$ are isomorphic to $q \times \mathbb{R}^{n+m}$.

**Definition 8** (Submersions [1]). Let $M$ and $N$ be manifolds. A map $h : M \to N$ is a **submersion** if it contains no singular points (that is, its Jacobian is always full rank).

**Definition 9** (Groups [6]). A **group** is a set $G$ and an operation $G \times G \to G$ mapping $(g, h) \mapsto gh$, along with the following axioms:

1. For all $g, h, k \in G$, $(gh)k = g(hk)$.
2. There exists an identity $e \in G$ so that for all $g \in G$ we have $eg = ge = g$.
3. For all $g \in G$ there is an inverse $h \in G$ so that $gh = e$

If additionally the group satisfies $gh = hg$ for all $g, h \in G$, the group is called **abelian**.

**Definition 10** (Group Homomorphism [6]). Let $G$ and $H$ be groups. A function $f : G \to H$ is a **homomorphism** if $f(g_1g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

**Definition 11** (Group Torsion [6]). An abelian group $G$ is a **torsion group** if for each $g \in G$, there exists $n \in \mathbb{N}$ so that $g^n = e$. If this is not the case, $G$ is said to be **torsion-free**.

**Definition 12** (Quotient Group [6]). Let $g \in G$ and $H \subset G$. Define $g \equiv g' (\text{mod } H)$ if and only if $g^{-1}g' \in H$. The set of equivalence classes mod $H$ is denoted $G/H$.

**Definition 13** (Group Commutator [6]). Let $G$ be a group. The **commutator subgroup**, denoted $[G, G]$, is the subgroup of $G$ generated by the elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$

**Definition 14** (Exact Sequence [6]). A sequence of abelian groups $\{G_1, G_2, \ldots\}$ and homomorphisms $\alpha_p : G_p \to G_{p-1}$

$$\cdots \to G_{p+1} \xrightarrow{\alpha_{p+1}} G_p \xrightarrow{\alpha_p} G_{p-1} \to \cdots$$

is **exact** if $\text{Image}(\alpha_{p+1}) = \text{Ker}(\alpha_p)$ for all $p$.

**Proposition 1** (Homotopy Groups [5]). *The homotopy group $\pi_n(X, x_0)$ of $n$-loops at $x_0$ is the set of equivalence classes of maps from $\mathbb{S}^n$ to $X$, along with the homotopy group operation (see [6]).*

**Notation.** The notation $g : (Y, C, y_0) \to (X, A, x_0)$ (used in [5]) means that $g : Y \to X$, $g(C) = A$, $y_0 \in C$ and $g(y_0) = x_0 \in A$.

**Notation.** Let $g : (D^n, \mathbb{S}^{n-1}, t_0) \to (X, A, x_0)$. The restriction of $g$ to the sphere $\mathbb{S}^{n-1}$ is denoted by $\partial g : (\mathbb{S}^{n-1}, t_0) \to (A, t_0)$ [5].

**Proposition 2** (Boundary Homomorphism [5]). *The function $\partial g : (\mathbb{S}^{n-1}, t_0) \to (A, t_0)$ defines a homomorphism $\partial_* : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$.*

**Definition 15** (Relative Homotopy Group [5]). The relative homotopy group $\pi_n(X, A, x_0)$ is the set of equivalence classes of the maps $g : (D^n, \mathbb{S}^{n-1}, t_0) \to (X, A, x_0)$ where $D^n$ is the $n$-disk.

**Definition 16** (Homotopy-Exact sequence [5])**.** Let $F \to E \xrightarrow{f} X$ be a locally trivial fibration and suppose $\iota : F \to E$ is the inclusion map of the fiber. The **homotopy-exact sequence** of this fibration is the exact sequence

$$\cdots \to \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{f_*} \pi_n(X) \xrightarrow{\partial_*} \pi_{n-1}(F) \to \cdots$$

**Definition 17** (Homology [5])**.** The **first homology group** $H_1(X)$ of a space $X$ the set characterized by the abelianization of the fundamental group $\pi_1(X)$:

$$H_1(X) \equiv \pi_1(X)/[\pi_1, \pi_1]$$

**Notation.** The notation $f : X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in [4] means that $f = \beta \circ \alpha$ where $\alpha : X \to Y$ and $\beta : Y \to Z$.

**Theorem 1** (Ehresmann's Theorem [8])**.** *Let $M$ and $N$ be smooth manifolds, with $M$ a compact. Let $f : M \to N$ be a surjective submersion. Then $f$ is a locally trivial fibration.*

# 3 Topology and the Robot Arm

This section covers a summary of [4]. It does not cover all proofs of the results from this paper. When possible, we will try to motivate why the results are true by using concrete examples.

## 3.1 The Global Inverse Kinematics Problem

At the start of the paper, the authors note that a robot arm can be represented by a series of links. Let $l_1$ be the first link in the robot arm, which is attached to the base and can rotate about the fixed line $y_1$ in 3D space. Then, attach a new link $l_2$ to the end of $l_1$ and pick a line $y_2$ around which $l_2$ will rotate. Building this up inductively, one creates a robot arm so that $y_{i+1}$ rotates about $y_i$. The orientation of the end-effector is described by a coordinate frame attached to the end of the link $l_n$, whose $x$-axis is colinear with $y_n$.

**Example 3.** The 2D planar robot from Figure 1 has $y_1 = y_2 = \hat{z}$.

Since the end-effector's pose can be represented by a point in $\mathbb{R}^3 \times SO(3)$, the authors define the "rotation map" $R : \mathbb{T}^n \to SO(3)$ as follows: given $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$, the map $R(\theta)$ rotates $l_i$ about $y_i$ by the angle $\theta_i$ and returns the orientation of the end-effector (ignoring its position in 3D-space). Here it is assumed that rods cannot collide with each other, which is a reasonable assumption as many robot arms are designed so that each limb can rotate completely without breaking the system.

Now we arrive at the first theorem of the paper, which characterizes the singular configurations of the map $R$ in terms of the physical representation of the axes.

**Theorem 2.** *A point $\theta \in \mathbb{T}^n$ is a singular configuration for $R$ if and only if the axes $\{y_1, \ldots, y_n\}$ are parallel to a plane.*

**Corollary.** *The set of singularities $S \subset \mathbb{T}^n$ is 2-dimensional.*

The phrasing of Theorem 2 is somewhat confusing when we look at the 2D planar robot: any configuration has the axes in some plane since $y_1$ and $y_2$ are connected by the link $l_1$. In this case, we need to add an additional "axis" (which cannot rotate) labelled $y_3$ at the end of the arm. If these three lines $y_1$, $y_2$, and $y_3$ are all in the same plane, the robot must have $\theta_2 = \pm\pi$ and it must be in a singular configuration. This is shown in Figure 2, where the lines $y_i$ are perpendicular to the green coordinate frames. To see that the set of singular configurations is 2D, observe that rotating by $y_1$ keeps the axes in a plane, as would any hypothetical rotation of the end-effector about $y_3$.

Next, the authors make the claim that we can create a homotopy of robot arms. Letting $\phi_i$ be the angle between $y_{i+1}$ and $y_i$, they construct the homotopy by shrinking each $\phi_i$ until we get an $n$-link planar robot arm. In this case, the coordinate frame on the end-effector has an $x$-axis which is fixed to be colinear with $y_1$ (as is the case with our 2-link planar arm). This gives us the homotopy $R \sim R_1$, where $R_1$ rotates the end-effector rotates about this fixed $x$-axis when any link is rotated. The amount of rotation is $\theta' = \theta_1 + \cdots + \theta_n \in \mathbb{S}^1$. This addition of angles is the group operation on $\mathbb{S}^1$, so we represent this by a map $\mu : \mathbb{T}^n \to \mathbb{S}^1$.

Letting $\alpha : \mathbb{S}^1 \to SO(3)$ be the rotation map around the $x$-axis, we can now state the next result.

**Theorem 3.** *A robot arm map $R : \mathbb{T}^n \to SO(3)$ is homotopic to the composition $\mathbb{T}^n \xrightarrow{\mu} \mathbb{S}^1 \xrightarrow{\alpha} SO(3)$ where $\mu$ is the group operation on $\mathbb{S}^1$ and $\alpha$ is a group generator for $\pi_1(SO(3))$.*

That $R$ is homotopic to the composition and $\mu$ is the group operation on $\mathbb{S}^1$ comes from the previous discussion. The proof that $\alpha$ is the group generator for the fundamental group of $SO(3)$, however, is more obtuse, and I do not understand the proof enough to explain it in this report.

Let now $w : SO(3) \to \mathbb{S}^2$ be the map which takes a rotated coordinate frame and returns its $x$-axis (in the sense that the tip of the $x$-axis vector lies along a sphere). Since $R$ is homotopic to $R_1$, $w \circ R : \mathbb{T}^n \to \mathbb{S}^2$ is homotopic to $w \circ R_1$, which is constant because the $x$-axis of $R_1$ is constant. The authors claim $w$ creates a fiber bundle $\mathbb{S}^1 \to SO(3) \xrightarrow{w} \mathbb{S}^2$, which is a fact we will take for granted. This leads us to one of the most important results in this paper.

**Corollary.** *There is no continuous cross-section to $R$ or to $w \circ R$.*

*Proof.* In this context, the cross-section is a continuous map $s : SO(3) \to \mathbb{T}^n$ so that $R \circ s = \text{id}$. Looking at the fundamental groups, existence of a cross-section would imply that $R_* \circ s_* = \text{id}_{\pi_1(SO(3))}$. This is impossible, since $\pi_1(\mathbb{T}^n)$ is torsion-free, while $\pi_1(SO(3))$ is the cyclyc group of order 2 (which has torsion). If $w \circ R$ had a cross-section $s'$, then $\text{id}_{\mathbb{S}^2}$ would be homotopic to a constant (which it is not). □

Finding a cross-section to $R$ is equivalent to finding a continuous map which gives joint angle $\theta \in \mathbb{T}^n$ for any end-effector orientation. In other words, this corollary tells us there is no global, continuous solution to the inverse kinematics problem.

## 3.2 Avoiding Singularities

The rest of the paper focuses on solving the inverse kinematics problem while avoiding singularities. The authors suggest finding some set $D$ and a map $\hat{f} : D \to \mathbb{T}^n - S$ so that $R \circ \hat{f}$ has no singularities.

**Proposition 3.** *If $D$ is a closed manifold, $R \circ \hat{f}$ will have singularities.*

*Proof.* Suppose $R \circ \hat{f}$ has no singularities. Then neither will $w \circ R \circ \hat{f} : D \to \mathbb{S}^2$, since $w$ is a projection map of a fiber bundle. By definition, this means $w \circ R \circ \hat{f}$ is a submersion. By Ehresmann's theorem, since $D$ and $\mathbb{S}^2$ are closed manifolds, $w \circ R \circ \hat{f}$ is a locally trivial fibration with some fiber $F$. Let $\Omega\mathbb{S}^2$ be the set of all loops based in X (the "loop space"). The authors claim $F$ must be a finite-dimensional manifold homotopic to $D \times \Omega\mathbb{S}^2$ because $w \circ R \circ \hat{f}$ is homotopic to the constant map $w \circ R_1 \circ \hat{f}$. I was unable to find a source on this claim, so we will take it as fact. Since homology groups are preserved under homotopy [6], and the homology group of finite-dimensional manifolds is finite dimensional, the homology of $D \times \Omega\mathbb{S}^2$ must also be finite dimensional. The authors claim this is not true (I could not find a good source proving this claim), giving a contradiction. □

Finally, let $E = TSO(3)$ be the (trivial) tangent bundle of $SO(3)$. Let $D = T(\mathbb{T}^n - S)$ be the tangent bundle of $\mathbb{T}^n - S$. Let $R^*E$ be the pullback bundle of $R|_{\mathbb{T}^n - S}$ onto $\mathbb{T}^n - S$. Then $D$ is the set of position-velocity pairs $(\theta, v_\theta)$ in joint space, and $R^*E$ is the set of pairs $(\theta, v_{R(\theta)})$ where $v_{R(\theta)} \in T_{R(\theta)}SO(3)$ is a velocity at the current orientation. Given a starting orientation $R(\theta_0)$, the existence of a cross-section $s : R^*E \to D$ would find the velocity in joint space required to produce a desired velocity and position in the orientation (all while avoiding singular configurations). By the results of [1], this cross-section exists, so the inverse kinematics problem can be solved.

# 4 Conclusion

This report compiles the relevant bacground information to understand the paper "Topology and the Robot Arm" [4] and summarizes the results of that paper. We attempted to explain the proofs, but many were beyond the level of comprehension attained by a student of MAT327. To summarize completely, the paper proves it is only possible to solve the inverse kinematics problem by omitting singular configurations from the joint space of the robot arm.

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